THE LINK TRANSMISSION MODEL: AN EFFICIENT IMPLEMENTATION OF THE KINEMATIC WAVE THEORY IN TRAFFIC NETWORKS

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Abstract. This paper describes a numerical solution method for a dynamic network loading model that is consistent with the first order kinematic wave theory. The proposed procedure, called link transmission model (LTM), only requires calculations at network nodes. Compared to the cell transmission model (CTM), the computational complexity of the LTM is about n times smaller for the same level of accuracy, where n is the mean amount of cells in a homogeneous network link.

1. Introduction

The kinematic wave theory, introduced by Lighthill and Whitham [6] and Richards [8], can be solved using analytical and graphical methods. In his simplified theory of kinematic waves, Newell [7] analytically and graphically determines cumulative vehicle numbers on links. His analysis is limited to a freeway stretch with a few entrance and exit ramps.

For complex networks however, these methods become very cumbersome, and a numerical procedure has to be used. Numerical methods were first introduced by Daganzo [1] and Lebacque [5]. A road is divided into cells and in successive time steps, the occupancy of each cell and the flows between the cells are calculated. Node models were developed for merges ([2],[5],[3]) and diverges ([2],[5]).

This paper proposes a numerical procedure for complex networks that as in Newell [7] only requires calculations at link boundaries (network nodes). For the same level of accuracy, the computational complexity of the link transmission method (LTM) is about n times smaller compared to the cell transmission model (CTM), where n is the mean amount of cells in a homogeneous network link.

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2. Solution method

The link transmission model determines the dynamic link travel times on a traffic network given a time-varying traffic demand and given the split proportions at each junction. The network consists of homogeneous links i, which start at place x_i^0 and end at place x_i^L. The links can have any length L_i and they are connected to each other via nodes. The evolution of traffic on the road network is represented by the cumulative number of vehicles $N(x,t)$ that pass the locations x_i^0 and x_i^L of each link i by time t. And since the FIFO discipline is ensured in each part of the network, this representation in terms of cumulative vehicle numbers allows for a simple derivation of densities and link travel times.

The numerical procedure divides the simulation time into time intervals Δt. For each time interval, the method involves two steps:

1) For each link, the traffic flow model determines the sending flow at the downstream end (i.e. the maximum amount of vehicles that can be sent to a next link in case of an unlimited outflow) and the receiving flow at the upstream end (i.e. the maximum amount of vehicles that can be received by the link in case of an infinite traffic demand).

2) The node models determine which parts of these sending and receiving flows can actually be sent and received. They transfer the vehicles from upstream (sending) to downstream (receiving) links and they update the cumulative vehicle numbers at the boundaries of the connected links.

3. Traffic flow model and determination of sending and receiving flows

Traffic propagates on a link as assumed in kinematic wave theory, which is based on the conservation of vehicles concept. Newell [7] shows that vehicles are neither created nor lost along a homogeneous link in a certain region of interest $(\Delta x, \Delta t)$, if there exists a cumulative vehicle number function $N(x,t)$ in this region $(\Delta x, \Delta t)$. The partial derivatives of $N(x,t)$ are the flow and density functions:

$$q(x,t) = \frac{\partial N(x,t)}{\partial t} \tag{1}$$

$$k(x,t) = \frac{-\partial N(x,t)}{\partial x} \tag{2}$$

Provided that the function $N(x,t)$ exists in a certain region $(\Delta x, \Delta t)$, as well as its first and second derivatives, the identity

$$\frac{\partial^2 N(x,t)}{\partial x \partial t} = \frac{\partial^2 N(x,t)}{\partial t \partial x} \tag{3}$$

together with (1) and (2) becomes:

$$\frac{\partial q(x,t)}{\partial x} + \frac{\partial k(x,t)}{\partial t} = 0 \tag{4}$$

which is the well-known form of the conservation law. Green's theorem yields:

$$N(x_2,t_2) - N(x_1,t_1) = \int_C dN(x,t) = \int_C \frac{\partial N(x,t)}{\partial t} dt + \frac{\partial N(x,t)}{\partial x} dx = \int_C q dt - k dx \tag{5}$$

for all C, where C is an arbitrary curve from (x_1,t_1) to (x_2,t_2) in a region $(\Delta x,\Delta t)$ in which vehicles are conserved.

