Low-Dimensional Cohomology of the Witt and the Virasoro Algebra

based on 1707.06106 [math.RA] & on-going work with Martin Schlichenmaier

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September 2017, PhD Away Days, Durbuy, Belgium

Outline

- Introduction : known results about the second cohomology group & continuous cohomology ; motivation
- The Witt and the Virasoro algebra : two of the most important infinite-dimensional Lie algebras (exple : Bosonic string)
- The cohomology of Lie algebras : Interpretation of low-dimensional cohomology groups with values in the adjoint module
- Warm-up : Proof of the vanishing of the first cohomology group
- Comments on the proof of the vanishing of the third cohomology group
- Outlook

Introduction

- Main aim : Prove the vanishing of the first and third cohomology groups with values in the adjoint module of the Witt and the Virasoro algebra by purely algebraic means $\Rightarrow$ $\boxed{\text{algebraic cohomology}}$
- Second cohomology group : Schlichenmaier [5, 6], see also Fialowski [1]
- Witt algebra $\leftrightarrow$ subalgebra of $Vect(S^1)$ $\Rightarrow$ compare to results from $\boxed{\text{continuous cohomology}}$ (Fialowski & Schlichenmaier[2]) :

$$H^*(Vect(S^1), Vect(S^1)) = \{0\}$$

- First three cohomology groups $\rightarrow$ interpretation in terms of important Lie algebra objects (outer derivations, deformations, obstructions)

Lie Algebra

Definition : Lie algebra

A Lie algebra $\mathcal{L}$ is a vector space over a field $\mathbb{K}$ with a bilinear product $[\cdot, \cdot]$ called Lie bracket satisfying (for $x, y, z \in \mathcal{L}$) :

- Skew-symmetry : $[x, y] = -[y, x]$
- Jacobi identity : $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$

The Witt algebra

- Witt algebra $\mathcal{W}$ generated as vector space over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ by the elements $\{e_n \mid n \in \mathbb{Z}\}$ satisfying the following Lie structure :

$$[e_n, e_m] = (m-n)e_{n+m}, \qquad n, m \in \mathbb{Z}$$

- $\mathbb{Z}$-graded Lie algebra : $deg(e_n) := n$
- Decomposition of $\mathcal{W}$: $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n$, with each $\mathcal{W}_n$ a 1-dimensional homogeneous subspace generated by e_n
- Internally graded : $[e_0, e_n] = ne_n = deg(e_n)e_n$, i.e. e_n is eigenvector of $ad_{e_0} := [e_0, \cdot]$ with eigenvalue n
- Algebraic realization : Lie algebra of derivations of Laurent polynomials $\mathbb{K}[Z^{-1}, Z]$
- Geometrical realization :
  - $\mathbb{K} = \mathbb{C}$, algebra of meromorphic vector fields on $\mathbb{CP}^1$ holomorphic outside of 0 and ∞, with $e_n = z^{n+1}\frac{d}{dz}$
  - Lie algebra of polynomial vector fields on S^1, with $e_n = e^{in\phi}\frac{d}{d\phi}$

The Virasoro algebra

- The Virasoro algebra $\mathcal{V}$ is the universal one-dimensional central extension of the Witt algebra
- As a vector space, $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$ generated by $\hat{e}_n := (0, e_n)$ and $t := (1, 0)$
- Lie structure equation :

$$\boxed{\begin{array}{l} [\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} - \frac{1}{12}(n^3 - n)\delta_n^{-m} t, \\ [\hat{e}_n, t] = [t, t] = 0 \end{array}}$$

- $deg(\hat{e}_n) := deg(e_n) = n$ and $deg(t) = 0 \Rightarrow \mathcal{V}$ is $\mathbb{Z}$-graded

