Assignment 2

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Introduction

In this assignment we present a mortality projection model for Belgian males and females. We download the mortality data from the Human Mortality Database[1] and focus on Belgian mortality data. We download the number of deaths and exposures (period data) for both males and females. All our calculations are made in `R`.

In the first question we elaborate an elementary descriptive analysis of the differences between mortality statistics for males and females. In the second question we build a Lee-Carter (Carter and Lee, 1992) mortality projection model for males and females separately. In the third question we implement a Lee-Carter mortality projection model for males and females jointly, the so-called Li-lee model (Li and Lee, 2005). In the final question we thoroughly discuss our preferred strategy. We present two criteria or tools to compare the approaches from question 2 and 3.

Question 1

We start with a short descriptive analysis of the differences between mortality statistics for males and females.[2] Figure 1 illustrates the central death rate μ_x (log scale) for Belgian males and females between 1960-2010. We observe that the death rate μ_x for males is obviously higher than for females which indicates that men have more chance to die compared to women. Death rates μ_x have increased from 1960 to 2010 for both males and females. Moreover, we notice that infant mortality has significantly decreased over time. Remark that for men, the accident hump on their early twenties has decreased much more over the years compared to other ages. We also notice that the older people get, the more the chance to die for men and women converges.

It is possible that the death rate is zero for certain ages in our data set. Because we use the log scale this possibly creates a problem. Therefore we replace the zeros by 0.1 in our algorithm to avoid this problem.

[1] www.mortality.org
[2] We use the files 'mltper 1x1 BE.txt' and 'fltper 1x1 BE.txt' for our calculations.

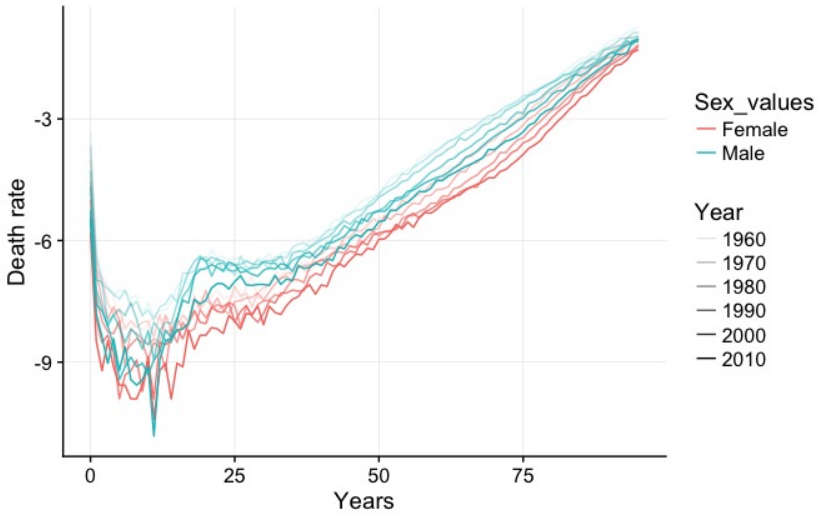

Figure 1: The central death rate (log scale) for Belgian males (red) and females (red) over the age range 0 to 101 for the set of years 1960,1970,1980,1990,2000 and 2010

Question 2

Calibration

We calibrate the parameters of the Lee Carter Model (A_x, B_x and K_t) using Maximum Likelihood Estimation (MLE). [3] We use the piecewise constant assumptions to calculate the force of mortality μ_x which entails that we can use the 'central death rate', $m_{x,t}$ for the force of mortality. We define $m_{x,t}$ as follows:

$$m_{x,t} = \frac{d_{x,t}}{E_{x,t}}. \tag{2}$$

Following the seminal paper by Brouhns et al. (2002) we assume a Poisson distribution for the number of deaths random variable $D_{x,t}$, with mean $E_{x,t} \cdot \mu_{x,t}$ where $E_{x,t}$ is the observed exposure-to-risk. Observed deaths to calibrate the Belgian trend, $d_{x,t}^{BE}$ and corresponding exposures-to-risk, $E_{x,t}^{BE}$ are obtained from the Human Mortality Database. We aggregate deaths and exposures using $\mathcal{X} = \{0, ..., 101\}$ and $\mathcal{T} = \{1960, ..., 2015\}$. The parameters A_x, B_x and K_t are determined by maximizing the Poisson likelihood function for the observed deaths with the given exposures:

$$\max_{A_x, B_x, K_t} = \prod_{x \in \mathcal{X}} \prod_{t \in T} (E_{x,t}^{BE} \cdot \mu_{x,t}^{BE})^{d_{x,t}^{BE}} \cdot \frac{\exp\left(-E_{x,t}^{BE} \cdot \mu_{x,t}^{BE}\right)}{d_{x,t}^{BE}!} \tag{3}$$

with $\mu_{x,t}^{BE} = \exp(A_x + B_x K_t)$. Normalisation is required to derive a unique specification of the three vectors. Therefore, we impose the usual Carter and Lee (1992) restrictions:

[3] We use the files Deaths 1x1 BE.txt and Exposures 1x1 BE.txt for out calculations.

