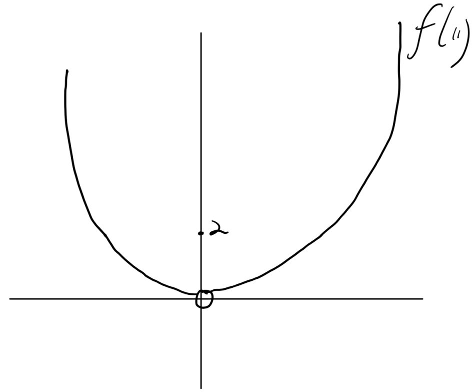
Elements of real analysis (uh oh)

Limits

Consider $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$



eg: $\lim_{x \to 3} f(x) = 9$

eg: $\lim_{x \to 0} f(x) = 0$

The limit describes the behaviour of $f(x)$ as x approaches a point 'a', but does not care what happens at that point.

If $f(x)$ approaches a definite value L as x approaches a, then we say that the limit exists and we write $\lim_{x \to a} f(x) = L$.

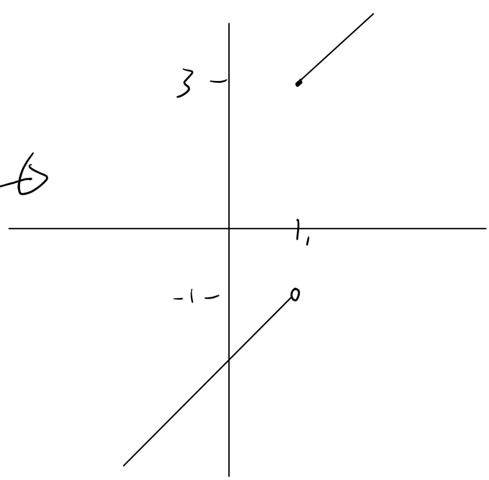
We can also write this as "$f(x) \to L$ as $x \to a$"

Examples

1.) $\lim_{x \to 2} x^2 = 4$

2.) $\lim_{x \to 0} |x| = 0$

3.) $\lim_{x \to 1} \begin{cases} x+2 & \text{if } x \geq 1 \\ x-2 & \text{if } x < 1 \end{cases}$ does not exist



Q: What is "$\lim_{x \to a} f(x) = L$" Mathematically?

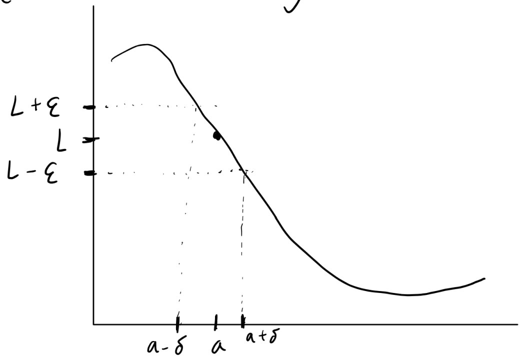
Roughly speaking we want $f(x)$ to be "boxed in", so $|f(x) - L| < \varepsilon$
(i.e. $L - \varepsilon < f(x) < L + \varepsilon$)

Whenever we take x close to a, so:

$$0 < |x - a| < \delta$$

without considering $x = a$ exactly

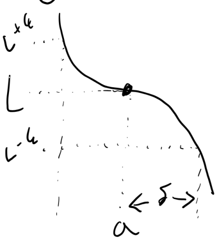
This boxing in should work for any ε, however small.

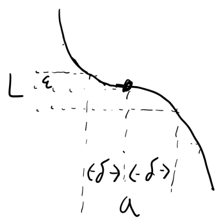


Definition: Consider a function $f(x)$ and $a, L \in \mathbb{R}$.
If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for **all** x with $0 < |x - a| < \delta$ then we say L is the limit of $f(x)$ as x approaches a, i.e.