The Lie algebra cohomology

- Let $\mathcal{L}$: Lie algebra ; M : $\mathcal{L}$-module and $C^q(\mathcal{L}, M)$: vector space of q-multilinear alternating maps with values in M, called q-cochains $(q \in \mathbb{N})$
  Convention : $C^0(\mathcal{L}, M) := M$
- Coboundary operators δ_q defined by :

$$\forall q \in \mathbb{N}, \qquad \delta_q : C^q(\mathcal{L}, M) \to C^{q+1}(\mathcal{L}, M) : \psi \mapsto \delta_q \psi \,,$$

$$\begin{aligned}(\delta_q\psi)(x_1, \dots x_{q+1}) : \quad &= \textstyle\sum_{1 \leq i < j \leq q+1} (-1)^{i+j+1} \; \psi([x_i, x_j] \,, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{q+1}) \\ &+ \textstyle\sum_{i=1}^{q+1} (-1)^i \; x_i \cdot \psi(x_1, \dots, \hat{x}_i, \dots, x_{q+1}) \,,\end{aligned}$$

  with $x_1, \dots, x_{q+1} \in \mathcal{L}$
- Adjoint module $M = \mathcal{L}$, $x \cdot m = [x, m]$; trivial module $M = \mathbb{K}$, $x \cdot m = 0$
- $\delta_{q+1} \circ \delta_q = 0 \; \forall \; q \in \mathbb{N} \to$ complex of vector spaces :

$$\{0\} \xrightarrow{\delta_{-1}} M \xrightarrow{\delta_0} C^1(\mathcal{L}, M) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{q-2}} C^{q-1}(\mathcal{L}, M) \xrightarrow{\delta_{q-1}} C^q(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} C^{q+1}(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} \dots \longrightarrow \dots$$

  where $\delta_{-1} := 0$

The Chevalley-Eilenberg cohomology

- *q-cocycles* : $Z^q(\mathcal{L}, M) :=$ ker δ_q
- *q-coboundaries* : $B^q(\mathcal{L}, M) :=$ im δ_{q-1}
- *q^{th} cohomology group* of $\mathcal{L}$ with values in M :

$$H^q(\mathcal{L}, M) := Z^q(\mathcal{L}, M)/B^q(\mathcal{L}, M)$$

- Chevalley-Eilenberg cohomology :

$$H^*(\mathcal{L}, M) := \bigoplus_{q=0}^{\infty} H^q(\mathcal{L}, M)$$

The degree of a homogeneous cochain

- $\mathcal{L}$ graded Lie algebra, M a graded $\mathcal{L}$-module, M internally graded with respect to the same grading element as the Lie algebra $\mathcal{L}$
- **Examples** : adjoint module $M{=}\mathcal{L}$; trivial module $M{=}\mathbb{K}$ with $\mathbb{K} = \bigoplus_{n\in\mathbb{Z}} \mathbb{K}_n$, $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_n = \{0\}$ for $n \neq 0$
- A q-cochain ψ is homogeneous of degree d if $\exists$ a $d \in \mathbb{Z}$ s.t. for all q-tuple $x_1, \ldots, x_q$ of homogeneous $x_i \in \mathcal{L}_{deg(x_i)}$, we have :

$$\psi(x_1, \ldots, x_q) \in M_n \text{ with } n = \sum_{i=1}^{q} deg(x_i) + d$$

$\rightsquigarrow$ decomposition of cohomology :

$$H^q(\mathcal{L}, M) = \bigoplus_{d\in\mathbb{Z}} H^q_{(d)}(\mathcal{L}, M)$$

- Result by Fuks [3] :

$$H^q_{(d)}(\mathcal{L}, M) = \{0\} \quad \text{for} \quad d \neq 0 \ ,$$
$$H^q(\mathcal{L}, M) = \mathrm{H}^q_{(0)}(\mathcal{L}, M)$$

Interpretation in case of the adjoint module

$H^1(\mathcal{L},\mathcal{L})$: Outer derivations

- Kernel of δ_1 :

$$(\delta_1\psi)(x_1,x_2) = \psi([x_1,x_2]) - [x_1,\psi(x_2)] - [\psi(x_1),x_2] = 0$$
$$\Leftrightarrow \ \psi([x_1,x_2]) = [x_1,\psi(x_2)] + [\psi(x_1),x_2]$$

