Single Mathematics A

Aim - how to calculate, why things work
- Mathematics from the ground up

Foundations (assumed)

Numbers - real numbers $\mathbb{R}$
- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Non-negative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$

Operations - $+$, $-$, $\times$ or $\cdot$ or nothing, division

$a+b$ $\quad a-b \quad a\times b / a\cdot b / ab \quad \frac{a}{b}$

- Do these for all $a, b \in \mathbb{R}$, Except $\frac{a}{b}$ only defined for $b \neq 0$

Shorthand - $a^n = \underbrace{a \cdot a \cdot a \cdots\cdots a}_{n \text{ times}}$

$(a^0 = 1)$

Algebraic Manipulation - Simplifying factorising etc.

Sums and the Σ notation

Q. What is $M = 1+2+3+\dots+99+100$

A. well $M = 100+99+98+\dots+2+1$

$\therefore 2M = \underbrace{101+101+101+\dots+101+101}_{100 \text{ times}}$

$\therefore = 101 \times 100$

So $M = 5050$

Q. What is $M(n) = 1+2+\dots+n$

A. Same trick

$2M(n) = (n+1)+(n+1)+\dots+(n+1)$

So $M(n) = \dfrac{n(n+1)}{2}$

Next example

$1^2 + 2^2 + 3^2 + \dots + n^2$

Notation - Σ

write this as $\sum_{k=1}^{n} k^2$

(The sum from $k=1$ to $k=n$ of k^2)

$1+2+3+\dots+100 = \sum_{k=1}^{100} k$

more generally $\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + \dots + f(b)$

for a, b integers and $b > a$

3 properties of Σ notation

① Independent of the label

- $\sum_{k=a}^{b} f(k) = \sum_{x=a}^{b} f(x) = \sum_{웃=a}^{b} f(웃)$

② We can peel off terms

b

- $\sum_{k=a} f(k) = \sum_{k=a}^{c} f(k) + \sum_{k=c+1}^{b} f(k)$

3. We can shift the label

- $$\sum_{k=0}^{10} f(k) = \sum_{L=55}^{65} f(L-55)$$

Generally: $$\sum_{m=a}^{a+10} f(m-a) = \sum_{L=0}^{10} f(L)$$

Lecture 2

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + \dots + f(b)$$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Q. What is $\sum_{k=1}^{n} k^2 = M(n)$

A. It is $\frac{n(2n+1)(n+1)}{6} = Q(n)$

Try Some Cases

n	$\mathcal{Q}(n)$	$M(n)$
0	0	$\sum_{k=0}^{0} k^2 = 0$
1	1	$\sum_{k=0}^{1} k^2 = 0^2+1^2 = 1$
2	$(2\times5\times3)/6 = 5$	$\sum_{k=0}^{2} k^2 = 0^2+1^2+2^2 = 5$
3	$(3\times7\times4)/6 = 14$	$\sum_{k=0}^{3} k^2 = 0^2+1^2+2^2+3^2 = 14$

A mathematical proof that $M(n) = \mathcal{Q}(n)$ for all $n \in \mathbb{N}$.

- It is true for $n = 0, 1, 2, 3$

Suppose $M(n) = \mathcal{Q}(n)$ for some unspecified value of n

Now look at the next value, $n+1$

$$M(n+1) = \sum_{k=0}^{n+1} k^2 = \sum_{k=0}^{n} k^2 + (n+1)^2 = M(n) + (n+1)^2$$

$$= \mathcal{Q}(n) + (n+1)^2$$

by assumption

$$= \frac{n(2n+1)(n+1)}{6} + (n+1)^2$$

$$= \frac{n+1}{6}\left[n(2n+1) + 6(n+1)\right]$$

$$= \frac{n+1}{6}\left(2n^2+7n+6\right)$$

now $\mathcal{C}(n+1) = \dfrac{(n+1)\left(2(n+1)+1\right)\left(n+1+1\right)}{6}$

Are equivalent

So $m(n+1) = \mathcal{C}(n+1)$

We have shown that if $m(n) = \mathcal{C}(n)$ then $m(n+1) = \mathcal{C}(n+1)$ for any n.

We know that $m(3) = \mathcal{C}(3)$ $\therefore$ we also know that $m(4) = \mathcal{C}(4)$ etc

So $m(n) = \mathcal{C}(n)$ for all $n \in \mathbb{N}$

(Proof by induction)

General template

We want to prove some claim $f(n) = g(n)$ for all $n \in \mathbb{N}$

Proof

- Show that $f(0) = g(0)$ (or perhaps $f(1) = g(1)$)
- Show that for any $n \in \mathbb{N}$:

If $f(n) = g(n)$ then $f(n+1) = g(n+1)$

- The claim follows by induction.

The Binomial Theorem

Definitions

- Factorial $n! = n(n-1)(n-2)\cdots(2)(1)$ for $n \in \mathbb{N}$

  (define $0! = 1$)

- Binomial Coefficient: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $n, k \in \mathbb{N}$, $n \geq k$

Examples

- $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$,

  $4! = 4(3!) = 4 \times 6 = 24$, $5! = 5(4!) = 5 \times 24 = 120$ etc

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
=
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \cdots & & & &
\end{array}
$$

Pascal's Triangle

$$\left(\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{2\times 2} = 6\right)$$

- Left-right Symmetry $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

Claim: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Pascal's Triangle

Proof:

$$\text{RHS} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!}{k!(n-k+1)!}\left(k + (n-k+1)\right)$$

$$\left(\text{using } k! = k(k-1)! \quad (n-k+1)! = (n-k+1)(n-k)!\right)$$

$$= \frac{n!}{k!(n-k+1)!}(n+1) = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

$= \text{LHS}$

Q. What is $(a+b)^n$ expanded?

$n=1 \quad (a+b)^1 = a+b$

$n=2 \quad (a+b)^2 = a^2+2ab+b^2$

$n=3 \quad (a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$n=4 \quad (a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

etc.