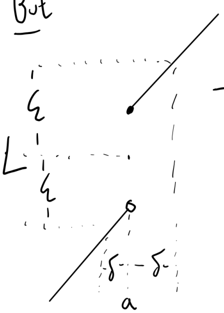
$$\lim_{x \to a} f(x) = L$$

Example 1:

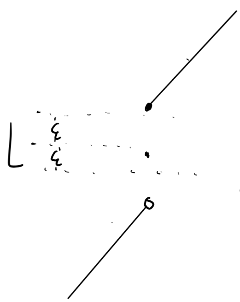




But



— Here it is ok as we can find a δ since ε was big.



- No δ exists for this ε

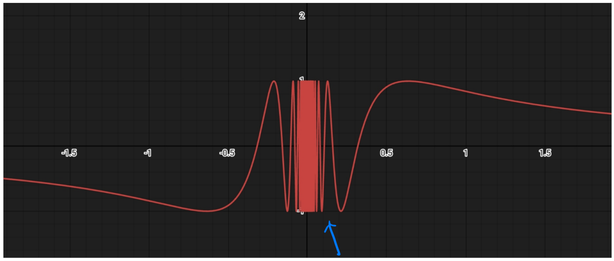
$\therefore$ δ does not exist for every $\varepsilon > 0$

$\therefore$ Limit does not exist

Examples

e. $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

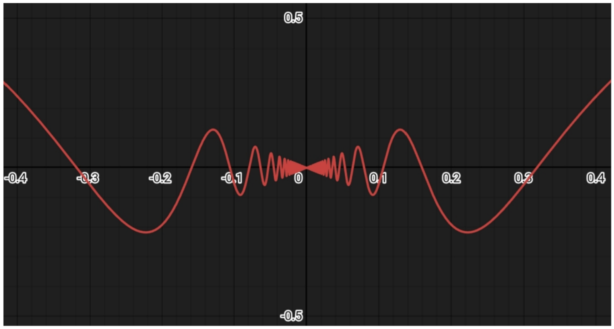
– does not exist



Infinite number of oscillations → no limit as doesn't converge

e. $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

$= 0$



$\lim_{x \to a} f(x) = L$

"The limit of $f(x)$ as x approaching a is L"

Practical Methods for Computing Limits

Method 1 (M1): If nothing 'special' happens then the limit is just the function value

i.e. no infinities or specially defined points. (Continuous)

Example: a) $\lim_{x \to 3} \frac{2x^2+3}{7x-1} = \frac{2 \times 3^2 + 3}{7 \times 3 - 1} = \frac{21}{20}$

nothing special happens

b.) $\lim_{x \to 0} x \sin \frac{1}{x} = ?$

$= 0 \sin\left(\frac{1}{0}\right)$ ← Special

So cannot just let $x = 0$

c.) $\lim_{x \to b} x \sin \frac{1}{x}$ for $b \neq 0$

$= b \sin\left(\frac{1}{b}\right)$

Method 2 (M2) If M1 gives $0/0$ try to cancel common factors until M1 works

e.g. a.) $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + 2x - 3}$ " $= \frac{0}{0}$ "

However, we can write:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + 2x - 3} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)(x+3)} = \lim_{x \to 1} \frac{x+2}{x+3}$$

$$= \frac{1+2}{1+3}$$

$$= \frac{3}{4}$$

b.) $\lim_{x \to 0} \frac{\sin 2x}{\sin x}$ M1 gives $\frac{0}{0}$

Use $\sin 2x = 2 \sin x \cos x$ to cancel common factors

$$= \lim_{x \to 0} \frac{2 \sin x \cos x}{\sin x} = \lim_{x \to 0} 2 \cos x$$

by M1 $= 2$

Method 3 (M3)

Think about $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$. Above methods don't work ☹

needs to be true for x near a, not for all x, or for $x=a$

Use "pinching theorem"

