

Review of Probability Distributions

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Pairing and Ordering of Objects

Unique pairing of objects

- given n objects, how many possible ways of selecting unique pairs, without caring about ordering ?
 - let's consider a matrix $n \times n$
 - every element in the matrix, except the leading diagonal, is a pairing
 - since the two parts on each side of the diagonal are identical (order does not count), we have

$$n_{\text{pairs}} = (n^2 - n)/2 = n(n - 1)/2$$

Unique ordering of objects

- given n objects, how many possible ways of ordering them ?
 - we have n options to select the first element
 - $n - 1$ for the second, $n - 2$ for the third, ...
 - therefore it is

$$n(n - 1)(n - 2) \dots 2 \cdot 1 = n!$$

Combinations and Permutations

- in the english language the word "*combination*" is used loosely, without specifying if the order of the object is relevant
- examples:
 - when buying an ice cream, we select a *combination* of mint, chocolate and stracciatella. We do not care about the order of the three flavours on the cone
 - the *combination* of my bike locker is 4-3-6-9. In this case, the order of the numbers really matters!
- when we select k elements from a set of n objects
 - if the order of selection is NOT important, we have a *combination*
 - but if the order matters, we have a *permutation*

→ a *permutation* is an ordered combination



Permutations - ORDER MATTERS

- there are two types of *permutations*

Repetition IS allowed

- given n objects, how many sequences of r elements ($r \leq n$) can be built ?
Example: given n letters, how many words of r characters can be built with those letters ?
 - each object (character) has n different possibilities, therefore it is

$$n^r$$

Repetition is NOT allowed

- given n objects, we select r elements ($r \leq n$) from the set
- how many unique selections are possible ?
 - there are n ways to select the first, $n - 1$ for the second, and $n - r + 1$ for the r -th
 - we get:

$$n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!} = {}^n P_r$$

- this is called permutations, ${}^n P_r$. Note that ${}^n P_n = n!$

- there are two types of combinations

Repetition is NOT allowed

- we now select r objects, as in the previous case, but we are not concerned about the order
- the number of ways of selecting r object from a set of n without regard to the order of selection is called combinations, nC_r

$${}^nC_r = \frac{{}^nP_r}{n!} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- this is the binomial coefficient, also called n choose r

Repetition IS allowed

- finally, the number of ways of choosing r objects from a set of n with replacement and without caring about the order is

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

- this is sometimes called n multichoose r

Application: the Birthday Paradox

The Problem

- in a large room, full of people, how many of them do you have to ask before there is a 50% chance that any of two, ore more, share a common birthday ?
- assuming $n = 365$ birthday/year and **equally probable**, we consider r people and we combine them so that they do not share a common birthday

$$A = n(n-1) \dots (n-r+1) = \frac{n!}{(n-r)!}$$

- the way of assigning n birthday to r people is $B = n^r$
- the probability of no common birthday is A/B
- therefore the probability of at least one birthday is

$$P(\text{birthday} \geq 1) = 1 - \frac{A}{B} = 1 - \frac{n!}{(n-r)!} \frac{1}{n^r}$$

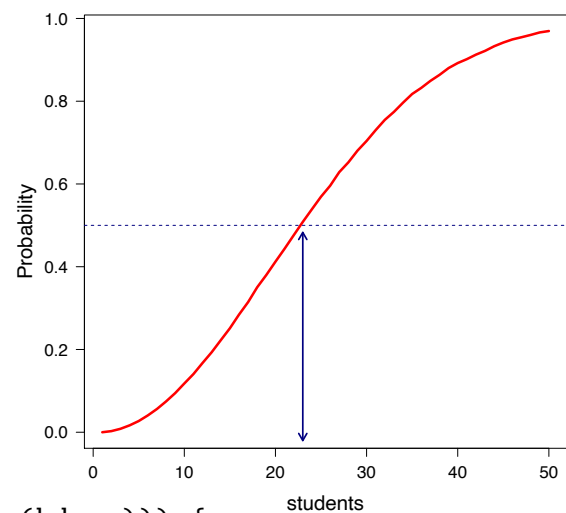
Computation of the birthday problem

First element with prob>0.5: 23

R code

```
n_people_tot <- 50
pbday <- rep(0, n_people_tot)
for (k in 2:n_people_tot) {
  n_tests = 1E5; cb <- 0
  for (i in 1:n_tests) {
    bdays <- sample(1:365, k,
                     replace=TRUE)
    if (length(bdays) > length(unique(bdays))) {
      cb = cb + 1
    }
  }
  pbday[k] <- cb/n_tests
  message(paste("k:", k, "pb(", k, "):", pbday[k]))
}
pfunc <- function(f, b) function(a) f(a,b)
p50_index <- Position(pfunc(`>`, 0.5), pbday)

message(paste("First element with prob>0.5:", p50_index))
```



R language note : closures

Anonymous functions

- can be used to create small function, not worth naming
- another important use is **to create closures**: functions written by functions

```
power <- function(exponent) function(x) x^exponent
```

```
square <- power(2)
square(2)
# [1] 4
```

```
cube <- power(3)
cube(2)
# [1] 8
```

```
pfunc <- function(f, b) {
  function(a) { f(a,b) }
}
p50 <- pfunc(`>`, 0.5)
```

```
x <- c(0.3, 0.51, 0.9)
p50(x)
# [1] FALSE TRUE TRUE
```

Probability Distributions

- two basic types: **discrete distributions** and **continuous distributions**
- **discrete** distribution : finite or countable set of possible outcomes of the random variable
- **continuous** distribution : a random variable can have outcomes in an interval of the real line
- probability densities are a way to specify probability distributions
- the cumulative distribution function (CDF) is defined by

$$F(x) = P(X \leq x)$$

- for **discrete distributions**:

$$F(x_j) = P(X \leq x_j) = \sum_j p_j$$

- while for **continuous distributions**:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

Probability Distributions

- with distribution functions, we compute the probability for intervals, $(c, d]$ as

$$P(c < X \leq d) = P(X \leq d) - P(X \leq c) = F(d) - F(c)$$

- the **expectation**, or expected value reflects the location of a distribution

$$E[X] = \sum_i x_i p(x_i) \quad E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

