Quantile Regression and Value at Risk

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Abstract

This paper studies quantile regression (QR) estimation of Value at Risk (VaR). VaRs estimated by the QR method display some nice properties. In this paper, different QR models in estimating VaRs are introduced. In particular, VaR

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estimations based on quantile regression of the QAR models, copula models, ARCH models, GARCH models, and the CaViaR models are systematically introduced. Comparing the proposed QR method with traditional methods based on distributional assumptions, the QR method has the important property in that it is robust to non-Gaussian distributions. Quantile estimation is only influenced by the local behavior of the conditional distribution of the response near the specified quantile. As a result, the estimates are not sensitive to outlier observations. Such a property is especially attractive in financial applications since many financial data like, say, portfolio returns (or log returns) are usually not normally distributed. To highlight the importance of the QR method in estimating VaR, we apply the QR techniques to estimate VaRs in International Equity Markets. Numerical evidence indicates that QR is a robust estimation method for VaR.

Keywords
ARCH • Copula • GARCH • Non-normality • QAR • Quantile regression • Risk management • Robust estimation • Time series • Value at risk

41.1 Introduction

The Value at Risk (VaR) is the loss in market value over a given time horizon that is exceeded with probability τ, where τ is often set at 0.01 or 0.05. In recent years, VaR has become a popular tool in the measurement and management of financial risk. This popularity is spurred both by the need of various institutions for managing risk and by government regulations (see Blankley et al., 2000; Dowd 1998, 2000; Saunders 1999). VaR is an easily interpretable measure of risk that summarizes information regarding the distribution of potential losses. In requiring publicly traded firms to report risk exposure, the Securities and Exchange Commission (SEC) lists VaR as a disclosure method "expressing the potential loss in future earnings, fair values, or cash flows from market movements over a selected period of time and with a selected likelihood of occurrence."

Estimation of VaR has attracted much attention from researchers (Duffie and Pan (1997); Wu and Xiao (2002); Guo et al. (2007)). Many existing methods of VaR estimation in economics and finance are based on the assumption that financial returns have normal (or conditional normal) distributions (usually with ARCH or GARCH effects). Under the assumption of a conditionally normal return distribution, the estimation of conditional quantiles is equivalent to estimating conditional volatility of returns. The massive literature on volatility modeling offers a rich source of parametric methods of this type. However, there is accumulating evidence that financial time series and return distributions are *not* well approximated by Gaussian models. In particular, it is frequently found that market returns display negative skewness and excess kurtosis. Extreme realizations of returns can adversely affect the performance of estimation and inference designed for Gaussian conditions; this is particularly true of ARCH and GARCH models whose estimation of variances is very sensitive to large innovations. For this reason, research attention has recently shifted toward the development of more robust estimators of conditional quantiles.

There is also growing interest in nonparametric estimation of conditional quantiles. However, nearest neighbor and kernel methods are somewhat limited in their ability to cope with more than one or two covariates. Other approaches to estimating VaR include the hybrid method and methods based on extreme value theory.

Quantile regression as introduced by Koenker and Bassett (1978) is well suited to estimating VaR. Value at Risk, as mandated in many current regulatory contexts, is a conditional quantile by definition. This concept is intimately linked to quantile regression estimation.

Quantile regression has now become a popular robust approach for statistical analysis. Just as classical linear regression methods based on minimizing sums of squared residuals enable one to estimate models for conditional mean, quantile regression methods offer a mechanism for estimating models for the conditional quantiles. These methods exhibit robustness to extreme shocks and facilitate distribution-free inference. In recent years, quantile regression estimation for time-series models has gradually attracted more attention. Koenker and Zhao (1996) extended quantile regression to linear ARCH models and estimate conditional quantiles by a linear quantile regression. Engle and Manganelli (1999) have suggested a nonlinear dynamic quantile model where conditional quantiles themselves follow an autoregression, and they call this a Conditional Autoregressive Value at Risk (CaViaR) specification. Computation of the CaViaR model is challenging and grid searching is conventionally used in practice. Koenker and Xiao (2006) investigate quantile autoregressive processes that can capture systematic influences of conditioning variables on the location, scale, and shape of the conditional distribution of the response and therefore constitute a significant extension of classical time-series models in which the effect of conditioning is confined to a location shift. Xiao and Koenker (2009) recently studied quantile regression estimation of GARCH models. GARCH models have proven to be highly successful in modeling financial data and are arguably the most widely used class of models in financial applications. However, quantile regression GARCH models are highly nonlinear and thus complicated to estimate. The quantile estimation problem in GARCH models corresponds to a restricted nonlinear quantile regression, and conventional nonlinear quantile regression techniques are not directly applicable, adding an additional challenge to the already complicated estimation problem. Koenker and Xiao (2009) propose a two-step approach for quantile regression on GARCH models. The proposed method is relatively easy to implement compared to other nonlinear estimation techniques in quantile regression and has good sampling performance in our simulation experiments.

VaRs estimated by the quantile regression approach display some nice properties. For example, they track VaRs estimated from GARCH volatility models well during normal market conditions. However, during market turmoils when large market price drops are followed by either further drops or rebounds, GARCH volatility models tend to predict implausibly high VaRs. This is due to the fact that volatility and VaRs are not synonymous. While large positive and negative return shocks indicate higher volatility, only large negative return shocks indicate higher Value at Risk. GARCH models treat both large positive and negative return shocks as indications of higher volatility. VaRs estimated by the ARCH/GARCH quantile regression model, while predicting higher volatility in the ARCH/GARCH component, assign a much bigger weight to the large negative return shock than the large positive return shock. The resulting estimated VaRs seem to be closer to reality.

In this chapter, we study quantile regression estimation of VaR. Different quantile models in estimating VaR are introduced in this paper. In particular, Value at Risk analysis based on quantile regression of the QAR models, copula models, ARCH models, GARCH models, and the CaViaR models is systematically introduced. To highlight the importance of quantile regression method in estimating VaR, we apply the quantile regression techniques to estimate VaR in International Equity Markets. Numerical evidence indicates that quantile regression is a robust estimation method for VaR.

This chapter is organized as follows. We introduce the traditional VaR estimation methods and quantile regression in Sect. 41.2. The quantile autoregression (QAR) models are given in Sect. 41.3; nonlinear QAR models based on copula and the CaViaR models are also introduced. Section 41.4 introduces quantile regression estimation on ARCH and GARCH models. Section 41.5 contains an empirical application of quantile regression estimation of VaRs. Section 41.6 concludes.

41.2 Traditional Estimation Methods of VaR

For a time series of returns on an asset, $\{r_t\}_{t=1}^{n}$, the τ (or $100\tau\%$) VaR at time t, denoted by VaR_t, is defined by

$$\Pr(r_t < -VaR_t | \mathcal{F}_{t-1}) = \tau \tag{41.1}$$

where $\mathcal{F}_{t-1}$ denotes the information set at time $t-1$, including past values of returns and possibly the value of some covariates X_t.

If we assume that the time series of returns are modeled by

$$r_t = \mu_t + \sigma_t \varepsilon_t,$$

where r_t is the return of an asset at time t and $\mu_t, \sigma_t \in \mathcal{F}_{t-1}$. The random variables ε_t are martingale difference sequences. The Conditional Value at Risk of r_t given $\mathcal{F}_{t-1}$ is

$$VaR_t(\tau) = \mu_t + \sigma_t Q_\varepsilon(\tau),$$

where $Q_\varepsilon(\tau)$ denotes the Unconditional Value at Risk of the error term ε_t. Assuming conditional normality, the 5 % VaR at time t can be computed as

$$VaR_t(0.05) = \mu_t + 1.65\sigma_t$$

where μ_t and σ_t are the conditional mean and conditional volatility for r_t.

