Course: Methods and algorithms for advanced process control
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Reference introduction

Thus far, we have only designed state feedback regulators, so, the control objective has been to bring the state of the system to zero. With these regulators, we have managed to stabilize an unstable system and to reject disturbances.

But, what about tracking? What if we want our system to follow a given reference or setpoint, what to do?
In order to achieve this, we have to modify the control law!

We are going to study two approaches to build tracking controllers, namely,

- Reference input – full state feedback
- Integral Control

1. Reference Input – Full state feedback

1.1 Discrete-time case

State space model:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k)$$

The control objective is to make sure that output of the system $\boldsymbol{y}(k) \in \mathbb{R}^p$ follows a reference signal $\boldsymbol{r}(k) \in \mathbb{R}^p$.

To ensure zero steady state error to a **step input** $\boldsymbol{r}(k)$, the feedback control law has to be modified as follows:

$$\boldsymbol{u}(k) = -\boldsymbol{K}\Big(\boldsymbol{x}(k) - \underbrace{\boldsymbol{N}_x\boldsymbol{r}(k)}_{\substack{\boxed{\text{Value of } \boldsymbol{x}(k) \text{ when } \boldsymbol{y}(k)=\boldsymbol{r}(k)} \\ \boldsymbol{x}_{ref}(k)}}\Big) + \underbrace{\boldsymbol{N}_u\boldsymbol{r}(k)}_{\substack{\boxed{\text{Value of } \boldsymbol{u}(k) \text{ when } \boldsymbol{y}(k)=\boldsymbol{r}(k)} \\ \boldsymbol{u}_{ref}(k)}}$$

How to compute $\boldsymbol{N}_x \in \mathbb{R}^{n\times p}$ and $\boldsymbol{N}_u \in \mathbb{R}^{m\times p}$?
(n = model order, m = number of inputs, p = number of outputs)

First, we have to calculate the steady state values of $\boldsymbol{x}(k) \in \mathbb{R}^n$ and $\boldsymbol{u}(k) \in \mathbb{R}^m$ for which the output of the system $\boldsymbol{y}(k)$ in steady state ($\boldsymbol{y}_{ss}$) is equal to the steady state value $\boldsymbol{r}_{ss}$ of the step reference.

When the discrete-time system is in steady state, $\boldsymbol{x}(k+1) = \boldsymbol{x}(k)$, we have

$$\boldsymbol{x}_{ss} = \boldsymbol{A}\boldsymbol{x}_{ss} + \boldsymbol{B}\boldsymbol{u}_{ss} \qquad (1)$$
$$\boldsymbol{r}_{ss} = \boldsymbol{C}\boldsymbol{x}_{ss} + \boldsymbol{D}\boldsymbol{u}_{ss} \qquad (2)$$

$\boldsymbol{x}_{ss} \rightarrow$ steady state value of $\boldsymbol{x}(k)$
$\boldsymbol{u}_{ss} \rightarrow$ steady state value of $\boldsymbol{u}(k)$

Let $\boldsymbol{x}_{ss} = \boldsymbol{N}_x r_{ss}$ and $\boldsymbol{u}_{ss} = \boldsymbol{N}_u r_{ss}$. If we plug these definitions into Equation (1) we get

$$\boldsymbol{N}_x \boldsymbol{r}_{ss} = \boldsymbol{A}\boldsymbol{N}_x \boldsymbol{r}_{ss} + \boldsymbol{B}\boldsymbol{N}_u \boldsymbol{r}_{ss}$$

$$\boldsymbol{0} = (\boldsymbol{A}\boldsymbol{N}_x - \boldsymbol{N}_x + \boldsymbol{B}\boldsymbol{N}_u)\boldsymbol{r}_{ss}, \quad \boldsymbol{r}_{ss} \neq \boldsymbol{0}$$

$$(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{N}_x + \boldsymbol{B}\boldsymbol{N}_u = \boldsymbol{0}, \qquad (3)$$

and into equation (2) we obtain

$$\boldsymbol{r}_{ss} = \boldsymbol{C}\boldsymbol{N}_x \boldsymbol{r}_{ss} + \boldsymbol{D}\boldsymbol{N}_u \boldsymbol{r}_{ss}$$

$$\boldsymbol{I}\boldsymbol{r}_{ss} = (\boldsymbol{C}\boldsymbol{N}_x + \boldsymbol{D}\boldsymbol{N}_u)\boldsymbol{r}_{ss}$$

$$\boldsymbol{C}\boldsymbol{N}_x + \boldsymbol{D}\boldsymbol{N}_u = \boldsymbol{I}. \qquad (4)$$

