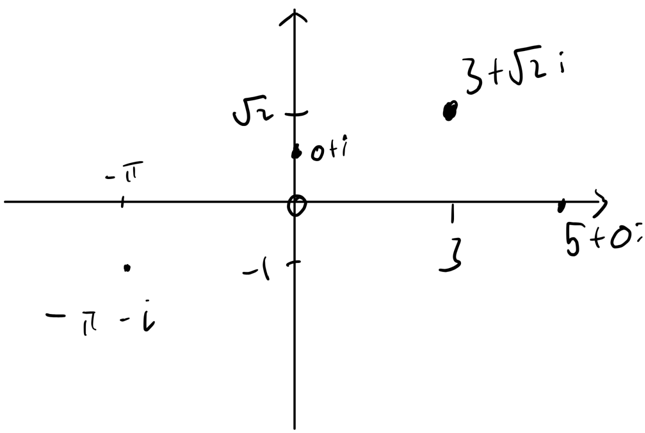
Chapter 2 : Complex Numbers

Definition - To every point (x, y) in the plane, we associate a complex number $x + iy$ $(x, y \in \mathbb{R})$ (i some indeterminate)

$\mathbb{C}$ = Set of Complex numbers

(x, y) plane is called the Complex or Argand plane



for $\mathbb{C}$, we have $+, -, \times, \div$

Addition and Subtraction

$$(3+4i)+(-6+2i) = 3-6+4i+2i$$
$$= -3+6i$$

$$2+i+3i = 2+4i$$

$$\pi+2i-(3+\sqrt{7}i) = (\pi-3)+(2-\sqrt{7})i$$

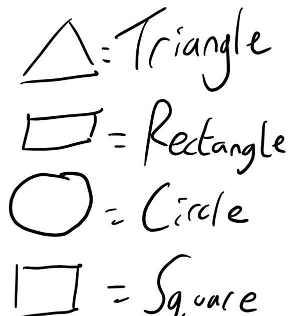
Multiplication

Also as expected, except 1 extra rule

$$i^2 = -1$$

For example:

$$(1+i)(2+i)$$
$$= 2+2i+i+i^2$$
$$= 2+3i-1$$
$$= 1+3i$$



$$(3-i)(4+2i)$$
$$= 12-4i+6i-2i^2$$
$$= 12+2i-2(-1)$$
$$= 14+2i$$

$$(1+i)^2 = (1+i)(1+i)$$
$$= 1+2i+i^2 = 2i$$
$$(1+i)^3 = (1+i)^2(1+i)$$
$$= 2i(1+i) = 2i+2i^2$$
$$= -2+2i$$

Notation - we usually use z or w to denote a complex number, so we write $z = x+iy$ $x,y \in \mathbb{R}$
or $w = u+iv$ $u,v \in \mathbb{R}$

So in equations $z+w = x+iy+u+iv$
$$= x+u+i(y+v)$$
$$zw = (x+iy)(u+iv) = xu+iyu+ixv+i^2yv$$
$$= xu-yv+i(yu+xv)$$

Examples 6. $z = 3-4i$, $w = 3-i$ Compute $z-w$, zw and w^3

$z \cdot w =$

A. $z-w = 3-4i-(3-i)$
$$= -3i$$

$$1 = (3-4i)(3-i)$$
$$= 9 - 15i + 4i^2$$
$$= 5 - 15i$$

$$w^3 = (3-i)(3-i)(3-i)$$
$$= (9-6i+i^2)(3-i)$$
$$= (8-6i)(3-i)$$
$$= 24 - 18i - 8i + 6i^2$$
$$= 18 - 26i$$

6. i^4, $i(-i)$, i^{121}

A. $i^4 = i^2 i^2 = (-1)(-1) = 1$

$$i(-i) = -i^2 = -(-1) = 1$$
$$i^{121} = i^{4\times 30+1} = (i^4)^{30}\, i$$
$$= 1^{30}\, i$$
$$= i$$

Division

by a real number: $\frac{a+bi}{c} = \frac{a}{c} + \frac{b}{c}i$

by a Complex number: $\frac{1-2i}{3+4i} = \frac{1-2i}{3+4i} \times \frac{3-4i}{3-4i}$

$$= \frac{(1-2i)(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{3-6i-4i+8i^2}{9+12i-12i-16i^2}$$

$$= \frac{-5-10i}{9+16} = \frac{-5-10i}{25}$$

$$= \frac{-1-2i}{5}$$

This trick always works because $(x+iy)(x-iy) = x^2 + ixy - ixy - i^2y^2$

$$= x^2 + y^2 \in \mathbb{R}$$

So we can divide by any Complex number except 0

$$\frac{1}{i} = \frac{1}{i} * \frac{-i}{-i} = \frac{-i}{1} = -i$$

Definitions

Let $z = x + iy \in \mathbb{C}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$

- $Re(z) = x$ ← real part of z
- $Im(z) = y$ ← Imaginary part of z
- $|z| = \sqrt{x^2 + y^2}$ ← Modulus of z
- $\bar{z} = x - iy$ ← Complex Conjugate of z (sometimes called z^*)

Example

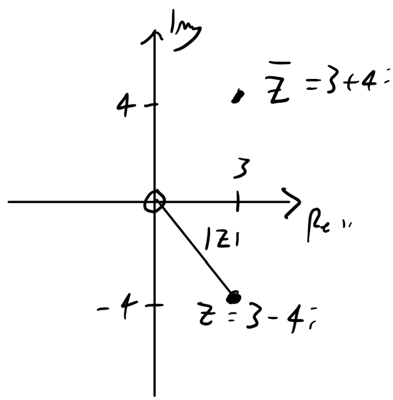
If $z = 3 - 4i$, then:

$Re(z) = 3$

$Im(z) = -4$

$|z| = \sqrt{3^2 + 4^2} = 5$

$\bar{z} = 3 + 4i$



If $Z = i(i+3)$

$= -1+3i$

$Re(Z) = -1$

$Im(Z) = 3$

$|Z| = \sqrt{1+3^2} = \sqrt{10}$

$\bar{Z} = -1-3i$

Definition

- If $Im(Z) = 0$, i.e. $Z = x + 0i$, then Z is real
- The x axis is called the real axis
- If $Re(Z) = 0$, i.e. $Z = 0 + yi$, then Z is imaginary
- The y axis is called the imaginary axis

Q. What is the modulus of a real number?

$$|z| = \sqrt{x^2+y^2} = \sqrt{x^2} = \begin{cases} x \rightarrow x \geq 0 \\ -x \rightarrow x < 0 \end{cases}$$

$y=0$ for real numbers

Note in the complex plane:

- $Re(z)$ is the x-coordinate
- $Im(z)$ is the y-coordinate
- $|z|$ is the distance between 0 and z
- $\bar{z}$ is a reflection of z in the real axis