RiskMetrics takes a simple and pragmatic approach to modeling the conditional volatility. The forecast for time t variance in RiskMetrics method is a weighted average of the previous forecast, using weight λ, and of the latest squared innovation, using weight $(1-\lambda)$:

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)r_{t-1}^2 \tag{41.2}$$

where the parameter λ is called the decay factor $(1 > \lambda > 0)$. Conceptually λ should be estimated using a maximum likelihood approach. RiskMetrics simply set it optimally at 0.94 for daily data and 0.97 for monthly data. Our analysis is on weekly data and we set λ at 0.95.

There are extensive empirical evidences supporting the use of ARCH and GARCH models in conditional volatility estimation. Bollerslev et al. (1992) provide a nice overview of the issue. Sarma et al. (2000) showed that at the 5 % level, an AR(1)-GARCH(1,1) model is a preferred model under the conditional normality assumption. The AR(1)-GARCH(1,1) model is specified:

$$\begin{aligned} r_{t+1} &= a_0 + a_1 r_t + \in_{t+1} \qquad\qquad \in_{t+1}\big|I_t \sim N\left(0, \sigma_t^2\right) \\ \sigma_t^2 &= \omega_0 + \omega_1\sigma_{t-1}^2 + \omega_2 \in_t^2. \end{aligned} \tag{41.3}$$

The conditional mean equation is modeled as an AR(1) process to account for the weakly autoregressive behavior of returns.

41.3 Quantile Regression

Quantile regression was introduced by Koenker and Bassett (1978) and has received a lot of attention in econometrics and statistics research in the past two decades. The *quantile function* of a scalar random variable Y is the inverse of its distribution function. Similarly, the *conditional quantile function* of Y given X is the inverse of the corresponding conditional distribution function, i.e.,

$$Q_Y\left(\tau|X\right) = F_Y^{-1}(\tau|X) = \inf\{y : F_Y(y|X) \geq \tau\},$$

where $F_Y(y|X) = P(Y \leq y|X)$. By definition, the τ VaR at time t is the τ-th conditional quantile of r_t giving information at time $t-1$.

Consider a random variable Y characterized by its distribution function $F(y)$, the τ-th quantile of Y is defined by

$$Q_Y(\tau) = \inf\{y|F(y) \geq \tau\}.$$

If we have a random sample $\{y_1, \ldots, y_n\}$ from the distribution F, the τ-th sample quantile can be defined as

$$\hat{Q}_Y(\tau) = \inf\left\{y \middle| \hat{F}(y) \geq \tau\right\},$$

where $\hat{F}$ is the empirical distribution function of the random sample. Note that the above sample quantile may be found by solving the following minimization problem:

$$\min_{b \in \Re}\left[\sum_{t \in \{t: y_t \geq b\}} \tau|y_t - b| + \sum_{t \in \{t: y_t < b\}} (1-\tau)|y_t - b|\right]. \tag{41.4}$$

Koenker and Bassett (1978) studied the analogue of the empirical quantile function for the linear models and generalized the concept of quantiles to the regression context.

If we consider the linear regression model

$$Y_t = \beta' X_t + u_t, \tag{41.5}$$

where u_t are iid mean zero random variables with quantile function $Q_u(\tau)$ and X_t are k-by-1 vector of regressors including an intercept term and lagged residuals, then, conditional on the regressor X_t, the τ-th quantile of Y is a linear function of X_t:

$$Q_{Y_t}(\tau|X_t) = \beta' X_t + Q_u(\tau) = \beta(\tau)' X_t$$

where $\beta(\tau)' = (\beta_1 + Q_u(\tau), \beta_2, \cdots, \beta_k)$. Koenker and Bassett (1978) show that the τ-th conditional quantile of Y can be estimated by an analogue of Eq. 41.4:

$$\hat{Q}_{Y_t}(\tau|X_t) = X_t' \hat{\beta}(\tau)$$

where

$$\hat{\beta}(\tau) = \arg\min_{\beta \in \Re^k}\left[\sum_{t \in \{t: y_t \geq x_t \beta\}} \tau\left|y_t - x_t' \beta\right| + \sum_{t \in \{t: y_t < x_t \beta\}} (1-\tau)\left|y_t - x_t' \beta\right|\right] \tag{41.6}$$

is called as the regression quantiles. Let $\rho_\tau(u) = u(\tau - I(u < 0))$, then

$$\hat{\beta}(\tau) = \arg\min_{\beta \in \Re^k} \sum_t \rho_\tau\left(y_t - x_t' \beta\right).$$

Quantile regression method has the important property that it is robust to distributional assumptions. This property is inherited from the robustness property of the ordinary sample quantiles. Quantile estimation is only influenced by the local behavior of the conditional distribution of the response near the specified quantile. As a result, the estimated coefficient vector $\hat{y}(\tau)$ is not sensitive to outlier observations. Such a property is especially attractive in financial applications since many financial data like, say, portfolio returns (or log returns) are usually heavy tailed and thus not normally distributed.

The quantile regression model has a mathematical programming representation which facilitates the estimation. Notice that the optimization problem (Eq. 41.6) may be reformulated as a linear program by introducing "slack" variables to represent the positive and negative parts of the vector of residuals (see Koenker and Bassett (1978) for a more detailed discussion). Computation of the regression quantiles by standard linear programming techniques is very efficient. It is also straightforward to impose the nonnegativity constraints on all elements of γ. Barrodale and Roberts (1974) proposed the first efficient algorithm for L_1-estimation problems based on modified simplex method. Koenker and d'Orey (1987) modified this algorithm to solve quantile regression problems. For very large quantile regression problems, there are some important new ideas which speed up the performance of computation relative to the simplex approach underlying the original code. Portnoy and Koenker (1997) describe an approach that combines some statistical preprocessing with interior point methods and achieves faster speed over the simplex method for very large problems.

41.4 Autoregressive Quantile Regression Models

41.4.1 The QAR Models

In many finance applications, the time-series dynamics can be more complicated than the classical autoregression where past information (Y_{t-j}) influences only the location of the conditional distribution of Y_t. For example, it is well known that the correlations tend to be larger in bear than in bull markets. Recognizing that the correlation is asymmetric is important for risk management and other financial applications. Any attempt to diagnose or forecast series of this type requires that a mechanism be introduced to capture the empirical features of the series.

An important extension of the classical constant coefficient time-series model is the quantile autoregression (QAR) model (Koenker and Xiao 2006). Given a time series $\{Y_t\}$, let $\mathcal{F}_t$ be the σ-field generated by $\{Y_s, s \leq t\}$; $\{Y_t\}$ is a p-th order QAR process if

$$Q_{Y_t}(\tau|\mathcal{F}_{t-1}) = \theta_0(\tau) + \theta_1(\tau)Y_{t-1} + \cdots + \theta_p(\tau)Y_{t-p}; \tag{41.7}$$

this implies, of course, that the right-hand side of Eq. 41.7 is monotonically increasing in τ. In the above QAR model, the autoregressive coefficients may be τ-dependent and thus can vary over different quantiles of the conditional distribution. Consequently, the conditioning variables not only shift the location of the distribution of Y_t but also may alter the scale and shape of the conditional distribution. The QAR models play a useful role in expanding the modeling territory of the classical autoregressive time-series models, and the classical AR(p) model can be viewed as a special case of QAR by setting $\theta_j(\tau)$ ($j = 1, \ldots, p$) to constants.

