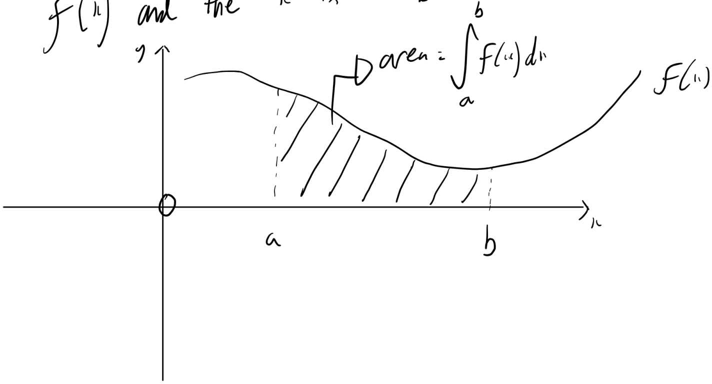
# Integration

## Integration in theory

Definition 1: the integral $\int_a^b f(x)\,dx$ measures the area between the graph of $f(x)$ and the $x$ axis between $x=a$ and $x=b$.

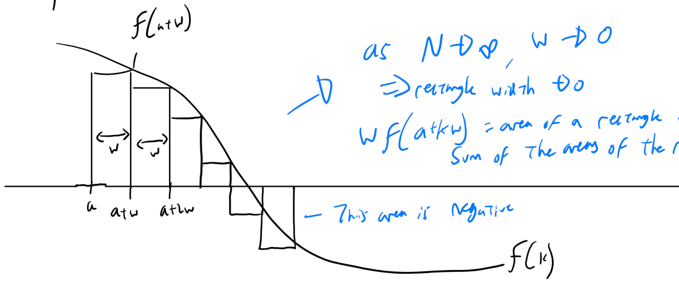



Definition 2: Let $a, b \in \mathbb{R}$, $f(x)$ be a real continuous function between a and b. Let $N \in \mathbb{N}$, $N > 0$ and $w = \dfrac{b-a}{N}$

We define:

$$\int_a^b f(x)\,dx = \lim_{N \to \infty} \sum_{k=1}^{N} w f(a+kw)$$

Explanation




as $N \to \infty$, $w \to 0$
$\Rightarrow$ rectangle width $\to 0$

$w f(a+kw)$ = area of a rectangle so the integral is the sum of the areas of the rectangles as $w \to 0$

Properties

- Swapping b and a gives: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$ $\left(w = \frac{b-a}{N} = 0 \text{ in this case}\right)$
- the integral does not depend on $x$ → dummy variable.

$$\int_a^b f(x)\,dx = \int_a^b f(s)\,ds = \int_a^b f(\text{웃})\,d\text{웃}$$

- we can split integrals

i.e. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

This is the Definition 2 above

Example
Use the Riemann Sum definition to show that:

$$\int_a^b 2x\,dx = b^2 - a^2$$

$$\int_a^b 2x\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} w\, 2(a+kw)$$

$f(x) = 2x$

$f(a+kw) = 2(a+kw)$

$$= \lim_{N\to\infty}\left(2awN + 2w^2\frac{N(N+1)}{2}\right)$$ — using Summation formulae.

Using $W = \frac{b-a}{N}$:

$$= \lim_{N\to\infty}\left(2a\frac{(b-a)}{N}N + 2\frac{(b-a)^2}{N^2}\cdot\frac{N(N+1)}{2}\right)$$

$$= \lim_{N\to\infty}\left(2a(b-a) + 2\frac{(b-a)^2}{2}\left(1+\frac{1}{N}\right)\right)$$

$$= 2a(b-a) + (b-a)^2$$ (by CoLT)

$$= b-a(b-a+2a)$$

$= (\quad a)(b+a)$
$= b^2 - a^2$

(we are probably never going to use this definition again ☺)

**Definition** Let $F(x)$ be differentiable.
If $F'(x)$ is $f(x)$ then $F(x)$ is an antiderivative of $f(x)$

**Example** If $f(x) = \cos x$ then $F(x) = \sin x$ is an antiderivative of $f$.
(Since $F'(x) = f(x)$)

but so is $G(x) = \sin x + 1$ and $\sin x + 2$ etc...
Since $G'(x) = \cos x = f(x)$