Further Properties

$(z = x+iy,\ w = u+iv)$

- $Re(z) = \frac{1}{2}(z+\bar{z}) \to \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(x+iy+x-iy) = x = Re(z)$
- $Im(y) = \frac{1}{2i}(z-\bar{z}) \to \frac{1}{2i}(z-\bar{z}) = \frac{1}{2i}(x+iy-x+iy) = y = Im(z)$
- $z\bar{z} = |z|^2 \to (x+iy)(x-iy) = x^2+iyx-iyx-i^2y^2 = x^2+y^2 = |z|^2$

Note $z\bar{z}$ always real and ≥ 0

- $\overline{z+w} = \bar{z}+\bar{w} \to \overline{z+w} = \overline{x+iy+u+iv} = \overline{x+u+i(y+v)} = x+u-i(y+v)$
$= x-iy+u-iv$
$= \bar{z}+\bar{w}$
- $\overline{zw} = \bar{z}\bar{w} \to \overline{zw} = \overline{(x+iy)(u+iv)} = \overline{xu-yv+i(yu+xv)}$
$= xu-yv-i(yu+xv)$
$\bar{z}\bar{w} = (x-iy)(u-iv) = xu-iyu-ixv+i^2yv$
$= xu-yv-i(yu+xv) =$
- $|zw| = |z||w| \to |zw|^2 = zw\overline{zw}$
$= z\bar{z}w\bar{w}$
$= |z|^2|w|^2$
And so $|zw| = |z||w|$ by taking square root.
(using $|x| \geq 0$)
- $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

Example: $z = 2+i$, $w = 1-3i$

$|z| = \sqrt{5}$ $|w| = \sqrt{10}$

$|z||w| = \sqrt{50}$

$zw = (2+i)(1-3i) = 2+i-6i-3i^2$

$= 2+3-5i$

$= 5-5i$

$|zw| = \sqrt{5^2+5^2} = \sqrt{50}$!!

Division

General rule: to compute $\frac{z}{w}$ we write $\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$ real

So $\mathbb{C}$ division has changed into $\mathbb{R}$ division.

$$\boxed{\frac{z}{w} = \frac{z\bar{w}}{|w|^2}}$$

Q. Compute $\text{Re}\left(\frac{2-i}{2+3i}\right)$ and $\text{Im}\left(\frac{2-i}{2+3i}\right)$

A. $\frac{2-i}{2+3i} = \frac{(2-i)(2-3i)}{2^2+3^2} = \frac{4-8i-3}{13} = \frac{1}{13} - \frac{8}{13}i$

$\text{Re}\left(\frac{2-i}{2+3i}\right) = \frac{1}{13}$, $\text{Im}\left(\frac{2-i}{2+3i}\right) = -\frac{8}{13}$

~~further~~ further properties

- $|\bar{z}| = |z|$ $\rightarrow$ $|\bar{z}| = |x-iy| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|$

- $\left|\frac{1}{w}\right| = \frac{1}{|w|}$ Same as below with $z=1$

$$-\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \rightarrow \left|\frac{z}{w}\right| = \left|\frac{z\bar{w}}{|w|^2}\right| = |z\bar{w}|\left|\frac{1}{|w|^2}\right|$$

$$= \frac{|z\bar{w}|}{|w|^2} = \frac{|z||\bar{w}|}{|w|^2} = \frac{|z||w|}{|w|^2} = \frac{|z|}{|w|}$$

Example $\left|\frac{2-i}{2+3i}\right| = \left|\frac{1}{13} - \frac{8}{13}i\right| = \sqrt{\left(\frac{1}{13}\right)^2 + \left(\frac{8}{13}\right)^2} = \sqrt{\frac{65}{169}} = \sqrt{\frac{5}{13}}$

Whereas $\frac{|2-i|}{|2+3i|} = \frac{\sqrt{5}}{\sqrt{13}} = \sqrt{\frac{5}{13}} =$

Example: find $Re\left(\frac{z}{w}\right)$ in terms of $Re(z)$, $Re(w)$, $Im(z)$, $Im(w)$

Let $z = x+iy$, $w = u+iv$ so $Re(z) = x$, $Im(z) = y$
$Re(w) = u$, $Im(w) = v$

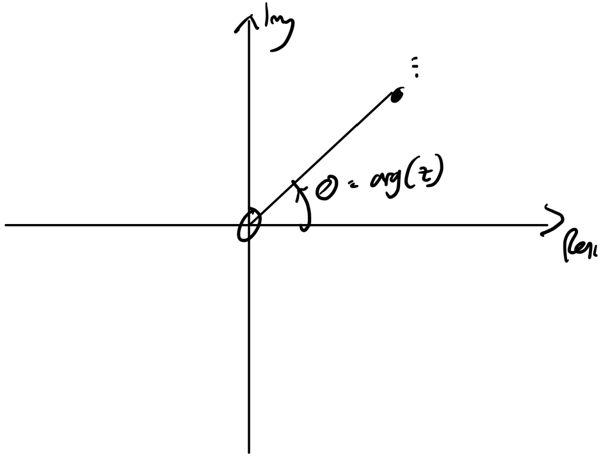
Now work out $\frac{z}{w} = \frac{z\bar{w}}{|w|^2} = \frac{(x+iy)(u-iv)}{u^2+v^2} = \frac{xu+yv+i(yu-xv)}{u^2+v^2}$

$$= \frac{xu+yv}{u^2+v^2} + i\,\frac{yu-xv}{u^2+v^2}$$

$$Re\left(\frac{z}{w}\right) = \frac{xu+yv}{u^2+v^2}$$

$$= \frac{Re(z)Re(w) + Im(z)Im(w)}{Re(w)^2 + Im(w)^2}$$

Definition - for a complex number z, we define the <u>argument</u> of z, denoted $\arg(z)$, as the angle between the x axis and line between 0 and z, measured anticlockwise (clockwise = negative)



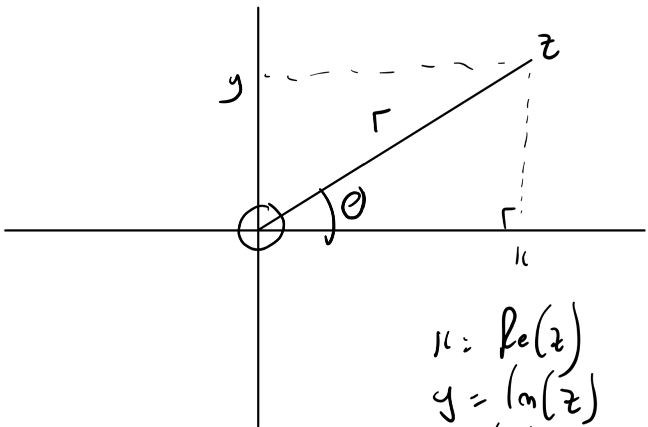
e.g. $\arg(1+i) = \frac{\pi}{4}$

$\arg(-3) = \pi$

$\arg(-2i) = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$

Usually $\underline{-\pi < \arg z \leq \pi}$

(but will not be marked wrong if not)