- Image of δ_0 :

$$(\delta_0\phi)(x) = -[x,\phi] = ad_\phi(x)\,,$$

with $\phi \in C^0(\mathcal{L},\mathcal{L}) = \mathcal{L}$

- $\rightsquigarrow$ first cohomology group :

$$H^1(\mathcal{L},\mathcal{L}) = \frac{\ker(\delta_1 : C^1(\mathcal{L},\mathcal{L}) \to C^2(\mathcal{L},\mathcal{L}))}{\operatorname{im}(\delta_0 : C^0(\mathcal{L},\mathcal{L}) \to C^1(\mathcal{L},\mathcal{L}))} = \frac{\{\text{derivations}\}}{\{\text{inner derivations}\}} = \{\text{outer derivations}\}$$

$H^2(\mathcal{L},\mathcal{L})$: Infinitesimal deformations

- Lie algebra $\mathcal{L}$ over field $\mathbb{K}$ with bracket $[\cdot,\cdot]$ expressed with anti-symmetric bilinear map ψ_0 :

$$\psi_0 : \mathcal{L}\times\mathcal{L}\to\mathcal{L}, \qquad (x_1,x_2)\mapsto\psi_0(x_1,x_2)=[x_1,x_2]$$

  $\mathcal{L}$ Lie algebra $\rightsquigarrow$ ψ_0 must fulfill Jacobi identity

- Family of Lie algebra structures :

$$\mu_t=\psi_0+\psi_1\ t+\psi_2\ t^2+\dots$$

  $\psi_i : \mathcal{L}\times\mathcal{L}\to\mathcal{L}$ anti-symmetric bilinear maps such that μ_t fulfills Jacobi identity $\rightsquigarrow$ $\mathcal{L}_t := (\mathcal{L},\mu_t)$ Lie algebra $\rightsquigarrow$ deformation of $(\mathcal{L},\mu_0)$

- Deformation parameter t :
  - t is a variable $\rightsquigarrow$ deformation over the affine line or the convergent power series $\rightsquigarrow$ geometric or analytic deformation
  - t is a formal variable $\rightsquigarrow$ deformation over the formal power series $\rightsquigarrow$ formal deformation
  - deformation over the quotient $\mathbb{K}[[X]]/(X^{n+1})$ $\rightsquigarrow$ n-deformation. In particular $n=1$ $\rightsquigarrow$ infinitesimal deformation (i.e. $t^2=0$)

- The family μ_t must fulfill the Jacobi identity up to all orders, i.e.

$$\mu_t(\mu_t(x_1,x_2),x_3) + \text{ cyclic permutations of } (x_1,x_2,x_3) = 0$$
$$\Leftrightarrow \sum_{i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3)\ t^{i+j} + \text{ cyclic permutations of } (x_1,x_2,x_3) = 0$$

- Infinitesimal deformations $t^2 = 0$:

$$t^0 : \text{ original Jacobi identity for } \psi_0 \text{ on } \mathcal{L}$$
$$t^1 : \psi_1([x_1,x_2],x_3) + \text{ cycl. perm. } + [\psi_1(x_1,x_2),x_3] + \text{ cycl. perm. } = 0$$

- $\boxed{\mu_t = \psi_0 + \psi_1 t \text{ is an infinitesimal deformation iff } \psi_1 \in Z^2(\mathcal{L},\mathcal{L})}$

- Notion of equivalence, two families $\mu_t' = \psi_0 + \psi_1' t$ and $\mu_t = \psi_0 + \psi_1 t$ equivalent $\leftrightarrow$ ψ_1' and ψ_1 are cohomologous

- $\Rightarrow$ $\boxed{\text{Elements of } H^2(\mathcal{L},\mathcal{L}) \text{ correspond to inf. def. up to equivalence}}$

$H^3(\mathcal{L},\mathcal{L})$: Obstructions

- Lift of infinitesimal deformation to formal deformation $\rightsquigarrow$ step n to step $n+1$ lifting property