Koenker and Xiao (2006) studied the QAR model. The QAR model can be estimated by

$$\hat{\theta}(\tau) = \min_{\theta} \sum_t \rho_\tau\left(Y_t - \theta^\top X_t\right), \tag{41.8}$$

where $X_t = (1, Y_{t-1}, \ldots, Y_{t-p})^\top$ and $\theta(\tau) = (\theta_0(\tau), \theta_1(\tau), \ldots, \theta_p(\tau))^\top$; they show that under regularity assumptions, the limiting distribution of the QAR estimator is given by

$$\sqrt{n}\left(\hat{\theta}(\tau) - \theta(\tau)\right) \Rightarrow \mathcal{N}\left(0, \tau(1-\tau)\Omega_1^{-1}\Omega_0\Omega_1^{-1}\right),$$

where $\Omega_0 = E(X_t X_t^\top)$ and $\Omega_1 = \lim n^{-1} \sum_{t=1}^n f_{t-1}[F_{t-1}^{-1}(\tau)]X_t X_t^\top$.

The QAR models expand the modeling options for time series that display asymmetric dynamics and allow for local persistency. The models can capture systematic influences of conditioning variables on the location, scale, and shape of the conditional distribution of the response and therefore constitute a significant extension of classical constant coefficient linear time-series models.

Quantile varying coefficients indicate the existence of conditional heteroskedasticity. Given the QAR process (Eq. 41.7), let $\theta_0 = \mathrm{E}[\theta_0(U_t)]$, $\theta_1 = \mathrm{E}[\theta_1(U_t)]$, ..., $\theta_p = \mathrm{E}[\theta_p(U_t)]$, and

$$V_t = \theta_0(U_t) - \mathrm{E}\theta_0(U_t) + [\theta_1(U_t) - \mathrm{E}\theta_1(U_t)]Y_{t-1} + \cdots + \left[\theta_p(U_t) - \mathrm{E}\theta_p(U_t)\right]Y_{t-p};$$

the QAR process can be rewritten as

$$Y_t = \theta_0 + \theta_1 Y_{t-1} + \cdots + \theta_p Y_{t-p} + V_t \tag{41.9}$$

where V_t is martingale difference sequence. The QAR process is a weak sense AR process with conditional heteroskedasticity.

What's the difference between a QAR process and an AR process with ARCH (or GARCH) errors? In short, the ARCH type model only focuses on the first two moments, while the QAR model goes beyond the second moment and allows for more flexible structure in higher moments. Both models allow for conditional heteroskedasticity and they are similar in the first two moments, but they can be quite different beyond conditional variance.

41.4.2 Nonlinear QAR and Copula-Based Quantile Models

More complicated functional forms with nonlinearity can be considered for the conditional quantile function if we are interested in the global behavior of the time series. If the τ-th conditional quantile function of Y_t is given by

$$Q_{Y_t}(\tau|\mathcal{F}_{t-1}) = H(X_t;\, \theta(\tau)),$$

where X_t is the vector containing lagged Ys, we may estimate the vector of parameters $\theta(\tau)$ (and thus the conditional quantile of Y_t) by the following nonlinear quantile regression:

$$\min_{\theta} \sum_t \rho_\tau(Y_t - H(X_t,\, \theta)). \tag{41.10}$$

Let $\varepsilon_{t\tau} = y_t - H(x_t, \theta(\tau)), \dot{H}_\theta(x_t, \theta) = \partial H(x_t; \theta)/\partial\theta$; we assume that

$$V_n(\tau) = \frac{1}{n}\sum_t f_t\big(Q_{Y_t}(\tau|X_t)\big)\dot{H}_\theta(X_t, \theta(\tau))\dot{H}_\theta(X_t, \theta(\tau))^\top \xrightarrow{P} V(\tau),$$

$$\Omega_n(\tau) = \frac{1}{n}\sum_t \dot{H}_\theta(X_t, \theta(\tau))\dot{H}_\theta(X_t, \theta(\tau))^\top \xrightarrow{P} \Omega(\tau),$$

and

$$\frac{1}{\sqrt{n}}\sum_t \dot{H}_\theta(x_t,\, \theta(\tau))Y_\tau(\varepsilon_{t\tau}) \Rightarrow N(0,\;\; \tau(1-\tau)\Omega(\tau)),$$

where $V(\tau)$ and $\Omega(\tau)$ are non-singular; then under appropriate assumptions, the nonlinear QAR estimator $\hat{\theta}(\tau)$ defined as solution of Eq. 41.10 is root-n consistent and

$$\sqrt{n}\Big(\hat{\theta}(\tau) - \theta(\tau)\Big) \Rightarrow N\Big(0, \tau(1-\tau)V(\tau)^{-1}\Omega(\tau)V(\tau)^{-1}\Big). \tag{41.11}$$

In practice, one may employ parametric copula models to generate nonlinear-in-parameters QAR models (see, e.g., Bouyé and Salmon 2008; Chen et al. 2009). Copula-based Markov models provide a rich source of potential nonlinear dynamics describing temporal dependence and tail dependence. If we consider, for example, a first-order strictly stationary Markov process, $\{Y_t\}_{t=1}^n$, whose probabilistic properties are determined by the joint distribution of Y_{t-1} and Y_t, say, $G^*(y_{t-1}, y_t)$, and suppose that $G^*(y_{t-1}, y_t)$ has continuous marginal distribution function $F^*(\cdot)$, then by Sklar's Theorem, there exists a unique copula function $C^*(\cdot, \cdot)$ such that

$$G^*(y_{t-1}, y_t) \equiv C^*(F^*(y_{t-1}),\;\; F^*(y_t)),$$

where the copula function $C^*(\cdot, \cdot)$ is a bivariate probability distribution function with uniform marginals. Differentiating $C^*(u, v)$ with respect to u and evaluating at $u = F^*(x)$, $v = F^*(y)$, we obtain the conditional distribution of Y_t given $Y_{t-1} = x$:

$$\Pr[Y_t < y|Y_{t-1} = x] = \left.\frac{\partial C^*(u,v)}{\partial u}\right|_{u=F^*(x),\, v=F^*(y)} \equiv C_1^*(F^*(x), F^*(y)).$$

For any $\tau \in (0, 1)$, solving $\tau = \Pr[Y_t < y|Y_{t-1} = x] \equiv C_1^*(F^*(x), F^*(y))$ for y (in terms of τ), we obtain the τ-th conditional quantile function of Y_t given $Y_{t-1} = x$:

$$Q_{Y_t}(\tau|x) = F^{*-1}\left(C_1^{*-1}(\tau; F^*(x))\right),$$

where $F^{*-1}(\cdot)$ signifies the inverse of $F^*(\cdot)$ and $C_1^{*-1}(\cdot;u)$ is the partial inverse of $C_1^*(u,v)$ with respect to $v = F^*(y_t)$.