$x = Re(z)$

$y = Im(z)$

$r = |z|$

$\theta = \arg(z)$

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\tan\theta = y/x$

So $Re(z) = |z|\cos(\arg z)$

$Im(z) = |z|\sin(\arg z)$

$|z| = \sqrt{Re(z)^2 + Im(z)^2}$

$\tan(\arg z) = \frac{Im(z)}{Re(z)}$

Note: $\tan\theta = \frac{y}{x}$ is subtle

For example: for $z = 1+i$ we find $\tan\theta = \frac{1}{1} = 1$ ✓

$\theta = \pi/4$

But for $z = -1-i$ $\Rightarrow$ $\tan\theta = \frac{-1}{-1} = 1$ but now $\theta = -\frac{3\pi}{4}$

$\therefore \tan\theta = y/x$ determines θ only partially because $\tan(\theta+\pi) = \tan\theta$

So draw a picture to see what θ really is !!

Q. $\arg(\bar{z}) = -\arg(z)$

$\arg(-z) = \pi - \arg(z)$ (or $\pi + \arg(z)$)

$$\arg(rz) = \begin{cases} \arg(z) & r > 0 \\ \pi + \arg(z) & r < 0 \\ \text{Undefined} & r = 0 \end{cases}$$

Polar Representation

Definition: The form $z = r(\cos\theta + i\sin\theta)$ (instead of $z = x + iy$)

is called the polar representation of z

It tells you immediately what $|z|$ and $\arg z$ are (r and θ)

Examples

Q: Write $z_1 = 1 - \sqrt{3}i$

$z_2 = 17(\sin\frac{\pi}{4} + i\cos\frac{\pi}{4})$

$z_3 = -7$

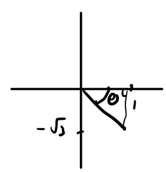
In the polar representation

A: find $|z|$ and $\arg z$ then write $z = r(\cos\theta + i\sin\theta)$

① $z_1 = 1 - \sqrt{3}\,i$

$\rightarrow |z| = \sqrt{1^2 + 3} = 2$

$\arg(z_1) = -\pi/3$



So $z_1 = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

2 $17\left(\sin\frac{\pi}{4} + i\cos\frac{\pi}{4}\right)$

$|z_2| = 17$

$\arg(z_2) = \frac{\pi}{4}$

So $z_2 = 17\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

3 $z_3 = -7 + 0i$

$r = |z| = 7$

$\arg z = \theta = \pi$ ← ew

So $z_3 = 7(\cos\pi + i\sin\pi)$

Q. What is $\arg(zw)$?

Use polar representation

$z = r(\cos\theta + i\sin\theta)$

$w = \rho(\cos\chi + i\sin\chi)$

then $zw = r\rho(\cos\theta + i\sin\theta)(\cos\chi + i\sin\chi)$

$= r\rho(\cos\theta\cos\chi - \sin\theta\sin\chi + i(\sin\theta\cos\chi + \cos\theta\sin\chi))$

$= r\rho(\cos(\theta+\chi) + i\sin(\theta+\chi))$

So we see $\boxed{\begin{aligned} |zw| &= r\rho = |z||w| \\ \arg(zw) &= \theta + \chi = \arg z + \arg w \end{aligned}}$

De Moivre's Theorem

For any integer $n \geq 1$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof by induction

True for $n=1$ $\Rightarrow$ $\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$

Assume true for some n

Then the next case $n+1$

$$(\cos\theta + i\sin\theta)^{n+1} = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^n$$ by assumption

$$= (\cos\theta + i\sin\theta)(\cos(n\theta) + i\sin(n\theta))$$

$$= \cos\theta\cos(n\theta) - \sin\theta\sin(n\theta) + i(\sin\theta\cos(n\theta) + \cos\theta\sin(n\theta))$$

$$= \cos(n\theta + \theta) + i\sin(n\theta + \theta)$$

$$= \cos((n+1)\theta) + i\sin((n+1)\theta)$$ ✓

Computing z^n quickly

$$(1+i)^{10} = \left(\sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)\right)^{10} = \sqrt{2}^{10}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)^{10}$$ by DM

$$= 32\left(\cos\tfrac{10\pi}{4} + i\sin\tfrac{10\pi}{4}\right)$$

$$= 32\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right)$$

$$= 32i$$

Application of De Moivre's Theorem

Proving Trig Identities

e.g. Q: $\cos 3\theta$ as a polynomial in $\cos\theta$

A: $\cos(3\theta) + i\sin(3\theta) = (\cos\theta + i\sin\theta)^3$ by de Moivre

$$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$ by binomial theorem

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

So $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$$= 4\cos^3\theta - 3\cos\theta$$

Claim: $\cos\theta + i\sin\theta = e^{i\theta}$ $\to$ Euler's formula

What is $e^{i\theta}$? Same as e^x but replace x with $i\theta$

Definition: $\exp(i\theta) = e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$

So $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$

$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots$

gives a complex number for every real θ

Note: $\left.\begin{array}{l} e^{i\theta} e^{ix} = e^{i(\theta + x)} \\ e^{in\theta} = (e^{i\theta})^n \end{array}\right\}$ Proofs the same as for real exponential

$e^{i0} = e^0 = 1 = e^{i\theta - i\theta} = e^{i\theta} e^{-i\theta}$

$\therefore e^{-i\theta} = \frac{1}{e^{i\theta}}$ as for real x

Proof of Euler's formula (not examinable)

Consider $\frac{d}{d\theta}\left(e^{-i\theta}(\cos\theta + i\sin\theta)\right)$

$= \frac{d}{d\theta} e^{-i\theta}(\cos\theta + i\sin\theta) + e^{-i\theta} \frac{d}{d\theta}(\cos\theta + i\sin\theta)$

$= -ie^{-i\theta}(\cos\theta + i\sin\theta) + e^{-i\theta}(-\sin\theta + i\cos\theta)$

$= 0$

$\therefore e^{-i\theta}(\cos\theta + i\sin\theta)$ is a constant (independent of θ)

Let $\theta = 0$

then $e^{-i0}(\cos 0 + i\sin 0) = 1 =$ constant

So $\cos\theta + i\sin\theta = e^{i\theta}$

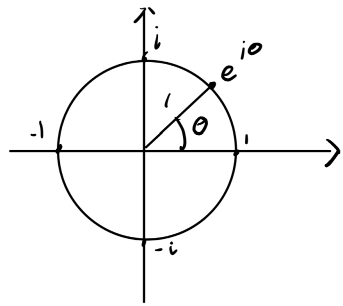
Consequences of $\cos\theta + i\sin\theta = e^{i\theta}$