Binomial Theorem

For any integer $n \geq 0$:

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

Proof: Use induction

- It is true for $n=0$ LHS: $(a+b)^0 = 1$

$$\text{RHS} = \sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^k$$

$$= \binom{0}{0} a^0 b^0 = 1 = \text{LHS}$$

- Let's just check $n=1$ (just in case)

LHS $= (a+b)^1 = a+b$

$$RHS = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^{k} = \binom{1}{0} a^{1-0} b^{0} + \binom{1}{1} a^{0} b^{1}$$

$$= a + b = LHS$$

So true for $n=0$ and $n=1$

- Now assume theorem holds for some n, then we can show it also holds for $n+1$

Consider LHS for $n+1$

$$(a+b)^{n+1} = (a+b)\underbrace{(a+b)^{n}}_{\text{by assumption}} = (a+b)\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}$$

Now manipulate RHS

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^{k} + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

Let $j = k+1$ (relabel)

$$= \text{— '' —} + \sum_{j=1}^{n+1} \binom{n}{j-1} a^{n-j+1} b^{j}$$

Now let $j = k$

$$= \text{— '' —} + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-k+1} b^k$$

$$= \sum_{k=0}^{n+1} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n-k+1} b^k$$

(need to check $k=0$ and $k=n+1$ also works - see lecture notes)

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

$=$ RHS with $n+1$

Shown the theorem holds for $n+1$

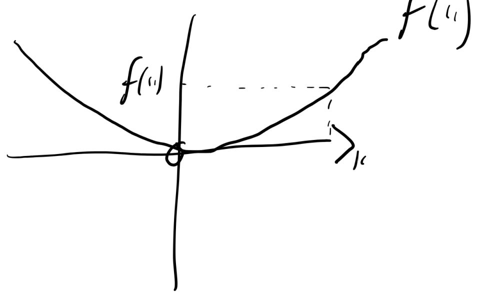
$\therefore$ by induction the theorem holds for all $n \in \mathbb{N}$

Functions and Inverse functions

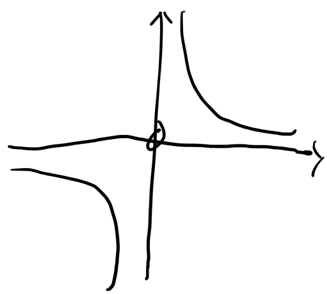
- A function f associates a value $f(x)$ to every number x in its domain

Examples

- $f(x) = x^2$
  domain : $\mathbb{R}$

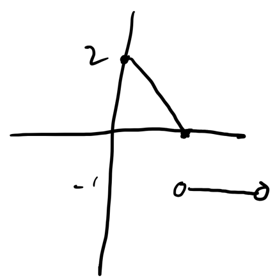


- $f(x) = 1/x$
  domain : $\mathbb{R}$ except 0
  $[\mathbb{R} \setminus \{0\}]$



- $f(n) = n!$
  domain : $\mathbb{N}$

- $f(x) = \begin{cases} -x+2 & 0 \leq x \leq 2 \\ -1 & 2 < x < 4 \end{cases}$
  domain $0 \leq x < 4$



- $f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{otherwise} \end{cases}$

Inverse functions

For a function $f(x)$ we can define an inverse function $f^{-1}(x)$ such that $f(f^{-1}(x)) = x$

So $f(f^{-1}(y)) = y$ so if I solve $f(x) = y$ for x then $x = f^{-1}(y)$

Practically to find $f^{-1}(x)$ just solve the equation $y = f(x)$ for x to find $f^{-1}(y)$

Example If $f(x) = \frac{1}{3x+4}$ what is $f^{-1}(x)$

Solve $y = f(x) = \frac{1}{3x+4}$ for x

$$3x+4 = \frac{1}{y}$$

$$x = \frac{1}{3}\left(\frac{1}{y} - 4\right)$$

$$x = \frac{1-4y}{3y}$$

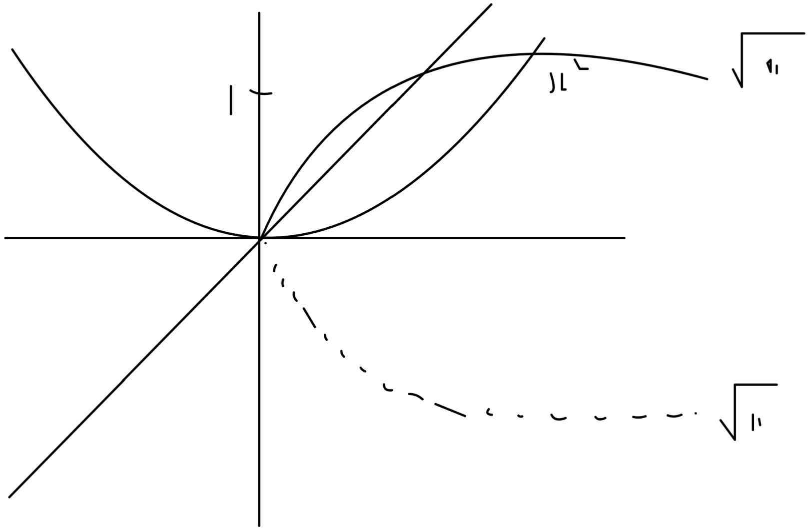
So this is $f^{-1}(y) = \frac{1-4y}{3y}$

$$\therefore f^{-1}(x) = \frac{1-4x}{3x}$$

The plot of $f^{-1}(x)$ is obtained from the plot of $f(x)$ by the reflection in the line $x=y$ (Since we're swapping x and y)

We can now define $\sqrt{x}$ as the inverse for x^2 that is also positive

<u>Definition</u> for every $x \geq 0$ $\sqrt{x}$ is the unique non-negative number such that $(\sqrt{x})^2 = x$



Note: $\sqrt{x} \geq 0$ always

$\sqrt{x}$ not defined when $x < 0$

Q. What is $\sqrt{2}$?

A. A number $y \geq 0$ such that $y^2 = 2$

y is > 1 : $1^2 = 1 < 2$

$y < 2$: $2^2 = 4 > 2$

$y < 1.5$: $1.5^2 = 2.25 > 2$

$y > 1.3$: $1.3^2 = 1.69 < 2$

($\sqrt{2} = 1.41421356\ldots$)