**Note** any two antiderivatives $F(x)$ and $G(x)$ differ at most by a constant.

**Theorem**: Fundamental Theorem of Calculus

Define $A(x) = \int_a^x f(s)\,ds$ then $A(x)$ is an antiderivative of $f(x)$

i.e. $A'(x) = f(x)$

**Consequence**: $\int_a^b f(t)\,dt = F(b) - F(a)$ for any antiderivative $F$ of $f$.

Proof of Consequence

$$\int_a^b f(s)\,ds = A(b) = F(b) + c$$

But then put $b = a$ so $\int_a^a f(s)\,ds = 0 = F(a) + c$, so $c = -F(a)$

$$\therefore \int_a^b f(s)\,ds = F(b) - F(a)$$

**Proof of FTC**, i.e. $A'(x) = f(x)$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(s)\,ds - \int_a^x f(s)\,ds}{h}$$

$$= \lim_{h \to 0} \int_x^{x+h} f(s)\,ds \cdot \frac{1}{h}$$

$w = \frac{b-a}{N}$

$$= \lim_{h \to 0} \lim_{N \to \infty} \sum_{k=1}^{N} w f(x + kw) \frac{1}{h} \qquad \text{with } w = \frac{(x+h) - x}{N} = \frac{h}{N}$$

$$= \lim_{h \to 0} \lim_{N \to \infty} \sum_{k=1}^{N} \frac{f\left(x + \frac{kh}{N}\right)}{N}$$

dodgy step (actually OK but needs proving)

$$= \lim_{N \to \infty} \lim_{h \to 0} \sum_{k=1}^{N} \frac{f\left(x + \frac{kh}{N}\right)}{N}$$

$$= \lim_{N \to \infty} \sum_{k=1}^{N} \frac{f(x)}{N} = \lim_{N \to \infty} f(x) = f(x) \quad \checkmark$$

## Integration In Practice

Two types of questions

1: "Compute $\int_a^b f(x)\,dx$" $\to$ definite integral
- Find an antiderivative $F(x)$ s.t. $F'(x) = f(x)$
- Compute $F(b) - F(a)$

2: "Compute $\int f(x)\,dx$" $\to$ Indefinite Integral
- find an antiderivative $F(x)$ s.t. $F'(x) = f(x)$
- write $\int f(x)\,dx = F(x) + C$

## Methods for finding antiderivatives

1. Guess / memorisation.

Q1: Compute $\int e^{2x}\,dx$

Q2: Compute $\int_1^3 e^{2x}\,dx$

A1: Antiderivative of $e^{2x}$?

$$\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$F(x) \quad \therefore \frac{d}{dx}\left(\frac{e^{2x}}{2}\right) = e^{2x} \leftarrow f(x)$

$$\therefore \int e^{2x}\,dx = \tfrac{1}{2}e^{2x} + c$$

A2: "$F(b) - F(a)$"

Useful notation: $F(x)\Big|_a^b = F(b) - F(a)$ ← C cancels

So here $\displaystyle\int_1^3 e^{2x}\,dx = \tfrac{1}{2}e^{2x}\Big|_1^3 = \tfrac{1}{2}e^{6} - \tfrac{1}{2}e^{2}$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

Proof - differentiate RHS + use FTC.

Guessing is always fine ☺

- always check by differentiating, i.e. $\int f(x)\,dx = G(x)$ can be verified by checking $G'(x) = f(x)$