- Suppose $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=L$ and $g(x)\leq h(x)\leq f(x)$ (h is between g and f)

  then $\lim_{x\to a} h(x)=L$

- Can be proven using ε, δ definition. (but we're not doing that :))

Example: (again) Q: $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

A: Use pinching theorem with $f(x)=|x|$, $g(x)=-|x|$

Indeed for all $x\neq 0$, $-1\leq \sin\left(\frac{1}{x}\right)\leq 1$,
so $-|x|\leq x\sin\left(\frac{1}{x}\right)\leq |x|$

$-1\leq \sin\left(\frac{1}{x}\right)$ so $-x\leq x\sin\left(\frac{1}{x}\right)$, for $x>0$.
$-x\geq x\sin\left(\frac{1}{x}\right)$, for $x<0$.
so $-|x|\leq x\sin\left(\frac{1}{x}\right)\leq |x|$ for $x\neq 0$

also $\lim_{x\to 0} -|x|$ and $\lim_{x\to 0}|x|=0$

So $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=0$ by pinching theorem.

In question it is fine to write:

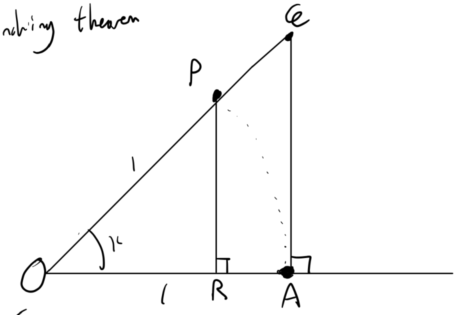
$-1\leq \sin\frac{1}{x}\leq 1$

$\Rightarrow -|x|\leq x\sin\left(\frac{1}{x}\right)\leq |x|$ $\therefore x\sin\left(\frac{1}{x}\right)\to 0$ as $x\to 0$ by pinching theorem.

$-|x|\to 0$, $|x|\to 0$ (as $x\to 0$)

Method 4 (m4): Use $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to find similar results

Proof: geometric + pinching theorem



Set $OP = OA = 1$

then $QA = \tan x$

$PR = \sin x$

Area $\Delta OAP = \frac{1}{2} \times 1 \times \sin x$

But area of sector $OAP = \frac{x}{2\pi} \times \pi 1^2 = \frac{x}{2}$

Area $OAQ = \frac{1}{2} \tan x$

geometrically we see that for $0 < x < \frac{\pi}{2}$:

$\frac{1}{2} \sin x < \frac{x}{2} < \frac{1}{2} \tan x$ (all shapes lie inside each other)

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x} = \frac{1}{\cos x}$ $\left(\times \frac{2}{\sin x} \right)$

$(\sin x > 0$ for $0 < x < \frac{\pi}{2})$

$\Rightarrow \underbrace{1 > \frac{\sin x}{x} > \cos x}$ (reciprocate)

True for $0 < x < \frac{\pi}{2}$, but also for $-\frac{\pi}{2} < x < 0$

$\cos x$ even

$\left.\begin{matrix} \sin x \text{ odd} \\ x \text{ odd} \end{matrix}\right\}$ $\frac{\sin x}{x}$ even

all good :)

Now use pinching theorem

As $x \to 0$, $\cos x \to 1$

$\therefore \frac{\sin x}{x} \to 1$ as $x \to 0$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Method 5 (M5): Use Calculus of limits theorem (CoLT)

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then: only if these limits exist

- $\lim_{x \to a} (f(x) + g(x)) = L + M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $\lim_{x \to a} f(x) g(x) = LM = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ (provided $M \neq 0$)

$f(x)$ $g(x)$

Examples

Examples

- $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$

$= \left(\lim_{x \to a} \frac{\sin x}{x}\right)\left(\lim_{x \to a} \frac{1}{\cos x}\right)$ ← Using M5

Using M4 Using M1

$= 1 \cdot 1 = \underline{\underline{1}}$

- $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 x}{x^2} = 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2$

$= 2 \cdot 1^2 = \underline{\underline{2}}$

Method 6 (M6) - Use a change of variables to simplify the limit

$\lim_{x \to a} f(x) = L$ does not depend on x, so we can change variables.