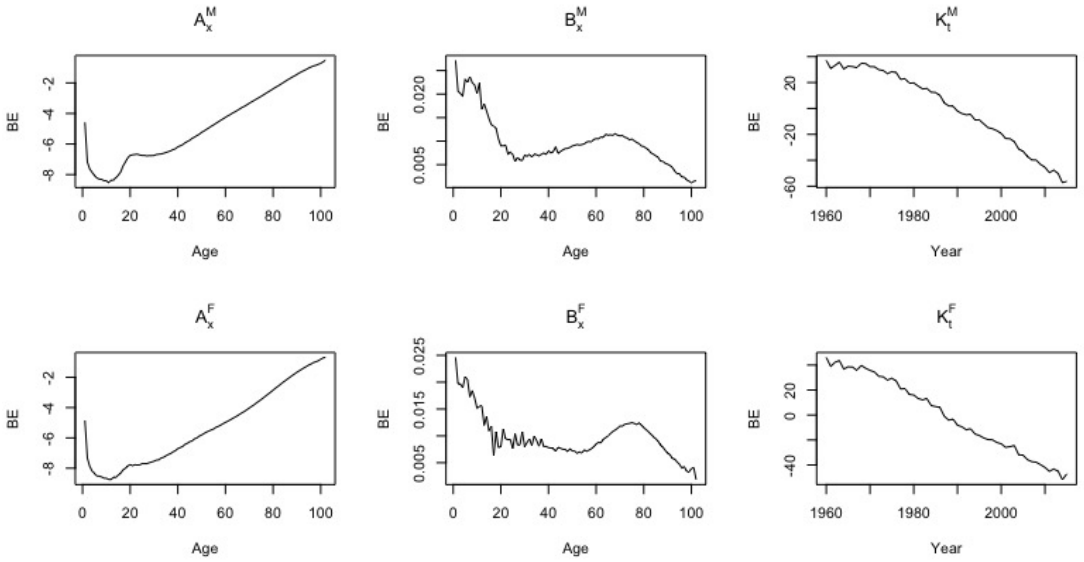

Figure 2: Results of the calibration process: parameters estimates for the Lee-Carter Model (A_x, B_x, K_t) for the Belgian males (top row) and females (bottom row). Data: Belgian data, ages 0-90, years 1960-2015.

$$\sum_{t \in \mathcal{T}} K_t = 0 \quad \text{and} \quad \sum_{x \in \mathcal{X}} B_x = 1. \tag{4}$$

We apply these constraints immediately in the iterative procedure after every update of both K_t and B_x using univariate NR steps (Pitacco et al., 2009) to obtain the MLEs. We visualize the resulting male (top row) and female (bottom row) parameters estimates in Figure 2. A_x^M clearly displays the effect relation between age and mortality (increasing over age) and K_t^M displays the effect of time on mortality (linearly decreasing over time). We observe the accident hump is more outspoken with Belgian males as with females.

Forecasting

In here we generate future scenarios of mortality for the years $\mathcal{T}* = \{2016, \ldots, 2055\}$. Therefore, we project the time dynamics separately. We fit an Arima(0,1,0) model and use the function `forecast()` of the `R`-package `forecast`. We simulate 1000 scenarios and present the corresponding fan charts (containing the 0.5% quantile, the median and the 99.5% quantile of the generated scenarios). We can clearly notice the linear trend in Figure 3. We notice that the time-dependent effect fitted is almost equal for men and women, the projection for women is slightly higher which means that the model results in a higher mortality rate due to evolution in time for women compared to men.

