$$J_1 = \frac{\delta H_u}{\delta c_u} \qquad \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}$$

$$J_1(m,i) = \frac{\delta}{\delta c_i} \int_\Omega r \frac{V_{mu} c_u}{(K_{mu}+c_u)\left(1+\frac{c_v}{K_{mv}}\right)} \varphi_m \, d\Omega$$

$$= V_{mu} \int_\Omega r \frac{1}{\left(1+\frac{c_v}{K_{mv}}\right)} \frac{\delta}{\delta c_i}\left[\frac{c_u}{K_{mu}+c_u}\right] \varphi_m \, d\Omega$$

$$= V_{mu} \int_\Omega r \frac{1}{\left(1+\frac{c_v}{K_{mv}}\right)} \frac{\varphi_i (K_{mu}+c_u) - \varphi_i c_u}{(K_{mu}+c_u)^2} \varphi_m \, d\Omega$$

$$= V_{mu} \int_\Omega r \frac{1}{\left(1+\frac{c_v}{K_{mv}}\right)} \frac{K_{mu} \varphi_i \varphi_m}{(K_{mu}+c_u)^2} \, d\Omega$$

<u>Contribution from one elemet:</u>

$$V_{mu} K_{mu} \int_T r \frac{\varphi_i \varphi_m \, dT}{\left(1+\frac{1}{K_{mv}}\left[c_{n+1}\varphi_1 + c_{n+2}\varphi_2 + c_{n+3}\varphi_3\right]\right)\left(K_{mu} + \sum_{j=1}^{3} c_j \varphi_j\right)^2}$$

$$= V_{mu} K_{mu} \int_{\tilde{T}} \frac{(r_1 \hat{N}_1 + r_2 \hat{N}_2 + r_3 \hat{N}_3)\, \hat{N}_i \hat{N}_m \, |J| \, d\tilde{T}}{\left(1 + \frac{1}{K_{mv}} \sum_{j=1}^{3} c_{M+j} \hat{N}_j\right)\left(K_{mu} + \sum_j^3 c_j \hat{N}_j\right)^2}$$

$$J_2 = \frac{\delta H_u}{\delta c_v} \quad \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}$$

$$J_2(m,i) = \frac{\delta}{\delta c_{M+i}} \int_\Omega r \frac{V_{mu}\, c_u}{(K_{mu} + c_u)\left(1 + \frac{c_v}{K_{mv}}\right)} \varphi_m \, d\Omega$$

$$= V_{mu} \int_\Omega r \frac{c_u}{(K_{mu} + c_u)} \frac{\delta}{\delta c_{M+i}} \left[\frac{1}{1 + \frac{c_v}{K_{mv}}}\right] \varphi_m \, d\Omega$$

$$= V_{mu} \int_\Omega r \frac{c_u}{(K_{mu} + c_u)} \frac{-1}{\left(1 + \frac{c_v}{K_{mv}}\right)^2} \cdot \frac{1}{K_{mv}} \varphi_i \, \varphi_m \, d\Omega$$

$$= \frac{-\cancel{K}_{mu}}{K_{mv}} \int_\Omega r \frac{c_u}{(K_{mu} + c_u)} \frac{1}{\left(1 + \frac{c_v}{K_{mv}}\right)^2} \varphi_i \, \varphi_m \, d\Omega$$

Contribution from one element

$$\frac{-V_{mu}}{K_{mv}} \int_T r \frac{\sum c_j \varphi_j}{\left(K_{mu} + \sum c_j \varphi_j\right)} \frac{1}{\left(1 + \frac{\sum c_{M+j} \varphi_j}{K_{mv}}\right)^2} \varphi_i \, \varphi_m \, d\Omega$$

$$= -\frac{V_{mu}}{K_{mv}} \int_{\tilde{T}} \frac{\left(r_1 \hat{N}_1 + r_2 \hat{N}_2 + r_3 \hat{N}_3\right)\left(\sum_{j=1}^{3} c_j \hat{N}_j\right) \hat{N}_i \, \hat{N}_m |J| \, d\xi \, d\eta}{\left(K_{mu} + \sum_{j=1}^{3} c_j \hat{N}_j\right)\left(1 + \frac{1}{K_{mv}} \sum_{j=1}^{3} c_{M+j} \hat{N}_j\right)^2}$$

$$J_3 = -r_q J_1 + J_5 \qquad \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}$$

$$J_5(m,i) = \frac{\delta}{\delta c_i} \int_\Omega -r \frac{V_{mfv}}{1 + \frac{c_u}{K_{mfu}}} \varphi_m \, d\Omega$$

$$= -V_{mfv} \int_\Omega r \frac{\delta}{\delta c_i} \left[\frac{1}{1 + \frac{c_u}{K_{mfu}}} \right] \varphi_m \, d\Omega$$

$$= -V_{mfv} \int r \frac{-1}{\left(1 + \frac{c_u}{K_{mfu}}\right)^2} \cdot \frac{1}{K_{mfu}} \cdot \varphi_i \varphi_m \, d\Omega$$

$$= \frac{V_{mfv}}{K_{mfu}} \int r \frac{\varphi_i \varphi_m}{\left(1 + \frac{c_u}{K_{mfu}}\right)^2} \, d\Omega$$

Contribution from one element:

$$\frac{V_{mfv}}{K_{mfu}} \int_T r \frac{\varphi_i \varphi_m}{\left(1 + \frac{1}{K_{mfu}} \sum_{i=1}^{3} c_j \varphi_j\right)^2} \, dT$$

$$= \frac{V_{mfv}}{K_{mfu}} \int_{\tilde{T}} \frac{(r_1 \hat{N}_1 + r_2 \hat{N}_2 + r_3 \hat{N}_3)\, \hat{N}_i \hat{N}_m |J|}{\left(1 + \frac{1}{K_{mfu}} \sum c_j \hat{N}_j\right)^2} \, d\xi \, d\eta$$

$$J_4 = \frac{\delta H_v}{\delta c_v} = -r_q J_2 + J_6 \qquad \begin{bmatrix} J_1 & T_2 \\ J_3 & J_4 \end{bmatrix}$$

$$J_6(m,i) = \frac{\delta}{\delta c_{M+i}} \int_\Omega -r \frac{V_{m\rho v}}{1+\frac{c_u}{K_{m\rho u}}} \varphi_m \, d\Omega$$

$$= 0$$

$$H_u = \int_\Omega r R_u(c_u, c_v) \varphi(r,z) \, d\Omega$$

$$H_u(m) = \int_\Omega r R_u(c_u, c_v) \varphi_m(r,z) \, d\Omega$$

$$= V_{mu} \int_\Omega r \frac{c_u}{(K_{mu}+c_u)\left(1+\frac{c_v}{K_{mv}}\right)} \varphi_m(r,z) \, d\Omega$$

<u>Contribution from 1 element:</u>

$$V_{mu} \int_T \frac{r\left(\sum_{j=1}^{3} c_j \varphi_j\right) \varphi_m}{\left(K_{mu} + \sum c_j \varphi_j\right)\left(1+\frac{1}{K_{mv}} \sum c_{M+j} \varphi_j\right)} \, dT$$

$$= V_{mu} \int_{\hat{T}} \frac{(r_1 \hat{N}_1 + r_2 \hat{N}_2 + r_3 \hat{N}_3)\left(\sum_1^3 c_j \hat{N}_j\right) \hat{N}_m \, |J|}{\left(K_{mu} + \sum c_j \hat{N}_j\right)\left(1+\frac{1}{K_{mv}} \sum c_{M+j} \hat{N}_j\right)} \, d\xi \, d\eta$$

$$H_v = \int_\Omega -r\, R_v(c_u, c_v)\, \varphi(r,z)\, d\Omega$$

$$= -r_q H_u + \underbrace{\int_\Omega -r \frac{V_{m\rho v}}{1 + \frac{c_u}{K_{m\rho u}}} \varphi(r,z)\, d\Omega}_{H_3}$$

$$H_v(m) = -r_q H_u(m) + H_3(m)$$

$$H_3(m) = -V_{m\rho v} \int_\Omega \frac{r\, \varphi_m}{1 + \frac{c_u}{K_{m\rho u}}}\, d\Omega$$

Contribution from one element: for H_3

$$-V_{m\rho v} \int_T r\, \varphi_m \frac{1}{1 + \frac{\sum c_j \varphi_j}{K_{m\rho u}}}\, d\Omega$$

$$= -V_{m\rho v} \int_{\bar{T}} \frac{(r_1 \hat{N}_1 + r_2 \hat{N}_2 + r_3 \hat{N}_3)\, \hat{N}_m\, |J|}{1 + \frac{1}{K_{m\rho u}} \sum c_j \hat{N}_j}\, d\xi\, d\eta$$