Example:

- $\lim_{x \to 0} \frac{\sin(3x)}{x}$

Let $y = 3x$, so as $x \to 0$, $y \to 0$

$$= \lim_{y \to 0} \frac{\sin y}{\frac{y}{3}} = \lim_{y \to 0} \frac{3 \sin y}{y}$$

$$= 3$$

- $\lim_{x \to 1} \frac{\sin(3x-3)}{x-1}$

Let $y = 3x - 3$

as $x \to 1$, $y \to 0$

$$= \lim_{y \to 0} \frac{\sin y}{y/3} = 3$$

Infinity in Limits

$\lim_{x \to \infty} f(x)$: behaviour for large positive x

$\lim_{x \to -\infty} f(x)$: behaviour for large negative x

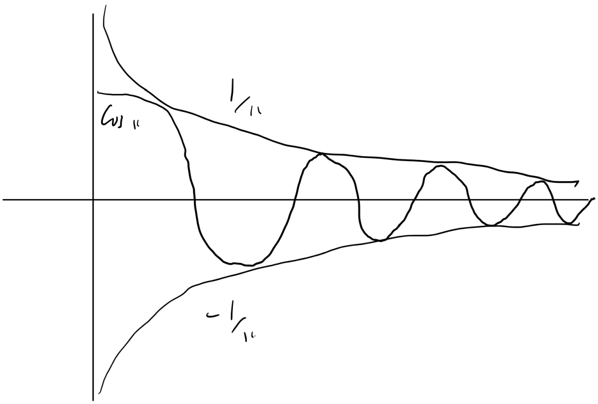
" In definition, replace " $|x-a| < \delta$ " with " $x > \frac{1}{\delta}$ " or " $x < -\frac{1}{\delta}$ "

Example

- $\lim_{x \to \pm\infty} \frac{1}{x} = 0$
- $\lim_{x \to -\infty} e^{x} = 0$
- $\lim_{x \to \infty} x^{-c} = 0$, $c > 0$
- $\lim_{x \to \infty} \frac{1}{\ln x} = 0$

- $\lim \frac{2x^2-7x+2}{3x^2+4} = \lim_{x\to\infty} \frac{2-\frac{7}{x}+\frac{2}{x^2}}{3+\frac{4}{x^2}} = \frac{2}{3}$

- $\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by pinching theorem



Proof $-1 \le \cos x \le 1$ for $x>0$

$$\underset{\downarrow\, 0}{-\frac{1}{x}} \le \frac{\cos x}{x} \le \underset{\downarrow\, 0}{\frac{1}{x}}$$

$\therefore$ by pinching theorem, $\frac{\cos x}{x} \to 0$ as $x\to\infty$

Claim: $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

divided through each term of e^x expansion by x

Proof: $\lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\dots\right)-1}{x} = \lim_{x\to 0}\left(1+\frac{x}{2}+\frac{x^2}{3!}+\frac{x^3}{4!}+\dots\right)$

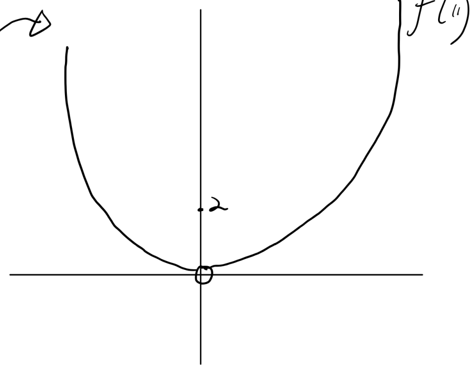
$= 1$

Continuity and Differentiability

Definition: A function $f(x)$ is continuous at a point $x=a$ if $\lim_{x\to a} f(x)$ exists and equals $f(a)$.