Figure 4 illustrates the calibrated and projected mortality rates with the model proposed above. Each row represents a different age and the left column illustrates the

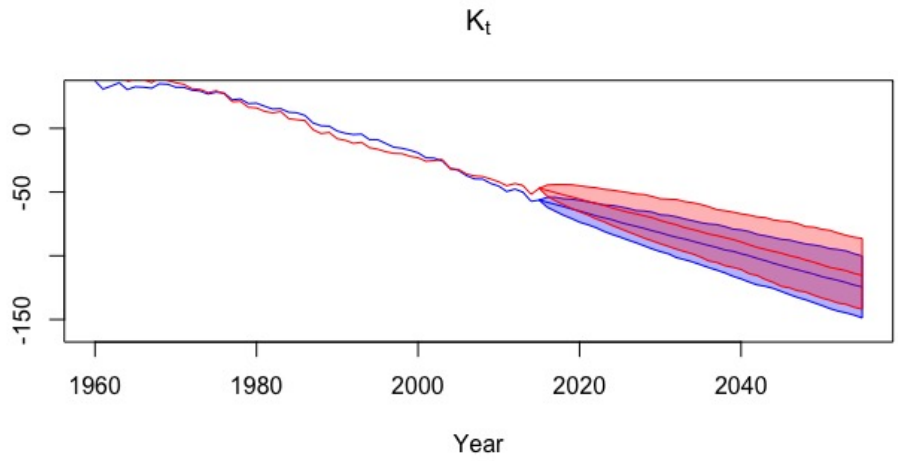

Figure 3: Projection of time dependent parameters: K_t for males (blue) and females (red). We plot the 0.5% quantile, the median and the 99.5% quantile obtained from 1000 simulations. Data: ages 0-101 and years 1960-2015 for Belgium.

males while the right column represent the females. We notice that the mortality rates are lower for women compared to men for every age. The fit seems very good because the dotted lines follows the black dots quite good.

Question 3

Calibration

Next we construct a mortality projection model for males and females jointly. We choose the (Li Lee) model (2005, Demography) as in the research of Antonio et al. (2017); Koninklijk Actuarieel Genootschap (2014). Similarly to the approach in Question 2, no cohorts effects are taken into account. We use both data of men and women to model the general trend and project the mortality of men or women as a deviation of these general trend. In this way we take the correlation between men and women into account. Nevertheless, you can only project mortality of one sex at a time. We model the force of mortality via the LL-model as follows:

$$\ln \mu_{x,t}^{(s)} = \ln \mu_{x,t}^{(M+F)} + \ln \tilde{\mu}_{x,t}^{(s)} \tag{5}$$

$$\ln \mu_{x,t}^{(M+F)} = A_x + B_x K_t \tag{6}$$

$$\ln \tilde{\mu}_{x,t}^{(s)} = \alpha_x^{(s)} + \beta_x^{(s)} \kappa_t^{(s)} \tag{7}$$

with $\mu_{x,t}^{(s)}$ denoting the force of mortality of the sex under investigation, $\mu_{x,t}^{(M+W)}$ the force of mortality of the whole data set and $\tilde{\mu}_{x,t}^{(s)}$ of the sex's deviation from the whole data set containing both sexes. The superscript $s \in \{M, F\}$ refers to the sex under consideration. We recognize a double Lee-Carter specification. Where A_x and $B_x \cdot K_t$ are the common factors. Meaning factors calibrated using both the female and male Belgian data. While $\alpha_x^{(s)}$ and $\beta_x^{(s)} \cdot \kappa_t^{(s)}$ are the gender specific deviations from the common part. Therefore,