Putting (3) and (4) together, we get the following system of equations:

$$\underbrace{\begin{bmatrix} \boldsymbol{A} - \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}}_{\text{size: } (n+p)\times(n+m)} \underbrace{\begin{bmatrix} \boldsymbol{N}_x \\ \boldsymbol{N}_u \end{bmatrix}}_{\text{size: } (n+m)\times p} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}$$

We have, $p(n+m)$ unknowns and $p(n+p)$ equations! Only when the number of inputs is equal to the number of outputs ($p = m$), the matrix in red is square and we can obtain $\boldsymbol{N}_x$ and $\boldsymbol{N}_u$ in this way,

$$\begin{bmatrix} \boldsymbol{N}_x \\ \boldsymbol{N}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}$$

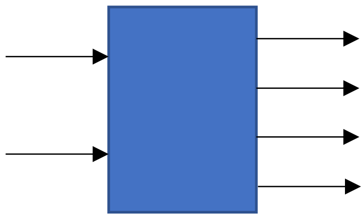
Let’s take a look at the following scenarios:

- If (number of inputs) = (number of outputs) or ($m = p$),

Square system

- Number of unknows = number of equations

- If (number of inputs) < (number of outputs) or ($m < p$),



Tall system
(underactuated)

- Number of unknows < number of equations
- Overdetermined system of equations ➔ least squares solution ➔ zero steady state error cannot longer be guaranteed!
- Lack of degrees of freedom to control the outputs

- If (number of inputs) > (number of outputs) or ($m > p$),

Fat system
(overactuated)

- Number of unknows > number of equations
- Underdetermined system of equations ➔ Infinite number of solutions
- Extra degrees of freedom to control the outputs

The adjectives “square”, “tall” and “fat” are due to the size/shape (number of rows and number of columns) of the transfer function matrix.

The reference introduction does not change the poles of the closed-loop system

- Closed-loop system without reference introduction: $\boldsymbol{x}(k+1) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(k)$
- Closed-loop system with reference introduction:
  - We can write the modified control law, $\boldsymbol{u}(k) = -\boldsymbol{K}\big(\boldsymbol{x}(k) - \boldsymbol{N}_x\boldsymbol{r}(k)\big) + \boldsymbol{N}_u\boldsymbol{r}(k)$, as follows:

$$\boldsymbol{u}(k) = -\boldsymbol{K}\boldsymbol{x}(k) + \underbrace{(\boldsymbol{K}\boldsymbol{N}_x + \boldsymbol{N}_u)}\boldsymbol{r}(k)$$

$$\boldsymbol{u}(k) = -\boldsymbol{K}\boldsymbol{x}(k) + \bar{\boldsymbol{N}}\boldsymbol{r}(k) \qquad (5)$$

  - If we plug (5) into the state equation, $\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$, we get

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}(-\boldsymbol{K}\boldsymbol{x}(k) + \bar{\boldsymbol{N}}\boldsymbol{r}(k))$$

$$\boldsymbol{x}(k+1) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(k) + \boldsymbol{B}\bar{\boldsymbol{N}}\boldsymbol{r}(k))$$

- The state matrix of the closed-loop system with the reference introduction is the same as the one without reference introduction ➔ The poles do not change!

1.2 Continuous-time case

State space model:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

Modified control law:

$$\boldsymbol{u}(t) = -\boldsymbol{K}\big(\boldsymbol{x}(t) - \boldsymbol{N}_x\boldsymbol{r}(t)\big) + \boldsymbol{N}_u\boldsymbol{r}(t)$$

We use the same reasoning as before to find $\boldsymbol{N}_x \in \mathbb{R}^{n\times p}$ and $\boldsymbol{N}_u \in \mathbb{R}^{m\times p}$.

When the continuous-time system is in steady state, $\dot{\boldsymbol{x}}(t) = \boldsymbol{0}$, we have

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{x}_{ss} + \boldsymbol{B}\boldsymbol{u}_{ss} \qquad (6)$$
$$\boldsymbol{r}_{ss} = \boldsymbol{C}\boldsymbol{x}_{ss} + \boldsymbol{D}\boldsymbol{u}_{ss} \qquad (7)$$

Let $\boldsymbol{x}_{ss} = \boldsymbol{N}_x r_{ss}$ and $\boldsymbol{u}_{ss} = \boldsymbol{N}_u r_{ss}$. If we plug these definitions into Equation (6) we get

