Implementation of [Control Barrier Function (CBF)]() for [SSM]()

$\forall p_h \in \mathcal{H}$ and $\forall p_r \in \mathcal{R}$ we can define

- $v_r(t)$ linear velocity of the robot point
- $v_{h\|}(t)$ linear velocity of the human point
- $d(t) = \|p_r(t) - p_h(t)\|$ distance from human to robot points
- $u_{rh}(t) = (p_r(t) - p_h(t))/d(t)$ versor pointing from human to robot points
- $v_{r\|}(t) = v_r(t) \cdot u_{rf}(t)$ projected velocity of the robot ($v_{r\|} > 0$ robot moves on the opposite side w.r.t. human point)
- $v_{h\|}(t) = v_h(t) \cdot u_{rf}(t)$ projected velocity of the robot ($v_{h\|} > 0$ robot moves towards the human point)
- $v_{r\|act} = v_{r\|}(0)$ actual value of $v_{r\|}$
- $v_{h\|act} = v_{h\|}(0)$ actual value of $v_{h\|}$
- $d_{act} = d(0)$ actual value of d

Choice of state χ

The state of the CBF is defined in Secchi as $\chi = [d_{act}, v_{r\|act}]$. It is worth noticing that the barrier function $h(\chi)$ should have as a single variable the state χ. If the human velocity changes (or it is not assumed constant) we need to augment the state. In that case we considered as state elements $p_r, h_r, v_{r\|act}, v_{h\|act}$ if $v_{h\|}$ is supposed to be constant and $p_r, h_r, v_r, v_h, \dot{v}_r, \dot{v}_h$, if $\dot{v}_h$ is supposed to be constant

$$\frac{\partial d}{\partial p_r} = u_{rh}$$

$$\frac{\partial d}{\partial p_h} = -u_{rh}$$

$$\frac{\partial d}{\partial v_r} = 0$$

$$\frac{\partial d}{\partial v_h} = 0$$

Define the projection matrix:

$$P = I - u_{rh} u_{rh}^T$$

Then, the partial derivatives are:

$$\frac{\partial v_{r||act}}{\partial v_r} = u_{rh}$$

$$\frac{\partial v_{r||act}}{\partial p_r} = \frac{v_r}{||p_r - p_h||} \left(I - u_{rh} u_{rh}^T\right) = \frac{v_r}{||p_r - p_h||} P$$

$$\frac{\partial v_{r||act}}{\partial p_h} = -\frac{v_r}{||p_r - p_h||} \left(I - u_{rh} u_{rh}^T\right) = -\frac{v_r}{||p_r - p_h||} P$$

$$\frac{\partial v_{r||act}}{\partial v_h} = 0$$

$$\frac{\partial v_{h||}}{\partial v_h} = u_{rh}$$

$$\frac{\partial v_{h||}}{\partial p_r} = \frac{v_h}{||p_r - p_h||} \left(I - u_{rh} u_{rh}^T\right) = \frac{v_h}{||p_r - p_h||} P$$

$$\frac{\partial v_{h||}}{\partial p_h} = -\frac{v_h}{||p_r - p_h||} \left(I - u_{rh} u_{rh}^T\right) = -\frac{v_h}{||p_r - p_h||} P$$

$$\frac{\partial v_{h||}}{\partial v_r} = 0$$

🔵 $v_h = constant$

Define $\chi = [p_r, h_r, v_r, v_h]$ and $\psi(\chi) = [d_{act}, v_{r||act}, v_{h||act}]$, it is possible to write

$$\frac{\partial \psi}{\partial \chi} = \begin{bmatrix} u^T & -u^T & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} \\ \frac{v_r}{d} P & -\frac{v_r}{d} P & u^T & \mathbf{0}_{1\times 3} \\ \frac{v_h}{d} P & -\frac{v_h}{d} P & \mathbf{0}_{1\times 3} & u^T \end{bmatrix}$$

🟢 $\dot{v}_h = constant$

Define $\chi = [p_r, h_r, v_r, v_h]$ and $\psi(\chi) = [d_{act}, v_{r\|act}, v_{h\|act}]$, it is possible to write

$$\frac{\partial\psi}{\partial\chi} = \begin{bmatrix} u^T & -u^T & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \frac{v_r}{d}P & -\frac{v_r}{d}P & u^T & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \frac{v_h}{d}P & -\frac{v_h}{d}P & \mathbf{0}_{1\times3} & u^T & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \end{bmatrix}$$

SSM requirement (🔵 $v_h = constant$)

Case $v_{r\|act} < 0$

if $v_{r\|act} < 0$ the constraint is $min(d(t)) \geq C$, where $min(d(t))$ is compute in the time interval between the actual time $t = 0$ and the stopping time. The stopping time is the some of the reaction time t_r and the deceleration time $t_{dec} = -\frac{v_{h\|}}{a}$.