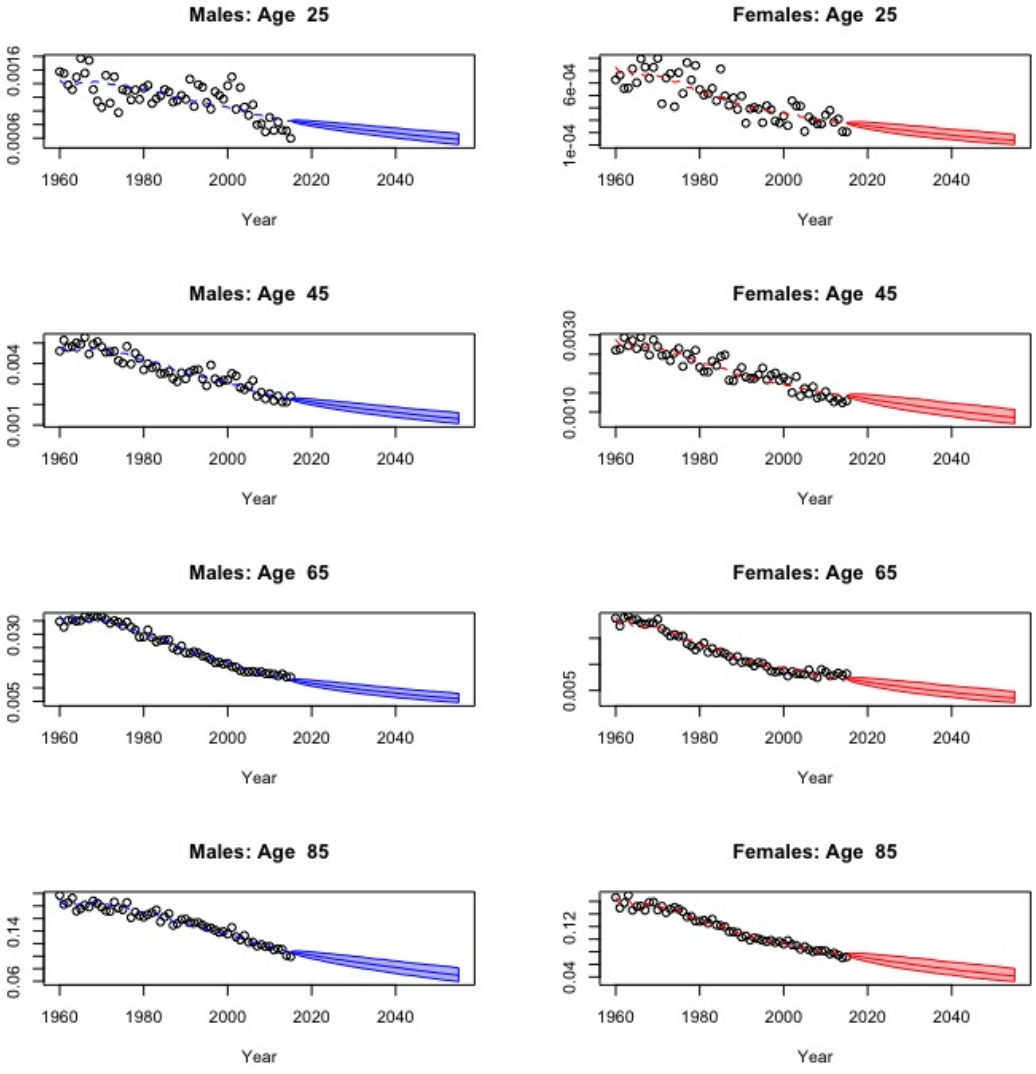

Figure 4: Observed (black dots), estimated (dashed line) and project (fan chart) mortality rates,$q_{x,t}$, for Belgian males (left in blue) and females (right in red). The first row present the projected mortality rates for a 25-year old, the second row for a 45-year old, the third row for a 65-year old and the bottom row for a 85-year old. Data: ages 0-101 and years 1960-2015.

we will need to calibrate the common factors only once. Whereas the gender specific factors will have to be estimated for females and males separately given the common factors. We use the following constraints in the calibration procedure:

$$\sum_{t\in\mathcal{T}} K_t = \kappa_t^{(s)} = 0 \quad \text{and} \quad \sum_{x\in\mathcal{X}} B_x = \beta_x^{(s)} = 1. \tag{8}$$

As calibration methodology we again use a maximum likelihood estimator with a Poisson likelihood. We use a two-step Newton-Rhapson approach to maximize the MLE. Where we first obtain results for the unisex trend 6. After which we obtain results for the gender specific parameters through conditional likelihood 7. This approach is thoroughly explained in the research of Antonio et al. (2017); Koninklijk Actuarieel Genootschap (2014). The results can be found in the graphs below. Figure 5 illustrates the fitted parameters. The first column shows the age-dependent parameter which increases for the general trend. For males we notice that the deviation corrects the general trend by adding a positive number to the mortality rate while the deviation for women corrects a little downwards for some ages. This seems logic as the mortality for males is in general higher compared to females. In the third column we see the time-dependent effect, which is decreasing for the common trend. For males, the deviation corrects downwards for the early years and upwards for more recent years while the female deviation corrects upwards for the oldest years and downwards for more recent years. Otherwise stated, The general mortality decreased slowly over the 60's while the pace of decreasing went upwards over the more recent years. The opposite is true for females.