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{N}_x \boldsymbol{r}_{ss} + \boldsymbol{B}\boldsymbol{N}_u \boldsymbol{r}_{ss}$$

$$\boldsymbol{0} = (\boldsymbol{A}\boldsymbol{N}_x + \boldsymbol{B}\boldsymbol{N}_u)\boldsymbol{r}_{ss}, \quad \boldsymbol{r}_{ss} \neq \boldsymbol{0}$$

$$\boldsymbol{A}\boldsymbol{N}_x + \boldsymbol{B}\boldsymbol{N}_u = \boldsymbol{0}, \qquad (8)$$

and into equation (7) we obtain

$$\boldsymbol{r}_{ss} = \boldsymbol{C}\boldsymbol{N}_x \boldsymbol{r}_{ss} + \boldsymbol{D}\boldsymbol{N}_u \boldsymbol{r}_{ss}$$

$$\boldsymbol{I}\boldsymbol{r}_{ss} = (\boldsymbol{C}\boldsymbol{N}_x + \boldsymbol{D}\boldsymbol{N}_u)\boldsymbol{r}_{ss}$$

$$\boldsymbol{C}\boldsymbol{N}_x + \boldsymbol{D}\boldsymbol{N}_u = \boldsymbol{I}. \qquad (9)$$

Putting (8) and (9) together, we get the following system of equations:

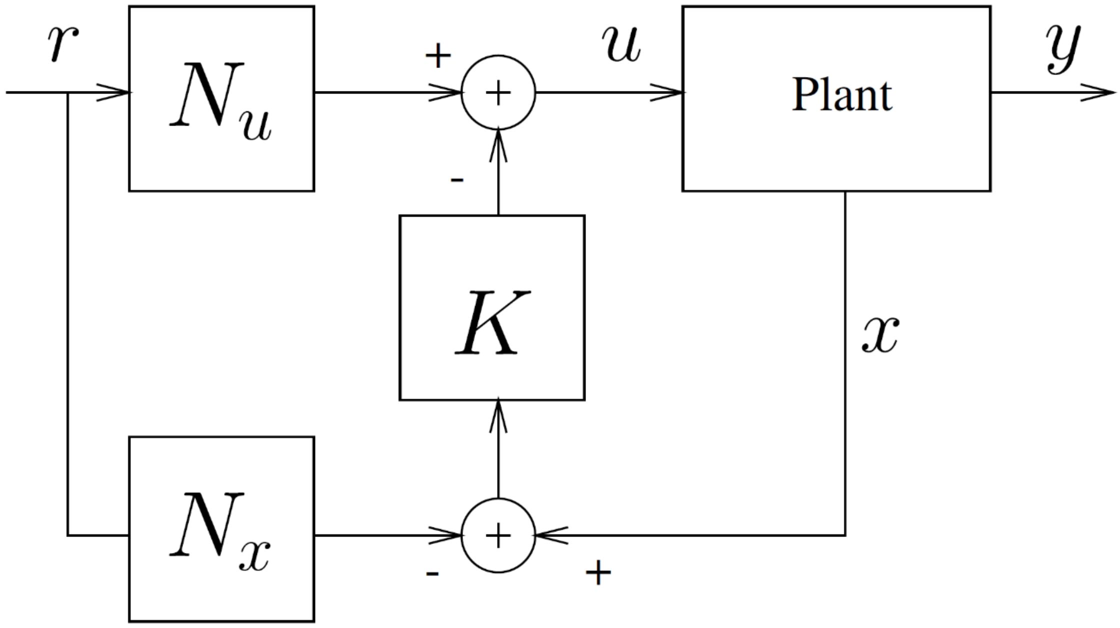
$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_x \\ \boldsymbol{N}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}$$

By solving this system of equations, we can find $\boldsymbol{N}_x$ and $\boldsymbol{N}_u$.