- Let $\mu_t = \sum_{i=0}^{n} \psi_i t^i$ be a n-deformation, i.e. the following holds :

$$\sum_{i+j=k, i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.} = 0 \qquad \boxed{0 \leq k \leq n}$$

- Extension to $n+1$-deformation $\rightsquigarrow$ Jacobi identity must be fulfilled also for $\boxed{k = n+1}$, i.e. :

$$\begin{aligned}
&\sum_{i+j=n+1, i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.} = 0 \\
\Leftrightarrow\ & [(\psi_0(\psi_{n+1}(x_1,x_2),x_3) + \psi_{n+1}(\psi_0(x_1,x_2),x_3)) + \text{ cycl. perm.}] \\
&+ \left[\sum_{i+j=n+1, i,j>0} (\psi_i(\psi_j(x_1,x_2),x_3)) + \text{ cycl. perm.}\right] = 0 \\
\Leftrightarrow\ & (\delta_2\psi_{n+1})(x_1,x_2,x_3) + \left[\sum_{i+j=n+1, i,j>0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.}\right] = 0
\end{aligned}$$

- 3-coboundary term plus an extra term called obstruction :

$$\boxed{\Psi_{n+1} := \sum_{i+j=n+1, i,j>0} \psi_i(\psi_j(x_1, x_2), x_3) + \text{ cycl. perm.}}$$

- We have $\Psi_{n+1} \in Z^3(\mathcal{L}, \mathcal{L})$

$\Rightarrow$ $\boxed{\text{a } n\text{-deformation can be extended to a } n+1\text{-deformation iff } [\Psi_{n+1}] = 0}$ in $H^3(\mathcal{L}, \mathcal{L})$

- If $H^3(\mathcal{L}, \mathcal{L}) = \{0\}$, all obstructions vanish at all levels

$\Rightarrow$ $\boxed{\text{every infinitesimal deformation can be extended to a formal deformation}}$

The vanishing of $H^1(\mathcal{W},\mathcal{W})$ and $H^1(\mathcal{V},\mathcal{V})$

Theorem

The first cohomology of the Witt algebra $\mathcal{W}$ and the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$ and values in the adjoint module vanishes, i.e.

$$\boxed{H^1(\mathcal{W},\mathcal{W}) = H^1(\mathcal{V},\mathcal{V}) = \{0\}}$$

- Fuks : $H^1_{(d)}(\mathcal{W},\mathcal{W}) = H^1_{(d)}(\mathcal{V},\mathcal{V}) = \{0\}$ for $d \neq 0$
- Need to prove $H^1_{(0)}(\mathcal{W},\mathcal{W}) = H^1_{(0)}(\mathcal{V},\mathcal{V}) = \{0\}$
- Here : Proof for $\mathcal{W}$; proof for $\mathcal{V}$ similar

Proof of $H^1_{(0)}(\mathcal{W},\mathcal{W}) = \{0\}$

- Let ψ be a degree zero 1-cocycle, i.e. $\psi(e_i) = \psi_i e_i$ with suitable $\psi_i \in \mathbb{K}$
- Consider 0-cochain $\phi = \psi_1 e_0$. Coboundary condition for ϕ :

$$(\delta_0\phi)(e_i) = [\phi, e_i] = i\psi_1 e_i$$

- Cohomological change $\psi' = \psi - \delta_0\phi \Rightarrow \boxed{\psi'_1 = 0}$
- Cocycle condition for ψ' on (e_i, e_j) :

$$0 = \psi'([e_i, e_j]) - [e_i, \psi'(e_j)] - [\psi'(e_i), e_j] \Leftrightarrow 0 = (j-i)\left(\psi'_{i+j} - \psi'_j - \psi'_i\right)$$

-
  - For $j = 1$ and $i = 0$: $\psi'_0 = 0$
  - For $j = 1$ and $i < 0$ decreasing : $\psi'_i = \psi'_{i+1} = 0$
  - For $j = 1$ and $i > 1$ increasing : $\psi'_{i+1} = \psi'_i = \psi'_2$
- Next, taking $j = 2$ and for example $i = 3$:

$$\psi'_5 - \psi'_2 - \psi'_3 = 0 \Leftrightarrow \psi'_2 - \psi'_2 - \psi'_2 = 0 \text{ as we have } \psi'_i = \psi'_2 \ \forall i > 1$$
$$\Leftrightarrow \psi'_2 = 0$$