Forecasting

Next we have to forecast the time dependent effects K_t and $\kappa_t^{(s)}$ using bivariate time series. Where

$$K_{t+1} = K_t + \theta^{(s)} + \epsilon_{t+1}^{(s)} \tag{9}$$

$$\kappa_{t+1}^{(s)} = \alpha^{(s)}\kappa_t^{(s)} + \delta_{t+1}^{(s)} \tag{10}$$

with $(\epsilon_{t+1}^{(s)}, \delta_{t+1}^{(s)})$ i.i.d bi-variate normal with mean (0,0) and covariance $V^{(s)}$. Where the $V^{(s)}$ is estimated using Seemingly Unrelated Regression (SUR) which is included in the `systemfit` package. Meaning we model the K_{t+1} as an ARIMA(0,1,0) and $\kappa_t^{(s)}$ as an AR(1). Which can be easily implemented using the `forecast` package in R. When setting up a scenario generator we simulate new $(\epsilon_{t+1}^{(s)}, \delta_{t+1}^{(s)})$ to obtain projections for $(K_t, \kappa_t^{(s)})$. The results are shown in Figure 6. We see that the K_t are very linear and thus the use of an ARIMA(0,1,0) is justified. But it is difficult to observe an auto regressive trend (AR(1)) in the κ_t^M and especially in the κ_t^F. Nevertheless, we chose an AR(1) instead of

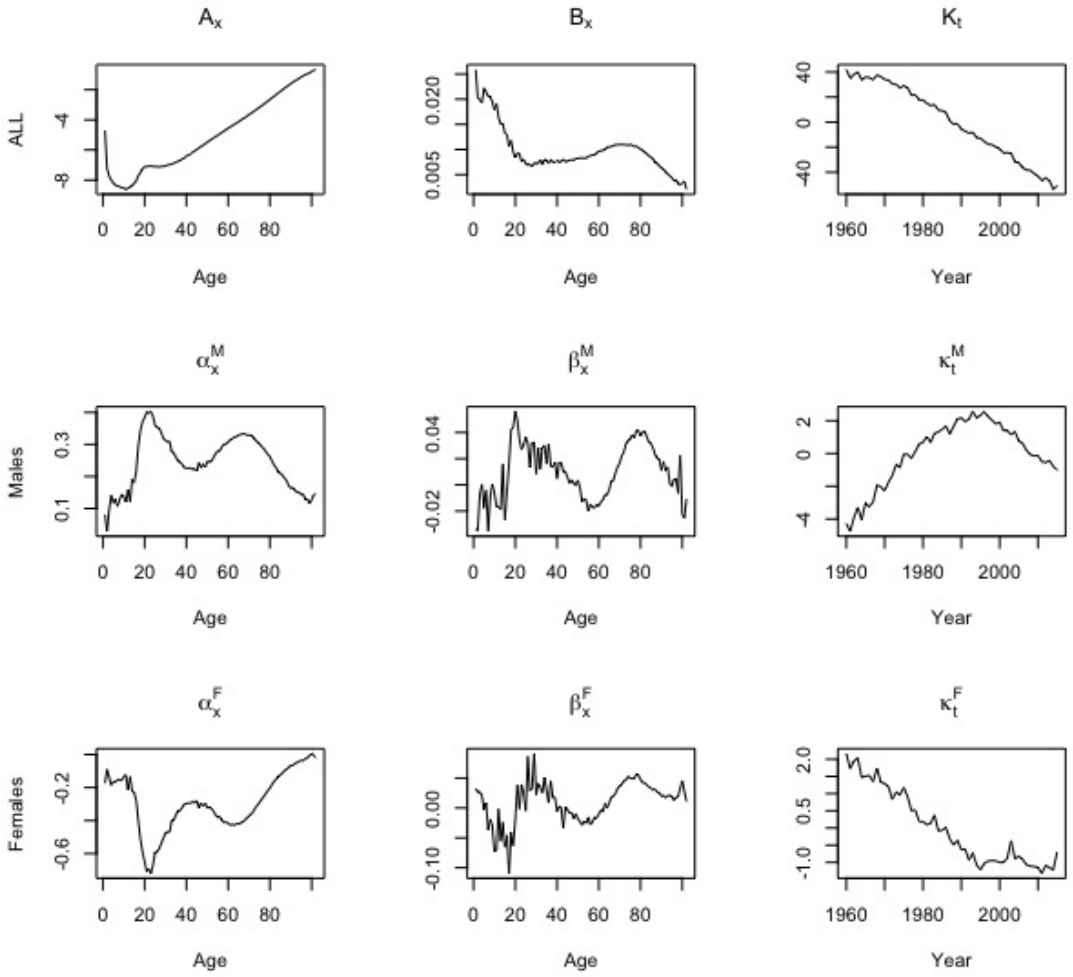

Figure 5: Results of the calibration process: (top row) parameters estimates for the common trend (A_x, B_x, K_t); (middle row) parameters estimates for the male parameters deviations (α_x^M, β_x^M and κ_t^M); (bottom row) parameters estimates for the female parameters deviations (α_x^F, β_x^F and κ_t^F). Data: Belgian data, ages 0-101, years 1960-2015.