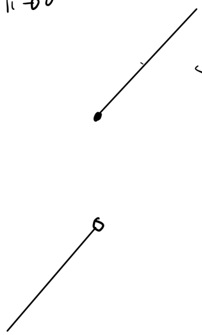
Examples:
- $\sqrt{x}$, $\sin x$, $\cos x$, e^x, $\ln|x|$, $|x|$, $\cosh x$, $\sinh x$, $\cos^{-1} x$ etc. are all continuous at every point in their domain.

- $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$



Is not continuous at $x=0$ but is everywhere else. In this case $\lim_{x \to 0} f(x) = 0$ but $f(0) = 2$

So $\lim_{x \to 0} f(x) \neq f(0)$

- 

In this case the limit doesnt exist at the jump. Is continuous everywhere else

- $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

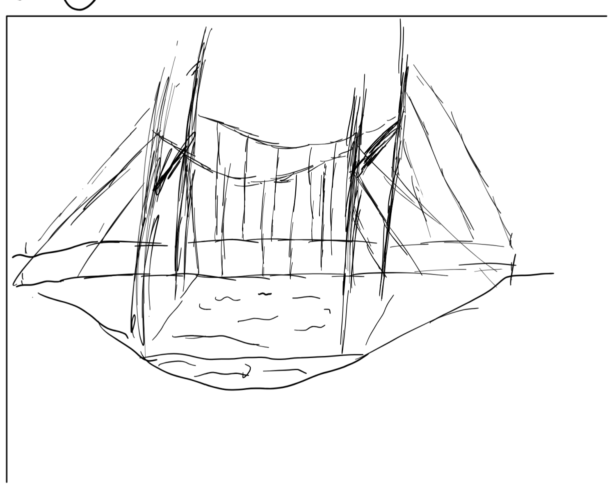
Not continuous since $\lim_{x \to 0} f(x)$ does not exist.

- $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{at } x = 0 \end{cases}$

$\lim_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$ so continuous at 0 (and everywhere else)

- $f(x) = \begin{cases} 0 & \text{if } x = \frac{a}{b},\ a, b \in \mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$

Not continuous anywhere as no limits defined

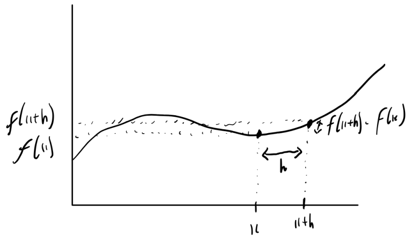


Definition: Consider $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

If this limit exists it is called the derivative of $f(x)$ at x, and is denoted $\frac{df}{dx}$ or $f'(x)$

↳ The function is said to be differentiable at x

Note: derivative measures the slope:



→ Slope $= \frac{f(x+h) - f(x)}{h}$

As $h \to 0$ will equal the slope of the tangent to point x.

* OMG DiffErEnTiAtIoN *

Q: derivative of x^2 as a limit

A: $\frac{d}{dx} x^2 = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h}$

$= \lim_{h \to 0} 2x + h = \underline{\underline{2x}}$

Q. Derivative of x^n as a limit.

A: As above but use binomial theorem: $(x+h)^n = \underbrace{x^n}_{} + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots$

all other terms will have a h in them after dividing by h

↳ cancels with the $f(x)$ in the differentiation formula

So $\frac{d}{dx} x^n = nx^{n-1}$

full derivation in lecture notes

Q. Derivative of $\frac{1}{x}$ as a limit.

$\frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x} = \lim_{h \to 0} \frac{-h}{h(x+h)x}$

$= \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$

Q. Derivative of $\sin x$ as a limit.

A: $\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \cos x \underbrace{\left(\frac{\sin h}{h}\right)}_{\to 1} + \lim_{h\to 0} \sin x \underbrace{\left(\frac{\cos h - 1}{h}\right)}_{\to 0}$$

$$= \cos x \cdot 1 + \sin x \cdot 0 \quad (\text{by COLT})$$

$$= \cos x$$

Note $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

$$\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right)$$

$$\therefore \frac{\cos h - 1}{h} = \frac{-2\sin^2 \frac{h}{2}}{h}$$

$$= -\underbrace{\underbrace{\frac{\sin \frac{h}{2}}{\frac{h}{2}}}_{\to 1} \cdot \underbrace{\sin \frac{h}{2}}_{\to 0}}_{\text{as } h\to 0}$$

$$\therefore \frac{\cos h - 1}{h} \to 0 \text{ as } h \to 0$$

Q: Derivative of e^x as a limit.

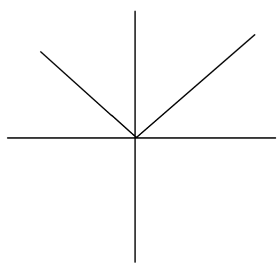
$$\frac{d}{dx} e^x = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x \frac{(e^h - 1)}{h} = e^x \cdot 1$$
$$= e^x$$

Claim (important but no proof ☹):

If $f'(x) = 0$ for all x, then $f(x) = c$, the constant function for some $c \in \mathbb{R}$

Subtle examples

- $f(x) = |x|$



Is continuous everywhere and is differentiable everywhere except at $x = 0$

Proof: $f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

So for $x > 0$: $\frac{df}{dx} = \lim_{h\to 0} \frac{|x+h| - |x|}{h}$ for h small enough, $x+h>0$

$$= \lim_{h\to 0} \frac{x+h-x}{h}$$

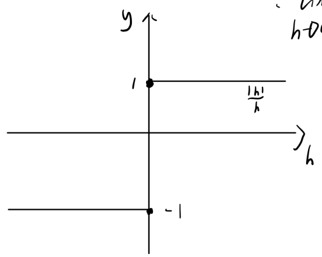
$$= \lim_{h\to 0} \frac{h}{h} = 1$$

for $x < 0$ similar $\to = \lim_{h\to 0} \frac{-(x+h) - (-x)}{h}$

$$= \lim_{h\to 0} \frac{-h}{h} = -1$$

But for $x=0$: $\dfrac{df}{dx} = \lim\limits_{h\to 0} \dfrac{|0+h| - 0}{h}$

$= \lim\limits_{h\to 0} \dfrac{|h|}{h} \to$ does not exist as $h \to 0$



- Claim: $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

is not differentiable at $x=0$

Proof: $\dfrac{df}{dx} = \lim\limits_{h\to 0} \dfrac{h \sin \frac{1}{h} - 0}{h} = \lim\limits_{h\to 0} \sin \frac{1}{h} \to$ does not exist

- Claim: $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

is differentiable everywhere

Proof: $\dfrac{df}{dx} = \lim\limits_{h\to 0} \dfrac{h^2 \sin \frac{1}{h} - 0}{h} = \lim\limits_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$

differentiating at $x=0$ so this equals $\dfrac{f(0+h) - f(0)}{h} \to$ only need to prove for 0.

Rules for Computing derivatives

Assume $f(x)$, $g(x)$ differentiable, then:

Sum rule

$$\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x)$$

Product rule

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

Quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \quad \text{for } g(x) \neq 0$$

Proofs (Sum and chain rule proofs in lecture notes)

- Product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left(\underbrace{g(x+h)}_{\downarrow\, g(x)} \cdot \underbrace{\frac{f(x+h)-f(x)}{h}}_{\downarrow\, f'(x)} + \underbrace{f(x)}_{\downarrow\, f(x)} \cdot \underbrace{\frac{g(x+h)-g(x)}{h}}_{\downarrow\, g'(x)} \right) \text{ as } h \to 0$$

COLT

$$= g(x)f'(x) + f(x)g'(x)$$

Example: $\frac{d}{dx}\left(\underbrace{x^2}_{f(x)} \underbrace{\sin x}_{g(x)}\right) = 2x\sin x + x^2\cos x$

Chain rule examples:

- $\frac{d}{dx} \sin(x^2) = \cos x^2 \cdot 2x = 2x\cos(x^2)$

  $(f(x) = \sin x,\ g(x) = x^2)$

  $(f'(x) = \cos x,\ g'(x) = 2x$

- $\frac{d}{dx} e^{-x} = -e^{-x}$

  $(f(x) = e^x,\ g(x) = -x)$

- $\frac{d}{dx} \cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$

- $\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

Quotient rule

Prove using product rule and d rule

Product rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)g(x)^{-1}\right) = \frac{df}{dx}\cdot g(x)^{-1} + f(x)\frac{d}{dx}\left(g(x)^{-1}\right)$$

Using $\frac{d}{dx}x^{-1} = -x^{-2}$

$$= \frac{f'(x)}{g(x)} + f(x)\left(-g(x)^{-2}g'(x)\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$ ✓

Examples

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\overset{f(x)}{\sin x}}{\underset{g(x)}{\cos x}} = \frac{\overset{g(x)}{\cos x}\,\overset{f'(x)}{\cos x} - \overset{g'(x)}{(-\sin x)}\,\overset{f(x)}{\sin x}}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of inverse functions

Examples:

Q: $\frac{d}{dx}\ln x$?

A: $\exp(\ln x) = x$

Differentiate both sides wrt x

Chain rule

$$\frac{d}{dx}\exp(\ln x) = \underset{f'(g(x))}{\exp(\ln x)}\cdot\underset{g'(x)}{\frac{d}{dx}\ln x} = x\frac{d}{dx}\ln x = \frac{d}{dx}x = 1$$

So $\frac{d}{dx}\ln x = \frac{1}{x}$

Q: $\frac{d}{dx}\arcsin x$

A: $\sin(\arcsin x) = x$

$\frac{d}{dx}$ both sides.

$\to \cos(\arcsin x)\frac{d}{dx}\arcsin x = 1$

$\therefore \frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)}$

But $\cos y = \pm\sqrt{1-\sin^2 y}$

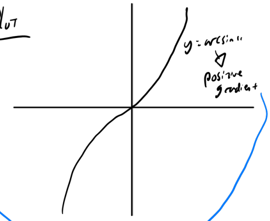
$\therefore \cos(\arcsin x) = \pm\sqrt{1-\sin(\arcsin x)^2} = \pm\sqrt{1-x^2}$

So $\frac{d}{dx}\arcsin x = \pm\frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{d}{dx}\arcsin x = +\frac{1}{\sqrt{1-x^2}}$

(Similar proof for arcsinh x)

Plot



we define arcsin x between $-\pi$ and π

$$\frac{d}{dx}x^r, \quad r\in\mathbb{R}$$

$$= \frac{d}{dx}e^{r\ln x} = e^{r\ln x}\cdot\frac{r}{x} = x^r\cdot\frac{r}{x} = rx^{r-1}$$

So $\frac{d}{dx}x^r = rx^{r-1}$ for any $r\in\mathbb{R}$

'Advanced' example

chain rule

$$\frac{d}{dx}\left(x^{(x^2)}\right) = \frac{d}{dx}e^{x^2\ln x} = e^{x^2\ln x}\cdot\frac{d}{dx}(x^2\ln x)$$

product rule

$$= \left(x^{x^2}\right)\left(x^2\cdot\frac{1}{x} + 2x\ln x\right)$$

$$= x^{x^2+1}(1+2\ln x)$$

More in exercises ☺

Claim: If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$

Proof: $f(x)$ is differentiable at $x=a \Rightarrow \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists

CoLT

$$\Rightarrow \lim_{h\to 0} h\,\frac{f(a+h)-f(a)}{h} = \left(\lim_{h\to 0}h\right)\cdot\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right)$$

$$= 0\cdot f'(a) = 0$$

But $\lim_{h\to 0} h \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0} f(a+h) - f(a)$

$\therefore \lim f(a+h) = f(a)$

So $f(x)$ continuous at a.