Quadratic equations

Solve $ax^2 + bx + c = 0$

$\times 4a$ $\quad 4a^2x^2 + 4abx = -4ac$

C.T.S $\quad (2ax + b)^2 - b^2 = -4ac$

So $(2ax + b)^2 = b^2 - 4ac$

Cases:

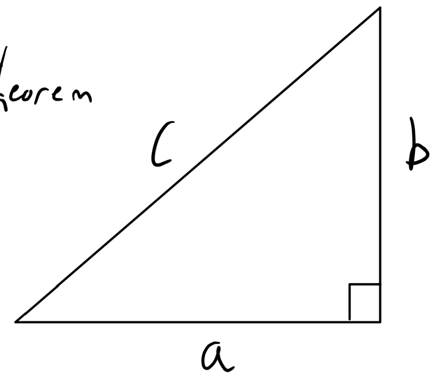
1. $b^2 - 4ac < 0$ – no real solutions
2. $b^2 - 4ac = 0 \Rightarrow 2ax + b = 0 \rightarrow x = \frac{-b}{2a}$
3. $b^2 - 4ac > 0 \rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac}$

   $\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(**Note** The equation $y^2 = x$ has 2 solutions when $x > 0 \Rightarrow y = \pm\sqrt{x}$)

Trigonometry

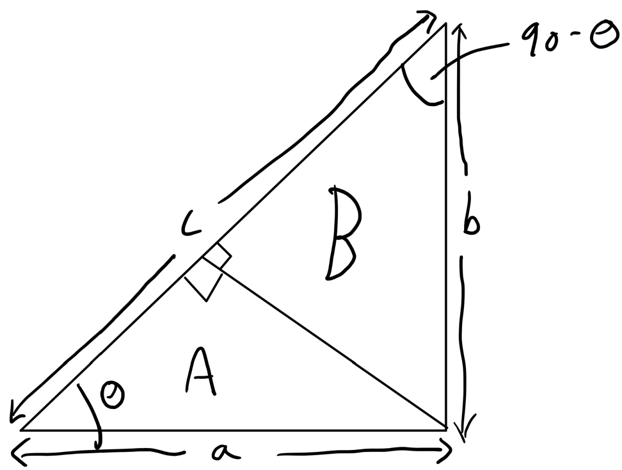
Pythagoras' Theorem

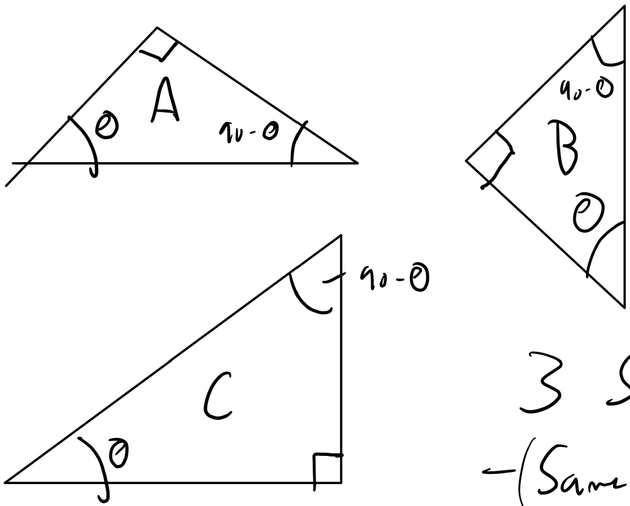


$c^2 = a^2 + b^2$

(121 proofs)

Proof





3 Similar Triangles

- (Same shape rotated and scaled)
- Similar shapes area is proportional to $(\text{Side length})^2$

Area A + area B = area C

Area A $= \lambda a^2$
Area B $= \lambda b^2$
Area C $= \lambda c^2$

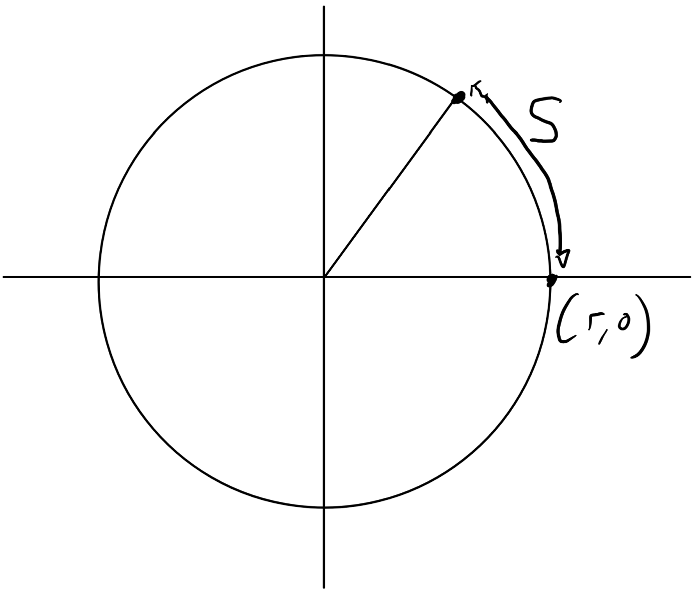
Constant of Proportionality

$$\lambda a^2 + \lambda b^2 = \lambda c^2$$
$$a^2 + b^2 = c^2$$

Trigonometry

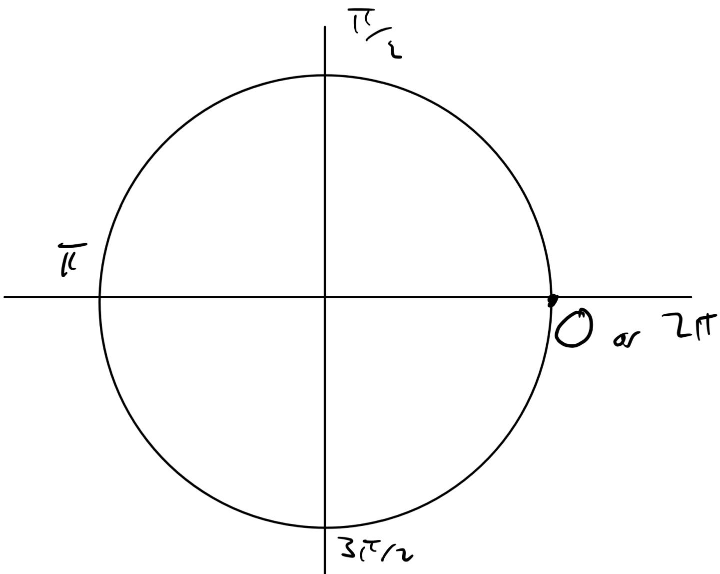
- Arc lengths and angles

Draw a circle with radius r



- To an arc of length S starting at $(r,0)$ and anti clockwise, we associate an angle, $\theta = \frac{S}{r}$ (radians)

- Right angles



Circumference $= 2\pi r$

$\Rightarrow$ angle $= 2\pi$

An arc in the clockwise direction corresponds to a negative angle

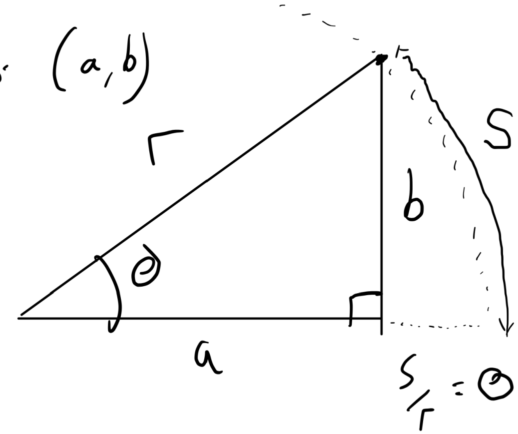
Trig functions

Suppose the endpoint has coordinates (a,b)

we define $\cos\theta = \frac{a}{r}$

$\sin\theta = b/r$

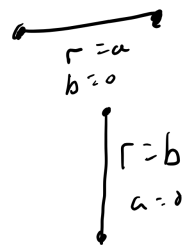
(a,b can be negative)



Observations

- If $\theta = 0$ then $a = r$, $b = 0$

  So $\cos 0 = 1$, $\sin 0 = 0$

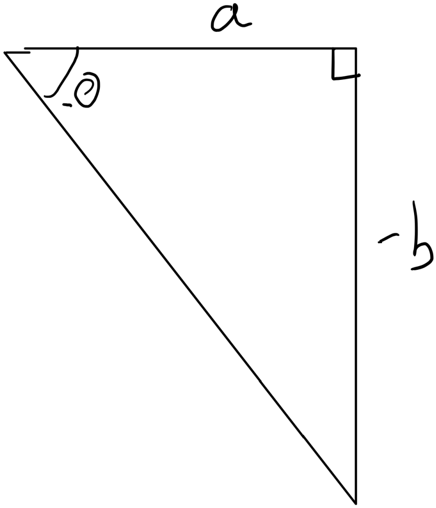


- $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$

- Pythagoras: $a^2 + b^2 = r^2$

  So $r^2\cos^2\theta + r^2\sin^2\theta = r^2$

  $\cos^2\theta + \sin^2\theta = 1$



- letting $\theta \to -\theta$

$$a \to a$$
$$b \to -b$$

So $\cos(-\theta) = \cos\theta$ even function

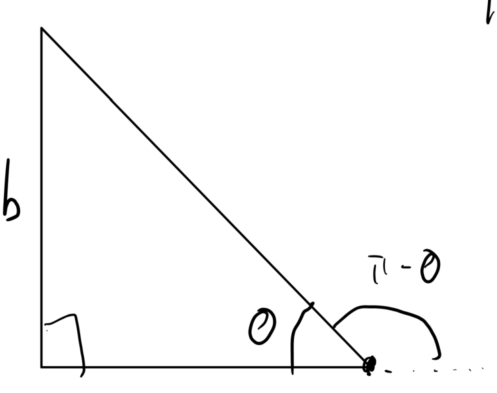
$\sin(-\theta) = -\sin\theta$ odd function

- letting $\theta \to \theta + 2\pi$

$$\cos(\theta + 2\pi) = \cos\theta$$
$$\sin(\theta + 2\pi) = \sin\theta$$

- flipping a, not b

$$a \to -a$$
$$b \to b$$
$$\theta \to \pi - \theta$$



$$\cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi - \theta) = \sin\theta$$

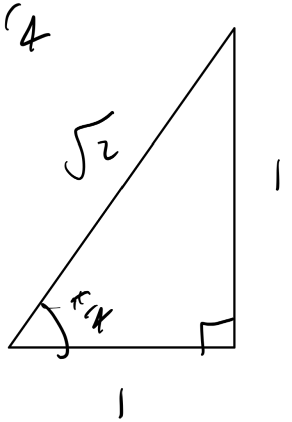
- Specific values

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
Sin	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
Cos	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$

Proof

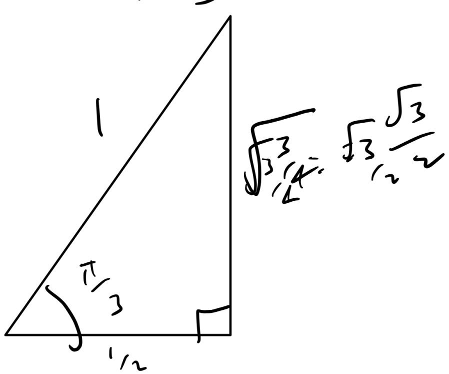
- $0, \pi/2$ we have already seen

- for $\pi/4$



$$\cos \pi/4 = 1/\sqrt{2} = \frac{\sqrt{2}}{2}$$
$$\sin \pi/4 = 1/\sqrt{2} = \frac{\sqrt{2}}{2}$$

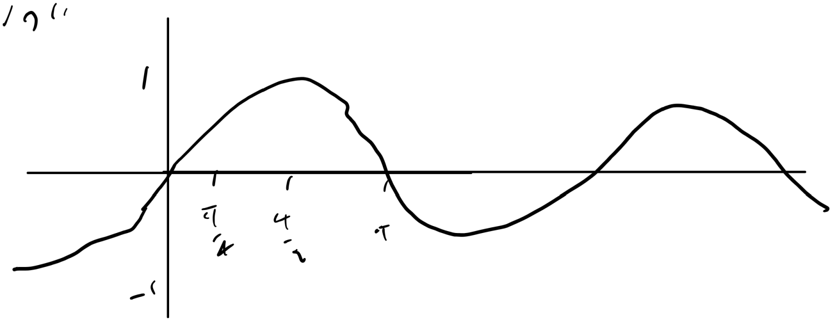
- for $\pi/6$, $\pi/3$



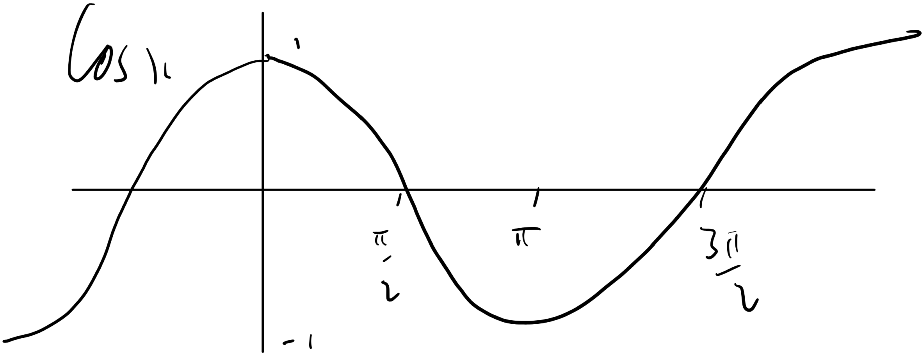
$$\cos \frac{\pi}{3} = \frac{1}{2} = \sin \frac{\pi}{6}$$
$$\sin \pi/3 = \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6}$$

Plots of Sin, Cos

Sin



Cos



Addition formulae

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ (1)

- Geometric proof in Lecture notes

From this we can derive $\cos\left(A - \frac{\pi}{2}\right) = \cos(A)\cos\left(-\frac{\pi}{2}\right) - \sin A \sin\left(-\frac{\pi}{2}\right)$

$$= \sin A$$

$$\therefore \cos\left(A - \tfrac{\pi}{2}\right) = \sin A \quad (2)$$

$A = \tfrac{\pi}{2}$

$$\cos\left(\tfrac{\pi}{2} - \tfrac{\pi}{2}\right) = \cos 0 = 1 = \sin \tfrac{\pi}{2}$$

Now set $A = \tfrac{\pi}{2} - \beta$

$$\cos\beta = \sin\left(\tfrac{\pi}{2} - \beta\right) \quad (3)$$

$$\therefore \sin(\alpha + \gamma) = \cos\left(\alpha + \gamma - \tfrac{\pi}{2}\right) = \cos\alpha\cos\left(\gamma - \tfrac{\pi}{2}\right)$$
$$- \sin\alpha\sin\left(\alpha - \tfrac{\pi}{2}\right)$$

$$\sin(\alpha + \gamma) = \cos\alpha\sin\gamma + \sin\alpha\cos\gamma \quad (4)$$

Example Questions

Q. What is $\sin\left(\frac{19\pi}{6}\right)$

$$= \sin\left(2\pi + \frac{7\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right)$$

$$= \sin\left(\pi + \frac{\pi}{6}\right)$$

$$= -\sin\frac{\pi}{6}$$

$$= -\frac{1}{2}$$

Q. Plot $\sin x + \cos x$

Let $\sin x + \cos x = R\sin(x + a)$ for some R, a

$$= R\sin x\cos a + R\cos x\sin a$$

$\therefore$ we need $R\cos a = 1$, $R\sin a = 1$ $\Rightarrow R^2\cos^2 a + R^2\sin^2 a = R^2$

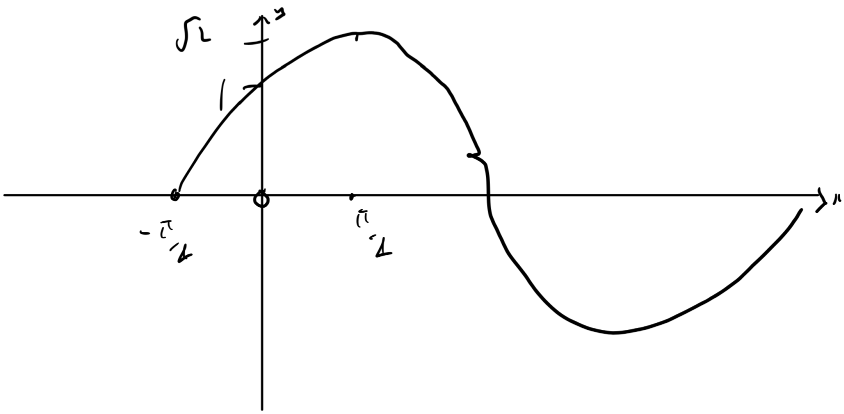
$$= 1^2 + 1^2 = 2$$

$$\therefore R = \sqrt{2}$$

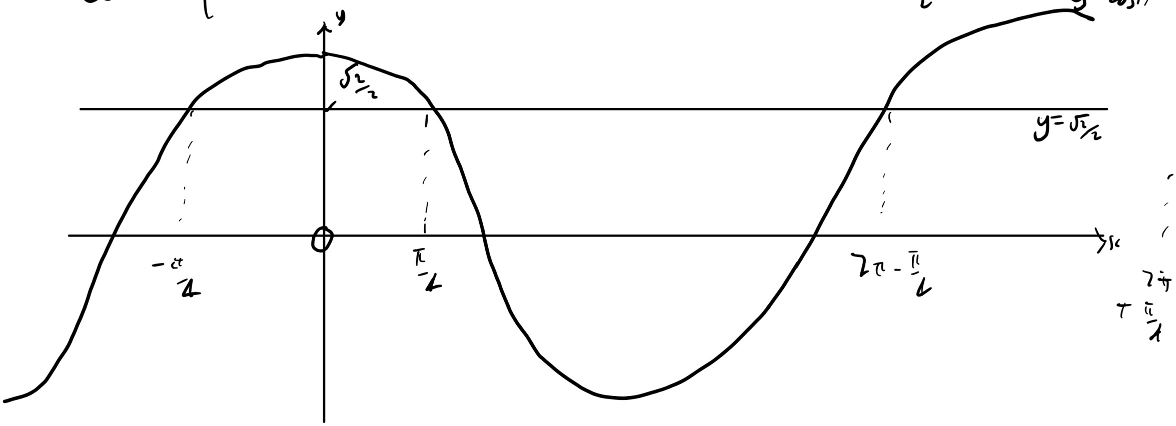
$$\cos a = \sin a = \frac{1}{\sqrt{2}}$$

$$a = \frac{\pi}{4}$$

So $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

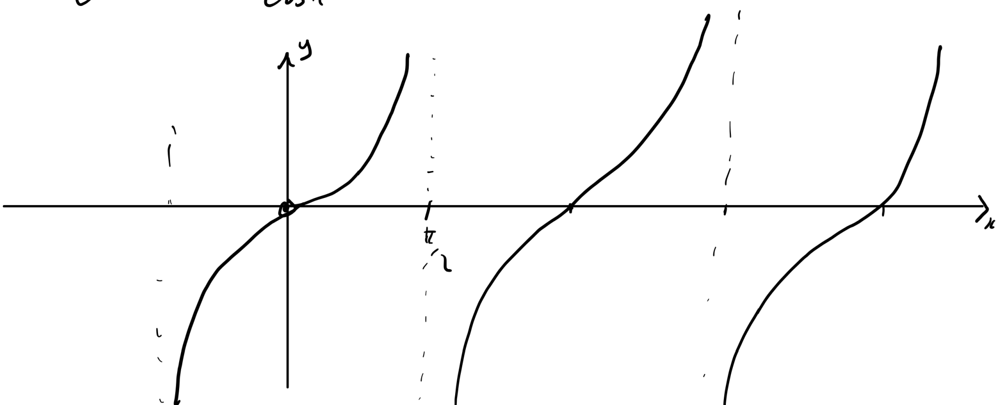


Q. Find all x such that $\cos x = \frac{\sqrt{2}}{2}$



$$\frac{\pi}{4} + 2\pi n \text{ or } -\frac{\pi}{4} + 2\pi n \text{ for } n \in \mathbb{Z}$$

$$\tan x = \frac{\sin x}{\cos x}$$



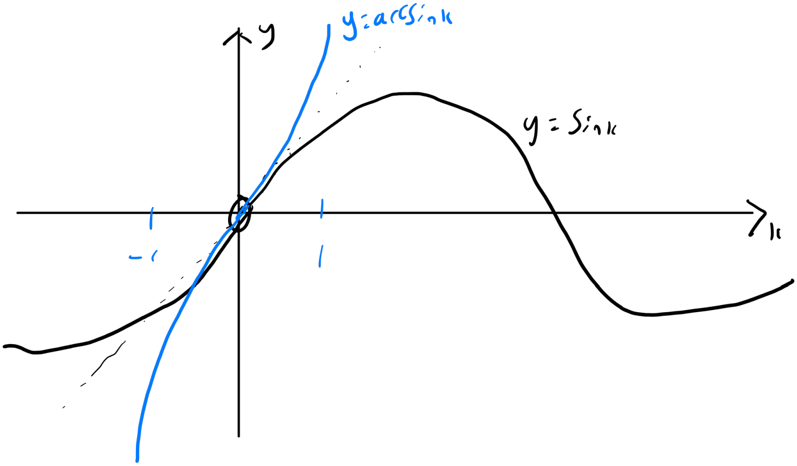
$$\operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

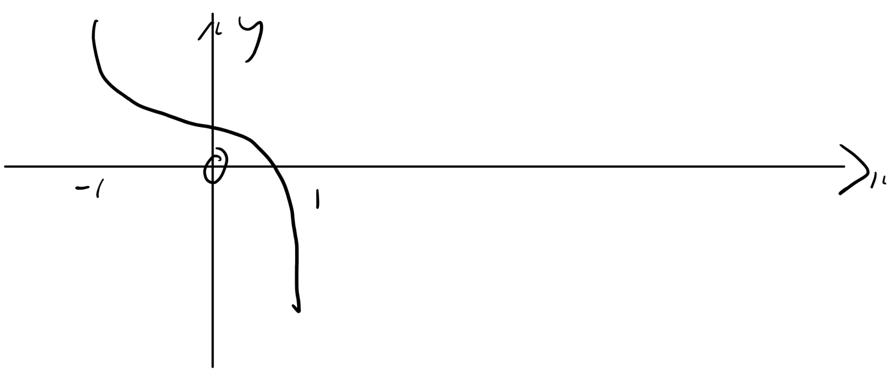
Inverse trig functions

$\arcsin x$ (inverse of $\sin$)

Define for $-1 \le x \le 1$, $\arcsin x$ is defined as the unique solution to the equation $\sin(\arcsin x) = x$ with $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$

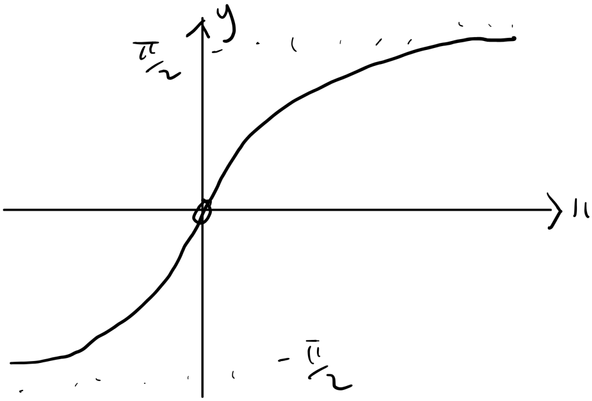


Similarly for $-1 \le x \le 1$ $\cos(\arccos x) = x$ with $0 \le \arccos x \le \pi$



Finally for $u \in \mathbb{R}$, $\tan(\arctan u) = u$ with

$$-\frac{\pi}{2} < \arctan u < \frac{\pi}{2}$$



Exp & ln

Definition for $x \in \mathbb{R}$ we define

$$\exp(x) := e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \frac{x^0}{0!} + \frac{x^1}{1!} + \dots$$

for example $e^0 = 1$

$$e^1 = e = 1 + \frac{1^2}{2} + \frac{1^3}{6} + \dots$$

$$= 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \dots$$

$$\approx 2.7182818\dots$$

Claim: $e^x e^y = e^{x+y}$

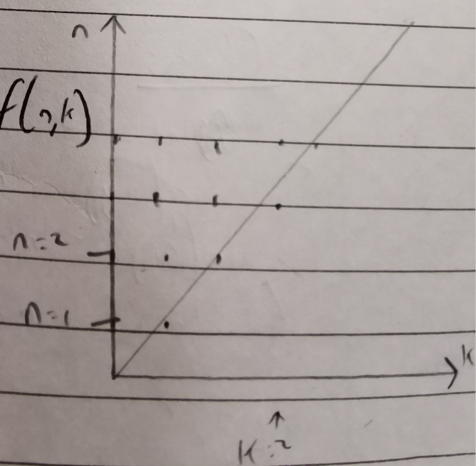
Proof $\exp(x+y) = \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n$ binomial Theorem

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} x^k y^{n-k}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} f(n,k) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} f(n,k)$$

$0 \leq k \leq n$



$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} x^k y^{n-k}$$

Let $m = n-k$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!} \times \frac{1}{m!} x^k y^m$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{m=0}^{\infty} \frac{y^m}{m!}$$

$$= \exp(x)\exp(y)$$

Therefore $1 = e^0 = e^{x-x} = e^x e^{-x}$ $\quad$ (e^x always > 0 for all x)

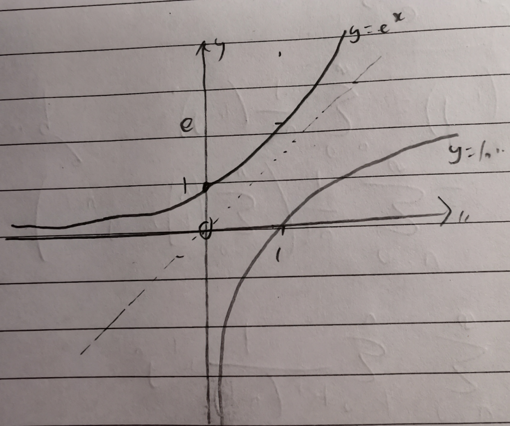
So $e^{-x} = \frac{1}{e^x}$

Definition - for $x > 0$ we define $\ln x$ such that

$\exp(\ln x) = x$ (inverse of e^x)

Note also $\ln(\exp(x)) = x$ for all $x \in \mathbb{R}$

Plot



Claims

1. $\ln 1 = 0$
2. $\ln(xy) = \ln x + \ln y$
3. $\ln(\frac{1}{x}) = -\ln x$
4. $\ln x^n = n \ln x$

Proofs

1. $\exp(0) = 0$
2. $\ln(\exp(\ln x + \ln y)) = \ln(\exp(\ln x)\exp(\ln y))$

$= \ln xy$

3. from ② put $y = \frac{1}{x}$

4. Do ② n times with $y = x$, e.g. $\ln(x^2) = \ln x + \ln x = 2\ln x$

$\ln(x^3) = \ln x^2 x = \ln x^2 + \ln x = 2\ln x + \ln x = 3\ln x$

Definition - for any real $x > 0$ and any real c, we define

$$\underline{x^c = \exp(c\ln x)}$$

Properties

- $\ln(x^c) = c\ln x$
- $(x^c)^d = \exp(d\ln(x^c))$
  $= \exp(dc\ln x)$
  $= x^{cd}$

- $x^c x^d = \exp(c\ln x)\exp(d\ln x)$
  $= \exp(c\ln x + d\ln x)$
  $= \exp((c+d)\ln x)$
  $= x^{c+d}$

Agrees with x^n when $c = n \in \mathbb{N}$?

$$x^n = \exp(n\ln x) = \exp(\underbrace{\ln x + \ln x + \ln x + \dots + \ln x}_{n \text{ times}})$$

$$= \exp(\ln x)\exp(\ln x)\dots\exp(\ln x)$$

$$= \underbrace{x\, x\, x \dots x}_{n \text{ times}}$$

So agrees with previous definition

- $x^1 = x$
- $x^0 = 1$

- $c = \frac{1}{2}$

Claim: $x^{\frac{1}{2}} = \sqrt{x}$

$\exp(\frac{1}{2}\ln x)$ Inverse of x^2 s.t. $\sqrt{x} > 0$

Proof we need to show $x^{\frac{1}{2}}$ is an inverse of x^2 and $x^{\frac{1}{2}} \geq 0$ for all $x \geq 0$

$\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2} \cdot 2} = x^1 = x$ So an inverse of x^2

and $x^{\frac{1}{2}} = \exp\left(\frac{1}{2}\ln x\right) > 0$ always so $x^{\frac{1}{2}} > 0$ also