- the **variance** reflects the dispersion of the distribution:

$$\text{var}(X) = E[X - E[X]]^2 = E[X^2] - (E[X])^2$$

- properties:

$$\begin{aligned} E[a + bX] &= a + bE[X] & \text{var}(a + bX) &= b^2 \text{var}(X) \\ E[X + Y] &= E[X] + E[Y] & \text{var}(X + Y) &= \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y) \end{aligned}$$

- with the **covariance** of the two variables

$$\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

Moments of a distribution

- they are analogous to the [center-of-mass](#) and to the [momentum of inertia](#)

Algebraic Moments

- the [moment of order \$k\$ about the origin](#) is

$$\mu'_k \equiv E[x^k] = \int x^k f(x) dx \quad \text{and} \quad \sum_j x_j^k p_j$$

Central Moments

- the [moment of order \$k\$ about the mean](#) are

$$\mu'_k \equiv E[(x - \mu)^k] = \int (x - \mu)^k f(x) dx \quad \text{and} \quad \sum_j (x_j - \mu)^k p_j$$

$$\begin{array}{ll} \mu'_0 = 1 & \mu_0 = 1 \\ \mu'_1 = \mu & \mu_1 = 0 \\ \mu'_2 = \mu + \sigma^2 & \mu_2 = \sigma^2 \end{array}$$

- the [higher order moments](#) become interesting only for studying the behavior of $f(x)$ for large $|x - \mu|$
- for a symmetric distribution, all odd central moments vanish → [non zero values](#) are a possible [measure of the skewness of a distribution](#)

Probability Distributions in R

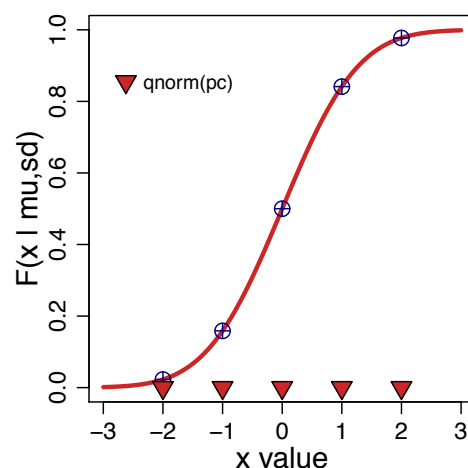
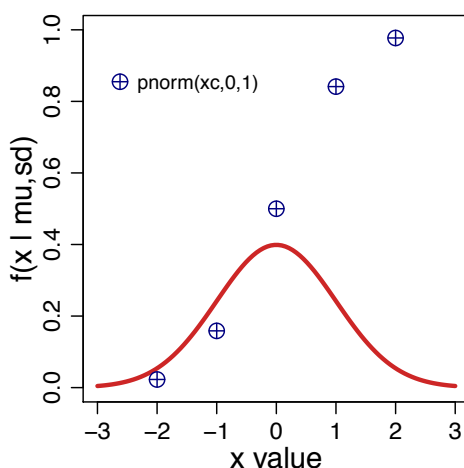
- [all standard distributions available](#)
- naming convention: a [core name](#) is associated with each distribution, and a [prefix](#) is [appended](#) to indicate the four basic associated functions:
 - d for the [probability density function](#) (pdf)
 - p for the [cumulative density function](#) (cdf)
 - q for the [quantile function](#)
 - r for the [sampling from the distribution](#)
- note that [pcore_name\(\)](#) and [qcore_name\(\)](#) are one the inverse of one another

```
xc <- seq(-2, 2, 1)
pc <- pnorm(xc, 0, 1)
qc <- qnorm(pc)
```

xc: -2 -1 0 1 2

pc: 0.023 0.159 0.5 0.841
0.978

qc: -2 -1 0 1 2

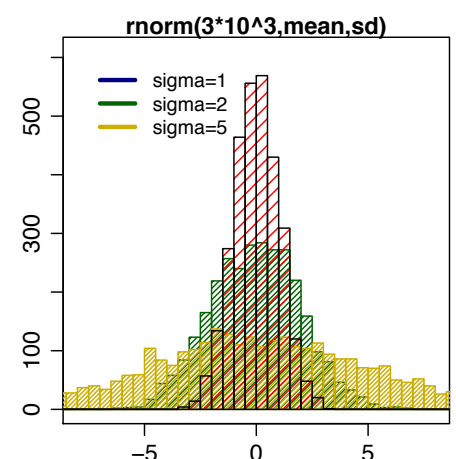
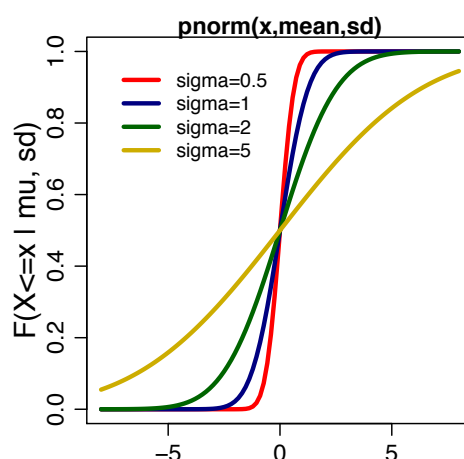
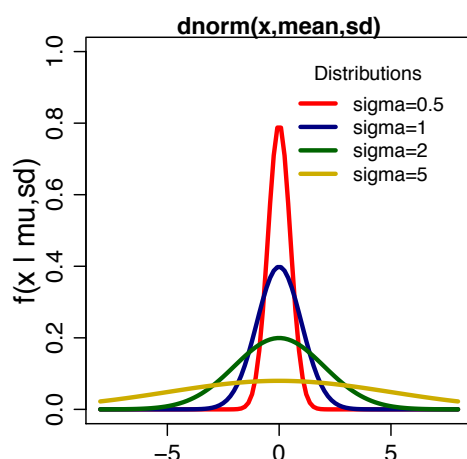


Standard Probability Distributions in R

Distribution	Core name	Parameters	Default values
Beta	<code>beta</code>	shape1, shape2	
Binomial	<code>binom</code>	size, prob	
Cauchy	<code>cauchy</code>	location, scale	0, 1
Chi-square	<code>chisq</code>	df	
Exponential	<code>exp</code>	1/mean	1
Fisher	<code>f</code>	df1, df2	
Gamma	<code>gamma</code>	shape, 1/scale	NA, 1
Geometric	<code>geom</code>	prob	
Hypergeometric	<code>hyper</code>	m, n, k	
Log-Normal	<code>lnorm</code>	mean, sd	0,1
Logistic	<code>logis</code>	location, scale	0,1
Normal	<code>norm</code>	mean, sd	0,1
Poisson	<code>pois</code>	lambda	
Student	<code>t</code>	df	
Uniform	<code>unif</code>	min, max	0,1
Weibull	<code>weibull</code>	shape	

Probability Distributions in R: normal distribution

- `dnorm(x, mean = 0, sd = 1)` gives a density of a normal distribution i.e. the pdf
- `pnorm(q, mean = 0, sd = 1)` returns the distribution function, i.e. the cdf
- `rnorm(n, mean = 0, sd = 1)` generates random numbers from a normal distribution function
- `qnorm(p, mean = 0, sd = 1)` is the quantile function



Standard Discrete Distributions

Bernoulli process

- it is a process with **only two possible outcomes**: **success** with **probability p** and **failure** with **probability $1 - p$** (also called q , since $q = 1 - p$)
- if we call the two outcomes, 0 and 1, we can define $x \in [0, 1]$, and

$$P(X = 1) = p$$

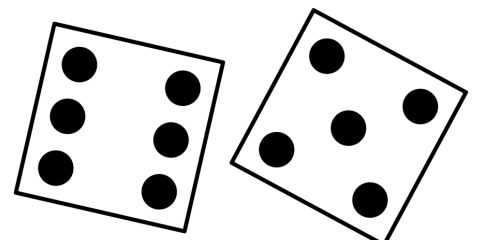
$$P(X = 0) = 1 - p = q$$

- the **expected value** and **variance** are

$$E[x] = p \quad \text{and} \quad \text{Var}(x) = p(1 - p)$$

Examples

- the toss of a coin
- the draw of a die



Binomial distribution

- the **sum of n independent Bernoulli trials**, follows a Binomial distribution

$$Bn(x | p, n) = \binom{n}{x} p^x (1 - p)^{n-x}$$

- it gives the **probability of x successes in n independent Bernoulli trials**
- the **expected value** and **variance** are

$$E[x] = np \quad \text{and} \quad \text{Var}(x) = np(1 - p)$$

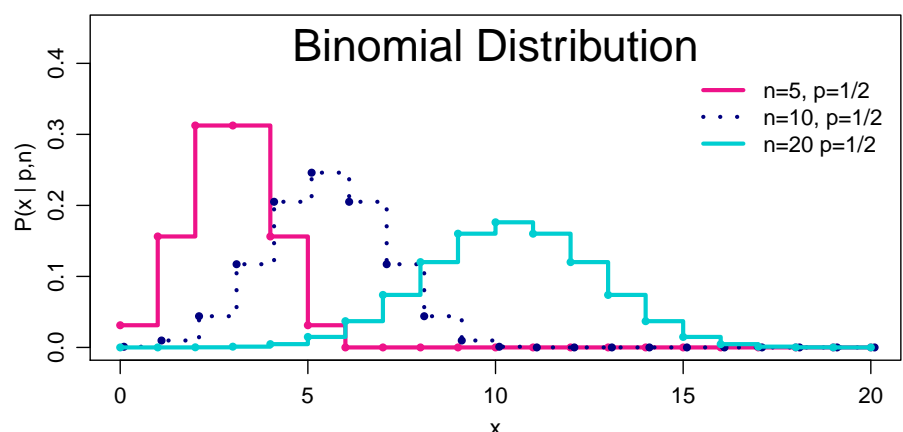
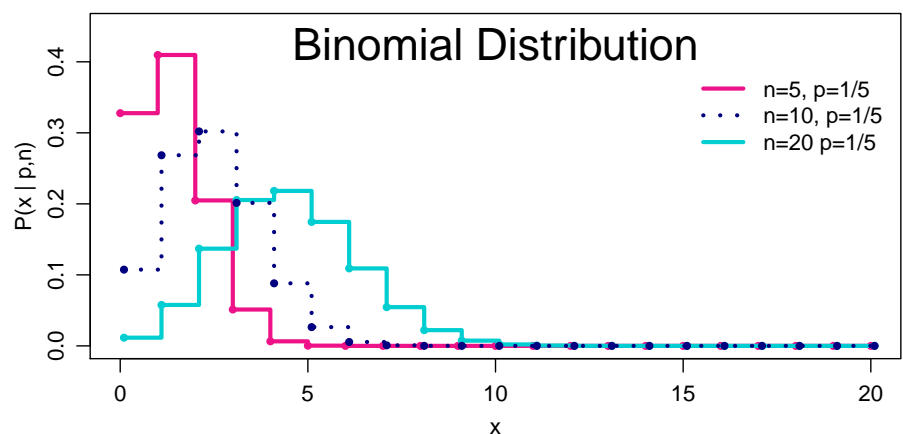
$$\sum_{j=0}^n \binom{n}{j} p^j (1 - p)^{n-j} = (p + 1 - p)^n = 1$$

Examples

- multiple toss of a coin, or coins
- draw of dice
- drawing x red balls from an urn with n red and white balls (the fraction of red balls is p). Draws are done with replacement (\rightarrow **p remains constant**)

Binomial distribution examples

- the distribution is **symmetric when $p = 1/2$** , and **otherwise skew**
- the distribution gets increasingly **symmetric for higher values of n**
- when **n becomes large**, it takes and approximate **Gaussian shape**



Binomial distribution - exercise

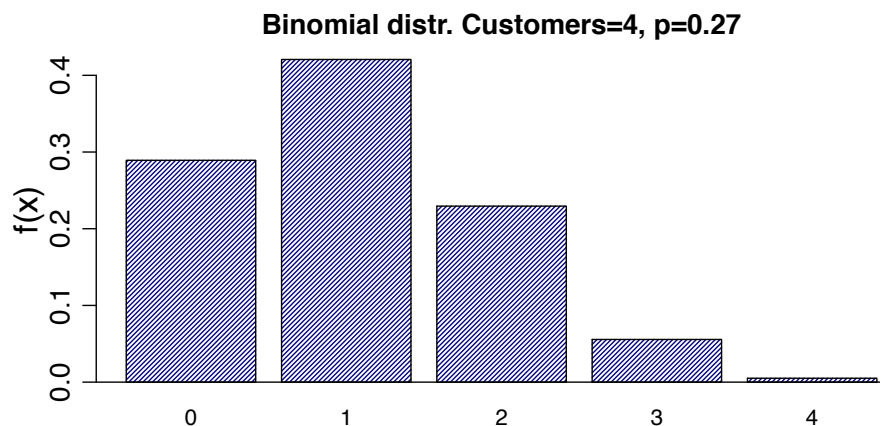
Problem

- in a restaurant 8 entrees of fish, 12 of beef and 10 of poultry are served
- what is the probability that 2 of the 4 next customers order fish entrees ?

Solution

```
cust <- 4
p <- 4/15
x <- 0:4
ap <- dbinom(x, cust, p)
barplot(ap, names=x, col='navy', xlab='x', ylab='f(x)', density=40,
        main = sprintf("Binomial_distr. Customers=%d, p=%.2f", cust, p),
        cex.lab=1.5, cex.axis=1.25, cex.main=1.25, cex.sub=1.5)
cat(paste(c("P(2|np) = ", ap[3], '\n')))
```

$P(2|np) = 0.229451851851852$



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Example: histogramming events

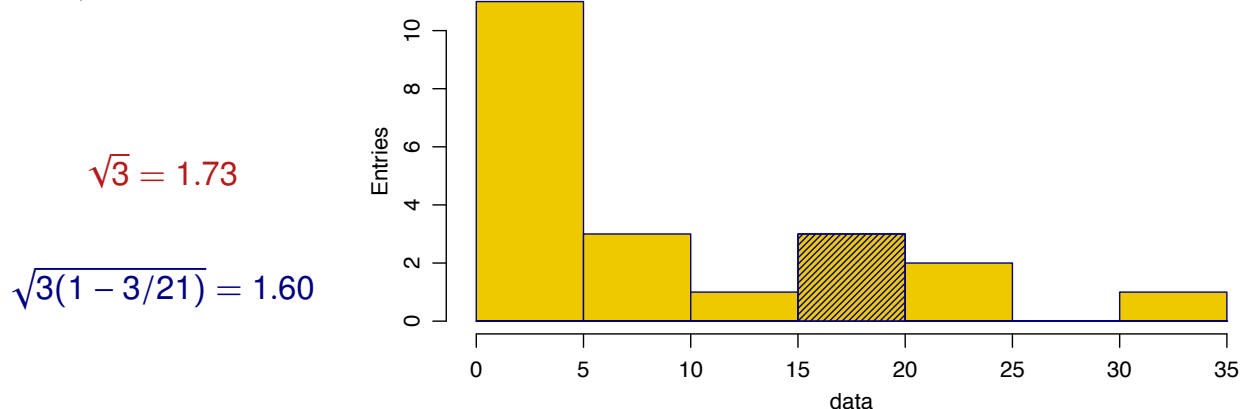
- we are interested in just the **events** contained in **one bin** of the histogram
 - A : we get the event of that particular bin (success)
 - \bar{A} : correspond to the events in any other bin (failure)
- the probability of having x_o out of n events in the bin follows a Binomial distribution:

$$E[x] = np \quad \text{and} \quad \text{Var}(x) = np(1 - p)$$

- p can be estimated as the ratio $p = x_o/n$:

$$E[x] = np = n \frac{x_o}{n} = x_o \quad \text{and} \quad \text{Var}(x) = x_o \left(1 - \frac{x_o}{n}\right)$$

- the error on the number of the events is not $\sqrt{x_o}$, but a smaller quantity, $\sqrt{x_o(1 - p)}$. Only in the limit $p \rightarrow 0$ (Poisson limit), $\sigma = \sqrt{x_o}$



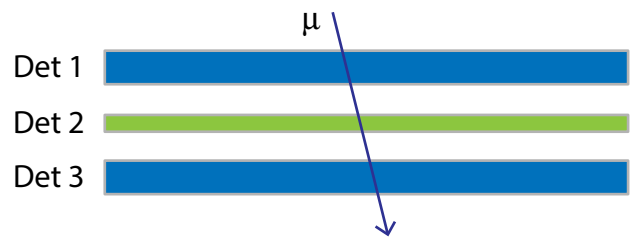
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Example: detection efficiency

- we want to compute the **efficiency of a detector** and evaluate the **uncertainty on the measurement**
- a muon-like signal has been registered by Det1 and Det3
- what is the detection efficiency of our Det2 ?
- detection is a **Bernoulli process**:



$$\epsilon_2 = \frac{N_{det2}}{N_{det1 \& det3}} \quad \text{with} \quad N_{det2} \subset N_{det1 \& det3}$$

- since we are interested in a relative number of success in a trial,

$$E\left[\frac{r}{n}\right] = \frac{1}{n}E[r] = p \quad \text{and} \quad \text{Var}\left(\frac{r}{n}\right) = \frac{1}{n^2}V(r) = \frac{p(1-p)}{n} = \frac{pq}{n}$$

- in our case, p is the ratio of events detected with Det2 with respect to those seen by both Det1 and Det3
- therefore:

$$\sigma(\epsilon_2) = \sqrt{\frac{\epsilon_2(1 - \epsilon_2)}{N_{det1 \& det3}}}$$

The drunk-man and the home keys problem

The **background information**

- a man comes back home pretty drunk
- he has **8 keys** and **tries them randomly** to unlock his door apartment
- after each trial he loses memory
- we watch him and **bet on the attempt** on which he will succeed
- $n_{try} = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots$
- on **which number would you bet ?**

The problem

- E_j : the **door** gets **unlocked in attempt j** , with $j = 1, 2, \dots$
- we know that: $P(E_j | \bigcup_{j < i} \bar{E}_j) = 1/8$
 $f(1) = P(E_1) = p = 1/8$
 $f(2) = P(E_2 \cdot \bar{E}_1) = P(E_2 | \bar{E}_1) \cdot P(\bar{E}_1) = p \cdot (1 - p)$
 $f(3) = P(E_3 \cdot \bar{E}_2 \cdot \bar{E}_1) = P(E_3 | \bar{E}_2 \cdot \bar{E}_1) \cdot P(\bar{E}_2 | \bar{E}_1) \cdot P(\bar{E}_1) = p \cdot (1 - p)^2$
 $f(x) = p \cdot (1 - p)^{x-1}$

Geometric distribution

- our probabilities follow a **geometric distribution** with $p = 1/8$

$$f(1) = p = 1/8 = 0.125 \quad \checkmark \text{ our best bet!}$$

$$f(2) = p(1 - p) = 1/8(7/8) = 0.109$$

$$f(3) = p(1 - p)^2 = 0.096$$

$$f(4) = p(1 - p)^3 = 0.084$$

...

- the **geometric distribution** gives the **number of trials to get the first success**

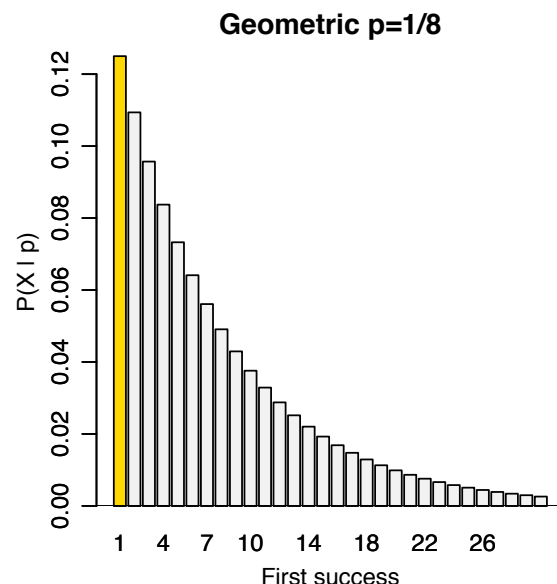
$$\text{Geo}(x|p) = p(1 - p)^x$$

- the **expected value** and **variance** are

$$E[x] = \frac{1}{p} \quad \text{and} \quad \text{Var}(x) = \frac{1 - p}{p^2}$$

- useful relations:

$$P(x \leq r) = 1 - (1 - p)^r = q^r \quad \text{and} \quad P(x > r) = 1 - q^r$$



Geometric distribution examples (1)

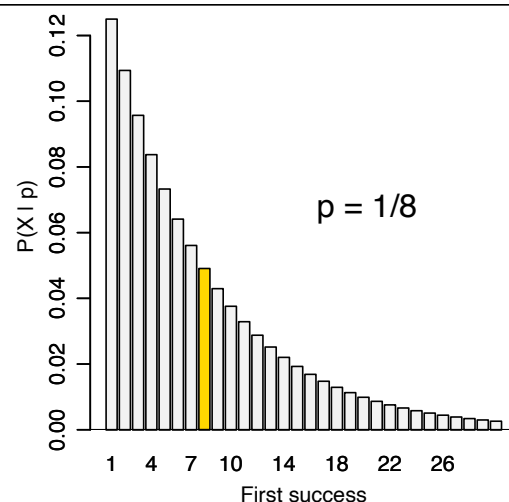
Drunk-man

- the **first trial** is the **most probable**
- but

$$E[X] = 1/p = 8$$

and

$$\sigma = \sqrt{(1 - p)/p^2} = 7.5$$

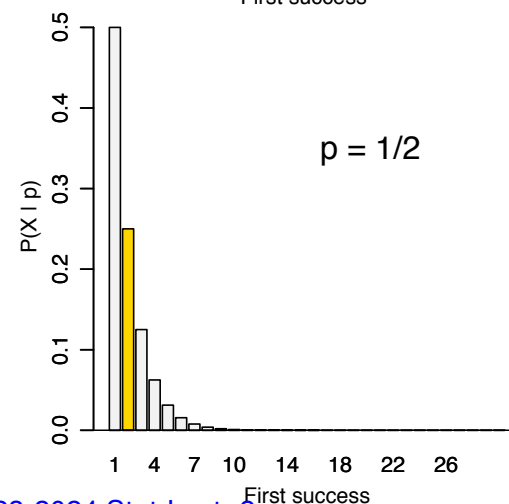


Coin tossing

- if we apply it to the tossing of one coin, we get
- $p_{\max} = p = 1/2$
- and $E[X] = 1/p = 2$

and

$$\sigma = \sqrt{(1 - p)/p^2} = 1.4$$



Geometric distribution examples (2)

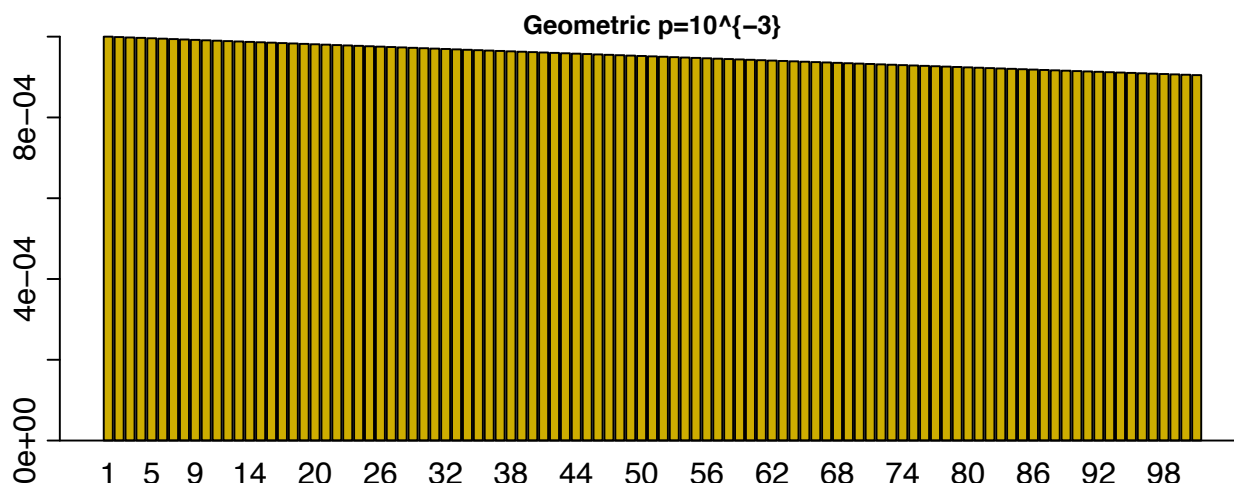
Rare Events

- let's decrease the probability of the event

$$E[X] = 1/p = 10^3$$

$$\text{Var}(X) = \frac{\sqrt{1-p}}{p} \xrightarrow{p \rightarrow 0} \frac{1}{p}$$

- rare moments might happen at any moment
(even if they have a negligible probability to happen at any moment)



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Geometric distribution in R

- given $x = \{1, 2, 3, \dots\}$ as the number of trials for the first success
an alternative representation uses
- $y = \{0, 1, 2, \dots\}$ as the number of failures before the first success
- the two representations are equivalent:

$$y = x - 1$$

$$\begin{aligned} f(x) &= p(1-p)^x = 1 - q^x \\ F(x) &= 1 - (1-p)^x = 1 - q^x \\ E[x] &= (1-p)/p \quad \text{Var}[x] = (1-p)/p^2 \end{aligned}$$

$$\begin{aligned} f(y) &= p(1-p)^y \\ F(y) &= 1 - (1-p)^{(y+1)} \\ E[y] &= (1-p)/p \quad \text{Var}[y] = (1-p)/p^2 \end{aligned}$$

- the geometric distribution in R

Geometric package:stats
The Geometric Distribution
Usage:

[R Documentation](#)

```
dgeom(x, prob, log = FALSE)
```

Arguments:

x , q : vector of quantiles representing the number of failures in a sequence of Bernoulli trials before success occurs.

Multinomial distribution

- it is a **generalization of the binomial distribution** to the case **with more than 2 possible outcomes**
- labeling the **disjoint outcomes** A_1, A_2, \dots, A_r , we define $P(A_j) = p_j$, with $1 \leq j \leq r$
- in **n independent trials**, x_j denotes the **number of times that A_j occurs**
- assuming, by construction, $n = x_1 + x_2 + \dots + x_r$, we have

$$P(X_1 = x_1, X_2 = x_2, \dots, X_r = x_r | p_1, p_2, \dots, p_r, n) = \frac{n!}{x_1! x_2! \dots x_r!} p_1^{x_1} p_2^{x_2} \dots p_r^{x_r}$$

Properties

- the **expectation** for class A_j is $E[x_j] = np_j$
- the **variance** for class A_j is $Var(x_j) = np_j(1 - p_j)$
- the **covariance** for classes A_i, A_j is $cov(x_i, x_j) = -n p_i p_j$
- when **n becomes large**, the distribution **tends to a multinormal distribution**

Multinomial distribution - exercise

Problem

- in a certain town, at 20:00, 30% of the TV audience watches the news, 25% a TV show, and the rest other programs
- What is the probability that, selecting 7 random viewers, exactly 3 watch the news and at least 2 watch the TV show ?

Solution

- the probabilities are $p_1 = 3/10$, $p_2 = 1/4$, $p_3 = 9/20$
- the sum of the trials $i + j + k = 7$
- we write

$$P(i, j, k | n = 7) = \frac{7!}{i! j! k!} \left(\frac{3}{10}\right)^i \left(\frac{1}{4}\right)^j \left(\frac{9}{20}\right)^k$$

- and we compute

$$\begin{aligned} P(i = 3, j \geq 2 | n = 7) &= P(3, 2, 2 | 7) + P(3, 3, 1 | 7) + P(3, 4, 0 | 7) \\ &= \frac{7!}{3! 2! 2!} \left(\frac{3}{10}\right)^3 \left(\frac{1}{4}\right)^2 \left(\frac{9}{20}\right)^2 + \frac{7!}{3! 3! 1!} \left(\frac{3}{10}\right)^3 \left(\frac{1}{4}\right)^3 \left(\frac{9}{20}\right) \\ &+ \frac{7!}{3! 4! 0!} \left(\frac{3}{10}\right)^3 \left(\frac{1}{4}\right)^4 \simeq 0.103 \end{aligned}$$

- let suppose we have a multinomial distribution $P(X_1, X_2, \dots, X_r)$ and we want to find the marginal probability $P(X_1)$

$$\begin{aligned}P(X_1) &= \sum_{x_2+x_3+\dots+x_r=n-x_1} \frac{n!}{x_1!x_2!\dots x_r!} p_1^{x_1} p_2^{x_2} \dots p_r^{x_r} \\&= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} \sum_{x_2+x_3+\dots+x_r=n-x_1} \frac{(n-x_1)!}{x_2!\dots x_r!} p_2^{x_2} \dots p_r^{x_r} \\&= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} (p_2 + \dots + p_r)^{n-x_1} \\&= \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1}\end{aligned}$$

- where the multinomial expansion has been used, and also the fact that $p_1 + p_2 + \dots + p_r = 1$
- the obtained distribution coincides with the binomial distribution

Poisson process

- let's consider an event that **might happen at a given time**, with the following conditions:
 - the probability of 1 count in Δt is proportional to Δt itself, with Δt a 'small' value
 - calling r , the **intensity of the process**,

$$p = P('1 \text{ count in } \Delta t') = r\Delta t$$

- moreover:
 - $P(\geq 2 \text{ counts}) \lll P(1 \text{ count})$
 - what happens in one interval does not depend on other intervals \rightarrow it has a memory-less property

Examples

- accidents occurring at an intersection
- γ -s emitted from a radioactive substance
- customers entering a post office
- earthquakes in Italy

Poisson distribution

- the **Poisson distribution** can be **derived** by the **Binomial distribution**, in the limit where the **rate of success, p** , is **very small**
- we divide a finite time interval, T , in n small intervals:

$$T = n \Delta T$$

- and we consider the possible occurrence of an event, an independent Bernoulli trial, in each small interval Δt

$$p = r \Delta T = r \frac{T}{n}$$

- if the number of trials is large, the total number of successes, np , is however considerable: $np = rT = \lambda$
- mathematically, in the limit $p \rightarrow 0$, $n \rightarrow \infty$ and $np = \lambda$ remaining constant, we get

$$\text{Bn}(r|n p) \rightarrow \text{Poi}(r|\lambda)$$

- λ depends only on the intensity of the process, r , and on the finite time of observation

$$\text{Poi}(r|\lambda) = \frac{\lambda^r}{r!} \exp(-\lambda)$$

Poisson distribution

- Given the Poisson distribution function:

$$\text{Poi}(r|\lambda) = \frac{\lambda^r}{r!} \exp(-\lambda)$$

- the **expected value** and **variance** are

$$E[x] = \lambda \quad \text{and} \quad \text{Var}(x) = \lambda$$

- Asymptotically, for growing λ values, the Poisson distribution becomes identical to the normal distribution
the **similarity** is rather close **already at $\lambda = 20$**
- an interesting property is:

$$\text{Poi}(r|\lambda) = \text{Poi}(r-1|\lambda) \frac{\lambda}{r}$$

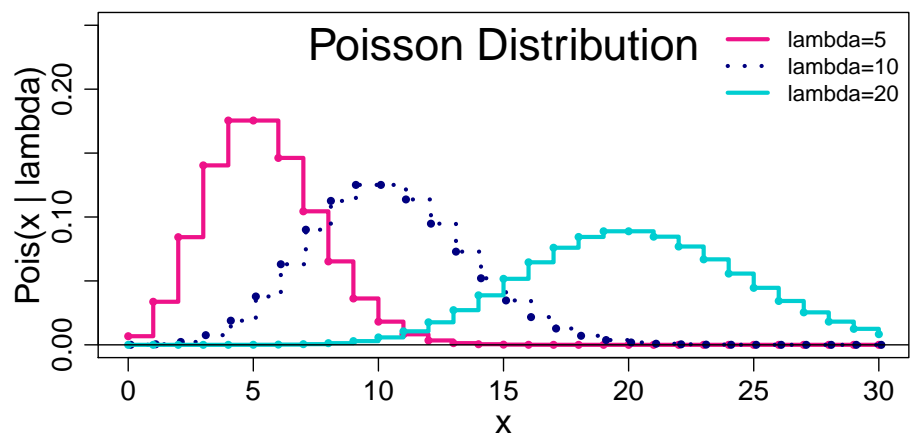
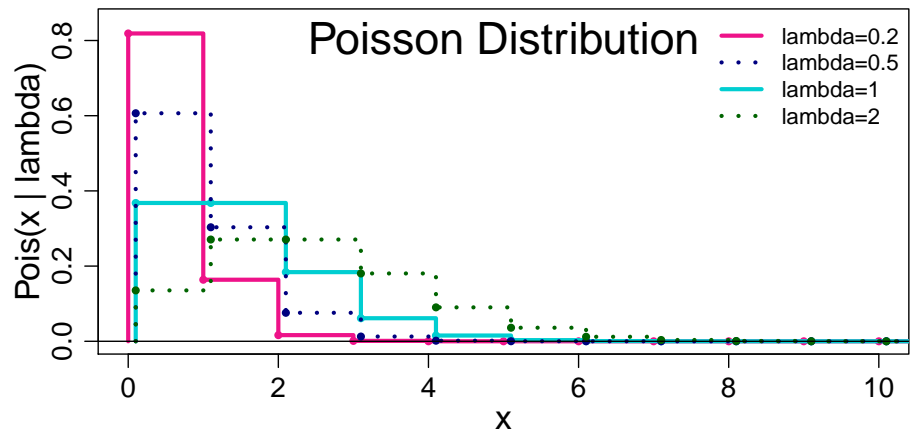
- it is possible to demonstrate that **the sum of any independent Poisson variables is itself a Poisson variable** with **mean value equal to the sum of the individual means**

Poisson distribution examples

- the distribution is **very asymmetric** for small λ and it has a **tail to the right of the mean**

- the distribution gets increasingly **symmetric** for higher values of λ

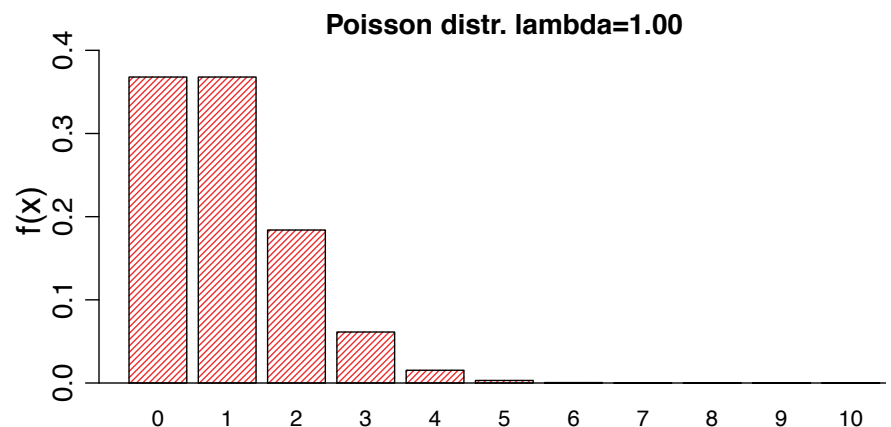
- already for $\lambda = 20$ is very **similar to the normal distribution** (but it has only integer values)



Poisson distribution - exercise 1

Problem

- the average number of received wrong phone calls per week is 7
- what is the probability to get, tomorrow, A) two wrong calls ? B) at least one wrong call ?



Solution

- assuming we get a large number of calls, the number of wrong calls follows, to a good approximation, a Poisson distribution
- we assume $\lambda = 1$

$$P(2 | \lambda) = 0.184$$

$$P(\geq 1 | \lambda) = 0.632$$

```
lambda <- 1
x <- 0:10
ap <- dpois(x, lambda)
barplot(ap, names=x, col='firebrick2', xlab='x', ylab='f(x)', density=30,
        main = sprintf("Poisson_distr._lambda=%.2f", lambda),
        ylim=c(0, 0.415),
        cex.lab=1.5, cex.axis=1.25, cex.main=1.25, cex.sub=1.5)
cat(paste(c("P(2|lambda)=", ap[3], "\n"))
cat(paste(c("P(>=1|lambda)=", 1 - ap[1], "\n"))
```

Poisson distribution - exercise 2

Problem

- a radioactive substance emits on average 3.9 α /s per gram
- compute the probability that, in the next second, the number of emitted alpha particles is
 - A) at most 6
 - B) at least 2
 - C) at least 3 and at most 6

Solution

- every gram of element has n atoms
- From the information we have, $E[X] = np = \lambda = 3.9$

$$P(x|\lambda) = \frac{\lambda^x}{x!} \exp(-\lambda)$$

$$A) \quad P(x \leq 6) = \sum_{x=0}^6 \frac{3.9^x}{x!} \exp(-3.9)$$

$$B) \quad P(x \geq 2) = 1 - P(x \leq 1) = 1 - \sum_{x=0}^1 \frac{3.9^x}{x!} \exp(-3.9)$$

$$C) \quad P(3 \leq x \leq 6) = \sum_{x=3}^6 \frac{3.9^x}{x!} \exp(-3.9)$$

Poisson distribution - exercise 2

```
P(<=6) = 0.899483035093612
P(>=2) = 0.900814633915558
P(2<X<=6) = 0.646357932463829
```

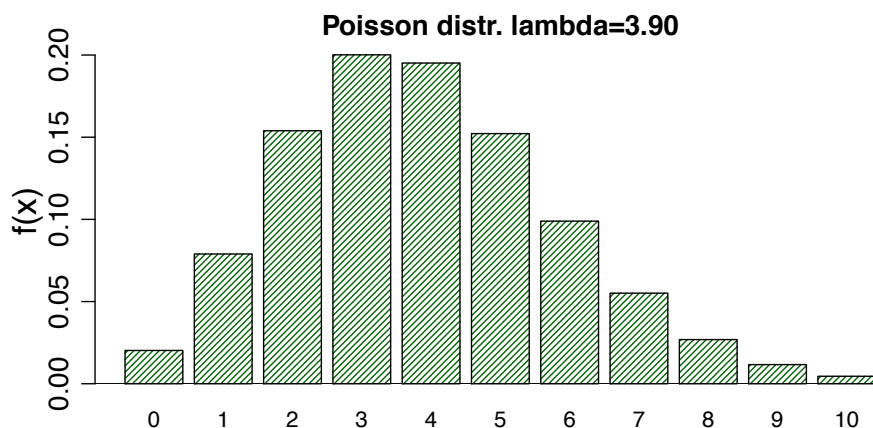
```
lambda <- 3.9
x <- 0:10
ap <- dpois(x, lambda)
```

```
barplot(ap, names=x, col='darkgreen', xlab='x', ylab='f(x)', density=30,
        main = sprintf("Poisson_distr._lambda=%.2f", lambda),
        ylim=c(0,0.21),
        cex.lab=1.5, cex.axis=1.25, cex.main=1.25, cex.sub=1.5)
abline(0,0)
```

```
P_6 = sum(ap[x<=6])
P_2 = 1 - sum(ap[x<=1])
```

```
cat(paste(c("P(<=6) = ", P_6, '\n'))))
cat(paste(c("P(>=2) = ", P_2, '\n'))))
```

```
pp <- ppois(x, lambda)
P_36 = pp[x==6] - pp[x==2]
cat(paste(c("P(2<X<=6) = ", P_36, '\n'))))
```



Pascal or Negative Binomial distribution

- the probability of obtaining the r -th success in n trials, is given by the Negative Binomial, or Pascal, distribution
- since in $n - 1$ trials we had $r - 1$ successes, the probability is given by the Binomial distribution:

$$Bn(r|n, p) = \binom{n-1}{r-1} p^{r-1} (1-p)^{n-1-r+1} = \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r}$$

- but we got the r -th success at the n -th trial, therefore

$$Bneg(r|n, p) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

- the expected value and variance are

$$E[x] = \frac{r}{p} \quad \text{and} \quad Var(x) = \frac{r(1-p)}{p^2}$$

Pascal distribution - exercise

Problem

- Ann and Maggie are playing cards until one of them wins 5 games
- suppose all games are independent and the probability that Ann wins is 58%
 - A) what is the probability that they complete in 7 games
 - B) if the series ends in 7 games, what is the probability that Ann wins ?

Solution to A

- X : number of games played until Ann wins 5 games
- Y : number of games played until Maggie wins 5 games
- both X and Y follow a Pascal distribution

$$P(X = 7, r = 5) = \binom{6}{4} 0.58^5 0.42^2 = 0.174$$

$$P(Y = 7, r = 5) = \binom{6}{4} 0.42^5 0.58^2 = 0.066$$

- we get $P(X = 7, r = 5) + P(Y = 7, r = 5) = 0.24$

Solution to B

- A: Ann wins
- B: the series ends in 7 games

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(X=7)}{P(X=7) + P(Y=7)} = \frac{0.17}{0.24} = 0.71$$

Solution with R

`dnbinom(x, size, prob, mu)`

The negative binomial distribution with 'size' = n and 'prob' = p
...
for x = 0, 1, 2, ..., n > 0 and 0 < p <= 1.

This represents the number of failures which occur in a sequence of Bernoulli trials before a target number of successes is reached. The mean is $\mu = n(1-p)/p$ and variance $n(1-p)/p^2$.

```
P_Ann <- dnbinom(2,5,0.58) # 0.173672
P_Maggie <- dnbinom(2,5,0.42) # 0.0659468
```

Exercise : Binomial/Poisson

Defective screws

- a company produces screws
- the probability of a screw to be defective is $p = 0.015$
- a box with $n = 100$ screws is packaged.

Compute:

- A) the probability that all screws are non defective
- B) the defective screws distribution comparing the Binomial and Poisson distributions
- C) how many extra screws should the box contain in order to have $n = 100$ non defective screws with probability greater than 80%