In practice, neither the true copula function $C^*(\cdot, \cdot)$ nor the true marginal distribution function $F^*(\cdot)$ of $\{Y_t\}$ is known. If we model both parametrically by $C(\cdot, \cdot; \alpha)$ and $F(y; \beta)$, then the τ-th conditional quantile function of Y_t, $Q_{Y_t}(\tau|x)$, becomes a function of the unknown parameters α and β, i.e.,

$$Q_{Y_t}(\tau|x) = F^{-1}\left(C_1^{-1}(\tau; F(x, \beta), \alpha), \beta\right).$$

Denoting $\theta = (\alpha', \beta')'$ and $h(x, \alpha, \beta) \equiv C_1^{-1}(\tau; F(x, \beta), \alpha)$, we will write

$$Q_{Y_t}(\tau|x) = F^{-1}(h(x, \alpha, \beta), \beta) \equiv H(x; \theta). \tag{41.12}$$

For example, if we consider the Clayton copula:

$$C(u, v; \alpha) = [u^{-\alpha} + v^{-\alpha} - 1]^{-1/\alpha}, \text{ where } \alpha > 0,$$

one can easily verify that the τ-th conditional quantile function of U_t given u_{t-1} is

$$Q_{U_t}\left(\tau|u_{t-1}\right) = \left[\left(\tau^{-\alpha/(1+\alpha)} - 1\right)u_{t-1}^{-\alpha} + 1\right]^{-1/\alpha}.$$

See Bouyé and Salmon (2008) for additional examples of copula-based conditional quantile functions.

Although the quantile function specification in the above representation assumes the parameters to be identical across quantiles, we may permit the estimated parameters to vary with τ, thus extending the original copula-based QAR models to capture a wide range of systematic influences of conditioning variables on the conditional distribution of the response. By varying the choice of the copula specification, we can induce a wide variety of nonlinear QAR(1) dependence, and the choice of the marginal enables us to consider a wide range of possible tail behavior as well. In many financial time-series applications, the nature of the temporal dependence varies over the quantiles of the conditional distribution. Chen et al. (2009) studied asymptotic properties of the copula-based nonlinear quantile autoregression.

41.4.3 The CaViaR Models

Quantile-based method provides a local approach to directly model the dynamics of a time series at a specified quantile. Engle and Manganelli (2004) propose the Conditional Autoregressive Value at Risk (CaViaR) specification for the τ-th conditional quantile of u_t:

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) = \beta_0 + \sum_{i=1}^{p} \beta_i Q_{u_{t-i}}(\tau|\mathcal{F}_{t-i-1}) + \sum_{j=1}^{q} \alpha_j \ell\left(X_{t-j}\right), \tag{41.13}$$

where $X_{t-j} \in \mathcal{F}_{t-j}$, $\mathcal{F}_{t-j}$ is the information set at time $t-j$. A natural choice of X_{t-j} is the lagged u. When we choose $X_{t-j} = |u_{t-j}|$, we obtain GARCH-type CaViaR models:

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) = \beta_0 + \sum_{i=1}^{p} \beta_i Q_{u_{t-i}}(\tau|\mathcal{F}_{t-i-1}) + \sum_{j=1}^{q} \alpha_j \left|u_{t-j}\right|.$$

If $X_{t-j} = 0$, we obtain an autoregressive model for the VaRs:

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) = \beta_0 + \sum_{i=1}^{p} \beta_i Q_{u_{t-i}}(\tau|\mathcal{F}_{t-i-1}).$$

Engle and Manganelli (2004) discussed many choices of $\ell(X_{t-j})$, leading to different specifications of the CaViaR model.

41.5 Quantile Regression of Conditional Heteroskedastic Models

41.5.1 ARCH Quantile Regression Models

ARCH and GARCH models have proven to be highly successful in modeling financial data. Estimators of volatilities and quantiles based on ARCH and GARCH models are now widely used in finance applications. Consider the following linear ARCH(p) process:

$$u_t = \sigma_t \cdot \varepsilon_t, \sigma_t = \gamma_0 + \gamma_1 |u_{t-1}| + \cdots + \gamma_p \left|u_{t-p}\right|, \tag{41.14}$$

where $0 < \gamma_0 < \infty$, $\gamma_1, \ldots, \gamma_p \geq 0$ and ε_t are iid(0,1) random variables with pdf $f(\cdot)$ and CDF $F(\cdot)$. Let $Z_t = (1, |u_{t-1}|, \ldots, |u_{t-q}|)^\top$ and $\gamma(\tau) = (\gamma_0 F^{-1}(\tau), \gamma_1 F^{-1}(\tau), \ldots, \gamma_q F^{-1}(\tau))^\top$; the conditional quantiles of u_t is given by

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) = \gamma_0(\tau) + \gamma_1(\tau)|u_{t-1}| + \cdots + \gamma_p(\tau)\left|u_{t-p}\right| = \gamma(\tau)^\top Z_t$$

and can be estimated by the following linear quantile regression of u_t on Z_t:

$$\min_\gamma \sum_t \rho_\tau\left(u_t - \gamma^\top Z_t\right), \tag{41.15}$$

where $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_q)^\top$. The asymptotic behavior of the above quantile regression estimator is given by Koenker and Zhao (1996). In particular, suppose that u_t is given by model (Eq. 41.14), f is bounded and continuous, and $f(F^{-1}(t)) > 0$ for any $0 < \tau < 1$. In addition, if $\mathrm{E}|u_t|^{2+\delta} < \infty$, then the regression quantile $\hat{\gamma}(\tau)$ of Eq. 41.15 has the following Bahadur representation:

$$\sqrt{n}(\hat{\gamma}(\tau) - \gamma(\tau)) = \frac{\Sigma_1^{-1}}{f\left(F^{-1}(\tau)\right)} \frac{1}{\sqrt{n}} \sum_{t=1}^n Z_t^\top \psi_\tau(\varepsilon_{t\tau}) + o_p(1)$$

where $\Sigma_1 = \mathrm{E}Z_t Z_t'/\sigma_t$ and $\varepsilon_{t\tau} = \varepsilon_t - F^{-1}(\tau)$. Consequently,

$$\sqrt{n}(\hat{\gamma}(\tau) - \gamma(\tau)) = N\left(0, \ \frac{\tau(1-\tau)}{f\left(F^{-1}(\tau)\right)^2} \Sigma_1^{-1}\Sigma_0\Sigma_1^{-1}\right), \ \text{with } \Sigma_0 = \mathrm{E}Z_t Z_t'.$$

In many applications, conditional heteroskedasticity is modeled on the residuals of a regression. For example, we may consider the following AR-ARCH model:

$$Y_t = \alpha' X_t + u_t \tag{41.16}$$

where $X_t = (1, Y_{t-1}, \ldots, Y_{t-p})^\top$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p)^\top$, and u_t is a linear ARCH(p) process given by model (Eq. 41.14). The conditional quantiles of Y_t is then given by

$$Q_{Y_t}(\tau|\mathcal{F}_{t-1}) = \alpha' X_t + \gamma(\tau)^\top Z_t. \tag{41.17}$$

One way to estimate the above model is to construct a *joint* estimation of α and $\gamma(\tau)$ based on *nonlinear* quantile regression. Alternatively, we may consider a two-step procedure that estimates α in the first step and then estimates $\gamma(\tau)$ based on the estimated residuals. The two-step procedure is usually less efficient because the preliminary estimation of α may affect the second-step estimation of $\gamma(\tau)$, but it is computationally much simpler and is widely used in empirical applications.

41.5.2 GARCH Quantile Regression Models

ARCH models are easier to estimate, but cannot parsimoniously capture the persistent influence of long past shocks comparing to the GARCH models. However, quantile regression GARCH models are highly nonlinear and thus complicated to estimate. In particular, the quantile estimation problem in GARCH models corresponds to a restricted nonlinear quantile regression, and conventional nonlinear quantile regression techniques are not directly applicable.