$|e^{i\theta}| = 1$ Since $e^{i\theta} = 1 \cdot (\cos\theta + i\sin\theta)$

$\arg(e^{i\theta}) = \theta$

$\mathrm{Re}(e^{i\theta}) = \cos\theta$

$\mathrm{Im}(e^{i\theta}) = \sin\theta$



$e^{i\theta}$ lives on the unit circle

$e^{i0} = 1$

$e^{i\pi/2} = i$

$e^{i\pi} = -1$ $\left(e^{i\pi} + 1 = 0\right) \to$ Euler's Identity

$e^{\frac{3i\pi}{2}} = -i$

Periodicity - $e^{i(\theta + 2\pi)} = e^{i\theta}$

$\overline{e^{i\theta}} = e^{-i\theta}$

$\arg(\overline{e^{i\theta}}) = -\theta \left(= \arg(e^{-i\theta}) = \arg(e^{i(-\theta)})\right)$

Applications

1. Save space when writing + $e^{i\theta}$ instead of $\cos\theta + i\sin\theta$
   - Polar representation is just $\boxed{z = re^{i\theta}}$ $(r > 0)$
   - Multiplication of complex numbers becomes:

     $z = re^{i\theta}$, $w = \rho e^{ix}$

     then $zw = r\rho e^{i(\theta + x)}$

- De Moivre's theorem - $(e^{i\theta})^n = e^{in\theta}$

2. Further Trig identities

Euler $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

Add

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\quad$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

Use to derive:

$$\cos^2\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 = \frac{(e^{i\theta})^2 + 2e^{i\theta}e^{-i\theta} + (e^{-i\theta})^2}{4}$$

$$= \frac{e^{2i\theta} + 2 + e^{-2i\theta}}{4}$$

$$= \frac{1}{2}\left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right) + \frac{1}{2}$$

$$= \frac{1}{2}\cos 2\theta + \frac{1}{2}$$

$$\cos^4\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 = \frac{1}{16}\left(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right)$$

$$= \frac{1}{16}\left(2\cos 4\theta + 8\cos 2\theta + 6\right)$$

$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

Definitions

An algebraic equation for a (Complex) variable z is a polynomial equation of the form $a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$ with coefficients $a_n, a_{n-1}, \dots, a_0 \in \mathbb{C}$

If $a_n \neq 0$ then the equation is of degree n

Plan: 1 - Degree 1

2 - Degree 2, real Coefficients

3 - First degree n example : $z^n = 1$

4 - Second degree n example : $z^n = a$, $a \in \mathbb{C}$

5 - More examples

6 - A theorem

1. Degree 1

Q: $az + b = 0$, $a, b \in \mathbb{C}$, $a \neq 0$

A: $z = \frac{-b}{a}$ (one Unique Solution)

2. Degree 2 with real Coefficients

$az^2 + bz + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$

A: multiply by $4a$

$4a^2z^2 + 4abz + 4ac = 0$

delta = discriminant (not Triangle !!)

$(2az + b)^2 - b^2 + 4ac = 0$

$(2az + b)^2 = b^2 - 4ac \quad (= \Delta)$

$$2az + b = \begin{cases} \pm\sqrt{b^2 - 4ac} & (\Delta > 0) \\ 0 & (\Delta = 0) \\ \pm i\sqrt{4ac - b^2} & (\Delta < 0) \end{cases}$$

Hence $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad b^2 - 4ac > 0$

$z = \dfrac{-b \pm i\sqrt{4ac - b^2}}{2a}$ $\quad b^2 - 4ac < 0$

$z = \dfrac{-b}{2a}$ $\quad b^2 - 4ac = 0$

Over $\mathbb{C}$, quadratic equations are always solvable.

E.g. $z^2 + 2z + 2 = 0$ $\xrightarrow{\text{normal}}$ $\Delta = b^2 - 4ac = 4 - 4\times 2 = -4 < 0$

$z = \dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$

$\Downarrow$ *magic*

$(z + (i+1))(z + (-i+1)) = 0$

$z = -1-i$ or $-1+i$

3. Q: $z^n = 1$

A: Use polar representation

$z = re^{i\theta}$, $r \geq 0$

$z^n = (re^{i\theta})^n = r^n e^{in\theta} = 1e^{i0} = 1e^{2\pi k}$

Compare modulus and argument of both sides

modulus: $r^n = 1$

Argument: $n\theta = 2\pi k$, $k \in \mathbb{Z}$

Hence $r = 1$ since $r \in \mathbb{R}$, $r \geq 0$

and $\theta = \frac{2\pi k}{n}$ So solutions are $z_k = 1 \cdot \exp\left(2\pi i \frac{k}{n}\right)$ for $k \in \mathbb{Z}$

Might seem to be infinitely many solutions, but

$z_{k+n} = \exp\left(2\pi i\left(\frac{k+n}{n}\right)\right) = \exp\left(\frac{2\pi i k}{n} + 2\pi i \frac{n}{n}\right)$

$= \exp\left(2\pi i \frac{k}{n}\right)\exp(2\pi i) = \exp\left(\frac{2\pi i k}{n}\right)$

$= z_k$

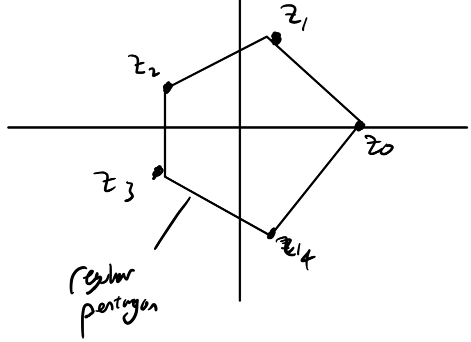
So the distinct solutions are only n in number

They are for example $z_1, z_2, \dots, z_n$

or $\underbrace{z_0, z_1, \dots, z_{n-1}}_{\text{nth roots of unity}}$ (Since $z_0 = z_n$)

E.g. $n=5$ gives $z_k = e^{\frac{2\pi i k}{5}}$, $z_{k+5} = e^{\frac{2\pi i}{5}(k+5)} = e^{\frac{2\pi i k}{5}} e^{2\pi i} = e^{\frac{2\pi i k}{5}} = z_k$

Distinct Solutions: $z_0 = 1$, $z_1 = e^{2\pi i/5}$, ...