Method 2: Integration by parts

Product rule of differentiation says:

$$\frac{d}{dx} u(x)v(x) = u'(x)v(x) + u(x)v'(x)$$

$$\therefore \int (u'v + v'u)\,dx = uv + c$$

$$\therefore \int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

For definite integrals:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx$$

Example: Q. $\int x e^x\,dx$

Let $u = x \quad u' = 1$ — Important to find the right $u(x)$, $v'(x)$

$v' = e^x \quad v = e^x$

So $\int x e^x\,dx = xe^x - \int 1 \cdot e^x\,dx = xe^x - e^x + c$

Q: $\int e^{2x}\cos(3x)\,dx$

A: use IBP

Let $u = \cos 3x \Rightarrow u' = -3\sin 3x$

$v' = e^{2x} \qquad v = \frac{1}{2}e^{2x}$

So $I = \frac{1}{2}\cos 3x\, e^{2x} + \frac{3}{2}\int \sin(3x)e^{2x}\,dx$

IBP again $u = \sin 3x \Rightarrow u' = 3\cos 3x$

$v' = e^{2x} \qquad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}\cos 3x\, e^{2x} + \frac{3}{2}\left(\sin 3x \frac{1}{2}e^{2x} - \frac{3}{2}\int \cos 3x\, e^{2x}\,dx\right)$

$I = \frac{1}{2}\cos 3x\, e^{2x} + \frac{3}{4}\sin 3x\, e^{2x} - \frac{9}{4}I + c$

$\frac{13}{4}I = \frac{1}{2}\cos 3x\, e^{2x} + \frac{3}{4}\sin 3x\, e^{2x}$

$\therefore I = \frac{2}{13}\cos 3x\, e^{2x} + \frac{3}{13}\sin 3x\, e^{2x} + C$

## Definition: Euler-Gamma function

For $x > 0$ we define $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$

Properties: for $x > 1$

$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$

Let $u = t^{x-1} \Rightarrow u' = (x-1)t^{x-2}$

$v' = e^{-t} \qquad v = -e^{-t}$

$\Gamma(x) = \underbrace{-t^{x-1}e^{-t}\Big|_{t=0}^{t=\infty}}_{\text{by magic this} = 0} - \int (x-1)t^{x-2}(-e^{-t})\,dt$

$= 0 + (x-1)\int t^{x-2}e^{-t}\,dt = (x-1)\Gamma(x-1)$

So $\Gamma(x) = (x-1)\Gamma(x-1)$

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1$$

$$\Gamma(2) = (2-1)\Gamma(1) = 1$$

$$\Gamma(3) = (3-1)\Gamma(2) = 2$$

$$\Gamma(4) = (4-1)\Gamma(3) = 6$$

$$\Gamma(5) = (5-1)\Gamma(4) = 24$$

$$\Gamma(n) = (n-1)!$$

So $\Gamma(x)$ is an extension of the factorial (minus one) to all reals $>0$

- only defined for $x>0$ by the integral. But $\Gamma(x) = (x-1)\Gamma(x-1)$

So $\Gamma(y) = \dfrac{\Gamma(y+1)}{y}$ letting $y = x-1$

Use to define $\Gamma$ for negative values.

So e.g. $\Gamma(-0.5) = \dfrac{\Gamma(0.5)}{-0.5}$ etc

4. Substitution

Claim: $\displaystyle\int_a^b f(u(x))\,\frac{du}{dx}\,dx = \int_{u(a)}^{u(b)} f(u)\,du$

Proof: Chain rule

$$\frac{d}{dx}\left(F(u(x))\right) = F'(u(x))\,\frac{du}{dx} \quad \text{where } F' = f.$$

$$C + F(u) = \int F'(u)\,du$$

$$C + F(u(x)) = \int F'(u(x))\,\frac{du}{dx}\,dx$$

$$F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} F'(u)\,du$$

Eg. $\int e^{\sin x} \cos x \, dx$ → trick is spotting the substitution $u(x)$

Q: $I = \int_0^{\pi/2} e^{\sin x} \cos x \, dx$

Set $u(x) = \sin x$ then $du = \frac{du}{dx} dx = \cos x \, dx$

Endpoints: $u(0) = \sin 0 = 0$

$u(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

So $I = \int_0^1 e^u \, du = e^u \Big|_{u=0}^{u=1} = e - 1$

Q: $I = \int_0^1 x^6 \sqrt{x^7 + 12} \, dx$

Set $u(x) = x^7 + 12$

$du = 7x^6 \, dx$ so $x^6 \, dx = \frac{1}{7} du$

Endpoints: $u(0) = 12$

$u(1) = 13$

So $I = \int_{12}^{13} \frac{1}{7} \sqrt{u} \, du = \frac{1}{7} \cdot \frac{2}{3} u^{3/2} \Big|_{12}^{13}$

$= \frac{2}{21} \left( 13^{3/2} - 12^{3/2} \right)$

For indefinite integrals do not forget to change back to a function of $x$.