L'Hôpital's rule

For '$\frac{0}{0}$' cases we have:

Method 7: Theorem - Suppose $f(x)$ and $g(x)$ are $\underbrace{\text{continuously differentiable}}_{\text{Derivatives exist and are continuous near } a.}$ at $x=a$

Suppose also that $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists and that $\underbrace{f(a) = g(a) = 0}_{}$, then

$$\underbrace{\lim_{x\to a} \frac{f(x)}{g(x)}}_{\text{This limit exists}} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Do not forget this → needs to be a '$\frac{0}{0}$' limit to work!

Examples

1. $\lim_{x\to 0} \dfrac{\sin x}{x}$ $= \frac{0}{0}$ → try L'hôpital $\xrightarrow{M7} = \lim_{x\to 0} \dfrac{\cos x}{1} \xrightarrow{M1} = 1$

$$\left[\begin{matrix} f(x) = \sin x, & f'(x) = \cos x \\ g(x) = x, & g'(x) = 1 \end{matrix}\right]$$

2. $\lim_{x\to 0} \dfrac{\sinh(3x)^2}{\sin^2 x} = \frac{0}{0}$ → try L'hôpital

$f'(x) = 2\sinh(3x)\cosh(3x)\cdot 3$

$g'(x) = 2\sin x\cos x$

$$= \lim_{x\to 0} \frac{3\cdot 2\sinh(3x)\cosh(3x)}{2\sin x\cos x} \overset{0/0}{=} \lim_{x\to 0} \frac{3\sinh(6x)}{\sin(2x)} \overset{\text{L'H again!!}}{=} \lim_{x\to 0} \frac{18\cosh(6x)}{2\cos(2x)}$$

double angle/double hyp. formulae.

M_1

$$= 9$$

(This is what NOT to do)

3. $\lim_{x\to 1} \dfrac{x^2+x-2}{\underbrace{2x^2+5x-3}_{}} = \lim_{x\to 1} \dfrac{2x+1}{4x+5} = \dfrac{3}{9}$ ✗

This is not $\frac{0}{0}$, so do not use hospital rule

$= 0$ by M_1

4. $\lim_{x\to 1} \dfrac{x^2+x-2}{2x^2-5x+3} = \lim_{x\to 1} \dfrac{2x+1}{4x-5} = \dfrac{3}{-1} = -3$ by L'hopital

$= \lim_{x\to 1} \dfrac{(x-1)(x+2)}{(x-1)(2x-3)} = \dfrac{3}{-1} = -3$ by M_2

This is $\frac{0}{0}$, but Could have used M_2 instead as has $(x-1)$ factor.

"proof" of L'Hôpital's rule

Let $f(a) = g(a) = 0$

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}} \overset{\text{by CoLT}}{=} \lim_{x\to a}\lim_{h\to 0} \frac{f(x+h)-f(x)}{g(x+h)-g(x)}$$

$$= \lim_{h\to 0}\lim_{x\to a} \frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \lim_{h\to 0} \frac{f(a+h)-\overset{0}{f(a)}}{g(a+h)-\underset{0}{g(a)}} = \lim_{h\to 0} \frac{f(a+h)}{g(a+h)}$$

f,g Continuous So can do $x=a$

Let $x = a+h$

$$\therefore = \lim_{x\to a} \frac{f(x)}{g(x)}$$

If $h\to 0$, $x\to a$ when $x = a+h$

Not quite a proof as swapping the order of limits is potentially problematic.
The swapping is ok when f, g continuously differentiable at a.