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présentée par

Pietro MESQUITA-PICCIONE

Autour de la conjecture de Yau–Tian–Donaldson pour une classe transcendante

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dirigée par Sébastien Boucksom et Tat Dat Tô

Soutenue le 3 octobre 2025 devant le jury composé de :

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... if not, let's thank the speaker again.

The Chair, *A talk*

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Résumé

L'objectif principal de cette thèse est d'étudier la direction réciproque de la conjecture de Yau–Tian–Donaldson dans le cas d'une classe de Kähler générale, sans supposer qu'elle soit rationnelle. C'est-à-dire, de prouver que la notion appropriée de K-stabilité dans ce cadre implique l'existence d'une métrique de Kähler à courbure scalaire constante.

Plus précisément, la thèse introduit une notion renforcée de K-stabilité pour les variétés de Kähler, appelée $\widehat{K}$-stabilité, et établit un critère valuatif pour cette dernière, analogue à ceux connus en géométrie algébrique projective. La thèse développe une théorie non-archimédienne pour les variétés complexes compactes non algébriques, construisant un espace topologique de valuations et étudiant ses propriétés fondamentales, notamment la densité d'un sous-ensemble naturel de valuations associées à des diviseurs sur X, appelées valuations divisorielles.

Sur cette base, la thèse construit la théorie pluripotentielle non-archimédienne associée aux variétés de Kähler compactes : elle définit des fonctions plurisousharmoniques non-archimédiennes, établit une comparaison détaillée avec la théorie pluripotentielle classique (y compris la démonstration des formules de pente pour les fonctionnels d'énergie), et prouve l'existence de solutions à l'équation de Monge–Ampère non-archimédienne, généralisant ainsi un résultat de Boucksom–Favre–Jonsson.

L'analyse approfondie de cette théorie pluripotentielle non-archimédienne permet ensuite de relier le fonctionnel de Mabuchi non-archimédien à la K-énergie analytique et de montrer que la $\widehat{K}$-stabilité implique l'existence de métriques de Kähler à courbure scalaire constante, généralisant un résultat de Chi Li. De plus, la résolution de l'équation de Monge–Ampère non-archimédienne permet de démontrer le critère valuatif pour la $\widehat{K}$-stabilité, tel qu'établi par Boucksom et Jonsson dans le cadre projectif.

Mots-clés

Géometrie Kählerienne, Métriques canoniques, K-stabilité, Géometrie non-Archimédienne

Around the Yau-Tian-Donaldson conjecture for a transcendental class

Abstract

The main objective of this thesis is to investigate the converse direction of the Yau–Tian–Donaldson conjecture for a general Kähler class, without assuming it to be rational. It aims to prove that a suitable notion of K-stability in this setting implies the existence of a Kähler metric with constant scalar curvature.

More precisely, the thesis introduces a strengthened notion of K-stability for Kähler manifolds, called $\widehat{K}$-stability, and establishes a valuative criterion for it analogous to those known in projective algebraic geometry. It develops a non-Archimedean theory for compact complex non-algebraic manifolds by constructing a topological space of valuations and studying its fundamental properties, notably the density of a natural subset of valuations associated to divisors over X, known as divisorial valuations.

Building on this, the thesis constructs the non-Archimedean pluripotential theory associated with compact Kähler manifolds: it defines non-Archimedean plurisubharmonic functions, provides a detailed comparison with classical pluripotential theory (including the proof of slope formulas for energy functionals), and proves the existence of solutions to the non-Archimedean Monge–Ampère equation, thereby generalizing a result of Boucksom–Favre–Jonsson.

An analysis of this non-Archimedean pluripotential theory then relates the non-Archimedean Mabuchi functional to the analytic K-energy and shows that $\widehat{K}$-stability implies the existence of constant scalar curvature Kähler metrics, extending a result of Chi Li. Furthermore, the resolution of the non-Archimedean Monge–Ampère equation enables the proof of the valuative criterion for $\widehat{K}$-stability, as established by Boucksom and Jonsson in the projective setting.

Keywords

Kähler geometry, Canonical metrics, K-stability, Non-Archimedean geometry

Contents

Introduction

In this thesis we study a non-Archimedean approach to the Yau–Tian–Donaldson conjecture for a Kähler manifold.

The main questions this thesis tries to answer are the following:

Question. *Let X be a compact Kähler manifold, and α a Kähler class on X. Is there an algebraic stability condition on α that ensures the existence of a constant scalar curvature Kähler metric representing α?*

Question. *Is there an effective way of checking this stability condition for a given Kähler class?*

While we give an affirmative answer to the first question by adapting to our transcendental non-Archimedean theory a result of Chi Li [Li22], the answer to the second question is not yet well understood.

To answer the first question we prove:

Theorem A. *Let (X, α) be a compact Kähler manifold that is uniformly $\widehat{K}$-stable. Then, α contains a unique cscK metric.*

To say a few words, $\widehat{K}$-stability is a notion strengthening K-stability, a stability concept in algebraic geometry dealing with degenerations of the manifold X and a numerical invariant.

Our attempt to answer the second question resides in giving a *valuative criterion* for $\widehat{K}$-stability by adapting a result of Boucksom and Jonsson [BJ23] to this setting. In the Fano setting, such a criterion —that involved computing invariants such as the log discrepancy and the volume of big classes— turned out to be a very effective method to check the stability of low dimensional Fano manifolds. Although our criterion involves computing the same quantities —log discrepancies and volumes—, ours is much more complicated than the one for Fanos. Our main result in this sense is then:

Theorem B. *There is a valuative criterion for $\widehat{K}$-stability that, up to taking derivatives, integrals and Legendre transforms, involves computing log discrepancies and volumes of big classes.*

For precise statements see Theorems 5.3.1 and 5.3.4.

While the main interest of this thesis lies in its geometric questions, the core of the new results resides in the development of a non-Archimedean theory for a possibly non-projective compact Kähler manifold. For the reader's convenience, right after the next Organization section, we will begin by giving an overview of the geometric context for the Yau–Tian–Donaldson conjecture. After that, we will explain the main contributions of the thesis.

0.1 Organization

In this first introductory chapter we propose an extensive discussion of the history of the Yau–Tian–Donaldson conjecture, and discuss some aspects of the pluripotential theory of Kähler manifolds. This motivates Section 0.5, in which we present the new results of the thesis, by giving an overview of the Kähler non-Archimedean pluripotential theory that we develop.

In Chapter 1, we introduce the basics of the non-Archimedean geometry of a compact complex manifold, analogous to the Berkovich analytic geometry of varieties over a non-Archimedean field. We introduce a notion of *valuations* on X and study the space of valuations $X^\beth$. We also give an interpretation of the space of such valuations $X^\beth$ as a limit of simplicial complexes that we call *dual complexes*. Finally, in Theorem 1.2.16 we prove that a special class of valuations –coming from prime divisors on bimeromorphic models of X– are dense in the space of all valuations.

Chapter 2 is devoted to developing a non-Archimedean pluripotential theory on $X^\beth$ when X is a compact Kähler manifold, in analogy with the projective non-Archimedean pluripotential theory of [BJ22]. In Section 2.1 we define the notion of plurisubharmonic (psh) functions in this context, and prove some basic properties. Section 2.2 is concerned with the behavior of psh functions on dual complexes. We prove that such functions are continuous and convex once restricted to a dual complex, and we develop an *energy pairing* formalism on Section 2.3 for this non-Archimedean setting. This allows us to define energy functionals, Monge–Ampère measures and more. This formalism is covered by the synthetic pluripotential theory of [BJ25a]. In Section 2.4 we establish important connections between the non-Archimedean pluripotential theory of $X^\beth$, and the geometry of the archimedean Kähler potentials of X. More precisely, arguing as in [BBJ21], we prove that there is a correspondence between the non-Archimedean psh functions of finite energy and the set of geodesic rays of potentials of finite energy on X up to parallelism.

We then proceed with the pluripotential theory of $X^\beth$ in Chapter 3. In Section 3.1 we use the results from the previous chapter –regarding the geodesic rays and non-Archimedean psh functions– to argue as in [BJ22] and [DXZ25] and prove Theorem 3.1.1, which states that the *psh envelope* of a continuous function f

$$\mathrm{P}_\alpha(f) \doteq \sup\{\varphi \in \mathrm{PSH}(\alpha) \mid \varphi \leq f\}$$

is continuous. In Section 3.2 we recall some definitions and results from the theory of big $(1,1)$-classes on Kähler manifolds, to give, in Section 3.3, an algebraic interpretation of the psh envelope of a special class of continuous functions on $X^\daleth$. This will allow us to prove Theorem 3.3.4, which states that for any continuous function f the psh envelope of f coincides with f μ-almost everywhere, where μ is the Monge–Ampère measure of the envelope of f.

These two results, once combined, enable us to apply all the synthetic pluripotential theory of [BJ25a], and ultimately to prove a non-Archimedean analogue of the Calabi–Yau theorem on Chapter 4, using the variational approach of [BFJ15]. In Section 4.1 we define the *energy* of a measure, while Section 4.2 is devoted to the study of a topology for the set of measures of finite energy. In Section 4.4, we prove Theorem 4.4.5, which states that for every measure of finite energy μ we can find a psh function of finite energy φ that solves the Monge–Ampère equation

$$\mathrm{MA}_\alpha(\varphi) = \mu.$$

Lastly, in Section 4.5, by adapting [BJ22], we study the regularity of solutions of the Monge–Ampère equation, and show that if the measure μ is supported on a dual complex then the solution of the equation must be continuous.

Finally, in Chapter 5 we use all the non-Archimedean pluripotential theory we developed to apply it to the study of K-stability and of the existence of constant scalar curvature Kähler metrics. Indeed, we translate K-stability into non-Archimedean terms, by asking that a certain functional, that we call the *non-Archimedean Mabuchi functional*, is positive on certain continuous plurisubharmonic test functions that we denote by $\mathcal{H}(\alpha)$ on $X^\daleth$. We then enlarge the set of test functions to introduce the notion of $\widehat{K}$*-stability* in Section 5.2. More precisely, for $\widehat{K}$-stability we ask that the non-Archimedean Mabuchi functional is positive on the set of envelopes of functions that are in the linear span of $\mathcal{H}(\alpha)$. Moreover, as in [BJ23] in the projective setting, we give a *valuative criterion* for $\widehat{K}$-stability in Section 5.3. In the last section of the thesis we adapt the arguments of [Li22] to obtain Theorem 5.4.4, that states that uniform $\widehat{K}$-stability of (X, α) implies the existence and uniqueness of a constant scalar curvature Kähler metric in α.

0.2 Short introduction to pluripotential theory

0.2.1 Electrostatics

In classical electrostatics in the plane, given a charge distribution $\rho\colon \mathbb{R}^2 \to \mathbb{R}$ on the plane $\mathbb{R}^2$, Coulomb's Law states that the electromagnetic force is proportional to the gradient of the *electric potential* V, that satisfies the following Poisson equation:

$$\Delta V = \frac{\rho}{\epsilon_0}, \tag{0.2.1.1}$$

where ϵ_0 is the *permittivity of free space*, a physical scaling constant. For the rest of this manuscript we will ignore physical constants.

A physically interesting problem is to understand the electric potential created by a single positively charged particle at a point $y \in \mathbb{R}^2$. To study this problem one then must consider singular charge distributions, which are described using *distributions*. More precisely, the charge distribution of a unit point charge at a point y is given by $\rho = \delta_y$, where δ_y is the *Dirac mass at* y. This means that the electric potential V of a positive unit point charge is such that when its Laplacian is averaged against any test function, we get the value of the test function at y.

Mathematically, it is easy to see that there are no smooth functions on $\mathbb{R}^2$ with this property, and hence this naturally leads us to consider potentials that are not necessarily smooth.

Therefore, in this generality, if the charge distribution is given by some signed measure μ, the (possibly singular) potential V then satisfies

$$\Delta V = \mu, \tag{0.2.1.2}$$

as measures.

In the case of a positive point charge at $y \in \mathbb{R}^2$, Poisson's equations boils down to

$$\Delta V = \delta_y.$$

In dimension 2, the classical solution is given by the *Green kernel*

$$G(x,y) \doteq \frac{1}{2\pi} \ln|x-y|,$$

that is, the potential is given by $V\colon x \mapsto G(x,y)$.

When in equation (0.2.1.2) the signed measure μ is actually a positive measure –physically this means to consider a distribution of only positive charges– we say that a solution of (0.2.1.2) is *subharmonic*.

Motivated by physics, the study of the Laplace and Poisson equations became an important and rich area of mathematics, which we refer to as *potential theory*.

0.2.2 Potential theory

We hope that by now it is clear that one of the main objectives of potential theory is to give a complete study of subharmonic functions.

These functions are analogues of convex functions in one complex variable: it is in fact easy to check that the Laplacian of V coincides with the complex Hessian $\frac{\partial^2 V}{\partial z \partial \bar{z}}$ as a distribution. Thus, in the same way that convex functions on $\mathbb{R}$ are defined as the functions satisfying

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \text{for all } x, y \in \mathbb{R} \text{ and } t \in [0,1]$$

in $\mathbb{R}^2 \simeq \mathbb{C}$ we can identify subharmonic functions with functions satisfying the submean value inequality:

$$V(a) \leq \frac{1}{\pi r^2} \int_{D(a,r)} V(x)\,\mathrm{d}\lambda(x), \tag{0.2.2.1}$$

for each disk $D(a,r)$ in their domain of definition.

We can extend this notion of "convexity" to $\mathbb{C}^n$ by saying that *plurisubharmonic functions* are the functions on open subsets of $\mathbb{C}^n$, whose restriction to complex lines are subharmonic. This, in turn, means that smooth plurisubharmonic functions U are exactly those for which the complex Hessian

$$\left(\frac{\partial^2 U}{\partial z_i \partial \overline{z_j}} \right)_{i,j} \geq 0$$

is positive semidefinite. We can easily check that plurisubharmonicity is preserved under pullback by holomorphic maps, which implies that plurisubharmonicity is a notion well defined on complex manifolds.

Let X be a complex manifold, $f \in \mathcal{M}_X$ a meromorphic function on X, and $\operatorname{div}(f) = \sum_j m_j Z_j$ the associated principal divisor on X. The *Lelong–Poincaré equation* tells us that the complex Hessian of $\log|f|$ is the current[1] given by

$$\frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log|f| = \sum_j m_j \delta_{Z_j},$$

where δ_{Z_j} refers to the current of integration along Z_j. In particular, if f is holomorphic, then $\log|f|$ is plurisubharmonic (or *psh* for short). For simplicity we will denote by $\mathrm{dd^c}$ the complex Hessian operator $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}$.

It follows from the maximum principle for psh functions that on a compact complex manifold X every plurisubharmonic function is constant. Hence, on compact manifolds one must allow some loss of positivity, and consider *quasi-plurisubharmonic functions.*

0.2.3 Global theory on compact complex manifolds

Let X be a compact complex manifold, we say that the function $u \colon X \to [-\infty, +\infty[$ is *quasi-plurisubharmonic* (or *quasi-psh*) if locally it is given by the sum of a plurisubhar-

[1] A distribution valued differential form.

monic function and a smooth function.

Moreover, if $\theta \in \mathcal{A}^{1,1}$ is a real smooth $(1,1)$-form, we say that a quasi-psh function u is *θ-psh* if the complex Hessian current of u satisfies

$$\mathrm{dd^c} u \geq -\theta.$$

Whenever θ is closed, by the Poincaré lemma, then θ is locally given by the complex Hessian of a smooth *local potential* f

$$\theta \underset{\text{loc}}{=} \mathrm{dd^c} f,$$

which in particular means that u is θ-psh if $u+f$ is psh for some (hence any) local potential f of θ. Moreover, it is easy to see —with a convolution argument— that locally every θ-psh function is a decreasing limit of locally defined smooth θ-psh functions. More interestingly, whenever X is a compact manifold and ω is a strictly positive $(1,1)$-form, we have the following global regularization result due to [BK07]:

Theorem 0.2.1. *An integrable usc function $u\colon X \to [-\infty, +\infty[$ is ω-psh if and only if u is the decreasing limit of smooth ω-psh functions.*

This shows that even though plurisubharmonicity is a local notion, we can see it globally once we fix the class of smooth psh functions. This is the key point for the non-Archimedean approach that we will discuss later.

Plurisubharmonic functions share a lot of properties with subharmonic functions, however there is a substantial difference in the case of complex dimension one: while the complex Hessian, a $(1,1)$-form, of a psh function is a volume form in dimension one, in higher dimensions this is not the case. The natural positive measure associated to a smooth θ-psh function u is then

$$\underbrace{(\theta + \mathrm{dd^c} u) \wedge \cdots \wedge (\theta + \mathrm{dd^c} u)}_{n \text{ times}} \in \mathcal{A}^{n,n}(X), \quad \text{for } n = \dim X. \tag{0.2.3.1}$$

When θ is closed, this is locally given by

$$\det \left(\partial_{z_i} \partial_{\bar{z}_j} (u + f) \right)_{i,j} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n,$$

where f is a local potential of θ. This is why we say that the measure associated to the volume form of Equation (4.4.0.1) is the *Monge–Ampère measure of u.*

A classical problem in pluripotential theory is to study the convergence of the Monge–Ampère measure of smooth psh functions. We will study this problem on a special class of complex manifolds, compact *Kähler manifolds*, the manifolds that admit a globally defined strictly positive $(1,1)$-form that is closed.

0.2.4 Kähler geometry

We say that $\omega \in \mathcal{A}^{1,1}$ is a *Kähler form* if ω is closed and positive; equivalently, if locally ω can be written as the complex Hessian of a smooth *strictly* psh function, that is, if

$$\omega \underset{\mathrm{loc}}{=} \mathrm{dd^c} u, \quad \text{for } u \text{ with } \mathrm{dd^c} u > 0 \text{ positive definite.}$$

Moreover, we say that the cohomology class $\alpha \doteq [\omega]$ is a *Kähler class*, and denote by $V_\alpha = V_\omega$ the integral $\int_X \omega^n \in \mathbb{R}$.

By the $\partial\bar{\partial}$-Lemma, any other smooth representative of $\theta \in \alpha$ differs from ω by the complex Hessian of a smooth function

$$\theta = \omega + \mathrm{dd^c} u, \quad \text{for some } u \in C^\infty(X, \mathbb{R}).$$

In particular, the set

$$\mathcal{H}(\omega) \doteq \{u \in C^\infty(X, \mathbb{R}) \mid \omega + \mathrm{dd^c} u > 0\}$$

parametrizes the Kähler forms in α, up to an additive constant.

Not all compact complex manifolds carry a Kähler form; those that do are said to be *Kähler*.

We will now fix X a compact Kähler manifold and ω a Kähler form on X. Let us now deal with the problem of finding a suitable topology to make the Monge–Ampère operator continuous. To do so we define the *Monge–Ampère energy*.

0.2.5 Finite energy potentials

For a *Kähler potential* $u \in \mathcal{H}(\omega)$ we define the *Monge–Ampère energy*[2]

$$\mathrm{E}_\omega(u) \doteq \frac{1}{V_\omega(n+1)} \sum_{j=0}^{n} \int_X u\,(\omega + \mathrm{dd^c} u)^j \wedge \omega^{n-j}.$$

Moreover, for v a (not necessarily smooth) ω-plurisubharmonic function we set

$$\mathrm{E}_\omega(v) \doteq \inf\{\mathrm{E}_\omega(u) \mid v \leq u \in \mathcal{H}(\omega)\} \in [-\infty, +\infty[\,,$$

and we say that v is of *finite energy* if $\mathrm{E}_\omega(v) > -\infty$. We denote the set of finite energy potentials by $\mathcal{E}^1(X, \omega)$. We endow $\mathcal{E}^1(X, \omega)$ with the coarsest refinement of the L^1 topol-

[2] It is not hard to check that the Monge–Ampère energy is an *Euler–Lagrange functional* of the Monge–Ampère measure, that is, if f is any smooth function then

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{E}_\omega(u + tf) = V_\omega^{-1} \int_X f\,(\omega + \mathrm{dd^c} u)^n.$$

ogy that makes the Monge–Ampère energy continuous. Finally, one can check that the Monge–Ampère operator extends continuously to $\mathcal{E}^1$.

After spending some time discussing pluripotential theory on compact Kähler manifolds, let us now study their geometry.

0.3 A less short introduction to complex differential geometry

The goal of this section is to study a classical problem in Riemannian geometry in this complex (Kählerian) setting: the problem of finding metrics of constant curvature.

Curvatures

In complex dimension one, Poincaré's uniformization theorem states that on every compact Riemann surface there exists a unique Hermitian (and therefore —for dimensional reasons— Kähler) metric with constant curvature.

In higher dimensions, there are several notions that generalize Gaussian curvature of a Kähler metric :

- Sectional curvature and its holomorphic variants;
- Ricci curvature;
- Scalar curvature.

For constant sectional curvature[3] metrics we know that there is a uniformization theorem analogous to the one in complex dimension one. However, there are plenty of examples of manifolds that cannot be uniformized by either complex projective space, the unit ball or $\mathbb{C}^n$, therefore it is too much to ask for a given Kähler manifold to carry a metric of constant sectional curvature.

Thus, a more natural question to ask would be whether, as in the case of Riemann surfaces, there always exists a Kähler metric with constant Ricci curvature, i.e. a *Kähler–Einstein metric*, on a given complex manifold.

0.3.1 Ricci curvature

On an orientable Riemannian manifold (M, g), there is a naturally defined volume form associated to the metric, which locally can be written as:

$$\sqrt{\det(g_{i,j})}\, \mathrm{d}x_1 \wedge \cdots \mathrm{d}x_n.$$

[3] or its Kähler variants

The *Ricci curvature* then appears as the first metric invariant when looking at the Taylor expansion in normal coordinates:

$$\sqrt{\det(g_{i,j}(x))} = 1 - \frac{1}{6} R_{i,j}(x_0) x^i x^j + O(|x - x_0|^3),$$

where $R_{i,j}$ represents the symmetric *Ricci tensor* in these coordinates.

Similarly, when X is complex manifold and $\omega \in \mathcal{A}^{1,1}(X)$ a smooth strictly positive $(1,1)$-form, we can define a volume form on X, by doing the following procedure: let h be the associated[4] *Hermitian metric* on TX. By taking the top exterior product we get a Hermitian metric on the holomorphic line bundle $\det TX$. Its curvature[5] is a closed smooth $(1,1)$-form on X that we denote by

$$\mathrm{Ric}(\omega) \in \mathcal{A}^{1,1}(X),$$

which locally is given by

$$\mathrm{Ric}(\omega) \underset{loc}{=} -\mathrm{dd^c} \log \omega^n.$$

By classical Chern–Weil theory, the *Ricci form* $\mathrm{Ric}(\omega)$ represents a characteristic class of X: the *first Chern class of X*, $\mathrm{c}_1(X)$.

Whenever ω is closed, i.e. a Kähler form, we can then prove that the Chern connection coincides with the complexified Levi–Civita connection of the associated[6] Riemannian metric g, and that the complexified Ricci tensor of g coincides with Ricci form of ω.

Before studying Kähler metrics of constant Ricci curvature, that is, Kähler metrics satisfying the Einstein equation:

$$\mathrm{Ric}(\omega) = \lambda\omega,$$

let us first delve into a more basic question of prescribing the Ricci form of a Kähler metric.

0.3.2 Calabi conjecture: prescribing a Ricci form

After observing that the Ricci form of any Kähler metric lies in the same cohomology class $\mathrm{c}_1(X)$, a natural question arises:

Question. *Given a compact Kähler manifold X, and $\rho \in \mathrm{c}_1(X)$ a smooth representative of the first Chern class of X, can we find a Kähler form ω whose Ricci form*

$$\mathrm{Ric}(\omega) = \rho$$

realizes ρ?

[4] If J represents the complex structure on X we may define $h(X,Y)$ to be $\omega(X, JY) - i\omega(X,Y)$.
[5] By curvature we mean the curvature of the associated Chern connection.
[6] As before, to a positive $(1,1)$-form ω we associate Riemannian metric $g(X,Y)$ given by $\omega(X,JY)$, and turns out that ω is Kähler if and only if the complex structure J is parallel with respect to the Levi–Civita connection of g.

Calabi conjectured that this was the case in [Cal57], and this became known as the *first Calabi conjecture.* Observe, in particular, that if the canonical bundle is trivial, i.e. $\mathrm{c}_1(X) = 0$, then Calabi's first conjecture implies that we can always find a Ricci flat Kähler metric on X.

It turns out that the Calabi conjecture is equivalent to solving a Monge–Ampère equation on X. Indeed, let ω be any fixed Kähler metric, and F a smooth function such that

$$\mathrm{Ric}(\omega) = \rho + \mathrm{dd^c}F,$$

we then have that $u \in \mathcal{H}(\omega)$ is such that $\mathrm{Ric}(\omega + \mathrm{dd^c}u) = \rho$ if and only if

$$(\omega + \mathrm{dd^c}u)^n = e^{-F}\omega^n, \tag{0.3.2.1}$$

up to normalization of F.

In 1978 Yau showed, [Yau77, Yau78], that in every Kähler class $\alpha = [\omega]$ of a Kähler manifold X, we can find a Kähler form $\omega + \mathrm{dd^c}u \in \alpha$ solving (0.3.2.1). Yau's ground breaking work awarded us the solution to Calabi's conjecture, and awarded him a Fields medal.

0.3.3 Kähler–Einstein metrics

We next proceed to study the existence of Kähler–Einstein metrics, that is, to find a Kähler form such that

$$\mathrm{Ric}(\omega) = \lambda\omega.$$

To do so let us first observe that if X is a compact Kähler manifold that carries a Kähler metric of definite signed Ricci curvature, then the class $\mathrm{c}_1(X) \doteq \mathrm{c}_1(TX) \in H^{1,1}(X)$ must have a sign constraint, that is :

1. If X admits a Kähler metric of zero Ricci curvature, then we must have $\mathrm{c}_1(X) = 0$, in this case we say that X is *Calabi–Yau.*

2. If X admits a Kähler metric of Ricci curvature that is negative, then minus the first Chern class $-\mathrm{c}_1(X)$ is a Kähler class, in this case we say that X is *canonically polarized.*

3. Lastly, if X admits a Kähler metric of positive Ricci curvature, we then must have that the first Chern class of X is a Kähler class, and we say that X is a *Fano* manifold.

In complex dimension one, the topology completely determines the first Chern class: if X is a compact Riemann surface of genus is zero, then X is biholomorphic to $\mathbb{P}^1$ and, in particular, Fano. If the genus of X is one, then X is an elliptic curve, which is easily verified to be Calabi–Yau. If the genus of X is greater than or equal to two, then X

is canonically polarized. The uniformization theorem tells us that one can always find a metric –in the conformal class given by the complex structure– of curvature constant equal to $+1, 0$, and -1 respectively.

As was with the Calabi conjecture, it is easy to see that finding a Kähler–Einstein metric is equivalent to solving a Monge–Ampère type equation

$$(\omega + \mathrm{dd^c} u)^n = e^{-\lambda u + c}\omega^n,$$

for some constant c. However, the complexity of finding solutions greatly changes according to the sign of λ. We will now explore what is known for these three very important classes of manifolds.

Calabi–Yau manifolds

Calabi–Yau manifolds, that is manifolds with trivial canonical bundle, always carry Ricci flat Kähler metrics.

Indeed, as mentioned before, this was proven by Yau as part of the solution to the Calabi conjecture.

Canonically polarized manifolds

Whenever X is canonically polarized, the solution of the Monge–Ampère equation associated to the Kähler–Einstein problem

$$(\omega + \mathrm{dd^c} u)^n = e^{f+u}\omega^n, \qquad \text{for some smooth function } f \in C^\infty(X, \mathbb{R}).$$

was shown to be simpler than the previous one, and, independently, Yau and Aubin proved in [Aub76, Aub78, Yau77, Yau78] that one can always solve it.

Fano manifolds

When $\mathrm{c}_1(X)$ is a Kähler class, the situation is more complicated. Unlike the previous cases, Kähler–Einstein metrics don't always exist.

The first obstructions to find Kähler–Einstein metrics were related to the existence of vector fields. By a theorem of Matsushima [Mat57] we know that on a Fano manifold that admits a Kähler–Einstein metric, the automorphism group should be *reductive*, that is, the complexification of a maximal compact subgroup given as the group of holomorphic isometries of the KE metric. In general, the Lie algebra of holomorphic vector fields with fixed points on a Kähler manifold bearing a constant scalar curvature Kähler metric is reductive.

Remark 0.3.1. *If the first Chern class of X has a sign, that is, if for some $\lambda \in \mathbb{R}^*$ we have that $\lambda\,\mathrm{c}_1(X)$ is a Kähler class, then any constant scalar curvature metric on $\lambda\,\mathrm{c}_1(X)$*

is actually of constant Ricci curvature. Indeed, if ω is such a metric, then

$$\mathrm{Ric}(\omega) - \lambda\omega = \mathrm{dd^c} f,$$

for some smooth function f on X. Moreover, taking the trace with respect to ω we get

$$\mathrm{Scal}(\omega) - \lambda n = \Delta_\omega f,$$

which by assumption on ω is zero, thus f is harmonic on a compact manifold, hence constant.

As just observed, on a Fano manifold a constant scalar curvature Kähler metric (which we will just call *cscK* for the rest of this manuscript) in $\mathrm{c}_1(X)$ is Kähler–Einstein. Therefore Kähler–Einstein metrics are exactly the zeroes of the following map μ:

$$\mathcal{H}(\omega) \ni u \mapsto \mu_u \doteq (\mathrm{Scal}(\omega + \mathrm{dd^c} u) - \underline{s})(\omega + \mathrm{dd^c} u)^n,$$

where $\omega \in \mathrm{c}_1(X)$ denotes a fixed Kähler metric on the Fano manifold X, and $\underline{s}$ denotes the average of the scalar curvature of ω.

Moreover, if G denotes the identity component of the automorphism group of X, $\mathrm{Aut}_0(X)$, we can define an action of G on $\mathcal{H}(\omega)$ that will make the map μ G-equivariant. Indeed, since G preserves the cohomology class of ω, we can define the action of G on $\mathcal{H}(\omega)$ by the induced action on Kähler forms, which will clearly make μ equivariant. More precisely, denoting by τ_g the smooth function[7] such that $g^*\omega = \omega + \mathrm{dd^c}\tau_g$ for any $g \in G$, we then can define a G action on $\mathcal{H}(\omega)$ by

$$G \times \mathcal{H}(\omega) \ni (g, u) \mapsto g^*u + \tau_g \in \mathcal{H}(\omega).$$

Turns out that μ admits a primitive, that is, a functional $F_\mu \colon \mathcal{H}(\omega) \to \mathbb{R}$ in such a way that derivative of F_μ at u along a smooth function f is given by integration against μ_u

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F_\mu(u + tf) = \int_X f\mu_u.$$

This was first introduced by Mabuchi [Mab86], who called it the *K-energy*. Moreover, in [Mab87] Mabuchi introduced a natural metric structure on $\mathcal{H}(\omega)$ that in particular makes F_μ a convex functional.

Since μ is G-equivariant, F_μ satisfies the following invariance property:

$$F_\mu(g \cdot u) - F_\mu(g \cdot v) = F_\mu(u) - F_\mu(v), \qquad \text{for any } u, v \in \mathcal{H}(\omega).$$

In particular, the quantity $\chi(g) \doteq F_\mu(g \cdot u) - F_\mu(u)$ does not depend on the choice of

[7] We can normalize τ_g so that $\mathrm{E}_\omega(\tau) = 0$, for more details see [BJT24].

potential u, and only on $g \in G$, and defines a group homomorphism $G \to \mathbb{R}$. Now, to any holomorphic vector field $\xi \in H^0(X, TX) = \operatorname{Lie} G$ we can associate the following quantity:

$$\mathrm{Fut}(\xi) \doteq \mathrm{d}_e\chi(\xi) \in \mathbb{R},$$

known as the *Futaki invariant*, defined by Futaki in [Fut83][8]. Concretely it is given by the Lie derivative

$$\mathcal{L}_\xi F_\mu(u) = \int_X (\dot{\tau}_\xi + \mathcal{L}_\xi u)\,\mu_u, \quad \text{for } \dot{\tau}_\xi \doteq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tau_{\exp(\xi t)}.$$

Clearly, this implies that if $u \in \mathcal{H}(\omega)$ defines a cscK metric, and hence a Kähler–Einstein metric, $\omega + \mathrm{dd^c}u$, then $\mu_u = 0$ and, in particular, $\mathrm{Fut}(\xi) = 0$ for any holomorphic vector field, creating an obstruction to the existence of a Kähler–Einstein metric.

Up until this point, all the obstructions we discussed, [Fut83, Mat57], are related to the automorphism group of X. In fact, until the 1980's it was believed that vector fields were the only obstruction for existence of such metrics.

Conjecture (Folklore conjecture)**.** *Let X be a Fano manifold with no holomorphic vector fields, $H^0(X, TX) = \{0\}$, then X admits a Kähler–Einstein metric.*

Indeed, in 1990 Tian [Tia90] proved the conjecture for surfaces.

However, in [Tia97] Tian realized that discreteness of the automorphism group was not enough to guarantee a KE metric. Indeed, he gave an example of a Fano manifold with discrete automorphism group (i.e. where there are no nonzero holomorphic vector fields) that still did not admit any Kähler–Einstein metric. On a previous work with Ding, [DT92], the authors had already proposed that the obstruction should be related to $\mathbb{C}^*$-equivariant degenerations of X, and gave a counterexample for the analogous Folklore conjecture for orbifolds Kähler–Einstein metrics. In the article [Tia97], Tian improves their result to the smooth setting. To better understand this direction, let us first delve in Tian's work.

In [Tia97], Tian proves that the *properness* of the Mabuchi's K-energy F_μ implies the existence of Kähler–Einstein metrics on a Fano manifold. This then suggests that if the "derivative at infinity" of the Mabuchi functional along some "testing" rays $u_t \in \mathcal{H}(\omega)$ is positive,

$$\liminf_{t \to +\infty} \frac{F_\mu(u_t)}{t} > 0,$$

then there should exists a Kähler–Einstein metric on X.

Remember, however, that if $\xi \in H^0(X, TX)$ is a holomorphic vector field, φ_t its flow, and $u \in \mathcal{H}(\omega)$ a Kähler potential, then $u_t \doteq \varphi_t^* u$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} F_\mu(u_t) = \mathrm{Fut}(\xi).$$

[8]Historically this not how it was found, Futaki's invariant predates the work of Mabuchi [Mab86] introducing the functional F_μ. In fact, when announcing his result in 1985 Mabuchi uses the preliminary title "A functional integrating Futaki's invariant" [Mab85].

This led Tian to introduce the notion of *K-stability*.

0.4 K-stability

Loosely, Tian's idea to the obstruction of Kähler–Einstein metrics on a Fano manifold was to consider the Futaki obstruction, not only on X, but also on special degenerations of X, that had the following form: If we embed X into some projective space $\mathbb{P}^N$[9], given any representation $\rho\colon \mathbb{C}^* \to \mathrm{GL}(n+1, \mathbb{C})$, we can define for $t \in \mathbb{C}^*$ the manifold

$$\mathcal{X}_t \doteq \rho(t)(X) \subseteq \mathbb{P}^n,$$

then taking the *flat limit* $\mathcal{X}_0 \doteq \lim_{t\to 0} \mathcal{X}_t$ gives us a projective scheme, our desired degeneration. Then, whenever $\mathcal{X}_0$ "not too singular", Tian considered the induced Futaki invariant $\mathrm{Fut}_{\mathcal{X}_0}$ for the restricted action $\mathbb{C}^* \curvearrowright \mathcal{X}_0$, and conjectured:

Conjecture (Tian)**.** *Let X be a Fano manifold such that for any mildly singular $\mathbb{C}^*$-degeneration $\mathcal{X}$ –as described above– we have*

$$\mathrm{Fut}_{\mathcal{X}_0} \geq 0,$$

then X admits a Kähler–Einstein metric.

Donaldson in [Don02] refined this conjecture, by giving an algebraic interpretation of the Futaki invariant for such "not so singular" degenerations. Indeed, Donaldson considered not only Fano manifolds, but assuming that the manifold X was polarized[10], he defined an algebraic invariant attached to $\mathbb{C}^*$-equivariant degenerations of X, that coincided with the Futaki invariant as above when the latter is defined.

This, in particular, allowed him to extend the definition of the Futaki invariant for degenerations that are much more singular, and is still the point of view taken today. More precisely:

Definition 0.4.1. *Let (X, L) be a polarized manifold, that is, X a compact complex manifold and $L \to X$ an ample line bundle. We say that $(\mathcal{X}, \mathcal{L})$ is a* test configuration *for (X, L) if*

1. *$\mathcal{L} \to \mathcal{X}$ is a line bundle over a normal projective variety, with a linearized $\mathbb{C}^*$ action on $\mathcal{L}$.*

2. *There exists a flat morphism $\pi\colon \mathcal{X} \to \mathbb{P}^1$, together with a $\mathbb{C}^*$-equivariant isomorphism*

$$\mathcal{X} \setminus \pi^{-1}(0) \xrightarrow{\rho} X \times (\mathbb{P}^1 \setminus \{0\}),$$

[9] Using sections of powers of $-K_X$.

[10] When the Kähler form ω is in the first Chern class $\mathrm{c}_1(L)$ of some ample line bundle $L \to X$

mapping $\mathcal{L}|_{\pi^{-1}(t)}$ *to* L *for any* $t \in \mathbb{P}^1 \setminus \{0\}$*.*

We say that $(\mathcal{X}, \mathcal{L})$ *is* positive *if* $\mathcal{L}$ *is* π*-ample.*

Donaldson then defined the refined version of the Futaki invariant for a test configuration that became known as the *Donaldson–Futaki invariant*, and that we explain next.

If $(\mathcal{X}, \mathcal{L})$ is a positive test configuration for (X, L), then the $\mathbb{C}^*$-action on $\mathcal{X}$ induces an action on the vector spaces $H^0(\mathcal{X}_0, k\mathcal{L}_0)$, and we denote $w(k)$ the *total weight* of the action.

Definition 0.4.2. *We say that the* Donaldson–Futaki invariant *of a positive test configuration* $\mathcal{L} \to \mathcal{X}$ *is the quantity:*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \doteq \frac{b_0 a_1 - a_0 b_1}{a_0^2},$$

for a_0, a_1, b_0, b_1 *the leading coefficients of the Hilbert polynomial,* $k \mapsto \dim H^0(X, kL)$*, and of the polynomial given by total weight* $w(k)$ *respectively, when* k *is sufficiently large.*

Donaldson in [Don02] then proved that, whenever the central fiber $\mathcal{X}_0$ of $(\mathcal{X}, \mathcal{L})$ is smooth, his invariant coincides with the Futaki invariant of the induced action $\mathbb{C}^* \curvearrowright (\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$.

Until this point, test configurations and the Donaldson–Futaki invariant are very algebraic and exclusively defined for polarized manifolds. However, Odaka in [Oda13] and Wang in [Wan12] gave the following "topological" interpretation of the Donaldson–Futaki invariant: they proved that it can be expressed in terms of intersection numbers

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \doteq \frac{n}{n+1}\mu(X, \alpha)\mathcal{L}^{n+1} + \mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1}, \tag{0.4.0.1}$$

where $\mu(X, \alpha) \doteq \frac{-K_X \cdot \alpha^{n-1}}{\alpha^n}$.

Then, since to compute the Donaldson–Futaki invariant one is only required to compute intersection numbers, one could try to make sense of *cohomological test configurations.* Moreover, if we let $(\mathcal{X}, \mathcal{L})$ be any test configuration with ρ a $\mathbb{C}^*$-equivariant morphism to $X \times \mathbb{P}^1$, we then have that the $(1,1)$-class $\mathcal{L} - \rho^* L$ is the class of a $\mathbb{C}^*$-invariant divisor $D \subseteq \mathcal{X}_0$, thus we can obtain $\mathcal{L}$ by adding $\rho^* L$ and D.

With this hint, Sjöström Dyrefelt and, independetly, Dervan and Ross [SD18, DR17] defined a test configuration for a general (non necessarily polarized) Kähler manifold (X, α), where α is a Kähler class, by considering test configurations for X as before, together with the data of $\mathbb{C}^*$-invariant divisor supported on the central fibers. The expression of (0.4.0.1) allows us to define the Donaldson–Futaki invariant for such test configurations, leading up to the following definition:

Definition 0.4.3. *We say that* (X, α) *is* K-stable *if, for every positive test configuration* $(\mathcal{X}, \mathcal{L})$*, we have*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0,$$

and equality only holds for product *test configurations* $\mathcal{X} \setminus \mathcal{X}_\infty \simeq X \times \mathbb{C}$.

The intersection theoretic formula (0.4.0.1) allowed Odaka [Oda12] to prove that Calabi–Yau and canonically polarized manifolds are all K-stable. Doing so he beautifully gave a new perspective to the conjecture we will describe next.

The *Yau–Tian–Donaldson conjecture* then predicts the following obstruction to constant curvature metrics:

Conjecture. *There exists a constant scalar curvature Kähler metric in the Kähler class* α *if and only if* (X, α) *is K-stable.*

In particular, a Fano manifold admits a Kähler–Einstein metric if and only if the pair $(X, -K_X)$ *is K-stable.*

In the Fano setting, the conjecture was proved by Chen–Donaldson–Sun in [CDS15a, CDS15b, CDS15c], Tian in [Tia15] and multiple other proofs became available at a later time. In particular, in [BBJ21] Berman–Boucksom–Jonsson developed a non-Archimedean approach solving the conjecture based on a variational method to find critical points of the Mabuchi functional as we have seen in this text. The general conjecture is still open, although significant progress has been made since the first time it appeared in the literature. Until this point, the non-Archimedean/variational approach of [BBJ21] ended up being the most successful in the polarized setting: In [BHJ19] the authors prove that the existence of cscK metrics implies K-stability. In [Li22] Chi Li proves, using the non-Archimedean approach, a partial converse: that a stronger version of K-stability (that in this manuscript we will call $\widehat{K}$-stability) implies the existence of cscK metrics.

In the Kähler setting, [SD18, DR17] independently prove that the existence of a cscK metric in α implies the stability of (X, α).

The goal of this thesis is to give a non-Archimedean interpretation of this conjecture, when (X, α) is Kähler and not necessarily algebraic. As an application of this non-Archimedean theory for Kähler manifolds we generalize a result of Chi Li [Li22] to the transcendental setting, and prove:

Theorem 0.4.4 (Theorem A)**.** *Let* (X, α) *be* $\widehat{K}$-stable*, then there exists a constant scalar curvature Kähler metric in* α*.*

As we will see next, the main idea is to use non-Archimedean pluripotential theory to give an intepretation for a "completion" of the set of test configurations, and $\widehat{K}$-stability consists in the positivity of the Donaldson–Futaki invariant on this completion. It will then be an algebraic notion of stability which is stronger than usual K-stability.

Recenlty, Boucksom and Jonsson announced the solution to a weaker form of a much more general *weighted Yau–Tian–Donaldson conjecture*, that in particular would imply that the existence of cscK metrics is equivalent to $\widehat{K}$-stability.

Before going into non-Archimedean geometry, let us first give an analogy with a finite dimensional problem that resembles this infinite dimensional setting of the Yau–Tian–Donaldson conjecture.

0.4.1 From K-stability to non-Archimedean geometry.

To define the Donaldson–Futaki invariant, Donaldson was deeply inspired by the connections between algebraic geometry with symplectic geometry. More precisely, the connections of *stability* coming from *Geometric Invariant Theory (GIT)*, and the symplectic theory of *symplectic reduction.*

In GIT, when considering a representation $G \to \mathrm{GL}(V)$ of a reductive group G on a vector space V, the *stability* of a point $0 \neq x \in V$ is equivalent to the *properness* of a given functional f which we describe next: Let N be a K-invariant Hermitian norm on V, where K is a maximal compact subgroup of G, then we define a map

$$g \mapsto f(g) \doteq \log(g^*N)(x) = \log N(g \cdot x),$$

that, by K-invariance, descends to the quotient

$$f\colon G/K \to \mathbb{R},$$

which is the functional we want to consider.

Furthermore, one can check that f is a *convex function* for the natural Riemannian structure on the symmetric space G/K, which we can consider to be a parametrization of a set of norms on V. Thus, by convexity, checking properness is equivalent to having

$$\lim_{t\to+\infty} \frac{f(h(t))}{t} > 0,$$

where h_t is any nontrivial geodesic ray of G/K.

More specifically, by the Hilbert–Mumford criterion, to check the stability of x it suffices to verify that for each algebraic one parameter subgroup $g\colon \mathbb{C}^* \to G$, the behaviour of f on real lines $g\colon \mathbb{R} = \mathbb{C}^*/S^1 \to G/K$ is “proper”, that is,

$$\lim_{t\to+\infty} \frac{f(t)}{t} \geq 0, \quad \text{for } f(t) = f \circ g_t.$$

Therefore, to check the stability of points one needs to study the behaviour of the norms

$(g_t^* N)^{1/t}$ when t goes to infinity. A simple computation gives us that the pointwise limit

$$\lim_{t\to\infty} (g_t^* N)^{1/t} \text{ exists and is a non-Archimedean norm on } V,$$

with respect to the trivial absolute value, i.e. a map $\chi\colon V \to [0,+\infty[$ such that

$$\chi(v+w) \leq \max\{\chi(v), \chi(w)\}, \quad \chi(\lambda v) = \chi(v), \quad \chi(v) = 0 \iff v = 0,$$

for any $v, w \in V$ and $\lambda \in \mathbb{C}^*$. Thus, in the finite dimensional setting of GIT, stability of a point on a vector space V can be checked using non-Archimedean norms on V.

In our Kähler setting, by the results of Chen–Cheng [CC21a, CC21b], to find a metric of constant scalar curvature we must check a properness type condition for the Mabuchi K-energy. More specifically, we can endow $\mathcal{E}^1(\omega)$ a natural metric structure[11], and extend the Mabuchi functional F_μ to $\mathcal{E}^1(\omega)$, and (neglecting the action of automorphisms) check that:

$$\lim_{t\to\infty} \frac{F_\mu(u_t)}{t} > 0, \qquad \text{for all unit speed geodesic rays } u_t \in \mathcal{E}^1(\omega).$$

When this happens we say that F_μ is *coercive.*

It is thus natural to wonder whether there is an equivalent of the Hilbert–Mumford criterion in this infinite dimensional setting, that is, if checking the linear growth condition of F_μ along $\mathbb{C}^*$ representations is enough to guarantee the properness F_μ. This is exactly in the spirit of the Yau–Tian–Donaldson conjecture. Furthermore, since in the finite dimensional setting the stability condition was closely related to the study of non-Archimedean norms on V, it is natural to imagine that stability in the infinite dimensional setting involves the study of an infinite dimensional analogue of a non-Archimedean metric on X.

In the projective setting this was accomplished by a cumulation of multiple works, see [BFJ15, BFJ16, BHJ17, BHJ19, BBJ21, BJ22, BJ25b, BJ23, BJ24] among others, which we will briefly recall now.

0.4.2 Non-Archimedean geometry: projective setting

Let X be a compact Kähler manifold, and $L \to X$ a very ample line bundle. By definition, we can embed X into projective space using sections of L

$$X \overset{i}{\hookrightarrow} \mathbb{P}(H^0(X, L)), \quad \text{s.t. } i^*\mathcal{O}(1) = L.$$

In turn, Chow's theorem says that every submanifold of projective space is algebraic –the zeroes of some homogeneous polynomials in projective space. This tells us, then, that

[11] By the work of Darvas [Dar15, Dar17] this metric structure comes from a noncomplete L^1 type Finsler metric on $\mathcal{H}(\omega)$, whose completion is isometric to $\mathcal{E}^1(\omega)$.

we can obtain X as the analytification of some integral separated projective scheme over $\mathbb{C}$, that we will denote by X^{sch}.

The key to what comes next is the following observation: in order to get X from X^{sch} we analytified X^{sch} with the Euclidean topology of the Archimedean field $(\mathbb{C}, |\cdot|)$. However, starting from X^{sch} we could have equally seen it as defined over the non-Archimedean field $(\mathbb{C}, |\cdot|_{\mathrm{triv}})$, where $|\cdot|_{\mathrm{triv}}$ denotes the trivial absolute value ($|\lambda|_{\mathrm{triv}} = 1$ for $\lambda \in \mathbb{C}^*$, and $|0|_{\mathrm{triv}} = 0$). Then, by taking the analytification with respect to this trivial absolute value, we obtain a Berkovich analytic space X^{an}, instead of the complex manifold X. We can proceed similarly and get an analytification of L, L^{an}.

Concretely, as a topological space, X^{an} is the set of *valuations* on X, that is, real valued valuations of the field of rational functions $\mathbb{C}(Y)$ of subvarities $Y \subseteq X$. An important definition for what comes next is the *center* of a valuation $v \in X^{\mathrm{an}}$: the *center of* v is the unique (scheme) point $c(v) \in X^{\mathrm{sch}}$ satisfying

$$v|_{\mathcal{O}_{X,c(v)}} \geq 0, \quad v|_{m_{c(v)}} > 0,$$

where $m_{c(v)}$ is the maximal ideal of the local ring $\mathcal{O}_{X,c(v)}$.

Non-Archimedean pluripotential theory of Boucksom, Favre and Jonsson.

For $s \in H^0(X, mL)$ a section of a power of L, we can define a function

$$\log|s| : X^{\mathrm{an}} \to [-\infty, 0], \quad v \mapsto -v(f),$$

where the function f is such that $s = fs_0$ for s_0 a local trivialization[12] of mL around $c(v)$. Moreover, we say that $\psi\colon X^{\mathrm{an}} \to [-\infty, +\infty[$ is a *Fubini–Study* function if there is some $m \in \mathbb{N}$ such that

$$\varphi = m^{-1} \max_{1 \leq i \leq \ell} \{\log|s_i| + \lambda_i\}, \quad \text{for } s_i \in H^0(X, mL), \text{ and } \lambda_i \in \mathbb{Q}.$$

We denote by $\mathrm{FS}_L(X^{\mathrm{an}})$ the set of Fubini–Study functions on X^{an}.

One can check that there is a 1 to 1 correspondence between Fubini–Study functions and (ample) test configurations for L, and moreover these functions are the basis of the non-Archimedean pluripotential theory on X^{an}. One can use them to define general L-psh functions by taking decreasing limits, Monge–Ampère equations and more.

The content of this thesis is to adapt this construction for X a compact Kähler manifold, not necessarily algebraic. We will describe next some of the non-Archimedean pluripotential theory in that context that of course apply to this context.

[12]We observe that since local trivializations around a point differ by a unit of the local ring at the point, this definition does not depend on the choice of local trivialization.

Most of the results in this thesis are analogues of results found in [BBJ21, BJ22, BJ24, BFJ15, BFJ15, BFJ16, BHJ19].

As we said before Theorem A is a transcendental analogue of the main result of [Li22]. The valuative criterion for $\widehat{K}$-stability we prove is based on the valuative criterion first proved in the projective setting by Boucksom and Jonsson in [BJ25b, BJ23]. Although not explicitly formulated this way, the proof of Theorem B follows the same strategy of those above mentioned papers.

The most important ingredient for this valuative criterion is the solution of the non-Archimedean Calabi–Yau theorem that we will explain below. That was first proven by Boucksom, Favre and Jonsson in [BFJ15], Theorem D follows their variational approach.

0.5 Non-Archimedean geometry of Kähler manifolds: new results

The main idea of this thesis is to develop a non-Archimedean theory to study K-stability of a non-algebraic compact Kähler manifold. Which would then allow us to use a non-Archimedean analogue of pluripotential theory to study this algebraic notion.

The basic idea is to associate to a given compact Kähler manifold, X, a compact topological space, that we will denote by $X^\beth$. We then define a natural class of "nice" continuous test functions on $X^\beth$, $\mathcal{D}$, that will parametrize –in a meaningful sense– the set of test configurations. We also define a partially ordered vector space of *closed (1,1)-forms*[13] that we denote by $\mathcal{Z}$, and an operator $\mathrm{dd^c}\colon \mathcal{D} \to \mathcal{Z}$ vanishing only on constants, which will allow us to define test *ω-psh functions*, for any positive $\omega \in \mathcal{Z}$. We do so in analogy with the complex setting by saying that a function $\varphi \in \mathcal{D}$ is ω-plurisubharmonic if $\omega + \mathrm{dd^c}\varphi \geq 0$. In turn, ω-psh functions will parametrize positive test configurations, the key object of study of K-stability.

The natural language for us to do this is the one of non-Archimedean geometry. When considering a polarized pair (X, L) we have discussed how this is done. The main difference from the polarized setting to the general Kähler one is that, if we suppose that we can find a holomorphic line bundle $L \to X$ whose first Chern class $\mathrm{c}_1(L)$ is a Kähler class α, then as we explained before, X is algebraic, and the non-Archimedean space $X^\beth$ corresponds to the Berkovich analytification of X as a scheme, and the "nice" test psh functions are given by Fubini-Study functions of $X^\beth$.

We cannot replicate the same definitions for a non-algebraic X. In practical terms, when X is algebraic, the Berkovich analytification is given by the set of valuations of the fields of meromorphic (hence rational by GAGA's theorem) functions of subvarieties of X. However, in general, when X is not algebraic, X may not admit non-trivial subvarieties,

[13] although denoted this way, this set does not correspond to differential forms, at this point this will be only an abstract vector space.

and the field of meromorphic functions of X, $\mathcal{M}(X)$ may be trivial ($=\mathbb{C}$).

To deal with this issue, we define a set of valuations that —instead of acting on the set of meromorphic functions— acts on the set, $\mathscr{I}_X$, of coherent ideal sheaves of X. This can be given a *semiring* structure and so we define:

Definition 0.5.1. *A non-constant map*

$$v\colon \mathscr{I}_X \to [0,+\infty]$$

is a valuation on X *if it satisfies*

$$v(I\cdot J) = v(I) + v(J), \quad v(I+J) = \min\{v(I), v(J)\},$$

for any two coherent ideals I, J. The set of all valuations will be denoted by $X^{\beth}$, and is endowed with the pointwise convergence topology.

A typical example of a valuation is a *divisorial valuation*:

Definition 0.5.2. *Let $F \subseteq Y \xrightarrow{\mu} X$ be a smooth prime divisor on Y, a bimeromorphic model of X. We define* $\mathrm{ord}_F \in X^{\beth}$ *to be the map that to an ideal it associates the minimal order of vanishing of all local generators of the ideal, that is*

$$\mathscr{I}_X \ni I \mapsto \mathrm{ord}_F(I) = \min\{\mathrm{ord}_F(f_1), \ldots, \mathrm{ord}_F(f_\ell)\},$$

where $f_1, \ldots, f_\ell$ are local generators of the pullback of I to Y at any point $p \in F$. We then define X^{div} to be the set of valuations of the form $r\cdot \mathrm{ord}_F$ for all positive rational numbers $r \in \mathbb{Q}_{>0}$, and all smooth prime divisors over X. We say that an element of X^{div} is a divisorial valuation.

We then have that $X^{\beth}$ is compact (Hausdorff). The first main result of this thesis states that, as in the projective case, it can be viewed as a compactification of the set of divisorial valuations:

Theorem C. *The set of divisorial valuations is dense in $X^{\beth}$.*

Moreover, $X^{\beth}$ comes equiped with the following natural set of continuous functions:

Definition 0.5.3. *We define the set* $\mathrm{PL}(X^{\beth}) \subseteq \mathrm{C}^0(X^{\beth})$ *of PL functions on $X^{\beth}$ as the $\mathbb{Q}$-vector space generated by all finite valued functions of the form:*

$$v \mapsto \min_{1\le j\le \ell}\{v(I_j) + k_j\},$$

for $I_1, \ldots, I_\ell \in \mathscr{I}_X$ coherent ideals, and $k_1, \ldots, k_\ell \in \mathbb{Z}$ integers.

This set of functions is what we will use to parametrize test configurations: As we observed before, test configurations are the data of a $\mathbb{C}^*$-equivariant flat morphism $\pi\colon \mathcal{X} \to \mathbb{P}^1$ of a normal variety $\mathcal{X}$ with a $\mathbb{C}^*$ action, that has all the fibers isomorphic to X, together with the data of a $\mathbb{C}^*$-invariant divisor D supported in the central fiber. Such an object can be obtained by considering the normalized blow-up of a $\mathbb{C}^*$-invariant ideal of $X \times \mathbb{P}^1$ supported on $X \times \{0\}$, together with its exceptional divisor. Finally, we will be able to construct such an ideal from a PL function by decomposing a $\mathbb{C}^*$-invariant ideal $\mathfrak{a}$ as

$$\mathfrak{a} = \sum_{\lambda \in \mathbb{Z}} \mathfrak{a}_\lambda t^\lambda \subseteq \mathcal{O}_{X \times \mathbb{P}^1}, \quad \text{for } \mathfrak{a}_\lambda \in \mathscr{I}_X.$$

As it turns out, this will give us a correspondence between test configurations up to pullback and PL functions. Given a test configuration $\mathcal{X}$ and a divisor $D \subseteq \mathcal{X}_0$, we denote by φ_D the associated PL function. The image under this correspondence of the set of positive test configurations will be denoted by $\mathcal{H}(\alpha)$.

In terms of what we discussed before, we will take $\mathcal{D}$ to be the set of PL functions, the set $\mathcal{Z}$ of "$(1,1)$-forms" will be

$$\varinjlim_{\mathcal{X} \text{ test conf.}} H^{1,1}(\mathcal{X}),$$

and $\mathrm{dd^c}\colon \mathcal{D} \to \mathcal{Z}$ will correspond to the map that associates to φ_D the class of the divisor D.

Comparing with Theorem 0.2.1, it is natural to define the set of α-psh functions by taking decreasing limits of functions in $\mathcal{H}(\alpha)$.

Definition 0.5.4. *Let $\alpha \in H^{1,1}(X)$ be a Kähler class, we say that a usc function $\varphi\colon X^\beth \to [-\infty, +\infty[$ is α-plurisubharmonic, or α-psh, if it is not identically $-\infty$ and there exists a decreasing net $\varphi_j \in \mathcal{H}(\alpha)$, such that*

$$\varphi_j \searrow \varphi.$$

We denote the set of α-psh functions by $\mathrm{PSH}(\alpha)$.

As we shall see, α-psh functions are finite valued on the set of divisorial valuations and we endow $\mathrm{PSH}(\alpha)$ with the *weak* topology of pointwise convergence of X^{div}.

Moreover, this description in terms of test configurations allows us to define the *Monge–Ampère energy* of $\varphi \in \mathcal{D}$[14]:

$$\mathrm{E}_\alpha(\varphi) \doteq \frac{V_\alpha^{-1}}{n+1}(\alpha + \mathrm{dd^c}\varphi)^{n+1}.$$

It is then easy to see that the energy is usc for the weak topology, and so we define the following in analogy with the complex setting:

[14] Observe that by the projection formula, the intersection number, does not depend on the choice of representative of $\mathrm{dd^c}\varphi$ in $\mathcal{Z}$.

Definition 0.5.5. *For any $\varphi \in \mathrm{PSH}(\alpha)$ we set*

$$\mathrm{E}_\alpha(\varphi) \doteq \inf\left\{\mathrm{E}_\alpha(\psi) \mid \varphi \leq \psi \in \mathcal{H}(\alpha)\right\} \in [-\infty, +\infty[\,.$$

We say that φ is a finite energy potential *if $\mathrm{E}_\alpha(\varphi) > -\infty$, and denote the set of all such potentials by $\mathcal{E}^1(\alpha)$, that we endow with the coarsest refinement of the weak topology that makes the Monge–Ampère energy continuous.*

0.5.1 Using non-Archimedean pluripotential theory to study K-stability

Using the description of PL functions in terms of test configurations, we can easily define the Donaldson–Futaki invariant of $\varphi \in \mathrm{PL}(X^\beth)$. However, in order to test stability, we will use a slightly different functional, the *non-Archimedean Mabuchi functional*, a variant of the Donaldson–Futaki invariant that replaces $K_{\mathcal{X}/\mathbb{P}^1}$ with its log version in the formula (0.4.0.1):

$$\begin{aligned}\mathrm{M}_\alpha(\varphi) &\doteq \mathrm{DF}(\mathcal{X}, \underbrace{\alpha + D}_{\mathcal{L}}) + V_\alpha^{-1}(\mathcal{X}_{0,red} - \mathcal{X}_0)\cdot(\underbrace{\alpha + D}_{\mathcal{L}})^n \\ &= \frac{n}{n+1}\mu(X,\alpha)\mathcal{L}^{n+1} + V_\alpha^{-1}\mathcal{L}^n \cdot K^{\log}_{\mathcal{X}/\mathbb{P}^1}\end{aligned}$$

where $K^{\log}_{\mathcal{X}/\mathbb{P}^1}$ is defined to be $K_{\mathcal{X}} + \mathcal{X}_{0,red} - \pi^*(K_{\mathbb{P}^1} + \{0\})$, $\mu(X,\alpha) \doteq \frac{-K_X\cdot\alpha^{n-1}}{\alpha^n}$, and $\varphi = \varphi_D$ for some $D \subseteq \mathcal{X}_0$. In this language, we then have:

Definition 0.5.6. *We say that (X,α) is* K-stable *if*

$$\mathrm{M}_\alpha(\varphi) \geq 0, \qquad \textit{for all } \varphi \in \mathcal{H}(\alpha),$$

where the equality holds only on constant functions.

To define our strengthened stability condition, $\widehat{K}$-stability, we will need to extend the non-Archimedean Mabuchi functional to $\mathcal{E}^1(\alpha)$. We do so by using the non-Archimedean analogue of the Chen–Tian formula. Before introducing it, we establish some notation: let ζ be the first Chern class of K_X; E^ζ_α a variant of the Monge–Ampère energy that, for $\varphi \in \mathcal{D}$, computes the intersection number[15] $V_\alpha^{-1}\zeta\cdot(\alpha + \mathrm{dd^c}\varphi)^n$; and lastly, denote by $A_X\colon X^\beth \to [0,+\infty]$ the *log discrepancy* function that we will define[16] in Chapter 1.

With all this notation established, the Chen–Tian formula for the non-Archimedean Mabuchi functional for $\varphi \in \mathcal{H}(\alpha)$ reads:

$$\mathrm{M}_\alpha(\varphi) \doteq n\frac{-\alpha^{n-1}\cdot\zeta}{V_\alpha}\,\mathrm{E}_\alpha(\varphi) + \mathrm{E}^\zeta_\alpha(\varphi) + \int_{X^\beth} A_X\,\mathrm{d}\mu_\varphi, \tag{0.5.1.1}$$

[15] Again, this is well defined by the projection formula.

[16] Usually, the log discrepancy function is defined for divisorial valuations, what we do in Section 1.3.1 is to extend it to the whole set of valuations.

where μ_φ is the unique probability measure[17] that extends the (positive) functional

$$\mathrm{PL} \ni \psi \mapsto V_\alpha^{-1} \mathrm{dd^c}\psi \cdot (\alpha + \mathrm{dd^c}\psi)^n \in \mathbb{R}.$$

Then, using the symmetry and multilinearity of the intersection product, we can, similarly to how we did for the Monge–Ampère energy, define $\mathrm{E}_\alpha^\zeta(\varphi)$, and μ_φ for $\varphi \in \mathcal{E}^1(\alpha)$. In this way, we can use the formula (0.5.1.1) to extend the non-Archimedean Mabuchi functional to $\mathcal{E}^1(\alpha)$, and we define:

Definition 0.5.7. *We say that* (X, α) *is* $\widehat{K}$-stable *if*

$$\mathrm{M}_\alpha(\varphi) \geq 0, \qquad \textit{for all } \varphi \in \mathcal{E}^1(\alpha),$$

where equality holds only for constants.

We may now state the second main result of this thesis, which extends a result of Chi Li, [Li22], to the transcendental setting:

Theorem 0.5.8 (Theorem A)**.** *Assume* (X, α) $\widehat{K}$-stable*. Then there exists a constant scalar curvature Kähler metric in* α*.*

As in [BBJ21], we show that the non-Archimedean pluripotential theory developed here interacts nicely with its complex counterpart: to each *psh ray* $u_t \in \mathrm{PSH}(\omega)$ we can associate a non-Archimedean α-psh function $U^\beth$. Conversely, for each non-Archimedean psh function of finite energy $\varphi \in \mathcal{E}^1(\alpha)$, there exists a psh ray v_t such that $V^\beth = \varphi$. Moreover, there is a one-to-one correspondence between *maximal rays* in $\mathcal{E}^1(\omega)$ and non-Archimedean potentials in $\mathcal{E}^1(\alpha)$.

Adapting [BBJ21] and using the results of [SD18, DR17], we also prove that to each maximal ray u_t there are slope formulas of the type:

$$\lim_{s\to\infty} \frac{\mathrm{E}_\omega(u_s)}{s} = \mathrm{E}_\alpha(U^\beth), \tag{0.5.1.2}$$

and the inequality:

$$\lim_{s\to\infty} \frac{F_\mu(u_s)}{s} \geq \mathrm{M}_\alpha(U^\beth),$$

where M_α is the non-Archimedean Mabuchi functional, the non-Archimedean counterpart of F_μ. These slope formulas provide the good inequality to the prove Theorem A. Indeed, recall that by a result of Chen-Cheng [CC21a, CC21b] the existence of cscK metrics is the same as a linear growth condition on F_μ, called the *coercivity* of F_μ.

Then, just as in the projective setting, if F_μ is not coercive we can find a geodesic ray $u_t \in \mathcal{E}^1$ such that $t \mapsto F_\mu(u_t)$ is decreasing for t large. We call such a ray a *destabilizing*

[17] This measure is what we will call the *Monge–Ampère measure of* φ, and will be discussed shortly in the next section.

geodesic ray. Analogously to [Li22], every destabilizing ray is maximal, that is, it comes from a non-Archimedean potential of finite energy, $\varphi \in \mathcal{E}^1(\alpha)$. In particular:

$$\lim_{t\to\infty} \frac{F_\mu(u_t)}{t} \geq \mathrm{M}_\alpha(\varphi), \tag{0.5.1.3}$$

which allows us to conclude that if (X, α) is $\widehat{K}$-stable then the term on the right hand side is positive, yielding a contradiction.

Non-Archimedean Monge–Ampère equations

As we briefly mentioned before, to each $\varphi \in \mathcal{H}(\alpha)$ we may define the following positive functional:

$$\mathrm{PL} \ni \psi \mapsto V_\alpha^{-1} \mathrm{dd^c}\psi \cdot (\alpha + \mathrm{dd^c}\varphi)^n \in \mathbb{R}.$$

By positivity, one can verify that this functional is continuous with respect to the C^0 topology, and thus, by the density of PL functions in $\mathrm{C}^0(X^\beth)$, we can show that it defines a probability measure on $X^\beth$, the *Monge–Ampère measure* of φ.

As in the complex setting, and using the synthetic pluripotential theory of [BJ25a], we prove that the operator

$$\mathrm{MA}_\alpha \colon \mathcal{H}(\alpha) \to \mathcal{P}(X^\beth)$$

admits a unique continuous extension to $\mathcal{E}^1(\alpha)$. Moreover, from the definitions we can show that for any $f \in \mathrm{PL}(X^\beth)$ and $\varphi \in \mathcal{H}(\alpha)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{E}_\alpha(\varphi + tf) = \int_{X^\beth} f\, \mathrm{MA}_\alpha(\varphi). \tag{0.5.1.4}$$

Having defined a Monge–Ampère operator, we can formulate and study Monge–Ampère equations in this non-Archimedean setting. The main result of Chapter 4 is that such equations always admit solutions: following [BFJ15, BJ25a] we define an energy functional in the set of measures

$$\mathrm{E}_\alpha^\vee \colon \left(\mathrm{C}^0(X^\beth)\right)^\vee \to \left]-\infty, +\infty\right], \quad \mu \mapsto \sup_{\varphi\in\mathcal{E}^1(\alpha)} \left\{ \mathrm{E}_\alpha(\varphi) - \int_{X^\beth} \varphi\, \mathrm{d}\mu \right\},$$

and we prove that for any probability *measure of finite energy* μ, we can find a potential of finite energy $\varphi \in \mathcal{E}^1(\alpha)$ such that

$$\mathrm{MA}_\alpha(\varphi) = \mu.$$

Theorem D. *The Monge–Ampère operator*

$$\mathrm{MA}_\alpha \colon \mathcal{E}^1(\alpha) \to \mathcal{M}^1$$

is surjective, where $\mathcal{M}^1$ denotes the set of finite energy measures.

By inspecting Equation 0.5.1.4 one expects that solutions of this Monge–Ampère equation should correspond exactly to critical points/maximizers of $\mathcal{E}^1(\alpha) \ni \varphi \mapsto \mathrm{E}_\alpha(\varphi) - \int \varphi \, \mathrm{d}\mu$. This is in fact the case, however, we must deal with the following caveat: in order to formalize the argument we must take any continuous function $f \in \mathrm{C}^0(X^\beth)$ and show that $t \mapsto \mathrm{E}_\alpha(\varphi + tf)$ is differentiable and, as in (0.5.1.4), the derivative is equal to $\int f \, \mathrm{MA}_\alpha(\varphi)$. However, unlike for functions in PL, the Monge–Ampère energy is not defined for a function like $\varphi + tf$, and thus we must consider a different functional. Let ψ_t denote the α*-psh envelope*

$$\mathrm{P}_\alpha(\varphi + tf) \doteq \sup\{\psi \in \mathrm{PSH}(\alpha) \mid \psi \leq \varphi + tf\},$$

that we will show in Chapter 3 to be α-psh. Then we can consider the map $t \mapsto \mathrm{E}_\alpha(\psi_t)$, which is now well defined. Let us study its differentiability on $t = 0$.

It is easy to see that $\mathrm{E}_\alpha(\psi_t) - \mathrm{E}_\alpha(\psi_0) = \mathrm{E}_\alpha(\psi_t) - \mathrm{E}_\alpha(\varphi)$ satisfies

$$\int (\psi_t - \varphi) \, \mathrm{MA}_\alpha(\psi_t) \leq \mathrm{E}_\alpha(\psi_t) - \mathrm{E}_\alpha(\varphi) \leq \int (\psi_t - \varphi) \, \mathrm{MA}_\alpha(\varphi) \leq t \int f \, \mathrm{MA}_\alpha(\varphi),$$

and thus that

$$\begin{aligned} \left| \mathrm{E}_\alpha(\psi_t) - \mathrm{E}_\alpha(\psi_0) - t \int_{X^\beth} f \, \mathrm{MA}_\alpha(\varphi) \right| &\leq \int (\varphi + tf - \psi_t) \, \mathrm{MA}_\alpha(\psi_t) \\ &\quad + t \int f(\mathrm{MA}_\alpha(\varphi) - \mathrm{MA}_\alpha(\psi_t)) \\ &\leq \int (\varphi + tf - \psi_t) \, \mathrm{MA}_\alpha(\psi_t) + 2t \sup|f|. \end{aligned}$$

Therefore, in order to conclude that $t \mapsto \mathrm{E}_\alpha(\mathrm{P}_\alpha(\varphi + tf))$ has derivative at $t = 0$ equal to $\int f \, \mathrm{MA}_\alpha(\varphi)$ we must have that

$$\int (\varphi + tf - \psi_t) \, \mathrm{MA}_\alpha(\psi_t) = 0.$$

This is exactly the content of the *Orthogonality property* that we will prove on Chapter 3. This property will allows us to conclude that the maximizers of $\mathrm{E}_\alpha(\varphi) - \int \varphi \, \mathrm{d}\mu$ are the solutions of the Monge–Ampère equation.

Moreover, in Chapter 3 we consider general α-psh envelopes

$$\mathrm{P}_\alpha(f) \doteq \sup\{\psi \in \mathrm{PSH}(\alpha) \mid \psi \leq f\},$$

and prove that, when $f \in \mathrm{C}^0$ is continuous, so is the envelope $\mathrm{P}_\alpha(f) \in \mathrm{C}^0(X^\beth)$. With this in hand, we proceed to prove that the set of the sup normalized α-psh functions is compact, and together with the estimates of [BJ25a] on Chapter 4 we conclude the proof of Theorem D.

A direct application of Theorem D is to allow us to study $\widehat{K}$-stability in terms of measures instead of potentials. That is, we will be able to define a functional $\beta_\alpha\colon \mathcal{M}^1 \to]-\infty, +\infty]$ such that

$$\beta_\alpha(\mathrm{MA}_\alpha(\varphi)) = \mathrm{M}_\alpha(\varphi), \quad \text{for any } \varphi \in \mathcal{E}^1(\alpha),$$

where M_α denotes the non-Archimedean Mabuchi functional.

The main advantage of this point of view, is that the set of measures of finite energy –as observed by Boucksom and Jonsson on [BJ25a]– does not depend on α.

Valuative criterion for $\widehat{K}$-stability

By following the strategy of [Li22] and [BJ23] we are then able to show that any *divisorial measure* μ, a probability measure supported on a finite set of divisorial valuations, is of finite energy, and moreover, can be approximated by a sequence $\mu_j \in \mathcal{M}^1$ such that

$$\mu_j \to \mu, \quad \text{and } \beta_\alpha(\mu_j) \to \beta_\alpha(\mu).$$

In turn, this implies that to check $\widehat{K}$-stability it is enough to compute the β invariant for divisorial measures.

Then, by using the slope formulas (0.5.1.2), and a formula of [DXZ25] for the energy of the Ross–Witt Nyström transform of a geodesic ray of $\mathcal{E}^1(\omega)$, on Chapter 5 we are able to prove the following more precise version of Theorem B:

Theorem 0.5.9 (Theorem B'). *Let $F_1, \ldots, F_\ell$ be prime divisors over X, and $\xi \in (\mathbb{R}_{\geq 0})^\ell$ such that $\sum \xi_j = 1$, and $\mu_\xi = \sum_j \xi_j \delta_{\mathrm{ord}_{F_j}}$ the associated divisorial measure. Then the beta invariant satisfies*

$$\beta_\alpha(\mu_\xi) = \sum_{i=1}^{\ell} \xi_i \cdot A_X(\mathrm{ord}_{F_i}) + \nabla_{K_X} \widehat{-f_\alpha}(-\xi)$$

for

$$f_\alpha(t) = \lambda + V_\alpha^{-1} \int_\lambda^{+\infty} \mathrm{vol}\left(\alpha - \sum_{i=1}^{\ell} (\lambda - t_i)_+ F_i\right) \mathrm{d}\lambda,$$

where λ is any constant strictly less then $\min\{t_1, \ldots, t_\ell\}$, and $(\lambda - t_i)_+$ denotes $\max\{0, \lambda - t_i\}$.

From the pluripotential theoretic side of things, using the regularity of the solution of the Monge–Ampère equation for a divisorial measure, we prove, as Chi Li did in [Li22], that $\widehat{K}$-stability is equivalent to

$$\mathrm{M}_\alpha(\mathrm{P}_\alpha(\varphi)) \geq 0, \quad \text{for all } \varphi \in \mathrm{PL}(X^{\beth}),$$

where $\mathrm{P}_\alpha(\varphi)$ denotes the α-psh envelope of φ.

This thesis is based on the articles [MP25, MPN25].

For this introduction the author was very inspired by the very good survey article [AS17], the thesis [Szé06], as well as –what from the point of view of the author were– inspiring conversations with his advisors Sébastien Boucksom and Tat Dat Tô.

Chapter 1

Non-Archimedean geometry of Kähler manifolds.

Notations and Conventions

Throughout this paper, a *ring* will always be unital and commutative. By an *analytic variety* we mean a reduced and irreducible complex analytic space. Given X an analytic variety we denote by $\widetilde{X}$ the normalization of X, cf. [GR12, Chapter 6]. Moreover, we simply call *ideal* a coherent ideal sheaf of $\mathcal{O}_X$. We call *flag ideal* a $\mathbb{C}^*$-invariant fractional coherent ideal of $\mathcal{O}_{X\times\mathbb{P}^1}$, supported on $X \times \{0\}$, and we denote the set of such ideals by $\mathcal{F}$.

A flag ideal $\mathfrak{a} \in \mathcal{F}$ can be written as a sum:

$$\mathfrak{a} = \sum_{\lambda\in\mathbb{Z}} \mathfrak{a}_\lambda t^\lambda, \tag{1.0.0.1}$$

for $(\mathfrak{a}_\lambda)_\lambda$ an increasing sequence of ideals of X such that $\mathfrak{a}_\lambda = 0$ for $\lambda \ll 0$ and $\mathfrak{a}_\lambda = \mathcal{O}_X$ for $\lambda \gg 0$.

An ideal on X will typically be denoted by I, J or K. An ideal on $X \times \mathbb{P}^1$ will typically be denoted either $\mathfrak{a}$ or $\mathfrak{b}$.

An ideal I on X is a *prime ideal* if given J, K ideals on X satisfying:

$$I \supseteq J \cdot K,$$

then either $I \supseteq J$, or $I \supseteq K$. An ideal is prime if, and only if, it is a radical ideal and the underlying analytic subspace is irreducible. That is, prime ideals of X are exactly the ideals attached to analytic subvarieties.

Let X be an analytic variety, I an ideal, and $q \in \mathbb{Q}$, a map $g\colon X \to [-\infty, +\infty[$ has *singularities of type* I^q if locally:

$$g(z) = q \log \sum_{i=1}^{k} |f_i| + O(1),$$

for $f_1, \ldots, f_k$ local generators of I.

If X is a Kähler manifold, we denote by $\mathcal{K}(X)$ the set of Kähler forms on X, by $\mathrm{Pos}(X) \subseteq H^{1,1}(X)$ the set of Kähler classes, and $\mathrm{Nef}(X)$ the set of nef classes, i.e. the closure $\overline{\mathrm{Pos}(X)} \subseteq H^{1,1}(X)$.

For $\omega \in \mathcal{K}(X)$, we denote by $\mathrm{Ric}(\omega)$ its *Ricci form.* The trace of $\mathrm{Ric}(\omega)$ is denoted by $\mathrm{Scal}(\omega)$, the *scalar curvature*, and the cohomological quantity:

$$n \cdot \frac{[\mathrm{Ric}(\omega)] \cdot [\omega]^{n-1}}{[\omega]^n},$$

the average of the scalar curvature, by $\underline{s}$.

For $x, y \in \mathbb{R}$, we write $x \lesssim y$ if there exists a uniform constant $C_n > 0$, depending only on an integer n given in the setup, such that $x \leq C_n y$.

1.1 Berkovich spectra as tropical spectra

1.1.1 Berkovich spectrum

Let X be an affine scheme, that is $X = \operatorname{Spec} R$ for some ring R.

If we consider R as a normed ring, with the trivial norm (the norm that to $a \in R$ associates $\|a\|_{\mathrm{triv}}$ that is 1 if $a \neq 0$, and 0 otherwise), then R can be seen as a Banach ring. In particular, we can associate to it a compact Hausdorff topological space, the *Berkovich spectrum of* R, first defined in [Ber90].

Definition 1.1.1. *The* Berkovich spectrum *associated to* $(R, \|\cdot\|_{\mathrm{triv}})$*, denoted by* $\mathcal{M}(R)$*, is the set of bounded multiplicative semi-norms on* R*, i.e.*

$$N \in \mathcal{M}(R) \iff N \leq \|\cdot\|_{\mathrm{triv}},$$

equipped with the Hausdorff topology of pointwise convergence.

The topology of pointwise convergence is the induced subspace topology given by natural inclusion

$$\mathcal{M}(R) \subseteq \prod_{a \in R} [0, 1], \tag{1.1.1.1}$$

as an easy consequence of Tychonoff's theorem $\mathcal{M}(R)$ is compact.

To a semi-norm $N \in \mathcal{M}(R)$ we can attach a *semivaluation* on R, by taking $-\log N$. Thus, equivalently, we can consider the Berkovich spectrum of R as the set of semivaluations of R with values on $[0, +\infty]$, which, following the notation of [Thu07], we will denote by $X^{\beth}$, that is:

$$X^{\beth} \doteq \left\{ v\colon R \to [0,+\infty] \;\middle|\; \begin{matrix} v(a \cdot b) = v(a) + v(b) \\ v(a+b) \geq \min\{v(a), v(b)\} \\ v(1) = 0 \;\; \& \;\; v(0) = +\infty \end{matrix} \right\}. \tag{1.1.1.2}$$

The construction is functorial, that is, given Y another affine scheme and a morphism $f\colon Y \to X$, we can associate a continuous map between $Y^{\beth}$ and $X^{\beth}$:

$$\begin{aligned} f^{\beth}\colon Y^{\beth} &\to X^{\beth} \\ v &\mapsto f^{\beth}(v)\colon a \mapsto v\left(f(a)\right), \end{aligned}$$

compatible with compositions.

The Berkovich spectrum comes with a natural class of continuous functions. Given $a \in R$, we associate $|a|\colon \mathcal{M}(R) \to \mathbb{R}_+$ by the formula $N \mapsto N(a)$, or equivalently:

$$\begin{aligned} \log|a|\colon X^{\beth} &\to [-\infty, 0] \\ v &\mapsto -v(a). \end{aligned}$$

The notation is so that $\exp(\log|a|) = |a|$.

There is an equivalent formulation of Berkovich spectrum of a trivially normed ring, namely the *tropical spectrum* of the semi-ring of its ideals of finite type.

Now we study such an object.

1.1.2 Tropical spectrum

For more details on this section see Appendix A.1.

Definition 1.1.2. *Let $(S, +, \cdot)$ be a semi-ring, and consider the semi-ring of the extended real line $(]-\infty, +\infty], \min, +)$, we define the* tropical spectrum of S *as the topological space given by the set of* tropical characters*, i.e. the semi-ring morphisms from S to $]-\infty, +\infty]$:*

$$\operatorname{TropSpec} S \doteq \left\{ \chi\colon S \to \,]-\infty, +\infty] \;\middle|\; \begin{matrix} \chi(a \cdot b) = \chi(a) + \chi(b) \\ \chi(0) = +\infty \\ \chi(a+b) = \min\{\chi(a), \chi(b)\} \end{matrix} \right\}, \tag{1.1.2.1}$$

endowed with the pointwise convergence topology. Moreover, if S has a multiplicative identity we ask $\chi(1) = 0$, this is equivalent χ not identically $+\infty$.

As before, with this topology $\operatorname{TropSpec} S$ is compact and Hausdorff.

The tropical spectrum comes with a natural order relation, and a natural order compatible $\mathbb{R}_{>0}$-action, the usual order of functions, and multiplication by scalar action. Moreover, the $\mathbb{R}_{>0}$ action on $\operatorname{TropSpec} S$, induces an action on $\mathrm{C}^0(\operatorname{TropSpec} S, \mathbb{R})$, for $t \in \mathbb{R}_{>0}$ we define:

$$(t \cdot \varphi)(v) = t\varphi(t^{-1}v). \tag{1.1.2.2}$$

Remark 1.1.3. *For any semi-ring S,* $\operatorname{TropSpec} S$ *is non-empty. Indeed, we define $\chi_{\mathrm{triv}} \colon S \to [0, +\infty]$, by the formula:*

$$\chi_{\mathrm{triv}}(a) = 0, \qquad \textit{for every } a \in S \setminus \{0\}.$$

It is easy to see that χ_{triv} is a semivaluation, and, moreover, it is a fixed point of the $\mathbb{R}_{>0}$-action.

Comparison with the Berkovich spectrum

Now, let R be a ring, and $\mathscr{I}(R)$ be the set of ideals of finite type of R. Together with the usual operations of sum and product $\mathscr{I}(R)$ can be seen as a semi-ring. Moreover, as an easy conseuqence of the algebraic structure of $\mathscr{I}(R)$, its tropical characters are all positive, that is:

$$\operatorname{TropSpec} \mathscr{I}(R) = \left\{ \chi \colon \mathscr{I}(R) \to [0, +\infty] \;\middle|\; \begin{array}{c} \chi(I \cdot J) = \chi(I) + \chi(J) \\ \chi(I + J) = \min\{\chi(I), \chi(J)\} \\ \chi(R) = 0 \;\;\&\;\; \chi(0) = +\infty \end{array} \right\},$$

see Appendix A.1 for more details.

There is a natural continuous map:

$$\operatorname{TropSpec} \mathscr{I}(R) \to (\operatorname{Spec} R)^{\beth},$$

that assigns to each tropical character $\chi \in \operatorname{TropSpec} \mathscr{I}(R)$ the semivaluation:

$$R \ni a \mapsto \chi(a \cdot R).$$

Proposition 1.1.4. *The natural map,* $\operatorname{TropSpec} \mathscr{I}(R) \to (\operatorname{Spec} R)^{\beth}$*, is a homeomorphism.*

Proof. Since $\operatorname{TropSpec} \mathscr{I}(R)$ is compact it is enough to check that the map is bijective.

The desired inverse function, $(\operatorname{Spec} R)^{\beth} \to \operatorname{TropSpec} \mathscr{I}(R)$, is the one that assigns to $v \in (\operatorname{Spec} R)^{\beth}$ the character:

$$\mathscr{I}(R) \ni I \mapsto \min_{f \in I} v(f),$$

where the minimum is achieved on any (finite) set of generators. □

With this "tropical" characterization of the Berkovich spectrum of a ring, we will extend this construction to locally ringed spaces.

1.1.3 Semivaluations on locally ringed spaces

Let $(X, \mathcal{O}_X)$ be a locally ringed topological space, and denote by $\mathscr{I}_X$ the set of $\mathcal{O}_X$-ideals locally of finite type. The set $\mathscr{I}_X$ has a semi-ring structure given by the usual addition and multiplication of ideal sheaves, and we define:

Definition 1.1.5. *Let X be a locally ringed space, we define the space of* semivaluations on X *as the tropical spectrum of* $\mathscr{I}_X$[1]*:*

$$X^{\beth} \doteq \operatorname{TropSpec} \mathscr{I}_X = \left\{ v\colon \mathscr{I}_X \to [0, +\infty] \;\middle|\; \begin{array}{c} v(I \cdot J) = v(I) + v(J) \\ v(I + J) = \min\{v(I), v(J)\} \\ v(\mathcal{O}_X) = 0 \;\;\&\;\; v(0_X) = +\infty \end{array} \right\} \tag{1.1.3.1}$$

Some geometrically relevant examples are as follows:

Example 1.1.6. *1. If X is a scheme locally of finite type over a field k, equiped with the trivial absolute value, $X^{\beth}$ is a subset of its Berkovich analytification, X^{an}, the set of semivaluations centered on X. This construction goes back to [Ber90] and [Thu07]. Whenever the scheme is proper, by the valuative criterion of properness $X^{\beth} = X^{\mathrm{an}}$.*

2. If X is a complex analytic space, $X^{\beth}$ is an analogue of the Berkovich analytification of algebraic varieties over $\mathbb{C}$. The study of $X^{\beth}$ will be done on Section 1.2, and will be the central object of study of the present paper.

3. If X is a Berkovich space we can also associate to it an "analytified" $X^{\beth}$.

Here again the construction is functorial: a morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, induces a mapping:

$$f^*\colon \mathscr{I}_Y \to \mathscr{I}_X$$

which in turn induces a continuous map:

$$f^{\beth}\colon X^{\beth} \to Y^{\beth}.$$

Remark 1.1.7. *If X is a proper algebraic variety over $\mathbb{C}$, the GAGA theorems –for the Berkovich and complex analytifications– allow us to compare $(X^{\mathrm{an}})^{\beth}$, $(X_{\mathrm{hol}})^{\beth}$ and $X^{\beth}$, where X_{hol} denotes the usual complex analytification.*

Indeed, the theorems provide us with morphisms of ringed spaces between X_{hol}, X and X^{an}, which induce a 1to1 correspondence on the set of coherent ideal sheaves. Therefore we have canonical homeomorphisms:

$$(X^{\mathrm{an}})^{\beth} \simeq X^{\beth} \simeq (X_{\mathrm{hol}})^{\beth}.$$

Moreover, as explained before, for a proper $\mathbb{C}$-scheme X^{an} coincides with $X^{\beth}$.

[1] Again by Appendix A.1 we have that all the tropical characters are positive.

Properties of the functor $\beth$

Let X be a locally ringed space, and $I \subseteq \mathcal{O}_X$ be an ideal of locally finite type. Consider $Y \doteq \operatorname{supp}(\mathcal{O}_X/I) \subseteq X$, together with the sheaf $\mathcal{O}_Y \doteq (\mathcal{O}_X/I)|_Y$ We thus have that the inclusion

$$i\colon (Y, \mathcal{O}_Y) \hookrightarrow (X, \mathcal{O}_X)$$

is a ringed space morphism, and moreover we have the following lemma.

Proposition 1.1.8. *Let $Y \subseteq X$ as above, then*

$$i^{\beth}\colon Y^{\beth} \to X^{\beth}$$

is an embedding.

Proof. Since $Y^{\beth}$ is compact it is enough to prove that $i^{\beth}$ is injective.

By definition, the morphism

$$i^*\colon \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Y$$

is surjective.

Therefore, if $v, u \in Y^{\beth}$, with $v \neq u$, then there exists an ideal locally of finite type $J \in \mathscr{I}_Y$ such that $v(J) \neq u(J)$. Moreover, $K \doteq (i^*)^{-1}(J) \subseteq \mathcal{O}_X$ is an ideal of locally finite type, and

$$i^{\beth}v(K) = v(J) \neq u(J) = i^{\beth}u(K).$$

□

Remark 1.1.9. *We have just seen that the $\beth$ functor preserves embeddings, however it does not preserve open mappings.*

In point of fact, if $U \subseteq X$ then $U^{\beth}$ will be a compact subset of $X^{\beth}$, even if U is open on X.

Examples of semivaluations in $X^{\beth}$

Example 1.1.10. *Let $p \in X$, we denote by X_p the affine scheme* $\operatorname{Spec} \mathcal{O}_{X,p}$.

We have a natural continuous map from the set of local semivaluations at p to the set of global semivaluations on X, that to each $v \in (X_p)^{\beth}$ assigns the valuation:

$$\mathscr{I}_X \ni I \mapsto v_p(I) \doteq v(I_p),$$

where I_p denotes the stalk of I at p.

Remark 1.1.11. *If we suppose that the local ring $\mathcal{O}_{X,p}$ is Noetherian (or at least its maximal ideal of finite type), the map*

$$v \mapsto v_p$$

is injective on the set of semivaluations centered at p, *that is the set of semivaluations such that* $v(m_p) > 0$.

Indeed, if v, v', *centered at* p, *are such that there exists an ideal* $J \subseteq \mathcal{O}_{X,p}$ *with*

$$v(J) \neq v'(J),$$

then, for every $k \in \mathbb{N}$ *consider the ideal*

$$J_k \doteq J + m_p^k \subseteq \mathcal{O}_{X,p}.$$

Since J_k *is primary at* p *it extends to a global ideal, by triviality at any other point. Hence*

$$\begin{aligned} v_p(J_k) &= v(J_k) = \min\{v(J), k \cdot v(m_p)\} \\ v'_p(J_k) &= v'(J_k) = \min\{v'(J), k \cdot v'(m_p)\}, \end{aligned}$$

for every k. *Letting* $k \to +\infty$, *it follows that*

$$\lim_{k\to\infty} v_p(J_k) = v(J), \quad \lim_{k\to\infty} v'_p(J_k) = v'(J).$$

We thus have $v_p(J_k) \neq v'_p(J_k)$ *for* $k \gg 1$, *and hence* $v_p \neq v'_p$.

When X is an analytic surface and $p \in X$ a regular point, $(X_p)^{\beth}$ was deeply studied in [FJ04].

1.1.4 PL functions

Just like in the affine case, the space $X^{\beth}$ comes with a natural class of functions, which we will call the *piecewise linear functions*. The terminology will become clear in Section 1.3.1.

Definition 1.1.12. *Let* $I \in \mathscr{I}_X$ *be an ideal locally of finite type, we have a function* $\log|I|\colon X^{\beth} \to [-\infty, 0]$ *that maps:*

$$X^{\beth} \ni v \mapsto \log|I|\,(v) \doteq -v(I).$$

Clearly, for every ideal I the function $\log|I|$ is monotonic decreasing (with respect to the natural partial order on $X^{\beth}$). Moreover:

$$\log|I \cdot J| = \log|I| + \log|J|, \quad \log|I + J| = \max\{\log|I|, \log|J|\}. \tag{1.1.4.1}$$

Definition 1.1.13. *The set of functions* $\psi\colon X^{\beth} \to \mathbb{R}$, *of the form*

$$v \mapsto \frac{1}{m} \max\{\log|I_i|(v) + k_i\} \tag{1.1.4.2}$$

for $I_1, \ldots, I_N$ ideals that are locally of finite type, integers $k_1, \ldots, k_N \in \mathbb{Z}$, and $m \in \mathbb{N}$, is denoted $\mathrm{PL}^+\left(X^\beth\right)$*, and the $\mathbb{Q}$-vector space it generates* $\mathrm{PL}(X^\beth) \subseteq \mathrm{C}^0\left(X^\beth, \mathbb{R}\right)$.

An element of $\mathrm{PL}(X^\beth)$ *will be called a* piecewise linear function*, we denote* $\mathrm{PL}_{\mathbb{R}}(X^\beth) \doteq \mathrm{PL}(X^\beth) \otimes_{\mathbb{Q}} \mathbb{R}$.

Lemma 1.1.14. $\mathrm{PL}^+(X^\beth)$ *separates points.*

Proof. If $v, w \in X^\beth$ are distinct, then there exists $I \in \mathscr{I}_X$ such that $v(I) \neq w(I)$, with no loss of generality we can assume $w(I) < v(I)$, and thus taking

$$\ell \in \left]\, w(I),\, v(I) \,\right[\bigcap \mathbb{Q},$$

and $\varphi \doteq \max\{\log|I|, -\ell\} \in \mathrm{PL}(X^\beth)$, we have $\varphi(v) = -\ell < -w(I) = \varphi(w)$. $\square$

Proposition 1.1.15. $\mathrm{PL}(X^\beth)$ *is dense in* $\mathrm{C}^0(X^\beth, \mathbb{R})$

Proof. Since PL is a $\mathbb{Q}$-linear subspace of C^0 stable by max, containing the ($\mathbb{Q}$-) constants, and separating points, the result follows from the lattice version of the Stone-Weierstrass Theorem. $\square$

1.2 Semivaluations on a complex space X

In this section we study $X^\beth$ attached to a compact analytic variety X. From now on X will always denote such a space.

For a complete reference on the more standard algebraic setting see [BJ22], most of the results proved here, are direct analogues of results found there.

We recall the defintion of $X^\beth$:

$$X^\beth = \left\{ v \colon \mathscr{I}_X \to [0, +\infty] \;\middle|\; \begin{array}{c} v(I \cdot J) = v(I) + v(J) \\ v(I + J) = \min\{v(I), v(J)\} \\ v(\mathcal{O}_X) = 0 \;\; \& \;\; v(0_X) = +\infty \end{array} \right\},$$

where $\mathscr{I}_X$ denotes the set of ideals locally of finite type, that for a complex space coincides with the set coherent ideals by Oka's theorem.

We call an element, $v \in X^\beth$, a *semivaluation.* For $D \subseteq X$ an effective divisor, and $v \in X^\beth$ we set:

$$v(D) \doteq v\left(\mathcal{O}_X(-D)\right).$$

1.2.1 Support and center of a semivaluation

Lemma 1.2.1. *Let $v \in X^\beth$ be a semivaluation, then there exist unique coherent ideals $I_s(v), I_c(v)$ that satisfy*

$$v(I_s) = \infty, \quad \textit{and} \quad v(I_c) > 0, \tag{1.2.1.1}$$

and are maximal with this property. Moreover, $I_s(v)$ and $I_c(v)$ are prime ideals.

Proof. Let $S \subseteq \mathscr{I}_X$ be the set of ideals on which v is infinite, C the set on which v is positive, and take:

$$I_s \doteq \sum_{J \in S} J, \quad \text{and} \quad I_c \doteq \sum_{J \in C} J.$$

By the strong noertherian property, I_s and I_c are coherent, and by construction satisfy (1.2.1.1).

For primality, if J an K are ideals on X, such that $J \cdot K \subseteq I_c(v)$, then $0 < v(I_c) \leq v(J \cdot K) = v(J) + v(K) \implies$ either $0 < v(J)$ or $0 < v(K)$, and hence either $J \subseteq I_c$ or $K \subseteq I_c$, which implies that I_c is prime. We proceed similarly for I_s. $\square$

Definition 1.2.2. *Let $v \in X^{\beth}$, we denote by $S_X(v) = S(v)$, the* support *of v, the subvariety of X given by I_s. In the same fashion, we denote by $Z_X(v) = Z(v)$, the* central variety *of v, the subvariety attached to I_c.*

Remark 1.2.3. *Since $I_s \subseteq I_c$ it follows that $Z(v) \subseteq S(v)$.*

Moreover, for any $p \in X$ the global ideal m_p is maximal, and thus

$$Z_X(v) = \{p\} \iff v(m_p) > 0, \quad \textit{and} \quad S_X(v) = \{p\} \iff v(m_p) = \infty.$$

It is easy to see that the support and the center are well-behaved under inclusions.

If v is finite valued on nonzero ideals, i.e. $S_X(v) = X$, then we will say that v is *valuation* on X. We will denote the set of valuations by X^{val}. The support thus give us the following decomposition

$$X^{\beth} = \bigsqcup_{Y \subseteq X} Y^{\mathrm{val}} \tag{1.2.1.2}$$

where Y ranges over all the subvarieties of X.

We can think the set of valuations centered at $\{p\}$ in terms of the Example 1.1.10:

Example 1.2.4. *The assignment*

$$(X_p)^{\beth} \ni v \mapsto v_p \in X^{\beth}$$

induces a bijection from the set of local semivaluations centered at p to the set of global semivaluations of X centered at p.

Indeed, by Remark 1.2.3 it is clear that the mapping sends semivaluations centered at p to semivaluations centered at p.

On the other hand, if $Z_X(v) = \{p\}$, $f \in \mathcal{O}_{X,p}$, and I_f the local ideal generated by f. We denote

$$I_k \doteq I_f + m_p^k$$

and observe:

1. *I_k can be extended to a global ideal, just by triviality outside p.*

2. $I_{k+1} = I_k + m_p^{k+1} \subseteq I_k$.

We define $\nu(f)$ as the decreasing limit:

$$\nu(f) \doteq \lim_{k\to\infty} v(I_k) = \lim_{k\to\infty} \min\{v(I_{k-1}), k \cdot v(m_p)\} \in [0, +\infty].$$

It is easy to see that ν is a valuation and that $\nu_p = v$.

In the next section we will use the above example to reconstruct $X^{\beth}$ from $(X_p)^{\beth}$ for every p, whenever X a smooth analytic curve.

Remark 1.2.5. *A locally ringed space $(X, \mathcal{O}_X)$ satisfies the* strong Noetherian property *if locally every increasing chain of $\mathcal{O}_X$-ideals, locally of finite type, is locally stationary.*

This is the property needed to define the center and support of a semivaluation, since Lemma 1.2.1 holds in this case.

Proposition 1.2.6. *If $\pi\colon Y \to X$ is a bimeromorphic morphism, the map*

$$\pi^{\beth}\colon Y^{\beth} \to X^{\beth}$$

induces a bijection $Y^{\mathrm{val}} \simeq X^{\mathrm{val}}$.

Proof. We first observe that $\pi^{\beth}$ maps valuations to valuations: if $v \in Y^{\beth}$ is finite valued on the set of non-zero ideals of Y, then we need to check that, for a non-zero ideal I of X, $\pi^{-1}(I)$ is not zero. Let $U \subseteq X$ be an open set such that I is not zero, then $\pi^{-1}(U)$ is an open set bimeromorphic to U, and

$$\pi^{-1}I\,(U) = \{0\} \iff \pi(U) \subseteq Z_I,$$

where Z_I is the zero locus of I, that has strictly positive codimension on U, therefore $\pi^{-1}I \neq 0$. Thus $\pi^{\beth}(v)(I) < +\infty$, for I a non-zero ideal on X.

Let's prove the that $\pi^{\beth}$ induces the desired bijection.

First suppose that there exists an anti-effective divisor $E \subseteq Y$ that is π-ample. Then for every ideal of Y, $J \in \mathscr{I}_Y$, choosing $m_J \in \mathbb{N}$ sufficiently large, $J \cdot \mathcal{O}_Y(E)^{m_J}$ is π-globally generated, that implies

$$\pi^{-1}I_J \cdot \mathcal{O}_Y = J \cdot \mathcal{O}_Y(E)^{m_J}, \tag{1.2.1.3}$$

for some ideal of X, $I_J \in \mathscr{I}_X$. Since $\mathcal{O}_Y(E)$ is also π-globally generated we can also find $K \in \mathscr{I}_X$ such that

$$\pi^{-1}K \cdot \mathcal{O}_Y = \mathcal{O}_Y(E).$$

Hence, given $v_X \in X^{\mathrm{val}}$ the function:

$$\mathscr{I}_Y \ni J \mapsto v_X\left(I_J\right) - m_J v_X\left(K\right)^{2} \tag{1.2.1.4}$$

is a valuation on Y, which we will denote by v_Y, such that $\pi^{\sqsupset}(v_Y) = v_X$.

If, for $v, w \in Y^{\mathrm{val}}$, $\pi^{\sqsupset}(v) = \pi^{\sqsupset}(w)$, then for every J ideal on Y:

$$\begin{aligned}
v(J) &= v\left(J \cdot \mathcal{O}_Y(E)^{m_J}\right) - m_J v\left(\mathcal{O}_Y(E)\right) \\
&= v(\pi^{-1} I_J \cdot \mathcal{O}_Y) - m_J v(\pi^{-1} K \cdot \mathcal{O}_Y) \\
&= \pi^{\sqsupset}(v)(I_J) - m_J \pi^{\sqsupset}(v)(K) \\
&= \pi^{\sqsupset}(w)(I_J) - m_J \pi^{\sqsupset}(w)(K) \\
&= w(\pi^{-1} I_J \cdot \mathcal{O}_Y) - m_J w(\pi^{-1} K \cdot \mathcal{O}_Y) \\
&= w\left(J \cdot \mathcal{O}_Y(E)^{m_J}\right) - m_J w\left(\mathcal{O}_Y(E)\right) = w(J),
\end{aligned}$$

which implies that $v = w$.

If Y does not admit a divisor E as above, by Hironaka, we can find Y' a smooth complex analytic space:

$$\begin{tikzcd}
Y' \arrow[dr, "\nu"] \arrow[dd, "\mu"'] & \\
& Y \arrow[dl, "\pi"] \\
X &
\end{tikzcd}$$

such that μ is a sequence of blow-ups of smooth center, and ν is a bimeromorphic morphism.

Thus taking E_μ the exceptional divisors of μ, we have –by the Negativity Lemma of [KM98, Lemma 3.39]– that E_μ will be anti-effective, and relatively ample, giving the bijection

$$\begin{tikzcd}
(Y')^{\mathrm{val}} \arrow[dr, "\nu^{\sqsupset}"] \arrow[dd, hook, two heads, "\mu^{\sqsupset}"'] & \\
& Y^{\mathrm{val}} \arrow[dl, "\pi^{\sqsupset}"] \\
X^{\mathrm{val}} &
\end{tikzcd} \tag{1.2.1.5}$$

which implies that $\pi^{\sqsupset}|_{Y^{\mathrm{val}}}$ is surjective on X^{val}, and that $\nu^{\sqsupset}|_{(Y')^{\mathrm{val}}}$ is injective. In turn, the same argument gives the surjection

$$\nu^{\sqsupset} \colon (Y')^{\mathrm{val}} \twoheadrightarrow Y^{\mathrm{val}}.$$

[2] Observe that we can take the difference because v_X is a finite valued.

Since $\mu^\daleth$ and $\nu^\daleth$ induce bijections on the set of valuations, so does $\pi^\daleth$. $\square$

1.2.2 Integral closure of an ideal and PL functions

In this section we can be in a slightly more general setting of either complex analytic spaces or excellent schemes of equi-characteristic 0. The two examples, we have in mind are complex analytic spaces, and the scheme $\operatorname{Spec} \mathcal{O}_{\mathbb{C}^n,0}$.

For simplicity of the exposure from here on X will be of normal singularities, see [GR12] for a complete reference on normal complex spaces.

Definition 1.2.7. *Let $I \in \mathscr{I}_X$, we consider $\overline{I}$ the* integral closure of I *to be the ideal given locally by all the elements $f \in \mathcal{O}_X$ that satisfy a polynomial equation*

$$f^d = \sum_{i=0}^{d-1} a_i f^i \tag{1.2.2.1}$$

for $a_i \in I$.

It turns out that for complex analytic spaces, or excellent schemes as above, the ideal $\overline{I}$ is a coherent ideal, and hence $\overline{I} \in \mathscr{I}_X$.

This follows indeed from the following geometric description of the integral closure of an ideal, that will be useful later, given by the following results:

Proposition 1.2.8. *Let $I \in \mathscr{I}_X$, $\nu \colon Y \to X$ the normalized blow-up of X along I, and $E \subseteq Y$ the exceptional divisor. We then have that $\nu_*(\mathcal{O}_Y(-E)) = \overline{I}$.*

Proof. See [Laz17, Proposition 9.6.6] $\square$

Corollary 1.2.9. *Let $\mu \colon Y \to X$ be a projective modification of X, $D \subset Y$ an effective divisor such that*

$$\mathcal{O}_Y(-D) = I \cdot \mathcal{O}_Y \tag{1.2.2.2}$$

for some I ideal of $\mathcal{O}_X$. Then $\mu_*(\mathcal{O}_Y(-D)) = \overline{I}$.

Proof. By Equation (1.2.2.2) μ factors through the normalized blow-up of X along I, $\widetilde{\operatorname{Bl}_I X} \to X$, and the result follows from Proposition 1.2.8. $\square$

Lemma 1.2.10. *If I is a coherent ideal then the associated function satisfies*

$$\log|I| = \log|\overline{I}| \tag{1.2.2.3}$$

on X^{val}.

Proof. Let $k \in \mathbb{N}$ such that $I \cdot \overline{I^k} = \overline{I} \cdot \overline{I^k}$ [3],

$$\log|I| + \log|\overline{I^k}| = \log|I \cdot \overline{I^k}| = \log|\overline{I} \cdot \overline{I^k}| = \log|\overline{I}| + \log|\overline{I^k}|$$
$$\implies \log|I| = \log|\overline{I}|.$$

□

The converse also holds, the valuative criterion of integral closedness gives us that if $\log|I| = \log|J|$, then $\overline{I} = \overline{J}$. Later, in Section 1.2.6, we will see a more general statement that will imply it.

1.2.3 Divisorial and monomial valuations

The trivial character described before, will be denoted v_{triv} on $X^{\beth}$, and called the *trivial valuation*:

$$v_{\text{triv}}(I) = 0, \quad \text{for every non-zero ideal } I.$$

More interestingly, to each irreducible divisor F, of a normal analytic variety $Y \xrightarrow{\mu} X$ bimeromorphic to X, we can associate a valuation on Y –and hence on X by Proposition 1.2.6– given by:

$$\mathscr{I}_Y \ni I \mapsto \operatorname{ord}_F(I),$$

where ord_F is given by the following procedure: choose any point $q \in F$, consider $\operatorname{ord}_{F_q} \in (Y_q)^{\beth}$ the order of vanishing along the germ of F at q, and denote by $\operatorname{ord}_F \in Y^{\text{val}}$ the induced global valuation of Example 1.1.10:

$$\operatorname{ord}_F \doteq (\operatorname{ord}_{F_q})_q.$$

It is classical that this construction does not depend on $q \in F$. More details will be given in the discussion below on monomial valuations, see Proposition 1.2.14.

Equivalently, $\operatorname{ord}_F(I) = k$ if and only if we can find a decomposition

$$I = \mathcal{O}_Y(-kF) \cdot J$$

where $F \not\subseteq Z_J$, the zero set of J.

Definition 1.2.11. *We denote by X^{div} the set of valuations of the form:*

$$r \cdot \operatorname{ord}_F \colon \mathscr{I}_X \setminus \{0_X\} \to \mathbb{Q},$$

for a rational number $r \in \mathbb{Q}_{\geq 0}$ and F a divisor as above.

[3] See [Bou18, Remark 8.7]

We say that an element of X^{div} *is a* divisorial valuation. *In particular, the trivial valuation* v_{triv} *is a divisorial valuation.*

Remark 1.2.12. *Divisorial valuations are a birational invariant: if* $f\colon Y \to X$ *is a bimeromorphic morphism, then* $f^{\beth}$ *maps divisorial valuations to divisorial valuations, moreover the restriction*

$$f^{\beth}|_{Y^{\mathrm{div}}} \colon Y^{\mathrm{div}} \to X^{\mathrm{div}}$$

is bijective.

The study of divisorial valuations will be of central importance for the following, as they encode a lot of the geometry of X.

For an analytic curve, $X^{\beth}$ can be completely described in terms of divisorial valuations, as we can see in the next example:

Example 1.2.13. *Let* X *be a smooth analytic curve, and* $v \in X^{\beth}$ *a semivaluation on* X*. The central variety of* v*,* $Z_X(v)$*, is an irreducible subvariety of* X*, therefore either* $Z(v) = X$ *or* $Z(v) = \{p\}$ *a point on* X*. Let's study the two options:*

1. *If* $Z(v) = X$ *then* $v = v_{\mathrm{triv}}$*.*

2. *If* $Z(v) = \{p\}$*, then*

$$v = t \cdot \mathrm{ord}_p$$

   for $t \doteq v(m_p)$*.*

   Indeed, v *being centered at* p *implies, by Example 1.2.4, that* $v = w_p$ *for some local semivaluation,* w*, centered at* p*. In addition, in dimension* 1*, the ring* $\mathcal{O}_{X,p}$ *is a discrete valuation ring and thus* $w = t \cdot \mathrm{ord}_p$*. Hence we get* $v = (t \cdot \mathrm{ord}_p)_p = t \cdot \mathrm{ord}_p$*.*

Before further studying the divisorial valuations, we will study a slightly more general class of examples of points on $X^{\beth}$, the so-called *monomial valuations.*

Monomial valuations and cone complexes

Let (Y, B) be a snc pair over X, i.e. a smooth bimeromorphic model of X together with $B = \sum_{i \in I} B_i$ a reduced simple normal crossing (snc) divisor. Define, for each subset $J \subseteq I$, the intersection $B_J \doteq \bigcap_{j \in J} B_j$. The connected components of each B_J are called the *strata* of B, and we usually denote a stratum of B, by the letter Z.

We can associate to the pair (Y, B) a *cone complex*, $\hat{\Delta}(Y, B)$, given by the following rule: for each $J \subseteq I$, and each connected component, Z, of B_J, we associate $\hat{\sigma}_Z \subset \mathbb{R}^I$, a cone identified with $(\mathbb{R}_+)^J$.

We give now a procedure for assigning to each point

$$w \in \hat{\sigma}_Z \subseteq \hat{\Delta}(Y, B)$$

a valuation on X:

Take any point $p \in Z$, and let's suppose for convenience that $J = \{1, \dots, k\}$, we can construct a monomial valuation on $\mathcal{O}_{X,p}$ with respect to the germs $B_1, \dots, B_k$ at p and weights $w_1, \dots, w_k$ following [JM12]. This gives us a valuation $\mathrm{val}_p(w) \in \mathrm{Val}(\mathcal{O}_{X,p}) = (X_p)^{\mathrm{val}}$. If we are given a coordinate open set $z\colon U \to \mathbb{D}^n$ around p, such that in U the divisor B_j is given by the local equation $z_j = 0$, for $j \in J$, then the valuation $\mathrm{val}_p(w)$ is given by:

$$\mathrm{val}_p(w)(f) = \min_{c_\alpha \neq 0} \langle \alpha, w \rangle,$$

for $f \in \mathcal{O}_{X,p}$, and $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$, in a possibly smaller open set. By some standard algebraic machinery, like in [MN15, JM12], $\mathrm{val}_p(w)$ does not depend on the coordinates chosen, and only on the divisors. For an analytic proof see Appendix A.2.

Just like in Example 1.1.10, consider the valuation $(\mathrm{val}_p(w))_p \in X^{\beth}$. We will now argue that $(\mathrm{val}_p(w))_p$ does not depend on the point chosen $p \in Z$[4], and therefore for $w \in \hat{\sigma}_Z \subseteq \hat{\Delta}(Y, B)$ we denote:

$$\mathrm{val}(w) \doteq (\mathrm{val}_p(w))_p. \tag{1.2.3.1}$$

Proposition 1.2.14. *For any $w \in \hat{\sigma}_Z$ and $p \in Z$, the image in X^{val} of $\mathrm{val}_p(w) \in (X_p)^{\mathrm{val}}$ is independent of the choice of p.*

Proof. By connectedness of Z, it is enough to show that the statement holds locally near a given $p \in X$.

Let $z\colon U \to \mathbb{D}^n$ be a local coordinate chart around p such that:

- $z(p) = 0$;
- $B_J \cap U = Z \cap U$;
- $B_j \cap U = (z_j = 0)$, for every $j \in J$.

Let $q \in z^{-1}\left(\mathbb{D}(\frac{1}{3})\right) \cap Z$, and $f \in \mathcal{O}_X(U)$, we'll prove that $\mathrm{val}_p(w)(f) = \mathrm{val}_q(w)(f)$. Before going on, we introduce some notation that will be useful. Denote by $z = (\underline{z}_1, \underline{z}_2)$ in such a way that $\underline{z}_1 \in \mathbb{C}^k$, and $\underline{z}_2 \in \mathbb{C}^{n-k}$.

Using the coordinate system z to identify U and $\mathbb{D}^n$, writing $q = (\underline{q}_1, \underline{q}_2)$, we have that $\underline{q}_1 = 0$, and if

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

[4] Again, the analytic point of view of the monomial valuations explored in Appendix A.2, give us the local independence of p, but for simplicity we will give here a more elementary approach.

is the expansion of f at 0, then:

$$\begin{aligned}
f(q+z') &= \sum c_\alpha (q+z')^\alpha = \sum c_{\alpha_1,\alpha_2}\,(\underline{q}_1 + \underline{z}'_1)^{\alpha_1}(\underline{q}_2+\underline{z}'_2)^{\alpha_2} \\
&= \sum c_{\alpha_1,\alpha_2}\,\underline{z}'^{\alpha_1}_1(\underline{q}_2+\underline{z}'_2)^{\alpha_2} \\
&= \sum c_{\alpha_1,\alpha_2}\,\underline{z}'^{\alpha_1}_1\left[\sum_j \binom{\alpha_2}{j}\underline{q}_2^j\underline{z}'^{\alpha_2-j}_2\right] \\
&= \sum\left[\sum_j c_{\alpha_1,\alpha_2+j}\,\underline{q}_2^j\binom{\alpha_2+j}{j}\right]\underline{z}'^{\alpha_1}_1\,\underline{z}'^{\alpha_2}_2,
\end{aligned}$$

denoting $c'_{\alpha_1,\alpha_2} \doteq \sum_j c_{\alpha_1,\alpha_2+j}\,\underline{q}_2^j\binom{\alpha_2+j}{j}$, the expansion of f around q becomes:

$$f(q+z') = \sum c'_{\alpha_1,\alpha_2}\underline{z}'^{\alpha^1}_1\,\underline{z}'^{\alpha_2}_2,$$

hence we get: $\mathrm{val}_q(w)(f) = \min_{c'_{\alpha_1,\alpha_2}\neq 0}\langle w,\alpha_1\rangle$, and

$$\begin{aligned}
\mathrm{val}_p(w)(f) &= \min_{c_{\alpha_1,\alpha_2}\neq 0}\langle w,\alpha_1\rangle \\
&= \min\left\{\langle w,\alpha_1\rangle \;\middle|\; \forall\,\alpha_1\in\mathbb{N}^k \text{ such that } \exists\,\alpha_2\in\mathbb{N}^{n-k} \text{ with } c_{\alpha_1,\alpha_2}\neq 0\right\}.
\end{aligned} \tag{1.2.3.2}$$

Now, if $c'_{\alpha_1,\beta}\neq 0$ for some $\alpha_1\in\mathbb{N}^k$ and $\beta\in\mathbb{N}^{n-k}$, there exists $\alpha_2\in\mathbb{N}^{n-k}$ such that $c_{\alpha_1,\alpha_2}\neq 0$. By Equation (1.2.3.2) we thus get that $\mathrm{val}_p(w)(f)\le \mathrm{val}_q(w)(f)$.

As long as q is sufficiently close to p, close enough to find a coordinate chart around q that contains p with the property that the divisors $B_1,\dots,B_k$ are given by the equations $(z_1=0),\dots,(z_k=0)$, we can exchange the role of p and q, and get the equality:

$$\mathrm{val}_p(w)(f) = \mathrm{val}_q(w)(f). \tag{1.2.3.3}$$

Thus, if we let I be a coherent ideal, there exists an open set U containing p, and local generators $f_1,\dots,f_\rho\in I(U)$ such that at any point $q\in U$, the germs $(f_1)_q,\dots,(f_\rho)_q$ generate I_q. By Equation 1.2.3.3, this implies that:

$$\mathrm{val}_p(w)(I_p) = \mathrm{val}_q(w)(I_q),$$

for $q\in U$. Therefore, we have just proved that for every I coherent ideal $(\mathrm{val}_p(w))_p(I)$ is locally independent of p, getting the desired result.

□

The above construction of monomial valuations gives us an embedding:

$$\mathrm{val}\colon \hat{\Delta}(Y,B)\hookrightarrow X^{\beth}.$$

In fact, if $w \neq w'$ both lie on $\hat{\sigma}_Z$, then there exists an irreducible component of B, $Z \subseteq B_i$, such that $\mathrm{val}(w)\,(B_i) \neq \mathrm{val}(w')\,(B_i)$.

An element of the image of the above mapping is called a *monomial valuation.*

Remark 1.2.15. *A monomial valuation associated to a rational point on $\hat{\Delta}(Y, B)$ is a divisorial valuation. As in the algebraic setting, this can be seen using weighted blowups.*

Back to divisorial valuations

As stated before, divisorial valuations will be of fundamental importance for our study of the geometry of X. As an example, one is able to prove that X^{div} alone can tell homogeneous PL functions apart. To see that, one proves that the homogeneous PL functions are essentially ($\mathbb{Q}$-Cartier) b-divisors over X, and the value on divisorial valuations will completely determine the associated b-divisor. This idea will be further developed in Section 1.2.6, when we'll prove that $\mathrm{PL}(X^{\beth})$ is isomorphic to a set of $\mathbb{C}^*$-equivariant $\mathbb{Q}$-Cartier b-divisors over $X \times \mathbb{P}^1$.

Now we will focus our attention to obtain the following theorem :

Theorem 1.2.16 (Theorem C)**.** *X^{div} is dense in $X^{\beth}$.*

Since $\mathrm{PL}(X^{\beth})$ is dense in $\mathrm{C}^0(X^{\beth}, \mathbb{R})$, $\mathrm{PL}(X^{\beth})$ separates points from closed sets, and hence to prove Theorem 1.2.16, it is enough to show that $\varphi(X^{\mathrm{div}}) = \{0\} \implies \varphi = 0$, for every $\varphi \in \mathrm{PL}\left(X^{\beth}\right)$.

Therefore we can restate Theorem 1.2.16, in the following way:

Statement. *If $\varphi \in \mathrm{PL}(X^{\beth})$ is such that*

$$\varphi|_{X^{\mathrm{div}}} = 0$$

then $\varphi = 0$.

General idea of the proof of Theorem 1.2.16. In order to prove Theorem 1.2.16, it will be useful to write a PL function as the evaluation function of some ideal.

Let's assume that it is the case, and $\varphi \in \mathrm{PL}$ is attached to an ideal I, i.e. $\varphi = \log|I|$, then if $\log|I|(v) = 0$ for every $v \in X^{\mathrm{div}}$, and if $\overline{I} \neq \mathcal{O}_X$, it would be enough to take an irreducible component of the exceptional divisor $F \subseteq \widetilde{\mathrm{Bl}_I X}$, and consider $\mathrm{ord}_F \in X^{\mathrm{div}}$. This would give us $0 = \log|I|(\mathrm{ord}_F) = -\mathrm{ord}_F(I) \neq 0$. $\square$

Just like in the algebraic trivially valued case of [BJ22], this lead us to consider $\mathbb{C}^*$-equivariant models of $X \times \mathbb{P}^1$, since we will be able to see $\mathrm{PL}^+\left(X^{\beth}\right)$ as the evaluation functions attached to $\mathbb{C}^*$-equivariant ideals.

1.2.4 $\mathbb{C}^*$-equivariant non-archimedean space

Recall that $\mathfrak{a}$ is a flag ideal of $X \times \mathbb{P}^1$ if it is $\mathbb{C}^*$-equivariant coherent ideal of $X \times \mathbb{P}^1$ whose support is contained in $X \times \{0\}$.

Let $\mathcal{F}$ be the set of fractional flag ideals of $X \times \mathbb{P}^1$, and consider the following subset of $(X \times \mathbb{P}^1)^{\beth}$:

$$(X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*} \doteq \left\{ v \colon \mathcal{F} \to \mathbb{R} \;\middle|\; \begin{array}{c} v(\mathfrak{a} \cdot \mathfrak{b}) = v(\mathfrak{a}) + v(\mathfrak{b}) \\ v(\mathfrak{a} + \mathfrak{b}) = \min\{v(\mathfrak{a}), v(\mathfrak{b})\} \\ v(t) = 1 \quad \& \quad v(\mathcal{O}_{X\times\mathbb{P}^1}) = 0 \end{array} \right\} \tag{1.2.4.1}$$

together with a continuous map, σ, the *Gauss extension map*

$$\sigma \colon X^{\beth} \to (X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*} \subseteq (X \times \mathbb{P}^1)^{\beth}$$

where $\sigma(v) \colon \mathcal{F} \to \mathbb{R}$, the *Gauss extension* of v, is given by

$$\sigma(v)(\mathfrak{a}) = \sigma(v)(\sum_{\lambda\in\mathbb{Z}} a_\lambda t^\lambda) \doteq \min_\lambda\{v(a_\lambda) + \lambda\} \in \mathbb{R}. \tag{1.2.4.2}$$

Remark 1.2.17. *As said before, we can see $(X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*}$ as a subset of $(X \times \mathbb{P}^1)^{\beth}$. Given $v \in (X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*}$, we can extend it to the set of all ideals $\mathscr{I}_{X\times\mathbb{P}^1}$, by setting*

$$v(I) \doteq \lim_{k\to\infty} v\left((t^k) + \sum_{\lambda\in\mathbb{C}^*} \lambda^* I\right)$$

for I an ideal in $X \times \mathbb{P}^1$.

Lemma 1.2.18. *The Gauss extension,*

$$\sigma \colon X^{\beth} \to (X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*},$$

is a bijection.

Proof. In fact, taking $r \colon (X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*} \to X^{\beth}$ to be the *restriction map*

$$r(v) \colon \mathscr{I}_X \ni I \mapsto v\left(I \cdot \mathcal{O}_{X\times\mathbb{P}^1}\right)$$

we get the inverse of σ. It is clear that, as defined, $r(v)$ is a semivaluation on X. So we are left to checking that for $\sigma(r(v)) = v \in (X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*}$, and $r(\sigma(v)) = v \in X^{\beth}$, which follows from:

$$\begin{aligned} \sigma(r(v))(\mathfrak{a}) &= \min\left\{r(v)(\mathfrak{a}_\lambda) + \lambda\right\} \\ &= \min\left\{v\left(\mathfrak{a}_\lambda \cdot \mathcal{O}_{X\times\mathbb{P}^1}\right) + \lambda\right\} \end{aligned}$$

$$= \min \left\{ v\left(\mathfrak{a}_\lambda \cdot \mathcal{O}_{X\times\mathbb{P}^1} \cdot (t^\lambda)\right) \right\}$$
$$= v\left(\sum_\lambda \mathfrak{a}_\lambda \cdot \mathcal{O}_{X\times\mathbb{P}^1} \cdot (t^\lambda)\right) = v(\mathfrak{a})$$

and

$$\begin{aligned} r(\sigma(v))(I) &= \sigma(v)(I \cdot \mathcal{O}_{X\times\mathbb{P}^1}) \\ &= \sigma(v)\left(\sum_{\lambda\geq 0} I \cdot t^\lambda\right) = \min\left\{v(I) + \lambda\right\} = v(I). \end{aligned}$$

□

For simplicity sometimes we identify $X^\beth$ and $(X\times\mathbb{P}^1)^\beth_{\mathbb{C}^*}$.

The set of PL functions has a nicer description in $(X\times\mathbb{P}^1)^\beth_{\mathbb{C}^*}$. If we denote

$$\varphi_\mathfrak{a} \doteq \log|\mathfrak{a}| \circ \sigma$$

we get the following result:

Proposition 1.2.19. *The set* $\{\varphi_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{F}\}$ $\mathbb{Q}$*-generates* $\mathrm{PL}(X^\beth)$, *the space of PL functions of* $X^\beth$*. Moreover*

$$\mathrm{PL}^+(X^\beth) = \left\{ \frac{1}{m}\varphi_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{F}, m \in \mathbb{Z} \right\}.$$

Proof. The proof follows form the description of flag ideals given in the equation (1.0.0.1). □

As we remarked at the end of Section 1.2.3, this is a step in the right direction in order to prove Theorem 1.2.16. But then two problems rise up:

1. The argument at the end of section 1.2.3 gives us a $\mathbb{C}^*$-equivariant divisorial valuation $v_E \in (X\times\mathbb{P}^1)^\beth_{\mathbb{C}^*}$, but *a priori* we don't know if v_E comes from a divisorial valuation over X, i.e. the restriction of v_E, $r(v_E)$, lies in X^{div}. Hence, we don't get a contradiction.

2. Even though the set PL^+ generates the PL functions, it is not enough to check that Theorem 1.2.16 is true for a PL^+ function. What we need to check is that if $\varphi_1(\mathrm{ord}_F) = \varphi_2(\mathrm{ord}_F)$ for every $\mathrm{ord}_F \in X^{\mathrm{div}}$ then $\varphi_1 = \varphi_2$.

Section 1.2.5 will deal with the first problem, and Section 1.2.6 with the second one.

1.2.5 $\mathbb{C}^*$-equivariant divisorial valuations

Definition 1.2.20 (Test configuration)**.** *We define a* test configuration for X *as the data of*

- *a normal compact Kähler space* $\mathcal{X}$*;*
- *a* $\mathbb{C}^*$*-action on* $\mathcal{X}$*;*
- *a* $\mathbb{C}^*$*-equivariant flat morphism* $\pi\colon \mathcal{X} \to \mathbb{P}^1$*;*
- *a* $\mathbb{C}^*$*-equivariant biholomorphism*

$$\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\}),$$

where $\mathcal{X}_0 \doteq \pi^{-1}(0)$*.*

Moreover, if $\mathcal{X}$*, and* $\mathcal{X}'$ *are test configuration,* $\mathcal{X}$ dominates $\mathcal{X}'$ *if the bimeromorphic map*

$$\mathcal{X} \dashrightarrow X \times \mathbb{P}^1 \dashrightarrow \mathcal{X}'$$

extends to a $\mathbb{C}^*$*-equivariant holomorphic map, which we say is a* morphism of test configurations*. When* $\mathcal{X}$ *dominates* $X \times \mathbb{P}^1$*, we say that* $\mathcal{X}$ *is* dominating.

Remark 1.2.21. *Test configurations of a given normal analytic variety* X *define a directed system. By the equivariant version of Hironaka's theorem, the set of test configurations that are projective over* $X \times \mathbb{P}^1$*, and of snc central fiber, are cofinal in all test configurations.*

Unless otherwise stated, we will consider always such test configurations.

An important class of examples of test configuration are given by flag ideals:

Example 1.2.22. *Let* $\mathfrak{a}$ *be a flag ideal on* $X \times \mathbb{P}^1$*, then*

$$\mathcal{X} \doteq \widetilde{\mathrm{Bl}_{\mathfrak{a}}(X \times \mathbb{P}^1)}$$

is a test configuration of X *that dominates* $X \times \mathbb{P}^1$*.*

For more examples see [DR17, Example 2.11].

Definition 1.2.23 ($\mathbb{C}^*$-invariant divisorial valuations)**.** *Let* $\mathcal{X}$ *be a test configuration of* X *that dominates* $X \times \mathbb{P}^1$*, and* $\mathcal{X}_0 = \sum b_E E$ *its decomposition into irreducible components. We can associate to* E *an element,* v_E*, of* $(X \times \mathbb{P}^1)^{\mathrm{div}} \cap (X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*}$ *by the formula:*

$$v_E(\mathfrak{a}) \doteq \frac{1}{b_E} \operatorname{ord}_E(\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}}). \tag{1.2.5.1}$$

We denote the set of such valuations, for all test configurations $\mathcal{X}$*, and all irreducible* $E \subseteq \mathcal{X}_0$ *irreducible components of the central fibers, by* $(X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*} \subseteq (X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*}$*.*

Now, we focus our attention to $(X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$, and prove the following theorem:

Theorem 1.2.24. *Let* $r\colon (X \times \mathbb{P}^1)^{\beth} \to X^{\beth}$ *be the restriction map of Lemma 1.2.18, we then have*

$$r((X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}) = X^{\mathrm{div}}.$$

This is known in the algebraic case, see [BHJ17, Theorem 4.6]. In that context, the proof relies on some valuative machinery that we don't have at our disposal here. More specifically, when X is a proper algebraic variety over $\mathbb{C}$, $X^{\beth}$ corresponds to the Berkovich analytification, whose points are (semi)valuations on the field of functions of (a subvariety of) X. Hence, we can associate to it invariants such as the rational rank, and the transcendental degree, that characterize the divisorial valuations completely.

To prove Theorem 1.2.24 in our transcendental setting we will reduce to the algebraic setting. In order to do that we use Proposition 1.2.29, which states that the order of vanishing along a smooth divisor can be computed locally. Hence it will be enough to consider "germs" of manifolds and divisors, that is to consider at any $p \in X$, the scheme $X_p = \operatorname{Spec} \mathcal{O}_{X,p}$, and do local computations.

A useful tool in the following will be the following 'GAGA'/base change theorem.

Theorem 1.2.25. *Given X an analytic variety, and $p \in X$ a point. There is an equivalence between the category of projective $\mathcal{O}_{X,p}$–schemes, and that of analytic spaces which are projective over the germ of X at p.*

Sketch of correspondence. Let's first build the functor on the objects.

Given a projective morphism $Y \xrightarrow{\pi} U$, $U \subseteq X$ an open set containing p, there exists an embedding $Y \overset{j}{\hookrightarrow} U \times \mathbb{P}^N_{\mathbb{C}}$ such that

$$\begin{tikzcd} Y \arrow[rr, hook, "j"] \arrow[dr, "\pi"'] & & U \times \mathbb{P}^N_{\mathbb{C}} \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes. This means that we can find a finite number of homogeneous polynomials $f_1, \dots, f_k \in \mathcal{O}_X(V)[t_1, \dots, t_{N+1}]$ that cut-out $Y|_V \doteq \pi^{-1}(V)$, for some V open neighborhood of p. Taking the germ of the coefficient of f_i at p we get $f_1, \dots, f_k \in \mathcal{O}_{X,p}[t_1, \dots, t_{N+1}]$, which defines a subvariety Y_p of $\mathbb{P}^N_{\mathcal{O}_{X,p}} \cong \operatorname{Spec} \mathcal{O}_{X,p} \times_{\operatorname{Spec} \mathbb{C}} \mathbb{P}^N_{\mathbb{C}}$ and hence we get a projective morphism $\pi_p \colon Y_p \to \operatorname{Spec} \mathcal{O}_{X,p}$ given by the diagram

$$\begin{tikzcd} Y_p \arrow[rr, "j"] \arrow[dr, "\pi_p"'] & & X_p \times_{\operatorname{Spec} \mathbb{C}} \mathbb{P}^N_{\mathbb{C}} \arrow[dl, "\mathrm{pr}_1"] \\ & X_p & \end{tikzcd}$$

where j is the inclusion.

To analytify a projective morphism over $\operatorname{Spec} \mathcal{O}_{X,p}$ the strategy is the same, see [JM14]. It is clear how the correspondence of the objects induce a correspondence on the morphisms. For more details see and [Bin76].

$\square$

Apart from the usual correspondence of sheaves, one important property of this 'GAGA' theorem is the following dimension compatibility result:

Proposition 1.2.26. *Let $U \subseteq X$ be an open set, and let $p \in U \subseteq X$, Z a complex analytic space that is projective over U, and $q \in Z$ in the fiber of p. Then the dimension at q of the germ of Z over p is equal to the dimension of Z.*

Proof. See [Bin76, Aussage 2.8]. □

Given a projective morphism $\varphi\colon Y \to X$ we say the *localization of φ at p* is its isomorphism class on the category of the projective morphisms over the germ of X at p.

Birational Geometry intermezzo

Before proving Theorem 1.2.24, we recall some basic facts of Birational Geometry.

Remark 1.2.27. *Let $v \in X^{\mathrm{div}}$ be a divisorial valuation, and $\mu\colon X' \to X$ a bimeromorphic morphism such that $Z \doteq Z_{X'}(v) \subseteq X'$ is a (irreducible) divisor, then $\mathrm{ord}_Z = v$.*

Indeed, if $F \subseteq Y \xrightarrow{\pi} X$ is chosen such that $v = \mathrm{ord}_F$, then it is enough to choose a bimeromorphic model Y' that dominates Y and X', together with an irreducible divisor, $F' \subseteq Y'$, making the diagram,

$$\begin{tikzcd} Z \subseteq X' \arrow[d, "\mu"] & F' \subseteq Y' \arrow[l, "\nu"'] \arrow[d, "\nu'"] \\ X & F \subseteq Y, \arrow[l, "\pi"'] \end{tikzcd} \tag{1.2.5.2}$$

commute, where $F' \doteq \nu^{-1}(Z)$ is the strict transform of Z by ν. Then $\mathrm{ord}_{F'} = \mathrm{ord}_Z$, and $\mathrm{ord}_{F'} = \mathrm{ord}_F$, in particular $\mathrm{ord}_F = \mathrm{ord}_Z$.

The next Lemma is a version of Zariski's Lemma, found on [Art86, Appendix: Prime Divisors, pg. 229], that will be important for the following.

Lemma 1.2.28. *Let X be an integral scheme, and v a divisorial valuation of X, then after blowing-up a finite number of times the center of v, $c(v)$, the latter will be the generic point of a divisor.* □

For more details see [Art86].

Back at the discussion of Theorem 1.2.24

Let's recollect the discussion on Section 1.2.3, when $F \subseteq X$ is a prime smooth divisor on X, and p a point in F, then the valuation $\mathrm{ord}_F \in X^{\mathrm{div}}$ is given by the following procedure:

1. Consider the valuation $\mathrm{ord}_{F_p} \in (X_p)^{\mathrm{val}}$ given by the germ of F at p

2. Then define
$$\operatorname{ord}_F(I) \doteq \operatorname{ord}_{F_q}(I_p)$$
where I_p denotes the germ of I at p. We saw that this definition does not depend on the point $p \in F$.

More generally:

Proposition 1.2.29. *Let $G \subseteq Y \xrightarrow{\mu} X$ be a (prime smooth) divisor, and μ a projective bimeromorphic morphism, consider $p \in Z_X(\operatorname{ord}_G) = \mu(G)$. Then, localizing at p we get a (prime smooth) divisor*
$$G_p \subseteq Y_p \xrightarrow{\mu_p} X_p$$
and the associated divisorial valuation on $X_p^{\sqsupset}$ satisfies
$$\operatorname{ord}_G(I \cdot \mathcal{O}_Y) = \operatorname{ord}_{G_p}(I_p \cdot \mathcal{O}_{Y_p}) \tag{1.2.5.3}$$
for I a coherent ideal of X, and I_p the germ of I at p.

Proof. Let I be an ideal on X and $k \doteq \operatorname{ord}_G(I)$, write:
$$I \cdot \mathcal{O}_Y = \mathcal{O}_Y(-kG) \cdot J,$$
with J an ideal such that $G \nsubseteq Z_J$. Localizing at p we get:
$$I_p \cdot \mathcal{O}_{Y_p} = \mathcal{O}_{Y_p}(-kG_p) \cdot J_p.$$
By primality of G we have that G_p is prime and smooth, in particular $G_p \nsubseteq Z_{J_p}$. Getting
$$\operatorname{ord}_{G_p}(I_p \cdot \mathcal{O}_{Y_p}) = k.$$
□

In this $\mathbb{C}^*$-equivariant setting we also have an analogue statement as of Remark 1.2.12, that is of key importance for Theorem 1.2.24.

Proposition 1.2.30. *Let $f\colon Y \to X$ be a bimeromorphic morphism, then the morphism $F \doteq (f, \mathrm{id})\colon Y \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ induces a bijection:*
$$F^{\sqsupset}|_{(Y\times\mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}}\colon (Y \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*} \to (X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}.$$

By Remark 1.2.12, $F^{\sqsupset}$ is a bijection between $(Y \times \mathbb{P}^1)^{\mathrm{div}}$ and $(X \times \mathbb{P}^1)^{\mathrm{div}}$, but since the valuations on $(X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$ (or $(Y \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$) are attached to divisors corresponding to ireducible components of $\mathbb{C}^*$-equivariant degenerations of X (or Y resp.), we need to check that, for $v_E \in (Y \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$, the divisorial valuation $F^{\sqsupset}(v_E)$ can be obtained from an irreducible component of the central fiber of a test configuration of X.

Proof of Proposition 1.2.30. Let's start proving that $F^{\sqsupset}$ maps $(Y \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$ to $(X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$.

If $\mathcal{Y}$ is a test configuration for Y, dominating $Y \times \mathbb{P}^1$, and E is a prime smooth vertical divisor, then we'll show that $F^{\sqsupset}(v_E) \in (X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$. That is, that there exists a test configuration $\mathcal{X}$ for X, together with an irreducible vertical divisor D and a $\mathbb{C}^*$-equivariant birational map $\mu_T \colon \mathcal{Y} \to \mathcal{X}$:

$$\begin{tikzcd}
Y \times \mathbb{P}^1 \arrow[d, "{(f,\mathrm{id})}"'] & \mathcal{Y} \supseteq E \arrow[l] \arrow[dl, "\mu_1"'] \arrow[d, dashed, "\mu_T"] \\
X \times \mathbb{P}^1 & \mathcal{X} \supseteq D, \arrow[l]
\end{tikzcd}$$

such that D and E generate the same divisorial valuation on $X \times \mathbb{P}^1$.

To prove this we first observe that $v \doteq F^{\sqsupset}(v_E)$ is a valuation on $X \times \mathbb{P}^1$, and thus, denoting $X \times \mathbb{P}^1$ by $\mathcal{X}^1$, the central variety $Z_1 \doteq Z(v, \mathcal{X}^1)$ is well defined and a $\mathbb{C}^*$-invariant irreducible set supported on the central fiber $X \times \{0\}$, given by the zeroes of $(\mu_1)_* \mathcal{O}_{\mathcal{Y}}(-E)$. Therefore, the blow-up $\mathcal{X}^2 \doteq \mathrm{Bl}_{Z_1} \mathcal{X}^1$ is a test-configuration for X, and the central variety, Z_2, of v on $\mathcal{X}^2$ is $\mathbb{C}^*$-invariant, and supported on the central fiber.

Inductively, the blow-up $b_{k+1} \colon \mathcal{X}^{k+1} \to \mathcal{X}^k$ of $\mathcal{X}^k$ along Z_k, is a test configuration and the center, Z_{k+1}, of v in $\mathcal{X}^{k+1}$ is $\mathbb{C}^*$-invariant and supoorted on central fiber:

$$\begin{tikzcd}
E \subseteq \mathcal{Y} \arrow[d, dashed, "\mu_k"] \arrow[dr, dashed, "\mu_{k+1}"] & \\
Z_k \subseteq \mathcal{X}^k & \mathcal{X}^{k+1} \supseteq Z_{k+1} = \overline{\mu_{k+1}(E)}, \arrow[l, "b_{k+1}"]
\end{tikzcd}$$

where μ_{k+1} is the bimeromorphic map defined by μ_k and b_{k+1}.

In the algebraic case, by a Lemma of Zariski after blowing up the center of the divisorial valuation a finite number of times we get that $Z(v, \mathcal{X}^k)$ is a divisor. But in our non-algebraic context, Zariski's result does not a priori apply. The strategy of our proof will be to localize at a point $p \in Z_1 \subseteq X \times \mathbb{P}^1$, use the version of Zariski's lemma given in Lemma 1.2.28, that applies in this local case, to get that, for some $k \gg 0$ sufficiently big, $(Z_k)_p$ is a divisor. By Proposition 1.2.26 this implies that Z_k is a divisor, and thus by Remark 1.2.27 we are done.

Taking $p \in Z_1 \subseteq X \times \mathbb{P}^1$, and localizing at p we get:

$$\begin{tikzcd}
E_p \subseteq \mathcal{Y}^p \arrow[d, "(\mu_1)_p"] & & & \\
(Z_1)_p \subseteq (\mathcal{X}^1)_p & (Z_2)_p \subseteq (\mathcal{X}^2)_p \arrow[l, "(b_2)_p"'] & \cdots \arrow[l] & (Z_k)_p \subseteq (\mathcal{X}^k)_p. \arrow[l] \arrow[lll, bend left, "b_p"]
\end{tikzcd} \tag{1.2.5.4}$$

The valuation v_{E_p} is a divisorial valuation on the $\mathcal{O}_{X,p}$-scheme $(\mathcal{X}^i)_p$ whose (scheme theoretic) center is the generic point of $(Z_i)_p$. Since $(\mathcal{X}^i)_p = \mathrm{Bl}_{(Z_{i-1})_p}(\mathcal{X}^{i-1})_p$, by Lemma 1.2.28 after a finite number of steps $(Z_k)_p$ becomes a divisor. By irreducibility of Z_k and Proposition 1.2.26, Z_k is a –global– divisor of $\mathcal{X}^k$.

The map $F^{\sqsupset}|_{(Y\times\mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}}$ is injective by Proposition 1.2.6, and it is easy to see that it is surjective. □

On to the proof of Theorem 1.2.24:

Proof of Theorem 1.2.24. Let's start proving that $X^{\mathrm{div}} \subseteq r((X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*})$, that is for every $F \subseteq Y \xrightarrow{\mu} X$, irreducible smooth divisor on a bimeromorphic model of X,

$$\mathrm{ord}_F \in r((X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}).$$

Let $\mathcal{Y}$ be the deformation to the normal cone of $F \subseteq Y$, that is the blow-up of $F \times \{0\}$ in $Y \times \mathbb{P}^1$, with exceptional divisor $E \subseteq \mathcal{Y} \xrightarrow{\mu} Y \times \mathbb{P}^1$, the irreducible divisor corresponding to the blow-up of Y along F, cf. [Ful98, Chapter 5]. Localizing at $p \in F$:

$$F_p \subseteq Y_p \quad \& \quad E_p \subseteq \mathcal{Y}_p = \mathrm{Bl}_{F_p \times \{0\}}(Y_p \times_{\mathrm{Spec}\,\mathbb{C}} \mathbb{P}^1).$$

Now, applying Proposition 1.2.29 we get that for any ideal I:

$$\mathrm{ord}_F(I) = \mathrm{ord}_{F_p}(I_p) = r(v_{E_p})(I_p) = v_{E_p}\left(I_p \cdot \mathcal{O}_{\mathcal{Y}_p}\right) = v_E(I \cdot \mathcal{O}_{\mathcal{Y}}), \tag{1.2.5.5}$$

where the second equality is given by [BHJ17, Theorem 4.8][5], and thus

$$\mathrm{ord}_F = r(v_E) \in r(\mathcal{Y}_{\mathbb{C}^*}).$$

By Proposition 1.2.30 we have stablished that $X^{\mathrm{div}} \subseteq r((X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*})$.

To prove that $X^{\mathrm{div}} \supseteq r((X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*})$, the strategy will be the same.

[5]The set up there is for a scheme of finite type over a field of characteristic zero, but the same arguments apply.

Let $\mathcal{X}$ be a test configuration that dominates $X \times \mathbb{P}^1$, and $E \subseteq \mathcal{X}_0$ an irreducible component. We then have:

$$\begin{tikzcd}
E \subseteq \mathcal{X} \arrow[d] & E_1 \subseteq \mathcal{Y}^1 \arrow[l, "f_1"'] \arrow[dd, "\rho_1"] \\
X \times \mathbb{P}^1 \arrow[d] & \\
Z_0 \subseteq X & Y_1 \doteq \mathrm{Bl}_{Z_0} X \arrow[l, "b_1"]
\end{tikzcd}$$

where $Z_0 \doteq Z(r(v_E), X)$ is the central variety of $r(v_E)$ on X, and $\mathcal{Y}^1$ is dominates Y_1 and $\mathcal{X}$, and f_1 is a bimeromorphism such that the strict transform $E_1 = f_1^{-1}E$ is an irreducible smooth divisor. We can define then $Z_1 \doteq Z(r(v_E), Y_1) = \overline{\rho_1(E_1)} \subseteq Y_1$.

Localizing at a point $p \in Z_0$ we get:

$$\begin{tikzcd}
E_p \subseteq \mathcal{X}_p \arrow[d] & (E_1)_p \subseteq (\mathcal{Y}^1)_p \arrow[l, "f_{1,p}"'] \arrow[dd, "\rho_{1,p}"] \\
(X \times \mathbb{P}^1)_p \arrow[d] & \\
(Z_0)_p \subseteq X_p & (Z_1)_p \subseteq (Y_1)_p \arrow[l, "b_{1,p}"]
\end{tikzcd}$$

and, as before, E_p defines a divisorial valuation on X_p –again by the same arguments as in [BHJ17, Theorem 4.8]– and its schematic center in X_p is the generic point of $(Z_0)_p$, similarly the center of v_{E_p} on $(Y_1)_p$ is the generic point $(Z_1)_p$. Repeating the construction we get:

$$\begin{tikzcd}
E \subseteq \mathcal{X} \arrow[d] & E_1 \subseteq \mathcal{Y}^1 \arrow[l, "f_1"'] \arrow[dd, "\rho_1"] & \cdots \arrow[l] & E_\ell \subseteq \mathcal{Y}^\ell \arrow[l] \arrow[dd] & \cdots \arrow[l] \\
X \times \mathbb{P}^1 \arrow[d] & & & & \\
Z_0 \subseteq X & Z_1 \subseteq Y_1 \arrow[l, "b_1"] & \cdot s \arrow[l] & Z_\ell \subseteq \mathrm{Bl}_{Z_{\ell-1}} Y_{\ell-1} \arrow[l] & \cdots, \arrow[l]
\end{tikzcd} \tag{1.2.5.6}$$

and

$$\begin{tikzcd}
E_p \subseteq \mathcal{X}_p \arrow[d] & (E_1)_p \subseteq (\mathcal{Y}^1)_p \arrow[l, "f_{1,p}"'] \arrow[dd, "\rho_{1,p}"] & \cdots \arrow[l] & (E_\ell)_p \subseteq (\mathcal{Y}^\ell)_p \arrow[l] \arrow[dd] & \cdots \arrow[l] \\
(X \times \mathbb{P}^1)_p \arrow[d] & & & & \\
(Z_0)_p \subseteq X_p & (Z_1)_p \subseteq (Y_1)_p \arrow[l, "b_{1,p}"] & \cdots \arrow[l] & (Z_\ell)_p \subseteq (Y_\ell)_p. \arrow[l] & \cdots . \arrow[l]
\end{tikzcd} \tag{1.2.5.7}$$

Again by Lemma 1.2.28, it exists $k \in \mathbb{N}$ such that $(Z_k)_p \subseteq (Y_k)_p = \mathrm{Bl}_{(Z_{k-1})_p}(Y_{k-1})_p$ is a prime divisor. Proposition 1.2.26, together with the irreducibility of Z_k, gives us that $Z_k \subseteq Y_k$ is a prime divisor. Moreover, by construction, Z_k is the central variety of v_E on Y_k, and therefore by Remark 1.2.27:

$$v_{E_p} = \mathrm{ord}_{(Z_k)_p} .$$

Thus

$$v_E(I \cdot \mathcal{O}_{\mathcal{X}}) = v_{E_p}(I_p \cdot \mathcal{O}_{\mathcal{X}_p}) = \mathrm{ord}_{(Z_k)_p}(I_p \cdot \mathcal{O}_{(Y_k)_p}) = \mathrm{ord}_{Z_k}(I \cdot \mathcal{O}_{Y_k}), \tag{1.2.5.8}$$

getting

$$r(v_E) = \mathrm{ord}_{Z_k} \in X^{\mathrm{div}}.$$

This shows that $r((X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}) \subseteq X^{\mathrm{div}}$, completing the proof.

□

1.2.6 PL functions as divisors

Definition 1.2.31. *Let $\mathcal{X}$ be a test configuration of X, we denote by* $\mathrm{VCar}(\mathcal{X})$ *the finite dimensional $\mathbb{Q}$-vector space given by the $\mathbb{C}^*$-invariant $\mathbb{Q}$-Cartier divisors supported on $\mathcal{X}_0$. An element of* $\mathrm{VCar}(\mathcal{X})$ *is called a* vertical $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$, *or simply,* vertical divisor of $\mathcal{X}$.

Now, since morphisms of test configurations induce linear mappings between the vertical $\mathbb{Q}$-Cartier divisors, we define *vertical Cartier b-divisors.*

Definition 1.2.32. *Consider the direct system given by $\langle \mathrm{VCar}(\mathcal{X}), \mu^*_{\mathcal{X},\mathcal{X}'} \rangle$, we then say that the elements of the directed limit:*

$$\varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X}),$$

are called vertical Cartier b-divisors.

For each test configuration $\mathcal{X}$ we can define a natural map:

$$\mathrm{PL}^+(X^{\beth}) \to \mathrm{VCar}(\mathcal{X}),$$

that assigns to each $\varphi \in \mathrm{PL}^+$ the vertical divisor given by:

$$\sum_{E \overset{\text{irred}}{\subseteq} \mathcal{X}_0} b_E\, \varphi(v_E) E, \tag{1.2.6.1}$$

where $\mathcal{X}_0 = \sum b_E\, E$ is the irrreducible decomposition. We show now that these maps glue well to define an universal one to the direct limit $\varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X})$.

Lemma 1.2.33. *The collection of the above mentioned maps induces the mapping:*

$$\mathrm{PL}^+(X^{\beth}) \to \varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X}).$$

Proof. Let $\varphi \in \mathrm{PL}^+(X^{\beth})$, we show that for a cofinal set of test configurations, and $\mu \colon \mathcal{X}' \to \mathcal{X}$ a morphism of such test configurations:

$$\mu^* \left(\sum_{E \overset{\text{irred}}{\subseteq} \mathcal{X}_0} b_E\, \varphi(v_E) E \right) = \sum_{E' \overset{\text{irred}}{\subseteq} \mathcal{X}'_0} b_{E'}\, \varphi(v_{E'}) E'.$$

After scaling, we may assume that $\varphi = \varphi_{\mathfrak{a}}$ for some flag ideal $\mathfrak{a}$. The set of test configurations that dominate the normalized blow-up of $X \times \mathbb{P}^1$ along $\mathfrak{a}$ is cofinal[6], and thus we will suppose that $\mathcal{X}$ is in this set. Let G be the effective divisor induced by $\mathfrak{a}$ on $\mathcal{X}$:

$$\mathcal{O}_{\mathcal{X}}(-G) = \mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}}. \tag{1.2.6.2}$$

Therefore,

$$\sum_{E \subseteq \mathcal{X}_0} b_E\, \varphi(v_E) E = -G.$$

Writing $\mathcal{X}'_0 = \sum_{E' \subseteq \mathcal{X}'_0} b_{E'}\, E'$ as the irreducible decomposition, it follows that:

$$\begin{aligned}
\mu^* \left(\sum_{E \subseteq \mathcal{X}_0} b_E\, \varphi(v_E) E \right) = \mu^*(-G) &= \sum_{E' \subseteq \mathcal{X}'_0} \mathrm{ord}_{E'}(\mathcal{O}(\mu^* G)) E' \\
&= \sum_{E' \subseteq \mathcal{X}'_0} - \mathrm{ord}_{E'}(\mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}'}) E' \\
&= \sum_{E' \subseteq \mathcal{X}'_0} -b_{E'} v_{E'}(\mathfrak{a}) E'
\end{aligned}$$

[6] Observe that more than that every test configuration that dominates an element of this cofinal set is also in the cofinal set.

$$= \sum_{E' \subseteq \mathcal{X}'_0} b_{E'}\, \varphi(v_{E'}) E',$$

concluding the proof. □

Theorem 1.2.34. *The above map induces an isomorphism*

$$\mathrm{PL}(X^{\beth}) \simeq \varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X}).$$

Proof. Since the map is additive, we have a unique linear extension:

$$\mathrm{PL}(X^{\beth}) \to \varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X}).$$

We now construct its inverse.

Let $\mathcal{X}$ be a test configuration of X. By Remark 1.2.21 we can suppose that there exists a morphism of test configurations $\mu\colon \mathcal{X} \to X \times \mathbb{P}^1$ that is projective, that is we can embed $\mathcal{X}$:

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, hook] \arrow[dr, "\mu"'] & & (X \times \mathbb{P}^1) \times \mathbb{P}^\ell \arrow[dl, "p_1"'] \arrow[d, "p_2"] \\
& X \times \mathbb{P}^1 & \mathbb{P}^\ell
\end{tikzcd}$$

making the diagram commute, where p_i is the i-th coordinate projection. Then the set:

$$\mathrm{AVCar}(\mathcal{X}) \doteq \{D \in \mathrm{VCar}(\mathcal{X}) \mid -D \text{ is } \mu\text{-very ample}\}$$

is non-empty, since $p_2^*(mL)|_{\mathcal{X}} \in \mathrm{AVCar}(\mathcal{X})$, for L an ample line bundle on $\mathbb{P}^\ell$ and for $m \gg 0$. Moreover, $\mathrm{AVCar}(\mathcal{X})$ is a semigroup that $\mathbb{Q}$-spans $\mathrm{VCar}(\mathcal{X})$.

For each $D \in \mathrm{AVCar}(\mathcal{X})$, $-D$ is μ-globally generated, which implies that there exists $\mathfrak{b}$, a fractional ideal sheaf of $\mathcal{O}_{X\times\mathbb{P}^1}$, such that

$$\mathcal{O}_{\mathcal{X}}(-D) = \mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}. \tag{1.2.6.3}$$

Since D is a vertical divisor implies that we can suppose $\mathfrak{b} \in \mathcal{F}$, and hence we define $\varphi_D \doteq -\varphi_{\mathfrak{b}}$. To see that φ_D is well defined, is enough to observe that if

$$\mathcal{O}_{\mathcal{X}}(-D) = \mathfrak{b}' \cdot \mathcal{O}_{\mathcal{X}} \tag{1.2.6.4}$$

then by Corollary 1.2.9 $\overline{\mathfrak{b}} = \overline{\mathfrak{b}'}$, which implies

$$\varphi_{\mathfrak{b}} = \varphi_{\overline{\mathfrak{b}}} = \varphi_{\overline{\mathfrak{b}'}} = \varphi_{\mathfrak{b}'}$$

by Lemma 1.2.10[7].

Let us now check that $\varphi\colon \operatorname{AVCar}(\mathcal{X}) \to \operatorname{PL}(X^{\beth})$ is additive. Pick $D, D' \in \operatorname{AVCar}(\mathcal{X})$ and write

$$\mathcal{O}_{\mathcal{X}}(-D) = \mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}} \quad \& \quad \mathcal{O}_{\mathcal{X}}(-D') = \mathfrak{c} \cdot \mathcal{O}_{\mathcal{X}},$$

which implies

$$\mathcal{O}_{\mathcal{X}}(-D - D') = \mathcal{O}_{\mathcal{X}}(-D) \cdot \mathcal{O}_{\mathcal{X}}(-D') = (\mathfrak{b} \cdot \mathfrak{c}) \cdot \mathcal{O}_{\mathcal{X}}$$

and thus

$$\varphi_{D+D'} = -\varphi_{\mathfrak{b}\cdot\mathfrak{c}} = -\varphi_{\mathfrak{b}} - \varphi_{\mathfrak{c}} = \varphi_D + \varphi_{D'}.$$

Again we can extend φ uniquely to $\operatorname{VCar}(\mathcal{X})$ by linearity.

Observe that this definition does not depend on $\mathcal{X}$, in the sense that if $\mu\colon \mathcal{X}' \to \mathcal{X}$ is a morphism of test configurations, and $D' \doteq \mu^* D \subseteq \mathcal{X}'$, then we have

$$\mathcal{O}_{\mathcal{X}'}(-D') = (\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}) \cdot \mathcal{O}_{\mathcal{X}'} = \mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}'}. \tag{1.2.6.5}$$

Hence $\varphi_{D'} = \varphi_D$.

This defines [8] a linear map

$$\varinjlim_{\mathcal{X}} \operatorname{VCar}(\mathcal{X}) \to \operatorname{PL}(X^{\beth})$$

which is the inverse of (1.2.6.1). □

We thus can conclude the proof of Theorem 1.2.16.

Proof of Theorem 1.2.16. If $\varphi \in \operatorname{PL}(X^{\beth})$ is such that

$$\varphi(v) = 0, \qquad \text{for every } v \in X^{\mathrm{div}},$$

then, using Theorem 1.2.24 we have that for all test configurations $\mathcal{X}'$ and all prime vertical divisors $E' \in \operatorname{VCar}(\mathcal{X}')$

$$\varphi(v_{E'}) = 0.$$

In particular, if $D \in \operatorname{VCar}(\mathcal{X})$ is a vertical divisor, such that $\varphi_D = \varphi$, writing

$$D = \sum_{E \overset{\text{irred}}{\subseteq} \mathcal{X}_0} \varphi_D(v_E) E,$$

we will have $D = 0$, and thus $\varphi = 0$.

[7] The proof of Lemma 1.2.10 applies, since for every flag ideal $\mathfrak{a}$ the function $\log|\mathfrak{a}|$ is finite valued on $(X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*}$.

[8] The set of test configurations obtained by a sequence of blow-ups is cofinal.

□

1.3 Dual complexes and log discrepancy

From this point on X will be a compact complex manifold.

1.3.1 Non-Archimedean as a limit of tropical

In the algebraic setting it is known that the Berkovich analytification corresponds to taking a limit of Tropical complexes, known as the *Dual Complexes*, associated to test configurations. See [BJ22, Appendix A] for a version in the trivially valued case. In this section we will show the analogous result in our transcendental setting.

Contruction of the dual complex

Let $\mathcal{X}$ a smooth snc test configuration for X. Let $\mathcal{X}_0 = \sum_i b_i E_i$ be the decomposition of the central fiber in its irreducible components.

Then $(\mathcal{X}, \mathcal{X}_{0,\mathrm{red}})$ is a snc reduced birational model of $\mathcal{X}_{\mathrm{triv}}$. Recall from Section 1.2.3 that we can then construct $\hat{\Delta}_{\mathcal{X}} \doteq \hat{\Delta}(\mathcal{X}, \mathcal{X}_{0,\mathrm{red}})$.

Now, we will construct a simplicial complex, $\Delta_{\mathcal{X}}$, as a sort of compact representative of $\hat{\Delta}_{\mathcal{X}}$. For each "cone" face, $\hat{\sigma}_Z \cong (\mathbb{R}_+)^J$, of $\hat{\Delta}_{\mathcal{X}}$ we will associate a "simplex" face, σ_Z, of $\Delta_{\mathcal{X}}$ given by the equation $\sum b_i\, w_i = 1$, that is:

$$\sigma_Z \doteq \left\{ w \in \hat{\sigma}_Z \cong (\mathbb{R}_+)^J \;\middle|\; \sum_{i \in J} b_i w_i = 1 \right\}.$$

Given a test configuration $\mathcal{X}$, we have a natural map:

$$p_{\mathcal{X}} \colon X^{\beth} \to \Delta_{\mathcal{X}} \tag{1.3.1.1}$$

defined by $p_{\mathcal{X}}(v) = (v(E_i)) \in \mathbb{R}_+^J$, where the latter that corresponds to the stratum Z, the smallest one that contains $Z(v, \mathcal{X})$ the central variety of v on $\mathcal{X}$.

Morphisms

Let $\mathcal{X}, \mathcal{X}'$ be test configurations of X, $\mu \colon \mathcal{X} \to \mathcal{X}'$ a test configuration morphism between them, and $\sum b_i E_i, \sum c_j E'_j$ be the decomposition in irreducible components of $\mathcal{X}_0, \mathcal{X}'_0$ respectively, then clearly:

$$\mathrm{Supp}(\mathcal{X}_0) \subseteq \mathrm{Supp}(\mu^* \mathcal{X}'_0).$$

In particular, we can write $\mu^* E'_j = \sum_i d^i_j E_i$, for $D_j = (d^1_j, \ldots, d^M_j) \in \mathbb{R}^M$, and we define the map:

$$r_{\mathcal{X},\mathcal{X}'} \colon \Delta_{\mathcal{X}} \longrightarrow \Delta_{\mathcal{X}'}$$
$$(\mathbb{R}_+)^J \cong \sigma_Z \ni w \mapsto r_{\mathcal{X},\mathcal{X}'}(w) \in \sigma_{Z'} \cong (\mathbb{R}_+)^{J'_w},$$

for $J'_w \doteq \{j \in I \mid d^i_j \neq 0 \text{ for some } i \in J\}$, given by:

$$r_{\mathcal{X},\mathcal{X}'}(w) \doteq \left(\sum d^i_j\, w_i\right)_{j \in J'_w}.$$

Since the snc test configurations form a directed poset we can take the projective limit,

$$\Delta \doteq \varprojlim_{\mathcal{X} \text{ snc}} \Delta_{\mathcal{X}},$$

and the family of maps $(p_{\mathcal{X}})_{\mathcal{X}}$ induces an injective[9] continuous map:

$$p \colon X^{\beth} \to \Delta.$$

Theorem 1.3.1. *The map $p \colon X^{\beth} \to \Delta$ is a homeomorphism.*

To get this, we'll see that, just like $X^{\beth}$, Δ has a PL structure, that will be isomorphic to the PL structure on $X^{\beth}$.

PL Functions

There is a natural class of functions defined on Δ, the ind-type set of *piecewise linear functions*. That is, the set of real valued functions that, on a complex $\Delta_{\mathcal{X}}$, are $\mathbb{Q}$-piecewise linear:

$$\mathrm{PL}(\Delta) \doteq \bigcup_{\mathcal{X}} (\pi_{\mathcal{X}})^* \, \mathrm{PL}(\Delta_{\mathcal{X}}),$$

for $\pi_{\mathcal{X}} \colon \Delta \to \Delta_{\mathcal{X}}$ the canonical projection.

After going to a higher model, we can assume that the functions are rationally affine on each face of the associated dual complex $\Delta_{\mathcal{X}}$, and hence we have:

$$\mathrm{PL}(\Delta) = \varinjlim_{\mathcal{X}} \mathrm{Aff}_{\mathbb{Q}}(\Delta_{\mathcal{X}}).$$

Now, observe that $\mathrm{VCar}(\mathcal{X})_{\mathbb{Q}} \cong \mathrm{Aff}_{\mathbb{Q}} \Delta_{\mathcal{X}}$, where the isomorphism is given by:

$$\mathrm{Aff}_{\mathbb{Q}} \Delta_{\mathcal{X}} \ni f \mapsto \sum b_i\, f(e_i)\, E_i. \tag{1.3.1.2}$$

[9] this is equivalent to $\mathrm{PL}(X^{\beth})$ separating points on $X^{\beth}$

Taking the limit we get:

$$\mathrm{PL}(\Delta) = \varinjlim_{\mathcal{X}} \mathrm{Aff}_{\mathbb{Q}}\, \Delta_{\mathcal{X}} \cong \varinjlim_{\mathcal{X}} \mathrm{VCar}(\mathcal{X}) \cong \mathrm{PL}(X^{\beth}). \tag{1.3.1.3}$$

Lemma 1.3.2. *The map*

$$p\colon X^{\beth} \to \Delta$$

is an isomorphism of PL structures[10].

Proof. We need to check that if $\eta\colon\ \mathrm{PL}(\Delta) \to \mathrm{PL}(X^{\beth})$ is the ismorphism of Equation (1.3.1.3), then, for $f \in \mathrm{PL}(\Delta)$ and $v \in X^{\beth}$:

$$\eta(f)(v) = f\left(p(v)\right). \tag{1.3.1.4}$$

By Theorem 1.2.16 it is enough to check (1.3.1.4) for $v \in X^{\mathrm{div}}$.

Now, given $f \in \mathrm{PL}(\Delta)$, and $v \in X^{\mathrm{div}}$, let $\mathcal{X}$ be a smooth test configuration such that:

- the function $f|_{\Delta_{\mathcal{X}}}$ is rationally affine, for $\Delta_{\mathcal{X}}$ the associated dual complex;
- decomposing the central fiber $\mathcal{X}_0 \doteq \sum_i b_i E_i$, we have $v = v_{E_1}$.

Then

$$f(p(v)) = f\left(v(E_1), v(E_2), \ldots, v(E_k)\right) = v(E_1) f(e_1) = f(e_1) = \eta(f)(v_{E_1})$$

where the last equality is given by (1.3.1.2) together with (1.2.6.1). □

Now, we will prove that the isomorphism (1.3.1.3) induces a homeomorphism

$$X^{\beth} \overset{p}{\cong} \Delta.$$

To do that, as mentioned before, we will use an analogue of the Gelfand transform, to show that $X^{\beth}$ and Δ can be seen as the "tropical spectra" of $\mathrm{PL}(X^{\beth})$ and $\mathrm{PL}(\Delta)$ respectively. Then, since the map

$$p\colon X^{\beth} \to \Delta$$

is an isomorphism of PL structures, p will be a homeomorphism.

Tropical Gelfand transform

Let's recall some definitions from Section 1.1.2, and from Appendix A.1.

Let $\mathcal{A}$ be a tropical algebra, that is a vector space together with a semi-ring operation, which we will denote by $\{\cdot,\cdot\}$, that makes $(\mathcal{A}, \{\cdot,\cdot\}, +)$ a semi-ring. The *tropical spectrum*

[10] See Appendix A.1.

of $\mathcal{A}$ is the topological space given by[11]

$$\operatorname{TropSpec}\mathcal{A} = \{\varphi \in \mathcal{A}^* | \, \varphi(\{f, g\}) = \max\{\varphi(f), \varphi(g)\}\}, \tag{1.3.1.5}$$

where $\mathcal{A}^*$ denotes the algebraic dual. We endow $\operatorname{TropSpec}\mathcal{A}$ with pointwise convergence topology.

Proposition 1.3.3. *Let K be a compact Hausdorff topological space, and $\mathcal{A} \subseteq \mathrm{C}^0(K, \mathbb{R})$ a dense linear subspace, containing all the constants, that is stable by* max*. Then $\mathcal{A}$ is subtroipcal algebra of $\mathrm{C}^0(K, \mathbb{R})$, and the map:*

$$\delta \colon K \to (\operatorname{TropSpec}\mathcal{A}), \quad \delta_x(f) = f(x),$$

induces the homeomorphism:

$$[\delta] \colon K \to (\operatorname{TropSpec}(\mathcal{A}) \setminus \{0\}) / \mathbb{R}_{>0}.$$

Proof. It is clear that $\mathcal{A}$ is tropical algebra with the max (or min as it also preserves minima) and sum as operations.

Moreover, the map δ is clearly continous and injective, therefore it suffices to prove that $[\delta]$ is surjective, since K is compact.

Let $\varphi \in \operatorname{TropSpec}\mathcal{A} \setminus \{0\}$, then

$$\varphi(|f|) = \varphi(\max\{f, -f\}) = \max\{\varphi(f), -\varphi(f)\} = |\varphi(f)| \tag{1.3.1.6}$$

hence $\varphi(1) > 0$, thus we can suppose that $\varphi(1) = 1$, and

$$\begin{aligned} |\varphi(f)| = \varphi(|f|) &\leq \max\{\varphi(|f|), \varphi(\|f\|_\infty)\} \\ &= \varphi\left(\max\{|f|, \|f\|_\infty\}\right) = \varphi(\|f\|_\infty) = \|f\|_\infty \cdot 1 \end{aligned}$$

Hence φ can be extended to a continuous linear functional on $\mathrm{C}^0(K)$. That is, φ is a signed measure on K. By (1.3.1.6), φ is actually a positive measure.

Let $x \in \operatorname{supp}\varphi$, we will show that $\varphi = \delta_x$. To do that we will just prove that $\ker\delta_x = \ker\varphi$, and the equality will follow since $\varphi(1) = 1 = \delta_x(1)$.

If $f \in \ker\varphi$, then, by Equation (1.3.1.6), $|f| \in \ker\varphi$. Since φ is a positive measure, and $|f| \geq 0$, we get $f = 0$ φ-almost everywhere. Therefore, since f is continuous, the restriction:

$$f|_{\operatorname{supp}\varphi} = 0.$$

In particular, $f(x) = 0$, and thus $f \in \ker\delta_x$. Since $\operatorname{codim}\ker\varphi = \operatorname{codim}\ker\delta_x$, we conclude. □

[11] cf. Lemma A.1.12

Proof of Theorem 1.3.1. The map

$$p\colon X^{\beth} \to \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$$

induces the isomorphism:

$$\eta\colon \varinjlim_{\mathcal{X}} \mathrm{Aff}_{\mathbb{Q}}\, \Delta_{\mathcal{X}} \to \mathrm{PL}(X^{\beth}),$$

and therefore we get a homeomorphism:

$$\eta^*\colon \mathrm{TropSpec}\left(\varinjlim_{\mathcal{X}} \mathrm{Aff}_{\mathbb{Q}}\, \Delta_{\mathcal{X}}\right) \to \mathrm{TropSpec}\left(\mathrm{PL}(X^{\beth})\right),$$

given by $\eta^*(\delta_x)(f) = \delta_x(\eta(f)) = \eta(f)(x) = f(p(x)) = \delta_{p(x)}(f)$ which means that the map

$$X^{\beth} \xrightarrow{\delta} \mathrm{TropSpec}\left(\mathrm{PL}(X^{\beth})\right) \xrightarrow{\eta^*} \mathrm{TropSpec}\left(\varinjlim_{\mathcal{X}} \mathrm{Aff}_{\mathbb{Q}}\, \Delta_{\mathcal{X}}\right) \xrightarrow{\delta^{-1}} \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}} \tag{1.3.1.7}$$

is given by $x \mapsto p(x)$. Hence, p is a homeomorphism. □

1.3.2 Log discrepancy on $X^{\beth}$

Log discrepancy over X

Let $F \subseteq Y \xrightarrow{\pi} X$ be an irreducible divisor, and $v = r\,\mathrm{ord}_F$ for some $r \in \mathbb{Q}_+$ We define the *log discrepancy of* v to be the quantity

$$A_X(v) \doteq r \cdot \left(1 + \mathrm{ord}_F(K_{Y/X})\right). \tag{1.3.2.1}$$

This gives us a function $A_X\colon X^{\mathrm{div}} \to \mathbb{Q}$, which will be called the *log discrepancy over* X.

If $r = 1$ we sometimes denote $A_X(F) \doteq A_X(\mathrm{ord}_F)$.

By some standard calculations, as in [Kol97, Section 3], the restriction of A_X to (the rational points of) each face of $\hat{\Delta}(Y,B) \subset X^{\beth}$ is linear, and hence we can extend it to $\hat{\Delta}(Y,B)$ by linearity.

Again by [Kol97, Section 3] if $(Y',B') \xrightarrow{\mu} (Y,B)$ is a snc reduced birational projective morphism over (Y,B), then the piecewise linear function induced by the pullback:

$$r_\mu\colon \hat{\Delta}(Y,B) \to \hat{\Delta}(Y',B')$$

satisfies the inequality

$$A_X \circ r_\mu \leq A_X. \tag{1.3.2.2}$$

Log discrepancy over $X \times \mathbb{P}^1$

The same log discrepancy defined on the previous section makes sense for $X \times \mathbb{P}^1$. We study now the relationship between A_X, and $A_{X\times\mathbb{P}^1} \circ \sigma$, where σ is the Gauss extension.

Let $F \subseteq Y \xrightarrow{\pi} X$ be a prime divisor, and ord_F the associated divisorial valuation. Then, consider the divisors:

$$\begin{array}{ccc} F \times \mathbb{P}^1 \subseteq Y \times \mathbb{P}^1 & \quad \& \quad & Y \times \{0\} \subseteq Y \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ X \times \mathbb{P}^1 & & X \times \mathbb{P}^1. \end{array}$$

A direct calculation gives us that $\sigma(\mathrm{ord}_F)$ is monomial with respect to $F \times \mathbb{P}^1$ and $Y \times \{0\}$, with associated weights $(1, 1)$. Therefore, using linearity of $A_{X\times\mathbb{P}^1}$, we get:

$$\begin{aligned} A_{X\times\mathbb{P}^1}(\sigma(\mathrm{ord}_F)) &= A_{X\times\mathbb{P}^1}(F \times \mathbb{P}^1) + A_{X\times\mathbb{P}^1}(Y \times \{0\}) \\ &= A_X(F) + 1 = A_X(\mathrm{ord}_F) + 1. \end{aligned} \tag{1.3.2.3}$$

Let $\mathcal{X}$ be a smooth test configuration. Like we did in Section 1.3.1, one can associate $\mathbb{C}^*$-equivariant monomial valuations on $(X \times \mathbb{P}^1)^{\beth}_{\mathbb{C}^*}$ to points $w \in \sigma_Z \subseteq \Delta_{\mathcal{X}}$. Again, the rational points on $\Delta_{\mathcal{X}}$ correspond to points in $(X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$. By the same the reasoning as before we get:

1. Let $w \in (\sigma_Z)_{\mathbb{Q}} \subseteq (\Delta_{\mathcal{X}})_{\mathbb{Q}}$, and $\mathrm{val}(w)$ the associated valuation satisfies:

$$A_{X\times\mathbb{P}^1}(\mathrm{val}(w)) = \sum w_i\, A_{X\times\mathbb{P}^1}(E_i).$$

2. For $\mathcal{X}' \xrightarrow{\mu} \mathcal{X}$ a morphism of test configurations, we have

$$A_{X\times\mathbb{P}^1} \circ p_{\mathcal{X}} \leq A_{X\times\mathbb{P}^1}$$

   on $(\Delta_{\mathcal{X}'})_{\mathbb{Q}}$.

Therefore, by (1) we can extend by linearity $A_{X\times\mathbb{P}^1}$ to $\Delta_{\mathcal{X}}$, and, by (2), define the limit:

$$A_{X\times\mathbb{P}^1} \colon X^{\beth} \to \mathbb{R} \cup \{+\infty\} \tag{1.3.2.4}$$

as the sup $A_{X\times\mathbb{P}^1}(v) \doteq \sup_{\mathcal{X}} A\,(p_{\mathcal{X}}(v))$.

The function on (1.3.2.4) will be called the *log discrepancy*, and from here on will be denoted by $A \colon X^{\beth} \to \mathbb{R} \cup \{+\infty\}$.

Remark 1.3.4. *It is clear to see that from the definition of the log discrepancy, if $\mathcal{X}$ is a test configuration, and $p_{\mathcal{X}}$ is the function defined in Section 1.3.1, then $A \circ p_{\mathcal{X}}$ is a PL function.*

Chapter 2

Non-Archimedean pluripotential theory

2.1 Non-Archimedean plurisubharmonic functions

From now on, X will be a compact Kähler manifold, with a fixed Kähler class $\alpha \in \mathrm{Pos}(X)$.

The goal of this section is to define and develop the theory non–Archimedean plurisubharmonic functions on $X^{\beth}$, in analogy with [BJ22] in the projective setting.

2.1.1 Plurisubharmonic PL functions

Let's denote by $\mathcal{X}_{\mathrm{triv}} \doteq X \times \mathbb{P}^1$ the trivial configuration, and by $p_1 \colon \mathcal{X}_{\mathrm{triv}} \to X$ the first projection. Given $\beta \in H^{1,1}(X)$, we then denote

$$\beta_{\mathcal{X}_{\mathrm{triv}}} \doteq p_1^* \beta \in H^{1,1}(X \times \mathbb{P}^1)$$

More generally, given any test configuration that μ-dominates $X \times \mathbb{P}^1$, we denote

$$\beta_{\mathcal{X}} \doteq \mu^* \beta_{\mathcal{X}_{\mathrm{triv}}}.$$

Remark 2.1.1. *In [SD18, DR17] the authors introduce, independently, the notion of* cohomological test configurations*, which are generalizations –to the transcendental setting– of the usual algebraic test configurations for a polarized manifold (X, L).*

For them, a cohomological test configuration is a test configuration $\mathcal{X}$ together with a $\mathbb{C}^*$-invariant Bott–Chern cohomology class $\mathcal{A} \in H^{1,1}_{\mathrm{BC}}(\mathcal{X})$ such that away from the central fiber:

$$\mathcal{A}|_{\mathcal{X}^*} = h^* \alpha_{\mathcal{X}},$$

for $h \colon \mathcal{X}^* \to X \times (\mathbb{P}^1 \setminus \{0\})$ the $\mathbb{C}^*$-equivariant biholomorphism.

By [SD18, Proposition 3.10], the data of a cohomological test configuration is the same *of a test configuration together with the choice of a vertical divisor D, i.e.*

$$\mathcal{A} = \alpha_{\mathcal{X}} + D \tag{2.1.1.1}$$

for $D \in \mathrm{VCar}\,(\mathcal{X})$.

Definition 2.1.2. *Let $\varphi \in \mathrm{PL}(X^{\beth})$ we say that φ is* α-plurisubharmonic *if given a snc dominating test configuration $\mathcal{X}$ with $D \in \mathrm{VCar}(\mathcal{X})$ such that $\varphi = \varphi_D$, we have*

$$\alpha_{\mathcal{X}} + D \textit{ is nef relatively to } \mathbb{P}^1, \tag{2.1.1.2}$$

that is, that $(\alpha_{\mathcal{X}} + D)|_E$ is nef for each E irreducible component of the central fiber $\mathcal{X}_0 = \sum_{E \subseteq \mathcal{X}_0} b_E E$. One can check using Demailly–Paun's criterion of nefness that this definition does not depend on $\mathcal{X}$.

We will denote the set of α-psh functions by $\mathrm{PL} \cap \mathrm{PSH}(\alpha)$. Moreover, we denote by $\mathcal{H}(\alpha)$ the set of functions $\varphi \in \mathrm{PL}(X^{\beth})$ such that there exists a dominating test configuration $\mathcal{X}$ and a vertical divisor $D \in \mathrm{VCar}(\mathcal{X})$ satisfying:

$$\varphi = \varphi_D, \quad \alpha_{\mathcal{X}} + D \textit{ is Kähler relatively to } \mathbb{P}^1.$$

Remark 2.1.3. *In the standard algebraic setting, the set of non-archimedean Fubini–Study metrics is usually denoted by $\mathcal{H}^{\mathrm{NA}}$. Here $\mathcal{H}(\alpha)$ will play the role of this set in our more general context. Note, however, that for an algebraic variety, $\mathcal{H}(\alpha)$ it is not the set Fubini–Study functions, the latter is only a subset: $\mathcal{H}^{\mathrm{NA}} \subsetneq \mathcal{H}(\alpha)$.*

Proposition 2.1.4. *Let $\varphi, \psi \in \mathrm{PL}(X^{\beth}) \cap \mathrm{PSH}(\alpha)$, and $f\colon X \to Y$ be a finite holomorphic map, then the following properties hold:*

1. $f^*\varphi \in \mathrm{PL} \cap \mathrm{PSH}(f^*\alpha)$;
2. $\varphi + c$, *and* $t \cdot \varphi$ *lie in* $\mathrm{PL} \cap \mathrm{PSH}(\alpha)$ *for* $c \in \mathbb{R}$ *and* $t \in \mathbb{Q}_{>0}$;
3. $\max\{\varphi, \psi\} \in \mathrm{PL} \cap \mathrm{PSH}(\alpha)$.

While proof of items (1) and (2) is essentially the same as in the algebraic trivially valued case, cf. [BJ22, Proposition 3.6], the proof of item (3) is different and relies on the analysis of singularities of psh functions of Lemma 2.4.2.

Proof. Let $\mathcal{X}$ be a test configuration μ-dominating the trivial one, such that

$$\varphi = \varphi_D \quad \& \quad \psi = \varphi_E$$

for some $D, E \in \mathrm{VCar}(\mathcal{X})$.

For (1) it is enough to observe that the pull-back of a PL function is PL and that pull-back of a nef class is nef.

For the first part of (2) it is enough to observe that:

- $\varphi + c = \varphi_{D+c\mathcal{X}_0}$;
- On the quotient
$$H^{1,1}(\mathcal{X})/\mu^*(H^{1,1}(\mathbb{P}^1))$$
we have $[\alpha + D] = [\alpha + D + c\mathcal{X}_0]$, and hence
$$D + \alpha + c\mathcal{X}_0 \text{ is nef rel. to } \mathbb{P}^1 \iff D + \alpha \text{ is nef rel. to } \mathbb{P}^1.$$

If $t \in \mathbb{N}$, then it is enough to observe that, like in the trivially valued case [BJ22, Proposition 3.6],
$$\varphi_{D_t} = t \cdot \varphi_D$$
where $D_t = \mu_t^* D$ is given by the base change
$$\begin{tikzcd} \mathcal{X}^t \arrow[r, "\mu_t"] \arrow[d, "\pi_t"] & \mathcal{X} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "z^t"] & \mathbb{P}^1. \end{tikzcd} \tag{2.1.1.3}$$

The result follows from (1), the general case $t \in \mathbb{Q}_{>0}$ follows from this one, since a cohomology class is nef iff its pullback by a finite branched covering is.

For item (3), let's suppose $\varphi, \psi \in \mathcal{H}(\alpha)$, the general case will follow from an approximation argument, cf. Theorem 2.1.5 below.

Now, let $c > 0$ be large enough so that $D' \doteq -D + c\mathcal{X}_0$ and $E' \doteq -E + c\mathcal{X}_0$ are effective, we have:
$$\max\{\varphi_{-D'}, \varphi_{-E'}\} = \max\{\varphi, \psi\} - c,$$
and thus it is α-psh iff $\max\{\varphi, \psi\}$ is α-psh.

Moreover, let $\mathcal{X}'$ be a test configuration ν-dominating $\mathcal{X}$, such that
$$\varphi_G = \max\{\varphi_{-D'}, \varphi_{-E'}\}$$
for $G \in \mathrm{VCar}(\mathcal{X}')$, then we observe that $\mathcal{O}_{\mathcal{X}'}(-G) = \nu^*[\mathcal{O}_{\mathcal{X}}(D') + \mathcal{O}_{\mathcal{X}}(E')]$, and the result follows from Lemma 2.4.2. □

Theorem 2.1.5. *Let $\varphi \in \mathrm{PL}(X^{\beth})$, and $(\varphi_\lambda)_{\lambda \in \Lambda}$ be a net of PL α-psh functions such that*
$$\varphi_\lambda(v) \to \varphi(v), \ \forall\, v \in X^{\mathrm{div}}$$
then φ is a PL α-psh function.

Proof. Let $\mathcal{X}_\lambda$, and $\mathcal{X}$ be snc test configurations together with morphisms of test configurations $\mu_\lambda\colon \mathcal{X}_\lambda \to \mathcal{X}$, and $\nu\colon \mathcal{X} \to X \times \mathbb{P}^1$, such that there exist vertical divisors $D \in \mathrm{VCar}(\mathcal{X})$, and $D_\lambda \in \mathrm{VCar}(\mathcal{X}_\lambda)$ satisfying:

$$\varphi = \varphi_D, \quad \text{and} \quad \varphi_\lambda = \varphi_{D_\lambda}.$$

To prove that $D + \alpha_{\mathcal{X}}$ is nef relatively to $\mathbb{P}^1$, it is enough to show that for every irreducible –hence smooth– component of the central fiber, $E \subseteq \mathcal{X}_0$, the restriction $(D + \alpha_{\mathcal{X}})_{|E}$ is in $\mathrm{Nef}(E)$. This follows from the simple observation that for $\tau \neq 0$ we have:

$$D_{|\mathcal{X}_\tau} = 0 \quad \& \quad (\alpha_{\mathcal{X}})_{|\mathcal{X}_\tau} = h_\tau^* \alpha \in \mathrm{Nef}(\mathcal{X}_\tau),$$

where $h_t\colon \mathcal{X}_\tau \to X$ is the biholomorphism provided by the $\mathbb{C}^*$-action.

Let then $Y^d \subseteq E$ be a d-dimensional subvariety of (the smooth complex manifold) E, and $\gamma \in \mathrm{Pos}(\mathcal{X})$ a Kähler class, by Demailly-Paun numeric characterization of nefness, it sufices to show that:

$$\int_Y (D + \alpha_{\mathcal{X}}) \wedge \gamma^{d-1} \geq 0. \tag{2.1.1.4}$$

To simplify notation, we rewrite the left hand side:

$$\int_Y (D + \alpha_{\mathcal{X}}) \wedge \gamma^{d-1} = [Y] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1}. \tag{2.1.1.5}$$

We can suppose that Y is invariant by the $\mathbb{C}^*$-action. Indeed, since $E \subseteq \mathcal{X}_0$ is irreducible, it is itself invariant. Therefore, denoting $Y_\tau \doteq \tau \cdot Y$, we get by compactness of each component of the space of effective cycles on E, cf. [HS74, Fuj78], that the limit $\lim_{\tau \to 0} Y_\tau$ exists as an effective cycle $\sum a_Z Z$, where the components Z are $\mathbb{C}^*$-invariant. Moreover, since $\mathbb{C}^*$ acts trivially on cohomology, we observe that $[Y_\tau] = [Y]$, and:

$$[Y] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1} = [Y_\tau] \cdot \rho(\tau)^*(D + \alpha_{\mathcal{X}}) \cdot \rho(\tau)^* \gamma^{d-1} \to \sum a_Z [Z] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1},$$

for ρ the $\mathbb{C}^*$-action on $\mathcal{X}$. Replacing Y for Z we get the $\mathbb{C}^*$-invariance.

Let $b\colon \mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along Y. Since Y is $\mathbb{C}^*$-invariant $\mathcal{X}'$ is a test configuration. Let $F \subseteq \mathcal{X}'$ be the exceptional divisor, and consider the positive current, of bi-dimension (d, d), given by:

$$T \doteq \delta_F \wedge \omega^{n-d},$$

where δ_F denotes the current of integration on F, and $\omega \in \mathcal{K}(\mathcal{X}')$ a Kähler form.

Consider now $b_*(T)$, by definition this is again a positive current of bi-dimension (d, d), with support $\operatorname{supp} b_*(T) \subseteq b(\operatorname{supp}(T)) = b(F) = Y$. By Demailly's support theorem every current of bi-dimension (d, d) supported on an irreducible cycle of dimension d must be a multiple of the current of integration over that cycle, which implies that the cohomology classes $[b_*(T)] = a[Y]$, for $a \geq 0$. Choosing η to be a Kähler form on $\mathcal{X}$ such that $\omega - b^*\eta$ is positive on the fibers of b, we have:

$$b_*T \cdot \eta = T \cdot b^*\eta \geq \int_F (\omega - b^*\eta)^{n-d} \wedge (b^*\eta)^d > 0,$$

thus $[b_*T] \neq 0 \implies a > 0$.

Hence, Equation (2.1.1.5) becomes:

$$\begin{aligned}\frac{1}{a}[b_*(T)] \cdot (D + \alpha_{\mathcal{X}}) \cdot \gamma^{d-1} &= \frac{1}{a}[T] \cdot b^*(D + \alpha_{\mathcal{X}}) \cdot b^*\gamma^{d-1} \\ &= \frac{1}{a}[F] \cdot \alpha^{n-d} \cdot (D_{\mathcal{X}'} + \alpha_{\mathcal{X}'}) \cdot \gamma_{\mathcal{X}'}^{d-1},\end{aligned}$$

where the second equality holds by the projection formula, and $D_{\mathcal{X}'} \doteq b^*D$.

Now, let $\mathcal{X}'_\lambda$ be a test configuration that dominates both $\mathcal{X}_\lambda$ and $\mathcal{X}'$

$$\begin{tikzcd} \mathcal{X}_\lambda \arrow[d, "\mu_\lambda"] & \mathcal{X}'_\lambda \arrow[l, "b_\lambda"'] \arrow[d, "\nu_\lambda"] \\ \mathcal{X} & \mathcal{X}'. \arrow[l, "b"] \end{tikzcd}$$

Denoting $F_\lambda \doteq \nu_\lambda^* F$, and $\omega_\lambda \doteq \nu_\lambda^*\omega$, we observe:

$$0 \leq \frac{1}{a}[F_\lambda] \cdot [\omega_\lambda]^{n-d} \cdot (D_{\mathcal{X}'_\lambda} + \alpha_{\mathcal{X}'_\lambda}) \cdot \gamma_{\mathcal{X}'_\lambda}^{d-1},$$

since F_λ is effective, and $D_\lambda + \alpha_{\mathcal{X}_\lambda}$, ω, and γ are nef –which implies that $D_{\mathcal{X}'_\lambda} + \alpha_{\mathcal{X}'_\lambda}$, $[\omega_\lambda]$, and $\gamma_{\mathcal{X}'_\lambda}$ are nef as well.

Again by the projection formula, we have:

$$\begin{aligned}0 &\leq \frac{1}{a}[F_\lambda] \cdot [\omega_\lambda]^{n-d} \cdot (D_{\mathcal{X}'_\lambda} + \alpha_{\mathcal{X}'_\lambda}) \cdot (\gamma_{\mathcal{X}'_\lambda})^{d-1} \\ &= \frac{1}{a}[F] \cdot \alpha^{n-d} \cdot [\nu_{\lambda *} D_{\mathcal{X}'_\lambda} + \alpha_{\mathcal{X}'}] \cdot \gamma_{\mathcal{X}'}^{d-1}.\end{aligned}$$

Now, since:

$$\nu_{\lambda *} D_{\mathcal{X}'_\lambda} = \sum_{G \overset{\text{irred}}{\subseteq} \mathcal{X}'_0} b_G \, \varphi_{D_\lambda}(v_G) G \longrightarrow \sum_{G \overset{\text{irred}}{\subseteq} \mathcal{X}'_0} b_G \, \varphi_D(v_G) G = D_{\mathcal{X}'}$$

it follows that:

$$\frac{1}{a}[F] \cdot \alpha^{n-d} \cdot [\nu_{\lambda *} D_{\mathcal{X}'_\lambda} + \alpha_{\mathcal{X}'}] \cdot \gamma_{\mathcal{X}'}^{d-1} \to \frac{1}{a}[F] \cdot \alpha^{n-d} \cdot (D_{\mathcal{X}'} + \alpha_{\mathcal{X}'}) \cdot \gamma_{\mathcal{X}'}^{d-1} \geq 0,$$

concluding the proof. $\square$

2.1.2 Non-Archimedean psh functions

We wil now define one of the most important objects of study of this paper the: *non-Archimedean psh functions.*

Definition 2.1.6. *A function*

$$\psi\colon X^{\beth} \to [-\infty, +\infty[$$

is α-psh *if* $\psi \not\equiv -\infty$*, and there exists a decreasing net* $(\varphi_\lambda)_{\lambda\in\Lambda} \in \mathrm{PL}\cap\mathrm{PSH}(\alpha)$ *such that*

$$\varphi_\lambda(v) \searrow \psi(v), \text{ for every } v \in X^{\beth}. \tag{2.1.2.1}$$

The set of all α*-psh functions will be denoted* $\mathrm{PSH}(\alpha)$*.*

Like for the PL functions we have the following properties: for $\varphi, \psi \in \mathrm{PSH}(\alpha)$, and $f\colon X \to Y$ a finite holomorphic map:

1. $f^*\varphi \in \mathrm{PSH}(f^*\alpha)$,
2. $\varphi + c$, and $t \cdot \varphi$ are α-psh for $c \in \mathbb{R}$ and $t \in \mathbb{Q}_{>0}$,
3. $\max\{\varphi, \psi\} \in \mathrm{PSH}(\alpha)$.

Next, an immediate consequence of the definition:

Lemma 2.1.7. *The intersection*

$$\bigcap_{\lambda>1} \mathrm{PSH}(\lambda\cdot\alpha) = \mathrm{PSH}(\alpha).$$

Proof. It is clear that $\bigcap_{\lambda>1}\mathrm{PSH}(\lambda\cdot\alpha) \supseteq \mathrm{PSH}(\alpha)$, we will prove now the other inclusion. Let $\varphi \in \mathrm{PSH}(\lambda\cdot\alpha)$ for every $\lambda > 1$, but subtracting a constant we can suppose that $\varphi \leq 0$. Now, it is enough to observe that the net $\varphi_\lambda \doteq \frac{1}{\lambda}\varphi \in \mathrm{PSH}(\alpha)$ decreases to φ as $\lambda \searrow 1$, which then implies that $\varphi \in \mathrm{PSH}(\alpha)$. □

The next result is well known in the algebraic case, see for instance [BJ22, Corollary 4.17] or [BFJ16], and the proof in our transcendental setting goes without a change. We follow the proof of [BFJ16, Section 6.1], which is added here for completeness.

Theorem 2.1.8. *If* $\psi \in \mathrm{PSH}(\alpha)$*, then*

$$\psi|_{X^{\mathrm{div}}} > -\infty.$$

Proof. Let v be a divisorial valuation, and $\mathcal{X}$ be a test configuration such that, decomposing the central fiber in irreducible components

$$\mathcal{X}_0 = \sum_{j=0}^{k} b_j\, E_j,$$

we have $v = v_{E_0}$.

Consider now, $\gamma \in \mathrm{Pos}(\mathcal{X})$, and $(\varphi_\lambda)_\lambda \in \mathrm{PL} \cap \mathrm{PSH}(\alpha)$ such that:

$$\varphi_\lambda(v) \searrow \psi(v), \text{ for every } v \in X^\beth.$$

We may assume $\sup \psi = 0$, and hence that $\sup \varphi_\lambda = 0 = \max\{\varphi_\lambda(v_{E_j})\}$.

If $\mathcal{X}_\lambda$ is a test configuration that μ_λ-dominates $\mathcal{X}$, with $D_\lambda \in \mathrm{VCar}(\mathcal{X}_\lambda)$ such that $\varphi_\lambda = \varphi_{D_\lambda}$, then, since $(\mu_\lambda)^* E_j$ is an effective divisor, and the classes

$$\alpha_{\mathcal{X}_\lambda} + D_\lambda,\ (\mu_\lambda)^*\gamma \text{ are relatively nef w.r.t. } \mathbb{P}^1,$$

we have:

$$0 \le (\mu_\lambda)^* E_j \cdot (\alpha_{\mathcal{X}_i} + D_\lambda) \cdot (\mu_\lambda)^*\gamma^{n-1} = E_j \cdot (\alpha_{\mathcal{X}} + (\mu_\lambda)_* D_\lambda) \cdot \gamma^{n-1}, \tag{2.1.2.2}$$

where the equality is given by the projection formula.

Since $(\mu_\lambda)_* D_\lambda = \sum_k b_k\, \varphi_{D_\lambda}(v_{E_k})\, E_k$, rewriting the inequality (2.1.2.2), it follows:

$$\sum_k b_k\, \varphi_{D_\lambda}(v_{E_k})\, (E_j \cdot E_k \cdot \gamma^{n-1}) \ge -E_j \cdot \alpha_{\mathcal{X}} \cdot \gamma^{n-1}.$$

Now, if $E_j \cap E_k \neq$, then $E_j \cdot E_k \cdot \gamma^{n-1} > 0$, and thus for all j:

$$\begin{aligned} b_j(E_j \cdot E_j \cdot \gamma^{n-1}) &= E_j \cdot (b_j E_j - \mathcal{X}_0) \cdot \gamma^{n-1} \\ &= -\sum_{k \neq j} b_k (E_j \cdot E_k \cdot \gamma^{n-1}) \le -1 \end{aligned}$$

where the first equality comes from flatness of $\pi \colon \mathcal{X} \to \mathbb{P}^1$, and the last inequality comes from $\mathcal{X}_0$ being connected with at least two irreducible components. Exactly like [BFJ16, Section 6.1], we get:

$$|\varphi_{D_\lambda}(v_{E_j})| \le C(\mathcal{X}, \alpha, \gamma), \tag{2.1.2.3}$$

for some constant C depending only on $\mathcal{X}, \alpha$ and γ. Hence:

$$C \ge \lim_i |\varphi_{D_\lambda}(v_{E_j})| = |\psi(v_{E_j})|,$$

concluding the proof. $\square$

Thanks to the above result, a natural topology to endow $\mathrm{PSH}(\alpha)$ will be the topology of pointwise convergence on divisorial valuations.

Below, we will also prove that:

$$\varphi \leq \psi \text{ on } X^{\mathrm{div}} \implies \varphi \leq \psi \text{ on } X^{\beth}$$

for $\varphi \in \mathrm{PSH}(\alpha)$ and $\psi\colon X^{\beth} \to [-\infty, +\infty[$ a usc function. To get this result we need the following description of divisorial valuations:

Definition 2.1.9. *Let $\mathfrak{a} \in \mathcal{F}$ be a flag ideal, the set $\Sigma_{\mathfrak{a}} \subseteq (X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*} = \sigma(X^{\mathrm{div}})$ of divisorial valuations given by the irreducible components of the exceptional divisor of the normalized blow-up $\mathrm{Bl}_{\mathfrak{a}} \widetilde{X \times \mathbb{P}^1}$ are called the* Rees valuations associated to $\mathfrak{a}$.

It is clear from definition that:

1. $\Sigma_{\mathfrak{a}} = \Sigma_{\overline{\mathfrak{a}}}$

2. $\Sigma_{\mathfrak{a}^m} = \Sigma_{\mathfrak{a}}$

3. $\bigcup_{\mathfrak{a} \in \mathscr{I}_X} \Sigma_{\mathfrak{a}} = (X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$

The next result is a generalization of [BJ22, Lemma 2.13], and the proof follows the same general lines.

Lemma 2.1.10. *Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}$, and $m \in \mathbb{N}$, then*

$$\sup_{X^{\beth}} \left\{ \frac{1}{m}\varphi_{\mathfrak{b}} - \varphi_{\mathfrak{a}} \right\} = \max_{\Sigma_{\mathfrak{a}}} \left\{ \frac{1}{m}\varphi_{\mathfrak{b}} - \varphi_{\mathfrak{a}} \right\}$$

Proof. After replacing $\mathfrak{a}$ for $\mathfrak{a}^m$, we can suppose that $m = 1$. Set

$$C \doteq \max_{\Sigma_{\mathfrak{b}}} \{\varphi_{\mathfrak{a}} - \varphi_{\mathfrak{b}}\} \tag{2.1.2.4}$$

Let $\mathcal{X} = \mathrm{Bl}_{\mathfrak{a}} \widetilde{X \times \mathbb{P}^1}$, and let $\mathcal{X}_0 = \sum b_i E_i$ be the decomposition into irreducible components of the central fiber, so that $\Sigma_{\mathfrak{a}} = \{v_{E_1}, \ldots, v_{E_k}\}$. Then, we observe that:

1. The ideal $\mathfrak{a}$ becomes invertible on $\mathcal{X}$.

2. We can then "subtract", and Equation 2.1.2.4 reads

$$\mathrm{ord}_{E_i}\left(\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}(D)\right) \geq 0, \qquad \text{for every } i = 1, \ldots, k \tag{2.1.2.5}$$

   for some divisor $D \in \mathrm{VCar}(\mathcal{X})$.

3. The polar variety of $\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}(D)$ is contained in the central fiber $\mathcal{X}_0$, hence Equation 2.1.2.5 implies that the polar variety is of codimension at least 2.

4. Since $\mathcal{X}$ is normal, $\mathfrak{b} \cdot \mathcal{O}_{\mathcal{X}}(D) \subseteq \mathcal{O}_{\mathcal{X}}$ hence proving the result.

□

As a consequence of the previous lemma, like in [BJ22, Lemma 4.26], we conclude that:

Proposition 2.1.11. *Let $\psi \in \mathrm{PL}(X^{\beth})$, then there exists a finite subset $\Sigma(\psi) = \Sigma \subseteq X^{\mathrm{div}}$ such that for every $\varphi \in \mathrm{PSH}(\alpha)$ we have:*

$$\sup_{X^{\beth}}(\varphi - \psi) = \max_{\Sigma}(\varphi - \psi).$$

□

Corollary 2.1.12. *Let $\psi \in \mathrm{C}^0\left(X^{\beth}, \mathbb{R}\right)$, then the function*

$$\mathrm{PSH}(\alpha) \to \mathbb{R}, \quad \varphi \mapsto \sup_{X^{\beth}}(\varphi - \psi)$$

is continuous.

Proof. Follows from the previous lemma, together with the density of $\mathrm{PL}(X^{\beth})$ in $\mathrm{C}^0(X^{\beth})$. □

Moreover, we now prove a partial converse result:

Lemma 2.1.13. *Let v be a divisorial valuation, then there exists a PL function $\varphi \in \mathrm{PL}$, such that*

$$\sup_{X^{\beth}}(\psi - \varphi) = \psi(v) - \varphi(v),$$

for every $\psi \in \mathrm{PSH}(\alpha)$.

This Lemma, once again, is valid in the projective setting, see [BJ22, Lemma 5.7]. The proof can be adapted from [BJ22] directly once established Proposition 2.1.14 below, which, however, has a different proof compared to the projective counterpart of Lemma 2.28 of [BJ22].

Proposition 2.1.14. *Let $\mathfrak{a}$ be a flag ideal, and $m, \ell \in \mathbb{N}$ sufficiently large. Then, for any $c \in \mathbb{Q}^{\Sigma_{\mathfrak{a}}}$, there exists $r \geq 1$ and $\rho \in \mathcal{H}(\alpha)$ such that: $\psi \doteq r\left(m\varphi_{\mathfrak{a}} - \ell\rho\right)$ satisfies:*

$$\psi(v) = c_v, \quad \forall v \in \Sigma_{\mathfrak{a}}.$$

Proof. Let $\mu \colon \mathcal{X} \to X \times \mathbb{P}^1$ be the normalized blow-up of $X \times \mathbb{P}^1$ along $\mathfrak{a}$, and let E_i be the prime components of the exceptional divisor of the blow-up, making $\Sigma_{\mathfrak{a}} = \{v_{E_i}\}_{i \in I}$. From now on, we denote by v_i the valuation v_{E_i}, and by c_i the corresponding rational number c_{v_i}.

We define, for $m \in \mathbb{N}$, and $j \in I$:

$$\begin{aligned}\mathfrak{b} &\doteq \left\{f \in \mathcal{O}_{X\times\mathbb{P}^1} \,|\, \mathrm{ord}_{E_i}(f) \geq \mathrm{ord}_{E_i}(\mathfrak{a}^m), \quad \forall i \in I\right\},\\ \mathfrak{b}_j &\doteq \left\{f \in \mathcal{O}_{X\times\mathbb{P}^1} \,|\, \mathrm{ord}_{E_i}(f) \geq \mathrm{ord}_{E_i}(\mathfrak{a}^m), \quad \forall i \in I\setminus\{j\}\right\},\end{aligned}$$

we have that clearly $\mathfrak{b}$ and $\mathfrak{b}_j$ are ideal sheaves of $\mathcal{O}_{X\times\mathbb{P}^1}$ that are $\mathbb{C}^*$-invariant and trivial away from the central fiber, i.e. $\mathfrak{b}$ and $\mathfrak{b}_j$ are flag ideals. Moreover, if we let $D \in \mathrm{VCar}(\mathcal{X})$ be such that $\mathcal{O}_{\mathcal{X}}(D) = \mathfrak{a}\cdot\mathcal{O}_{\mathcal{X}}$, we have that for m big enough:

$$\mathcal{O}_{\mathcal{X}}(mD)\cdot\mathcal{O}_{\mathcal{X}}(E_j)$$

is μ-globally generated, and $\mathfrak{b} = \mu_*\mathcal{O}_{\mathcal{X}}(mD)$, $\mathfrak{b}_j = \mu_*[\mathcal{O}_{\mathcal{X}}(mD)\cdot\mathcal{O}_{\mathcal{X}}(E_j)]$. In particular, this gives that $\mathfrak{b}_j \supsetneq \mathfrak{b}$.

By [Laz17, Example 9.6.3, Remark 9.6.4] we have that $\mathfrak{b}$ and $\mathfrak{b}_j$ are integrally closed, and actually $\mathfrak{b} = \overline{\mathfrak{a}^m}$.

Observe now that for $\ell > 0$ large enough $\varphi_j \doteq \varphi_{\mathfrak{b}_j}$ is in $\mathcal{H}(\ell\alpha)$ for every j. Indeed, if $\mu_j\colon \mathcal{X}_j \to X\times\mathbb{P}^1$ is the normalized blow-up of $X\times\mathbb{P}^1$ along $\mathfrak{b}_j$, and D_j the vertical divisor such that $\mathcal{O}_{\mathcal{X}}(D_j) = \mathfrak{b}_j\cdot\mathcal{O}_{\mathcal{X}}$, then D_j is μ_j-ample, in particular $D_j + \ell_j\alpha_{\mathcal{X}_j}$ is Kähler relatively to $\mathbb{P}^1$ for ℓ_j sufficiently big, it suffices to take $\ell = \max \ell_j$.

By definition, we then have:

- $\varphi_j(v_j) > \frac{m}{\ell}\varphi_{\mathfrak{a}}(v_j)$;
- $\varphi_j(v_i) \leq \frac{m}{\ell}\varphi_{\mathfrak{a}}(v_i)$ for $i \neq j$.

Let's denote the difference $\varphi_j(v_j) - \frac{m}{\ell}\varphi_{\mathfrak{a}}(v_j) > 0$ by ϵ_j. For $r \in \mathbb{Q}_{>0}$, we set for $k \in I$:

$$\Phi_k \doteq \frac{rm}{\ell}\varphi_{\mathfrak{a}} - r\varphi_k + r\epsilon_k + c_k.$$

Now, we observe that $\Phi_j(v_j) = -r\epsilon_j + r\epsilon_j + c_j = c_j$, and for $i \neq j$:

$$\begin{aligned}\Phi_i(v_j) &= \frac{rm}{\ell}\varphi_{\mathfrak{a}}(v_j) - r\varphi_i(v_j) + r\epsilon_i + c_i\\ &\geq 0 + r\epsilon_i + c_i \geq c_j,\end{aligned}$$

if $r \gg 1$ is big enough. Therefore, the function

$$\psi \doteq \min_{k\in I}\Phi_k$$

is such that $\psi(v) = c_v$ for $v \in \Sigma_{\mathfrak{a}}$, and moreover, denoting

$$\rho \doteq \max_{k\in I}\{\varphi_k - \epsilon_k - c_k/r\} \in \mathcal{H}(\alpha),$$

we get that $\psi = \frac{rm}{\ell}\varphi_{\mathfrak{a}} - r\rho = \frac{r}{\ell}(m\varphi_{\mathfrak{a}} - \ell\rho)$, as we wanted. $\square$

After stablishing this result, we add now, for completenes, the proof of Lemma 2.1.13:

Proof of Lemma 2.1.13. Let $\mathfrak{a}$ be a flag ideal such that $v \in \Sigma_{\mathfrak{a}}$. Then, from the proof of Theorem 2.1.8 –which is itself based on the arguments of [BFJ16, Section 6.1], we get an uniform constant $C > 0$ such that for every $\psi \in \mathrm{PSH}(\alpha)$ we have:

$$|\psi(w)| \leq C, \quad \forall\, w \in \Sigma_{\mathfrak{a}}.$$

By the previous Lemma, we can then find a PL function $\varphi = q(\varphi_{\mathfrak{a}} - m\rho)$, for $\rho \in \mathcal{H}(\alpha)$ such that:

$$\varphi(v) = 2C + 1, \quad \varphi(w) = 0 \text{ for all } w \in \Sigma_{\mathfrak{a}} \setminus \{v\}.$$

Now, using Proposition 2.1.11, we conclude that for every $\psi \in \mathrm{PSH}(\alpha)$:

$$\sup_{X^{\beth}} (\psi - \varphi) = \max_{\Sigma_{\mathfrak{a}}} (\psi - \varphi).$$

Moreover, by the definition of φ, we have that $\psi(v) - \varphi(v) = \max_{\Sigma_{\mathfrak{a}}}(\psi - \varphi)$. □

We now prove the following important feature of non–Archimedean pluripotential theory:

Theorem 2.1.15. *Let $\varphi \in \mathrm{PSH}(\alpha)$, and $\psi \colon X^{\beth} \to \mathbb{R} \cup \{-\infty\}$ a usc function, we then have:*

$$\varphi \leq \psi \text{ on } X^{\mathrm{div}} \iff \varphi \leq \psi \text{ on } X^{\beth}.$$

Proof. If $\psi \in \mathrm{PL}$ this is an easy consequence of Proposition 2.1.11.

Since every continuous function is a uniform limit of PL functions, the same result holds if $\psi \in \mathrm{C}^0(X^{\beth})$.

Lastly, if ψ is a decreasing limit of the net $(\psi_\lambda)_\lambda \in \mathrm{C}^0(X^{\beth})$, we have that $\varphi \leq \psi \leq \psi_\lambda$ on X^{div}, and hence by the previous case $\varphi \leq \psi_\lambda$, and finally this implies that $\varphi \leq \psi$. □

Observe that this implies, in particular, that the values on divisorial valuations completely determines a psh function.

Remark 2.1.16. *Darvas, Xia, and Zhang developed on [DXZ25] a notion of transcendental non-archimedean psh metrics. They use the formalism of Ross–Witt Nystrum of* test curves *on the complex manifold X, and call a* non-archimedean psh metric, *a* maximal *test curve.*

Their approach is more general to define non-archimedean β-psh metrics for a transcendetal big class β. Bellow, in Section 2.4.2, we compare the present approach with theirs. As we will see they coincide when β is Kähler, and the metric is of finite energy.

2.1.3 Non-pluripolar points

In this section we will generalize results of [BJ22, BFJ16].

Motivated by Theorem 2.1.8, we will now study the set of point where all psh function are finite, that is the set of *non-pluripolar* points of $X^\beth$. These points have an interesting metric structures that will be important later, when considering the regularity of psh functions, see Theorem 2.2.2.

The rest of the section is adapted from [BJ22] from the Sections 4.5, 4.6 and 11. We add proofs here for completeness, but they are formal and follow from the shared statements found here and in [BJ22].

Definition 2.1.17. *We say that $v \in X^\beth$ is* non-pluripolar *if for every $\varphi \in \mathrm{PSH}$*

$$\varphi(v) > -\infty.$$

We denote the set of non-pluripolar points by X^{np}.

By Theorem 2.1.8 we know that $X^{\mathrm{div}} \subseteq X^{\mathrm{np}}$. We will now describe a metric structure on X^{np}:

Definition 2.1.18. *Let $v, w \in X^{\mathrm{np}}$ denote by $d_\infty(v, w)$ the quantity:*

$$\sup_{\varphi \in \mathrm{PSH}} |\varphi(v) - \varphi(w)|.$$

Moreover, we denote by $\mathrm{T}(v) \doteq d_\infty(v, v_{\mathrm{triv}})$.

Proposition 2.1.19. *The function d_∞ is a metric distance on the set of non-pluripolar points of $X^\beth$.*

Proof. It is clear that

- $d_\infty(v, w) = d_\infty(w, v)$;
- $d_\infty(v, w) \leq d_\infty(v, u) + d_\infty(u, w)$

for all $v, u, w \in X^{\mathrm{np}}$.

To show that d_∞ is finite valued it is then enough to check that $d_\infty(v, v_{\mathrm{triv}}) < \infty$ for $v \in X^{\mathrm{np}}$.

For that, suppose that by contradiction that for some v there exists a sequence $\varphi_m \in \mathrm{PSH}$ such that

$$|\varphi_m(v_{\mathrm{triv}}) - \varphi(v)| > 2^m.$$

By adding a constant we may suppose that $\varphi_m(v_{\mathrm{triv}}) = 0$, and therefore the convex combination $\psi_m \doteq 2^m \cdot 0 + \sum_{k=1}^m 2^{-k}\varphi_k \in \mathrm{PSH}$ is psh, decreaing with m, and satisfies:

$$\psi_m(v_{\mathrm{triv}}) = 0, \quad \text{and } \psi_m(v) \leq \sum_{k=1}^m 2^{-k} \cdot (-2^k) = -m.$$

Therefore, the decreasing limit $\psi \doteq \lim \psi_m$ is PSH, and $\psi(v) = -\infty$.

Lastly, if $d_\infty(v, w) = 0$, then for every $\varphi \in \mathcal{H}$:

$$\varphi(v) = \varphi(w).$$

Since the linear span of $\mathcal{H}$ is PL, that is dense in $\mathrm{C}^0(X^\beth)$, then this implies that $v = w$. $\square$

Example 2.1.20. *Let $V \doteq \{v_1, \ldots, v_k\}$ be a finite set of divisorial valuations, then A is bounded for d_∞.*

Indeed let $\mathcal{X}$ be a snc test configuration of X such that the set of vertices of the associated dual complex $\Delta_\mathcal{X} \hookrightarrow X^\beth$, viewed as subset of the set of valuations, include V. Then, this follows directly from the proof of Theorem 2.1.8.

2.1.4 Negligible points

As in the complex setting we define

Definition 2.1.21. *We say that $p \in X^\beth$ is a* negligible point *if there exists a family of psh functions $\varphi_i \in \mathrm{PSH}(\alpha)$ such that*

$$(\sup \varphi_i)^\star(p) \neq \sup \varphi_i(p),$$

where $\star$ refers to the upper semicontinuous envelope of a function. Moreover, p will be said to be nonneglegible *if it is not negligible.*

The main result of this section is to prove that divisorial valuations are nonneglegible. We argue as in [BJ22, Theorem 5.6], to get our Proposition 2.1.22. The key ingredient is Lemma 2.1.13 which is itself a consequence of Proposition 2.1.14. From this Lemma to the nonnegligibility of divisorial points, the proof is formal, and carries exactly as in [BJ22].

Proposition 2.1.22 (Divisorial points are nonnegligible)**.** *Let (ψ_k) be a family of α-psh functions that is bounded above. Then, for every $v \in X^{\mathrm{div}}$,*

$$(\sup \psi_k)^\star(v) = \sup \psi_k(v).$$

Proof. Let v be a divisorial valuation, and $\varphi \in \mathrm{PL}$ the function given by Lemma 2.1.13. Since for each k, $\psi_k - \varphi \leq \psi_k(v) - \varphi(v)$, taking supremum over k gives

$$\sup_k \psi_k \leq (\sup_k \psi_k(v) - \varphi(v)) + \varphi.$$

The right hand side is then continuous, and from this it follows that

$$(\sup \psi_k)^\star \leq (\sup \psi_k(v) - \varphi(v)) + \varphi.$$

Plugging v at the inequality we obtain

$$\sup_k \psi_k(v) \geq (\sup \psi_k)^\star(v) \implies \sup_k \psi_k(v) = (\sup \psi_k)^\star(v),$$

completing the proof. $\square$

Moreover, arguing as in [BJ22, Proposition 5.3] we have:

Lemma 2.1.23. *Every nonneglegible point is non-pluripolar.*

2.2 Psh functions and dual complexes

In this section we will prove that if $\varphi \in \mathrm{PSH}(\alpha)$, then the restriction of φ to a dual complex $\Delta_{\mathcal{X}} \hookrightarrow X^{\beth}$ is continuous and convex. The main inspiration for this section is [BFJ16].

Let us start with a simpler statement:

Lemma 2.2.1. *Let $\varphi \in \mathrm{PSH}(\alpha)$, and $\mathcal{X}$ be a snc test configuration. Then, denoting $p_{\mathcal{X}} \colon X^{\beth} \to \Delta_{\mathcal{X}}$ the retraction into the dual complex $\Delta_{\mathcal{X}}$, we have:*

$$\varphi \leq \varphi \circ p_{\mathcal{X}}.$$

The original version of this Lemma in the projective setting can be found in [BFJ16, Proposition 5.9]. The proof presented here is based on the latter; however, here we explore the description of monomial valuations as Lelong numbers, as in Appendix A.2.

Proof of Lemma 2.2.1. Up by taking decreasing limits, it is enough to reduce for when $\varphi \in \mathcal{H}(\alpha)$, which in particular gives $\varphi = \varphi_{\mathfrak{a}}$ for some flag ideal $\mathfrak{a}$ on $X \times \mathbb{P}^1$. Indeed, let $\mathcal{X} \xrightarrow{\mu} X \times \mathbb{P}^1$ be a dominating test configuration, with $D \in \mathrm{VCar}(\mathcal{X})$ such that:

$$\varphi = \varphi_D, \quad \text{and} \quad D + \alpha_{\mathcal{X}} \text{ is Kähler relative to } \mathbb{P}^1.$$

We then have that D is μ-ample relative to $\mathbb{P}^1$, implying that

$$\mathcal{O}_{\mathcal{X}}(D) = \mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}},$$

for some flag ideal $\mathfrak{a}$ on $X \times \mathbb{P}^1$, which gives $\varphi = \varphi_{\mathfrak{a}}$.

Then, φ and $\varphi \circ p_{\mathcal{X}}$ are continuous, and it is enough to check that

$$\varphi(v) \leq \varphi \circ p_{\mathcal{X}}(v), \quad \text{for every } v \in X^{\mathrm{div}}.$$

Let v then be a divisorial valuation, and $E' \subseteq \mathcal{X}'$ be an irreducible component of the central fiber of a snc test configuration such that $\mathcal{X}'$ dominates $\mathcal{X}$ with $v = v_{E'}$.

Let $g\colon X\times\mathbb{P}^1\to[-\infty,+\infty[$ be a quasi-psh function with analitic singularities along $\mathfrak{a}$, then by Appendix A.2 it follows that given the morphisms:

$$\begin{tikzcd}
\mathcal{X}' \arrow[dd, "\mu'"'] \arrow[dr, "\rho"] & \\
& \mathcal{X} \arrow[dl, "\mu"] \\
X\times\mathbb{P}^1 &
\end{tikzcd}$$

and $w=(v(E_1),\dots,v(E_k))$ for $E_1,\dots,E_k$ the irreducible components of $\mathcal{X}_0$ we have

$$\varphi(v)=-\nu(g\circ\mu',E'),\quad\text{and}\quad\varphi(p_{\mathcal{X}}(v))=-\nu_w(g\circ\mu,p),$$

for any generic choice of $p\in Z(v,\mathcal{X})$. Then the matter reduces to a question on Lelong numbers that is easy to check.

Indeed, let $q\in E'$ be any point not contained in any other irreducible component of $\mathcal{X}'_0$, and $p\doteq\rho(q)$. The claim then reduces to show that

$$g\circ\mu\circ\rho\le\nu_w\log|z'|+O(1),\qquad\text{for}\qquad\nu_w\doteq\nu_w(g\circ\mu,p),$$

and z' a local equation of E' around q. Let $z_1,\dots,z_n$ be local coordinates around p such that $E_1,\dots,E_k$ are locally given by $z_1,\dots,z_k$. Since for every i we have $g\circ\mu\le\nu_w\frac{\log|z_i|}{w_i}$ precomposing with ρ we get

$$g\circ\mu\circ\rho\le\nu_w\frac{\log|(z')^{w_i}|}{w_i}+O(1)=\nu_w\log|z'|+O(1).$$

□

Let us state now the main theorem of this section:

Theorem 2.2.2. *Let $\varphi\in\mathrm{PSH}(\alpha)$, and $\Delta_{\mathcal{X}}$ be a dual complex associated with a test configuration, then the restriction:*

$$\mathrm{val}^*\varphi\colon\Delta_{\mathcal{X}}\to\mathbb{R}$$

is continuous and convex, where $\mathrm{val}\colon\Delta_{\mathcal{X}}\to X^{\beth}$ is the map that identifies the dual complex $\Delta_{\mathcal{X}}$ with the corresponding monomial valuations in $X^{\beth}$.

Once more, this is the transcendental analogue of [BFJ16, Proposition 7.5]. Our proof adapts and mixes the strategies of [BFJ16, Proposition 7.5] and of results from Section 11 of [BJ22].

For the proof of Theorem 2.2.2 let us first prove the following simpler statement:

Lemma 2.2.3. *Let $\varphi \in \mathcal{H}(\alpha)$, then the restriction of φ to a dual complex $\Delta \hookrightarrow X^{\beth}$ is convex.*

Proof. As in the proof of Lemma 2.2.1, we can write $\varphi = \varphi_{\mathfrak{a}}$ for some flag ideal $\mathfrak{a}$ of $X \times \mathbb{P}^1$.

We are left to check that, for every face $\sigma_Z \subseteq \mathcal{X}$, the map:

$$\sigma_Z \ni w \mapsto \varphi_{\mathfrak{a}}(\mathrm{val}(w)) = -\mathrm{val}(w)(\mathfrak{a})$$

is convex. This follows now, by the definition of $\mathrm{val}(w)$, cf. Proposition 1.2.14, if $p \in Z$ and $f_1, \ldots, f_\ell$ are local generators of $\mathfrak{a}_p$, then

$$-\mathrm{val}(w)(\mathfrak{a}) = \max_{1 \leq j \leq \ell} -\mathrm{val}(w)(f_j).$$

By Equation 1.2.3.2 is easy to see that $-\mathrm{val}(w)(f_j)$ is piecewise linear function that is convex, and the result follows. □

Since every psh function is the pointwise limit of functions in $\mathcal{H}$, its restriction to a dual complex is convex and therefore continuous on the interior of each face. To establish continuity at the vertices, we will the adapt [BJ22, Theorem 11.12] to get the following:

Proposition 2.2.4. *Let $K \subseteq X^{\mathrm{np}}$ be a bounded set for the d_∞ distance, then for every φ psh function we have that*

$$\varphi|_K \in \mathrm{C}^0(K)$$

is continuous for the topology of $X^{\beth}$.

We will prove this statement in Section 4.5.2, once we stablished some uniform estimates on the value of psh functions on divisorial valuations.

With this statement we can then prove Theorem 2.2.2.

Proof of Theorem 2.2.2. Let's start proving that $\varphi|_{\Delta_{\mathcal{X}}}$ is convex. Let $\varphi_i \in \mathcal{H}(\alpha)$ be a decreasing sequence converging to φ. Then, by Lemma 2.2.3 $\varphi|_{\Delta_{\mathcal{X}}}$ is convex. Since convexity is preserved under pointwise limits, we then have that the restriction $\varphi|_{\Delta_{\mathcal{X}}}$ is convex.

Now, the restriction $\varphi|_{\mathring{\sigma}_Z}$ to the interior of any face $\sigma_Z \subseteq \Delta_{\mathcal{X}}$ is continuous by convexity. Therefore, it is enough to prove that φ is continuous on every vertex $v \in \Delta_{\mathcal{X}}$. Let $V \doteq \{v_1, \ldots v_\ell\}$ be the set of vertices of $\Delta_{\mathcal{X}}$, the divisorial valuations attached to irreducible components of $\mathcal{X}$. By Example 2.1.20 the set V is d_∞ bounded, and hence we conclude by applying Proposition 2.2.4. □

2.3 The energy pairing

In this section we will develop an *energy pairing* formalism for the transcendental non–Archimedean psh functions that we studied in the previous sections. This will allow us to define non–Archimedean analogues of Monge–Ampère measures, Monge–Ampère energy, finite energy potentials and more.

2.3.1 PL Monge–Ampère operator and energy pairing

In this section we will give the definition of the PL Monge–Ampère operator and, more generally, the PL energy pairing. We also state a few important properties and results.

The notions of pluripotential theory for $X^{\beth}$, introduced in this thesis, are under the synthetic formalism developed in [BJ25a]. In particular, every result from Section 1 and 2 of Boucksom–Jonsson's synthetic approach holds in our case.

Monge–Ampère measure of a PL function

Let $\beta \in H^{1,1}(X)$ be a cohomology class of positive volume, i.e. $V_\beta \doteq \int_X \beta^n > 0$, and $\varphi \in \mathrm{PL}(X^{\beth})$ a PL function, we can associate to the pair (β, φ) a signed measure on $X^{\beth}$, called *Monge–Ampère measure*, given by the construction:

- Let $\mathcal{X}$ be a snc test configuration with a vertidcal divisor $D \in \mathrm{VCar}(\mathcal{X})$ satisfying $\varphi_D = \varphi$.
- Denote $\mathcal{X}_0 = \sum b_E\, E$ the decomposition in irreducible components of the central fiber, and let c_E be the constant given by
$$\frac{b_E}{V_\beta}\left((\beta_{\mathcal{X}} + D)|_E\right)^n .$$
- Define the signed measure as:
$$\mathrm{MA}_\beta(\varphi) \doteq \sum_{E \overset{\text{irred}}{\subseteq} \mathcal{X}_0} c_E\, \delta_{v_E}.$$

With the projection formula one checks that this definition does not depend on the choice of test configuration. Moreover, if $\beta \in \mathrm{Pos}(X)$ is a positive class and $\varphi \in \mathcal{H}(\beta)$ the above construction get us a probability measure.

Definition 2.3.1. *Let $\alpha \in \mathrm{Pos}(X)$ be Kähler class, and $\mathcal{P}(X^{\beth})$ the set of Radon probability measures on $X^{\beth}$. We call the operator:*
$$\mathcal{H}(\alpha) \ni \varphi \overset{\mathrm{MA}}{\mapsto} \mathrm{MA}_\alpha(\varphi) \in \mathcal{P}(X^{\beth}),$$
the Monge–Ampère operator*. Whenever α is it clear by context we write $\mathrm{MA}(\varphi)$ for $\mathrm{MA}_\alpha(\varphi)$.*

Using the identification of $X^{\beth}$ with the limit of dual complexes, we can observe that if $\varphi \in \mathrm{Aff}_{\mathbb{Q}}(\Delta_{\mathcal{X}})$, then $p_{\mathcal{X}}^{*}\varphi \in \mathrm{PL}(\Delta)$ is such that $\mathrm{MA}(p_{\mathcal{X}}^{*}\varphi)$ is supported on $\Delta_{\mathcal{X}} \subseteq X^{\beth}$, where $p_{\mathcal{X}}$ is the retraction $X^{\beth} \to \Delta_{\mathcal{X}}$.

Remark 2.3.2. *A borelian measure on a compact Hausdorff topological space K is completely determined by its values on a dense subset of $\mathrm{C}^0(K)$, therefore the probability measure $\mathrm{MA}(\varphi)$ is completely determined by its values on the set $\mathrm{PL}(X^{\beth})$. We will use this approach to generalize the Monge–Ampère measure (see Remark 2.3.17), and construct the* mixed Monge–Ampère energy *(see next section).*

Energy pairing for PL functions

Let $\varphi_0, \dots, \varphi_n \in \mathrm{PL}_{\mathbb{R}}$, and $\beta_0, \dots, \beta_n \in H^{1,1}(X)$.

Definition 2.3.3. *Let $\mathcal{X}$ be a test configuration dominating $X \times \mathbb{P}^1$, such that there exist $D_0, \dots, D_n \in \mathrm{VCar}(\mathcal{X})$, with the property that for every $i = 0, \dots, n$ we have*

$$\varphi_i = \varphi_{D_i}$$

then we define the energy pairing

$$(\beta_0, \varphi_0) \cdot (\beta_1, \varphi_1) \cdots (\beta_n, \varphi_n) \doteq (\beta_{0,\mathcal{X}} + D_0) \cdots (\beta_{n,\mathcal{X}} + D_n) \in \mathbb{R}, \tag{2.3.1.1}$$

where, in the right hand side of the inequality, the intersection product is against the fundamental class of $\mathcal{X}$. As before, the projection formula guarantees that the energy pairing is well defined.

We also refer to the energy pairing as the energy coupling*, or as* mixed Monge–Ampère energy*, see item (iv) of Proposition 2.3.4 below.*

Clearly the pairing is a symmetric multi-linear form, which further satisfies the following properties:x

Proposition 2.3.4. *Let $\varphi_0, \dots, \varphi_n \in \mathrm{PL}_{\mathbb{R}}$, and $\mathcal{X}$ a test configuration such that there exist $D_0, \dots, D_n \in \mathrm{VCar}_{\mathbb{R}}(\mathcal{X})$ with*

$$\varphi_i = \varphi_{D_i},$$

and let $\beta_0, \dots, \beta_n \in H^{1,1}(X)$, $t \in \mathbb{Q}_{>0}$, and $\alpha \in \mathrm{Pos}(X)$ then we have:

(i) $(0, 1) \cdot (\beta_1, \varphi_1) \cdots (\beta_n, \varphi_n) = \beta_1 \cdots \beta_n$

(ii) $(\beta_0, 0) \cdots (\beta_n, 0) = 0$

(iii) $(\beta_0, t \cdot \varphi_0) \cdots (\beta_n, t \cdot \varphi_n) = t(\beta_0, \varphi_0) \cdots (\beta_n, \varphi_n)$

(iv) For $\psi \in \mathrm{PL}_{\mathbb{R}}$, and $V_\alpha \doteq [X] \cdot \alpha^n$, we have

$$\frac{1}{V_\alpha}(0,\psi) \cdot (\alpha, \varphi_0)^n = \int_{X^\beth} \psi \,\mathrm{MA}_\alpha(\varphi_0)$$

(v) More generally, for $\psi \in \mathrm{PL}_{\mathbb{R}}$ we have

$$(0,\psi) \cdot (\beta_1, \varphi_1) \cdots (\beta_n, \varphi_n) = \sum_{E \subseteq \mathcal{X}_0} b_E \,\psi(v_E)(\beta_1 + D_1)|_E \cdots (\beta_n + D_n)|_E$$

for E irreducible component and $b_E = v_E(\mathcal{X}_0)$.

Proof. (i) Follows from the remark that, as cohomology classes, $[\mathcal{X}_0] = [\mathcal{X}_1]$. Indeed, both of them can be written as $\pi^*([0])$ and $\pi^*([1])$ respectively, where $[0]$ and $[1]$ represent the cohomology classes of $0, 1 \in \mathbb{P}^1$ in $H^2(\mathbb{P}^1, \mathbb{C}) \cong \mathbb{C}$, but since $[0] = [1]$, the result follows from the flatness of $\pi\colon \mathcal{X} \to \mathbb{P}^1$.

(ii) Observe that:

$$[\mathcal{X}_0] \cdot \beta_{0,\mathcal{X}} \cdot \cdots \cdot \beta_{n,\mathcal{X}} = [X] \cdot \beta_0 \cdot \cdots \cdot \beta_n = 0.$$

(iii) It is enough to check that for each $d \in \mathbb{Z}_{>0}$ we have $(\beta_0, d \cdot \varphi_0) \cdots (\beta_n, d \cdot \varphi_n) = d(\beta_0, \varphi_0) \cdots (\beta_n, \varphi_n)$, but then again $d \cdot \varphi_D = \varphi_{D_d}$, where D_d is the pullback of D under the normalized base change:

$$\begin{tikzcd}
E_d \subseteq \widetilde{\mathcal{X}_d} \arrow[d] & \\
\mathcal{X}_d \arrow[r] \arrow[d] & E \subseteq \mathcal{X} \arrow[d] \\
\mathbb{P}^1 \arrow[r, "t^d"] & \mathbb{P}^1,
\end{tikzcd} \tag{2.3.1.2}$$

in addition if $\alpha_0, \ldots, \alpha_{n+1} \in H^{1,1}(\mathcal{X})$, then $(\alpha_0)_{\mathcal{X}_d} \cdots (\alpha_{n+1})_{\mathcal{X}_d} = d\,\alpha_0 \cdots \alpha_{n+1}$ and the result follows.

(iv) Follows from (v).

(v) Let $\mathcal{X}'$ be a test configuration μ-dominating $\mathcal{X}$ such that there exists $G \in \mathrm{VCar}_{\mathbb{R}}(\mathcal{X}')$ with

$$\psi = \varphi_G,$$

then we have:

$$\begin{aligned}(0,\psi)\cdot(\beta_1,\varphi_1)\cdots(\beta_n,\varphi_n) &= G\cdot\big(\beta_{1,\mathcal{X}'}+\mu^*D_1\big)\cdots\big(\beta_{n,\mathcal{X}'}+\mu^*D_n\big)\\ &= \mu_*G\cdot(\beta_{1,\mathcal{X}}+D_1)\cdots(\beta_{n,\mathcal{X}}+D_n)\\ &= \sum_{E\subseteq\mathcal{X}_0}\operatorname{ord}_E(G)\,E\cdot(\beta_1+D_1)\cdots(\beta_n+D_n)\\ &= \sum_{E\subseteq\mathcal{X}_0} b_E\,\varphi_G(v_E)(\beta_1+D_1)|_E\cdots(\beta_n+D_n)|_E\\ &= \sum_{E\subseteq\mathcal{X}_0} b_E\,\psi(v_E)(\beta_1+D_1)|_E\cdots(\beta_n+D_n)|_E,\end{aligned} \tag{2.3.1.3}$$

where the second equality is given by the projection formula. □

Corollary 2.3.5. *Let $\beta_1,\dots\beta_n\in\mathrm{Pos}(X)$ and for each i a β_i-psh function $\varphi_i\in\mathrm{PL}\cap\mathrm{PSH}(\beta_i)$. If $\gamma\in H^{1,1}(X)$, and $\psi,\psi'\in\mathrm{PL}$ are such that $\psi\le\psi'$, then*

$$(\gamma,\psi)\cdot(\beta_1,\varphi_1)\cdots(\beta_n,\varphi_n)\le(\gamma,\psi')\cdot(\beta_1,\varphi_1)\cdots(\beta_n,\varphi_n)$$

Proof. Follows directly from equation 2.3.1.3. □

Lemma 2.3.6 (Zariski's Lemma)**.** *Let ψ be a PL function, and, for $i=2,\dots,n$, let $\varphi_i\in\mathrm{PL}\cap\mathrm{PSH}(\beta_i)$. Then*

$$(0,\psi)^2\cdot(\beta_2,\varphi_2)\cdots(\beta_n,\varphi_n)\le 0. \tag{2.3.1.4}$$

Proof. Let $\mathcal{X}$ be a test configuration that dominates $X\times\mathbb{P}^1$ with $D,D_2,\dots,D_n\in\mathrm{VCar}(\mathcal{X})$ such that $\psi=\varphi_D$ and $\varphi_i=\varphi_{D_i}$.

The energy pairing induces a bilinear form on the finite dimension vector space $\mathrm{VCar}_{\mathbb{R}}(\mathcal{X})$ as the map:

$$(G_1,G_2)\mapsto(0,\varphi_{G_1})\cdot(0,\varphi_{G_2})\cdot(\beta_2,\varphi_2)\cdots(\beta_n,\varphi_n).$$

Therefore, we must prove that this bilinear form is negative semidefinite.

The strategy will be to apply Lemma A.3.1. Let (E_i) be the irreducible components of $\mathcal{X}_0$, and note that they form a basis of VCar. Then:

1. For every i,j, $(E_i,E_j)=E_i\cdot E_j\cdot(\beta_2+D_2)\cdots(\beta_n+D_n)$ is non-negative, since β_K+D_k is nef, for $k=2,\dots n$.

2. Consider $\mathcal{X}_0$ as an element of $\mathrm{VCar}(\mathcal{X})$, then $(\mathcal{X}_0,E_i)=\mathcal{X}_0\cdot E_i\cdot(\beta_2+D_2)\cdots(\beta_n+D_n)=0$, since every component is supported on $\mathcal{X}_0$.

Therefore, the bilinear form is negative semi-definite, and the result follows. □

This version of the Zariski Lemma that we have just proved is what allow us to use all the synthetic pluripotential theory of [BJ25a], that will provide us with important a priori estimates for the mixed energy. For more details see Appendix A.4.

Now we will recall some notions from –the above mentioned– synthetic pluripotential theory restricting it to our case.

Recall of some synthetic facts of Boucksom–Jonsson

If we let $\varphi_k \in \mathrm{PL} \cap \mathrm{PSH}(\beta_k)$ for $k = 1, \dots, n-1$, and denote the symbol $(\beta_1, \varphi_1) \cdots (\beta_{n-1}, \varphi_{n-1})$ by Γ, we can associate a semi-norm:

$$\|\psi\|_\Gamma \doteq \sqrt{-(0, \psi)^2 \cdot \Gamma},$$

for $\psi \in \mathrm{PL}_\mathbb{R}$.

Remark 2.3.7. *Since the (positive semi-definite) quadratic form $-(0, \psi)^2 \cdot \Gamma$ comes from a bilinear form we have an associated Cauchy-Schwarz inequality:*

$$|(0, \psi_1) \cdot (0, \psi_2) \cdot \Gamma| \leq \|\psi_1\|_\Gamma \|\psi_2\|_\Gamma,$$

that is the base of the synthetic estimates of [BJ25a].

Definition 2.3.8. *Let $\varphi, \psi \in \mathrm{PL} \cap \mathrm{PSH}(\alpha)$, and denote:*

$$\mathrm{J}_\alpha(\varphi, \psi) \doteq \frac{1}{V_\alpha} \sum_{j=1}^{n} \frac{j}{n+1} \|\varphi - \psi\|_{(\alpha,\varphi)^{j-1} \cdot (\alpha,\psi)^{n-j}}; \tag{2.3.1.5}$$

$$\mathrm{I}_\alpha(\varphi, \psi) \doteq \frac{1}{V_\alpha} \sum_{j=1}^{n} \|\varphi - \psi\|_{(\alpha,\varphi)^{j-1} \cdot (\alpha,\psi)^{n-j}}. \tag{2.3.1.6}$$

Theorem 1.33 of [BJ25a] gives that these functionals define equivalent quasi-metrics.

We also recall the following definition:

Definition 2.3.9. *We define the* Monge–Ampère energy *as the functional* $\mathrm{E}_\alpha \colon \mathrm{PL}(X^\beth) \to \mathbb{R}$ *given by the expression:*

$$\mathrm{PL} \ni \varphi \mapsto \frac{1}{(n+1)V_\alpha} (\alpha, \varphi)^{n+1}.$$

If $\varphi, \psi \in \mathrm{PL}(X^{\beth})$, the variation of E_α is given by:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{E}_\alpha\left(t\psi + (1-t)\varphi\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{1}{(n+1)V_\alpha}(\alpha, t\psi + (1-t)\varphi)^{n+1} \\
&= \frac{1}{V_\omega}(\alpha, \varphi)^n \cdot (0, \psi - \varphi) \qquad (2.3.1.7)\\
&= \int_{X^{\beth}} (\psi - \varphi)\, \mathrm{MA}_\alpha(\varphi),
\end{aligned}$$

justifying its name.

The energy E_α restricts to a concave functional on $\mathrm{PL} \cap \mathrm{PSH}(\alpha)$, and thus for $\varphi, \psi \in \mathrm{PL} \cap \mathrm{PSH}(\alpha)$ we get that:

$$\mathrm{E}_\alpha(\psi) \leq \mathrm{E}_\alpha(\varphi) + \int_{X^{\beth}} (\psi - \varphi)\, \mathrm{MA}_\alpha(\varphi), \qquad (2.3.1.8)$$

and the difference

$$\mathrm{E}_\alpha(\varphi) - \mathrm{E}_\alpha(\psi) + \int_{X^{\beth}} (\psi - \varphi)\, \mathrm{MA}_\alpha(\varphi)$$

coincides with $\mathrm{J}_\alpha(\varphi, \psi)$.

2.3.2 Extending the energy pairing

In this section we will extend the energy pairing to general psh functions. Unlike section 2.3.1, the synthetic approach of Boucksom-Jonsson does not cover this singular case, even though similar generalizations can be done.

Here we will follow closely Section 7 of [BJ22].

Let $\alpha_0, \ldots, \alpha_n \in \mathrm{Pos}(X)$, and $\varphi_i \in \mathrm{PSH}(\alpha_i)$ for $i = 0, \ldots, n$, we define:

$$(\alpha_0, \varphi_0) \cdots (\alpha_n, \varphi_n) \doteq \inf\left\{(\alpha_0, \psi_0) \cdots (\alpha_n, \psi_n) : \psi_i \in \mathcal{H}(\alpha_i), \psi_i \geq \varphi_i\right\}. \qquad (2.3.2.1)$$

Lemma 2.3.10. *The energy pairing,*

$$\begin{aligned}
&\prod_{i=0}^{n} \mathrm{PSH}(\alpha_i) \to \mathbb{R} \cup \{-\infty\} \\
&(\varphi_0, \ldots, \varphi_n) \mapsto (\alpha_0, \varphi_0) \cdots (\alpha_n, \varphi_n),
\end{aligned}$$

is upper semi-continuous.

It is clear that the energy pairing is increasing in each variable. Hence, together with the previous lemma, we conclude that the energy pairing is continuous along decreasing nets.

Proof of Lemma 2.3.10. The proof follows from Corollary 2.1.12. For more details see [BJ22, Theorem 7.1].

□

Just like in the algebraic setting, Corollary 7.11 of [BJ22], we have the following result:

Proposition 2.3.11. *Let $\alpha_0, \ldots, \alpha_n \in \operatorname{Pos}(X)$, and for $i = 0, \ldots, n$ $\varphi_i \in \operatorname{PSH}(\alpha_i)$, with $\varphi_i \leq 0$, then*

$$(\alpha_0, \varphi_0) \cdots (\alpha_n, \varphi_n) \gtrsim t^{n^2} \min_i \left\{(\alpha_i, \varphi_i)^{n+1}\right\} \tag{2.3.2.2}$$

for $t \in \mathbb{R}$ sufficiently large in order to satisfy $\alpha_i \leq t\alpha_j$ for every $i, j \in \{0, \ldots, n\}$.

Proof. Theorem 1.18 of [BJ25a] gives the inequality for $\varphi_i \in \mathrm{PL} \cap \operatorname{PSH}(\alpha_i)$, by taking decreasing sequences we conclude. □

We now extend the Monge–Ampère energy functional to the class of α-psh functions.

Definition 2.3.12. *Let $\alpha \in \operatorname{Pos}(X)$, and $V_\alpha = \int_X \alpha^n$, we define the* Monge–Ampère energy functional *to be*

$$\begin{aligned} \mathrm{E}_\alpha \colon \operatorname{PSH}(\alpha) &\to \mathbb{R} \cup \{-\infty\} \\ \varphi &\mapsto \frac{V_\alpha^{-1}}{n+1} (\alpha, \varphi)^{n+1} \end{aligned}$$

we define the set of finite energy non-archimedean potentials *to be the set*

$$\mathcal{E}^1(\alpha) \doteq \{\varphi \in \operatorname{PSH}(\alpha) : \mathrm{E}_\alpha(\varphi) > -\infty\}$$

when is clear by context we may ommit α.

Moreover,

$$\mathcal{E}^1_{abs} \doteq \bigcup_{\alpha \in \operatorname{Pos}(X)} \mathcal{E}^1(\alpha)$$

As a direct consequence of Proposition 2.3.11, we have:

Proposition 2.3.13. *Let $\alpha_0, \ldots, \alpha_n \in \operatorname{Pos}(X)$, and $\varphi_i \in \mathcal{E}^1(\alpha_i)$, then*

$$(\alpha_0, \varphi_0) \cdots (\alpha_n, \varphi_n) \in \mathbb{R} \tag{2.3.2.3}$$

is finite. □

Remark 2.3.14. *This allow us to extend the J_α and the I_α functionals, defined in Section 2.3.1, to $\mathcal{E}^1(\alpha)$ by the formulas of Equations (2.3.1.5) and (2.3.1.6) respectively.*

Moreover, the quasi-triangular inequality, and quasi-symmetry of J_α for $\mathrm{PL} \cap \operatorname{PSH}(\alpha)$ functions, pass through, taking decreasing limits, to $\mathcal{E}^1(\alpha)$.

We can also see that the pairing of Proposition 2.3.13 is additive on the forms, and hence can be extended by linearity to $H^{1,1}(X)$.

Indeed, fix $\alpha_1, \dots, \alpha_n \in \operatorname{Pos}(X)$ and $\varphi_i \in \mathcal{E}^1(\alpha_i)$, for $i = 1, \dots, n$, finally denote $\Gamma \doteq (\alpha_1, \varphi_1) \cdots (\alpha_n, \varphi_n)$, we then define:

Definition 2.3.15. *Let $\beta \in H^{1,1}(X)$, $\varphi \in \mathcal{E}^1_{abs}$, and $\alpha_0, \tilde{\alpha}_0 \in \operatorname{Pos}(X)$, such that*

$$\beta = \alpha_0 - \tilde{\alpha}_0$$

We define the energy pairing $(\beta, \varphi) \cdot (\alpha_1, \varphi_1) \cdots (\alpha_n, \varphi_n)$ by the formula:

$$(\beta, \varphi) \cdot (\alpha_1, \varphi_1) \cdots (\alpha_n, \varphi_n) \doteq (\alpha_0 + \alpha, \varphi) \cdot \Gamma - (\tilde{\alpha}_0 + \alpha, 0) \cdot \Gamma \tag{2.3.2.4}$$

where $\varphi \in \mathcal{E}^1(\alpha) \subseteq \mathcal{E}^1(\alpha + \alpha_0) \subseteq \mathcal{E}^1_{abs}$, for some $\alpha \in \operatorname{Pos}(X)$.

Similarly, we get the energy pairing defined on $\prod_{k=0}^n \left(H^{1,1}(X) \times \mathcal{E}^1_{abs}\right)$.

In particular, we define another functional whose importance will become apparent in Section 5.4, which will be a twisted version of the Monge–Ampère energy.

Definition 2.3.16. *Let $\alpha \in \operatorname{Pos}(X)$, $V_\alpha = \int_X \alpha^n$, and $\beta \in H^{1,1}(X)$, we define the* Monge–Ampère twisted energy *to be the functional*

$$\begin{aligned} \mathrm{E}^\beta_\alpha \colon \mathcal{E}^1(\alpha) &\to \mathbb{R} \\ \varphi &\mapsto V_\alpha^{-1} (0, \beta) \cdot (\alpha, \varphi)^n. \end{aligned}$$

Remark 2.3.17. *We can also extend the Monge–Ampère operator to the set $\mathcal{E}^1(\alpha)$. We associate to $\varphi \in \mathcal{E}^1(\alpha)$ the probability measure $\mathrm{MA}_\alpha(\varphi)$ satisfying:*

$$\mathrm{PL}(X^\beth) \ni \psi \mapsto \int_{X^\beth} \psi \, \mathrm{MA}_\alpha(\varphi) \doteq \frac{1}{V_\alpha} (0, \psi) \cdot (\alpha, \varphi)^n.$$

2.4 From complex to non-Archimedean geometry

In this section we will develop some connections of the differential geometry of Kähler metrics on X, and the non–Archimedean pluripotential theory on $X^\beth$ that we developed so far. Once again (X, ω) is a compact Kähler manifold, and $\alpha \in H^{1,1}(X)$ the cohomology class defined by ω. We also will only consider smooth test configurations dominating $X \times \mathbb{P}^1$, with snc central fiber.

Basic Kähler Geometry tools

Let I be a coherent ideal of X, and $\beta \in H^{1,1}(X)$, we say that $I \otimes \beta$ is *nef*, if $-F + \beta$ is nef, for $F \subseteq Y$ a log resolution of I, and F the effective divisor induced by I.

Proposition 2.4.1. *Let I be an ideal such that there exists $U \colon X \to \mathbb{R} \cup \{-\infty\}$ a ω-psh function of singularity type I, then $\alpha \otimes I$ is nef.*

Proof. Let $\mu\colon Y \to X$ be a log resolution of I, and $F \subseteq Y$ the effective divisor induced by I. We have, by Siu's decomposition theorem, that:

$$0 \leq \mu^*\omega + \mathrm{dd^c}(U \circ \mu) = \delta_F + T \tag{2.4.0.1}$$

for T a positive current of bounded potential, that is,

$$T = -\eta_F + \mu^*\omega + \mathrm{dd^c}\psi$$

for $\psi \in L^\infty$, and η_F a smooth representative of $\mathrm{c}_1(\mathcal{O}(F))$. Hence

$$\psi \in \mathrm{PSH}(-\eta_F + \mu^*\omega) \cap L^\infty.$$

By a classical result due to Demailly $[-\eta_F + \mu^*\omega]$ is nef. □

Lemma 2.4.2. *Let $D, E \subseteq X$ effective irreducible divisors and $\beta \in H^{1,1}(X)$, such that $\beta - D$ and $\beta - E$ admit a smooth representatives which are semi-positive. Then $\beta \otimes \{\mathcal{O}(-D) + \mathcal{O}(-E)\}$ is nef.*

Proof. Let h_D (h_E resp.) be a smooth metric on $\mathcal{O}_X(D)$ ($\mathcal{O}_X(E)$ resp.) such that the associated curvature θ_D (θ_E resp.) is a smooth form with $\eta - \theta_D$ ($\eta - \theta_E$ resp.) semi-positive, for η a smooth representative of β.

Let s_D be the canonic section of $\mathcal{O}_X(D)$, and s_E of $\mathcal{O}_X(E)$, then $\psi_D \doteq \log|s_D|_{h_d}$ and $\psi_E \doteq \log|s_E|_{h_E}$ are such that:

$$\mathrm{dd^c}\psi_D + \theta_D = [D] \geq 0, \quad \text{and} \quad \mathrm{dd^c}\psi_E + \theta_E = [E] \geq 0,$$

and thus θ_D (θ_E resp.)-psh functions.

Now, since $\eta - \theta_D$ ($\eta - \theta_E$ resp.) is semi-positive, both ψ_D and ψ_E are η-psh. In particular, $\psi \doteq \max\{\psi_D, \psi_E\}$ is η-psh, and has the singularity type of $\mathcal{O}(-D) + \mathcal{O}(-E)$, which by Proposition 2.4.1 implies that $\beta \otimes \{\mathcal{O}(-D) + \mathcal{O}(-E)\}$ is nef. □

An important observation that will be important later is the following relative to global statement:

Theorem 2.4.3. *Let $\mathcal{X}$ be a test configuration for X, $D \in \mathrm{VCar}(\mathcal{X})$ an effective vertical divisor such that*

$$\beta \doteq \alpha + D \text{ is nef relative to } \mathbb{P}^1,$$

then $\beta$is globally nef in $\mathcal{X}$.

Proof. Let γ be a Kähler class on $\mathcal{X}$ and $Z \subseteq \mathcal{X}$ a d-dimensional subvariety, we want to check that

$$\beta \cdot \gamma^{d-1} \cdot [Z] = [D + \alpha] \cdot \gamma^{d-1} \cdot [Z] \geq 0.$$

We have two options:

1. Either Z is contained in a fiber, $Z \subseteq \mathcal{X}_t$ for some $t \in \mathbb{P}^1$;
2. or Z is contained in none, $\forall\, t \in \mathbb{P}^1$ $Z \nsubseteq \mathcal{X}_t$.

In the first case, by the relative nefness of β and the numerical criterion applied on the fiber $\mathcal{X}_t$, we have $\beta \cdot \gamma^{d-1} \cdot [Z] \geq 0$.

In the second case, the key observation is that we can write β as the sum of an effective divisor with a nef class. By the numerical criterion α –the nef part of β– has non-negative intersection with $\gamma^{d-1} \cdot [Z]$. Hence, it is enough to check that the intersection with the effective part is non-negative, i.e. that

$$[D] \cdot [\gamma]^{d-1} \cdot [Z] \geq 0.$$

Since, Z is not contained in any fiber, in particular it is no contained in the support of D, therefore the intersection $D \cdot Z$ is an effective $d-1$-cycle, which implies that

$$[D] \cdot [Z] \cdot [\gamma]^{d-1} \geq 0.$$

□

2.4.1 Geodesic rays and non-archimedean psh functions

The goal of this section is to get the analogues of Theorem 6.2 and Theorem 6.6 from [BBJ21] in our transcendental setting. These results are essential for the non-Archimedean approach for the YTD conjecture developed by Berman–Boucksom–Jonsson, of which [Li22] and the present paper rely on.

Remember that we have fixed a Kähler form ω on X, and its cohomology class $\alpha = [\omega]$.

Quick recall on geodesic rays

In this section we will use the conventions of [BBJ21].

We define a *psh ray* as a map $U\colon \mathbb{R}_{\geq 0} \to \mathrm{PSH}(\omega)$ such that the associated S^1-invariant function,

$$U\colon X \times \mathbb{D}^* \to [-\infty, +\infty[\,, \qquad U(x,\tau) \doteq U_{-\log|\tau|}, \tag{2.4.1.1}$$

is $p_1^*\omega$-psh.

Whenever a psh ray has image in $\mathcal{E}^1(\omega)$ and $t \mapsto \mathrm{E}_\omega(U_t)$ is affine, we say that U is a *psh geodesic ray*.

Moreover, a psh ray U has *linear growth*, if there exist $C, D > 0$ such that:

$$U_t \leq C\,t + D.$$

Every psh geodesic has linear growth, cf. [BBJ21, Proposition 4.1].

Remark 2.4.4. *Darvas proves in [Dar17, Theorem 2] that psh geodesic rays are –a distinguished class of– actual geodesic rays for the Darvas metric d_1, and in [Dar15] that for U_0 and U_1 finite energy potentials, there always exists a psh geodesic joining them.*

We will study now the relationship between –archimedean– rays of functions on X, with non-archimedean functions on $X^{\beth}$.

Definition 2.4.5. *A S^1-invariant function, $U\colon X\times\mathbb{D}^*\to\mathbb{R}\cup\{-\infty\}$, is C^∞ (resp. L^∞)-compatible with $D\in\mathrm{VCar}(\mathcal{X})$, for $\mathcal{X}$ a μ-dominating test configuration, if:*

$$U\circ\mu+\log|f_D| \text{ locally extends to a smooth (resp. bounded) function (across } \mathcal{X}_0\text{)},$$

for f_D a local equation of D.

Furthermore, if U is a compatible (either smoothly, or boundedly) with the vertical divisor D we write:

$$U^{\beth}\doteq\varphi_D. \tag{2.4.1.2}$$

Using this new terminology, we adapt Proposition 2.4.1 to this language.

Lemma 2.4.6. *Let U be a ω-psh ray L^∞-compatible with a vertical divisor $D\in\mathrm{VCar}(\mathcal{X})$, then $U^{\beth}=\varphi_D$ is α-psh.*

Proof. We first observe that we can suppose D effective, otherwise consider

$$\varphi_{D+c\mathcal{X}_0}=\varphi_D+c$$

for $c\gg 0$, that is α-psh iff φ_D is. Let $\mu\colon\mathcal{X}\to X\times\mathbb{P}^1$ be a morphism of test configurations, then by Siu's decomposition formula we have:

$$0\leq\omega_{\mathcal{X}}+\mathrm{dd^c}(U\circ\mu)=-\delta_D+T,$$

for T a positive current of bounded potential, that is

$$0\leq T=\eta_D+\omega_{\mathcal{X}}+\mathrm{dd^c}\psi$$

with $\psi\in L^\infty$, and η a smooth representative of $c_1(\mathcal{O}_{\mathcal{X}}(D))$.

Consider the irreducible decomposition $\mathcal{X}_0=\sum b_kE_k$, since T is of bounded potential, we can restrict T to a bounded positive current supported on E_k:

$$0\leq T|_{E_k}$$

And hence by a result of Demailly we have:

$$[T|_{E_k}] = [T]|_{E_k} = (D + [\omega_{\mathcal{X}}])|_{E_k} \text{ is nef.}$$

Therefore $U^{\beth}$ is α-psh. $\square$

More generally, for any $U\colon \mathbb{R}_{\geq 0} \to \mathrm{PSH}(\omega)$ psh ray of linear growth there is an induced "non-archimedean" map:

$$U^{\mathrm{NA}}\colon X^{\mathrm{div}} \to \mathbb{R}$$

given by the following procedure:

1. Let $E \subseteq \mathcal{X} \xrightarrow{\mu} X \times \mathbb{P}^1$ be a prime vertical divisor.

2. Consider the function $V \doteq U \circ \mu\colon \mu^{-1}(X \times \mathbb{D}^*) \to [-\infty, +\infty[$, where U is like in Equation 2.4.1.1.

3. Define

$$U^{\mathrm{NA}}(v_E) \doteq -\nu(V, E)$$

where ν denotes the generic Lelong number along E.

The goal of the next result is to extend the above construction of U^{NA} to an α-psh function on $X^{\beth}$, generalizing Lemma 2.4.6 for a more general singularity type. This result is an analogue of Theorem 6.2 of [BBJ21], the proof here follows the same strategy as in [BBJ21] but we use directly a regularization result of Demailly, [Dem92, Proposition 3.7], without passing by the Castelnuovo-Mumford criterion of global generation (remember that in the projective case $\alpha = c_1(L)$).

Theorem 2.4.7. *Let $U\colon \mathbb{R}_{>0} \to \mathrm{PSH}(\omega)$ be a psh ray of linear growth, then*

$$U^{\mathrm{NA}}\colon X^{\mathrm{div}} \to \mathbb{R}$$

extends to a α-psh function

$$U^{\beth}\colon X^{\beth} \to \mathbb{R} \cup \{-\infty\}.$$

Remark 2.4.8. *By Theorem 2.1.15 if such an extension exists it is unique.*

Proof of Theorem 2.4.7. We will show that there exists a sequence $\varphi_m \in \mathcal{H}(\alpha)$ such that:

1. $(\varphi_m)_m$ is decreasing

2. $\varphi_m(v_E) \searrow U^{\mathrm{NA}}(v_E)$ for every $v_E \in (X \times \mathbb{P}^1)^{\mathrm{div}}_{\mathbb{C}^*}$

Hence $U^{\beth}(v) \doteq \lim \varphi_m(v)$ will be the desired function.

By [Dem92, Proposition 3.7], for every $\lambda > 1$, there is a sequence of S^1-invariant functions:

$$V_m \colon X \times \mathbb{D}_{1-\delta} \subseteq X \times \mathbb{D} \to \mathbb{R} \cup \{-\infty\},$$

that are $(\lambda \cdot \omega)$-psh and of analytic singularities of type $\mathcal{J}(mU)^{\frac{1}{m}}$, where $\mathbb{D}_{1-\delta}$ the disk of radius $1-\delta$.

Therefore, $U_m \doteq \max\{V_m, \log|\tau|\}$ is $(\lambda \cdot \omega)$-psh, and has analytic singularities of type $(\mathfrak{a}_m)^{\frac{1}{m}}$ for $\mathfrak{a}_m \doteq \mathcal{J}(mU) \cdot \mathcal{O}_{X \times \mathbb{P}^1} + (t^m)$, a flag ideal.

Moreover, if we let $\mu_m \colon \mathcal{X}_m \to X \times \mathbb{P}^1$ be the test configuration given by the normalized blow-up of $X \times \mathbb{P}^1$ along $\mathfrak{a}_m$, and $E_m \subseteq \mathcal{X}_m$ be the exceptional divisor, then the function $U_m \circ \mu_m$:

1. is $\mu_m^*(\lambda \cdot \omega)$-psh;
2. has divisorial singularities along $\frac{1}{m}E_m$.

Hence, U_m is a psh ray L^∞-compatible with $\frac{1}{m}E_m$, and by Lemma 2.4.6

$$\varphi_m \doteq U_m^{\beth} = \varphi_{\mathfrak{a}_m}$$

is $(\lambda \cdot \alpha)$-psh.

The item (ii) of Proposition 3.7 of [Dem92] gives us that the Lelong numbers of V_m along divisors over the central fiber $X \times \{0\}$ approach the Lelong numbers of U over the same divisors, in particular, the Lelong numbers of U_m have the same property. Thus in non-archimedean terms:

$$\varphi_m|_{X^{\mathrm{div}}} \to U^{\mathrm{NA}}.$$

Moreover, by the subadditivity of multiplier ideals –like in [BBJ21, Lemma 5.7]– the sequence $(\varphi_{2^m})_m$ is decreasing. Applying Lemma 2.1.7 we conclude the proof. □

Now, we will prove a result in the converse direction of the above theorem. For that we remember the following definition:

Definition 2.4.9. *A psh geodesic ray U in* $\mathrm{PSH}(\omega)$ *is* maximal *if for every other geodesic ray V with $U_0 \geq V_0$ and $U^{\beth} \geq V^{\beth}$ we have $U_s \geq V_s$ for every $s \in \mathbb{R}_{\geq 0}$.*

Maximal geodesic rays are in correspondence with the non-archimedean potentials of finite energy, as we will see on Section 4.3, where we prove an analogue of [BBJ21, Theorem 6.6] in the Kähler setting. Next, we prove a partial result in this direction:

Lemma 2.4.10. *Let $\varphi \in \mathcal{H}(\alpha)$, and $\mathcal{X}$ a smooth dominating test configuration with $D \in$* $\mathrm{VCar}(\mathcal{X})$ *such that $\varphi = \varphi_D$, and*

$$D + \alpha_{\mathcal{X}} \text{ is Kähler relatively to } \mathbb{P}^1.$$

Then,

(i) there exists a psh ray, starting from $u \in \mathcal{H}(\omega)$, which is C^∞-compatible with $(\mathcal{X}, D)$.

(ii) The envelope usc of rays like in (i), is a maximal psh geodesic and is L^∞-compatible with $(\mathcal{X}, D)$.

Proof. For (i) see [SD18, Lemma 4.4]. For (ii) we observe that in the terminology of [Ber16, Proposition 2.7], a positively curved metric ϕ on a test configuration $(\mathcal{X}, \mathcal{L})$, $\mathcal{L} = L_{\mathcal{X}} + D$, it is a psh ray, being locally bounded it is equivalent to ours L^∞-compatibility with $(\mathcal{X}, D)$, and

$$(\mathrm{dd^c}\varphi)^{n+1} = 0$$

is equivalent –under the positivity condition– of being a psh geodesic. Translating to our language their proof follows with no change. □

Remark 2.4.11. *Let $\varphi \in \mathcal{H}(\alpha)$, and U a maximal psh geodesic s.t. $U^\daleth = \varphi$, like in the previous lemma. Then by [SD18, Remark 4.11] we have that:*

$$\mathrm{E}_\omega(U_t) = \mathrm{E}_\omega(U_0) + t \cdot \mathrm{E}_\alpha(\varphi).$$

Moreover, we now prove an analogue of Theorem 6.4 of [BBJ21].

Theorem 2.4.12. *Let $U : \mathbb{R}_{>0} \to \mathcal{E}^1(\omega)$ be a psh ray of linear growth, then $U^\daleth$ is a finite energy potential.*

Proof. Since $U^\daleth$ is α-psh we can find a sequence $\varphi_j \in \mathcal{H}(\alpha)$ decreasing to $U^\daleth$. Let U_j be a maximal geodesic ray associated to φ_j.

By maximality we must have that $U_s \leq U_{j,s}$. And thus $\mathrm{E}_\omega(U_s) \leq \mathrm{E}_\omega(U_{j,s}) = \mathrm{E}_\omega(U_{j,0}) + t\mathrm{E}_\alpha(\varphi_j)$ by Remark 2.4.11.

Therefore,

$$\lim_{s \to +\infty} \frac{\mathrm{E}_\omega(U_s)}{s} \leq \mathrm{E}_\alpha(\varphi_j),$$

for every j. We observe that the left hand side limit exists for the convexity of $s \mapsto \mathrm{E}_\omega(U_s)$, which itself is a consequence of U being a psh ray. Thus, $\lim_{s\to+\infty} \frac{\mathrm{E}_\omega(U_s)}{s} \leq \mathrm{E}_\alpha(U^\daleth)$, that concludes our proof. □

2.4.2 Comparison with Darvas–Xia–Zhang non-archimedean metrics

On [DXZ25] and [Xia25] the authors develop a theory of non-archimedean plurisubharmonic functions attached to a compact Kähler manifold.

Their approach is to define a non-archimedean α-psh function as a *test curve* on the manifold X. Test curves are the Ross–Witt Nyström transforms of maximal geodesic rays.

Following the strategy of [DXZ25, Theorem 3.17], together with Theorem 4.3.1 ahead, for $\beta \in \mathrm{Pos}(X)$ a Kähler class, one can gets a correspondence from their β-psh functions of finite energy to ours associating to every $\mathcal{I}$-maximal test curve, ψ_τ, the "beth" of its Ross–Witt Nyström transform $(\check{\psi}_t)$:

$$\mathcal{R}^1_{\mathcal{I}}(\theta) \ni (\psi_\tau) \mapsto (\check{\psi})^{\beth} \in \mathcal{E}^1(\beta)$$

for a smooth Kähler representative θ of β.

Remark 2.4.13. *As mentioned before, their theory remains more general since they can consider the case when β is big. On the other hand, our theory is a direct analogue of the algebraic setting, which for instance enables us to associate a Monge–Ampère measure to a β-psh function.*

The comparison of β-psh functions –without the energy assumption– is more delicate even in the algebraic case, and we refer to [DXZ25, Theorem 3.14] for more details.

2.4.3 Asymptotics for the mixed energy

The goal of this section is to prove Theorem 2.4.17, which will be of central importance to relate the variational cscK problem with non-archimedean geometry. We begin with an useful lemma.

Lemma 2.4.14. *Let U_s be a maximal geodesic ray, and $\varphi \in \mathcal{E}^1(\alpha)$ the associated non-Archimedean potential of finite energy. Then*

$$\frac{V_\omega^{-1}}{s}\int_X U_s\,\omega^n \xrightarrow{s\to\infty} \varphi(v_{\mathrm{triv}}) = \int_{X^\beth} \varphi\,\mathrm{MA}_\alpha(0).$$

Proof. Observe that, since $0 \le \sup U_s - V_\omega^{-1}\int_X U_s\,\omega^n \le T_\omega$, we are left to prove that $\frac{\sup U_s}{s} \to \varphi(v_{\mathrm{triv}}) = \sup\varphi$. Let $\varphi_m \in \mathcal{H}(\alpha)$ be a decreasing sequence converging to φ, and U^m_s the associated maximal geodesic rays. Let's assume for simlplicity that $U_0 = 0 = U^m_0$, by [BBJ21, Proposition 1.10] we have that:

$$\sup U_s = \ell\cdot s,\quad \sup U^m_s = \ell_m\cdot s,$$

for some real number $\ell \in \mathbb{R}$.

By Theorem B of [SD18], it follows that $\ell_m = \sup\varphi_m = \varphi_m(v_{\mathrm{triv}})$, hence

$$\ell = \frac{\sup U_s}{s} \swarrow \frac{\sup U^m_s}{s} = \varphi_m(v_{\mathrm{triv}}) \searrow \varphi(v_{\mathrm{triv}}),$$

concluding the proof. □

We will now recall Theorem 3.6 from [BJ25a] that will be useful later. Here we will state only the complex analytic version of the theorem.

Lemma 2.4.15. *Let $\eta_0, \dots, \eta_n$ be smooth closed (1,1)-forms, and for $i = 0, \dots n$ consider $U_i, V_i \in \mathcal{E}^1(\omega)$ normalized for $\int U_i\omega^n = 0 = \int V_i\omega^n$, then*

$$|(\eta_0, U_0)\cdots(\eta_n, U_n) - (\eta_0, V_0)\cdots(\eta_n, V_n)| \lesssim A\left(\max_i \mathrm{J}_\omega(U_i, V_i)^q \cdot \max_i \{\mathrm{J}_\omega(U_i) + T_\omega\}^{1-q}\right),$$

for

$$q \doteq 2^{-n}, \quad A \doteq V_\omega \prod_i (1 + 2\|\eta_i\|_\omega), \quad \text{and} \quad T_\omega \doteq \sup_{f \in \mathrm{PSH}(\omega)\cap C^\infty} \left\{\sup f - V_\omega^{-1}\int f\,\omega^n\right\}.$$

Remark 2.4.16. *The quantity T_ω is well known to be finite, see for instance [BJ25a, Theorem 1.26].*

Proof of Lemma 2.4.15. Whenever U^i, V^i are smooth functions the result follows from [BJ25a, Theorem 3.6].

In the general case, it suffices to take decreasing sequences of smooth potentials converging to U^i and V^i respectively, and to observe that the bound proved for smooth functions is uniform. Thus, since the energy pairing is continuous along decreasing sequences, taking limits on both sides of the inequality we conclude.

□

The following statement is a generalization to singular metrics of [SD18, Theorem B], and of [DR17, Theorem 4.15] -that after a small modification can be adapted to general pairings-, and to more general functionals of [Li22, Theorem 4.1] –where they only consider the twisted Monge–Ampère energy estimate. It is the key ingredient to relate the non-archimedean pluripotential theory with the complex analytic one, which will be essential to prove Theorem 5.4.4.

For the next theorem, let $\eta_0, \dots, \eta_k$ be smooth closed forms.

Theorem 2.4.17 (Slope Formula)**.** *Let $\varphi_0, \dots, \varphi_k \in \mathrm{PL}$, and $\varphi_{k+1}, \dots, \varphi_n \in \mathcal{E}^1(\alpha)$, and U_i a smooth ray C^∞-compatible with φ_i if $i \leq k$, and a maximal geodesic ray compatible with φ_i if $i > k$, we have:*

$$\frac{1}{s}(\eta_0, U_{0,s})\cdots(\eta_n, U_{n,s}) \overset{s\to\infty}{\longrightarrow} ([\eta_0], \varphi_0)\cdots([\eta_n], \varphi_n).$$

Proof. When $k = n$ the result corresponds to [SD18, Theorem B]. We restrict ourselves to the case when $k = n-1$ the general case, when $k \leq n-1$, will be similar.

Moreover, we observe that we can suppose that $\eta_n = \omega$, otherwise

$$(\eta_n, U_{n,s}) \cdot \Gamma_s = (\omega, U_{n,s}) \cdot \Gamma_s - (\omega - \eta_n, 0) \cdot \Gamma_s$$

where $\Gamma_s \doteq (\omega_0, U_{0,s}) \cdots (\omega_{n-1}, U_{n-1,s})$. Therefore, denoting $[\Gamma] \doteq ([\eta_0], \varphi_0) \cdots ([\eta_{n-1}], \varphi_{n-1})$, and applying the result for $\eta_n = \omega$, we have:

$$\begin{aligned}\frac{1}{s}(\eta_n, U_{n,s}) \cdot \Gamma_s &\to (\alpha, \varphi_n) \cdot [\Gamma] - ([\omega - \eta], 0) \cdot [\Gamma]\\ &= ([\eta_n], \varphi_n) \cdot [\Gamma],\end{aligned}$$

which, by symmetry, implies the result.

Now, let's prove the result. Let $\varphi_n^m \searrow \varphi_n$ be a decreasing sequence of functions in $\mathcal{H}(\alpha)$, and U_n^m the associated maximal geodesic ray.

By [SD18, Theorem B]:

$$\frac{1}{s}(\eta_0, U_{0,s}) \cdots (\eta_{n-1}, U_{n-1,s}) \cdot (\omega, U_{n,s}^m) \overset{s\to\infty}{\longrightarrow} ([\eta_0], \varphi_0) \cdots ([\eta_{n-1}], \varphi_{n-1})(\alpha, \varphi_n^m), \quad (2.4.3.1)$$

and as $m \to \infty$ the right hand side converges to $([\eta_0], \varphi_0) \cdots (\alpha, \varphi_n)$. Thus, to complete the proof we need to check that

$$\lim_{m\to\infty} \lim_{s\to\infty} \frac{1}{s}(\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}^m) = \lim_{s\to\infty} \frac{1}{s}(\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}).$$

What we will do next is then to study the difference:

$$\begin{aligned}(\star)_s &\doteq |(\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}) - (\eta_0, U_{0,s}) \cdots (\omega, U_{n,s}^m)|\\ &= |(\eta_0, U_{0,s}) \cdots (0, U_{n,s} - U_{n,s}^m)|.\end{aligned}$$

Denoting by $a_{i,s}$ and $a_{n,s}^m$ the averages $\int_X U_{i,s}\,\omega^n$ and $\int_X U_{i,s}^m\,\omega^n$ respectively, and by $V_{i,s}$ and $V_{n,s}^m$ the normalized potentials

$$U_{i,s} - a_{i,s}, \qquad U_{n,s}^m - a_{n,s}^m,$$

we observe that:

$$(\star)_s = |(\eta_0, V_{0,s}) \cdots (\eta_{n-1}, V_{n-1,s}) \cdot (0, U_{n,s} - U_{n,s}^m)|$$

and by the triangle inequality it follows that:

$$\begin{aligned}(\star)_s &\le |(\eta_0, V_{0,s}) \cdots (0, V_{n,s} - V_{n,s}^m)| + |(\eta_0, V_{0,s}) \cdots (0, a_{n,s}^m - a_{n,s})|\\ &= |(\eta_0, V_{0,s}) \cdots (0, V_{n,s} - V_{n,s}^m)| + a_{n,s}^m - a_{n,s}.\end{aligned}$$

By the Lemma 2.4.14 we have that taking the slope at infinity and letting m tend to infinity the second term vanishes. Next we will focus our attention on the first term.

Note that we can suppose that, for $i \le n-1$, φ_i is in $\mathcal{H}(\alpha)$ and $U_{i,s}$ is a smooth psh ray C^∞-compatible with φ_i. If it is not the case we write $\varphi_i = \psi_i' - \psi_i''$ the difference of

$\mathcal{H}(\alpha)$ functions, then we consider U_i' and U_i'' psh-rays that are smoothly compatible with ψ_i' and ψ_i'' respectively, and the difference

$$U_{i,s} \doteq U_s' - U_s''$$

will be smoothly compatible with φ_i, and the result follows from linearity of the pairing.

Consequently, it follows, by Lemma 2.4.15, that, for $q = 2^{-n}$:

$$\begin{aligned} |(\eta_0, V_{0,s}) \cdots (\omega, V_{n,s}) - (\eta_0, V_{0,s}) \cdots \big(\omega, V_{n,s}^m\big)| &\lesssim \mathrm{J}_\omega(V_{n,s}, V_{n,s}^m)^q \cdot \max_i \{\mathrm{J}_\omega(V_{i,s}) + T_\omega\}^{1-q} \\ &= \mathrm{J}_\omega(U_{n,s}, U_{n,s}^m)^q \cdot \max_i \{\mathrm{J}_\omega(U_{i,s}) + T_\omega\}^{1-q} \\ &\lesssim d_1(U_{n,s}, U_{n,s}^m)^q \cdot (s + T_\omega)^{1-q}, \end{aligned}$$

where the equality follows from the constant invariance of the J functional, and the last inequality by linear growth of U_i, since it implies that $\mathrm{J}(U_{i,s}) \lesssim s$.

Moreover, we observe that:

By maximality $U_{n,s} \le U_{n,s}^m$, and

$$\begin{aligned} d_1(U_{n,s}, U_{n,s}^m) &= \mathrm{E}_\omega(U_{n,s}^m) - \mathrm{E}_\omega(U_{n,s}) \\ &= \big(\mathrm{E}_\omega(U_{n,1}^m) - \mathrm{E}_\omega(U_{n,0}^m)\big)\, s - (\mathrm{E}_\omega(U_{n,1}) - \mathrm{E}_\omega(U_{n,0}))\, s + C_m \\ &= \big(\mathrm{E}_\omega(U_{n,1}^m) - \mathrm{E}_\omega(U_{n,1})\big)\, s - \big(\mathrm{E}_\omega(U_{n,0}^m) - \mathrm{E}_\omega(U_{n,0})\big)\, s + C_m, \end{aligned}$$

for $C_m \doteq \mathrm{E}_\omega(U_{n,0}^m) - \mathrm{E}_\omega(U_{n,0})$.

Therefore, taking the slope at infinity, and letting m tend to infinity we have the desired result. □

We have already seen the non-archimedean version of some classical functionals arising from pluripotential theory. We recall their archimedean –original– version. If $\omega \in \mathcal{K}(X)$ is a Kähler form, and η is any closed $(1,1)$-form we have for $u \in \mathcal{E}^1(\omega)$:

$$\begin{aligned} \mathrm{E}_\omega(u) &\doteq \frac{1}{n+1} V_\omega^{-1} (\omega, u)^{n+1} \\ \mathrm{E}_\omega^\eta(u) &\doteq V_\omega^{-1}(\eta, 0) \cdot (\omega, u)^n \\ \mathrm{J}_\omega(u) &\doteq V_\omega^{-1}(\omega, u) \cdot (\omega, 0)^n - \mathrm{E}_\omega(u). \end{aligned}$$

By Theorem 2.4.17, we can relate the above functionals with their non-archimedean counterpart. If U a maximal geodesic ray and $\varphi \in \mathcal{E}^1(\alpha)$ the associated non-Archimedean potential, we have:

$$\lim_{s\to\infty} \frac{\mathrm{E}_\omega(U_s)}{s} = \mathrm{E}_\alpha(\varphi), \quad \lim_{s\to\infty} \frac{\mathrm{E}_\omega^\eta(U_s)}{s} = \mathrm{E}_\alpha^\beta(\varphi), \text{ and } \quad \lim_{s\to\infty} \frac{\mathrm{J}_\omega(U_s)}{s} = \mathrm{J}_\alpha(\varphi),$$

for $\beta = [\eta]$.

Chapter 3

Currents, volumes and non-Archimedean envelopes

In this chapter we will push the non-Archimedean pluripotential theory further by applying results of Section 2.4, and by establishing new connections between the non-Archimedean pluripotential theory of $X^{\beth}$ and the study of currents on X and on test configurations $\mathcal{X}$.

3.1 Continuity of envelopes

We recall again that X is a compact Kähler manifold and ω a Kähler form of class α.

The goal of this section is to prove Theorem 3.1.1 below, commonly known as the Continuity of Envelopes Property, which says that if f is a continuous function on $X^{\beth}$, then the α-psh envelope is continuous and α-psh.

Theorem 3.1.1. *Let $f \in \mathrm{C}^0(X^{\beth})$ then the function:*

$$P_\alpha(f) \doteq \sup \{\varphi \in \mathrm{PSH}(\alpha) \mid \varphi \leq f\}$$

is continuous and α-psh.

To prove it we will rely on the key results of Section 2.4. The key ingredient is the fact that to each ω-psh ray of linear growth U, the associated function $U^{\mathrm{NA}} : X^{\mathrm{div}} \to \mathbb{R}$, can be extended to a α-psh function on $X^{\beth}$.

To begin, if f is a function on $X^{\beth}$ we let $U_{\leq f}$ be defined as the usc regularization of the supremum of all psh rays $U \leq 0$ such that $U^{\beth} \leq f$. It follows from standard pluripotential theory results that $U_{\leq f}$ itself is a psh ray.

Proposition 3.1.2. *If $f \leq 0$ is usc, then $(U_{\leq f})^{\beth} \leq f$, and for $\varphi \in \mathcal{H}(\alpha)$ with $\varphi \leq 0$ we get that $U_{\leq \varphi}$ is the maximal geodesic ray associated to φ .*

Proof. By Theorem 2.1.15 it is enough to check that $(U_{\leq f})^{\beth}|_{X^{\mathrm{div}}} \leq f|_{X^{\mathrm{div}}}$.

Let $v_{E_j} \in X^{\mathrm{div}}$ and let $(U_i)_{i\in I}$ be the family of all psh rays such that $U_i \leq 0$ and $U_i^{\beth} \leq f$. We then have that

$$(U_{\leq f})^{\beth}(v_E) = -b_E^{-1}\nu_E(U_{\leq f}) = -b_E^{-1}\inf_i \nu_E(U_i) = \sup_i U_i^{\beth}(v_E) \leq f(v_E),$$

where the second equality is given classical pluripotential theory. If $f = \varphi \in \mathcal{H}(\alpha)$ then by Lemma 2.4.10 there is a U_i with $U_i^{\beth} = \varphi$, which shows that $U_{\leq\varphi} = \varphi$. □

Proof of Theorem 3.1.1. We start by showing that $P_\alpha(f)$ is lower semicontinuous. Here we follow the argument in [BJ22, Lemma 5.17].

Let ψ be α-psh such that $\psi \leq f$, and let φ_i be a net in $\mathcal{H}(\alpha)$ which decreases to ψ. Then since $X^{\beth}$ is compact and f is continuous, for every $\epsilon > 0$ we can find a φ_i such that $\varphi_i - \epsilon \leq f$. This shows that

$$P_\alpha(f) = \sup\{\varphi \in \mathcal{H}(\alpha) \mid \varphi \leq f\},$$

and hence $P_\alpha(f)$, being the supremum of continuous functions, is lower semicontinuous.

Thus we are left to prove that $P_\alpha(f)$ is α-psh, since it is then also upper semicontinuous.

Let $U_{\leq f}$ be the psh ray defined above. By Theorem 2.4.7 we have that $(U_{\leq f})^{\beth} \in \mathrm{PSH}(\alpha)$ and $(U_{\leq f})^{\beth} \leq f$. This means that $(U_{\leq f})^{\beth}$ is a candidate for the envelope $P_\alpha(f)$ and thus $(U_{\leq f})^{\beth} \leq P_\alpha(f)$.

We now want to prove the reverse inequality, as that would show that $P_\alpha(f) = (U_{\leq f})^{\beth} \in \mathrm{PSH}(\alpha)$.

Let $\varphi \in \mathcal{H}(\alpha)$ be such that $\varphi \leq f \leq 0$. By Proposition 3.1.2 $(U_{\leq\varphi})^{\beth} = \varphi \leq f$, and thus $U_{\leq\varphi}$ is a candidate for the envelope $U_{\leq f}$. Hence $U_{\leq\varphi} \leq U_{\leq f}$, and by the monotonicity of Lelong numbers we get that $\varphi = (U_{\leq\varphi})^{\beth} \leq (U_{\leq f})^{\beth}$ on X^{div}. Since $(U_{\leq f})^{\beth} - \varphi$ is usc the inequality extends to $X^{\beth}$. This finally proves that $P_\alpha(f) \leq (U_{\leq f})^{\beth}$, and hence that $P_\alpha(f) = (U_{\leq f})^{\beth} \in \mathrm{PSH}(\alpha)$. □

Let us now see a simple consequence of the continuity of envelopes. The next result follows from the convexity of psh functions restricted to dual complexes, together with Lemma 2.2.1, and will be useful later in Section 4.5.4. In the algebraic setting this can be found in [BFJ15, Lemma 8.5], we follow their proof.

Lemma 3.1.3. *Let $\{v_1, \ldots, v_\ell\} \subseteq X^{\mathrm{div}}$ be a set of divisorial points, and $t \in \mathbb{R}^\ell$. Then there exists a PL function $f \in \mathrm{PL}$ such that*

$$P_\alpha(f) = \sup\{\varphi \in \mathrm{PSH}(\alpha) \mid \varphi(v_i) \leq t_i\}.$$

Furthermore, if $\Delta_{\mathcal{X}} \hookrightarrow X^{\beth}$ is a dual complex containing the set $\{v_1, \ldots, v_\ell\}$ as vertices, then f can be taken to be φ_D, for the vertical divisor $D = \sum_i t_i E_i$, where $v_{E_i} = v_i$.

Proof. It is clear that if $\varphi \in \mathrm{PSH}(\alpha)$ satisfies $\varphi \leq f = \varphi_D$ as above, then $\varphi(v_i) \leq t_i$. Moreover, if $\varphi \circ p_{\mathcal{X}}(v_i) = \varphi(v_i) \leq t_i$ by convexity of $\varphi \circ p_{\mathcal{X}}$, we have

$$\varphi \circ p_{\mathcal{X}} \leq f \circ p_{\mathcal{X}} = f,$$

which by Lemma 2.2.1 implies $\varphi \leq f$, concluding the proof. □

We will now prove a different pluripotential theoretic property that will be very relevant for what comes next, cf. Chapter 4

3.1.1 Stating Orthogonality

Let $f \in \mathrm{C}^0(X^{\beth})$ and consider the envelope

$$P_\alpha(f) \doteq \sup \{\varphi \leq f \mid \varphi \in \mathrm{PSH}(\alpha)\}.$$

By the continuity of envelopes we have:

$$P_\alpha(f) = \sup \{\varphi \leq f \mid \varphi \in \mathcal{H}(\alpha)\} \in \mathrm{CPSH}(\alpha).$$

The goal of this section is to prove *orthogonality*, that is to show that for every PL function $f \in \mathrm{PL}$:

$$O_\alpha(f) \doteq \int_{X^{\beth}} (P_\alpha(f) - f)(\alpha + \mathrm{dd^c} P_\alpha(f))^n = 0. \tag{3.1.1.1}$$

Since O_α is invariant under translation by constants, it is enough to prove (3.1.1.1) for $f \in \mathrm{PL}_{>0}$, that is when $f > 0$.

Let $f \in \mathrm{PL}_{>0}(X^{\beth})$ be a PL function, and $\mathcal{X}$ a smooth dominating test configuration together with an effective vertical divisor $D \in \mathrm{VCar}(\mathcal{X})$ such that

$$f = \varphi_D. \tag{3.1.1.2}$$

Moreover, we denote $\beta = D + \alpha$ and observe that

$$\begin{aligned} P_\alpha(f) &= f + \sup \left\{\varphi_E \mid E \leq 0,\ \text{and } E + D + \alpha \text{ nef rel. to } \mathbb{P}^1\right\} \\ &= f + P_\beta(0), \end{aligned} \tag{3.1.1.3}$$

and thus:

$$O_\alpha(f) = \int_{X^{\beth}} P_\beta(0)\, \mathrm{MA}(P_\alpha(f)).$$

As a consequence of Theorem 2.4.3, we get the following result:

Corollary 3.1.4. *Let β be as before. Then:*

$$P_\beta(0) = \sup \{\varphi_E \mid E \leq 0,\ \textit{and } E + \beta \textit{ nef}\}.$$

Proof. Clearly

$$P_\beta(0) \geq \sup\{\varphi_E \mid E \leq 0, \text{ and } E+\beta \text{ nef}\},$$

moreover, by (3.1.1.3), we have that

$$P_\beta(0) = P_\alpha(\varphi_D) - \varphi_D.$$

Since D is effective, it follows that the constant function zero is a candidate in the envelope $0 \leq P_\alpha(\varphi_D)$, and thus we may find a maximizing sequence $\psi_i \in \mathcal{H}(\alpha)$ such that $\psi_i \geq 0$, and $\psi_i \nearrow P_\alpha(\varphi_D)$. If we let G_i be effective vertical divisors defining ψ_i, then we have that $G_i - D$ is anti-effective, and $G_i - D + \beta = G_i + \alpha$ is nef relative to $\mathbb{P}^1$, which by Theorem 2.4.3 implies that $G_i - D + \beta$ is globally nef. Hence,

$$\psi_i - \varphi_D \leq \sup\{\varphi_E \mid E \leq 0, \text{ and } E+\beta \text{ nef}\}.$$

Letting i tend to infinity we obtain:

$$P_\beta(0) \leq \sup\{\varphi_E \mid E \leq 0, \text{ and } E+\beta \text{ nef}\}.$$

□

3.2 Big classes: volumes and valuations

We have seen in Section 2.4 that a psh ray of linear growth induces a non-Archimedean psh function by looking at Lelong numbers along S^1-invariant irreducible divisors over $X \times \mathbb{D}$. To prove Orthogonality we will need to follow a similar strategy, but instead of dealing with psh rays we will consider positive currents induced by a (1,1) big class. Next, we will recall some of the theory of transcendental big classes developed for instance in [Bou17, CT22].

3.2.1 Recalling some positivity in analytic geometry

In this section we will quickly recall some notions of positivity for $(1,1)$ classes on a Kähler manifold, for a complete exposition see [Dem12, Bou04].

Definition 3.2.1. *A current T is said to be of* bidegree $(1,1)$ *if it can be locally written as*

$$\sum_{i,j} T_{i,j}\, \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j},$$

where $T_{i,j}$ are distributions. Moreover, T is positive *if for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ the distribution*

$$\sum_{i,j} T_{i,j}\, \xi_i \overline{\xi_j}$$

is a positive measure.

Finally, we say that T is a Kähler current *if there exists $\epsilon > 0$ such that*

$$T - \epsilon\omega \geq 0,$$

is positive, for some Kähler form ω of X.

As was for differential forms, there is a Dolbeault-Grothendieck lemma for currents, which means that each closed current T of type $(1,1)$ defines a cohomology class in bidegree $(1,1)$ that we denote by $[T] \in H^{1,1}(X)$. Moreover, we define

Definition 3.2.2. *We say that a class $\theta \in H^{1,1}(X,\mathbb{R})$ is* pseudoeffective (res. big) *if there is a closed positive (resp. Kähler) current T defining θ.*

Proposition 3.2.3. *The set of big classes in $H^{1,1}(X,\mathbb{R})$ is a cone, and is the interior of the cone of pseudoeffective classes.*

Currents with minimal singularities

Let u, v be quasi-psh functions on X, we say that u is *less singular than* v if there exists a constant $C > 0$ such that $v \leq u + C$. In the same spirit, $T_1, T_2 \in \theta$ are positive currents representing the same pseudoeffective class θ, we say that T_1 is *less singular than* T_2 if for some smooth representative η of θ we have:

$$T_1 = \eta + \mathrm{dd^c}u, \quad T_2 = \eta + \mathrm{dd^c}v,$$

for u less singular than v.

Definition 3.2.4. *We say that $T \in \theta$ is a* current with minimal singularities*, if it is less singular than any other positive current defining θ.*

It is well known that on any pseudoeffective class one can find positive currents with minimal singularities.

Non positive loci

Let T be a closed positive $(1,1)$ current, we define the *Lelong number of T at x*

$$\nu(T,x)$$

to be the Lelong number of u at x where u is a quasi psh function satisfying

$$T = \eta + \mathrm{dd^c}u, \quad \text{where } \eta \text{ is a smooth form in } [T].$$

We then define

Definition 3.2.5. *Let θ be a big class on X and $T \in \theta$ a current with minimal singularities, we say that the* non-nef locus of θ *is the set*

$$E_{nn}(\theta) \doteq \{x \in X \mid \nu(T, x) > 0\}.$$

Moreover, we say that the non-Kähler locus of θ *is the set*

$$E_{nK}(\theta) \doteq \bigcap_{S\, Kähler} \{x \in X \mid \nu(S, x) > 0\},$$

where S varies through all the Kähler currents in θ.

After having recalled these definitions let us now proceed studying *positive intersection theory* of big classes that will be important for what is next.

3.2.2 Asymptotic intersection numbers

Let Y be a n-dimensional compact Kähler manifold and $\theta \in H^{1,1}(Y)$ a big class. Let D be a prime divisor on Y, we define following [ELM+09]:

Definition 3.2.6. *If $D \nsubseteq E_{nK}(\theta)$ we define:*

$$\|\theta^{n-1} \cdot D\| \doteq \sup_{\mu\ bimero} \left\{ \tilde{D} \cdot \rho^{n-1} \mid \rho \text{ nef, } E = \mu^*\theta - \rho \geq 0 \text{ and } \tilde{D} \nsubseteq \operatorname{supp} E \right\},$$

where $\tilde{D}$ denotes the strict transform of D, and E is an effective divisor whose support does not contain $\tilde{D}$.

The next lemma is the transcendental version of Proposition 4.6 of [BFJ09], although Y is assumed to be projective for them, the proof works with no change in the non algebraic case. The key observation is that the intersection product is homogeneous, which remains valid in the transcendental case.

Lemma 3.2.7 (Proposition 4.6 of [BFJ09])**.** *Let α be a Kähler class, and $D \nsubseteq E_{nK}(\theta)$, then:*

$$\lim_{\epsilon \downarrow 0} \|(\theta + \epsilon\alpha)^{n-1} \cdot D\| = \|\theta^{n-1} \cdot D\|$$

This allows us to extend Definition 3.2.6. If $D \nsubseteq E_{nn}(\theta)$ for every $\epsilon > 0$ the divisor $D \nsubseteq E_{nK}(\theta + \epsilon\alpha)$, and hence we define:

$$\|\theta^{n-1} \cdot D\| \doteq \lim_{\epsilon \downarrow 0} \|(\theta + \epsilon\alpha)^{n-1} \cdot D\|,$$

if furthermore $D \subseteq E_{nn}(\theta)$ we say that $\|\theta^{n-1} \cdot D\| \doteq 0$.

The next result is a generalization to the transcendental setting of Proposition 2.11 of [ELM+09], once again the proof works just the same, we add the scheme of proof for completeness.

Lemma 3.2.8. *Let $D \nsubseteq E_{nK}(\theta)$, we then have:*

$$\|\theta^{n-1} \cdot D\| = \sup_{\mu \text{ bimero}} \left\{ \tilde{D} \cdot \eta^{n-1} \mid \eta \text{ Kähler, } E = \mu^*\theta - \eta \geq 0 \text{ and } \tilde{D} \nsubseteq \operatorname{supp} E \right\}.$$

Proof. One inequality is clear, for the other let $\rho_m \in H^{1,1}(Y_m)$ be a sequence of nef classes such that $\tilde{D} \cdot \rho_m^{n-1} \to \|\theta^{n-1} \cdot D\|$, with $\pi_m \colon Y_m \to Y$ a bimeromorphic morphism. Since $D \nsubseteq E_{nK}(\theta)$, we have $\tilde{D} \nsubseteq E_{nK}(\rho_m)$, and therefore possibly after a blow-up we can suppose that there exists an effective divisor E_m such that:

- $\eta_m \doteq \rho_m - E_m$ is a Kähler class;
- $\tilde{D} \nsubseteq \operatorname{supp} E_m$.

Then it is enough to observe that:

- $\eta_{m,k} \doteq \frac{1}{k}\eta_m + \frac{k-1}{k}\rho_m$ is a Kähler class, and $\pi_m^\theta - \eta_{m,k}$ is the class of an effective divisor;
- $\lim_{k\to\infty} \eta_{m,k}^{n-1} \cdot \tilde{D} = \rho_m^{n-1} \cdot \tilde{D}$.

□

3.2.3 Restricted volumes

Following [CT22] we define:

Definition 3.2.9. *Let D be a prime divisor on Y not contained in the non-nef locus of θ, we define the* restricted volume of θ on D *by the quantity:*

$$\langle \theta^{n-1} \rangle_{Y|D} \doteq \lim_{\epsilon \downarrow 0} \int_D \langle T_{\min,\epsilon}|_D \rangle^{n-1},$$

where $T_{\min,\epsilon}$ is a positive current with minimal singularities in $\theta + \epsilon\alpha$, for α a given Kähler class, and the integrating term is the non-pluripolar Monge–Ampère operator.

If otherwise $D \subseteq E_{nn}(\theta)$ the restricted volume is defined to be zero.

The following result is a simple consequence of [CT22, Theorem 5.3], and Lemma 3.2.7.

Proposition 3.2.10. *We have that for every divisor D:*

$$\langle \theta^{n-1} \rangle_{Y|D} = \|\theta^{n-1} \cdot D\|. \tag{3.2.3.1}$$

Proof. When $D \subseteq E_{nn}(\theta)$, by definition $\langle \theta^{n-1} \rangle_{Y|D} = 0 = \|\theta^{n-1} \cdot\|$ and (3.2.3.1) is trivially satisfied.

When $D \nsubseteq E_{nn}(\theta)$, to get an inequality we proceed as follows: we apply Theorem 5.3 of [CT22] to obtain that for a fixed Kähler class α, and every $\epsilon > 0$, we can find a bimeromorophic model $\mu\colon \tilde{Y} \to Y$, and a nef class $\rho_\epsilon \in H^{1,1}(\tilde{Y})$ such that

$$0 \leq \mu^*(\theta + \epsilon\alpha) - \rho_\epsilon$$

is an effective divisor, and moreover

$$\tilde{D} \cdot \rho_\epsilon^{n-1},$$

where $\tilde{D}$ denotes the strict transform of D on $\tilde{Y}$. By definition, we have $\tilde{D} \cdot \rho_\epsilon^{n-1} \leq \|(\theta + \epsilon\alpha)^{n-1} \cdot D\|$, and hence for every $\epsilon > 0$

$$\langle \theta^{n-1} \rangle_{Y|D} \leq \|(\theta + \epsilon\alpha)^{n-1} \cdot D\|,$$

letting ϵ tend to 0 we have:

$$\langle \theta^{n-1} \rangle_{Y|D} \leq \|\theta^{n-1} \cdot D\|.$$

For equality, it is enough to observe that, for any fixed $\epsilon > 0$ and α Kähler, $D \nsubseteq E_{nK}(\theta + \epsilon\alpha)$, thus we apply Lemma 3.2.8 to get sequences $\mu_m\colon Y_m \to Y$ of bimeromorphic morphisms and $\eta_m \in H^{1,1}(Y_m)$ Kähler classes such that $\tilde{D} \cdot \rho_m^{n-1} \to \|\theta^{n-1} \cdot D\|$. Since $\rho_m = \mu_m^*(\theta + \epsilon\alpha) - N$, where N is an effective divisor, we can find a positive current S_m in $\mu_m^*(\theta + \epsilon\alpha)$ with singularities along N, such that

$$\tilde{D} \cdot \rho_m = \int_{\tilde{D}} \langle S_m \rangle^{n-1} \leq \int_{\tilde{D}} \langle T_m \rangle^{n-1} = \langle (\theta + \epsilon\alpha)^{n-1} \rangle_{Y|D},$$

for T_m a positive current with minimal singularities of $\mu_m^*(\theta + \epsilon\alpha)$, letting m go to infinity, and ϵ to zero we conclude that:

$$\|\theta^{n-1} \cdot D\| \leq \langle \theta^{n-1} \rangle_{Y|D}.$$

□

3.2.4 Minimal vanishing orders

Here we try to get a transcendental analogue of [BJ24, Section 4.2]. As above, let Y be a compact Kähler manifold and $\theta \in H^{1,1}(Y)$ big.

Let $v = r\,\mathrm{ord}_F \in Y^{\mathrm{div}}$ be a divisorial valuation, and $F \subseteq \tilde{Y} \xrightarrow{\mu} Y$ an associated prime divisor. We define:

$$v(\theta) \doteq r \cdot \nu(\mu^* T_{\min}, F) \geq 0,$$

for $T_{\min}$ a current with minimal singularities on θ, and ν the generic Lelong number along

F.

Adapting Theorem 2.4.7, we can actually extend this definition, and make sense of $v(\theta)$ for every $v \in Y^{\beth}$.

Lemma 3.2.11 (Proposition 3.6 of [Bou04]). *Let α be a Kähler class,*

$$\lim_{\epsilon \to 0^+} v(\theta + \epsilon\alpha) = v(\theta).$$

Proof. This follows directly from [Bou04, Proposition 3.6]. □

Proposition 3.2.12. *Let D be an effective divisor such that $\theta - D$ is nef, then:*

$$v(D) \geq v(\theta).$$

Proof. Let α be a Kähler class, then for every $\epsilon > 0$: $\theta - D + \epsilon\alpha$ is a Kähler class, thus we can find a positive current T in $\theta + \epsilon\alpha = (\theta - D + \epsilon\alpha) + D$ with divisorial singularities along D. Hence for every $\epsilon > 0$:

$$\nu(\mu^* T, F) = v(D) \geq v(\theta + \epsilon\alpha),$$

letting ϵ tend to 0 by Lemma 3.2.11 we have the desired result. □

Definition 3.2.13. *Let $\theta \in H^{1,1}(Y)$ be a big class we denote by $\log|\theta|$ the function:*

$$Y^{\beth} \ni v \mapsto -v(\theta).$$

3.3 Zariski decomposition and psh functions

The goal of this section is to give an interpretation to the envelopes considered in Section 3.1 using a birational Zariski decomposition of big classes. As a consequence we will give an explicit description of the Monge–Ampère measure of the envelope of a PL function, and this in turn will give us a proof of the Orthogonality property.

Let β be as before, we will prove next that the envelope $P_\beta(0)$, also denoted by V_β can be computed using currents:

Proposition 3.3.1. *Let β be the sum of an effective vertical divisor D of some test configuration with $\alpha_{\mathcal{X}}$, we then have:*

$$P_\beta(0) = \varphi_\beta.$$

Proof. By Theorem 2.4.7 we know that $\varphi_\beta \in \mathrm{PSH}(\beta)$, and since $\varphi_\beta \leq 0$ we get

$$\varphi_\beta \leq P_\beta(0).$$

Moreover, let ψ be a PL β-psh function in the definition of the envelope of $P_\beta(0)$. By Corollary 3.1.4 we can suppose that there exists an anti-effective vertical divisor E defining ψ such that $E+\beta$ is nef. By Proposition 3.2.12 we then have that for every valuation v: $v(-E) \geq v(\beta)$, or equivalently

$$\psi \leq \varphi_\beta,$$

proving the desired inequality.

□

3.3.1 Restricted volumes and Monge–Ampère measures

Let f be a positive PL function, $D \in \mathrm{VCar}(\mathcal{X})$ an effective vertical divisor on a test configuration $\mathcal{X}$ such that $f = \varphi_D$, let $\mathcal{X}_0 = \sum b_E E$ be the irreducible decomposition, and $\beta = D + \alpha$. Let G_k be a decreasing sequence of effective vertical divisors such that $\beta - G_k$ is nef, and $\varphi_{G_k} \to P_\beta(0)$.

Denote by ρ_k the nef class $\beta - G_k$, for $G_k \subseteq \mathcal{X}_k$ as before. The following result will be useful:

Lemma 3.3.2. *Let E be an irreducible component of $\mathcal{X}_0$, such that either $E \nsubseteq E_{nK}(\beta)$ or $E \subseteq E_{nn}(\beta)$, then*

$$\liminf_{k\to\infty} \tilde{E} \cdot \rho_k^n \geq b_E \langle \beta^n \rangle_{\mathcal{X}|E},$$

where $\tilde{E}$ denotes the strict transform of E on $\mathcal{X}_k$.

Proof. Let's first suppose that $E \nsubseteq E_{nK}(\beta)$, in this case we have:

$$\liminf \tilde{E} \cdot \rho_k^n = \lim \tilde{E} \cdot \rho_k^n = b_E \|\beta^n \cdot E\|. \tag{3.3.1.1}$$

Indeed, let $\mu\colon \mathcal{X}' \to \mathcal{X}$ be a morphism of test configurations and ρ be a nef class such that $H \doteq \mu^*\beta - \rho$ is an effective $\mathbb{Q}$-divisor, then there exists a $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have $H \geq G_k$, thus $\rho_k = \rho + H - G_k$. Now, since E is not contained in the non-nef locus of β we have that the support of $\tilde{E}$ is not contained in the support of H, and therefore it is not contained in the support of $H - G_k$. In particular, the restriction $(H-G_k)|_{\tilde{E}}$ is effective. We conclude that the restriction $\beta_k|_{\tilde{E}}$ differs to $\rho|_{\tilde{E}}$ by an effective divisor, and hence:

$$\tilde{E} \cdot \rho^n = (\rho|_{\tilde{E}})^n \leq (\rho_k|_{\tilde{E}})^n = \tilde{E} \cdot \rho_k^n,$$

obtaining (3.3.1.1). We then apply Proposition 3.2.10, and get the desired result.

If $E \subseteq E_{nn}(\beta)$, then $\langle \beta^n \rangle_{\mathcal{X}|E} = 0$, and for every $m \in \mathbb{N}$ the divisor $\tilde{E}$ being effective and ρ_m being nef implies $0 \leq \tilde{E} \cdot \rho_m^n$, and the result follows.

□

With this result we have the following generalization of [Li23, Theorem 1.1 (i)]:

Theorem 3.3.3. *Let f be a positive PL function, and $\mathcal{X}$ a test configuration, such that there exists a divisor $D \in \mathrm{VCar}(\mathcal{X})$, satisfying $f = \varphi_D$, let $\mathcal{X}_0 = \sum b_E E$ be the irreducible decomposition, we then have*

$$\mathrm{MA}_\alpha(\mathrm{P}_\alpha(f)) = \frac{1}{V_\alpha} \sum_{E \subseteq \mathcal{X}_0} b_E \langle (\alpha + D)^n \rangle_{\mathcal{X}|E}\, \delta_{v_E}.$$

Proof. We can argue as [Li23, Theorem 1.1 (i)]. Let D_m be the vertical divisor $D - \frac{1}{m} G_m \subseteq \mathcal{X}_m$ and consider the increasing sequence $\psi_m \doteq \varphi_{D_m} \in \mathcal{H}(\alpha)$, finally denote by μ_m the Monge–Ampère measure of ψ_m. By construction $\lim \psi_m = P_\alpha(f)$, moreover, by the continuity of the envelope and Dini's theorem, the convergence is uniform, hence we have weak convergence of measures $\mu_m \rightharpoonup \mu \doteq \mathrm{MA}(\mathrm{P}_\alpha(f))$.

Let $E \subseteq \mathcal{X}$ be an irreducible component of the central fiber of $\mathcal{X}$ By Portmanteau's theorem this implies that

$$\limsup \mu_m(\{v_E\}) \leq \mu(\{v_E\}).$$

Since ψ_m is a PL function we have an explicit formula for $\mu_m(v_E)$:

$$V_\alpha \cdot \mu_m(\{v_E\}) = \tilde{E} \cdot (\alpha + D_m)^n = \tilde{E} \cdot \rho_m^n$$

where $\tilde{E}$ is the strict transform of E in $\mathcal{X}_m$.

If we suppose that for every irreducible component E is either not contained in the non-Kähler locus of $\alpha + D$ or contained in the non-nef locus by Proposition 3.2.10, we then have that

$$b_E \langle (\alpha + D)^n \rangle_{\mathcal{X}|E} \leq \limsup \mu_m(\{v_E\}) \leq \mu(\{v_E\}).$$

Then, this implies:

$$\frac{1}{V_\alpha} \sum_{\substack{\text{irred.} \\ E \subseteq \mathcal{X}_0}} b_E \langle (\alpha + D)^n \rangle_{\mathcal{X}|E}\, \delta_{v_E} \leq \mu. \tag{3.3.1.2}$$

Otherwise, if there exists a component $E \nsubseteq E_{nn}(\theta)$ contained in the non-Kähler locus, we proceed as follows: Let $g \in \mathcal{H}(\alpha)$, after going to a higher test configuration we may suppose that both f and g are defined by divisors D, and G, such that $\mathcal{A} \doteq G + \alpha + H$ is a Kähler class in $\mathcal{X}$. We then have that for every ϵ positive $E \nsubseteq E_{nK}(\beta + \epsilon \mathcal{A})$. By the previous argument this implies that:

$$\frac{1}{V} \sum_{\substack{\text{irred.} \\ E \subseteq \mathcal{X}_0}} b_E \langle (\beta + \epsilon \mathcal{A})^n \rangle_{\mathcal{X}|E}\, \delta_{v_E} \leq \mathrm{MA}_{(1+\epsilon)\alpha}(P_{(1+\epsilon)\alpha}(f + \epsilon g)),$$

letting ϵ tend to zero, we obtain our result.

By [Nys24, Vu23], we then have that the total mass of the measure on the LHS of (3.3.1.2) is 1, and hence we have the desired equality. $\square$

3.3.2 Proving Orthogonality

Theorem 3.3.4. *Let* $f \in \mathrm{PL}_0(X^{\beth})$, *then*

$$O_\alpha(f) = 0.$$

Proof. Let $\mathcal{X}$ be a test configuration, $0 \leq D \in \mathrm{VCar}(\mathcal{X})$ such that $f = \varphi_D$, and $\beta \doteq D + \alpha$, we need to check that

$$\int_{X^{\beth}} \mathrm{P}_\beta(0)\, \mathrm{MA}_\alpha(\mathrm{P}_\alpha(f)) = 0.$$

By Theorem 3.3.3, we then have

$$\int_{X^{\beth}} \mathrm{P}_\beta(0)\, \mathrm{MA}_\alpha(\mathrm{P}_\alpha(f)) = \frac{1}{V} \sum_{\substack{\text{irred.} \\ E \subseteq \mathcal{X}_0}} b_E \langle \beta^n \rangle_{\mathcal{X}|E}\, \mathrm{P}_\beta(0)(v_E).$$

Now, by Proposition 3.3.1, we have that the last quantity equals

$$\frac{1}{V} \sum_{\substack{\text{irred.} \\ E \subseteq \mathcal{X}_0}} b_E \langle \beta^n \rangle_{\mathcal{X}|E}\, \varphi_\beta(v_E).$$

We then observe that if $\varphi_\beta(v_E) \neq 0$, i.e. a current with minimal singularities of β has positive Lelong number along a prime divisor E, then $\langle \beta^n \rangle_{\mathcal{X}|E} = 0$.

$\square$

Chapter 4

Calabi–Yau Theorem

Until this point we have extensively studied the non-Archimedean pluripotential theory of the α-psh functions on $X^\beth$. This will be essential for studying K-stability as we will see in Chapter 5.

One of the main results of this thesis is to give a *valuative criterion* for the existence of constant scalar curvature Kähler metrics. This, however, relies on a dual point of view from the one we had so far: in order to obtain this valuative criterion we will have to study the pluripotential theory of *measures* on $X^\beth$. This is what we will do now, we will define an *energy functional* on the set of measures on $X^\beth$, and prove that the Monge–Ampère operator gives us a correspondence between measures of finite energy and potentials of finite energy.

4.1 The dual point of view

In this section we recall some notions introduced in [BJ25a], under a broader setting of synthetic pluripotential theory, that applies in this transcendental non-Archimedean setting.

Let μ be a Radon probability measure on $X^\beth$, we define the *(dual) energy of* μ to be

$$\mathrm{E}^\vee_\alpha(\mu) \doteq \sup_{\varphi\in\mathcal{E}^1}\left\{\mathrm{E}_\alpha(\varphi) - \int_{X^\beth}\varphi\,\mathrm{d}\mu\right\} \in [0,+\infty]\,.$$

Definition 4.1.1. *We say that μ is a* measure of finite energy *if*

$$\mathrm{E}^\vee_\alpha(\mu) < +\infty.$$

We denote the set of all such measures by $\mathcal{M}^1$, and endow it with the coarsest topology, finer than the weak topology of measures, that makes $\mathrm{E}^\vee$ continuous.

Remark 4.1.2. *In [BJ25a], the authors prove*[1] *that, as a topological space, $\mathcal{M}^1$ does not depend on the choice of class α, which justifies the notation. This is one of the main differences between the pluripotential theory of finite energy potentials $\mathcal{E}^1$ and the dual approach with measures of finite energy.*

As a simple consequence of the definition, if $\mu \in \mathcal{M}^1$ is a measure of finite energy, then for every Kähler class α, and each potential of finite energy $\varphi \in \mathcal{E}^1(\alpha)$, we have $\varphi \in L^1(\mu)$.

Example 4.1.3. *Let $v \in X^{\mathrm{div}}$ be a divisorial valuation. Then, the Dirac mass δ_v is of finite energy.*

Indeed, the energy of δ_v is given by

$$\begin{aligned} \mathrm{E}^\vee_\alpha(\delta_v) = \sup_{\varphi\in\mathcal{E}^1(\alpha)} \left\{ \mathrm{E}_\alpha(\varphi) - \int \varphi\, \delta_v \right\} &= \sup_{\varphi\in\mathcal{E}^1(\alpha)} \{\mathrm{E}_\alpha(\varphi) - \varphi(v)\} \\ &\leq \sup_{\varphi\in\mathcal{E}^1(\alpha)} \{\sup \varphi - \varphi(v), \} \end{aligned}$$

which is finite by the proof of Theorem 2.1.8.

Moreover, following the same strategy as in Proposition 2.1.19 we obtain that for every $v \in X^{\mathrm{np}}$ we have:

$$\frac{1}{n+1}\mathrm{T}(v) \leq \mathrm{E}^\vee(\delta_v) \leq \mathrm{T}(v). \tag{4.1.0.1}$$

For more details, see the proof in the algebraic setting, [BJ22, Proposition 11.1], that applies with no change.

Proposition 4.1.4. *Monge–Ampère measures are of finite energy. That is, for every $\varphi \in \mathcal{E}^1(\alpha)$*

$$\mathrm{MA}_\alpha(\varphi) \in \mathcal{M}^1.$$

Proof. The result is obtained from the following observation:

$$\mathrm{E}^\vee(\mathrm{MA}(\varphi)) = \mathrm{E}(\varphi) - \int_{X^\beth} \varphi\, \mathrm{MA}(\varphi).$$

This, in particular, gives that $\mathrm{E}^\vee(\mathrm{MA}(\varphi)) < +\infty$.

Indeed, from the concavity of E_α, see (2.3.1.8) itself based on the estimates of [BJ25a], we have that, for every $\psi \in \mathcal{E}^1(\alpha)$,

$$\mathrm{E}_\alpha(\psi) \leq \mathrm{E}_\alpha(\varphi) + \int (\psi - \varphi)\, \mathrm{MA}_\alpha(\varphi).$$

□

As in [BJ25a], we now define a different functional on the set of measures of finite

[1] In the article they assume the Orthogonality property, that we have proved in our setting in the previous chapter.

energy $\mathcal{M}^1$. If $\varphi \in \mathcal{E}^1(\alpha)$ we define $\mathrm{J}_\alpha(\cdot, \varphi) \colon \mathcal{M}^1 \to [0, +\infty[$ to be the functional

$$\mathcal{M}^1 \ni \mu \mapsto \mathrm{J}_\alpha(\mu, \varphi) \doteq \mathrm{E}^\vee_\alpha(\mu) - \mathrm{E}_\alpha(\varphi) + \int \varphi \, \mathrm{d}\mu.$$

As a consequence of [BJ25a, Proposition 2.2] we have the following basic properties for $\mathrm{J}_\alpha(\cdot, \varphi)$.

Lemma 4.1.5. *Let $\varphi, \psi \in \mathcal{E}^1$ be potentials of finite energy, and let $\mu \in \mathcal{M}^1$ a measure of finite energy. Then,*

1. $\mathrm{J}_\alpha(\mathrm{MA}_\alpha(\psi), \varphi) = \mathrm{J}_\alpha(\psi, \varphi)$
2. $\mathrm{J}_\alpha(\varphi, \psi) \lesssim \mathrm{J}_\alpha(\mu, \varphi) + \mathrm{J}_\alpha(\mu, \psi)$.
3. $\mathrm{J}(\mu, \cdot)$ *is continuous under decreasing limits.*

Proof. For the first point, it is enough to observe that we have shown that $\mathrm{E}^\vee_\alpha(\mathrm{MA}_\alpha(\varphi)) = \mathrm{E}_\alpha(\varphi) - \int \varphi \, \mathrm{MA}_\alpha(\varphi)$. For the second, consider the decreasing sequences $\varphi_j, \psi_j \in \mathcal{H}(\alpha)$ converging to φ and ψ respectively. From Proposition 2.2 of [BJ25a], we get

$$\mathrm{J}_\alpha(\varphi_j, \psi_j) \lesssim 2\mathrm{E}^\vee_\alpha(\mu) - \mathrm{E}_\alpha(\varphi_j) - \mathrm{E}_\alpha(\psi_j) + \int (\psi_j + \varphi_j) \, \mathrm{d}\mu,$$

from the continuity of the energy and the monotone convergence theorem we then get

$$\mathrm{J}_\alpha(\varphi, \psi) \lesssim 2\mathrm{E}^\vee_\alpha(\mu) - \mathrm{E}_\alpha(\varphi) - \mathrm{E}_\alpha(\psi) + \int (\varphi + \psi) \, \mathrm{d}\mu,$$

as desired.

For the last point, observe that the energy $\mathrm{E}_\alpha \colon \mathcal{E}^1(\alpha)$ is continuous along decreasing sequences, and so is the integral $\mathcal{E}^1(\alpha) \ni \varphi \mapsto \int \varphi \, \mathrm{d}\mu$ by the monotone convergence theorem.

□

4.2 Strong topologies

Having proved Orthogonality and the Envelope properties, we can develop more of the pluripotential theory of $\mathcal{E}^1(\alpha)$ and of $\mathcal{M}^1$ that we started before.

The next estimate is a direct consequence of [BJ25a, Theorem 3.6].

Proposition 4.2.1. *Let $\theta_0, \ldots, \theta_n \in H^{1,1}(X)$, and $\varphi_0, \ldots \varphi_n, \psi_0, \ldots, \psi_n \in \mathcal{E}^1(\alpha)$, such that $\sup \varphi_i = 0 = \sup \psi_i$ for every $i \in \{1, \ldots, n\}$, then we have*

$$|(\theta_0, \varphi_0) \cdots (\theta_n, \varphi_n) - (\theta_0, \psi_0) \cdots (\theta_n, \psi_n)| \lesssim V_\alpha (1 + \lambda)^{n+1} \max_i \mathrm{J}_\alpha(\varphi_i, \psi_i)^q \cdot \max_i \mathrm{J}_\alpha(\varphi_i)^{1-q}$$

for $q = 2^{-n}$, $\lambda > 0$ such that $-\lambda\alpha \leq \theta_i \leq \lambda\alpha$ for every $i \in \{1, \ldots, n\}$.

Proof. If $\varphi_i, \psi_i \in \mathcal{H}(\alpha)$, this is an immediate consequence of [BJ25a, Theorem 3.6]. For singular potentials, it is enough to observe that both sides of the inequality are continuous under decreasing sequences, and hence, it is enough to consider the decreasing sequences $\varphi_{i,k}, \psi_{i,k} \in \mathcal{H}(\alpha)$ converging to φ_i and ψ_i respectively. □

Corollary 4.2.2. *Let $\varphi, \varphi', \psi, \psi' \in \mathcal{E}^1(\alpha)$ be potentials of finite energy normalized so that* $\sup = 0$, $\mu \doteq \mathrm{MA}_\alpha(\psi)$, $\nu \doteq \mathrm{MA}_\alpha(\psi')$ *the Monge–Ampère measures of ψ and ψ' respectively, then we have:*

$$\left|\int \varphi \, \mathrm{d}\mu - \int \varphi' \, \mathrm{d}\nu\right| \lesssim \max\{\mathrm{J}_\alpha(\varphi, \varphi'), \mathrm{J}_\alpha(\psi, \psi')\}^q \cdot J^{1-q},$$

for $q \doteq 2^{-n}$, and $J \doteq \max\{\mathrm{J}_\alpha(\psi), \mathrm{J}_\alpha(\psi'), \mathrm{J}_\alpha(\varphi), \mathrm{J}_\alpha(\varphi')\}$.

Proof. This follows directly from the last result. □

As a consequence, we get:

Lemma 4.2.3. *The Monge–Ampère operator*

$$\mathrm{MA}_\alpha \colon \mathcal{E}^1(\alpha) \to \mathcal{M}^1$$

is continuous along decreasing sequences.

Proof. Let $\varphi_i \in \mathcal{E}^1(\alpha)$ be a decreasing sequence converging to $\varphi \in \mathcal{E}^1(\alpha)$. Denote by μ_i the sequence of measures $\mathrm{MA}_\alpha(\varphi_i)$, and μ the measure $\mathrm{MA}_\alpha(\varphi)$. Since the energy pairing is continuous along decreasing sequences, we have that:

1. $\mathrm{E}_\alpha(\varphi_i)$ tends to $\mathrm{E}_\alpha(\varphi)$, and $\mathrm{J}_\alpha(\varphi_i)$ to $\mathrm{J}_\alpha(\varphi)$ as i tends to ∞.

2. $\mathrm{J}_\alpha(\varphi_i, \varphi)$ tends to 0 as $i \to \infty$.

3. For every PL function $f \in \mathrm{PL}$

$$\int f \, \mathrm{d}\mu_i \to \int f \, \mathrm{d}\mu.$$

By density we have that (3) implies the weak convergence of measures $\mu_i \xrightarrow{w} \mu$.

To get the strong convergence, it is enough to observe that

$$\mathrm{E}^\vee_\alpha(\mu_i) = \mathrm{E}_\alpha(\varphi_i) - \int \varphi_i \, \mathrm{d}\mu_i \quad \text{and} \quad \mathrm{E}^\vee_\alpha(\mu) = \mathrm{E}_\alpha(\varphi) - \int \varphi \, \mathrm{d}\mu.$$

Thus by Corollary 4.2.2

$$\begin{aligned}\left|\mathrm{E}^\vee_\alpha(\mu_i)-\mathrm{E}^\vee_\alpha(\mu)\right| &\le |\mathrm{E}_\alpha(\varphi_i)-\mathrm{E}_\alpha(\varphi)|+|\sup\varphi_i-\sup\varphi|\\ &+\left|\int(\varphi_i-\sup\varphi_i)\,\mathrm{d}\mu_i-\int(\varphi-\sup\varphi)\,\mathrm{d}\mu\right|\\ &\lesssim |\mathrm{E}_\alpha(\varphi_i)-\mathrm{E}_\alpha(\varphi)|+|\sup\varphi_i-\sup\varphi|\\ &+C\mathrm{J}_\alpha(\varphi_i,\varphi)^q,\end{aligned}$$

for $q \doteq 2^{-n}$, and some constant $C>0$ independent of i. Together with (2) and (3) this gives the convergence needed. □

The next result is a direct consequence of [BJ25a, Theorem 2.23 (iv)].

Proposition 4.2.4. *Let $\varphi,\psi\in\mathcal{E}^1$, $\mu\in\mathcal{M}^1$, and let ν denote $\mathrm{MA}_\alpha(\tau)$, we then have:*

$$\left|\int(\varphi-\psi)(\mathrm{d}\mu-\mathrm{d}\nu)\right|\lesssim \mathrm{J}(\varphi,\psi)^q\mathrm{J}(\mu,\tau)^{\frac12}R^{\frac12-q},$$

for

$$q\doteq 2^{-n},\quad \textit{and}\quad R\doteq\max\{\mathrm{J}(\varphi),\mathrm{J}(\psi),\mathrm{E}^\vee(\mu),\mathrm{E}^\vee(\nu)\}.$$

Proof. If $\varphi,\psi\in\mathrm{PL}$ this follows from [BJ25a, Theorem 2.23]. Otherwise, consider decreasing sequences $\varphi_i,\psi_i\in\mathrm{PL}\cap\mathrm{PSH}$ converging to φ and ψ respectively. Then, by the monotone convergence theorem, the LHS of the inequality converges, while the RHS converges by the continuity of J along decreasing sequences. □

To conclude, we will need the following result that, in the algebraic setting, can be found in [BJ22, Proposition 9.19]. The proof given there applies directly to our setting. We will add it here for completeness.

Lemma 4.2.5. *Let $\varphi_i\in\mathrm{PSH}(\alpha)$ be a sequence of α-psh functions converging weakly to $\varphi\in\mathrm{PSH}(\alpha)$, such that $\mathrm{J}_\alpha(\varphi_i)$ is bounded. Then, for any $\mu\in\mathrm{MA}_\alpha\big(\mathcal{E}^1(\alpha)\big)$ we have*

$$\int|\varphi_i-\varphi|\,\mathrm{d}\mu\to 0.$$

Proof. Since $\varphi_i\overset{w}{\rightharpoonup}\varphi$, we have $\sup\varphi_i=\varphi_i(v_{\mathrm{triv}})\to\varphi(v_{\mathrm{triv}})=\sup\varphi$. Hence, we can suppose, by subtracting the supremum, that $\sup\varphi_i=0=\sup\varphi$.

Let $\psi\in\mathcal{E}^1(\alpha)$ be such that $\mathrm{MA}_\alpha(\psi)=\mu,\sup\psi=0$, and let $\psi_j\in\mathcal{H}(\alpha)$ be a sequence decreasing to ψ. Also, let us denote by $\tilde{\psi}_j$ the normalized functions $\psi_j-\sup\psi_j$, and $\mu_j\doteq\mathrm{MA}_\alpha(\psi_j)=\mathrm{MA}_\alpha(\tilde{\psi}_j)$.

We will start proving that

$$\int(\varphi_i-\varphi)\,\mathrm{d}\mu\to 0.$$

Indeed, if we let $\epsilon > 0$, using the triangle inequality we obtain:

$$\left|\int (\varphi_i - \varphi)\, \mathrm{d}\mu\right| \leq \left|\int \varphi_i\, (\mathrm{d}\mu - \mathrm{d}\mu_j)\right| + \left|\int (\varphi_i - \varphi)\, \mathrm{d}\mu_j\right| + \left|\int \varphi\, (\mathrm{d}\mu - \mathrm{d}\mu_j)\right|,$$

which, by weak convergence, implies that we have:

$$\begin{aligned}\left|\int (\varphi_i - \varphi)\, \mathrm{d}\mu\right| &\leq \left|\int \varphi_i\, (\mathrm{d}\mu - \mathrm{d}\mu_j)\right| + \epsilon + \left|\int \varphi\, (\mathrm{d}\mu - \mathrm{d}\mu_j)\right| \\ &\lesssim \epsilon + 2\mathrm{J}_\alpha(\tilde{\psi}_j, \psi)^q \cdot \max\{C, \mathrm{J}_\alpha(\psi), \mathrm{J}_\alpha(\tilde{\psi}_j)\}^{1-q} \\ &= \epsilon + 2\mathrm{J}_\alpha(\psi_j, \psi)^q \cdot \max\{C, \mathrm{J}_\alpha(\psi), \mathrm{J}_\alpha(\psi_j)\}^{1-q},\end{aligned}$$

for all i large enough and $q \doteq 2^{-n}$. The second inequality is given by Corollary 4.2.2.

Now, since the energy pairing is continuous along decreasing sequences, for j sufficiently large we have:

$$\left|\int (\varphi_i - \varphi)\, \mathrm{d}\mu\right| \leq 3\epsilon.$$

As in [BJ22, Proposition 9.19], we can then apply the same strategy to $\tilde{\varphi}_i \doteq \max\{\varphi_i, \varphi\}$, and observe that

$$|\varphi - \varphi| = 2(\tilde{\varphi}_i - \varphi) + (\varphi_i - \varphi)$$

to conclude. □

As a direct consequence, we obtain the following result:

Corollary 4.2.6. *Let $\varphi_i \to \varphi$ be a converging sequence on $\mathcal{E}^1(\alpha)$, then*

$$\mathrm{J}_\alpha(\varphi, \varphi_i) \to 0.$$

With Orthogonality, we can utilize all the pluripotential machinery of the space of measures of finite energy $\mathcal{M}^1$, developed in [BJ25a]. What we will do next is to describe the *quasi metric structure* on $\mathcal{M}^1$, that Boucksom–Jonsson developed for the synthetic formalism.

4.2.1 $\mathcal{M}^1$ as a quasi metric space

We start with some very general definitions that we will use.

Definition 4.2.7. *We say that a continuous map $\delta\colon M \times M \to [0, +\infty[$ is a* quasi metric *for the topological space M if there exist constants $C, D > 0$ such that*

$$\begin{aligned}\delta(p,q) = 0 &\iff p = q \\ C \cdot \delta(p,q) &\leq \delta(p,t) + \delta(t,q) \\ D^{-1} \cdot \delta(p,q) \leq \delta(q,p) &\leq D \cdot \delta(p,q),\end{aligned}$$

and the topology generated by all the sets $\{p \in M \mid \delta(p,q) \le r\}$ *coincides with the topology of* M*.*

The next result is a direct consequence of [BJ25a, Theorems 2.23 and 2.25].

Theorem 4.2.8. *Let* α *be a Kähler class on* X*, then there exists a unique quasi metric on* $\mathcal{M}^1$

$$\delta_\alpha \colon \mathcal{M}^1 \times \mathcal{M}^1 \to [0,\infty[$$

such that:

1. *For all* $\varphi \in \mathcal{E}^1(\alpha)$
$$\delta_\alpha(\mu, \mathrm{MA}_\alpha(\varphi)) = \mathrm{J}_\alpha(\mu,\varphi).$$
2. *For all* $\varphi,\psi \in \mathcal{E}^1$*,* $\mu,\nu \in \mathcal{M}^1$
$$\left|\int (\varphi-\psi)(\mathrm{d}\mu - \mathrm{d}\nu)\right| \lesssim \mathrm{J}_\alpha(\varphi,\psi)^q \cdot \delta_\alpha(\mu,\nu)^{\frac{1}{2}} \cdot R^{\frac{1}{2}-q},$$
for
$$q \doteq 2^{-n}, \quad \textit{and} \quad R \doteq \max\{\mathrm{J}(\varphi),\mathrm{J}(\psi),\mathrm{E}^\vee(\mu),\mathrm{E}^\vee(\nu)\}.$$
3. *For all* $\mu,\mu',\nu,\nu' \in \mathcal{M}^1$
$$\left|\delta_\alpha(\mu,\nu) - \delta_\alpha(\mu',\nu')\right| \lesssim \max\{\delta_\alpha(\mu,\mu'),\delta_\alpha(\nu,\nu')\}^q \cdot S^{1-q},$$
for $S \doteq \max\{\mathrm{E}^\vee_\alpha(\mu),\mathrm{E}^\vee_\alpha(\mu'),\mathrm{E}^\vee_\alpha(\nu),\mathrm{E}^\vee_\alpha(\nu')\}$.

Proof. By [BJ25a, Theorems 2.23] there exists a unique quasi metric δ_α as before that satisfies $\delta_\alpha(\mu,\mathrm{MA}_\alpha(\varphi)) = \mathrm{J}_\alpha(\mu,\varphi)$ for all $\varphi \in \mathcal{H}(\alpha)$.

To get (1), it is then enough to observe that if $\varphi \in \mathcal{E}^1(\alpha)$ is a finite energy potential and $\varphi_i \in \mathcal{H}(\alpha)$ is decreasing sequence converging to φ then $\mathrm{MA}_\alpha(\varphi_i)$ converges strongly to $\mathrm{MA}_\alpha(\varphi)$ and $\mathrm{J}_\alpha(\mu,\varphi_i)$ converges to $\mathrm{J}_\alpha(\varphi)$ by Lemma 4.2.3 and Lemma 4.1.5 respectively.

In order to prove (2) we observe that both the LHS and RHS of the inequality are continuous under decreasing limits, and then apply [BJ25a, Theorem 2.23] for φ and ψ in $\mathcal{H}(\alpha)$.

The third point is a direct consequence of [BJ25a, Theorem 2.22]. □

As a direct consequence we get the continuity of Monge–Ampère operator with the strong topologies, this is an important step to “translate" the pluripotential of $\mathcal{E}^1(\alpha)$ to the one on $\mathcal{M}^1$.

Theorem 4.2.9. *The Monge–Ampère operator*

$$\mathrm{MA}_\alpha \colon \mathcal{E}^1(\alpha) \to \mathcal{M}^1, \quad \varphi \mapsto \mathrm{MA}_\alpha(\varphi)$$

is continuous.

Proof. Let $\varphi_i \in \mathcal{E}^1(\alpha)$ be a sequence that converges (strongly) to $\varphi \in \mathcal{E}^1(\alpha)$. Then

$$\delta_\alpha(\mathrm{MA}_\alpha(\varphi_i), \mathrm{MA}_\alpha(\varphi)) = \mathrm{J}_\alpha(\varphi_i, \varphi) \to 0,$$

by Corollary 4.2.6. Thus by Theorem 4.2.8 we conclude that $\mathrm{MA}_\alpha(\varphi_i)$ converges strongly to $\mathrm{MA}_\alpha(\varphi)$. □

4.2.2 $\mathcal{E}^1(\alpha)$ as a quasi metric space

As we did with $\mathcal{M}^1$, we will now give a quasi metric structure on $\mathcal{E}^1(\alpha)$. In the algebraic setting this is done in [BJ22, Section 12.1].

Definition 4.2.10. *Let $\varphi, \psi \in \mathcal{E}^1(\alpha)$ be finite energy potentials, we denote*

$$\partial_\alpha(\varphi, \psi) \doteq \mathrm{J}_\alpha(\varphi, \psi) + |\sup \varphi - \sup \psi| \in [0, +\infty[\,.$$

The next result is the transcendental version of [BJ22, Theorem 12.4]

Theorem 4.2.11. *The map*

$$\partial_\alpha \colon \mathcal{E}^1(\alpha) \times \mathcal{E}^1(\alpha) \to [0, +\infty[$$

is a continuous quasi metric on $\mathcal{E}^1(\alpha)$ that defines its topology.

Proof. By Corollary 4.2.6 the map ∂_α is continuous. Let us prove now that ∂_α is a quasi metric.

It is easy to see that ∂_α is quasi symmetric, and the quasi triangle inequality follows from Lemma 4.1.5. Hence, in order to prove that ∂_α is a quasi metric, it remains to check that $\partial_\alpha(\varphi, \psi) = 0 \iff \varphi = \psi$.

Let φ, ψ be such that $\partial_\alpha(\varphi, \psi) = 0$, we will then prove that $\varphi = \psi$. Observe also that, since psh functions are completely determined on the set of divisorial valuations, cf. Theorem 2.1.15, it is enough to prove that

$$\varphi|_{X^{\mathrm{div}}} = \psi|_{X^{\mathrm{div}}}.$$

Let $v \in X^{\mathrm{div}}$ be a divisorial valuation. By Example 4.1.3, the measure δ_v is of finite energy, which in turn implies that

$$0 \le |\varphi(v) - \psi(v)| = \left| \int (\varphi - \psi)\,(\delta_v - \delta_{v_{\mathrm{triv}}}) \right| \lesssim 0,$$

by Theorem 4.2.8.

We then prove that if $\varphi_i, \varphi \in \mathcal{E}^1(\alpha)$ are such that $\partial_\alpha(\varphi_i, \varphi) \to 0$, then $\varphi_i \xrightarrow{s} \varphi$. For that, first observe that if $\partial_\alpha(\varphi_i, \varphi) \to 0$, then

1. $|\sup \varphi_i - \sup \varphi| \to 0$;

2. $\mathrm{J}_\alpha(\varphi_i, \varphi) \to 0$. In particular, $\mathrm{J}_\alpha(\varphi_i)$ is a bounded sequence by Lemma 4.1.5.

We now prove that φ_i converges weakly to φ. Let $v \in X^{\mathrm{div}}$ be a divisorial valuation, we estimate as before

$$\begin{aligned}|\varphi_i(v) - \varphi(v)| &\leq |\sup \varphi_i - \sup \varphi| + \left| \int (\varphi_i - \varphi)\, (\delta_v - \delta_{v_{\mathrm{triv}}}) \right| \\ &\lesssim |\sup \varphi_i - \sup \varphi| + C \cdot \mathrm{J}_\alpha(\varphi_i, \varphi)^q \to 0,\end{aligned}$$

where $q \doteq 2^{-n}$, and $C > 0$ is a constant independent of i, and we get $\varphi_i \xrightarrow{w} \varphi$.

To conclude, Lemma 4.2.5 gives the convergence of the integrals $\int \varphi_i \, \mathrm{MA}_\alpha(\varphi)$ to the integral $\int \varphi \, \mathrm{MA}_\alpha(\varphi)$, and thus

$$|\mathrm{E}_\alpha(\varphi_i) - \mathrm{E}_\alpha(\varphi)| \leq \mathrm{J}_\alpha(\varphi, \varphi_i) + \left| \int (\varphi_i - \varphi)\, \mathrm{MA}_\alpha(\varphi) \right| \to 0,$$

and we are done. $\square$

As a consequence we get, as in [BJ22, Corollary 10.4] in the algebraic setting, we prove:

Proposition 4.2.12. *Let $\varphi, \psi \in \mathcal{E}^1$, such that $\mathrm{MA}(\varphi) = \mathrm{MA}(\psi)$, then $\varphi - \psi$ is constant.*

Proof. The Monge–Ampère operator is invariant by translation, thus up by adding a constant we can suppose that $\sup \varphi = \sup \psi$. We will then prove that $\varphi = \psi$.

Since $\mathrm{MA}(\varphi) = \mathrm{MA}(\psi)$, we get

$$\mathrm{J}(\varphi, \psi) + \mathrm{J}(\psi, \varphi) = \int (\varphi - \psi)\, (\mathrm{MA}(\psi) - \mathrm{MA}(\varphi)) = 0,$$

which by non-negativity of the Dirichlet functional implies $\mathrm{J}(\varphi, \psi) = 0 = \mathrm{J}(\psi, \varphi)$. This implies that $\partial_\alpha(\varphi, \psi) = 0$, and by Theorem 4.2.11 we get $\varphi = \psi$. $\square$

In the same spirit we prove

Proposition 4.2.13. *Let $\varphi, \psi \in \mathcal{E}^1(\alpha)$ be finite energy potentials, satisfying*

$$\varphi \geq \psi \quad \textit{and} \quad \mathrm{E}_\alpha(\psi) \geq \mathrm{E}_\alpha(\varphi),$$

then $\varphi = \psi$.

Proof. Recall that $0 \leq \mathrm{J}_\alpha(\varphi, \psi) = \mathrm{E}_\alpha(\varphi) - \mathrm{E}_\alpha(\psi) + \int (\psi - \varphi)\, \mathrm{MA}_\alpha(\varphi)$, which is nonpositive by hypothesis. Therefore, we have $\mathrm{J}_\alpha(\varphi, \psi) = 0$.

Which implies that denoting $\psi' \doteq \psi + \sup\varphi - \sup\psi$ we will have

$$\mathrm{J}_\alpha(\varphi, \psi') = 0, \quad \text{and} \quad \sup\varphi = \sup\psi'.$$

Applying Theorem 4.2.11 we then have that $\varphi = \psi'$, and thus

$$\mathrm{E}(\varphi) = \mathrm{E}_\alpha(\psi') = \mathrm{E}_\alpha(\psi) + \sup\varphi - \sup\psi \geq \mathrm{E}_\alpha(\psi) \geq \mathrm{E}_\alpha(\varphi) + \sup\varphi - \sup\psi.$$

This implies that $\sup\psi \geq \sup\varphi \geq \sup\psi$ that gives $\psi' = \psi = \varphi$ as wanted. $\square$

4.3 The weak topology

Let us start by recalling that $\mathrm{PSH}(\alpha)$ is endowed with the *weak topology* of pointwise convergence on X^{div}.

In this section we will prove that the set of psh functions normalized to have $\sup = 0$ is compact. To do so, we will rely on the comparison of non-Archimedean psh functions of finite energy with Archimedean rays on $\mathcal{E}^1(\omega)$.

Theorem 4.3.1. *Let $\varphi \in \mathcal{E}^1(\alpha)$ be a non-archimedean potential, and $u \in \mathcal{E}^1(\omega)$ a reference metric, then there exists a unique maximal geodesic ray $U\colon [0, +\infty[\to \mathcal{E}^1(\omega)$ starting at u, such that:*

$$U^{\beth} = \varphi.$$

Proof. The proof is like the one [BBJ21, Theorem 6.6]. It relies on the following observations:

- If $u \in \mathcal{H}(\omega)$, and $\varphi \in \mathcal{H}(\alpha)$, we apply Lemma 2.4.10, and get a maximal geodeisc ray connecting u and φ.
- Now, if $u \in \mathcal{E}^1(\omega)$, and $\varphi \in \mathcal{E}^1(\alpha)$, we can take “smoothing" decreasing sequences $\varphi_m \in \mathcal{H}$, and $u_m \in \mathcal{H}(\omega)$.
- Using the first point there exists maximal geodesic ray, $U_{m,s}$, uniting u_m and φ_m.
- By maximality $U_{m+1} \leq U_m$, and hence there exists the limit $\lim U_m(x)$, which we will denote by $U(x)$.
- Since the energy is affine on these maximal geodesic rays, by Remark 2.4.11, and decreasing on decreasing sequences, we have an uniform lower bound on the energy of $U_{m,s}$. This implies that U_s is of finite energy, and a psh geodesic ray.
- Moreover, from $U \leq U_m$ we get $U^{\beth} \leq \varphi_m$, and thus $U^{\beth} \leq \varphi$, but also the previous estimate gives us that $\mathrm{E}_\alpha(U^{\beth}) = \mathrm{E}_\alpha(\varphi)$, and therefore $U^{\beth} = \varphi$ by Proposition 4.2.13.

- Finally, if V is a psh ray of linear growth, such that $V_0 \leq u \leq u_m$, and $V^{\beth} \leq U^{\beth} = \varphi \leq \varphi_m$, by maximality of U_m we have $V \leq U_m$, and thus $V \leq U$, getting maximality of U and concluding the proof.

□

Now, we denote:

$$\mathrm{PSH}_{\sup}(\alpha) \doteq \{\varphi \in \mathrm{PSH}(\alpha) \mid \sup \varphi = 0\}.$$

The goal of the first part of this section is to prove the following result, that we will sometimes call the *Envelope property.*

Theorem 4.3.2. $\mathrm{PSH}_{\sup}(\alpha)$ *is compact in the weak topology.*

To accomplish this we will need the following proposition:

Proposition 4.3.3. *For every family $(\psi_i)_{i\in I}$ of α-psh functions that is uniformly bounded from above, we have that $(\sup_i \psi_i)^{\star}$ is α-psh.*

Proof. Without loss of generality we can assume that each $\psi_i \leq 0$. We also start by assuming that each $\psi_i \in \mathcal{E}^1(\alpha)$.

Let U_i be the maximal geodesic ray associated to ψ_i. Also let $V := (\sup_i U_i)^{\star}$ and $\psi := V^{\beth} \in \mathrm{PSH}(\alpha)$. By monotonicity of Lelong numbers we get that for all i: $\psi \geq \psi_i$ on X^{div}, and by Theorem 2.1.15 we get that $\psi \geq \psi_i$ on the whole of $X^{\beth}$. Thus $\psi \geq \sup_i \psi_i$, and since ψ is usc we get that $\psi \geq (\sup_i \psi_i)^{\star}$. If $v_E \in X^{\mathrm{div}}$ we now have that

$$\psi(v_E) = -b_E^{-1}\nu_E(V) = -b_E^{-1} \inf_i \nu_E(U_i) = \sup_i U_i^{\mathrm{NA}}(v_E) = \sup_i \psi_i(v_E).$$

which clearly implies that

$$(\sup_i \psi_i)^{\star}(v_E) \geq \sup_i \psi_i(v_E) = \psi(v_E) \geq (\sup_i \psi_i)^{\star}(v_{E_i}).$$

This implies that $\psi = (\sup_i \psi_i)^{\star}$ on X^{div}. It now follows from Theorem 2.1.15 that $\psi \leq (\sup_i \psi_i)^{\star}$ on the whole of $X^{\beth}$, and hence $(\sup_i \psi_i)^{\star} = \psi \in \mathrm{PSH}(\alpha)$. This concludes the proof in the case where each $\psi_i \in \mathcal{E}^1_A$.

In the general case, we observe that for any constant C:

$$\max((\sup_i \psi_i)^{\star}, C) = (\sup_i \max(\psi_i, C))^{\star},$$

and $\max(\psi_i, C) \in \mathcal{E}^1_A$. Thus $(\sup_i \psi_i)^{\star}$ is the decreasing limit of A-psh functions and hence A-psh. We similarly have that $\max((\sup_i \psi_i), C) = \sup_i(\max(\psi_i, C))$, which lets us conclude that $\sup_i \psi_i = (\sup_i \psi_i)^{\star}$ on X^{div} also in the general case.

□

We are now ready to prove Theorem 4.3.2, following the argument in [BJ22, Theorem 5.11].

Proof of Theorem 4.3.2. Let $\varphi_i \in \mathrm{PSH}_{\sup}(\alpha)$ be a family of sup normalized α-psh functions.

If $x \in X^{\mathrm{div}}$ we get by Theorem 2.1.8 that the sequence of real numbers $\varphi_i(x)$ is bounded. Thus by Tychonoff's theorem there exists a subsequence that converges to a function $\varphi\colon X^{\mathrm{div}} \to \mathbb{R}$, and we will let φ_i denote the subsequence.

Let $\psi_i \doteq (\sup_{j\geq i}\varphi_j)^\star$. By Proposition 4.3.3 ψ_i is α-psh, and by definition $(\psi_i)_i$ is decreasing. Also by Proposition 4.3.3 $\psi_i = \sup_{j\geq i}\varphi_j$ on X^{div}. In particular $\sup_{X^\beth}\psi_i = \psi_i(v_{\mathrm{triv}}) = \sup_{j\geq j}\varphi_j(v_{triv}) = 0$, and hence $\psi \doteq \lim_i \psi_i \in \mathrm{PSH}_{\sup}(\alpha)$. Since $\varphi_j(x) \to \varphi(x)$ for $x \in X^{\mathrm{div}}$ we clearly also get that $\psi(x) = \varphi(x) = \lim_j \varphi(x)$, i.e. that φ_j converges weakly to ψ. ☐

4.4 Measures of finite energy and the Calabi–Yau theorem

The goal of this section will be to prove the Calabi–Yau theorem for $X^\beth$, that is that for every measure of finite energy $\mu \in \mathcal{M}^1$ there exists a potential of finite energy $\varphi \in \mathcal{E}^1(\alpha)$ that solves the equation

$$\mathrm{MA}_\alpha(\varphi) = \mu. \tag{4.4.0.1}$$

Moreover, we will prove that if $\mathbb{R}$ acts on $\mathcal{E}^1(\alpha)$ by translations we have

Theorem 4.4.1. *The Monge–Ampère operator*

$$\mathrm{MA}_\alpha\colon \mathcal{E}^1(\alpha)/\mathbb{R} \to \mathcal{M}^1$$

is an homeomorphism for the strong topologies.

The key addition is that with Orthogonality we can apply all the results developed in [BJ25a].

4.4.1 Scheme of proof

We separate the proof in three key steps:

1. We have shown in Theorem 4.2.9 and Proposition 4.2.12 that the Monge–Ampère is continuous and up by translating by a constant is injective.

2. From Theorem 4.4.3, itself based in [BJ25a, Theorem 2.22], we get that the Monge–Ampère has dense image in $\mathcal{M}^1$.

3. The Envelope Property in turn implies that, given an approximation $\mathrm{MA}_\alpha(\varphi_j)$ of a measure $\mu \in \mathcal{M}^1$, we can take a weak limit of φ_j, and then prove that the limit φ_∞ is of finite energy and satisfies

$$\mathrm{MA}_\alpha(\varphi_\infty) = \mu.$$

To check that for a given measure $\mu \in \mathcal{M}^1$, a potential φ solves the Monge–Ampère equation (4.4.0.1) we, as in [BFJ15, BJ22], prove that there is a variational interpretation for (4.4.0.1). As seen before, we can define the *energy* of a measure μ to be the quantity

$$\mathrm{E}^\vee_\alpha(\mu) \doteq \sup_{\psi \in \mathcal{E}^1(\alpha)} \{\mathrm{E}_\alpha(\psi) - \int \psi \, \mathrm{d}\mu\}$$

and we prove

$$\mathrm{E}^\vee_\alpha(\mu) = \mathrm{E}_\alpha(\varphi) - \int \varphi \, \mathrm{d}\mu \iff \mathrm{MA}_\alpha(\varphi) = \mu. \tag{4.4.1.1}$$

This is the key step to show that the Monge–Ampère has dense image. More concretely, as in [BJ25a] we define:

Definition 4.4.2. *A sequence $\varphi_i \in \mathcal{E}^1(\alpha)$ is a* maximizing sequence for μ *if*

$$\mathrm{E}_\alpha(\varphi_i) - \int \varphi \, \mathrm{d}\mu \to \mathrm{E}^\vee_\alpha(\mu).$$

Given a measure $\mu \in \mathcal{M}^1$, and a sup normalized maximizing sequence $\varphi_j \in \mathrm{PSH}_{\sup}$, by the Envelope Property we have that there exists a weak limit $\varphi_j \to \varphi_\infty \in \mathrm{PSH}_{\sup}$. We then show that, since φ_j is maximizing, we have strong convergence $\varphi_j \to \varphi_\infty \in \mathcal{E}^1(\alpha)$, and the result follows from the continuity of the Monge–Ampère operator.

4.4.2 Calabi–Yau Theorem

Proposition 4.4.3. *Let $\mu \in \mathcal{M}^1$, and $\varphi_j \in \mathcal{E}^1(\alpha)$ a maximizing sequence, then*

$$\mathrm{MA}_\alpha(\varphi_j) \xrightarrow{s} \mu$$

converges strongly.

Proof. Since

$$\delta_\alpha(\mu, \mathrm{MA}_\alpha(\varphi_j)) = \mathrm{E}^\vee_\alpha(\mu) - \mathrm{J}_\alpha(\mu, \varphi_j) \to 0$$

Theorem 4.2.8 implies that $\mathrm{MA}_\alpha(\varphi_j)$ converges strongly to μ. □

Lemma 4.4.4. *Let $\mu \in \mathcal{M}^1$ be a probability measure of finite energy, and $\varphi \in \mathcal{E}^1$ such that*

$$\mathrm{E}^\vee(\mu) = \mathrm{E}(\varphi) - \int \varphi \mathrm{d}\mu,$$

then $\mathrm{MA}(\varphi) = \mu$.

Proof. If φ maximizes the energy $\mathrm{E}^\vee(\mu)$, then tautologically the constant sequence $\varphi_i = \varphi$ is a maximizing sequence, and by weak convergence of the Monge–Ampère measures of Proposition 4.4.3 the result follows. $\square$

For the next theorem we will follow the strategy of [BJ22, Theorem 12.8].

Theorem 4.4.5 (Calabi–Yau theorem)**.** *The Monge–Ampère operator*

$$\mathrm{MA}_\alpha \colon \mathcal{E}^1(\alpha) \to \mathcal{M}^1$$

is surjective.

Proof. Let $\mu \in \mathcal{M}^1$ be a measure of finite energy, and let $\varphi_i \in \mathcal{H}(\alpha)$ be a sup normalized maximizing sequence of μ. In Theorem 4.3.2 we have shown that $\mathrm{PSH}_{\sup}(\alpha)$ is weakly compact, thus, up by passing to a subsequence, we can suppose that φ_i converges weakly to some function $\varphi \in \mathrm{PSH}_{\sup}$. We will now prove that $\varphi \in \mathcal{E}^1(\alpha)$, and moreover $\mathrm{MA}_\alpha(\varphi) = \mu$.

Indeed, by Lemma 4.1.5, the sequence $\mathrm{J}_\alpha(\varphi_i)$ is bounded. By lower semicontinuity of J in the weak topology, this implies that $\mathrm{J}_\alpha(\varphi)$ is finite, which in turn implies that $\varphi \in \mathcal{E}^1$ is of finite energy.

Now, we claim that $\int \varphi_i \,\mathrm{d}\mu \to \int \varphi \,\mathrm{d}\mu$. If it was the case, by the weak upper semicontinuity of the energy functional E_α, we would have:

$$\mathrm{E}_\alpha^\vee(\mu) \geq \mathrm{E}_\alpha(\varphi) - \int \varphi \,\mathrm{d}\mu \geq \limsup \left\{ \mathrm{E}_\alpha(\varphi_i) - \int \varphi_i \,\mathrm{d}\mu \right\} = \mathrm{E}_\alpha^\vee(\mu)$$

This would imply that φ is maximizing and by Lemma 4.4.4 we are done.

Therefore, let us prove the claim. Let ψ_j be any maximizing sequence of μ, and μ_j its Monge–Ampère measure. Consider $R \gg 0$ be sufficiently large so that

$$\mathrm{J}_\alpha(\varphi_i) < R, \text{ and } \mathrm{E}^\vee(\mu) < R,$$

this, in particular, implies that $\mathrm{J}_\alpha(\varphi) \leq R$ and $\mathrm{E}_\alpha^\vee(\mu_j) \leq R$ for sufficiently large j. Applying Theorem 4.2.8 we then have

$$\begin{aligned} \left| \int (\varphi - \varphi_i) \,\mathrm{d}\mu \right| &\leq \left| \int (\varphi - \varphi_i)(\,\mathrm{d}\mu - \,\mathrm{d}\mu_j) \right| + \left| \int (\varphi_i - \varphi) \,\mathrm{d}\mu_j \right| \\ &\lesssim \mathrm{J}_\alpha(\mu, \mu_j)^q \cdot R^{1-q} + \left| \int (\varphi_i - \varphi) \,\mathrm{d}\mu_j \right|, \end{aligned}$$

where $q \doteq 2^{-n}$. For j sufficiently big we then have

$$\left| \int (\varphi - \varphi_i) \,\mathrm{d}\mu \right| \lesssim \epsilon + \left| \int (\varphi_i - \varphi) \,\mathrm{d}\mu_j \right|,$$

that by Lemma 4.2.5 tends to ϵ as i tends to ∞, as $\epsilon > 0$ was arbitrary we conclude. □

4.4.3 Monge–Ampère operator as a homeomorphism

Let us denote by $\mathcal{E}^1_{\sup}(\alpha)$ the set $\{\varphi \in \mathcal{E}^1 \mid \sup \varphi = 0\}$ with the induced subspace topology. The goal of this section is to prove the following transcendental version of [BJ22, Theorem 12.8].

Theorem 4.4.6. *The Monge–Ampère operator*

$$\mathrm{MA}\colon \mathcal{E}^1_{\sup} \hookrightarrow \mathcal{M}^1$$

is a homeomorphism for the strong topologies.

Proof. Observe that the Monge–Ampère operator MA_α is surjective by Theorem 4.4.5 and restricts to a quasi-isometry between $\mathcal{E}^1_{\sup}(\alpha)$ and $\mathcal{M}^1$, therefore it is a homeomorphism. □

4.5 Regularity of solutions

In this section we will prove an analogue of [BJ22, Theorem 12.12], that is we will prove that if $\mu \in \mathcal{M}^1$ is a measure of finite energy supported on a single dual complex $\Delta_{\mathcal{X}}$, then the solution of $\mathrm{MA}(\varphi) = \mu$ is a continuous psh function.

The proof goes without change to the algebraic setting, and is a formal consequence of the comparison principle of Theorem 4.5.1 together with Theorem 2.2.2, that establishes the continuity of a psh function once restricted to a dual complex.

4.5.1 Comparison principle

The next result is an analogue of [BJ22, Theorem 7.40], the proof goes exactly the same. We will add here for completeness.

Theorem 4.5.1. *For every $\varphi, \psi \in \mathcal{E}^1(\alpha)$ define $A \doteq \{x \in X^{\beth} \mid \varphi(x) > \psi(x)\}$, we then have:*

$$\mathbb{1}_A \,\mathrm{MA}(\max\{\varphi, \psi\}) = \mathbb{1}_A \,\mathrm{MA}(\varphi).$$

Proof. Let us first suppose that $\varphi, \psi \in \mathrm{PL} \cap \mathrm{PSH}$. In this case, we let $\mathcal{X}$ be a smooth snc test configuration with $D_1, D_2, G \in \mathrm{VCar}(\mathcal{X})$ vertical divisors such that:

$$\varphi = \varphi_{D_1}, \quad \psi = \varphi_{D_2}, \quad \max\{\varphi, \psi\} = \varphi_G.$$

Since we are dealing PL functions, we know how to explicitely compute their Monge–Ampère measure: let $\mathcal{X}_0 = \sum b_E E$ be the irreducible decomposition of the central fiber, we then have

$$V_\alpha \operatorname{MA}(\varphi) = \sum_E b_E(\alpha + D)^n \cdot E\, \delta_{v_E},$$

and similar formulas hold for ψ and $\max\{\varphi, \psi\}$. Therefore we need to show that if $v_E \in A$, i.e. $\varphi(v_E) > \psi(v_E)$, then $(\alpha + D_1)^n \cdot E = (\alpha + G)^n \cdot E$. To do that we will show that D_1 and G coincide in a neighborhood of E.

Indeed, we will show that if $v_E \in A$, then for every E' irreducible component of $\mathcal{X}_0$ that intersects E, we have $\varphi(v_{E'}) \geq \psi(v_{E'})$. This is in fact a general fact about piecewise linear functions, that we will prove only in this specific case. Now, observe that $G - D_1 \geq 0$ is an effective divisor, and since $v_E \in A$ we have

$$\begin{aligned} \varphi(v_E) = \max\{\varphi, \psi\}(v_E) &\implies \max\{\varphi, \psi\}(v_E) - \psi(v_E) > 0 \\ &\implies \operatorname{ord}_E(G - D_2) > 0 \end{aligned}$$

in particular E is the support of $G - D_2$. Hence, for every divisorial valuation v centered in $\mathcal{X}$ on E, we will have:

$$\max\{\varphi, \psi\}(v) > \psi(v) \implies \varphi(v) = \max\{\varphi, \psi\}(v).$$

Since $\mathcal{X}_0$ is snc and each irreducible component is $\mathbb{C}^*$-invariant, the intersection $Z \doteq E \cap E'$ is a $\mathbb{C}^*$-invariant submanifold. Thus, by blowing-up Z, we obtain a new test configuration $\mathcal{X}'$. If we denote by v the divisorial valuation associated with the strict transform of Z on $\mathcal{X}'$, the center of v on $\mathcal{X}$ then becomes

$$Z(v, \mathcal{X}) = Z.$$

Hence, $\varphi(v) = \max\{\varphi, \psi\}(v)$, as we discussed above. This implies that G and D_1 coincide in Z, which in turn gives that G and D_1 coincide on E'.

For the general case, consider decreasing sequences φ_j and ψ_j in $\mathrm{PL} \cap \mathrm{PSH}$ that converge to φ and ψ respectively, then we follow this scheme:

- We can replace the indicator function in the statement for $f = \max\{\varphi - \psi, 0\}$
- Applying the argument above we then have

  $$f_j \operatorname{MA}(\max\{\varphi_j, \psi_j\}) = f_j \operatorname{MA}(\varphi_j)$$

  for $f_j = \max\{\varphi_j - \psi_j, 0\}$.
- Since the Monge–Ampère operator is continuous along decreasing sequences, we have $\operatorname{MA}(\max\{\varphi_j, \psi_j\}) \to \operatorname{MA}(\max\{\varphi, \psi\})$, that together with $f_j \to f$ implies the result.

$\square$

Corollary 4.5.2. *Let $\varphi, \psi \in \mathcal{E}^1$, and $K \doteq \operatorname{supp} \operatorname{MA}(\varphi)$. If*

$$\varphi|_K \geq \psi|_K,$$

we then have $\varphi \geq \psi$.

Proof. Let $\epsilon > 0$ be small, and let φ_ϵ denote the function $\varphi + \epsilon$, and $A_\epsilon \doteq \{\varphi_\epsilon > \psi\}$, observe that $A_\epsilon \supseteq K$. By Theorem 4.5.1, we have:

$$\begin{aligned}\mathbb{1}_{A_\epsilon} \operatorname{MA}(\max\{\varphi_\epsilon, \psi\}) = \mathbb{1}_{A_\epsilon} \operatorname{MA}(\varphi_\epsilon) &= \mathbb{1}_{A_\epsilon} \operatorname{MA}(\varphi) \\ &= \operatorname{MA}(\varphi),\end{aligned}$$

which implies $\operatorname{MA}(\max\{\varphi_\epsilon, \psi\}) = \operatorname{MA}(\varphi)$. By Proposition 4.2.12 we then get that there exists a constant c_ϵ such that

$$\varphi + c_\epsilon = \max\{\varphi_\epsilon, \psi\} = \max\{\varphi, \psi - \epsilon\} + \epsilon,$$

since $\varphi = \max\{\varphi, \psi - \epsilon\}$ at some point –for instance at any point of the support of $\operatorname{MA}(\varphi)$– we get that $c_\epsilon = \epsilon$, and the result follows. □

The next result is an analogue of the "easy direction" of [BJ24, Theorem 8.10], the proof is exactly the same and we add here for completeness.

Corollary 4.5.3. *Let $\varphi \in \operatorname{CPSH}$ be a continuous psh function such that its Monge–Ampère measure $\mu \doteq \operatorname{MA}(\varphi)$ is supported on a dual complex $\Delta_{\mathcal{X}} \hookrightarrow X^\beth$, then*

$$\varphi = \operatorname{P}(\varphi \circ p_{\mathcal{X}}).$$

Proof. By Lemma 2.2.1 $\varphi \leq \varphi \circ p_{\mathcal{X}}$, hence we have $\varphi \leq \operatorname{P}(\varphi \circ p_{\mathcal{X}})$.

On the other hand, on $\Delta_{\mathcal{X}} \hookrightarrow X^\beth$ the restriction $\varphi|_{\Delta_{\mathcal{X}}}$ coincides with $\operatorname{P}(\varphi \circ p_{\mathcal{X}})|_{\Delta_{\mathcal{X}}}$. In particular, $\operatorname{P}(\varphi \circ p_{\mathcal{X}}) \leq \varphi$ in the support of μ. By the Comparison Principle of Corollary 4.5.2 we get $\operatorname{P}(\varphi \circ p_{\mathcal{X}}) \leq \varphi$ everywhere, and the result follows. □

4.5.2 Continuity of solutions

We will now prove the continuity of solutions of the Monge–Ampère equation, once the measure μ is supported on a single dual complex $\Delta_{\mathcal{X}}$.

To do so, one key ingredient is Theorem 2.2.2, whose proof is still not complete since we are missing the proof of Proposition 2.2.4, which we will do now. The next proof follows exactly the same strategy as Theorem 11.12 of [BJ22].

Proof of Proposition 2.2.4. Let's first observe that:

- Since $\exp\colon]-\infty, 0] \to \mathbb{R}$ is convex and has derivative $0 \leq \exp' \leq 1$, we have that $\psi \doteq \exp(\varphi - \sup \varphi) \in \mathrm{PSH}(\alpha)$ is a bounded psh function, and thus is a potential of finite energy.

- $\varphi|_K$ is continuous if and only if $\psi|_K$ is continuous.

Hence it enough to prove the result for finite energy potentials. Let $\psi_i \in \mathcal{H}$ be a decreasing sequence to ψ, we then have:

$$0 \leq \mathrm{J}(\psi_i, \psi) = \mathrm{E}(\psi) - \mathrm{E}(\psi_i) + \int (\psi_i - \psi)\,\mathrm{MA}(\psi) \leq \int (\psi_i - \psi)\,\mathrm{MA}(\psi) \to 0,$$

where the second inequality is given by $\psi_i \geq \psi$, and the limit by the monotone convergence theorem. In particular, $\mathrm{J}(\psi_i)$ is uniformly bounded.

Let $C > 0$ be large enough such that $\mathrm{J}(\psi_i) \leq C$, and that $K \subseteq \{v \in X^{\beth} \mid T(v) \leq C\}$. Then, for every $v \in K$

$$\begin{aligned} |\psi_i(v) - \psi(v)| &\leq |\psi_i(v) - \psi_i(v_{\mathrm{triv}}) - \psi(v) + \psi(v_{\mathrm{triv}})| + |\psi_i(v_{\mathrm{triv}}) - \psi(v_{\mathrm{triv}})| \\ &= \left| \int (\psi_i - \psi)(\delta_v - \delta_{v_{\mathrm{triv}}}) \right| + |\psi_i(v_{\mathrm{triv}}) - \psi(v_{\mathrm{triv}})| = (\star), \end{aligned}$$

which, by Proposition 4.2.4 together with Proposition 4.1.0.1, implies

$$(\star) \leq \mathrm{J}(\psi_i, \psi)^q \cdot C^{1-q} + |\psi_i(v_{\mathrm{triv}}) - \psi(v_{\mathrm{triv}})| \to 0, \tag{4.5.2.1}$$

where $q \doteq 2^{-n}$. Since the RHS of (4.5.2.1) is uniform on v, this implies that $\psi_i|_K$ converges uniformly to $\psi|_K$, which in turn implies that $\psi|_K$ is continuous. $\square$

Theorem 4.5.4. *Let $\mu \in \mathcal{M}^1$ be a measure of finite energy, and $K \doteq \operatorname{supp} \mu$ be its support. If there exists a test configuration $\mathcal{X}$, whose associated dual complex $\Delta_{\mathcal{X}}$ contains K, then there exists a contiuous function $\varphi \in \mathrm{CPSH}(\alpha)$ such that*

$$\mathrm{MA}(\varphi) = \mu.$$

Proof. Let $\varphi \in \mathcal{E}^1$ be a solution to $\mathrm{MA}(\varphi) = \mu$.

Consider now a decreasing sequence $\varphi_k \in \mathcal{H}$ converging to φ. By Theorem 2.2.2, we have that $\varphi|_{\Delta_{\mathcal{X}}} \in \mathrm{C}^0$ is continuous. Hence, by Dini's theorem, $\varphi_k|_{\Delta_{\mathcal{X}}}$ converges uniformly to $\varphi|_{\Delta_{\mathcal{X}}}$.

Therefore, for every $\epsilon > 0$, if $k \gg 0$ is sufficiently large, we have:

$$\varphi|_{\Delta_{\mathcal{X}}} \leq \varphi_k|_{\Delta_{\mathcal{X}}} \leq \epsilon + \varphi|_{\Delta_{\mathcal{X}}}.$$

In particular, $\varphi_k|_K \leq \epsilon + \varphi|_K$, for K the support of $\mu = \mathrm{MA}(\varphi)$.

By the Comparison Principle of Corollary 4.5.2, we have:

$$\varphi \leq \varphi_k \leq \epsilon + \varphi,$$

which implies that φ_k converges uniformly to φ on $X^{\beth}$, and thus φ is continous. □

4.5.3 Smoothing measures

We have just seen that given a measure μ, supported on a dual complex associated to a test configuration, the solutions of

$$\mathrm{MA}_\alpha(\varphi) = \mu$$

are continuous. In general, a measure of finite energy is not supported on a dual complex, however we can define a natural "smoothing procedure". Let $\mu \in \mathcal{M}^1$ be a measure of finite energy, for each snc test configuration $\mathcal{X}$, we may define the pushforward measure

$$\mu_{\mathcal{X}} \doteq (p_{\mathcal{X}})_* \mu.$$

The measure $\mu_{\mathcal{X}}$ is a measure in $\Delta_{\mathcal{X}} \hookrightarrow X^{\beth}$ which we identify with a subset of $X^{\beth}$, by using monomial valuations. Hence, we see $\mu_{\mathcal{X}}$ as a probability measure on $X^{\beth}$, supported on the dual complex $\Delta_{\mathcal{X}}$.

With the dual complex description of $X^{\beth}$, it is easy to see that the net

$$\mu_{\mathcal{X}} \rightharpoonup \mu$$

converges weakly.

The next result is adapted from a preprint version of [BJ22], it gives the strong convergence of $\mu_{\mathcal{X}} \xrightarrow{s} \mu$.

Lemma 4.5.5. *Let $\mu \in \mathcal{M}^1$ be a measure of finite energy, then the net $\mathrm{E}^\vee(\mu_{\mathcal{X}})$ is eventually increasing. Moreover,*

$$\lim_{\mathcal{X}} \mathrm{E}^\vee(\mu_{\mathcal{X}}) = \mathrm{E}^\vee(\mu).$$

Proof. Recall that for every $\nu \in \mathcal{M}^1$ the energy $\mathrm{E}^\vee(\nu) = \sup\{\mathrm{E}(\varphi) - \int \varphi \mathrm{d}\nu \mid \varphi \in \mathcal{E}^1\}$.

By Lemma 2.2.1, we have that $(\varphi \circ p_{\mathcal{X}})_{\mathcal{X}}$ is a decreasing net that converges to φ, cf. Section 1.3. Therefore, the integral

$$\int \varphi \, \mathrm{d}\mu_{\mathcal{X}} = \int (\varphi \circ p_{\mathcal{X}}) \, \mathrm{d}\mu$$

is decreasing as well, and by the monotone convergence theorem the result follows. □

Before going on let us quickly recall

We now define an important functional on the set of measures of $X^\beth$, the non-Archimedean analogue of the *entropy* of a measure.

Definition 4.5.6. *Let μ be a Radon measure on $X^\beth$ we define the* entropy of μ *to be*

$$\begin{aligned}\mathrm{Ent}(\mu) &\doteq \int_{X^\beth} A_X(x)\,\mathrm{d}\mu(x)\\ &= \sup_{\mathcal{X}} \int (A\circ p_{\mathcal{X}})\mathrm{d}\mu,\end{aligned}$$

where A_X denotes de log discrepancy function.

We then have:

Lemma 4.5.7. *Let $\mu \in \mathcal{M}^1$ be a finite energy measure on $X^\beth$, then*

$$\sup_{\mathcal{X}} \mathrm{Ent}(\mu_{\mathcal{X}}) = \mathrm{Ent}(\mu).$$

Proof. This follows directly from the definition of the entropy, together with the identity $\int (A\circ p_{\mathcal{X}})\,\mathrm{d}\mu = \int A\,\mathrm{d}\mu_{\mathcal{X}}$. □

Corollary 4.5.8. *Let $\mu \in \mathcal{M}^1$ be a measure of finite energy, we can find a sequence of measures μ_j supported on dual complexes $\Delta_{\mathcal{X}^j} \hookrightarrow X^\beth$ converging strongly to μ such that*

$$\mathrm{Ent}(\mu_j) \to \mathrm{Ent}(\mu).$$

Proof. By the previous lemma we can pick a sequence of test configurations, $\mathcal{X}^j$, such that $\mu_j \doteq \mu_{\mathcal{X}_j}$ converges weakly to μ, and such that

$$\lim_{j\to\infty} \mathrm{Ent}(\mu_j) = \mathrm{Ent}(\mu).$$

Since the energy $\mathrm{E}^\vee(\mu_{\mathcal{X}})$ is increasing by Lemma 4.5.5, we then directly have that $\mathrm{E}^\vee(\mu_j) \to \mathrm{E}^\vee(\mu)$. □

The following lemma is an easy consequence of definitions that will be used later.

Lemma 4.5.9. *Let $\mu_j, \mu \in \mathcal{M}^1$ be measures of finite energy such that $\mu_j \rightharpoonup \mu$ weakly. Then, for every test configuration $\mathcal{X}$*

$$\mathrm{Ent}(\mu_{j,\mathcal{X}}) \to \mathrm{Ent}(\mu_{\mathcal{X}}).$$

Proof. The entropy $\mathrm{Ent}(\mu_{j,\mathcal{X}})$ is defined to be

$$\int A\,\mathrm{d}\mu_{j,\mathcal{X}} = \int (A\circ p_{\mathcal{X}})\,\mathrm{d}\mu_j.$$

Since $A \circ p_{\mathcal{X}}$ is a PL function, in particular continuous, by weak convergence we have

$$\int (A \circ p_{\mathcal{X}}) \mathrm{d}\mu_j \to \int (A \circ p_{\mathcal{X}}) \mathrm{d}\mu = \int A \, \mathrm{d}\mu_{\mathcal{X}}$$

that coincides with $\mathrm{Ent}(\mu_{\mathcal{X}})$. □

4.5.4 Divisorial measures and envelopes

We have seen that a measure supported on a dual complex must have a continuous potential. In this section we will see that if furthermore the measure is supported on a finite set of divisorial valuations, then the potential of the measure must be given by the envelope of a PL function.

Definition 4.5.10. *Let $v_1, \ldots, v_N \in X^{\mathrm{div}}$ be a finite collection of divisorial valuations, we say that the measure $\mu \in \mathcal{M}^1$ given by the rational convex combination of $\delta_{v_1}, \ldots, \delta_{v_n}$ is a* divisorial measure.

We denote by $\mathcal{M}^{\mathrm{div}}_{\mathbb{Q}} \subseteq \mathcal{M}^1$ the set divisorial measures of X, and moreover by $\mathcal{M}^{\mathrm{div}}$ the tensor product $\mathcal{M}^{\mathrm{div}}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

The purpose of this section is to provide a description of the solution of the Monge-Ampère equation

$$\mathrm{MA}(\varphi) = \mu,$$

when $\mu \in \mathcal{M}^{\mathrm{div}}$ is a divisorial measure.

The next result is a non-algebraic analogue of [BFJ15, Proposition 8.6].

Proposition 4.5.11. *Let $\mu \in \mathcal{M}^{\mathrm{div}}$ be a divisorial measure, then there exists a PL function $f \in \mathrm{PL}$, such that:*

$$\mathrm{MA}(\mathrm{P}(f)) = \mu.$$

Proof. Let $\mathcal{X}$ be a snc test configuration such that the support of μ is contained in the set of vertices $\{v_1, \ldots v_\ell\}$ of the associated dual complex $\Delta_{\mathcal{X}}$. Moreover let $\sum_{i=1}^{\ell} b_i E_i = \mathcal{X}_0$ be the irreducible decomposition. By Theorem 4.5.4, we can find a continuous functions $\psi \in \mathrm{CPSH}(\alpha)$ satisfying:

$$\mathrm{MA}(\psi) = \mu.$$

Now, consider the vertical $\mathbb{R}$-divisor on $\mathcal{X}$ given by:

$$G \doteq \sum_{i=1}^{\ell} \psi(v_i) \, b_i \, E_i,$$

and denote by f the associated PL function φ_G. We will now prove that $\psi = \mathrm{P}(f)$.

For that, observe that for each $i \in \{1, \ldots, \ell\}$ we have $\mathrm{P}(f)(v_i) \leq f(v_i) = \psi(v_i)$. Therefore $\mathrm{P}(f) \leq \psi$ on the support of $\mu = \mathrm{MA}(\psi)$, by the Comparison Principle of

Corollary 4.5.2 we obtain $\mathrm{P}(f) \leq \psi$ everywhere. On the other hand, applying Lemma 3.1.3 we have that

$$\mathrm{P}(f) = \sup\{\varphi \in \mathrm{PSH} \mid \varphi(v_i) \leq f(v_i)\}. \tag{4.5.4.1}$$

The result then follows by observing that ψ is a candidate on the supremum of the RHS of (4.5.4.1). $\square$

Chapter 5

Constant scalar curvature Kähler metrics and K-stability

Until this point, we have only developed extensively the non-Archimedean pluripotential theory on $X^{\beth}$. In this section, we will apply all these results to study different notions of K-stability, and how they relate to the existence of constant scalar curvature Kähler metrics.

5.1 Non-Archimedean interpretation of K-stability

Before introducing K-stability in the language of non-Archimedean pluripotential theory, let us first, for the convenience of the reader, quickly recall some important definitions:

Log discrepancy and entropy functional

To any prime divisor $F \subseteq Y \xrightarrow{\mu} X$ on a birational model of X, we may define the *log discrepancy of* F to be the quantity

$$A_X(F) \doteq 1 + \mathrm{ord}_F(K_{Y/X}).$$

More generally, if $v = r \cdot \mathrm{ord}_F$ is a divisorial valuation, we define the log discrepancy of v by

$$A_X(v) \doteq r A_X(F).$$

As before, we can extend $A_X \colon X^{\beth} \to [0, +\infty]$ the log discrepancy to more general semi-valuations.

Definition 5.1.1. *If μ is a Radon measure on $X^{\beth}$, we define the* entropy of μ *by the following integral*

$$\mathrm{Ent}(\mu) \doteq \int_{X^{\beth}} A_X \, \mathrm{d}\mu.$$

We denote by $\mathrm{H}_\alpha(\varphi)$ *the entropy of the Monge–Ampère measure* $\mathrm{Ent}(\mathrm{MA}_\alpha(\varphi))$ *of* $\varphi \in \mathcal{E}^1(\alpha)$.

Energy functionals

We recall that if α is a Kähler class, we define the *Monge–Ampère energy* to be the functional:

$$\mathcal{E}^1(\alpha) \ni \varphi \mapsto \mathrm{E}_\alpha(\varphi) \doteq \frac{V_\alpha^{-1}}{n+1}(\alpha, \varphi)^{n+1} \in \mathbb{R},$$

if ζ is any $(1,1)$-class, we then define the *twisted Monge–Ampère energy* to be the functional

$$\mathcal{E}^1(\alpha) \ni \varphi \mapsto \mathrm{E}_\alpha^\zeta(\varphi) \doteq \frac{1}{V_\alpha}(\zeta, 0) \cdot (\alpha, \varphi)^n \in \mathbb{R}.$$

Finally, we will introduce the notion of K-stability.

The non-Archimedean Mabuchi functional

We now introduce a variant of the Donaldson–Futaki invariant that is invariant under base change, called the *non-Archimedean Mabuchi functional.* This functional will let us study the K-stability of (X, α).

Let $\alpha \in H^{1,1}(X)$ be a Kähler class on X, $\zeta \doteq \mathrm{c}_1(K_X)$ be the first Chern class of the canonical bundle, and $\underline{s} \doteq -n\frac{\alpha^{n-1}\cdot\zeta}{V_\alpha}$. We define the *non-Archimedean Mabuchi functional* M_α to be

$$\mathrm{M}_\alpha(\varphi) \doteq \underline{s}\,\mathrm{E}_\alpha(\varphi) + \mathrm{E}_\alpha^\zeta(\varphi) + \mathrm{H}_\alpha(\varphi),$$

for $\varphi \in \mathcal{E}^1(\alpha)$.

Denoting by $\mathrm{J}_\alpha \colon \mathcal{E}^1(\alpha) \times \mathcal{E}^1(\alpha) \to [0, +\infty[$ be the difference

$$\mathrm{J}_\alpha(\varphi, \psi) \doteq \mathrm{E}_\alpha(\varphi) - \mathrm{E}_\alpha(\psi) + \int (\psi - \varphi)\, \mathrm{MA}_\alpha(\varphi),$$

and simply $\mathrm{J}_\alpha(\varphi)$ the quantity $\mathrm{J}_\alpha(0, \varphi)$, we define:

Definition 5.1.2. *We say that* (X, α) *is* uniformly K-stable *if there exists* $\delta > 0$ *such that*

$$\mathrm{M}_\alpha(\varphi) \geq \delta \mathrm{J}_\alpha(\varphi), \qquad \textit{for every } \varphi \in \mathcal{H}(\alpha).$$

What we will do next is to take advantage of the fact that test configurations, in this non-Archimedean language, correspond to functions in $X^\beth$, in order to strengthen the notion of K-stability, by enlarging the set of functions where K-stability is tested.

5.2 Introducing $\widehat{K}$-stability

In this section we will define this stronger notion of stability that we call $\widehat{K}$-stability, and use the non-Archimedean pluripotential theory we developed to the study this notion.

As a direct consequence of solving the non-Archimedean Monge–Ampère equation, arguing like [Li22, Proposition 6.3], we get:

Proposition 5.2.1. *The following are equivalent:*

1. *There exits $\delta > 0$ such that*
$$\mathrm{M}_\alpha(\varphi) \geq \delta \mathrm{J}_\alpha(\varphi) \textit{ for every } \varphi \in \mathcal{E}^1(\alpha).$$

2. *There exits $\delta > 0$ such that*
$$\mathrm{M}_\alpha(\varphi) \geq \delta \mathrm{J}_\alpha(\varphi) \textit{ for every } \varphi \in \mathrm{CPSH}(\alpha).$$

Proof. Since $\mathrm{CPSH}(\alpha) \subseteq \mathcal{E}^1(\alpha)$ one implication is clear.

For the other implication, we will prove that for every $\varphi \in \mathcal{E}^1$ there exists a sequence $\varphi_j \in \mathrm{CPSH}$ such that
$$\varphi_j \xrightarrow{s} \varphi, \text{ and also } \mathrm{M}(\varphi_j) \to \mathrm{M}(\varphi).$$
Also, since both J_α and M_α are translation invariant, we can suppose that $\sup \varphi = 0 = \sup \varphi_i$.

Let μ be the Monge–Ampère measure of φ. By Corollary 4.5.8, we can find a sequence of measures μ_j supported on dual complexes $\Delta_{\mathcal{X}^j}$ converging strongly to μ such that

$$\mathrm{Ent}(\mu_j) \to \mathrm{Ent}(\mu). \tag{5.2.0.1}$$

Applying Theorem 4.5.4, we obtain a sequence of continuous functions φ_j satisfying

$$\mathrm{MA}(\varphi_j) = \mu_j,$$

which, by Theorem 4.4.6, implies that φ_j converges strongly to φ. By the entropy convergence of (5.2.0.1), we get:
$$\mathrm{M}(\varphi_j) \to \mathrm{M}(\varphi).$$

□

Definition 5.2.2. *If α satisfy any of the above conditions we say that (X, α) is* uniformly $\widehat{K}$-stable.

In Chi Li's paper [Li22], he introduces the notion of *K-stability for models* for a polarized manifold, we will study its analogue in this transcendental setting.

5.2.1 Chi Li's K-stability for models

Definition 5.2.3. *We say that* (X, α) *is* uniformly K-stable for models in the sense of Chi Li, *if there exists* $\delta > 0$ *such that for every* $f \in \mathrm{PL}$ *we have:*

$$\mathrm{M}_\alpha\left(\mathrm{P}_\alpha(f)\right) \geq \delta \mathrm{J}_\alpha\left(\mathrm{P}_\alpha(f)\right).$$

Once again, this result an analogue of part of [Li22, Proposition 6.3].

Proposition 5.2.4. *Uniform* $\widehat{K}$*-stability is equivalent to uniform K-stability for models in the sense of Chi Li. This means that the following assertions are equivalent:*

- *There exits* $\delta > 0$ *such that*

$$\mathrm{M}_\alpha(\varphi) \geq \delta \mathrm{J}_\alpha(\varphi) \textit{ for every } \varphi \in \mathrm{CPSH}(\alpha).$$

- *There exits* $\delta > 0$ *such that*

$$\mathrm{M}_\alpha(\mathrm{P}_\alpha(f)) \geq \delta \mathrm{J}_\alpha(\mathrm{P}_\alpha(f)) \textit{ for every } f \in \mathrm{PL}.$$

Proof. By the continuity of envelopes of Theorem 3.1.1, we have that one implication is clear.

For the other implication we will argue like in Proposition 5.2.1. We will prove that for every $\varphi \in \mathrm{CPSH}$, whose Monge–Ampère measure $\mu \doteq \mathrm{MA}(\varphi)$ is supported on a dual complex $\Delta_{\mathcal{X}}$, we can find a sequence $f_j \in \mathrm{PL}$ such that:

$$\mathrm{P}(f_j) \xrightarrow{s} \varphi, \quad \text{and } \mathrm{M}(\mathrm{P}(f_j)) \to \mathrm{M}(\varphi). \tag{5.2.1.1}$$

By Corollary 4.5.3, we have that $\varphi = \mathrm{P}(\varphi \circ p_{\mathcal{X}})$.

Since $\varphi \circ p_{\mathcal{X}}$ is a continuous function, that is invariant by $p_{\mathcal{X}}$, we can find a sequence of PL functions $f_j \in \mathrm{PL}$ such that:

- The sequence converges f_j uniformly to $\varphi \circ p_{\mathcal{X}}$

$$f_j \rightrightarrows \varphi \circ p_{\mathcal{X}}.$$

- We can find a test configuration $\mathcal{X}^j$, together with a vertical divisor $D_j \in \mathrm{VCar}(\mathcal{X}^j)$, such that $f_j = \varphi_{D_j}$, whose dual complex $\Delta_{\mathcal{X}^j}$ is just a subdivision of $\Delta_{\mathcal{X}}$. That is, $\mathcal{X}^j$ can be obtained by a sequence of blow-ups of smooth centers of $\mathcal{X}$.

As a consequence of the first point, we have $\mathrm{P}(f_j) \rightrightarrows \mathrm{P}(\varphi \circ p_{\mathcal{X}}) = \varphi$. In particular, $\mathrm{P}(f_j)$ converges strongly to φ.

Moreover, by Theorem 3.3.3 we have that $\mu_j \doteq \mathrm{MA}(\mathrm{P}(f_j))$ is supported on the set $\Delta_{\mathcal{X}^j} \hookrightarrow X^{\beth}$. Since the dual complex $\Delta_{\mathcal{X}^j}$ is a subdivision of $\Delta_{\mathcal{X}}$, as valuations they coincide

$$\mathrm{val}(\Delta_{\mathcal{X}^j}) = \mathrm{val}(\Delta_{\mathcal{X}}) \subseteq X^{\beth}.$$

Therefore μ_j is supported on $\Delta_{\mathcal{X}} \hookrightarrow X^{\beth}$ as well. By Lemma 4.5.9, we have

$$\mathrm{Ent}(\mu_j) \to \mathrm{Ent}(\mu),$$

that, together with the strong convergence $\mu_j \xrightarrow{s} \mu$, gives (5.2.1.1). □

We will now give a valuative criterion for $\widehat{K}$-stability. We begin by introducing valuative criteria for different notions of stability:

5.3 Valuative criteria for K-stability

When X is a Fano manifold, and $\alpha = \mathrm{c}_1(X)$, Fujita and Li [Fuj19, Li17], relying on the work of Li and Xu [LX14], gave a *valuative criterion for K-stability*. They introduced a numerical invariant $\beta\colon X^{\mathrm{div}} \to \mathbb{R}$, which for each divisorial valuation $v = \mathrm{ord}_F$ associates a real number given by:

$$\beta(v) = \beta(F) \doteq A_X(v) - \int_0^{+\infty} \mathrm{vol}(-K_X + \lambda F)\mathrm{d}\lambda.$$

They then proved that the positivity of this invariant in a special class of divisorial valuations is equivalent to K-stability of X. More concretely, if $\beta(v) > 0$, for all the divisorial valuations v that arise from a *dreamy divisor* –a divisor that induces an ample (special) test configuration[1]– then $(X, -K_X)$ is K-stable. Conversely, if X is K-stable, then $\beta(v) > 0$ for all such v. This point of view has turned out very effective when checking K-stability of low dimensional Fano varieties.

Dervan and Legendre in [DL23], extended this invariant to a general polarized (X, L), and proved that K-stability with respect to test configurations with integral central fiber is equivalent to the positivity of their β-invariant in this more general setting.

Furthermore, in the polarized setting, Boucksom and Jonsson in [BJ23] developed a valuative criterion for K-stability for models in the sense of Chi Li. This criterion, however, is different from the one in [DL23, Fuj19, Li17], since instead of dealing with the positivity of the β-invariant of single divisor, it deals with positivity of some invariant (that we will also call the β-invariant) on the set of divisorial measures, that is the data of a weighted combination of divisorial valuations. The objective of this section is to generalize the latter

[1] This can be formulated in the non-Archimedean dialect by saying that there exists $\varphi \in \mathcal{H}(\alpha)$ such that $\mathrm{MA}_\alpha(\varphi) = \delta_v$.

for Kähler manifolds. We also give a formula for the measure theoretic β-invariant, that involves computing only log discrepancies and integrals of volumes of the form

$$\operatorname{vol}\left(\alpha + \sum s_i F_i\right), \qquad \text{for } s_i \geq 0.$$

5.3.1 A valuative criterion for $\widehat{K}$-stability

To each measure of finite energy $\mu \in \mathcal{M}^1$, we define:

$$\begin{aligned}\beta_\alpha(\mu) &\doteq \operatorname{Ent}(\mu) + \nabla_{K_X} \mathrm{E}^\vee_\alpha(\mu) \\ &= \operatorname{Ent}(\mu) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{E}^\vee_{\alpha+tK_X}(\mu).\end{aligned}$$

We observe that the above expression is well defined, since the differentiability of the energy functional was established in [BJ25a].

Moreover, by [BJ25a, Proposition 4.5], we also find that for every $\varphi \in \mathcal{H}(\alpha)$ the beta invariant of the Monge–Ampère measure

$$\beta_\alpha(\mathrm{MA}_\alpha(\varphi)) = \mathrm{M}_\alpha(\varphi) \tag{5.3.1.1}$$

is equal to the Mabuchi energy of φ. Also, it is easy to see that the same argument, as the one in [BJ25a], proves that Equation (5.3.1.1) remains true for $\varphi \in \mathcal{E}^1(\alpha)$, and we prove

Theorem 5.3.1 (Theorem B)**.** *The pair (X, α) is uniformly $\widehat{K}$-stable if and only if there exists a $\delta > 0$ such that for every divisorial measure $\mu \in \mathcal{M}^{\mathrm{div}}$ we have*

$$\beta_\alpha(\mu) \geq \mathrm{E}^\vee_\alpha(\mu).$$

Proof. This follows directly from Proposition 4.5.11 and Proposition 5.2.4, together with Equation (5.3.1.1). □

Corollary 5.3.2. *Uniform $\widehat{K}$-stability is an open condition in the Kähler class $\alpha \in H^{1,1}(X)$.*

Proof. This follows from Theorem 5.3.1 together with [BJ25a, Theorem 5.5]. □

5.3.2 Computing the beta invariant

The main goal of this section is to give a description of how to compute the β-invariant of a divisorial measure, the key object to study the above mentioned pluridivisorial stability.

Let $\{F_1, \dots, F_\ell\}$ be a finite set of prime divisors over X, $\{v_1, \dots, v_\ell\}$ the associated divisorial valuations. Also let μ_d be the measure

$$\mu_d \doteq \sum_{i=1}^{\ell} d_i \, \delta_{v_i},$$

for $(d_1, \ldots, d_\ell) \in \{x \in (\mathbb{Q}_{\geq 0})^\ell \mid \sum x_i = 1\}$, we will denote μ_d by simply μ in the following. We recall that

$$\begin{aligned} \beta_\alpha(\mu) &= \mathrm{Ent}(\mu) + \nabla_{K_X} \mathrm{E}_\alpha^\vee(\mu) \\ &= \sum_{i=1}^{\ell} d_i \cdot A(F_i) + \nabla_{K_X} \mathrm{E}_\alpha^\vee(\mu). \end{aligned}$$

Hence, —up by computing the log discrepancy of the divisors $F_1, \ldots, F_\ell$— to calculate the β-invariant of μ, it is enough to compute the energy $\mathrm{E}_\zeta^\vee(\mu)$ explicitly for Kähler classes $\zeta \in H^{1,1}(X)$. More precisely, to check $\widehat{K}$-stability of (X, α), it enough to compute $\mathrm{E}_{\alpha + tK_X}^\vee(\mu)$, for t sufficiently small.

Let $\alpha \in H^{1,1}(X)$ be a Kähler class, we will now describe how to compute energy $\mathrm{E}_\alpha^\vee(\mu)$. To do so, we start by recalling that $\mathrm{E}_\alpha^\vee(\mu)$ is defined as

$$\sup \left\{ \mathrm{E}_\alpha(\varphi) - \int \varphi \, \mathrm{d}\mu \mid \varphi \in \mathcal{E}^1(\alpha) \right\}.$$

If $t = (t_1, \ldots, t_\ell) \in \mathbb{R}^\ell$, we denote by φ_t the potential

$$\varphi_t \doteq \{\varphi \in \mathrm{PSH}(\alpha) \mid \varphi(v_i) \leq t_i \text{ for every } i = 1, \ldots, \ell\} \in \mathrm{CPSH}(\alpha). \tag{5.3.2.1}$$

By Proposition 4.5.11 and Equation (4.5.4.1) there exists $t^\star \in \mathbb{R}^\ell$ such that the associated potential $\varphi_{t^\star}$ maximizes the energy

$$\mathrm{E}_\alpha^\vee(\mu) = \mathrm{E}_\alpha(\varphi_{t^\star}) - \int \varphi_{t^\star} \, \mathrm{d}\mu$$

which, as we saw before, is equivalent to solving $\mathrm{MA}_\alpha(\varphi_{t^\star}) = \mu$. Moreover, by Orthogonality we get

$$\mathrm{E}_\alpha^\vee(\mu) = \mathrm{E}_\alpha(\varphi_{t^\star}) - \sum d_i \cdot t_i^\star.$$

In particular,

$$\begin{aligned} \mathrm{E}_\alpha^\vee(\mu_d) &\geq \sup \left\{ \mathrm{E}_\alpha(\varphi_t) - \int \varphi_t \, \mathrm{d}\mu \mid t \in \mathbb{R}^\ell \right\} \\ &\geq \sup \left\{ \mathrm{E}_\alpha(\varphi_t) - d \cdot t \mid t \in \mathbb{R}^\ell \right\} \\ &\geq \mathrm{E}_\alpha(\varphi_{t^\star}) - d \cdot t^\star = \mathrm{E}_\alpha^\vee(\mu_d). \end{aligned} \tag{5.3.2.2}$$

In this way, we obtain the following finite dimensional optimization description of the energy $\mathrm{E}_\alpha^\vee(\mu)$: denoting by $f_\alpha\colon \mathbb{R}^\ell \to \mathbb{R}$ the function

$$t \mapsto \mathrm{E}_\alpha(\varphi_t),$$

we get

$$\mathrm{E}_\alpha^\vee(\mu_d) = \sup_{t\in\mathbb{R}^\ell} \{f_\alpha(t) - d\cdot t\}.$$

What we will do next is to compute f_α in terms of the expressions of the form

$$\mathrm{vol}(\alpha - \sum_{i=1}^{\ell} s_i F_i), \text{ for } s \in (\mathbb{R}_+)^\ell. \tag{5.3.2.3}$$

In such a way that we will see that f_α is convex, and then, using Equation (5.3.2.2), we see that the Legendre transform of $-f$ at the point $-d$ computes the energy of μ_d

$$\mathrm{E}_\alpha^\vee(\mu_d) = \widehat{-f_\alpha}(-d).$$

Computing $f(t)$

In order to compute the function $f(t)$ we will do as follows:

1. Use Theorem 2.4.17 to get a maximal geodesic ray U associated to φ_t such that

$$U^\daleth = \varphi_t, \quad \text{and} \quad \lim_{s\to\infty} \frac{\mathrm{E}_\omega(U_s)}{s} = \mathrm{E}_\alpha(\varphi_t) = f(t).$$

2. Give an explicit description of Ross–Witt Nyström transform of U, so we can apply [DXZ25, Theorem 2.6 (iii)] to compute the energy of U in terms of expressions of the form of (5.3.2.3).

Let us fix $t \in \mathbb{R}^\ell$, and denote by $\mathcal{F}_\lambda$ the set

$$\{u \in \mathrm{PSH}_{\sup}(\omega) \mid \nu(u, F_i) \geq \lambda - t_i \text{ for every } i = 1, \dots, \ell\},$$

where $\mathrm{PSH}_{\sup}$ denotes the set of ω-psh functions with supremum normalized to be zero, and G_λ denotes the ω-psh function $\sup\{u \in \mathcal{F}_\lambda\}$.

It is clear that $(G_\lambda)_\lambda$ is a *relative test curve*, that is for every $x \in X$ the map $\lambda \mapsto G_\lambda(x)$ is

- Decreasing: if $\lambda \leq \mu$ then $\mathcal{F}_\mu \subseteq \mathcal{F}_\lambda$, and thus $G_\mu \leq G_\lambda$.
- Concave: for any $t \in [0,1]$ and any two functions $u_1 \in \mathcal{F}_\lambda$, $u_2 \in \mathcal{F}_\mu$, we have $t\cdot u_1 + (1-t)\cdot u_2 \in \mathcal{F}_{t\lambda+(1-t)\mu}$, implying $t \cdot G_\lambda + (1-t)\cdot G_\mu \leq G_{t\lambda+(1-t)\mu}$.
- Upper semicontinuous: This follows from the fact that $\bigcap_{\epsilon>0} \mathcal{F}_{\lambda+\epsilon} = \mathcal{F}_\lambda$.

It is also clear that by definition G_λ is $\mathcal{I}$-*maximal*, that is

$$\mathrm{P}[G_\lambda]_{\mathcal{I}} = G_\lambda.$$

We also observe that since Lelong numbers are bounded, there exists $\lambda_0 \in \mathbb{R}$ such that for every $\lambda \geq \lambda_0$ the set $\mathcal{F}_\lambda$ is empty, we denote $\lambda_{\max}$ the smallest of such λ_0. Let us also denote by $\lambda_{\min}(t)$, or simply $\lambda_{\min}$, the quantity $\min\{t_1, \dots, t_\ell\}$, and point out that for every $\lambda \leq \lambda_{\min}$ the function G_λ is constant equal to zero.

Lemma 5.3.3. *The Ross–Witt Nyström transform of G_λ is the maximal geodesic ray starting from 0 associated to φ_t.*

Proof. Let U be a maximal geodesic ray associated to φ_t. As in [DXZ25, Proposition 3.1] —whose proof, originally for the projective setting, applies as is to our transcendental setting— we have that for every divisorial valuation $v = r \cdot \mathrm{ord}_F$

$$\sup\{-r\,\nu(\check{U}_\lambda, F) + \lambda \mid \lambda \leq \lambda_{\max}\} = U^{\beth}(v) = \varphi_t(v).$$

If we normalize so that $U_0 = 0$, then we observe that

$$0 = U_0 = \sup_{\lambda\in\mathbb{R}} \check{U}_\lambda + \lambda \cdot 0 = \sup_{\lambda\in\mathbb{R}} \check{U}_\lambda.$$

Thus, for every $\lambda \in \mathbb{R}$, the supremum

$$\sup \check{U}_\lambda \leq 0.$$

Moreover, we have for every $i = 1, \dots, \ell$ and every $\lambda \leq \lambda_{\max}$

$$-\nu(\check{U}_\lambda, F_i) + \lambda \leq \varphi_t(\mathrm{ord}_{F_i}) \leq t_i, \iff \nu(\check{U}_\lambda, F_i) \geq \lambda - t_i.$$

This, in turn implies that $\check{U}_\lambda \leq G_\lambda$. Since G_λ is $\mathcal{I}$-maximal (in particular maximal), the Ross–Witt Nyström transform of G_λ is a psh geodesic ray, which satisfies

$$U_s \leq \hat{G}_s, \quad \text{and for every } i = 1, \dots, \ell \quad \hat{G}^{\beth}(\mathrm{ord}_{F_i}) \leq t_i,$$

where, again, the second inequality is obtained by using [DXZ25, Proposition 3.1]. Combining the previous inequalities we have

$$\varphi_t = U^{\beth} \leq \hat{G}^{\beth} \leq \varphi_t,$$

which, by maximality of U, implies that $U = \hat{G}$. □

As a consequence we compute $f(t)$:

Theorem 5.3.4. *For every $t \in \mathbb{R}^\ell$ the energy of φ_t*

$$f_\alpha(t) = \lambda_{\min} + V_\alpha^{-1} \int_{\lambda_{\min}}^{+\infty} \mathrm{vol}\left(\alpha - \sum_{i=1}^{\ell} (\lambda - t_i)_+ F_i\right) \mathrm{d}\lambda,$$

where $\lambda_{\min}$ is any constant strictly less than $\min\{t_1, \dots, t_\ell\}$.

Proof. Once again, let U be the maximal geodesic ray associated to φ_t starting from 0, and recall that by the previous Lemma, its Ross–Witt Nyström transform is given by the family $\lambda \mapsto G_\lambda$.

By [DXZ25, Theorem 2.6 (iii)], we get

$$f(t) = \frac{\mathrm{E}_\omega(U_s)}{s} = \lambda_{\max} + \int_{-\infty}^{\lambda_{\max}} \left(-1 + V_\alpha^{-1} \int_X \langle \omega + \mathrm{dd^c} G_\lambda \rangle^n \right) \mathrm{d}\lambda. \tag{5.3.2.4}$$

Let us now compute the last energy term. Let θ_{F_i} be a smooth form representing the first Chern class of the line bundle associated to F_i, $\mathrm{c}_1(\mathcal{O}(F_i))$, and let ψ_{F_i} be a potential satisfying

$$\theta_{F_i} + \mathrm{dd^c} \psi_{F_i} = \delta_{F_i},$$

where δ_{F_i} represents the current of integration along F_i. For simplicity, denote by θ_F^λ the smooth form $\sum_{i=1}^{\ell} (\lambda - t_i)_+ \theta_{F_i}$. Moreover, denote by ψ_F^λ the function $\sum_{i=1}^{\ell} (\lambda - t_i)_+ \psi_{F_i}$, and by δ_F^λ the measure $\sum_{i=1}^{\ell} (\lambda - t_i)_+ \delta_{F_i}$. Then, the current

$$\omega - \theta_F^\lambda + \mathrm{dd^c} \left[G_\lambda - \psi_F^\lambda \right] = \omega + \mathrm{dd^c} G_\lambda - \delta_F^\lambda \in \alpha - \sum_{i=1}^{\ell} (\lambda - t_i)_+ F_i$$

is of minimal singularities in $\alpha - \sum_{i=1}^{\ell} (\lambda - t_i)_+ F_i$.

Furthermore, the integral of the nonpluripolar product satisfies:

$$\int_X \langle \omega + \mathrm{dd^c} G_\lambda \rangle^n = \int_X \langle \omega + \mathrm{dd^c} G_\lambda - \delta_F^\lambda \rangle^n.$$

Since the latter is of minimal singularities, we then have

$$\int_X \langle \omega + \mathrm{dd^c} G_\lambda \rangle^n = \mathrm{vol}\left(\alpha - \sum_{i=1}^{\ell} (\lambda - t_i)_+ F_i\right).$$

Adding up everything together we obtain the desired formula:

$$
\begin{aligned}
f(t) = \mathrm{E}_\alpha(\varphi_t) &= \lambda_{\max} + \int_{-\infty}^{\lambda_{\max}} \left(-1 + V_\alpha^{-1} \int_X \langle \omega + \mathrm{dd^c} G_\lambda \rangle^n \right) \mathrm{d}\lambda \\
&= \lambda_{\max} + \int_{\lambda_{\min}}^{\lambda_{\max}} \left(-1 + V_\alpha^{-1} \int_X \langle \omega + \mathrm{dd^c} G_\lambda \rangle^n \right) \mathrm{d}\lambda \\
&= \lambda_{\min} + \int_{\lambda_{\min}}^{\lambda_{\max}} \left(V_\alpha^{-1} \int_X \langle \omega + \mathrm{dd^c} G_\lambda \rangle^n \right) \mathrm{d}\lambda \\
&= \lambda_{\min} + V_\alpha^{-1} \int_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{vol} \left(\alpha - \sum_{i=1}^{\ell} (\lambda - t_i)_+ F_i \right) \mathrm{d}\lambda.
\end{aligned}
\tag{5.3.2.5}
$$

where $\lambda_{\min} < \min\{t_1, \ldots, t_\ell\}$. $\square$

Specializing the previous result for the case $\ell = 1$, we get the known formula:

$$
\mathrm{E}_\alpha(\varphi_v) = V_\alpha^{-1} \int_0^{\tau_{\mathrm{psef}}} \mathrm{vol}(\alpha - \lambda F)\, \mathrm{d}\lambda.
$$

Corollary 5.3.5. *The function $f(t)$ is concave, and the Legendre transform of $-f(t)$ is given by*

$$
\widehat{-f}\,(-\xi) = \sup_{t \in \mathbb{R}^\ell} \{ -\langle \xi, t \rangle + f(t) \} = \mathrm{E}_\alpha^\vee(\mu_\xi),
$$

where $\mu_\xi \doteq \sum_{i=1}^{\ell} \xi_i \delta_{v_i}$ is the divisorial measure associated to ξ and $F_1, \ldots, F_\ell$.

If we denote by $g_\alpha(\xi)$ the Legendre transform $\widehat{-f}\,(-\xi)$, for $f_\alpha(t)$ as in Theorem 5.3.4 we get the following result:

Corollary 5.3.6. *Let μ_ξ be as before, then*

$$
\beta_\alpha(\mu_\xi) = \sum_{i=1}^{\ell} \xi_i \cdot A_X(F_i) + \nabla_{K_X}\, g_\alpha(\xi).
$$

As a moral conclusion, we have that, in order to compute the beta invariant of a divisorial measure with support on $\mathrm{ord}_{F_1}, \ldots, \mathrm{ord}_{F_\ell}$, it is enough to be able to compute the log discrepancy of $F_1, \ldots, F_\ell$ and the integral of volumes of the form

$$
\mathrm{vol}(\alpha + \sum_{i=1}^{\ell} s_i F_i), \qquad \text{for } s \in (\mathbb{R}_+)^\ell.
$$

In the next section we will prove that uniform $\widehat{K}$-stability implies the existence of cscK metrics.

5.4 CscK metrics and $\widehat{K}$-stability

In this section we will generalize a result of Chi Li on the existence of cscK metrics to the transcendental setting. We recall that (X,ω) is again a compact Kähler manifold, $\alpha = [\omega]$ the cohomology class of ω, $\eta \doteq -\operatorname{Ric}(\omega)$ minus the Ricci form of ω, ζ its cohomology class, and $\underline{s}$ the cohomological constant $n \cdot \frac{[\operatorname{Ric}(\omega)]\cdot[\omega]^{n-1}}{[\omega]^n}$.

We begin by recalling some functionals introduced in Section 2.4, let $u \in \mathcal{E}^1(\omega)$ be an *Archimedean* potential of finite energy we define the *Monge–Ampère energy:*

$$\mathrm{E}_\omega(u) \doteq \frac{V_\alpha^{-1}}{n+1}(\omega,u)^{n+1} = \frac{V_\alpha^{-1}}{n+1}\sum_{j=0}^{n}\int_X u\,(\omega+\mathrm{dd^c}u)^j\wedge\omega^{n-j},$$

the *twisted Monge–Ampère energy*, also known as the *Ricci energy*

$$\mathrm{E}^\eta_\omega(u) \doteq \frac{1}{V_\alpha}(\eta,0)\cdot(\omega,u)^n = \frac{1}{V_\alpha}\sum_{j=0}^{n-1}\int_X u\,\eta\wedge(\omega+\mathrm{dd^c}u)^{n-j-1}\wedge\omega^j,$$

and lastly the *entropy* of the Monge–Ampère measure of u

$$\mathrm{H}_\mu(u) \doteq V_\alpha^{-1}\int_X \log\left(\frac{(\omega+\mathrm{dd^c}u)^n}{\omega^n}\right)(\omega+\mathrm{dd^c}u)^n.$$

Let us now discuss on the existence of constant scalar curvature Kähler (cscK) metrics:

5.4.1 The variational approach to the cscK problem

In Kähler geometry we say that the cscK problem is the one to look for Kähler metrics $\omega' \in \alpha$ such that their scalar curvature is constant. This then translates to finding a potential u solving the *cscK equation:*

$$\operatorname{Scal}(\omega+\mathrm{dd^c}u) = \underline{s}.$$

Luckily, there is a variational interpretation of this equation. This means that the cscK equation is the Euler–Lagrange equation for the *Mabuchi functional*:

$$\mathrm{M}_\omega = \underline{s}\,\mathrm{E}_\omega + \mathrm{E}^\eta_\omega + \mathrm{H}_\mu, \tag{5.4.1.1}$$

where μ is the probability measure associated to ω^n, and $\mathrm{H}_\mu(u)$ is the entropy of the Monge–Ampère measure of u with respect to μ as we have just seen.

By the work of Chen–Cheng, [CC21a, CC21b], there exist a unique cscK metric in α if, and only if, the Mabuchi functional is coercive, that is:

$$\mathrm{M}_\omega \geq \delta\,\mathrm{J}_\omega - C$$

for some $\delta, C > 0$.

As the notation indicates, we have already studied the non-Archimedean counterpart of the Mabuchi energy, M_α, what we will do next is then to use the slope formulas of Section 2.4 to prove that the coercivity of M_ω follows from the –non-Archimedean– *coercivity over* $\mathcal{E}^1$ of M_α, which is what we call $\widehat{K}$-stability.

Before comparing the non-Archimedean version of the entropy functional with the present one, we recall the following Legendre transform formula for the Archimedean entropy: if $u \in \mathcal{E}^1(\omega)$ we have

$$\mathrm{H}_\mu(u) = \sup_{f \in \mathrm{C}^0(X)} \left\{ \int_X f \ \mathrm{MA}_\omega(u) - \log \int_X \exp(f)\, \mathrm{d}\mu \right\}. \tag{5.4.1.2}$$

Proposition 5.4.1. *Let $\psi \in \mathcal{E}^1(\alpha)$, consider $V_s \in \mathcal{E}^1(\omega)$ the maximal geodesic ray associated to ψ. Then, it holds that*

$$\mathrm{H}_\alpha(\psi) \leq \lim_{s \to +\infty} \frac{\mathrm{H}_\mu(V_s)}{s}. \tag{5.4.1.3}$$

Proof. Let $\psi \in \mathcal{E}^1(\alpha)$, and consider $\mathcal{X}$ a snc test configuration. As seen before $A_X \circ p_{\mathcal{X}}$ is a PL function, let's denote it φ. We can write $A_\alpha^{\mathcal{X}}$ in terms of φ:

$$\begin{aligned} H_\alpha^{\mathcal{X}}(\psi) = \int_{X^\beth} (A \circ p_{\mathcal{X}})\, \mathrm{MA}_\alpha(\psi) &= \int_{X^\beth} \varphi\, \mathrm{MA}_\alpha(\psi) \\ &= (0, \varphi) \cdot (\alpha, \psi)^n. \end{aligned}$$

Then, by Theorem 2.4.17:

$$(0, \varphi) \cdot (\alpha, \psi)^n = \lim_{s \to +\infty} \frac{1}{s} (0, U_s) \cdot (\alpha, V_s)^n, \tag{5.4.1.4}$$

for U a smoothly compatible ray with φ.

On the other hand, for $f = U_s$ we have:

$$\begin{aligned} \frac{1}{s} H_\mu(V_s) &\geq \frac{1}{s} \left\{ \int_X f \omega_{V_s}^n - \log \int_X \exp(f)\, \mathrm{d}\mu \right\} \\ &= \frac{1}{s}(0, U_s) \cdot (\omega, V_s)^n - \frac{1}{s} \log \int_X \exp(U_s)\, \mathrm{d}\mu \longrightarrow (0, \varphi) \cdot (\alpha, \psi)^n - 0 = H_\alpha^{\mathcal{X}}(\psi) \end{aligned}$$

where in the limit we make use of Lemma 3.11 of [BHJ19], to get:

$$\log \int_X \exp(U_s)\, \mathrm{d}\mu = O\left(\log(s)\right).$$

Therefore,

$$\mathrm{H}_\alpha(\psi) = \sup_{\mathcal{X}} H_\alpha^{\mathcal{X}}(\psi) \leq \lim_{s \to +\infty} \frac{1}{s} \mathrm{H}_\mu(V_s),$$

concluding the proof. □

Corollary 5.4.2. *Let $\varphi \in \mathcal{E}^1(\alpha)$, and $U_s \in \mathcal{E}^1(\omega)$ the maximal geodesic ray associated. Then,*

$$\mathrm{M}_\alpha(\varphi) \leq \lim_{s\to+\infty} \frac{\mathrm{M}_\omega(U_s)}{s}. \tag{5.4.1.5}$$

Proof. This follows from Theorem 2.4.17, together with the Proposition 5.4.1. □

5.4.2 Main theorem

Proposition 5.4.3 (Theorem 1.2 from [Li22])**.** *Let $U_s \in \mathcal{E}^1$ be a geodesic ray such that the slope*

$$\lim_{s\to+\infty} \frac{\mathrm{M}_\omega(U_s)}{s} < +\infty$$

is finite. Then, U is maximal.

Proof. The proof goes without change as in the projective setting.

It is based on a local integrability result for the exponential of a difference of psh functions in the same singularity class, and a clever use of Jensen's inequality. For more details see [Li22, Theorem 1.2]. □

We recall that we say that (X, α) is uniformly $\widehat{K}$-stable if there exists $\delta > 0$ such that

$$\mathrm{M}_\alpha(\varphi) \geq \delta \mathrm{J}_\alpha(\varphi), \qquad \text{for every } \varphi \in \mathcal{E}^1(\alpha). \tag{5.4.2.1}$$

Now, we will prove Theorem A, our last main result of the thesis.

Theorem 5.4.4 (Theorem A)**.** *Let (X, α) be a compact Kähler manifold that is uniformly $\widehat{K}$-stable. Then, α contains a unique cscK metric.*

Proof. By [CC21b] the existence, and uniqueness of a cscK metric is equivalent to the coercivity of the Mabuchi functional M_ω, i.e. the existence of $C, \delta > 0$ such that

$$\mathrm{M}_\omega \geq \delta \mathrm{J}_\omega - C.$$

We will proceed by contradiction.

Suppose that M_ω is not coercive, then, by [BBJ21, Li22, CC21b], we can find a geodesic ray emanating from zero, $U_s \in \mathcal{E}^1(\omega)$, normalized so that $\sup U_s = 0$, such that:

$$\lim_{s\to+\infty} \frac{1}{s}\mathrm{M}_\omega(U_s) \leq 0.$$

By Proposition 5.4.3, U is maximal, therefore it is associated to a non-Archimedean potential $\varphi \in \mathcal{E}^1(\alpha)$.

Corollary 5.4.2 gives:
$$0 \geq \lim_{s \to +\infty} \frac{1}{s} \mathrm{M}_\omega(U_s) \geq \mathrm{M}_\alpha(\varphi),$$
but since (X, α) is uniformly $\widehat{K}$-stable , there exists a $\delta > 0$ such that:
$$\mathrm{M}_\alpha(\varphi) \geq \delta \mathrm{J}_\alpha(\varphi) > 0,$$
yielding a contradiction. □

In particular, as in the Fano setting, we get a valuative criterion for existence of cscK metrics on a Kähler manifold.

Appendix A

Some extra things

A.1 Semi-rings and tropical algebras

A.1.1 Semi-rings

Definition A.1.1. *A triple $(S, +, \cdot)$ is a commutative* semi-ring *if the following conditions hold:*

- *$(S, +)$ is a commutative monoid, with identity element denoted by 0_S*[1]*;*
- *$(S, \cdot)$ is a commutative semi-group;*
- *For every $a, b, c \in S$*

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \quad \textit{and} \quad 0_S \cdot a = 0.$$

A morphism of semi-rings *is a function*

$$\phi\colon (S, +, \cdot) \to (R, +, \cdot)$$

mapping 0_S to 0_R that satisfies

$$\phi(a + b) = \phi(a) + \phi(b), \quad \textit{and} \quad \phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

Whenever S and R have a multiplicative identity we ask

$$\phi(1_S) = 1_R$$

We denote the set of morphisms from $(S, +, \cdot)$ to $(R, +, \cdot)$ by $\hom(S, R)$

Now some examples

[1] A monoid is a semi-group with an identity element.

Example A.1.2. *1. Let $S \doteq \mathbb{R} \cup \{+\infty\}$, considered with* min *as the sum, and the usual sum, +, as the semi-ring multiplication is a semi-ring. Here S has a multiplicative identity given by:*

$$0_S = +\infty, \quad \textit{and} \quad 1_S = 0$$

Equivalently, $S = (\mathbb{R} \cup \{-\infty\}, \max, +)$ is isomorphic to $(\mathbb{R} \cup \{+\infty\}, \min, +)$.

2. *The subset $([0, +\infty], \min, +)$ is also a semi-ring.*

3. *Let X be a topological space, $(\mathrm{C}^0(X, \mathbb{R}) \cup \{-\infty\}, \max, +)$ is a semi-ring.*

4. *Let A be a commutative ring, and denote by $\mathscr{I}(A)$ the set of ideals of finite type of A, then together with the usual sum and multiplication of ideals, $\mathscr{I}(A)$ is a semi-ring with neutral elements given by:*

$$0_S = \{0\}, \quad \textit{and} \quad 1_S = A$$

A semi-ring S comes equipped with a natural order relation, we say

$$a \leq b, \quad \text{if} \quad a = b + a.$$

Whenever S is unital we denote by S_+ the set:

$$S_+ \doteq \{a \in S \mid a \geq 1_S\}$$

Lemma A.1.3. *Let S be a semi-ring with multiplicative unit, then S_+ inherits a semi-ring structure restricting the operations.*

Proof. Let $a, b \in S_+$, then $1_S = a + 1_S$ and $1_S = b + 1_S$, hence $a + b + 1_S = a + (b + 1_S) = a + 1_S = 1_S$. Moreover

$$a \cdot b + 1_S = a \cdot b + 1_S + b = (a + 1_S) \cdot b + 1_S = 1_S \cdot b + 1_S = b + 1_S = 1_S$$

□

Example A.1.4. *Using the notation of Example A.1.2 we have:*

1. *$\mathscr{I}(A)_+ = \mathscr{I}(A)$, since for every $I \in \mathscr{I}(A)$ we have*

$$I + A = A$$

2. *If $S = \mathrm{C}^0(X, \mathbb{R}) \cup \{+\infty\}$, then*

$$S_+ = \mathrm{C}^0(X, \mathbb{R}_+) \cup \{+\infty\}$$

Definition A.1.5. *A semi-ring* $(S, +, \cdot)$ *is* idempotent *if for every* $a \in S$

$$a + a = a$$

Remark A.1.6. *Every semi-ring of Example A.1.2 is idempotent.*

The order relation for idempotent semi-rings reads slightly more general, $a \leq b$ if, and only if, we can decompose

$$a = b + c$$

for some $c \in S$.

A.1.2 Tropical spectrum and restrictions

Let S be a semi-ring, we recall that the *tropical spectrum of* S is the set

$$\operatorname{TropSpec} S \doteq \hom(S, \mathbb{R} \cup \{+\infty\})$$

with the pointwise convergence topology.

Lemma A.1.7. *Let* S *be a unital semi-ring, the restriction induces a map*

$$\operatorname{TropSpec} S \to \hom(S_+, [0, +\infty])$$

Proof. Indeed, let $\chi \in \operatorname{TropSpec}$, and $f \in S_+$, we then have that $1_S + f = 1_S$, and hence

$$0 = \chi(1_S) = \chi(f + 1_S) = \min\{\chi(f), \chi(1_S)\} \leq \chi(f)$$

□

Corollary A.1.8. *Let* S *be a semi-ring such that* $S = S_+$*, then*

$$\operatorname{TropSpec} S = \hom(S, [0, +\infty])$$

□

Definition A.1.9. *We define a* $\mathbb{R}$-tropical algebra*,* $\mathcal{A}$*, as a* $\mathbb{R}$*-vector space together with an operation* $\{\cdot, \cdot\}$ *such that* $S = (\mathcal{A} \cup \{\infty\}, \{\cdot, \cdot\}, +)$ *is a semi-ring, with*

$$0_S = \infty, \qquad \textit{and} \qquad 1_S = 0$$

satisfying

$$0 \leq f \implies f \leq \lambda f$$

for $\lambda \geq 1$ *a real number.*

Remark A.1.10. *Tropical algebras are unital, and admit multiplicative inverses.*

Moreover, if $\mathcal{A} \cup \{\infty\}$ is idempotent then every element of $\mathcal{A}$ can be written as a difference of elements of $\mathcal{A}_+$. Indeed, if $f \in \mathcal{A}$ we can write

$$f = -\{-f, 0\} - (-\{f, 0\})$$

and we have

$$\{0, -f\} = \{0, \{0, -f\}\} \implies 0 = \{-\{0, -f\}, 0\} \tag{A.1.2.1}$$

which implies $0 \leq -\{0, -f\}$ and therefore $-\{0, -f\} \in \mathcal{A}_+$, we proceed similarly for $-\{f, 0\}$.

Example A.1.11. *The two main examples are:*

- $\mathbb{R}$ *is a tropical algebra.*
- $\mathrm{C}^0(K, \mathbb{R})$ *is a tropical algebra.*

Lemma A.1.12. *Let $\mathcal{A}$ be an idempotent tropical algebra, then*

$$\mathrm{TropSpec}(\mathcal{A} \cup \{\infty\}) = \{\varphi \in \mathcal{A}^* \mid \varphi(\{f, g\}) = \max\{\varphi(f), \varphi(g)\}\},$$

where $\mathcal{A}^*$ denotes the algebraic dual.

Proof. For a max commuting linear functional $\varphi \in \mathcal{A}^*$, we can define $\varphi(\infty)$ as $+\infty$ and $\varphi \in \mathrm{TropSpec}(\mathcal{A} \cup \{\infty\})$.

On the other hand, if $\chi \in \mathrm{TropSpec}(\mathcal{A} \cup \{\infty\})$, we observe that taking $f \in \mathcal{A}$ we have

$$0_{\mathbb{R}} = \chi(0_{\mathcal{A}}) = \chi(f + (-f)) = \chi(f) + \chi(-f)$$

getting that χ is finite and $\mathbb{Q}$-linear on $\mathcal{A}$.

We are left to prove that χ is $\mathbb{R}$-linear. Indeed if $\lambda \in \mathbb{R}_{>0}$, $p_n \in \mathbb{Q}_{>0}$ an increasing sequence, $q_n \in \mathbb{Q}_{>0}$ a decreasing sequence, both converging to λ, and $f \in \mathcal{A}_+$, we then have:

$$0 \leq p_n f \leq p_{n+1} f \leq \lambda f \leq q_{n+1} f \leq q_n f,$$

thus getting

$$p_n \chi(f) \leq \chi(\lambda f) \leq q_n \chi(f),$$

taking the limit we get:

$$\chi(\lambda f) = \lambda \chi(f).$$

Applying Remark A.1.10 and the $\mathbb{Q}$-linearity we get the desired result. $\square$

A.1.3 PL spaces

Let K be a compact Hausdorff topological space.

Definition A.1.13. *A* PL structure on K *is a $\mathbb{Q}$-linear subspace of the set of continuous functions,* $\mathrm{PL}(K) \subseteq \mathrm{C}^0(K, \mathbb{R})$*, such that:*

- *It separates points;*
- *It contains all the $\mathbb{Q}$-constants;*
- *It is stable by* max.

We refer to the pair $(K, \mathrm{PL}(K))$ *as a* PL space.

A map $f\colon K_1 \to K_2$ *is a* morphism of PL spaces *if it is continuous and*

$$f^*\colon\ \mathrm{C}^0(K_2, \mathbb{R}) \to \mathrm{C}^0(K_1, \mathbb{R})$$

maps $\mathrm{PL}(K_2)$ *to* $\mathrm{PL}(K_1)$*. Moreover, it is an* ismorphism of PL structures *if the induced map* $f^*\colon\ \mathrm{PL}(K_2) \to \mathrm{PL}(K_1)$ *is bijective.*

Remark A.1.14. *If* $\mathrm{PL}(K)$ *is a PL structure on* K*, then* $(\mathrm{PL}, \max, +)$ *is an idempotent semiring,* $\mathrm{PL}_{\mathbb{R}}(K) \doteq \mathrm{PL}(K) \otimes \mathbb{R}$ *is a subtropical algebra of* $\mathrm{C}^0(K, \mathbb{R})$*, and by Proposition 1.3.3*

$$K \simeq (\operatorname{TropSpec} \mathrm{PL}_{\mathbb{R}}(K)) \setminus \{0\}/\mathbb{R}_{>0}.$$

In particular, an isomorphism of PL structures $f\colon K_1 \to K_2$ *is also a homeomorphism.*

A.2 Monomial valuations and Lelong–Kiselman numbers

A.2.1 The Lelong–Kiselman number

Let $\Delta \subseteq \mathbb{C}^n$ be the unit polydisk centered at 0, $T \doteq (S^1)^n$ the compact torus, and $w \in \mathbb{R}^n_+$. Consider $\varphi\colon \Delta^* \to \mathbb{R}_{\leq 0}$ a psh function.

Definition A.2.1. *For* $w \in (\mathbb{R}_+)^n$ *the* Lelong–Kiselman number of φ at 0 with weight w *is given by*

$$\nu_w(\varphi, 0) \doteq \sup\left\{\delta > 0 \;\middle|\; \exists U \text{ open neighborhood of } 0, \varphi(z) \leq \delta \max_{w_i \neq 0} \frac{\log|z_i|}{w_i},\ \forall z \in U\right\}$$

More generally, given a complex manifold X*,* $p \in X$ *and* $\psi\colon X \to \mathbb{R}_{\leq 0} \cup \{-\infty\}$ *a quasi-psh function on* X *we can define* $\nu_w(\psi, p)$ *similarly.*

We will show now that if $p \in \cap_{w_i \neq 0}\{z_i = 0\} \subseteq \Delta$, then $\nu_w(\varphi, 0) = \nu_w(\varphi, p)$. To do that, we'll see that it is locally independent of p, more precisely we'll show that if there exists a open neighborhood U of 0, such that $\varphi(z) \leq \nu_w(\varphi, 0) \min_{w_i \neq 0} \frac{\log|z_i|}{w_i}$ for every $z \in U$, then the inequality

$$\varphi(z) \leq \nu_w(\varphi, 0) \max_{w_i \neq 0} \frac{\log|z_i|}{w_i}$$

holds for every $z \in \Delta$.

For that, define

$$\begin{aligned} \tilde{\varphi} \colon \Delta^* &\to \mathbb{R}_{\leq 0} \\ z &\mapsto \sup_{\xi \in T} \varphi(\xi \cdot z) \end{aligned}$$

By the maximum principle $\tilde{\varphi}(z) = \sup_{|\alpha_i| \leq 1} \varphi(\alpha_1 z_1, \ldots, \alpha_n z_n)$.

Now, since $\tilde{\varphi}$ is T-invariant, there exists a convex function $\chi \colon (\mathbb{R}_+)^n \to \mathbb{R}_{\leq 0}$, such that

$$\tilde{\varphi}(z) = \chi\left(-\log|z_1|, \ldots, -\log|z_n|\right) \tag{A.2.1.1}$$

where χ is decreasing in each variable, in particular it is also decreasing on rays, i.e. for every $w \in (\mathbb{R}_+)^n$ the map $t \mapsto \chi(t \cdot w)$ is decreasing.

Now, since every bounded above decreasing convex function has a finite slope at infinity we can define:

$$\chi'_\infty(w) \doteq \lim_{t \to +\infty} \frac{\chi(tw)}{t}. \tag{A.2.1.2}$$

It is clear that $\chi'_\infty(w) = -\nu_w(\varphi, 0)$.

We have, for $t > 0$, $\frac{\chi(tw) - \chi(0)}{t} \leq \chi'_\infty(w)$, hence

$$\chi(tw) \leq t \cdot \chi'_\infty(w) + \chi(0) \leq t \cdot \chi'_\infty(w) \tag{A.2.1.3}$$

For $t = -\max_{w_i \neq 0} \frac{\log|z_i|}{w_i} = \min_{w_i \neq 0} \frac{-\log|z_i|}{w_i}$ gives us the following inequality:

$$\begin{aligned} \varphi(z) \leq \tilde{\varphi}(z) = \chi(-\log|z_1|, \ldots, -\log|z_n|) \leq \\ \chi(tw) \leq \chi'_\infty(w) \min_{w_i \neq 0} \frac{-\log|z_i|}{w_i} \\ = \nu_w(\varphi, 0) \max_{w_i \neq 0} \frac{\log|z_i|}{w_i} \end{aligned} \tag{A.2.1.4}$$

for every $z \in \Delta$.

A.2.2 An analytic interpretation of monomial valuations

Now, let $B = \sum_{i \in I} B_i$ be a reduced snc divisor, Z a connected component of the intersection, and $p \in Z \subseteq X$.

Let $I \in \mathscr{I}_X$, and $f = \log|I|$ a quasi-psh function $f \colon X \to \mathbb{R} \cup \{-\infty\}$, with singularities along I.

Then the monomial valuation defined by Equation (1.2.3.1) satisfies:

$$v_{w,Z,p}(I) = \nu_w(f,p). \tag{A.2.2.1}$$

In particular, by the discussion the previous section, the right hand side does not depend on locally on $p \in Z$. Hence the left hand side also does not depend locally on $p \in Z$, therefore retrieving Proposition 1.2.14.

A.3 Basic linear algebra of bilinear forms

Lemma A.3.1. *Let V be a real finite dimensional vector space, and $B\colon V \times V \to \mathbb{R}$ a symmetric bilinear form, such that*

1. *There exists a base $e_i \in V$ such that $B(e_i, e_j) \geq 0$*

2. *There exist an element on the kernel, $v = \sum_i v^i e_i \in V$, that is for every i:*

$$B(v, e_i) = 0,$$

such that $v^j > 0$ for every j.

Then B is negative semi-definite.

Proof. After changing bases we can suppose that $v = \sum_i e_i$, and then the second item reduces to

$$B(e_i, e_i) = -\sum_{j \neq i} B(e_j, e_i)$$

and thus taking $x = \sum_j x^j e_j \in V$,

$$\begin{aligned} B(x,x) &= \sum_i (x^i)^2 B(e_i, e_i) + \sum_{i \neq j} x^i x^j B(e_i, e_j) \\ &= \sum_{i \neq j} \left(x^i x^j - (x^i)^2\right) B(e_i, e_j) \\ &= \sum_{i \neq j} \left(x^i x^j - (x^j)^2\right) B(e_i, e_j) \end{aligned}$$

where the last equality is given by symmetry. Thus, by changing the roles of i and j,

$$2B(x,x) = -\sum_{i \neq j} \left(x^i - x^j\right)^2 B(e_i, e_j) \leq 0$$

we get the desired result. $\square$

A.4 A synthetic comment

In this paper we assume the synthetic pluripotential theory developed in [BJ25a], for $X^{\beth}$, where X is a compact Kähler manifold.

In fact,

- The set $X^{\beth}$ is underlying the compact Hausdorff topological space.
- The “smooth" test functions $\mathcal{D}$ are the set $\mathrm{PL}_{\mathbb{R}}(X^{\beth}) \simeq \varinjlim_{\mathcal{X}} \mathrm{VCar}_{\mathbb{R}}(\mathcal{X})$, which is dense in $\mathrm{C}^0(X^{\beth}, \mathbb{R})$ by Proposition 1.1.15.
- The vector space $\mathcal{Z}$ in our case corresponds to $\varinjlim_{\mathcal{X}} H^{1,1}(\mathcal{X}/\mathbb{P}^1)$.
- The $\mathrm{dd^c}\colon \mathcal{D} \to \mathcal{Z}$ operator assigns:

$$\mathrm{PL}_{\mathbb{R}} \ni \varphi_D \mapsto [\mathrm{c}_1\left(\mathcal{O}_{\mathcal{X}}(D)\right)]$$

- For $\beta \in \mathcal{Z}$

$$\beta \geq 0,$$

  if $\beta_{\mathcal{X}}$ is nef rel. to $\mathbb{P}^1$, for some snc determination $\beta_{\mathcal{X}}$.
- The *dimension* of $X^{\beth}$ is defined to be $\dim X$.
- The assignment $\mathcal{Z}^n \to \mathrm{C}^0(X^{\beth})^{\vee}$, $(\beta_1, \ldots, \beta_n) \mapsto \beta_1 \wedge \cdots \wedge \beta_n$, is given by

$$\mathrm{C}^0(X^{\beth}) \ni f \mapsto \inf\left\{(0,\varphi)\cdot(0,\beta_1)\cdots(0,\beta_n) \mid \varphi \geq f\right\},$$

  that satisfies all the required properties by all the results on Section 2.3.1, in particular the version of Zariski's Lemma of Lemma 2.3.6 gives the seminegativity of

$$\mathcal{D}\times\mathcal{D} \ni (\varphi,\psi) \mapsto \int_{X^{\beth}} \varphi\, \mathrm{dd^c}\psi \wedge \beta_1 \wedge \cdots \wedge \beta_n,$$

  for $\beta_i \geq 0$.

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