When $v_{r\|act} < 0$ we define the barrier function $h = min(d(t)) - C$, when $h > 0$ the SSM requirement is fulfilled. When $v_{r\|act} \geq 0$ there are no SSM requirements.

Distance $d(t)$ changes in the interval $[0, t_r + t_{dec}]$, the minimum could be:

- when $\dot{d}(t) = 0$
- in $t = 0$ if $\dot{d}(t) > 0 \, \forall t \in [0, t_r + t_{dec}]$
- in $t = t_r + t_{dec}$ if $\dot{d}(t) < 0 \, \forall t \in [0, t_r + t_{dec}]$

The projected robot velocity $v_{r\|}(t)$ follow the trend

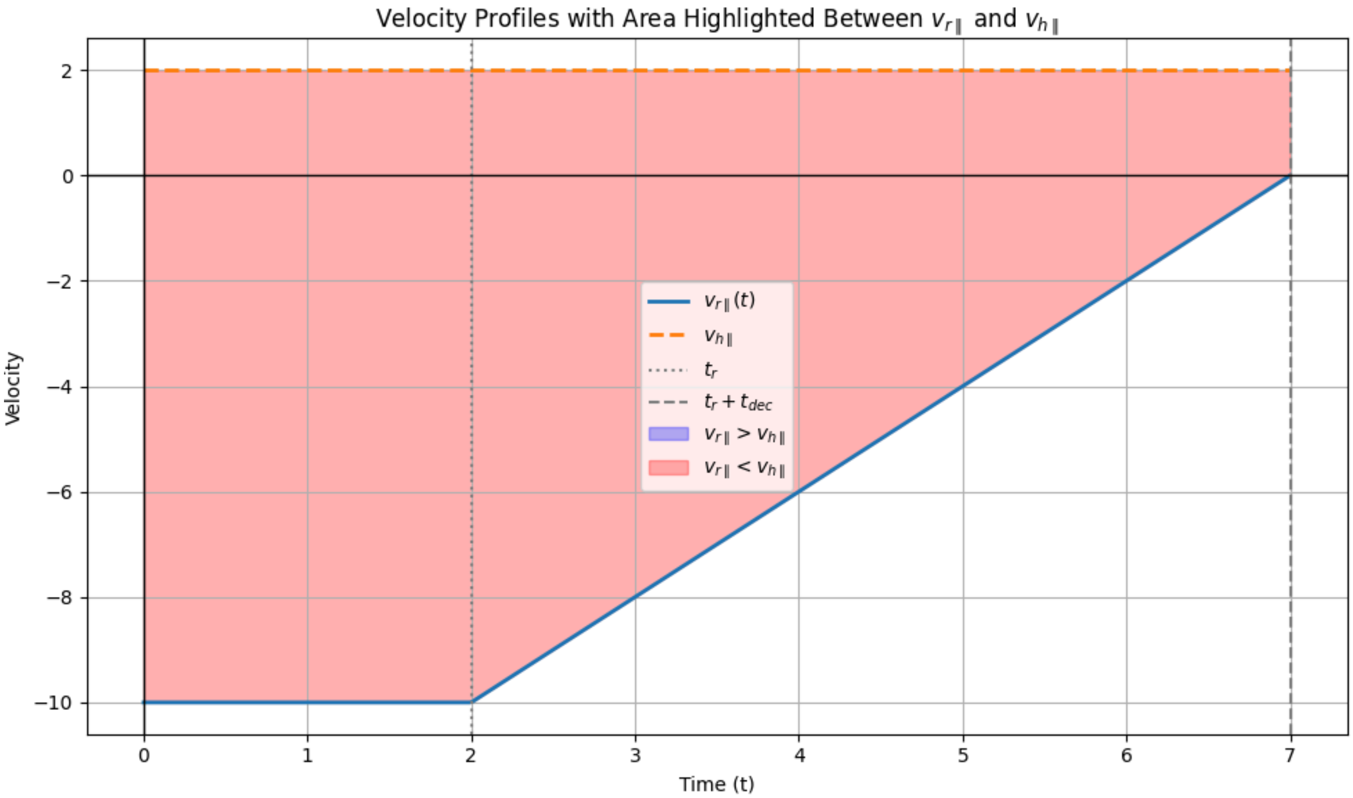
$$v_{r\|}(t) = \begin{cases} v_{r\|act} & \text{when } t < t_r \\ v + a(t - t_r) & \text{when } t_r \leq t \leq t_r + t_{dec} \end{cases}$$

The projected human velocity $v_{h\|}(t) = v_{h\|}$ is supposed constant during the time interval

The derivative of the distance is defined as $\dot{d}(t) = v_{r\|}(t) - v_{h\|}(t)$.

subcase $v_{r\|act} < 0$ and $v_{h\|} > 0$

in this case we got $\dot{d}(t) = v_{r\|}(t) - v_{h\|}(t) < 0$, the minimum is in $t = t_r + t_{dec}$



Therefore $min(d(t)) = d(t_r + t_{dec})$
where

$$d(t_r + t_{dec}) = d_{act} + (v_{r\|act} - v_{h\|act})t_r - v_{h\|}t_{dec} + v_{r\|act}t_{dec} + \frac{1}{2}at_{dec}^2$$

or equally

$$d(t_r + t_{dec}) = d_{act} + (v_{r\|act} - v_{h\|act})t_r + \frac{v_{h\|act}v_{r\|act}}{a} - \frac{1}{2}\frac{v_{r\|act}^2}{a}$$

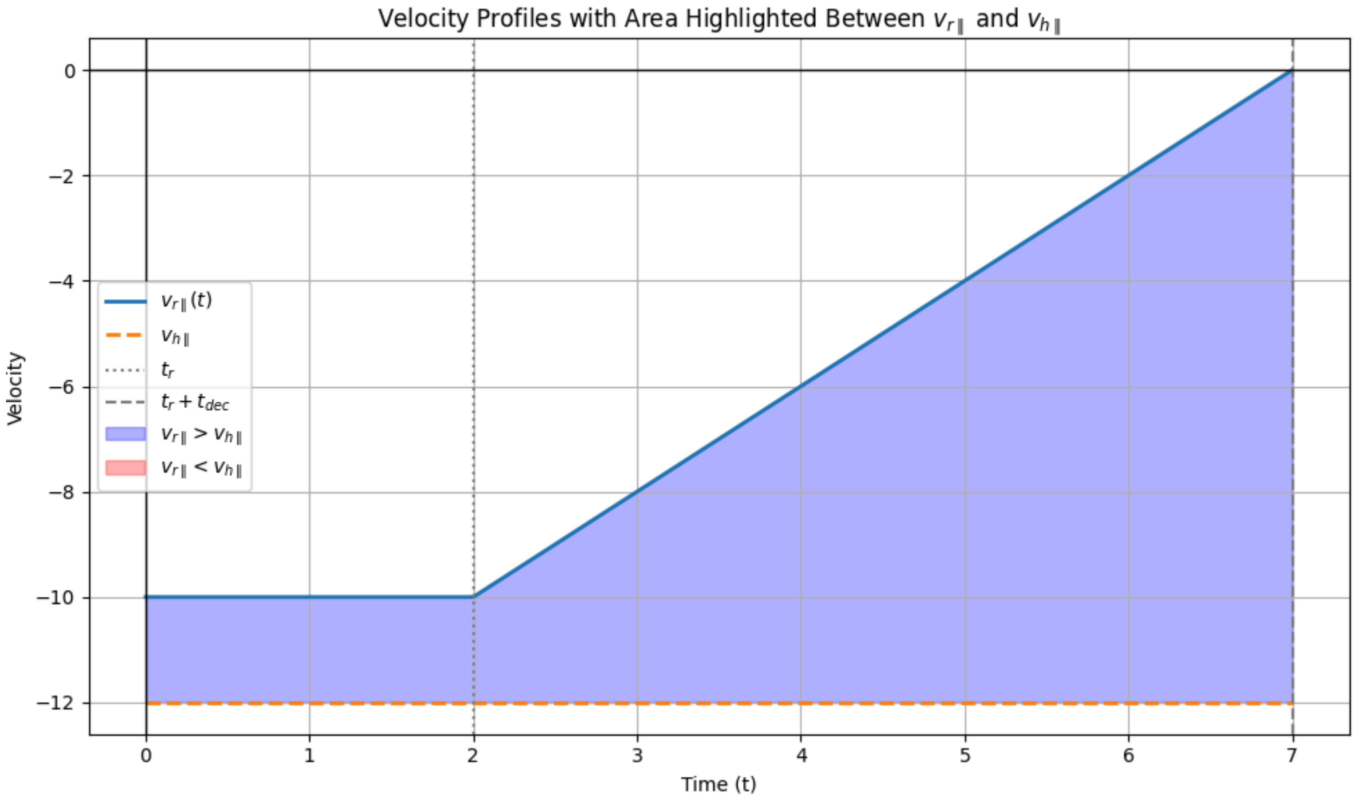
Recalling $h(t) = min(d(t)) - C$, we get

$$h = d_{act} + (v_{r\|act} - v_{h\|})t_r + \frac{v_{h\|}v_{r\|act}}{a} - \frac{1}{2} \cdot \frac{v_{r\|act}^2}{a} - C$$

when $v_{r\|act} \to 0$, the threshold value of d_{act} where $h = 0$ is $d_{th} = C + v_{h\|act}t_r$

subcase $v_{r\|act} < 0$ and $v_{h\|act} < v_{r\|act}$

in this case we got $\dot{d}(t) = v_{r\parallel}(t) - v_{h\parallel}(t) > 0$, the minimum is in $t = 0$