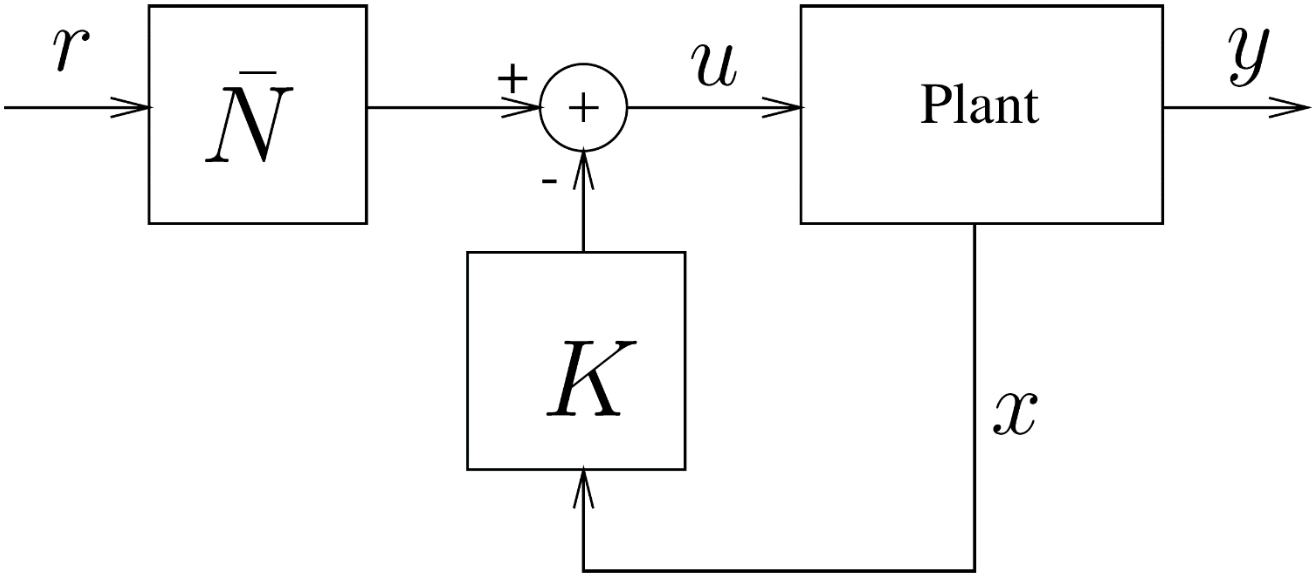
1.3 Types of interconnections

There are two types of interconnections for reference introduction with full state feedback (for continuous and discrete time systems):

Type I: $\boldsymbol{u}(k) = -\boldsymbol{K}\big(\boldsymbol{x}(k) - \boldsymbol{N}_x \boldsymbol{r}(k)\big) + \boldsymbol{N}_u \boldsymbol{r}(k)$



Type II: $\boldsymbol{u}(k) = -\boldsymbol{K}\boldsymbol{x}(k) + \bar{\boldsymbol{N}}\boldsymbol{r}(k)$



<u>From the official course notes:</u> "For a type II interconnection, the control law $\boldsymbol{K}$ used in the feedback ($\boldsymbol{u}(k) = -\boldsymbol{K}\boldsymbol{x}(k)$) and in the reference feedforward ($\bar{N} = \boldsymbol{K}\boldsymbol{N}_x + \boldsymbol{N}_u$) should be exactly the same, otherwise there is a steady-state error. There is no such problem in type I. → Type I is more ROBUST to parameter errors than Type II."

2. Integral control

- Problem of the previous approach:
  - **We need to have an exact model of the plant!** Any little change of the model parameters will lead to steady state error.

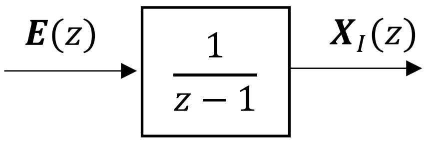
- Integral control allows robust tracking of step references!
  - Remember that the integral action of a PID controller gets rid of the steady state error!

- **Idea:** Add integrators in the control loop.

2.1 Discrete-time case

In this approach, we first augment the model of the plant by adding integrators (they will integrate the output error), and then we compute the feedback gain $\boldsymbol{K}$ of the augmented system by using pole placement or optimization techniques (LQR).

In discrete-time, the integrators have the following form:



Difference equation:

$$\boldsymbol{x}_I(k+1) = \boldsymbol{x}_I(k) + \boldsymbol{e}(k)$$

where $\boldsymbol{e}(k)$ is the tracking error $\boldsymbol{e}(k) = \boldsymbol{y}(k) - \boldsymbol{r}(k)$.