Q: $I = \int x^6 \sqrt{x^7 + 12} \, dx$

(from before)

$$= \frac{1}{7} \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{21} \left(x^7 + 12\right)^{3/2} + C$$

Do not forget to sub $x$ back in ☺

Q: $I = \int \frac{\ln^2 x}{x} \, dx$

Let $u = \ln x$

$u' = \frac{1}{x}$

$du = u' dx = \frac{dx}{x}$

So $I = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\ln^3 x}{3} + C$

Q: $I_n = \int_0^1 (-\log x)^n \, dx$

($\log x = \ln x$)

Let $u = -\log x$

$\Rightarrow x = e^{-u}$

$du = -\frac{1}{x} dx$ So $dx = -x \, du = -e^{-u} du$

Endpoints: $x = 0 \to u = \infty$

$x = 1 \to u = 0$

So $I_n = -\int_\infty^0 u^n e^{-u} \, du$

$$= \int_0^\infty u^n e^{-u} \, du = \Gamma(n+1)$$

# Integrating Rational Functions

We can develop a general algorithm for integrating any rational function

i.e. $\int \frac{p(x)}{q(x)}\,dx$ where $p, q$ are polynomials

Step 1: Polynomial division

Step 2: Factorise $q(x)$

Step 3: Partial fractions

Step 4: Integrate

## Step 1: Polynomial Division

If our integral $\left(\int \frac{p(x)}{q(x)}\,dx\right)$ has degree $p(x) \geq$ degree $q(x)$ → "improper rational function" → polynomial division needed.

- like fractions, eg $\frac{4}{3} = 1 + \frac{1}{3}$

Imp. Int. Prop.

- Here $\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$

i.e. $p(x) = q(x)s(x) + r(x)$ where degree of $r$ < degree of $q$

- We can do this by long division

eg $\frac{x^3+2x^2-x+3}{x^2-2x+1}$ → deg $p = 3$, deg $q = 2$

p

q

$$
\begin{aligned}
x^3+2x^2-x+3 &= (x^2-2x+1)x + 4x^2-2x+3 \\
&= (x^2-2x+1)x + (x^2-2x+1)\cdot 4 + 6x - 1 \\
&= (x^2-2x+1)(x+4) + 6x - 1
\end{aligned}
$$

So $\dfrac{x^3+2x^2-x+3}{x^2-2x+1} = x+4 + \dfrac{6x-1}{x^2-2x+1}$

easy to integrate :)

e.g. 2 $\dfrac{x^2+1}{2x^2-1}$ → write $x^2+1 = (2x^2-1)\times \tfrac{1}{2} + \tfrac{3}{2}$

So $\dfrac{x^2+1}{2x^2-1} = \dfrac{1}{2} + \dfrac{3}{2(2x^2-1)}$

Fund. Theorem of Algebra – every real polynomial has real or quadratic factors.

If $b^2-4ac<0$

## Step 2: factorise $q(x)$ into linear or quadratic real factors

e: factorise $q(x) = (x-1)(x^4-1)$ over reals

$$x^4-1 = (x^2)^2-1 = (x^2+1)(x^2-1) = (x+1)(x-1)(x^2+1)$$

leave as quadratic → $b^2-4ac<0$

So $q(x) = (x-1)^2(x+1)(x^2+1)$

The general factorised form for $q(x)$:

$$q(x) = c(x-w_1)^{m_1}(x-w_2)^{m_2}\cdots(x-w_n)^{m_n}\times (x^2+b_1x+c_1)^{v_1}(x^2+b_2x+c_2)^{v_2}\cdots(x^2+b_mx+c_m)^{v_m}$$

e: Factorise $q(x) = \underbrace{(-x^2+2x-2)}(x-1)$

$b^2-4ac$
$= 2^2-4(-1)(-2) = -4<0$

Already factorised :)