Therefore $min(d(t)) = d(0)$
where

$$d(0) = d_{act}$$

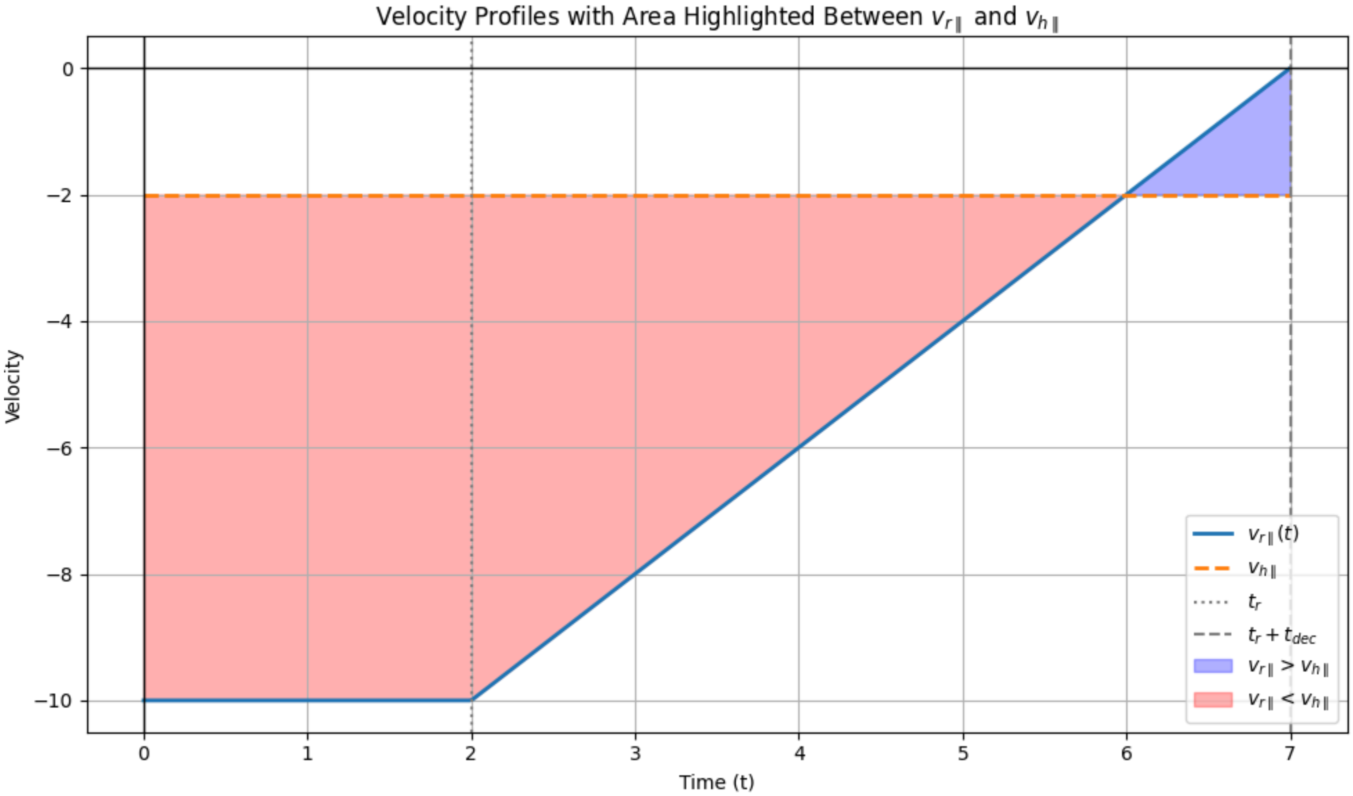
Recalling $h = min(d(t)) - C$, we get

$$h = d_{act} - C$$

when $v_{r\parallel act} \to 0$, the threshold value of d_{act} where $h = 0$ is $d_{th} = C$

subcase $v_{r\parallel act} < 0$ and $v_{r\parallel act} \leq v_{h\parallel act} \leq 0$

in this case we got $\dot{d}(t) = v_{r\parallel}(t) - v_{h\parallel}(t) = 0$, the minimum is t^* where $\dot{d}(t) = 0$, thus $v_{r\parallel}(t) = v_{h\parallel}(t)$



where $t^* = t_r + \dfrac{v_{h\|act} - v_{r\|act}}{a}$
and

$$d(t^*) = d_{act} + (v_{r\|act} - v_{h\|act})t_r - \frac{(v_{h\|act} - v_{r\|act})^2}{2a}$$

