Introduction to random variables and an example of random walk

PhD Away Days 2017

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1. **Coin tossing**
   possible outcomes: $\Omega = \{H \text{ (head) }, T \text{ (tail) }\}$
2. **Dice rolling**
   possible outcomes: $\Omega = \{1, 2, 3, 4, 5, 6\}$

To any possible outcome we associate a value $\rightarrow$ random variable

A **random variable** is a function

$$X : \Omega \to E, \omega \mapsto X(\omega) \ ,$$

where

- Ω is the set of possible outcomes,
- E is a set (e.g. $E = \mathbb{R}$)

Back to examples:

1. **Coin tossing**
   Define $X : \{H, T\} \to \{0, 1\}$
   $$X(H) = 0, X(T) = 1$$
2. **Dice rolling**
   Define $X : \{1, \dots, 6\} \to \{1, \dots, 6\}$
   $$X(k) = k \ , \ \forall k = 1, \dots, 6$$

assign probabilities to random events $\to$ probability distribution

Let $X : \Omega \to E$ be a random variable.

The **probability distribution of X** is the map

$$\begin{array}{rccl} \mathbb{P}_X : & \mathcal{P}(E) & \to & [0,1], \\ & A & \mapsto & \mathbb{P}_X(A) := \mathbb{P}(X(\omega) \in A) \ , \end{array}$$

given by

$$\begin{aligned} \mathbb{P}(X(\omega) \in A) &= \int_\Omega \mathbb{1}_{\{\omega : X(\omega) \in A\}} \mathrm{d}\mathbb{P}(\omega) \\ &= \sum_{x \in A} \mathbb{P}(X(\omega) \in \{x\}) \ . \end{aligned}$$

The map $\mathbb{P}_X$ is a **probability measure**, i.e. satisfies

- $\mathbb{P}_X(\Omega) = 1, \mathbb{P}_X(\emptyset) = 0$
- for any countable collection $\{A_i\}_{i\in I}$ of disjoint sets,

$$\mathbb{P}_X(\bigcup_{i\in I} A_i) = \sum_{i\in I} \mathbb{P}_X(A_i)$$

A gambler starts with initial fortune of k euros. In each game, he

- wins 1 euro with probability p
- loses 1 euro with probability $q = 1 - p$

Objective: reach a total fortune of $K > k$ euros before running out of money (ruin).

The game stops either when he reaches K euros or when he is ruined.

- $X_n \in \{-1, 1\}, n \geq 1$ denote the gain of the n-th game,

$$\mathbb{P}(X_n = 1) = p \ , \mathbb{P}(X_n = -1) = q = 1 - p$$

- $S_n, n \geq 0$ denote the gambler's fortune after the n-th game,

$$S_0 = k \ ,$$
$$S_n = k + X_1 + X_2 + \ldots + X_n \ , n \geq 1$$

- τ_k the time when the game stops,

$$\tau_k := \min\{n \geq 0 : S_n = 0 \text{ or } S_n = K | S_0 = k\}$$

Question: What is the probability that the gambler is ruined

$$p_k := \mathbb{P}(S_{\tau_k} = 0) \ ?$$

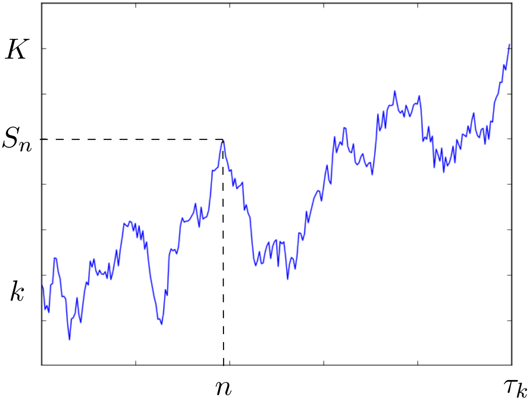
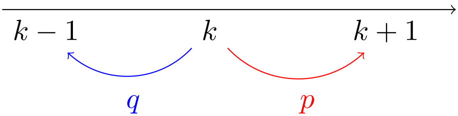


Figure: Random walk

We have

$$p_0 = \mathbb{P}(S_{\tau_0} = 0) = 1$$
$$p_K = \mathbb{P}(S_{\tau_K} = 0) = 0$$

Computation of $p_k = \mathbb{P}(S_{\tau_k} = 0)$:



$$p_k = p_{k-1} \times q + p_{k+1} \times p$$

We obtain a recurrence relation of order 2,

$$pp_{k+1} - p_k + qp_{k-1} = 0 \ , \quad 0 < k < K \ ,$$
$$p_0 = 1, p_K = 0 \ .$$

Associated characteristic equation: $px^2 - px + q = 0$
Discriminant: $(2p-1)^2$

$\rightarrow$ distinguish cases $p = 1/2$ and $p \neq 1/2$

Rec. relation: $\begin{cases} pp_{k+1} - p_k + qp_{k-1} = 0 \ , 0 < k < K \\ p_0 = 1, p_K = 0 \end{cases}$

Char. equation: $px^2 - px + q = 0$

$p = 1/2$	$p \neq 1/2$
$x = 1$	$x = \frac{1 \pm \lvert 2p-1 \rvert}{2p} \in \{1, q/p\}$
$p_k = a + bk$	$p_k = a + b(q/p)^k$
$p_k = 1 - k/K$	$p_k = \frac{(q/p)^k - (q/p)^K}{1-(q/p)^K}$

$\rightarrow$ **random walks** describe a path consisting of a succession of random steps

Some applications of random walks:

- finance: stock market prices
- physics: random movement of molecules in a liquid (Brownian motion)
- etc.

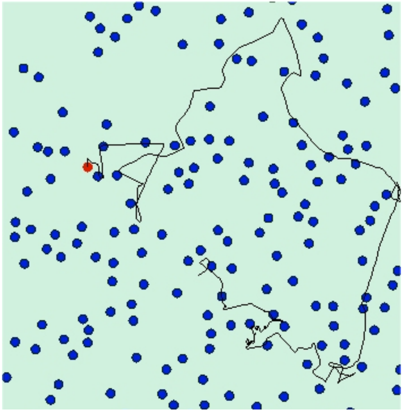


Figure: Simulation of a *Brownian motion*

Thank you for your attention!