On quasitrivial symmetric nondecreasing associative operations

PhD Seminar

Jimmy Devillet

in collaboration with Gergely Kiss and Jean-Luc Marichal

University of Luxembourg

Connectedness

Let X be a nonempty set and let $F\colon X^n \to X$

Definition

- The points $\mathbf{x}, \mathbf{y} \in X^n$ are *connected for F* if

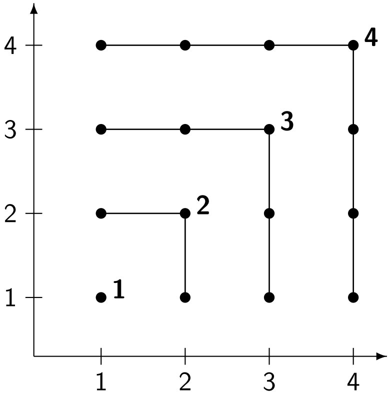
$$F(\mathbf{x}) \ = \ F(\mathbf{y})$$

- The point $\mathbf{x} \in X^n$ is *isolated for F* if it is not connected to another point in X^n

Connectedness

For any integer $n \geq 1$, let $L_n = \{1, ..., n\}$ endowed with $\leq$

Example. $F(x, y) = \max\{x, y\}$ on L_4



Quasitriviality and Idempotency

Definition

$F\colon X^n \to X$ is said to be

- *quasitrivial* if

$$F(x_1, \ldots, x_n) \ \in \ \{x_1, \ldots, x_n\}$$

- *idempotent* if

$$F(n \cdot x) \ = \ x$$

where $n \cdot x = x, \ldots, x$ (n times)

Associativity

Definition

$F\colon X^n \to X$ is said to be *associative* if

$$
\begin{aligned}
& F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) \\
& \quad = \quad F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1})
\end{aligned}
$$

A class of associative operations

Let $(X, \leq)$ be a chain, e.g., $X = L_n = \{1, \ldots, n\}$, $n \geq 1$, endowed with $\leq$

We are interested in the class of operations $F\colon X^n \to X$, that are

- quasitrivial
- symmetric
- nondecreasing in each variable
- associative

A first characterization

Let $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$

Theorem

$F\colon X^n \to X$ ($n \geq 3$) is quasitrivial, symmetric, nondecreasing and associative iff there exists a quasitrivial and nondecreasing operation $G\colon X^2 \to X$ such that

$$F(x_1, \ldots, x_n) \;=\; G\left(\bigwedge_{i=1}^n x_i\,, \bigvee_{i=1}^n x_i\right) \tag{1}$$

Moreover, G is unique, symmetric, and associative

Derivability

Definition (Dudek-Mukhin, 2006) Assume $F\colon X^n \to X$ and $H\colon X^2 \to X$ are associative. F is said to be *derived from* H if

$$F(x_1, \ldots, x_n) = x_1 \circ \cdots \circ x_n$$

where $x \circ y = H(x, y)$

Corollary

If $F\colon X^n \to X$ is of the form (1), where $G\colon X^2 \to X$ is quasitrivial, symmetric, and nondecreasing, then F is associative and derived from G

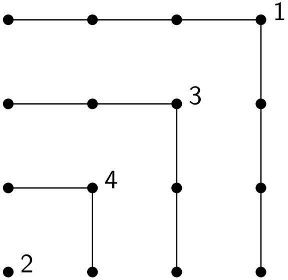
A result of Ackerman

Proposition (Ackerman, To appear)

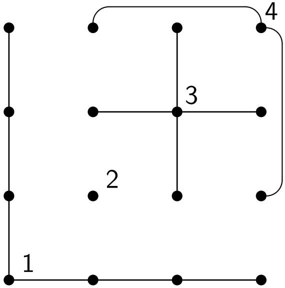
$F\colon X^n \to X$ is quasitrivial, symmetric, associative, and derived from a quasitrivial and associative operation $H\colon X^2 \to X$ iff there exists a linear ordering $\preceq$ on X such that F is the maximum operation on $(X, \preceq)$, i.e.,

$$F(x_1, \ldots, x_n) \;=\; x_1 \vee_{\preceq} \cdots \vee_{\preceq} x_n \tag{2}$$

A result of Ackerman



$\mathbf{2} \prec \mathbf{4} \prec \mathbf{3} \prec \mathbf{1}$



$\mathbf{1} < \mathbf{2} < \mathbf{3} < \mathbf{4}$

Single-peaked linear orderings

Definition. Let $(X, \leq)$ and $(X, \preceq)$ be chains. The linear ordering $\preceq$ is said to be *single-peaked w.r.t.* $\leq$ if for any $a, b, c \in X$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$

Example. The ordering $\preceq$ on

$$L_4 = \{1 < 2 < 3 < 4\}$$

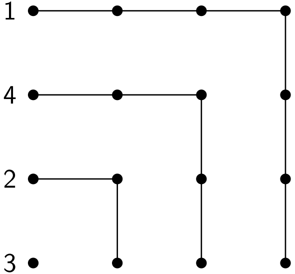
defined by

$$3 \prec 2 \prec 4 \prec 1$$

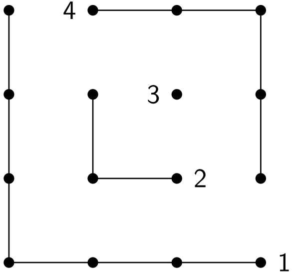
is single-peaked w.r.t. $\leq$

Note : There are exactly 2^{n-1} single-peaked linear orderings on L_n.

Single-peaked linear orderings



$\mathbf{3} \prec \mathbf{2} \prec \mathbf{4} \prec \mathbf{1}$



$\mathbf{1} < \mathbf{2} < \mathbf{3} < \mathbf{4}$

Single-peaked linear orderings

Proposition

Let $(X, \leq)$ and $(X, \preceq)$ be chains and let $F\colon X^n \to X$ be of the form (2). Then F is nondecreasing w.r.t. $\leq$ iff $\preceq$ is single-peaked w.r.t. $\leq$

A second characterization

Theorem

Let $(X, \leq)$ be a chain and let $F\colon X^n \to X$. The following assertions are equivalent.

(i) F is quasitrivial, symmetric, nondecreasing, and associative (associativity can be ignored when $n = 2$)

(ii) F is of the form (1), where $G\colon X^2 \to X$ is quasitrivial, symmetric, and nondecreasing

(iii) F is of the form (2) for some linear ordering $\preceq$ on X that is single-peaked w.r.t. $\leq$

If any of these assertions holds, then G is associative and F is derived from G

Thank you for your attention!

Selected references

N. L. Ackerman. A characterization of quasitrivial n-semigroups. To appear in Algebra Universalis

S. Berg and T. Perlinger. Single-peaked compatible preference profiles: some combinatorial results. Social Choice and Welfare, 27(1):89–102, 2006

D. Black. On the rationale of group decision-making. J Polit Economy, 56(1):23–34, 1948

D. Black. The theory of committees and elections. Kluwer Academic Publishers, Dordrecht, 1987

M. Couceiro, J. Devillet, and J.-L. Marichal. Characterizations of idempotent discrete uninorms. Fuzzy Sets and Systems. In press. https://doi.org/10.1016/j.fss.2017.06.013

J. Devillet, G. Kiss, and J.-L. Marichal. Characterizations of quasitrivial symmetric nondecreasing associative operations. Submitted for publication. arXiv:1705.00719.

W. A. Dudek and V. V. Mukhin. On n-ary semigroups with adjoint neutral element. Quasigroups and Related Systems, 14:163–168, 2006