4. Q: $z^n = a$, $a \in \mathbb{C}$, $a \neq 0$

A: Use polar representation again, here write $a = \rho e^{i\alpha}$, $z = re^{i\theta}$

So $r^n e^{in\theta} = \rho e^{i\alpha} = \rho e^{i(\alpha + 2\pi k)}$

$\Updownarrow$

$|z| = r = \sqrt[n]{\rho}$ $\quad \rho = |a| > 0$

$\arg z = \theta = \frac{\alpha}{n} + \frac{2\pi k}{n} \quad k \in \mathbb{Z}$

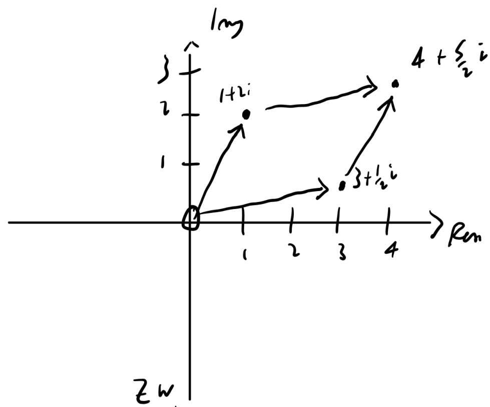
So solutions are $\sqrt[n]{\rho}\, e^{i\left(\frac{\alpha}{n} + \frac{2\pi k}{n}\right)}$

Distinct solutions are $z_0, z_1, \dots, z_{n-1}$

Addition geometrically

e.g. $(1+2i) + (3 + \frac{i}{2}) = 4 + \frac{5}{2}i$

$\mathbb{C}$ addition is geometrically vector addition



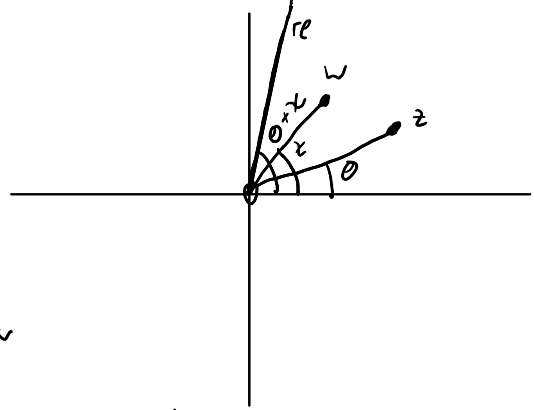
zw ...

Multiplication geometrically

Let $z = re^{i\theta}$

$w = \rho e^{i\alpha}$

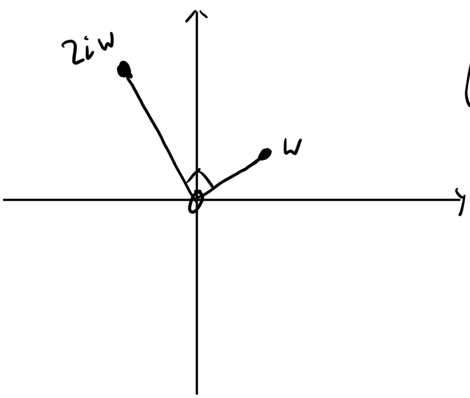
then $zw = r\rho e^{i(\theta+\alpha)}$



Multiplication by a complex number $z = re^{i\theta}$:

- rotates anticlockwise by angle $\theta = \arg(z)$
- expands from origin by a factor of $r = |z|$

eg multiply w by $2i$



(means rotate by $\pi/2$ and expand by 2)

Algebraic equations

1. $az+b=0$
2. degree 2
3. $z^n = 1 \to n$ solutions of the form $z_k = e^{\frac{2\pi i k}{n}}$, $k \in \mathbb{Z}$

   Only $z_0, \ldots, z_{n-1}$ are distinct
4. $z^n = a$

   $z^n = a$

   let $a = \rho e^{i\alpha} \to z_k = \sqrt[n]{\rho}\, e^{i\left(\frac{\alpha}{n} + \frac{2\pi k}{n}\right)}$

   $k = 0, \ldots, n$

Example Q: Solve $z^4 = 1+\sqrt{3}i$

A: Polar form

Let $z = re^{i\theta}$

$|1+\sqrt{3}i| = 2$

$\arg(1+\sqrt{3}i) = \pi/3$

So $1+\sqrt{3}i = 2e^{i\pi/3}$

So equation becomes:

$z^4 = r^4 e^{4i\theta} = 2e^{i\pi/3}$

$\Downarrow$

2 equations - mod and arg $\Rightarrow$ remember angles $+2\pi k$, $k \in \mathbb{Z}$

modulus
$r^4 = 2 \Rightarrow r = \sqrt[4]{2}$

argument
$4\theta = \pi/3 + 2\pi k \Rightarrow \theta = \frac{\pi}{12} + \frac{\pi k}{2}$, $k \in \mathbb{Z}$

So general solution $z = re^{i\theta} = \sqrt[4]{2}\, e^{i\left(\frac{\pi}{12} + \frac{\pi k}{2}\right)}$

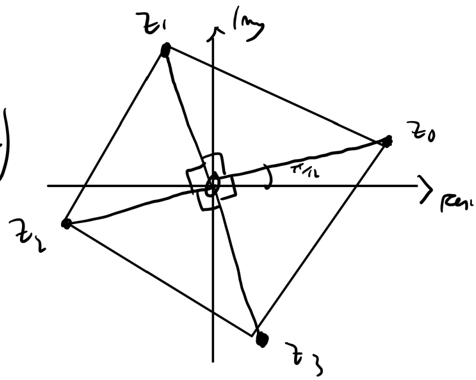
Only 4 are distinct solutions

$z: z_0 = \sqrt[4]{2}\, e^{i\frac{\pi}{12}}$

$z_1 = \sqrt[4]{2}\, e^{i\frac{7\pi}{12}} \left(= z_0 e^{i\frac{\pi}{2}}\right)$

$z_2 = \sqrt[4]{2}\, e^{i\frac{13\pi}{12}}$

$z_3 = \sqrt[4]{2}\, e^{i\frac{19\pi}{12}}$



Q: Solve $z^6 - 2z^3 + 2 = 0$

$\Downarrow$

This is a quadratic ☺

Let $w = z^3$ So $w^2 - 2w + 2 = 0$

Discriminant $b^2 - 4ac = 4 - 8 = -4 < 0$

$\therefore w = \dfrac{-b \pm i\sqrt{4ac - b^2}}{2a}$

$= \dfrac{2 \pm i\sqrt{4}}{2} = 1 \pm i = z^3$

Now need to solve $z^3 = 1+i$, and $z^3 = 1-i$

- Both are in the form $z^n = a$

Polar $\rightarrow$ $z = re^{i\theta}$

$1+i = \sqrt{2}e^{i\frac{\pi}{4}}$ $\qquad$ $1-i = \sqrt{2}e^{-i\frac{\pi}{4}}$

So $r^3 e^{3i\theta} = \sqrt{2}e^{i(\frac{\pi}{4} + 2\pi k)}$ or $r^3 e^{3i\theta} = \sqrt{2}e^{i(-\frac{\pi}{4} + 2\pi k)}$

$\rightarrow r = 2^{1/6}$

and $\theta = \frac{\pi}{12} + \frac{2\pi k}{3}$ or $-\frac{\pi}{12} + \frac{2\pi k}{3}$ for $k \in \mathbb{Z}$

With unique solutions for $k = 0, 1, 2$, so we get 6 solutions:

$2^{1/6}e^{i\pi/12}$, $2^{1/6}e^{i\frac{9\pi}{12}}$, $2^{1/6}e^{i\frac{17\pi}{12}}$, $2^{1/6}e^{-i\pi/12}$, $2^{1/6}e^{i\frac{7\pi}{12}}$, $2^{1/6}e^{i\frac{15\pi}{12}}$,

Can simplify

$2^{1/6}e^{\pm i\frac{\pi}{12}}$, $2^{1/6}e^{\pm\frac{3\pi}{4}i}$, $2^{1/6}e^{\pm\frac{7\pi}{12}i}$, Note - We can also do quadratic equations with complex coefficients

Solve by completing the square in the usual way

$$\left(2az+b\right)^2 = b^2-4ac$$

Now solve by letting $w = 2az+b \Rightarrow w^2 = b^2-4ac$ and write in polar

Polynomial Division $\Rightarrow$ for any polynomial $p(z)$, any $a \in \mathbb{C}$ we can write

$$\underbrace{p(z)}_{\text{degree } n} = (z-a)\underbrace{q(z)}_{\text{degree } (n-1)} + c$$

(degree: maximum power of z)

Example $p(z) = z^3+z^2+z+1$

$$= (z-1)(z^2+2z+3)+4$$

Consequence: if w is a root of $p(z)$ (i.e. $p(w)=0$), then $\underline{p(z) = (z-w)q(z)}$

Fundamental Theorem of Algebra

Every algebraic equation of degree $n \geq 1$ $(p(z)=0)$ has at least one solution in $\mathbb{C}$.

Consequence: If w_1 is a solution of $p(z)=0$, then $p(z) = (z-w_1)q(z)$ where $q(z)$ has degree $n-1$

w_1 always exists by F.T.A.

But now $q(z)=0$ must also have a solution, w_2, by F.T.A.

$\therefore q(z) = (z-w_2)r(z)$ where $r(z)$ has degree $n-2$

So $p(z) = (z - w_1)(z - w_2) r(z)$

But $r(z)$ also has a solution, w_3, by FTA

So $p(z) = (z - w_1)(z - w_2)(z - w_3) \dots (z - w_n) a_n$

$p(z) = a_n z^n + \dots + a_0$
So constant must be a_n to get $a_n z^n$

Corollary of FTA → every polynomial $p(z) = a_n z^n + \dots + a_0$ can be factorised completely over $\mathbb{C}$, i.e. can be written as $(z - w_1)(z - w_2)(z - w_3) \dots (z - w_n) a_n$

In this sense a degree n polynomial has exactly n roots in $\mathbb{C}$ (when counted with multiplicities, i.e. repeated roots).

Compare $\mathbb{R}$ to $\mathbb{C}$:

- $z^4 = 1$ → 2 real solutions $z = \pm 1$ but 4 complex solutions $\pm 1, \pm i$

- $az^2 + bz + c = 0$ → If $b^2 - 4ac < 0$, no real solutions, but always 2 solutions in $\mathbb{C}$

- $z^n = a$ → n roots (in $\mathbb{C}$)
- $z^6 - 2z^3 + 2 = 0$ → 6 solutions (in $\mathbb{C}$)

Corollary $\mathbb{R}$ → every real polynomial $p(x) = a_n x^n + \dots + a_0$ with $a_0, a_1, \dots, a_n \in \mathbb{R}$ can be completely factorised into linear and quadratic real factors.

e.g. $x^4 - 1$
$= (x-1)(x+1)\underbrace{(x^2+1)}_{\text{quadratic factor}}$

If complex: $z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$ ← F.T.A.

Proof: Consider the complex polynomial equation $p(z)=0$. FTA says

$$p(z) = (z-w_1)\cdots(z-w_n)a_n$$

Note if w_1 is real then no problem

If w_1 is not real, then $\overline{w_1}$ is also a root

Why?
$$p(w_1)=0 \Rightarrow a_n w_1^n + a_{n-1} w_1^{n-1} + \dots + a_0 = 0$$
$$\Rightarrow \overline{a_n w_1^n + \dots + a_0} = 0 \quad \leftarrow \text{Complex Conjugate both sides}$$
$$\Rightarrow \bar{a}_n \bar{w}_1^n + \dots + \bar{a}_0 = 0$$
$$\Rightarrow a_n \bar{w}_1^n + \dots + a_0 = 0 \quad \leftarrow \text{Comp. Conj. of real number is itself (Coefficients all real)}$$
$$\Rightarrow p(\bar{w}_1) = 0$$

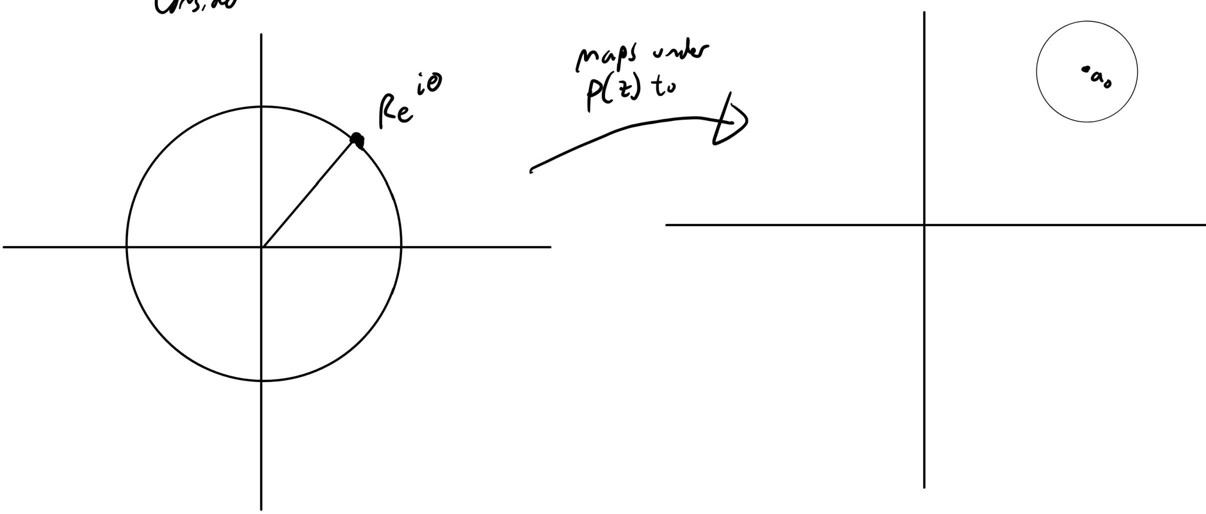
But then
$$(z-w_1)(z-\bar{w}_1) = z^2 - (w_1+\bar{w}_1)z + w_1\bar{w}_1$$
$$= z^2 - 2\,\mathrm{Re}(w_1)z + |w_1|^2$$

This is a real quadratic

Sketch proof of FTA

$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$

Consider the circle $z = Re^{i\theta}$ where R is fixed, θ varies from 0 to 2π



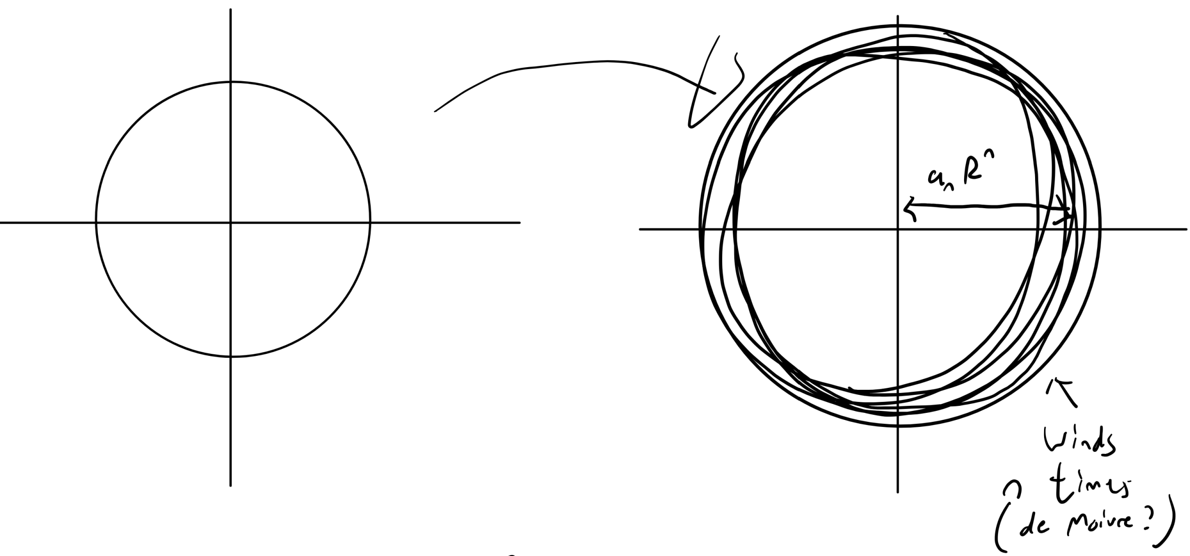
- When R is very small, the circle maps to a small closed curve near a_0 since $P(z) = P(Re^{i\theta}) = a_0 + a_1 Re^{i\theta} + a_2 R^2 e^{2i\theta}$

since R small

$\approx a_0$

- When R is very large:

$$P(z) = a_0 + a_1 Re^{i\theta} + \dots + a_n R^n e^{in\theta}$$
$$= a_n R^n e^{ni\theta}\left(1 + \frac{a_n}{a_{n-1}} \cdot \frac{1}{R} e^{i\theta} + \dots\right)$$
$$\approx a_n R^n e^{ni\theta}$$



Now considering varying R from very small to very large, the closed curve must pass through 0 somewhere. At that point, $P(z) = 0$.

Algebraic Equations:

Need to know how to solve:
- Linear
- quadratic
- $z^n = a$

know that any algebraic equation has solution (F.T.A)
(Proof not needed)

- Consequence: $P(z)$ can be completely factorised over $\mathbb{C}$
factorised into quadratic or linear factors over $\mathbb{R}$

The solutions to $P(z) = 0$ need not be able to be written down exactly, i.e. in terms of radicals $\sqrt[n]{\dots}$ - for quintic equations and beyond it is not possible to write down the exact solution.

Functions of a $\mathbb{C}$ variable

A function $f(z)$ takes a complex number z and outputs a complex number $f(z)$

e.g. $f(z) = z^2 + 3z^3$
Input z, output $z^2 + 3z^3$

New functions

Definitions: for any complex z, we define:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

As with $z = x$ and $z = i\theta$ we have that:

$$e^0 = 1, \quad e^z e^w = e^{z+w}$$

proof same as with reals

So for example, if $z = x + iy$:

$$e^z = e^{x+iy} = e^x e^{iy}$$

So $e^z = e^x(\cos y + i\sin y)$

Example $e^{1+i\frac{\pi}{4}} = e^1 e^{i\frac{\pi}{4}} = e\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}(1+i)$

Q: What are $Re(e^z)$, $Im(e^z)$, $|e^z|$, $\arg(e^z)$?

$$Re(e^z) = e^x \cos y = e^{Re(z)} \cos(\ (z))$$

$$Im(e^z) = e^x \sin y = e^{Re(z)} \sin(Im(z))$$

$$|e^z| = e^x$$

$$\arg(e^z) = y = Im(z) \quad \text{(only defined mod } 2\pi)$$

$$\overline{e^z} = \overline{e^x(\cos y + i\sin y)} = e^x(\cos y - i\sin y) = e^x e^{-iy} = e^{x-iy} = e^{\bar{z}}$$

Definition: recall $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Same thing !!

We define for any complex z:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

Q. In terms of $Re(z)$ and $Im(z)$ what are:

- $Re(\sin z)$?
- $|\cosh z|$?

Let $z = x + iy$ then $\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$

multiply by $-i$

$$= \frac{-i}{2}\left(e^{ix}e^{-y} - e^{-ix}e^{y}\right)$$

$$= \frac{-i}{2}\left(\left(\cos x + i\sin x\right)e^{-y} - \left(\cos x - i \sin x\right)e^{y}\right)$$

$$\sin z = \frac{-i}{2}\cos x\, e^{-y} + \frac{1}{2}\sin x\, e^{-y} + \frac{i}{2}\cos x\, e^{y} + \frac{1}{2}\sin x\, e^{y}$$

$$\text{So } \operatorname{Re}(\sin z) = \sin x\left(\frac{e^{y}+e^{-y}}{2}\right)$$

$$= \sin x \cosh y$$

$$\text{So } \operatorname{Re}(\sin z) = \sin(\operatorname{Re}(z))\cosh(\operatorname{Im}(z))$$

$$|\cosh z|^2 = \cosh z\,\overline{\cosh z} = \frac{e^{z}+e^{-z}}{2}\cdot\overline{\frac{e^{z}+e^{-z}}{2}}$$

$$= \frac{1}{4}\left(e^{z}+e^{-z}\right)\left(e^{\bar z} - e^{-\bar z}\right)$$

$$= \frac{1}{4}\left(e^{z+\bar z} + e^{-z+\bar z} + e^{z-\bar z} + e^{-(z+\bar z)}\right)$$

$$= \frac{1}{2}\left(\frac{e^{z+\bar z}+e^{-(z+\bar z)}}{2} + \frac{e^{z-\bar z}+e^{-(z-\bar z)}}{2}\right)$$