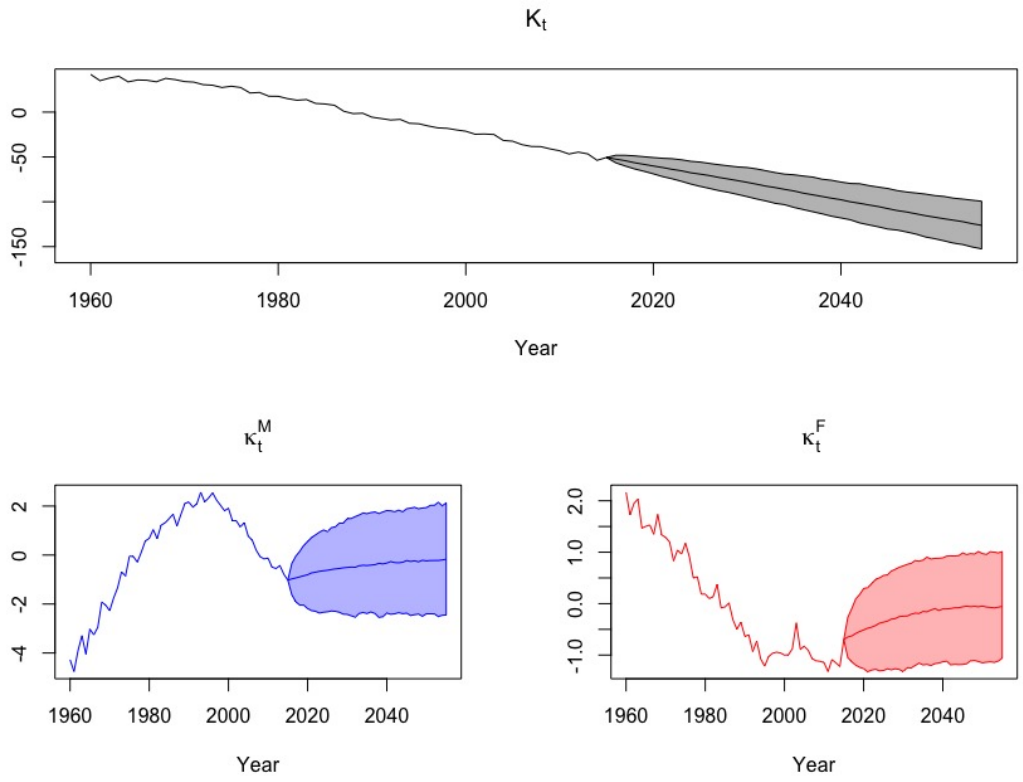

Figure 6: Projection of time dependent parameters: K_t (row 1) and $\kappa_t^{(s)}$ (row 2) for males (blue) and females (red). We plot the 0.5% quantile, the median and the 99.5% quantile obtained from 1000 simulations. Data: ages 0-101 and years 1960-2015 for Belgium.

an other time series due to biological reasons. An AR(1) process ensures that the gender specific mortality rate converges to the general trend.
Given the $(K_t, \kappa_t^{(s)})$ we can calculate $\mu_{x^{(s)},t}$ and $q_{x^{(s)},t}$. We show the mortality rates in Figure 7. We show the observed and the projected mortality rates for both males and females across different ages. We observe that older ages are much more stable and do not display as much volatility as younger ages. The younger ages tend to fluctuate more around the fitted dashed line.