## Step 3: Partial fractions

3.a. Write an 'ansatz' of the following form:

$$\frac{r(x)}{q(x)} = \sum_{i=1}^{n}\sum_{m_i'=1}^{m_i}\frac{A_{ij}}{(x-w_i)^{m_i'}} + \sum_{i=1}^{m}\sum_{n_i'=1}^{v_i}\frac{B_{ij}x+C_{ij}}{(x^2+b_ix+c_i)^{n_i'}}$$

Basically if you have $(x-w)^m$ in $q(x)$ you need $\dfrac{A}{x-w}+\dfrac{B}{(x-w)^2}+\dots+\dfrac{X}{(x-w)^m}$

(Similar for quadratic factors but need $\dfrac{Bx+C}{x^2+bx+c}$)

Example: $\dfrac{3x+4}{(x-1)(x+2)} = \dfrac{A}{x-1}+\dfrac{B}{x+2}$

$$\frac{3x+4}{(x-1)^3(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+2}$$

$$\frac{1}{(x+1)(x^2+x+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

3.b. Fix the coefficients of the anzats

e.g.

$$\frac{3x+4}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$\Rightarrow 3x+4 = A(x+2) + B(x-1)$$

$$= (A+B)x + 2A - B \quad \text{for all } x$$

$$\Rightarrow A+B=3, \quad 2A-B=4$$

$$\Rightarrow 3A = 7$$

$$A = 7/3 \qquad B = 2/3$$

$$\frac{1}{(x^2+2x+2)(x-1)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x-1}$$

$$1 = (Ax+B)(x-1) + C(x^2+2x+2)$$

$$1 = x^2(A+C) + x(-A+B+2C) - B + 2C$$

So $A+C=0$

$-A+B+2C=0 \quad \Rightarrow \quad A = -1/5$

$2C - B = 1 \qquad B = -3/5$

$C = 1/5$

By "cover up rule"

Claim: If $q(x)$ has a non-repeated linear factor, so $\frac{r(x)}{q(x)} = \frac{r(x)}{(x-a)\tilde{q}(x)} = \frac{A}{x-a} + \dots$

no other terms with $(x-a)$ factor

Then $A = \frac{r(a)}{\tilde{q}(a)}$

Proof: $\frac{r(x)}{(x-a)\tilde{q}(x)} = \frac{A}{x-a} + \dots$

$$\Rightarrow \frac{r(x)}{\tilde{q}(x)} = A + \underbrace{(x-a)\times(\cdots)}_{\text{all other terms have } (x-a) \text{ factor}}$$

now let $x = a$

$$\Rightarrow \frac{r(x)}{\tilde{q}(x)} = A + \underbrace{(a-a)\cdot(\cdots)}_{0}$$

$$= A$$

e.g. (previous example)

$$\frac{3x+4}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

"cover up" $x-1$: set $x=1$

$$A = \frac{3\cdot 1+4}{1+2} = \frac{7}{3}$$

"cover up" $x+2$: set $x=-2$

$$B = \frac{3\cdot(-2)+4}{-2-1} = \frac{2}{3}$$

Same as before

## Step 4: Integrate

Key integrals:

$$\int \frac{dx}{x-a} = \ln|x-a| + C$$

$$\int \frac{dx}{(x-a)^n} = \frac{1}{(1-n)}\cdot\frac{1}{(x-a)^{n-1}} + C \quad \text{for } n>1$$

quadratic factors $\Rightarrow$ complete square $\rightarrow$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

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Full example:

Q: Integrate $I = \int \frac{3x^3+5x^2+1}{x^3-1}\,dx$

A: Step 1 : $3x^3+5x^2+1 = (x^3-1)3 + 5x^2+4$

So $I = \int 3 + \frac{5x^2+4}{x^3-1}\,dx$

$= 3x + \int \frac{5x^2+4}{x^3-1}\,dx$

Step 2 Factorise $x^3-1$

↳ $x-1$ is a factor since $1^3-1=0$

So $x^3-1 = (x-1)\underbrace{(x^2+x+1)}_{b^2-4ac = 1-4<0}$

So $I = 3x + \int \frac{5x^2+4}{(x-1)(x^2+x+1)}\,dx$

Step 3 partial fractions

$$\frac{5x^2+4}{(x-1)(x^2+x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$$