Recalling $h = min(d(t)) - C$, we get

$$h = d(t^*) = d_{act} + (v_{r\|act} - v_{h\|act})t_r - \frac{(v_{h\|act} - v_{r\|act})^2}{2a} - C$$

when $v_{r\|act} \to 0$, the threshold value of d_{act} where $h = 0$ is $d_{th} = C$
Remark $v_{r\|act} \to 0$ and $v_{r\|act} \leq v_{h\|act} \leq 0$, so $v_{h\|act} \to 0$

Remark

$$d_{th} = \begin{cases} C + v_{h\|act}t_r & \text{when } v_{h\|act} > 0 \\ C & \text{otherwise} \end{cases}$$

Case $v_{r\|act} \geq 0$

Philosophical issue: the robot should actively act to keep the distance even if it is already moving away from the human (*i.e.* increase speed to keep the minimum distance)?

Cesare says no, Federico says yes

Option 1: robot does not actively keep the distance. All the first quadrant respect the constraint ($h > 0$)

subcase $d_{act} \leq d_{th}$

The distance is below the threshold value compute when $v_{r\|act} = 0^-$. Thus the velocity $v_{r\|act}$ should not decrease below the zero value. We define the barrier function to be linearly proportional to $v_{r\|act}$

$$h = \kappa v_{r\|act}$$

where $\kappa > 0$ is a scaling factor.

subcase $d_{act} > d_{th}$

In this case, the robot can change direction. We define h as the distance from the point $\{d_{act} = d_{th}, v_{r\|act} = 0\}$.
The $2-$norm is not derivable when $h = 0$, for this reason we use the $1-$ norm:

$$h = \|[d_{act} - d_{th}, \kappa v_{r\|act}]\|_1$$

since $d0 > d_{th}$ and $v_{r\|act} > 0$, the $1-$norm simply becomes:

$$h = d_{act} - d_{th} + \kappa v_{r\|act}$$

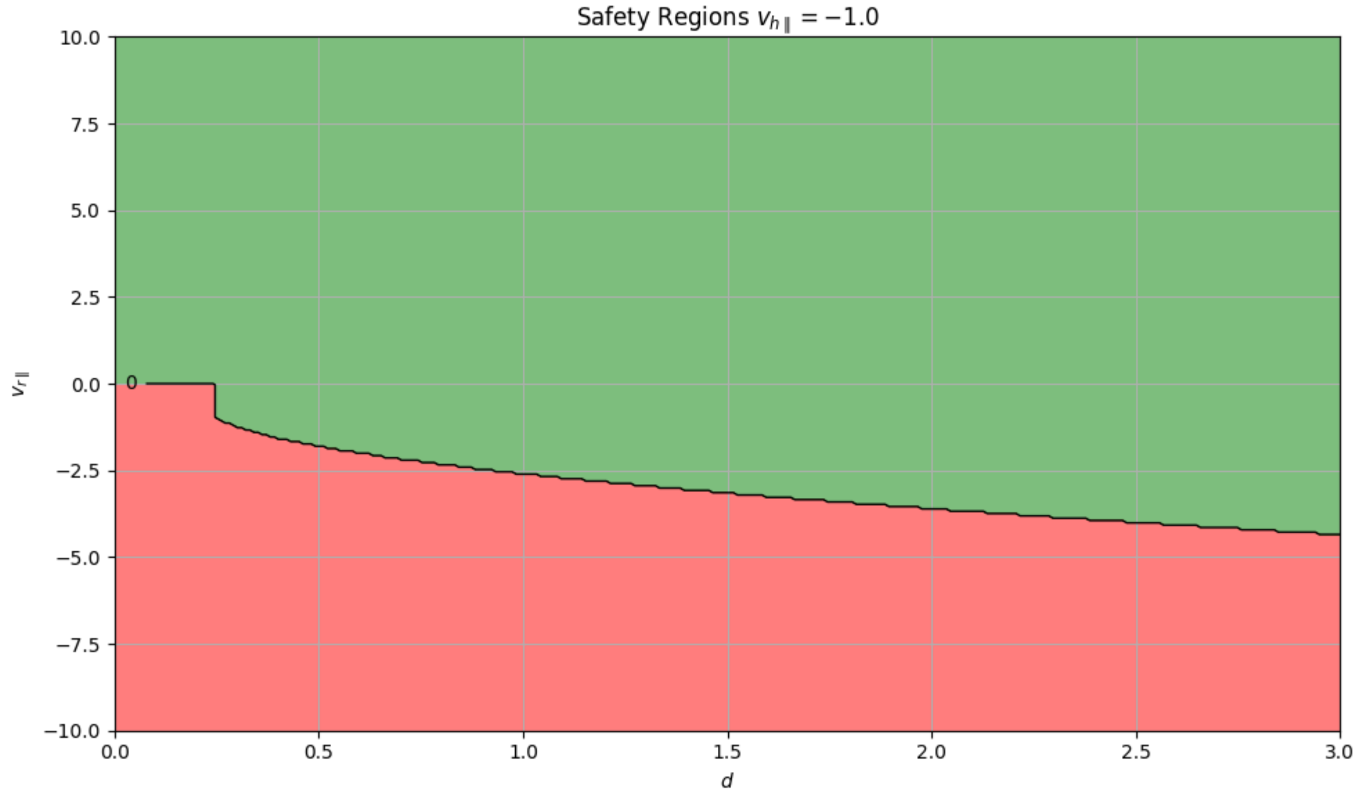
Remark (Continuity):
when $d_{act} \geq d_{th}$, h is continuous in $v_{r\|act} = 0$. In fact

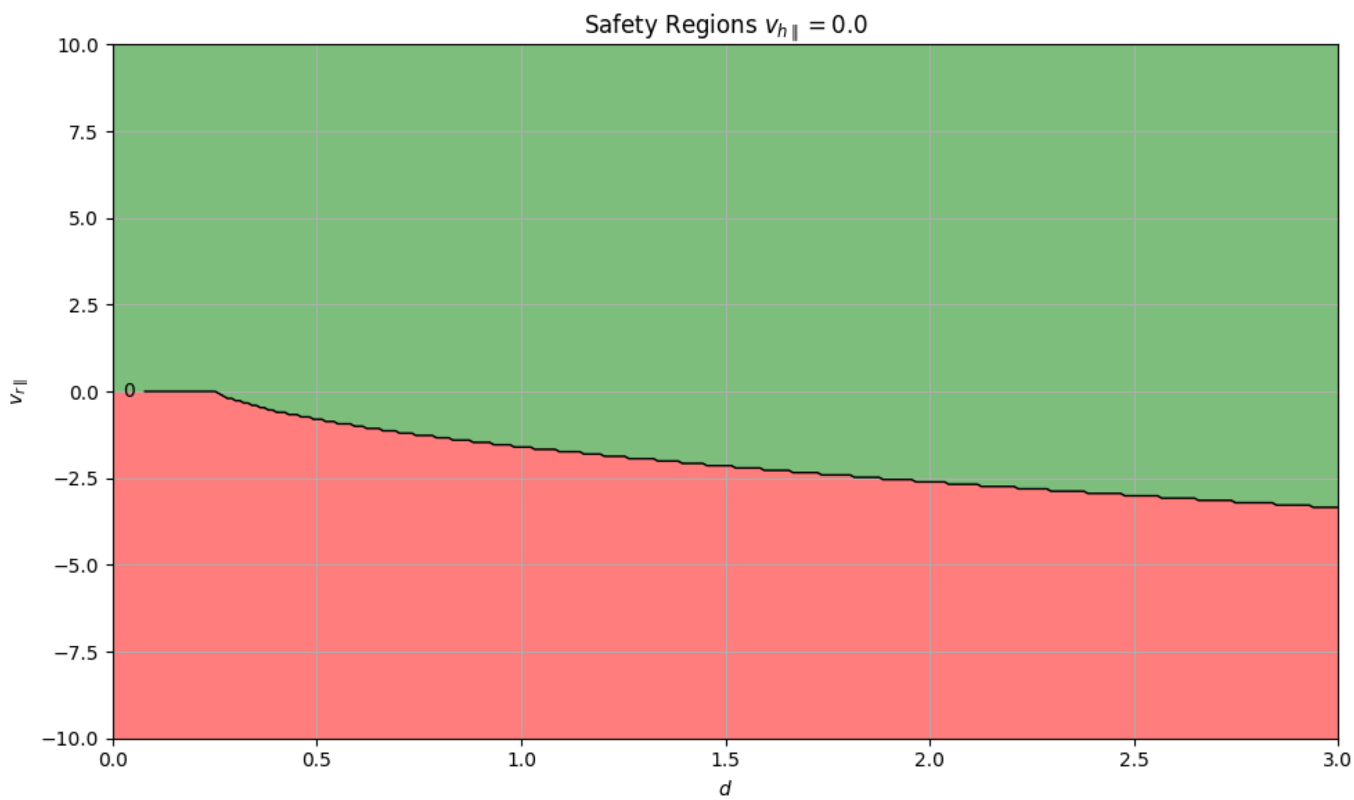
- $v_{r\|act} = 0^- \rightarrow h = d_{act} - d_{th}$
- $v_{r\|act} = 0^+ \rightarrow h = d_{act} - d_{th}$

when $v_{r\|act} \geq 0$, h is continuous in $d_{act} = d_{th}$. In fact

- $d_{act} = d_{th}^{-} \rightarrow h = \kappa v_{r\|act}$
- $d_{act} = d_{th}^{+} \rightarrow h = \kappa v_{r\|act}$

Safefy regions


Safety Regions $v_{h\parallel} = -1.0$
$v_{r\parallel}$
d
0
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0


Safety Regions $v_{h\parallel} = 0.0$
$v_{r\parallel}$
d
0
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0.0
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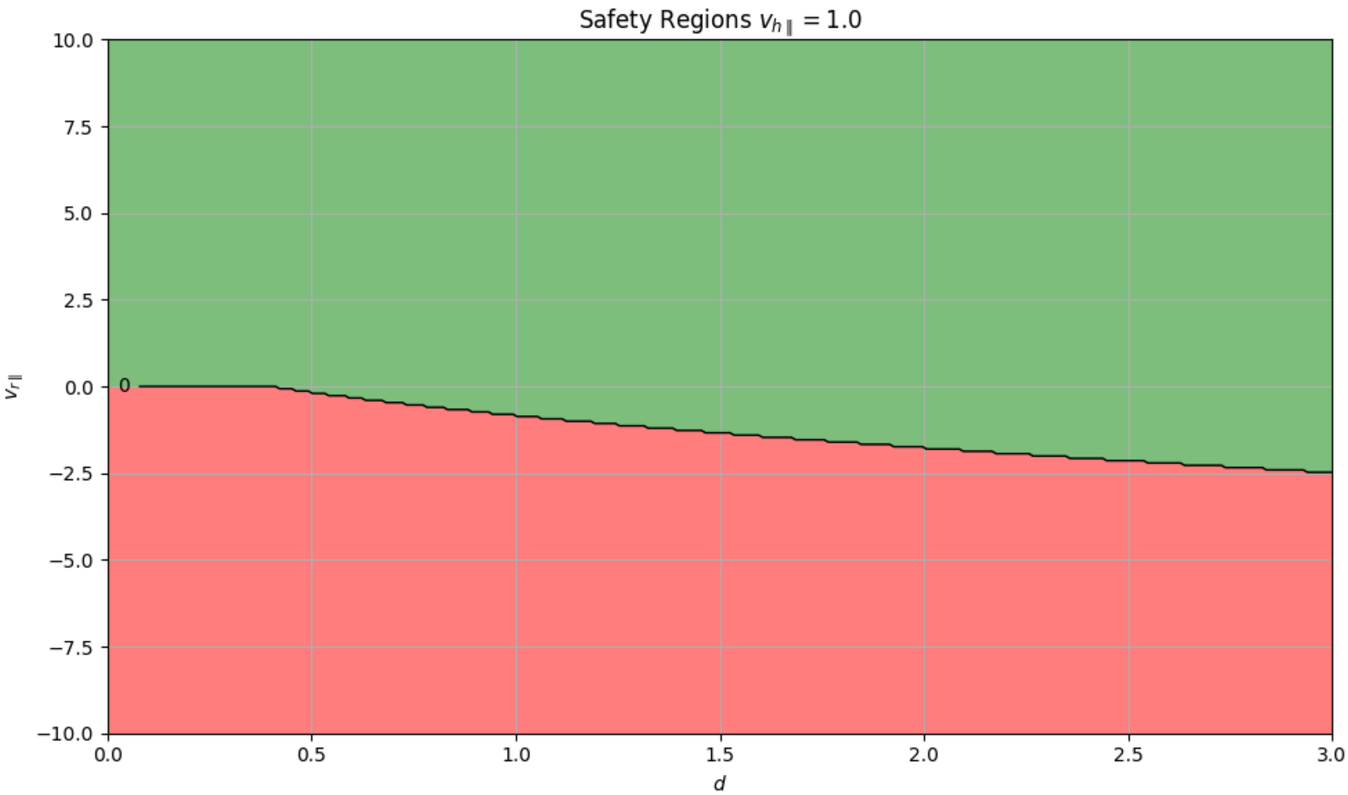
1.0
1.5
2.0
2.5
3.0