The augmented model of the plant will be given by the model of the plant,

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$$

and the equation of the integrators (additional states):

$$\boldsymbol{x}_I(k+1) = \boldsymbol{x}_I(k) + \boldsymbol{y}(k) - \boldsymbol{r}(k)$$

$$\boldsymbol{x}_I(k+1) = \boldsymbol{x}_I(k) + \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) - \boldsymbol{r}(k)$$

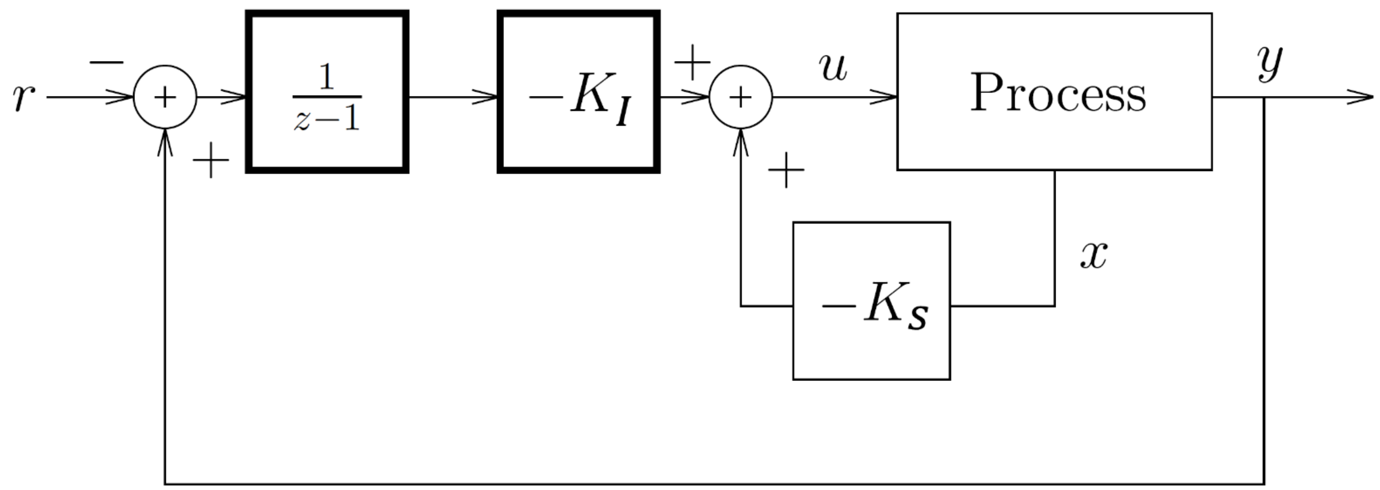
The augmented state equations can be compactly written as follows:

$$\begin{bmatrix} \boldsymbol{x}_I(k+1) \\ \boldsymbol{x}(k) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_I(k) \\ \boldsymbol{x}(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{bmatrix} \boldsymbol{u}(k) + \begin{bmatrix} -\boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{r}(k)$$

The control law that stabilizes the system is given by

$$\boldsymbol{u}(k) = -\underbrace{[\boldsymbol{K}_I \quad \boldsymbol{K}_s]}_{\boldsymbol{K}} \begin{bmatrix} \boldsymbol{x}_I(k) \\ \boldsymbol{x}(k) \end{bmatrix}$$

The block diagram of the closed-loop system looks like

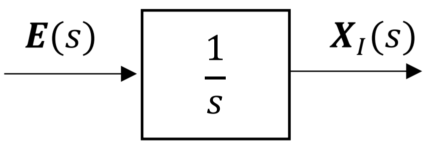


Typically, the states of the plant are estimated by means of an Observer or state estimator.

Note that pole placement or LQR might not work since the augmented system is NOT always stabilizable and, in this case, integral control cannot be used!

2.2 Continuous-time case

In continuous time, the integrators have the following form:



Differential equation:

$$\dot{\boldsymbol{x}}_I(t) = \boldsymbol{e}(t)$$

where $\boldsymbol{e}(t)$ is the tracking error $\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{r}(t)$.

As in the discrete-time case, the augmented model of the plant will be given by the model of the plant,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

and the equation of the integrators (additional states):

$$\dot{\boldsymbol{x}}_I(t) = \boldsymbol{y}(t) - \boldsymbol{r}(t)$$

$$\dot{\boldsymbol{x}}_I(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) - \boldsymbol{r}(k)$$

The augmented state equations can be compactly written as follows:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_I(t) \\ \dot{\boldsymbol{x}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_I(k) \\ \boldsymbol{x}(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{bmatrix} \boldsymbol{u}(k) + \begin{bmatrix} -\boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{r}(k)$$

The control law that stabilizes the system is given by

$$\boldsymbol{u}(t) = -\underbrace{[\boldsymbol{K}_I \quad \boldsymbol{K}_s]}_{\boldsymbol{K}} \begin{bmatrix} \boldsymbol{x}_I(t) \\ \boldsymbol{x}(t) \end{bmatrix}$$

Finally, the block diagram of the closed-loop system is as follows: