Classical applications of the h-Cobordism theorem

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Definition

A (differentiable) n-manifold with boundary is a Hausdorff topological space andmitting a countable basis, together with a smooth structure $\mathcal{S}$. $\mathcal{S}$ is a collection of pairs (U, φ), where

- each (U, h) consists of an open set $U \subset W$ and a homeomorphism $\varphi \colon U \to \varphi(U)$ onto an open subset of

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$$

- if (U_1, h_1) and (U_2, h_2) belong to $\mathcal{S}$, then

$$h_1 \circ h_2^{-1} : h_2(U_1 \cap U_2) \longrightarrow h_1(U_1 \cap U_2) \subset \mathbb{R}^n_+$$

  is smooth.

Definition

A triple (W, V, V') is a triad of dimension n if W is a compact smooth n-manifold and V, V' are two disjoint open sets of ∂W such that $V \cup V' = \partial W$.

Given two triads (W, V_0, V_1), (W', V_0', V_1') and a diffeomorphism $\varphi \colon V_1 \to V_0'$, we can glue them obtaining

$$(W \cup_\varphi W', V_0, V_1') = (W, V_0, V_1) \cup_\varphi (W', V_0', V_1')$$

A triad (W, V, V') is trivial if it is diffeo to $(V \times [0,1], V \times 0, V \times 1)$.

Gluing a trivial triple is a trivial operation.

Homology and Cohomology

Given X a topological space and $A \subset X$ a subset, we can define, for every $k \in \mathbb{N}$, the abelian groups $H_k(X, A)$, $H^k(X, A)$ called the h-homology and k-cohomology groups, resp.

Given $f : (X, A) \to (Y, B)$ a map, it induces for every $k \in \mathbb{N}$

$$f_* : H_k(X, A) \longrightarrow H_k(Y, B) \qquad f^* : H^k(Y, B) \longrightarrow H^k(X, A)$$

If f, g are homotopic, they induce the same map on H_k and H^k for every $k \in \mathbb{N}$.

$$H_k(S^n) \cong H^{n-k}(S^n) \cong \begin{cases} \mathbb{Z} & \textit{if } k = 0, n \\ 0 & \textit{otherwise} \end{cases} \quad H_k(pt) \cong \begin{cases} \mathbb{Z} & \textit{if } k = 0 \\ 0 & \textit{otherwise} \end{cases}$$

The main tools that we need are

- the exact sequence of the couple

$$\cdots \longrightarrow H_k(A) \longrightarrow H_k(X) \longrightarrow H_k(X,A) \longrightarrow H_{k+1}(A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{k-1}(A) \longrightarrow H^k(X,A) \longrightarrow H^k(X) \longrightarrow H^k(A) \longrightarrow \cdots$$

- the excision property: if A, Z are subsets of X such that $\mathrm{int}(A) \supset \overline{Z}$, then

$$H_\bullet(X,A) = H_\bullet(X \setminus Z, A \setminus Z)$$

- the Poincaré duality theorem: if M is an oriented n-manifold with boundary $\partial M = A \cup B$, then

$$H_k(M,A) \cong H^{n-k}(M,B)$$

Theorem (Smale, 1962)

Let (W, V, V') be a triad verifying:

- $\dim W = n \geq 6$;
- $H_\bullet(W, V) = 0$;
- W, V, V' *are simply connected.*

Then (W, V, V') is diffeomorphic to $(V \times [0,1], V \times 0, V \times 1)$.

Theorem (A)

Let W be a n-manifold with connected boundary ∂W, with $\dim \geq 6$. *Assume that both W and ∂W are simply connected. Then the following are equivalent:*

1. *W is diffeo to D^n;*
2. *W is homeo to D^n;*
3. *W is contractible;*
4. *W has the homology of a point.*

Proof:

Clearly $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. We will prove $4 \Rightarrow 1$. Let D_0 be a smooth n-disc embedded in $\mathrm{int}(W) = W \setminus \partial W$. By excision we have

$$H_\bullet(W \setminus \mathrm{int}(D_0), \partial D_0) = H_\bullet(W, D_0)$$

The exact sequence of the couple

$$\cdots \longrightarrow H_k(D_0) \longrightarrow H_k(W) \longrightarrow H_k(W, D_0) \longrightarrow H_{k+1}(D_0) \longrightarrow \cdots$$

tells us that $H_\bullet(W \setminus \mathrm{int}(D_0), \partial D_0) = H_\bullet(W, D_0) = 0$.

$W, \partial D_0, \partial W$ are simply connected, so we can apply the h-Cobordism theorem obtaining that

$$(W \setminus \mathrm{int}(D_0), \partial W, \partial D_0) \cong_\varphi (\partial D_0 \times [0,1], \partial D_0 \times 0, \partial D_0 \times 1)$$

Finally we see that

$$\begin{aligned}(W, \partial W, \emptyset) &= (W \setminus \mathrm{int}(D_0), \partial W, \partial D_0) \cup_i (D_0, \partial D_0, \emptyset) \\ &= (\partial D_0 \times [0,1], \partial D_0 \times 0, \partial D_0 \times 1) \cup_{\partial\varphi} (D_0, \partial D_0, \emptyset) \\ &= (D_0, \partial D_0, \emptyset)\end{aligned}$$

where in the last step we are using the collar lemma. $\square$

Theorem (B)

Let M be a compact n-manifold without boundary which is homotopically equivalent to the n-sphere S^n. Then M is homeomorphic to S^n.

Proof:

Take D_0 a smooth disc inside W and observe that

$$\begin{aligned} H_i(M \setminus \mathrm{int}(D_0)) &\cong H^{n-i}(M \setminus \mathrm{int}(D_0), \partial D_0) && \text{(Poincaré)} \\ &= H^{n-i}(M, D_0) && \text{(Excision)} \end{aligned}$$

Using the exact sequence of the couple

$$\cdots \longrightarrow H^{n-i-1}(D_0) \longrightarrow H^{n-i}(W, D_0) \longrightarrow H^{n-i}(W) \longrightarrow H^{n-i}(D_0) \longrightarrow \cdots$$

we see that $H_i(M \setminus \mathrm{int}(D_0)) \cong H^{n-i}(W, D_0) \cong H^{n-i}(W) = \delta_{0i}\mathbb{Z}$.

Then we have shown that $M \setminus \mathrm{int}(D_0)$ has the same homology of a point.

$M \setminus \mathrm{int}(D_0)$ is simply connected then, by applying **Theorem A**, we deduce that $M \setminus \mathrm{int}(D_0)$ is diffeo to a disk D_1.

Hence M is diffeo to the union of two disks D_0 and D_1 glued by a diffeomorphism of the boundaries. In particular it is homeo to S^n. $\square$

What is known and what is not:

- the h-Cobordism theorem is trivial or vacuous for $n = 0, 1, 2$. For $n = 3$, it follows from the classical Poincaré conjecture, proved by G. Perelman (where the conclusion is *diffeomorphic to* S^3);
- the Poincaré conjecture for dimension $n = 4$ has been proved by M. Freedman. It is still not known if we can strengthened the resul by asking *diffeomorphic to* S^4;
- for $n = 5, 6$ in fact the conclusion of the Poincaré's conjecture can be strengthened with *diffeomorphic to* S^n;
- for $n \geq 7$ we cannot ask for more than *homeomorphic to* S^n. For example for $n = 7$ there exist exactly 28 different structures on the sphere S^7;

Thank you for your attention!