$z+\bar z = 2\operatorname{Re}(z)$

$z-\bar z = 2i\operatorname{Im}(z)$

$$= \frac{1}{2}\left(\frac{e^{2\operatorname{Re}(z)}+e^{-2\operatorname{Re}(z)}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\operatorname{Im}(z)}+e^{-2i\operatorname{Im}(z)}}{2}\right) \qquad z$$

$$= \frac{1}{2}\cosh(2\operatorname{Re}(z)) + \frac{1}{2}\cos(2\operatorname{Im}(z)) = |\cosh z|^2$$

$$\therefore |\cosh z| = \sqrt{\frac{1}{2}\cosh(2\operatorname{Re}(z)) + \frac{1}{2}\cos(2\operatorname{Im}(z))}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cosh z = \frac{e^{z}+e^{-z}}{2} \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

Notice

$$\cosh(iz) = \cos z$$
$$\cos(iz) = \cosh z$$
$$\sinh(iz) = \frac{e^{iz}-e^{-iz}}{2} = i\sin z$$
$$\sin(iz) = \frac{e^{i^2 z}-e^{-i^2 z}}{2i} = \frac{e^{-z}-e^{z}}{2i} = \frac{-\sinh z}{i} = i\sinh z$$

by one line computations, one can easily prove:

- $\sin^2 z + \cos^2 z = 1$ for any $z \in \mathbb{C}$

Proof:

$$\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \quad (2i)^2 = -4 \text{ so minus}$$

$$= \frac{1}{4}\left(e^{2iz}+2+e^{-2iz} - \left(e^{2iz}-2+e^{-2iz}\right)\right)$$

$$= 1$$

- $\cosh^2 z - \sinh^2 z = 1$

proof similar to above.

- $\sin(z+w) = \sin z\cos w + \cos z \sin w$ for all $z, w \in \mathbb{C}$

Proof: RHS $= \sin z \cos w + \cos z \sin w$

$$= \frac{1}{4i}\left(e^{iz}-e^{-iz}\right)\left(e^{iw}+e^{-iw}\right) + \frac{1}{4i}\left(e^{iw}-e^{-iw}\right)\left(e^{iz}+e^{-iz}\right)$$

$$= \frac{1}{4i}\left(e^{i(z+w)} - e^{-i(z-w)} + e^{i(z-w)} - e^{-i(z+w)} + e^{i(z+w)} + e^{-i(z-w)} - e^{i(z-w)} - e^{-i(z+w)}\right)$$

$$= \frac{1}{4i}\left(2e^{i(z+w)} - 2e^{-i(z+w)}\right)$$

$$= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z+w)$$

(Similar proofs for $\cos(z+w)$, $\sinh(z+w)$, etc) all the same as in real case

Periodicity - for any $n \in \mathbb{Z}$:

- $\sin(z+2\pi n) = \sin z$
- $\cos(z+2\pi n) = \cos z$
- $\sinh(z+2\pi i n) = \sinh z$
- $\cosh(z+2\pi i n) = \cosh z$

$\Big\}$ Periodic in imaginary direction

All follow from the fact that $e^{i(z+2\pi n)} = e^{iz}\left(e^{2\pi}\right)^n = e^{iz}$

or $e^{z+2\pi i n} = e^{z}$

Q: In terms of $Re(z)$ and $Im(z)$, what is $Re(\sin z)$

A: Use angle addition formula

Let $z = x+iy$

$$\sin z = \sin(x+iy)$$
$$= \sin x \cos(iy) + \cos x \sin(iy)$$
$$= \sin x \cosh y + \cos x \cdot i \sinh y$$

So $Re(\sin z) = \sin(Re(z)) \cosh(im(z))$

Transcendental Equations $\Rightarrow$ not polynomial

Q: Solve $e^z = -1$

(method similar to $z^n = a$ $\Rightarrow$ use polar representation and equate mod and arg of both sides, but $+2\pi k$ to arg, $k \in \mathbb{Z}$) ← ignore this (?)

But here Cartesian z $\Rightarrow$ polar rep e^z

Let $z = x + iy$ then $e^z = e^x e^{iy}$

Cartesian (for $z = x + iy$); polar (for $e^x e^{iy}$)

-1 in polar rep $= e^{i\pi}$

So $e^z = -1$

becomes $e^x e^{iy} = e^{i\pi}$

modulus $\Rightarrow$ $e^x = 1$

$x = 0$

arg $\quad y = \pi + 2\pi k$

So $z = \pi i(1+2k)$ for $k \in \mathbb{Z}$

↑ Infinite number of solutions

Q: Solve $e^z = 0$

A: Set $z = x + iy$ then $e^x e^{iy} = 0$

modulus $e^x = 0$ $\Rightarrow$ no solutions

So $e^z = 0$ has no solutions.

The number of solutions to a transcendental equations can not be prior determined (unlike algebraic equations)

Note - you will get questions involving $e^z = a$ but they may be disguised.

Q: Solve $e^{2z} + 2e^z - 3 = 0$

A: Quadratic in e^z

$$(e^z + 3)(e^z - 1) = 0$$

$\therefore e^z = -3$, $e^z = 1$

Let $z = x + iy$, $-3 = 3e^{i\pi}$

$$e^x e^{iy} = 3e^{i(\pi + 2\pi k)}$$

$e^x = 3$ $\quad y = \pi + 2\pi k$

$x = \ln 3$

or $\quad e^x e^{iy} = e^0$

$e^x = 1$ $\quad y = 2\pi k$

$x = 0$

So $z = x + iy$
$= \ln 3 + i\pi(1 + 2k)$ $\quad$ or $z = 2\pi k i$, $\quad k \in \mathbb{Z}$

Q: Solve $\cos z = \sin z$ for complex z

A: $$\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2i}$$

$$i(e^{iz} + e^{-iz}) = e^{iz} - e^{-iz}$$

$$(i-1)e^{iz} = -(i+1)e^{-iz}$$

$$e^{2iz} = \frac{1+i}{1-i}$$

$\frac{1+i}{1-i}$ in polars?

$1+i = \sqrt{2}\,e^{i\pi/4}$ So $\frac{1+i}{1-i} = e^{i\pi/2}$

$1-i = \sqrt{2}\,e^{-i\pi/4}$

Need to solve $e^{2iz} = e^{-2y}e^{2ix} = e^{i\pi/2}$

using $z = x+iy$

$\therefore$ modulus $e^{-2y} = 1 \Rightarrow y = 0$

argument $2x = \frac{\pi}{2} + 2\pi k$

$\therefore x = \frac{\pi}{4} + \pi k$

So $z = x+iy = \frac{\pi}{4} + \pi k$