"Cover up rule" $C = \frac{5+4}{1+1+1} = \frac{9}{3} = 3$

now $5x^2+4 = (Ax+B)(x-1) + 3(x^2+x+1)$

$5x^2+4 = Ax^2+Bx-Ax-B+3x^2+3x+3$

Coeff $x^2$: $5 = A+3 \Rightarrow A=2$

Coeff $x$: $0 = B-A+3$

$B = A-3 = 2-3$

$= -1$ ✓

Coeff $x^0$: $4 = -B+3$

$I = 3x + \int \frac{2x-1}{x^2+x+1}\,dx + \int \frac{3}{x-1}\,dx$

Step 4 Integrate derivative of $x^2+x+1$

$I = 3x + \int \frac{2x+1}{x^2+x+1} - \frac{2}{x^2+x+1}\,dx + 3\ln|x-1|$

$$I = 3x + 3\ln|x-1| + \int \frac{2x+1}{x^2+x+1}\,dx - 2\int \frac{dx}{x^2+x+1}$$

let $u = x^2+x+1$

$du = 2x+1\,dx$

$$= \text{''} \qquad \text{''} \int \frac{du}{u} - 2\int \frac{dx}{(x+1/2)^2 + 3/4}$$

let $u = x + 1/2$

$du = dx$

$$= \text{''} \qquad \text{''} + \ln|x^2+x+1| - 2\int \frac{du}{u^2 + 3/4}$$

$$= 3x + 3\ln|x-1| + \ln|x^2+x+1| - \frac{2}{\sqrt{3/4}}\arctan\left(\frac{x+1/2}{\sqrt{3/4}}\right) + C$$

## Tricks for Trig Integrals

- Consider $I_{n,m} = \int \cos^n x \sin^m x\,dx$ ($m, n$ integer)
  - If $n$ is odd: use $u = \sin x$, $du = \cos x\,dx$

    $\cos^2 x = 1 - \sin^2 x = 1 - u^2$

    So $I_{n,m} = \int \cos^{\overbrace{n-1}^{\text{even}}}(x) \sin^m x \underbrace{\cos x\,dx}_{du}$

    $= \int (1-u^2)^{\frac{n-1}{2}} u^m\,du$ = Integral of a polynomial

    $\frac{n-1}{2}$ integer since $n-1$ even

  - If $m$ is odd: use $u = \cos x$ instead

Example: $I_{3,3} = \int \cos^3 x \sin^3 x\,dx$

Let $u = \cos x$ (could have used $u = \sin x$ since 3 is odd)

$du = -\sin x\,dx$

$$I_{3,3} = \int \cos^3 x \sin^2 x \sin x\,dx$$

$$= -\int u^3(1-u^2)\,du$$

$$= \int u^5 - u^3\,du$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$$

- If $m, n$ both even and for most other integrals involving trig functions, use complex exponentials to reduce to single trig function (i.e. no powers or products)

E.g. $\int \cos^4 x \, dx =$

$$\cos^4 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^4 = \frac{1}{16}\left(e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\right)$$

$$= \frac{1}{16}\left(2\cos 4x + 8\cos 2x + 6\right)$$

$$I = \int \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8} \, dx$$

$$= \frac{1}{32}\sin 4x + \tfrac{1}{4}\sin 2x + \tfrac{3}{8}x + C$$

E.g. 2. $\int \cos 3x \cos 5x \, dx$

$$= \int \frac{e^{3ix}+e^{-3ix}}{2} \cdot \frac{e^{5ix}+e^{-5ix}}{2} \, dx$$

$$= \int \frac{e^{8ix}+e^{-8ix}+e^{2ix}+e^{-2ix}}{4} \, dx$$

$$= \int \tfrac{1}{2}\cos 8x + \tfrac{1}{2}\cos 2x \, dx$$

$$= \frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C$$