Q. Solve $\ln(x^2+2) = \ln x + 3$ for x

A. Exp both sides

$$x^2 + 2 = \exp\left(\ln x + 3\right) = e^{\ln x} e^3 = xe^3$$

So $x^2 - e^3 x + 2 = 0$

So $x = \frac{1}{2}\left(e^3 \pm \sqrt{e^6 - 8}\right)$

Q. Solve $\left(x^x\right)^x = x^{\left(x^x\right)}$

$$x^c = \exp(c \ln x) \qquad x > 0,\ c \in \mathbb{R}$$

Solve $(x^x)^x = x^{(x^x)}$ — $x = 1$ — $x = 2$

$$(x^x)^x = \left(e^{x \ln x}\right)^x = e^{x^2 \ln x}$$

$$x^{(x^x)} = e^{x^x \ln x}$$

$$\Rightarrow x^2 \ln x = x^x \ln x$$

$$\Rightarrow \ln x \left(x^2 - x^x\right) = 0$$

$\therefore$ either $\ln x = 0$

$x = 1$

Or $x^2 = x^x$

take ln of both sides $\rightarrow$ $x^2 = \exp(x \ln x)$

$2 \ln x = x \ln x$

either $\ln x = 0$ or $x = 2$

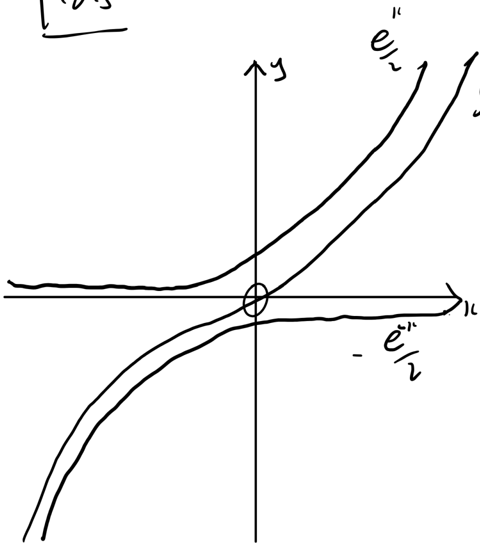
So either $\underline{x = 1 \text{ or } x = 2}$

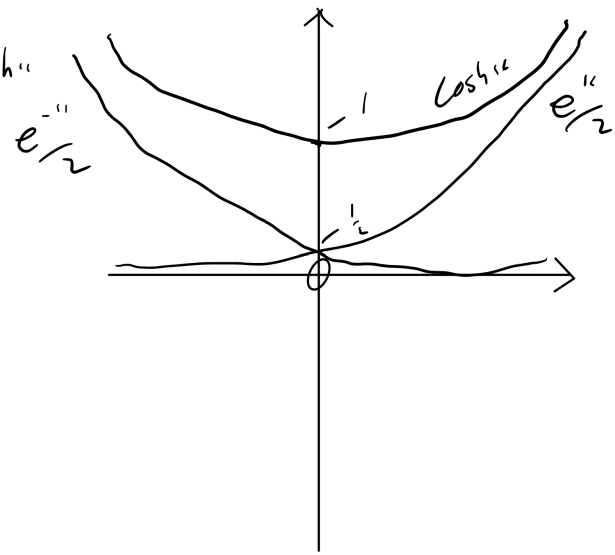
Hyperbolic Functions

For any $x \in \mathbb{R}$ we define

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Plots





Properties (derived from properties of exp)

$\sinh 0 = 0$

$\cosh 0 = 1$

$\sinh(-x) = -\sinh x \rightarrow$ odd function

$\cosh(-x) = \cosh x \rightarrow$ even function

$\cosh x + \sinh x = e^{x}$

$\cosh x - \sinh x = e^{-x}$

$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x)$
$= e^{-x} e^{x} = 1$

$\cosh x \cosh y + \sinh x \sinh y = \cosh(x+y) \rightarrow \cosh 2x = \cosh^2 x + \sinh^2 x$

$\sinh x \cosh y + \cosh x \sinh y = \sinh(x+y) \rightarrow \sinh 2x = 2 \sinh x \cosh x$

Similar to trig formulae but not exactly the same

Proof of Cosh addition formula

$$\text{LHS} = \cosh x \cosh y + \sinh x \sinh y$$

$$= \frac{(e^{x}+e^{-x})(e^{y}+e^{-y}) + (e^{x}-e^{-x})(e^{y}-e^{-y})}{4}$$

$$= \frac{e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y}}{4}$$

$$= \frac{e^{x+y} + e^{-x-y}}{2} = \cosh(x+y)$$

Other Proofs Similar

Inverses

Define $\operatorname{arcsinh} x \to \sinh(\operatorname{arcsinh} x) = x$, for all $x \in \mathbb{R}$

$\operatorname{arccosh} x \to \cosh(\operatorname{arccosh} x) = x$, for all $x \geq 1$

and $\operatorname{arccosh} x \geq 0$

Q. Find an expression for $\operatorname{arcsinh} x$ in terms of ln.

Let $y = \operatorname{arsinh} x$ then $\sinh y = x$

So $\frac{1}{2}(e^y - e^{-y}) = x$

$\times 2e^y$

$e^{2y} - 1 = 2xe^y$

$e^{2y} - 2xe^y - 1 = 0$

Let $z = e^y$

$z^2 - 2xz - 1 = 0$

So $z = x \pm \sqrt{x^2+1}$

$\sqrt{x^2+1} > x$ (since $> \sqrt{x^2}$)

So $x - \sqrt{x^2+1} < 0$

So this cannot equal $e^y > 0$

$\therefore$ take +ve root

$$\Rightarrow e^y = x + \sqrt{x^2+1}$$

$$y = \ln\left(x + \sqrt{x^2+1}\right) = \operatorname{arsinh} x$$

Q. $\operatorname{arcosh}\left(\left(e + \frac{1}{e}\right)\frac{1}{2}\right)$

Notice $\cosh 1 = \dfrac{e^1 + e^{-1}}{2} = \left(e + \frac{1}{e}\right)\frac{1}{2}$

So $\operatorname{arcosh}\left(\left(e + \frac{1}{e}\right)\frac{1}{2}\right) = 1$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$