Question 4

In this question we explain which model we prefer. We search for the optimal model using the the Bayesian information criterion (BIC) of Schwarz (1978). The BIC takes the goodness of fit and model complexity into account and is defined as follows:

$$\text{BIC} = -2 \cdot \log(\mathcal{L}) + \log(n) \cdot \text{EDF} \tag{11}$$

where log L is the log-likelihood of the mortality model, n is the number of observations in the data-set, and EDF represents the effective degrees of freedom which corresponds to the number of parameters. Lower BIC values indicate better models. Notice we use 258 parameters in Q2 while in Q3 there exist 516 parameters. This is because in Q3 we

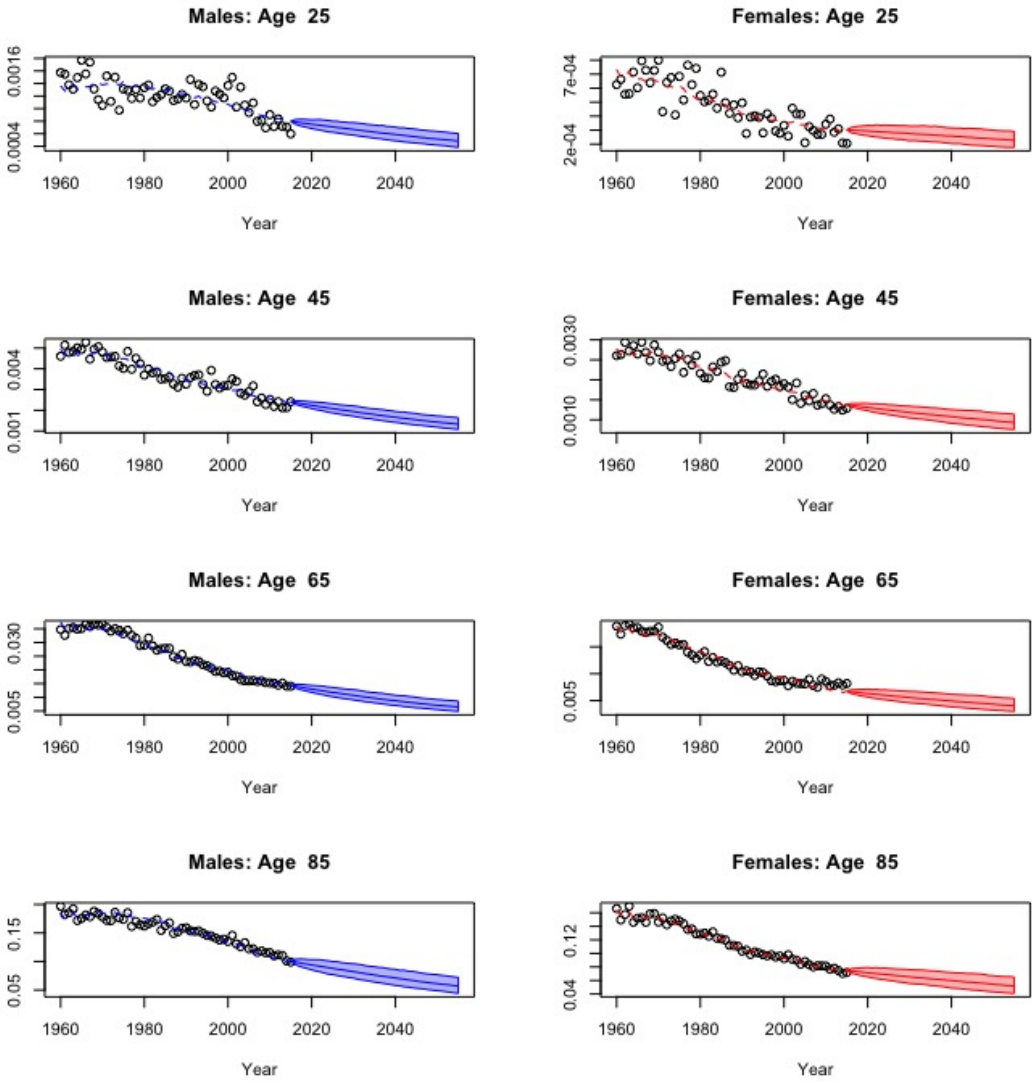

Figure 7: Observed (black dots), estimated (dashed line) and project (fan chart) mortality rates,$q_{x,t}$, for Belgian males (left in blue) and females (right in red). The first row present the projected mortality rates for a 25-year old, the second row for a 45-year old, the third row for a 65-year old and the bottom row for a 85-year old. Data: ages 0-101 and years 1960-2015.

