1) Linear Stability Analysis of the deterministic logistic equation

Suppose we want to model the evolution in population of a bacteria, such as E. Coli, which has a doubling time $T = \frac{\ln 2}{\mu}$ (i.e. $N(t+T) = 2N(t)$), where μ is the growth rate. Moreover, we want to introduce a carrying capacity K to express the limit to the increasing in population set by having a finite amount of resources.
Therefore, we obtain the logistic equation:

$$\boxed{\frac{dN}{dt} = \mu N\left(1 - \frac{N}{K}\right)} \qquad (1.1)$$

whose solution is referred to as logistic growth:

$$\boxed{N(t) = \frac{N(0)\, e^{\mu t}}{\left(1 + \frac{N(0)}{K}(e^{\mu t} - 1)\right)}} \qquad (1.2)$$

Stability Analysis

We first apply a small perturbation δN to the system, moving it from a stationary point N^*:

$$\boxed{\delta N(t) = \delta N - N^*} \qquad (1.3)$$

and if we take the derivative of that:

$$\boxed{\delta \dot{N}(t) = \dot{\delta N}} \qquad (1.4)$$

where J is the associated Jacobian matrix.
Now, for a 1-dimensional problem, the Jacobian matrix corresponds to the single derivative, which we can compute.

$$\left.\frac{\delta \dot{N}}{\delta N}\right|_{N=N^*} = J = \mu\left(1 - \frac{2N^*}{K}\right) \qquad (1.5)$$

Then, we look at the stationary solutions of Eq.(1.1) (stability condition).

$$\begin{cases} N^* = 0 \\ N^* = K \end{cases} \qquad (1.6)$$

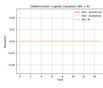
If we plug them into Eq.(1.5) we complete the stability analysis:

$$\begin{cases} N^* = 0 \implies \left.\frac{\delta \dot{N}}{\delta N}\right|_{N=N^*} = \mu \\ N^* = K \implies \left.\frac{\delta \dot{N}}{\delta N}\right|_{N=N^*} = -\mu \end{cases} \qquad (1.6)$$

Therefore, we see that, for a positive growth rate, $N^* = K$ is a stable solution while $N^* = 0$ is not, and viceversa for negative values of μ.
Moreover, a numerical simulation of the problem was performed, using the following parameters' values:

- $\mu = 1$
- $K = 10$

The resulting plots are the following:



The results clearly agree with what is predicted by the theory seen above. We see the two stationary points $N^* = 0$ and $N^* = K$ and how the first derivate is in accord with that.
We can also appreciate how the numerical integration coincides with the analytical one, taking into account the achievable precision.

2) QSA of the CRM

In the Consumer-Resource Model (CRM) we study the time evolution of both population and resources (respectively, $N(t)$ and $R(t)$).
This leads to the following differential equations:

$$\boxed{\frac{dN}{dt} = (\gamma c\, g(R) - d)N(t)} \qquad (2.1)$$

$$\boxed{\frac{dR}{dt} = \mu(R) - c\, g(R)N(t)} \qquad (2.2)$$

where:

- c is the efficiency of the uptake rate.
- γ is the yield coefficient.
- d is the death rate.

In the case of abiotic resources, the equations above become:

$$\begin{cases} \frac{dN}{dt} = (\gamma c R_t - d)N_t \\ \frac{dR}{dt} = S - c R_t N_t \end{cases} \qquad \begin{matrix} (2.3) \\ (2.4) \end{matrix}$$

where $N_t = N(t)$, $R_t = R(t)$ and s is the supply rate.
The Quasi-Stationary Approximation (QSA) is based on the assumption that the timescale of the resources is much faster than the population's one. This allows us to consider R to be fixed ($R_t = R$).
Now we apply the QSA and set Eq.(2.4) equal to zero:

$$\boxed{\frac{dR}{dt} = S - cRN_t = 0} \qquad (2.5)$$

We find the solution of the above equation to be:

$$\boxed{R^* = \frac{S}{cN_t}} \qquad (2.6)$$

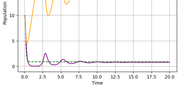
By substituting this result into Eq.(2.3) we get:

$$\boxed{\frac{dN}{dt} = \gamma S - dN_t} \qquad (2.7)$$

Now, we perform numerical simulations of $N(t)$ and $R(t)$ both in the general case and in QSA. For the simulations, the following parameters' values have been used:

- $\gamma = 0.3$
- $d = 10$
- $s = 70$
- $c = 0.9$

The results are shown in the figure below:



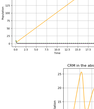
As expect, the QSA is a good approximation for the stationary regime, but fails to express the oscillating nature of $N(t)$. On another note, it can be seen how $N(t)$ and $R(t)$ have a similar behaviour: first oscillating and then stabilizing.
We can also introduce the Monod function:

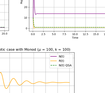
$$\boxed{\mu(R) = \mu_{max}\frac{R}{K + R}} \qquad (2.8)$$

and the equations become:

$$\begin{cases} \frac{dN}{dt} = (\gamma c\, \mu(R) - d)N_t \\ \frac{dR}{dt} = S - cRN_t \end{cases} \qquad \begin{matrix} (2.9) \\ (2.10) \end{matrix}$$

At this point, we perform numerical analysis on these equation, using the same parameters' values as before and different values for μ_{max} (from 1 to 10 μ) and K. The results can be inspected below:





The results show how the QSA behaves similarly also when considering the addition of the Monod function, except for the parameters' regime where $\mu \gg K$, where the stationary solutions differ.