The kinematic Wave theory further assumes a functional relation between the traffic flow q and density k, also known as the fundamental diagram of traffic flow. As in Newell (x), we will approximate this flow-density relation for reasons of simplicity by a triangular shaped fundamental diagram, defined by three parameters: a fixed free-flow speed (v_f), a maximum flow or capacity (q_M) occurring at critical density k_M and a jam density (k_J) (see figure 1). For $k < k_M$, vehicles travel with a fixed free-flow speed v_f, and for $k > k_M$, traffic is congested.

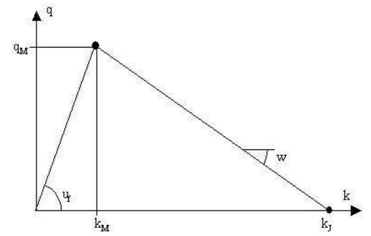

Figure 1. Triangular shaped fundamental diagram.

The sending flow of link i at time t is determined as the outflow out of this link between time t and $t + \Delta t$ in the case that there are no boundary conditions at x_i^L. The sending flow is restricted by the link's upstream boundary conditions if they include a free-flow traffic state. A free-flow traffic state at the upstream boundary will travel forward with wave speed v_f. It will reach the downstream boundary L_i/v_f time units later. The change in cumulative vehicle number between these points of interest equals (using equation (5), with C a curve in $(\Delta x, \Delta t)$ along which k ($k \le k_M$) and q are constant):

$$N(x_i^0, t+\Delta t-\frac{L_i}{v_f}) - N(x_i^L, t+\Delta t) = q(-\frac{L_i}{v_f}) - k(-L_i) = (L_i)(-\frac{q}{v_f}+k) = 0 \tag{6}$$

The maximum amount of vehicles that can be sent by link i during Δt in this case is restricted to:

$$S_{i,boundary}(t) = N(x_i^0, t+\Delta t-\frac{L_i}{v_f}) - N(x_i^L, t) \tag{7}$$

The sending flow of link i is also restricted by its link properties. The maximum flow of link i equals q_M (figure 1). The maximum amount of vehicles that can leave link i during Δt therefore equals:

$$S_{i,link}(t) = q_M \cdot \Delta t \tag{8}$$

In general, the sending flow of link i will be the most restrictive of these two conditions:

$$S_i(t) = \min(S_{i,boundary}(t), S_{i,link}(t)) \tag{9}$$

For the receiving flow, we find:

$$R_{i,boundary}(t) = N(x_i^L, t+\Delta t+\frac{L_i}{w}) + k_j L_i - N(x_i^0, t) \tag{10}$$

$$R_{i,link}(t) = q_M \cdot \Delta t \tag{11}$$

$$R_i(t) = \min(R_{i,boundary}(t), R_{i,link}(t)) \tag{12}$$

4. Node models

A node model determines which parts of the sending and receiving flows can actually be sent and received. To do this, a node always obeys to the conservation of vehicles concept. Besides that, each node holds some particular priority-rule or split proportion. A node model transfers the traffic from upstream (sending) to downstream (receiving) links and it updates the cumulative vehicle counts at the link boundaries it is connected to. Below, we discuss this procedure for the different node types that are necessary to model highway networks (Daganzo [2]).

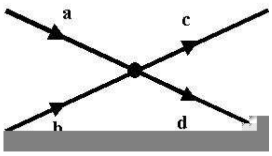

Figure 2. A general node connecting links a, b, c and d

4.1. Inhomogenous node (connecting link *a* with link *c*)

An inhomogeneous node connects an incoming link to an outgoing link. It can be used to model a change in capacity. The flow through an inhomogeneous node is the maximum that can be sent by the upstream link unless prevented to do so by the downstream link:

$$N(x_a^L, t+\Delta t) = N(x_a^L, t) + \min(S_a(t), R_c(t)) \tag{13}$$

4.2. Merge node (connecting links *a* and *b* with link *c*)

A merge node connects two incoming links (*a* & *b*) to only one outgoing link (*c*). It can be used to model an on-ramp. We will discuss three kinds of merge models.