All in all, we conclude $\boxed{\psi'_i = 0 \ \forall i \in \mathbb{Z}}$

The main result : the vanishing of $H^3(\mathcal{W},\mathcal{W})$

Theorem

The third cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)$= 0$ and values in the adjoint module vanishes, i.e.

$$\boxed{H^3(\mathcal{W},\mathcal{W}) = \{0\}}$$

- **Step 1** : Recall : $H^3_{(d)}(\mathcal{W},\mathcal{W}) = \{0\}$ for $d \neq 0$. Need to prove $\boxed{H^3_{(0)}(\mathcal{W},\mathcal{W}) = \{0\}} \rightsquigarrow \psi(e_i, e_j, e_k) = \psi_{i,j,k} e_{i+j+k}$ with suitable $\psi_{i,j,k} \in \mathbb{K}$
- **Step 2** : Find ϕ to perform cohomological change $\psi' = \psi - \delta_2\phi$ s.t. as many $\psi'_{i,j,k}$ as possible are zero
- **Step 3** : Use the fact that ψ' is a cocycle $\rightarrow$ cocycle conditions $\rightarrow$ all $\psi'_{i,j,k}$ are zero.
- Computation in six steps : show that $\psi'_{i,j,1}$, $\psi'_{i,j,0}$, $\psi'_{i,j,-1}$, $\psi'_{i,j,2}$ and $\psi'_{i,j,-2}$ vanish $\forall\ i,j \in \mathbb{Z}$, then use induction on the remaining index k in $\psi'_{i,j,k}$

Tools : Coboundary and cocycle conditions

- Coboundary condition on (e_i, e_j, e_k) :

$$
\begin{aligned}
(\delta_2\phi)_{i,j,k} =& (j-i)\phi_{i+j,k} + (k-j)\phi_{k+j,i} + (i-k)\phi_{i+k,j} \\
& - (j+k-i)\phi_{j,k} + (i+k-j)\phi_{i,k} - (i+j-k)\phi_{i,j}
\end{aligned}
$$

- Cocycle conditions on (e_i, e_j, e_k, e_l) :

$$
\begin{aligned}
(\delta_3\psi)_{i,j,k,l} =& (j-i)\psi_{i+j,k,l} - (k-i)\psi_{i+k,j,l} + (l-i)\psi_{i+l,j,k} \\
& + (k-j)\psi_{k+j,i,l} - (l-j)\psi_{l+j,i,k} + (l-k)\psi_{l+k,i,j} \\
& - (j+k+l-i)\psi_{j,k,l} + (i+k+l-j)\psi_{i,k,l} \\
& - (i+j+l-k)\psi_{i,j,l} + (i+j+k-l)\psi_{i,j,k} = 0
\end{aligned}
$$

- Main difficulty : $\lightning$ Poles $\lightning$ appearing in recurrence relations $\rightsquigarrow$ roundabout way $\Rightarrow$ long+technical

Outlook : $H^3(\mathcal{V}, \mathcal{V})$

Conjecture

The third cohomology groups of the Witt algebra $\mathcal{W}$ and the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the trivial module are one-dimensional, i.e.

$$\boxed{dim(H^3(\mathcal{W}, \mathbb{K})) = dim(H^3(\mathcal{V}, \mathbb{K})) = 1} \tag{1}$$

Numerical evidence and $dim(H^3(Vect(S^1), \mathbb{R})) = 1$ in the case of continuous cohomology, see Gelfand and Fuks [3, 4].

Theorem

Under the assumption that Conjecture 1 is true, it follows that the third cohomology group of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the adjoint module is one-dimensional, i.e.

$$\boxed{dim(H^3(\mathcal{V}, \mathcal{V})) = 1}$$

Thank you for your attention !

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