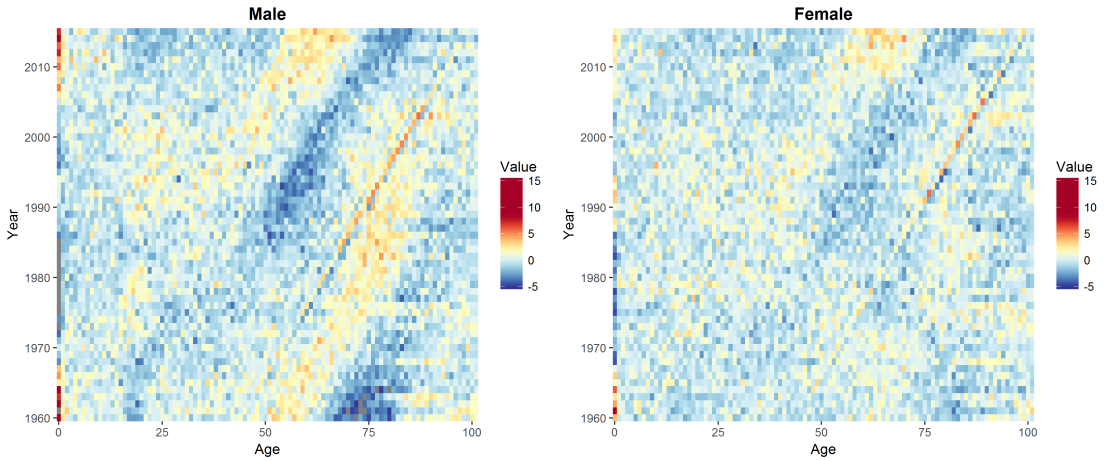

Figure 8: Pearson residual heatmaps for the proposed model in Question 2. Data: The Belgian males (left) and females (right); ages: 0-101, years: 1960-2015.

use in total two models. One to model the general trend and one to model the deviation of the gender to the general trend. The BIC penalty is quite severe compared with other information criteria like the AIC.

Table 1 gives an overview of the information criteria for the different models. The preferred model for males is the one in Q3. Although the mortality model is more complex it is also more flexible and allows for a more adequate fit. For females the BIC value attains a lower value for the model in question 2. Therefore we prefer the model in Q3 for modelling the mortality of males and the model in Q2 for females according to the BIC.

BIC	Male	Female
Q2	56,934.99	50,051.67
Q3	55,950.46	53,114.74

Table 1: Comparison of statistical performance of the Lee Carter Model (Q2) Li Lee model (Q3) using BIC values

A second tool to evaluate the and compare the approaches in Q2 and Q3 is to investigate the Pearson residuals of the mortality projection models. Figure 8 illustrate a heat map of the Pearson residuals for the proposed model in Q2. Figure 9 plots the residuals for the models for Q3. We observe that the cohort effects for the models in Q2 and Q3 are almost identical. Another observation is that the Pearson residuals for males are much higher in Q2 compared in Q3. The residual values for females increase in mortality projection model of Q3 compared to Q2.

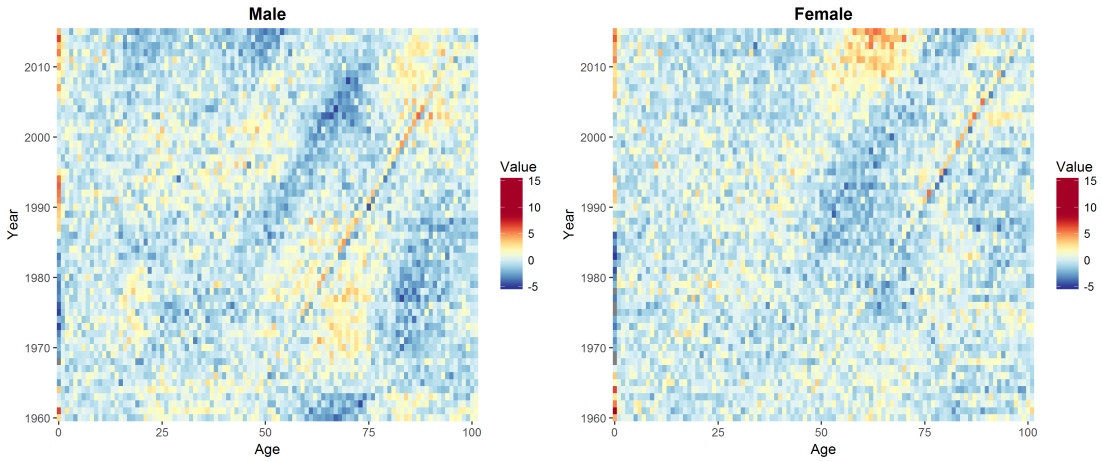

Figure 9: Pearson residual heatmaps for the proposed model in Question 3. Data: The Belgian males (left) and females (right); ages: 0-101, years: 1960-2015.

Literature

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