Xiao and Koenker (2009) studied quantile regression estimation of the following linear GARCH(p, q) model:

$$u_t = \sigma_t \cdot \varepsilon_t, \tag{41.18}$$

$$\sigma_t = \beta_0 + \beta_1\sigma_{t-1} + \cdots + \beta_p\sigma_{t-p} + \gamma_1|u_{t-1}| + \cdots + \gamma_q|u_{t-q}|. \tag{41.19}$$

Let $\mathcal{F}_{t-1}$ represents information up to time $t-1$; the τ-th conditional quantile of u_t is given by

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) = \theta(\tau)^\top Z_t, \tag{41.20}$$

where $Z_t = (1, \sigma_{t-1}, \ldots, \sigma_{t-p}, |u_{t-1}|, \ldots, |u_{t-q}|)^\top$ and $\theta(\tau)^\top = (\beta_0, \beta_1, \ldots, \beta_p, \gamma_1, \ldots, \gamma_q)F^{-1}(\tau)$.

Since Z_t contains $\sigma_{t-k}(k = 1, \cdots, q)$ which in turn depends on unknown parameters $\theta = (\beta_0, \beta_1, \ldots, \beta_p, \gamma_1, \ldots, \gamma_q)$, we may write Z_t as $Z_t(\theta)$ to emphasize the nonlinearity and its dependence on θ. If we use the following nonlinear quantile regression

$$\min_\theta \sum_t \rho_\tau\left(u_t - \theta^\top Z_t(\theta)\right), \tag{41.21}$$

for a fixed τ in isolation, consistent estimate of θ cannot be obtained since it ignores the global dependence of the σ_{t-k}'s on the entire function $\theta(\cdot)$. If the dependence structure of u_t is characterized by (1) and (1), we can consider the following restricted quantile regression instead of Eq. 41.21:

$$\left(\hat{\pi}, \hat{\theta}\right) = \begin{cases} \arg\min_{\pi,\theta} \sum_i \sum_t \rho_{\tau_i}\left(u_t - \pi_i^\top Z_t(\theta)\right) \\ \text{s.t. } \pi_i = \theta(\tau_i) = \theta F^{-1}(\tau_i). \end{cases}$$

Estimation of this global restricted nonlinear quantile regression is complicated. Xiao and Koenker (2009) propose a simpler two-stage estimator that both incorporates the global restrictions and also focuses on the local approximation around the specified quantile. The proposed estimation consists of the following two steps: (i) The first step considers a global estimation to incorporate the global dependence of the latent σ_{t-k}'s on θ. (ii) Then, using results from the first step, we focus on the specified quantile to find the best local estimate for the conditional quantile. Let

$$A(L) = 1 - \beta_1 L - \cdots - \beta_p L^p, \quad B(L) = \gamma_1 + \cdots + \gamma_q L^{q-1};$$

under regularity assumptions ensuring that $A(L)$ is invertible, we obtain an ARCH(∞) representation for σ_t:

$$\sigma_t = a_0 + \sum_{j=1}^{\infty} a_j |u_{t-j}|. \tag{41.22}$$

For identification, we normalize $a_0 = 1$. Substituting the above ARCH(∞) representation into (1) and (1), we have

$$u_t = \left(a_0 + \sum_{j=1}^{\infty} a_j |u_{t-j}| \right) \varepsilon_t, \tag{41.23}$$

and

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) = \alpha_0(\tau) + \sum_{j=1}^{\infty} \alpha_j(\tau) |u_{t-j}|,$$

where $\alpha_j(\tau) = a_j Q_{\varepsilon_t}(\tau), j = 0, 1, 2, \ldots$.

Let $m = m(n)$ be a truncation parameter; we may consider the following truncated quantile autoregression:

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) \approx a_0(\tau) + a_1(\tau)|u_{t-1}| + \cdots + a_m(\tau)|u_{t-m}|.$$

By choosing m suitably small relative to the sample size n, but large enough to avoid serious bias, we obtain a sieve approximation for the GARCH model.

One could estimate the conditional quantiles simply using a sieve approximation:

$$\check{Q}_{u_t}(\tau \mid \mathcal{F}_{t-1}) = \hat{a}_0(\tau) + \hat{\alpha}_1(\tau) \mid u_{t-1} \mid + \cdots + \hat{a}_m(\tau) \mid u_{t-m} \mid,$$

where $\hat{a}_j(\tau)$ are the quantile autoregression estimates. Under regularity assumptions

$$\check{Q}_{u_t}(\tau \mid \mathcal{F}_{t-1}) = Q_{u_t}(\tau \mid \mathcal{F}_{t-1}) + O_p(m/\sqrt{n}).$$

However, Monte Carlo evidence indicates that the simple sieve approximation does not directly provide a good estimator for the GARCH model, but it serves as an adequate preliminary estimator. Since the first step estimation focuses on the global model, it is desirable to use information over multiple quantiles in estimation.

Combining information over multiple quantiles helps us to obtain globally coherent estimate of the scale parameters.

Suppose that we estimate the m-th order quantile autoregression

$$\widetilde{\alpha}(\tau) = \arg\min_{\alpha} \sum_{t=m+1}^{n} \rho_\tau\left(u_t - \alpha_0 - \sum_{j=1}^{m} \alpha_j |u_{t-j}|\right) \tag{41.24}$$

at quantiles $(\tau_1, \ldots, \tau_K)$ and obtain estimates $\widetilde{\alpha}(\tau_k)$, $k = 1, \ldots, K$. Let $\widetilde{\alpha}_0 = 1$ in accordance with the identification assumption. Denote

$$\mathbf{a} = [a_1, \ldots, a_m, q_1, \ldots, q_K]^\top, \quad \boldsymbol{\overline{\pi}} = \left[\widetilde{\alpha}(\tau_1)^\top, \ldots, \widetilde{\alpha}(\tau_K)^\top\right]^\top,$$

where $q_k = Q_{\varepsilon_t}(\tau_k)$, and

$$\phi(\mathbf{a}) = g \otimes \alpha = [q_1, a_1 q_1, \ldots, a_m q_1, \ldots, q_K, a_1 q_K, \ldots, a_m q_K]^\top,$$

where $g = [q_1, \ldots, q_K]^\top$ and $\alpha = [1, a_1, a_2, \ldots, a_m]^\top$; we consider the following estimator for the vector a that combines information over the K quantile estimates based on the restrictions $\alpha_j(\tau) = a_j Q_{\varepsilon_t}(\tau)$:

$$\widetilde{\mathbf{a}} = \arg\min_{\mathbf{a}} (\boldsymbol{\overline{\pi}} - \phi(\mathbf{a}))^\top A_n (\boldsymbol{\overline{\pi}} - \phi(\mathbf{a})), \tag{41.25}$$

where A_n is a $(K(m + 1)) \times (K(m + 1))$ positive definite matrix. Denoting $\widetilde{\mathbf{a}} = (\widetilde{a}_0, \ldots, \widetilde{a}_m)$, σ_t can be estimated by

$$\widetilde{\sigma}_t = \widetilde{a}_0 + \sum_{j=1}^{m} \widetilde{a}_j |u_{t-j}|.$$

In the second step, we perform a quantile regression of u_t on $\widetilde{Z}_t = \left(1, \widetilde{\sigma}_{t-1}, \ldots \widetilde{\sigma}_{t-p}, |u_{t-1}|, \ldots, |u_{t-q}|\right)^\top$ by

$$\min_{\theta} \sum_{t} \rho_\tau\left(u_t - \theta^\top \widetilde{Z}_t\right); \tag{41.26}$$

the two-step estimator of $\theta(\tau)^\top = (\beta_0(\tau), \beta_1(\tau), \ldots, \beta_p(\tau), \gamma_1(\tau), \ldots, \gamma_q(\tau))$ is then given by the solution of Eq. 41.26, $\widehat{\theta}(\tau)$, and the τ-th conditional quantile of u_t can be estimated by

$$\hat{Q}_{u_t}(\tau | \mathcal{F}_{t-1}) = \hat{\theta}(\tau)^\top \widetilde{Z}_t.$$

Iteration can be applied to the above procedure for further improvement.

Let $\widetilde{\alpha}(\tau)$ be the solution of Eq. 41.24; then under appropriate assumptions, we have

$$\|\widetilde{\alpha}(\tau)-\alpha(\tau)\|^2=O_p(m/n), \tag{41.27}$$

and for any $\lambda\in\mathcal{R}^{m+1}$,

$$\frac{\sqrt{n}\lambda^\top(\widetilde{\alpha}(\tau)-\alpha(\tau))}{\sigma_\lambda}\Rightarrow N(0,1),$$

where $\sigma_\lambda^2=f_\varepsilon(F_\varepsilon^{-1}(\tau))^{-2}\lambda^\top D_n^{-1}\sum_n(\tau)D_n^{-1}\lambda$, and

$$D_n=\left[\frac{1}{n}\sum_{t=m+1}^{n}\frac{x_tx_t^\top}{\sigma_t}\right],\Sigma_n(\tau)=\frac{1}{n}\sum_{t=m+1}^{n}x_tx_t^TY_\tau^2(u_{t\tau}),$$

where $x_t=(1,|u_{t-1}|,\ldots,|u_{t-m}|)^\top$.

Define

$$G=\left.\frac{\partial\phi(\mathbf{a})}{\partial\mathbf{a}^\top}\right|_{\mathbf{a}=\mathbf{a}_0}=\dot{\phi}(\mathbf{a}_0)=[g\otimes J_m\ \vdots\ I_K\otimes\alpha_0],g_0=\begin{bmatrix}Q_{\varepsilon_t}(\tau_1)\\ \cdots\\ Q_{\varepsilon_t}(\tau_K)\end{bmatrix},$$

where g_0 and α_0 are the true values of vectors $g=[q_1,\ldots,q_K]^\top$ and $\alpha=[1,a_1,a_2,\ldots,a_m]^\top$, and

$$J_m=\begin{bmatrix}0&\cdots&0\\1&\cdots&0\\ \vdots&\ddots&\vdots\\0&\cdots&1\end{bmatrix}$$

is an $(m+1)\times m$ matrix and I_K is a K-dimensional identity matrix; under regularity assumptions, the minimum distance estimator $\widetilde{\mathbf{a}}$ solving (Eq. 41.25) has the following asymptotic representation:

$$\sqrt{n}(\hat{\mathbf{a}}-\mathbf{a}_0)=\left[G^\top A_nG\right]^{-1}G^\top A_n\sqrt{n}(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi})+o_p(1)$$

where

$$\sqrt{n}(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi})=-\frac{1}{\sqrt{n}}\sum_{t=m+1}^{n}\begin{bmatrix}\left(D_n^{-1}x_t\dfrac{y_{\tau_1}(u_{t\tau_1})}{f_\varepsilon(F_\varepsilon^{-1}(\tau_1))}\right)\\ \cdots\\ \left(D_n^{-1}x_t\dfrac{y_{\tau_m}(u_{t\tau_m})}{f_\varepsilon(F_\varepsilon^{-1}(\tau_m))}\right)\end{bmatrix}+o_p(1),$$

and the two-step estimator $\hat{\theta}(\tau)$ based on Eq. 41.26 has asymptotic representation:

$$\sqrt{n}\left(\hat{\theta}(\tau)-\theta(\tau)\right)=-\frac{1}{f_{\varepsilon}\left(F_{\varepsilon}^{-1}(\tau)\right)}\Omega^{-1}\left\{\frac{1}{\sqrt{n}}\sum_{t}Z_{t}y_{\tau}(u_{t\tau})\right\}+\Omega^{-1}\Gamma\sqrt{n}(\widetilde{\alpha}-\alpha)+o_{p}(1),$$

where $a=[a_1, a_2, \ldots, a_m]^{\top}$, $\Omega = E[Z_t Z_t^{\top}/\sigma_t]$, and

$$\Gamma=\sum_{k=1}^{p}\theta_k C_k, C_k=\mathrm{E}\left[(|u_{t-k-1}|,\ldots,|u_{t-k-m}|)\frac{Z_t}{\sigma_t}\right].$$

41.6 An Empirical Application

41.6.1 Data and the Empirical Model

In this section, we apply the quantile regression method to five major world equity market indexes. The data used in our application are the weekly return series, from September 1976 to June 2008, of five major world equity market indexes: the US S&P 500 Composite Index, the Japanese Nikkei 225 Index, the UK FTSE 100 Index, the Hong Kong Hang Seng Index, and the Singapore Strait Times Index. The FTSE 100 Index data are from January 1984 to June 2008. Table 41.1 reports some summary statistics of the data.

The mean weekly returns of the five indexes are all over 0.1 % per week, with the Hang Seng Index producing an average return of 0.23 % per week, an astonishing

Table 41.1 Summary statistics of the data

	S&P 500	Nikkei 225	FTSE 100	Hang Seng	Singapore ST
Mean	0.0015	0.0010	0.0017	0.0023	0.0012
Std. Dev.	0.0199	0.0253	0.0237	0.0376	0.0291
Max	0.1002	0.1205	0.1307	0.1592	0.1987
Min	−0.1566	−0.1289	−0.2489	−0.5401	−0.4551
Skewness	−0.4687	−0.2982	−1.7105	−3.0124	−1.5077
Excess kurtosis	3.3494	2.9958	12.867	9.8971	19.3154
AC(1)	−0.0703	−0.0306	0.0197	0.0891	0.0592
AC(2)	0.0508	0.0665	0.0916	0.0803	0.0081
AC(3)	0.0188	0.0328	−0.0490	−0.0171	0.0336
AC(4)	−0.0039	−0.0418	−0.0202	−0.0122	0.0099
AC(5)	−0.0189	−0.0053	−0.0069	−0.0386	0.0519
AC(10)	−0.0446	−0.0712	0.0138	−0.0345	−0.0227

This table shows the summary statistics for the weekly returns of five major equity indexes of the world. AC(k) denotes autocorrelation of order k. The source of the data is the online data service Datastream increase in the index level over the sample period. In comparison, the average return of Nikkei 225 index is only 0.1 %. The Hang Seng's phenomenal rise does not come without risk. The weekly sample standard deviation of the index is 3.76 %, the highest of the five indexes. In addition, over the sample period the Hang Seng suffered four larger than 15 % drop in weekly index level, with maximum loss reaching 35 %, and there were 23 weekly returns below −10 %! As has been documented extensively in the literature, all five indexes display negative skewness and excess kurtosis. The excess kurtosis of Singapore Strait Times Index reached 19.31, to a large extent driven by the huge 1 week loss of 47.47 % during the 1987 market crash. The autocorrelation coefficients for all five indexes are fairly small. The Hang Seng Index seems to display the strongest autocorrelation with the AR(1) coefficient equal to 0.0891.

We consider an AR-linear ARCH model in the empirical analysis. Thus, the return process is modeled as

$$r_t = \alpha_0 + \alpha_1 r_{t-1} + \cdots + \alpha_s r_{t-s} + u_t, \tag{41.28}$$

where

$$u_t = \sigma_t \varepsilon_t, \sigma_t = \gamma_0 + \gamma_1 |u_{t-1}| + \cdots + \gamma_q |u_{t-q}|,$$

and the τ Conditional VaR of u_t is given by

$$Q_{u_t}(\tau|\mathcal{F}_{t-1}) = \gamma(\tau)' Z_t$$

$$\gamma(\tau)' = (\gamma_0(\tau), \gamma_1(\tau), \ldots, \gamma_q(\tau)), \text{ and } Z_t = (1, |u_{t-1}|, \ldots, |u_{t-q}|)'.$$

For each time series, we first conduct model specification analysis and choose the appropriate lags for the mean equation and the quantile ARCH component. Based on the selected model, we use Eq. 41.28 to obtain a time series of residuals. The residuals are then used in the ARCH VaR estimation using a quantile regression.

41.6.2 Model Specification Analysis

We conduct sequential tests for the significance of the coefficients on lags. The inference procedures we use here are asymptotic inferences. For estimation of the covariance matrix, we use the robust HAC (Heteroskedastic and Autocorrelation Consistent) covariance matrix estimator of Andrews (1991) with the data-dependent automatic bandwidth parameter estimator recommended in that paper. First of all, we choose the lag length in the autoregression,

$$r_t = \alpha_0 + \alpha_1 r_{t-1} + \cdots + \alpha_s r_{t-s} + u_t,$$

using a sequential test of significance on lag coefficients. The maximum lag length that we start with is $s = 9$, and the procedure is repeated until a rejection occurs. Table 41.2 reports the sequential testing results for the S&P 500 index. The t-statistics of all the coefficients are listed for nine rounds of the test. We see that the t-statistic of the coefficient with the maximum number of lags does not become significant until $s = 1$, the ninth round. The preferred model is an AR(1) model. The selected mean equations for all five indexes are reported in Table 41.4.

Our next task is to select the lag length in the ARCH effect

$$u_t = \left(\gamma_0 + \gamma_1|u_{t-1}| + \cdots + \gamma_q\left|u_{t-q}\right|\right)\varepsilon_t.$$

Again, a sequential test is conducted. To calculate the t-statistic, we need to estimate $\omega^2 = \tau(1 - \tau)/f(F^{-1}(\tau))^2$. There are many studies on estimating $f(F^{-1}(t))$, including Siddiqui (1960), Bofinger (1975), Sheather and Maritz (1983), and Welsh (1987). Notice that

$$\frac{dF^{-1}(t)}{dt} = \frac{1}{f\left(F^{-1}(t)\right)}; \tag{41.29}$$

following Siddiqui (1960), we may estimate (Eq. 41.29) by a simple difference quotient of the empirical quantile function. As a result,

$$f\left(\widehat{F^{-1}(t)}\right) = \frac{2h_n}{\hat{F}^{-1}(t + h_n) - \hat{F}^{-1}(t - h_n)} \tag{41.30}$$

where $\hat{F}^{-1}(t)$ is an estimate of $F^{-1}(t)$ and h_n is a bandwidth which goes to zero as $n \to \infty$. A bandwidth choice has been suggested by Hall and Sheather (1988) based on Edgeworth expansion for studentized quantiles. This bandwidth is of order $n^{-1/3}$ and has the following representation:

$$h_{HS} = z_\alpha^{2/3}\left[1.5s(t)/s''(t)\right]^{1/3} n^{-1/3},$$

where z_α satisfies $\Phi(z_a) = 1 - \alpha/2$ for the construction of $1-\alpha$ confidence intervals. In the absence of additional information, $s(t)$ is just the normal density. Starting with $q_{\max} = 10$, a sequential test was conducted and results for the 5 % VaR model of the S&P 500 Index are reported in Table 41.3. We see that in the fourth round, the t-statistic on lag 7 becomes significant. The sequential test stops here, and it suggests that ARCH(7) is appropriate.

Based on the model selection tests, we decide to use the AR(1)-ARCH(7) regression quantile model to estimate 5 % VaR for the S&P 500 index. We also conduct similar tests on the 5 % VaR models for other four indexes. To conserve space we do not report the entire testing process in the paper. Table 41.4 provides a summary of the selected models based on the tests. The mean equations

Table 41.2 VaR model mean specification test for the S&P 500 Index

Round	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
α_0	3.3460	3.3003	3.2846	3.3248	3.2219	3.7304	3.1723	3.0650	3.8125
α_1	−1.6941	−1.7693	−1.8249	−1.9987	−1.9996	−2.0868	−2.1536	−2.097	2.2094
α_2	1.2950	1.3464	1.1555	1.0776	0.0872	1.3106	1.2089	1.0016	
α_3	−0.9235	−0.9565	−0.9774	1.5521	−0.8123	−0.8162	−0.9553		
α_4	−1.0414	−1.0080	−0.9947	−1.0102	−0.9899	−0.1612			
α_5	−0.7776	−0.7642	−0.7865	−0.8288	−0.7662				
α_6	0.2094	0.5362	0.7166	−0.8931					
α_7	−1.5594	−1.5426	−1.5233						
α_8	−0.8926	−0.8664							
α_9	−0.3816								

This table reports the test results for the VaR model mean equation specification for the S&P 500 Index. The number of lags in the AR component of the ARCH model is selected according to the sequential test. The table reports the t-statistic for the coefficient with the maximum lag in the mean equation

Table 41.3 5 % VaR model ARCH specification test for the S&P 500 Index

Round	1st	2nd	3rd	4th
γ_0	−16.856	−15.263	−17.118	−15.362
γ_1	2.9163	3.1891	3.2011	3.1106
γ_2	1.9601	2.658	2.533	2.321
γ_3	1.0982	1.0002	0.9951	1.0089
γ_4	0.6807	0.8954	1.1124	1.5811
γ_5	0.7456	0.8913	0.9016	0.9156
γ_6	0.3362	0.3456	0.4520	0.3795
γ_7	1.9868	2.0197	1.8145	2.1105
γ_8	0.4866	0.4688	1.5631	
γ_9	1.2045	1.0108		
γ_{10}	1.1326			

This table reports the test results for the 5 % VaR model specification for the S&P 500 Index. The number of lags in the volatility component of the ARCH model is selected according to the test. The table reports the *t*-statistic for the coefficient with the maximum lag in the ARCH equation

Table 41.4 ARCH VaR models selected by the sequential test

Index	Mean Lag	5 % ARCH Lag
S&P 500	1	6
Nikkei 225	1	7
FTSE 100	1	6
Hang Seng	3	6
Singapore ST	2	7

This table summarizes the preferred ARCH VaR models for the five global market indexes. The number of lags in the mean equation and the volatility component of the ARCH model is selected according to the test

generally have one or two lags, except the Hang Seng Index, which has a lag of 3 and displays more persistent autoregressive effect.

For the ARCH equations, at least six lags are needed for the indexes.

41.6.3 Estimated VaRs

The estimated parameters for the mean equations for all five indexes are reported in Table 41.5. The constant term for the five indexes is between 0.11 % for the Nikkei and 0.24 % for the Hang Seng. As suggested by Table 41.1, the Hang Seng seems to display the strongest autocorrelation, and this is reflected in the four lags chosen by the sequential test. Table 41.6 reports the estimated quantile regression ARCH parameters for the 5 % VaR model:

USA – S&P 500 Index. The estimated 5 % VaRs generally range between 2.5 % and 5 %, but during very volatile periods they could jump over 10 %, as what happened in October 1987. During high-volatility periods, there is high variation in estimated VaRs.

Table 41.5 Estimated mean equation parameters

Round	S&P 500	Nikkei 225	FTSE 100	Hang Seng	Singapore ST
α_0	0.0019	0.0011	0.0022	0.0024	0.0014
	(0.0006)	(0.0006)	(0.0008)	(0.001)	(0.0009)
α_1	−0.0579	−0.0827	0.0617	0.1110	0.0555
	(0.0233)	(0.0305)	(0.0283)	(0.0275)	(0.0225)
α_2				0.0796	0.0751
				(0.0288)	(0.0288)
α_3				−0.0985	
				(0.0238)	

This table reports the estimated parameters of the mean equation for the five global equity indexes. The standard errors are in parentheses under the estimated parameters

Table 41.6 Estimated ARCH equation parameters for the 5 % VaR model

Parameter	S&P 500	Nikkei 225	FTSE 100	Hang Seng	Singapore ST
γ_0	0.0351	0.0421	0.0346	0.0646	0.0428
	(0.0016)	(0.0023)	(0.0013)	(0.0031)	(0.0027)
γ_1	0.2096	0.0651	0.0518	0.1712	0.1119
	(0.0711)	(0.0416)	(0.0645)	(0.0803)	(0.0502)
γ_2	0.1007	0.1896	0.0588	0.0922	0.1389
	(0.0531)	(0.0415)	(0.0665)	(0.0314)	(0.0593)
γ_3	−0.0101	0.1109	0.0311	0.2054	0.0218
	(0.0142)	(0.0651)	(0.0242)	(0.0409)	(0.0379)
γ_4	0.1466	0.0528	0.0589	0.0671	0.1102
	(0.0908)	(0.0375)	(0.0776)	(0.0321)	(0.0903)
γ_5	0.0105	0.0987	−0.0119	0.0229	0.1519
	(0.0136)	(0.0448)	(0.0123)	(0.0338)	(0.0511)
γ_6	0.0318	0.0155	0.0876	0.0359	0.0311
	(0.0117)	(0.0297)	(0.0412)	(0.0136)	(0.0215)
γ_7		0.2323			0.1123
		(0.0451)			(0.0517)

This table reports the estimated parameters of the ARCH equation for the 5 % VaR model for the five global indexes. The standard errors are in parentheses under the estimated parameters

Japan – Nikkei 225 Index. The estimated VaR series is quite stable and remains at the 4 % and the 7 % level from 1976 till 1982. Then the Nikkei 225 Index took off and appreciated about 450 % over the next 8 years, reaching its highest level at the end of 1989. This quick rise in stock value is accompanied by high risk, manifested here by the more volatile VaR series. In particular, the VaRs fluctuated dramatically, ranging from a low of 3 % to a high of 15 %. This volatility in VaR may reflect both optimistic market outlook at times and worry about high valuation and the possibility of a market crash. That crash did come in 1990, and

10 years later, the Nikkei 225 Index still hovers around at a level which is about half off the record high in 1989. The 1990s is far from a rewarding decade for investors in the Japanese equity market. Average weekly 5 % VaR is about 5 %, and the variation is also very high.

UK – FTSE 100 Index. The 5 % VaR is very stable and averages about 3 %. They stay very much within the 2–4 % band, except on a few occasions, such as the 1987 global market crash.

Hong Kong – Hang Seng Index. The Hang Seng Index produces an average return of 0.23 % per week. The Hang Seng's phenomenal rise does not come without risk. We mentioned above that the weekly sample standard deviation of the index is 3.76 %, the highest of the five indexes. In addition, the Hong Kong stock market has had more than its fair share of the market crashes.

Singapore – Strait Times Index. Interestingly, the estimated VaRs display a pattern very similar to that of the UK FTSE 100 Index, although the former is generally larger than the latter. The higher risk in the Singapore market did not result in higher return over the sample period. Among the five indexes, the Singapore market suffered the largest loss during the 1987 crash, a 47.5 % drop in a week. The market has since recovered much of the loss. Among the five indexes, the Singapore market only outperformed the Nikkei 225 Index over this period.

41.6.4 Performance of the ARCH Quantile Regression Model

In this section we conduct an empirical analysis to compare VaRs estimated by RiskMetrics and regression quantiles and those by volatility models with the conditional normality assumption. There are extensive empirical evidences supporting the use of the GARCH models in conditional volatility estimation. Bollerslev et al. (1992) provide a nice overview of the issue. Therefore, we compare VaR estimated based on RiskMetrics and GARCH(1,1) model and quantile regression based on ARCH.

To measure the relative performance more accurately, we compute the percentage of realized returns that are below the negative estimated VaRs. The results are reported in Table 41.7. The top panel of the table presents the percentages for the VaRs estimated by the ARCH quantile regression model, the middle panel for the VaRs estimated by the GARCH model with the conditional normal return distribution assumption, and the bottom panel for the VaRs estimated by the RiskMetrics method. We estimate VaRs using these methods at 1 %, 2 %, 5 %, 10 %. Now we have a total of four percentage levels. The regression quantile method produces the closest percentage in general. Both the RiskMetrics method and the GARCH method seem to underestimate VaRs for the smaller percentages and overestimate VaRs for the larger percentages.

The five indexes we analyzed are quite different in their risk characteristics as discussed above. The quantile regression approach seems to be relatively robust and can consistently produce reasonably good estimates of the VaRs at different

Table 41.7 VaR model performance comparison

% VaR	1 %	2 %	5 %	10 %
Quantile regression				
S&P 500	1.319	1.925	5.3108	9.656
Nikkei 225	1.350	2.011	5.7210	10.56
FTSE 100	0.714	1.867	5.6019	9.016
Hang Seng	0.799	2.113	4.9011	9.289
GARCH				
S&P 500	1.3996	1.7641	4.0114	7.6151
Nikkei 225	1.4974	1.7927	4.3676	8.4098
FTSE 100	1.1980	1.6133	3.3891	6.7717
Hang Seng	1.8962	2.8658	3.6653	7.6439
RiskMetrics				
S&P 500	0.3790	0.5199	1.1180	3.2563
Nikkei 225	0.5877	0.9814	1.358	4.1367
FTSE 100	0.2979	0.5796	0.9984	3.5625
Hang Seng	0.7798	0.9822	1.4212	4.1936

This table reports the coverage ratios, i.e., the percentage of realized returns that is below the estimated VaRs. The top panel reports the performance of the VaRs estimated by the quantile regression model. The middle panel reports the results for VaRs estimated by the GARCH model based on the conditionally normal return distribution assumption. The bottom panel reports the results for VaRs estimated by the RiskMetrics method

percentage (probability) levels. The GARCH model with the normality assumption, being a good volatility model, is not able to produce good VaR estimates. The quantile regression model does not assume normality and is well suited to hand negative skewness and heavy tails.

41.7 Conclusion

Quantile regression provides a convenient and powerful method of estimating VaR. The quantile regression approach not only provides a method of estimating the conditional quantiles (VaRs) of existing time-series models; it also substantially expands the modeling options for time-series analysis. Estimating Value at Risk using the quantile regression does not assume a particular conditional distribution for the returns. Numerical evidence indicates that the quantile-based methods have better performance than the traditional J. P. Morgan's RiskMetrics method and other methods based on normality. The quantile regression based method provides an important tool in risk management.

There are several existing programs for quantile regression applications. For example, both parametric and nonparametric quantile regression estimations can be implemented by the function **rq**() and **rqss**() in the package **quantreg** in the computing language **R**, and **SAS** now has a suite of procedures modeled closely on the functionality of the R package quantreg.

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