- Optimal merge model (Daganzo [2])

$$G_a = \text{median}\,(S_a, R_c - S_b, p_a R_c) \tag{14}$$

$$G_b = \text{median}\,(S_b, R_c - S_a, p_b R_c) \tag{15}$$

- Fairness model (Jin and Zhang [3])

$$G_j = \min\left[S_j, R_c \frac{S_j}{S_a + S_b}\right] \qquad \text{for } j = a,b \tag{16}$$

- Fractioned-off merge model (Lebacque [5])

$$G_j = \min\,(S_j, \alpha_j R_c) \qquad \text{for } j = a,b \tag{17}$$

Each merge model determines the transition flows G_j for the upstream links. The cumulative curves at the link boundaries are updated as follows:

$$N(x_a^L, t+\Delta t) = N(x_a^L, t) + G_a(t) \tag{18}$$

$$N(x_b^L, t+\Delta t) = N(x_b^L, t) + G_b(t) \tag{19}$$

$$N(x_c^0, t+\Delta t) = N(x_c^0, t) + G_a(t) + G_b(t) \tag{20}$$

4.3. Diverge node (connecting link *a* with links *c* and *d*)

A diverge node connects one incoming link (a) to two outgoing links (c & d). It can be used to model an off-ramp. We study two types of diverge nodes, the FIFO and the parallel diverge. Both types work with predefined split factors that are assumed to be constant over the studied period. The parameters p_c and p_d denote the proportions of traffic going to respectively link c and link d ($p_c + p_d = 1$).

- FIFO model with split factors (Daganzo [2])

$$G_j = p_j \min\left[S, \min_{i=c,d}\left(R_i / p_i\right)\right] \qquad \text{for } j = c,d \tag{21}$$

- Parallel method with split factors (Lebacque [5])

$$G_j = \min(p_j S, R_j) \qquad \text{for } j = c,d \tag{22}$$

Both diverge models determine the transition flows that are used to update the cumulative curves as follows:

$$N(x_{a+1}, t + \Delta t) = N(x_{a+1}, t) + G_c(t) + G_d(t) \tag{23}$$

$$N(x_{c+1}, t + \Delta t) = N(x_{c+1}, t) + G_c(t) \tag{24}$$

$$N(x_{d+1}, t + \Delta t) = N(x_{d+1}, t) + G_d(t) \tag{25}$$

4.4. General node (connecting links *a* and *b* with links *c* and *d*)

For intersections with two or more upstream and downstream links, we can combine the merge and diverge models. As in Zhang [4], we combine the Zhang merge and the Daganzo diverge. The equations become:

$$G_{ij} = p_{ij} \min(\min(\frac{R_j S_i}{\sum_{q=a,b} p_{qj} S_q}), S_i) \qquad \text{for } i = a,b \text{ ; for } j = c,d \tag{26}$$

5. Accuracy and computational issues

In his CTM, Daganzo [1] and [2] proposes a system of finite difference equations to approximate the partial differential equations of the kinematic wave model. This method assumes that the road has been divided into homogeneous sections (cells), whose lengths equal the distance travelled by free-flowing traffic in one time interval. For each time interval, the CTM calculates sending and receiving flows for each cell and it determines the flows through the cell boundaries by a minimum rule. Afterwards, the number of vehicles contained in each cell are recalculated, allowing for an update of the cumulative vehicle numbers at the cell or link boundaries.

In the link transmission procedure, the sending and receiving flows are determined for the whole link. Determination of flows and cumulative vehicle numbers happens only at the link boundaries. For the same length of the time interval Δt, the cell transmission procedure requires more computational effort compared to the link transmission procedure. The computational complexity of the CTM is about n times higher, where n is the mean amount of cells in a homogeneous link.

On the other hand, we show that for the same length of the time interval Δt, the link transmission procedure has at least the same accuracy. In the case where w/v is an integer, the LTM is even able to fully reconstruct the analytical solutions, whereas the CTM suffers from numerical dispersion. In general, the LTM requires considerably less computational effort for the same level of accuracy, or, the LTM is considerably more accurate for the same level of computational effort.

6. Conclusions and outlook

The link transmission model is a numerical solution method for a kinematic wave dynamic network loading model. Compared to the cell transmission model, the computational complexity is considerably smaller for the same level of accuracy. The presented model assumes that the split proportions are known at all nodes. In [9] we present a procedure for the case where the routes to be followed are known at all nodes. The node models in this multicommodity LTM are more complicated.

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