Derivative of h with respect to $\psi = [d_{act}, v_{r\|act}, v_{h\|act}]$

case $v_{r\|act} < 0$

subcase $v_{h\|act} > 0$

$$h = d_{act} + (v_{r\|act} - v_{h\|act})t_r + \frac{v_{h\|act}v_{r\|act}}{a} - \frac{1}{2} \cdot \frac{v_{r\|act}^2}{a} - C$$

$$\frac{\partial h}{\partial d_{act}} = 1$$

$$\frac{\partial h}{\partial v_{r\|act}} = t_r + \frac{v_{h\|} - v}{a}$$

$$\frac{\partial h}{\partial v_{h\|acc}} = -t_r + \frac{v_{r\|act}}{a}$$

subcase $v_{h\|act} < v_{r\|act}$

$$h = d_{act} - C$$

$$\frac{\partial h}{\partial d_{act}} = 1$$

$$\frac{\partial h}{\partial v_{r\|act}} = 0$$

$$\frac{\partial h}{\partial v_{h\|act}} = 0$$

subcase $v_{r\|act} \leq v_{h\|act} \leq 0$

$$h = d(t^*) = d_{act} + (v_{r\|act} - v_{h\|act})t_r - \frac{(v_{h\|act} - v_{r\|act})^2}{2a} - C$$

$$\frac{\partial h}{\partial d_{act}} = 1$$

$$\frac{\partial h}{\partial v_{r\|act}} = t_r + \frac{v_{h\|act} - v_{r\|act}}{a}$$

$$\frac{\partial h}{\partial v_{h\|}} = -t_r - \frac{v_{h\|act} - v_{r\|act}}{a}$$

Case $v_{r\|act} \geq 0$

subcase $d_{act} \leq d_{th}$

$$h = \kappa v_{r\|act}$$

$$\frac{\partial h}{\partial d_{act}} = 0$$

$$\frac{\partial h}{\partial v_{r\|act}} = \kappa$$

$$\frac{\partial h}{\partial v_{h\|act}} = \frac{\partial \kappa}{\partial v_{h\|act}} v_{r\|act}$$

subcase $d_{act} > d_{th}$

$$h = d_{act} - d_{th} + \kappa v_{r\|act}$$

$$\frac{\partial h}{\partial d_{act}} = 1$$

$$\frac{\partial h}{\partial v_{r\|act}} = \kappa$$

$$\frac{\partial h}{\partial v_{h\|act}} = \frac{\partial \kappa}{\partial v_{h\|act}} v_{r\|act}$$

Remark (Choice of κ)
We choose κ to try keeping $\frac{\partial h}{\partial v_{r\|act}}$ continuous when $d_{act} > d_{th}$ and $v_{r\|act} = 0$
When $v_{r\|act} = 0^-$ and $v_{h\|act} \geq 0 > v_{r\|act}$

$$\frac{\partial h}{\partial v_{r\|act}} = t_r + \frac{v_{h\|act}}{a}$$

When $v_{r\|act} = 0^-$ and $v_{h\|} < v_{r\|act} < 0$

$$\frac{\partial h}{\partial v_{r\|act}} = 0$$

Thus, we choose

$$\kappa = \begin{cases} t_r + \frac{v_{h\|act}}{a} & \text{when } v_{h\|act} > 0 \\ t_r & \text{othercase} \end{cases}$$

Note: When $v_{r\|act} = 0^-$ and $v_{h\|act} < v_{r\|act} < 0$, it is not possible to keep $\frac{\partial h}{\partial v_{r\|act}}$ continuous since κ is strickly positive. Thus, we choose $\kappa = t_r$ to avoid jumps when $v_{h\|act}$ change sign.

$$\frac{\partial \kappa}{\partial v_{h\|act}} = \begin{cases} \frac{1}{a} & \text{when } v_{h\|act} > 0 \\ 0 & \text{othercase} \end{cases}$$

Dynamics of χ

The state $\chi = [p_r, h_r, v_r, v_h]$ evolution is

$$\dot{\chi} = f(\chi) + g(\chi)u$$

where $u = \dot{v}_r$
we got

$$\dot{p}_r = v_r$$

$$\dot{p}_h = v_h$$

$$\dot{v}_r = u$$

$$\dot{v}_h = \mathbf{0}_{3\times1}$$

therefore

$$f(\chi) = \begin{bmatrix} v_r \\ v_h \\ \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} \end{bmatrix}$$

and

$$g(\chi) = \begin{bmatrix} \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} \\ \mathbf{I}_{3\times3} \\ \mathbf{0}_{3\times3} \end{bmatrix}$$

Note

$$\dot{\psi} = \frac{\partial \psi}{\partial \chi}(f(\chi) + g(\chi)u))$$

and

$$\dot{h}(\psi(\chi)) = \frac{\partial h}{\partial \psi} \cdot \frac{\partial \psi}{\partial \chi} \cdot (f(\chi) + g(\chi)u) = L_f h(\psi(\chi)) + L_g h(\psi(\chi)) \cdot u$$

Multiple tuples

How to manage multiple tuples p_h, p_r?

- For each tuple, there is a different Jacobian $J_r(q) \to$ each one has a constraint (one row) in the optimization problem

❗ TODO ❗ [CBF ARTICLES]