Signals And Systems by Alan V. Oppenheim: Notes

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Contents

Chapter 1

Introduction

1.1 Signal Energy and Power

Motivation and definition
In many but not all, applications, the signals considered directly related to physical quantities capturing power and energy in a physical system. (for instance v^2/R for the power across a resistor)

As such it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal, denoted $x(t)$, or any discrete-time signal $x[n]$. In this case, the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x; see that the time-averaged signal can be obtained by dividing by $(t_2 - t_1)$. Similarly for a discrete signal $x[n]$ over the interval $n_1 \leq n \leq n_2$ the total energy is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

with the average power calculated by dividing by $(n_2 - n_1 + 1)$.

It is important to remember that the terms 'power' and 'energy' are used here *independently* of their relation to physical energy (they clearly don't correlate since their units or scalings would differ). Nevertheless we will find it convenient to use these terms in a general fashion.

Power and energy over infinite intervals

Considering signals over an infinite time interval, meaning for $-\infty < t < +\infty$ or $-\infty < n < +\infty$. Here we define the total energy as the limits of the aforementioned equations increase without bound; in continuous time,

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Note that these expressions may not converge; for instance say $x(t)$ or $x[n]$ equal some nonzero constant for all time: such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.
(next page)

Cont.
Analagously, we can define the time-averaged power over an infinite interval as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

and

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

In continuous and discrete time respectively.

Intuition
See that with these definitions, we can identify three classes of signals: first those with finite total energy, meaning $E_\infty < \infty$. See that such a signal would have zero average power:

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

Second would be signals with finite average power P_∞; see from the above expression that for $P_\infty > 0$, this requires that $E_\infty = \infty$.

Last would be signals for which neither P_∞ nor E_∞ are finite. An example of this might be $x(t) = t$.

Note on discrete signals
It is important to note that the discrete-time signal $x[n]$ is defined *only* for *integer* values of the independent variable.

1.2 Even and Odd signals

Definition
A continuous-time signal is *even* if

$$x(-t) = x(t)$$

while a discrete-time signal is *even* if

$$x[-n] = x[n]$$

These signals are referred to as *odd* if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

Note that an odd signal must be 0 at $t = 0$ or $n = 0$ since the equations require that $x(0) = -x(0)$ and $x[0] = -x[0]$.

Decomposition
An important fact is that any signal can be broken into a sum of two signals, where one is even and the other odd. To see this, consider

$$\mathrm{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\mathrm{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

See that $x(t)$ is the sum of the two. Exactly analagous definitions hold in the discrete time case.

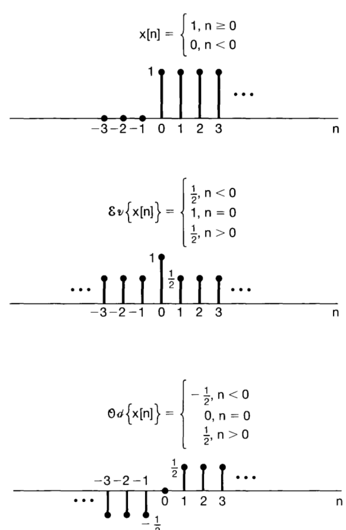

Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

1.3 Differences between continuous and discrete periodic complex exponentials

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers. An important class of complex exponentials is obtained by constraining a to be purely imaginary:

$$x(t) = e^{i\omega t}$$

Periodicity and harmonic relations (purely imaginary power)
An important property of this signal is that it is periodic; recall that $x(t)$ will be periodic with period T if

$$e^{i\omega t} = e^{i\omega(t+T)}$$

this means

$$e^{i\omega(t+T)} = e^{i\omega t}e^{i\omega T} \implies e^{i\omega T} = 1$$

If $\omega = 0$ then this is satisfied for any T. If $\omega \neq 0$, see that the *fundamental period* T_0 of $x(t)$—that is, the smallest positive value of T for which this holds—is

$$T_0 = \frac{2\pi}{|\omega|}$$

(the signals $e^{i\omega 0t}$ and $e^{-i\omega t}$ have the same fundamental period) Naturally, there is a set of exponentials periodic to a common period T_0. These are said to be *harmonically related* complex exponentials; the necessary condition they satisfy is

$$e^{i\omega T_0} = 1$$

which implies that

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

(next page)

Cont.
We had

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

if we define

$$\omega_0 = \frac{2\pi}{T_0}$$

this means that the harmonic frequencies ω must be integer multiples of ω_0:

$$\phi_k(t) = e^{ik\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Each $\phi_k(t)$ itself defines a fundamental frequency and a corresponding fundamental period. (see that $|k|\omega_0 \cdot T_0/|k| = 2\pi$, so this scaled down period is the corresponding period for this scaled up frequency. Each frequency is unique, point here is that they are also periodic with T_0, but with fundamental periods getting proportionally smaller.)

Note that the kth harmonic $\phi_k(t)$ is still periodic with T_0; it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0. (the term 'harmonic' is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency)
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Discrete case
As in continuous time, an important signal in discrete time is the *complex exponential signal*, defined as

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. See that this could also be expressed as

$$x[n] = Ce^{\beta n}$$

where $\alpha = e^{\beta}$. See that we can constrain β to be purely imaginary:

$$x[n] = e^{i\omega_0 n}$$

Periodicity properties of Discrete-time complex exponentials
While there are many similarities between continuous and discrete-time signals, there are a number of important differences. For the continuous time signal $e^{i\omega_0 t}$, we know that

- The larger the magnitude of ω_0, the higher the rate of oscillation of the signal
- $e^{i\omega_0 t}$ is periodic for any value of ω_0

These properties are different in the discrete-time case.

Given the first property, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{i(\omega_0+2\pi)n} = e^{i2\pi n}e^{i\omega_0 n} = e^{i\omega_0 n}$$

(see that this is a direct result of the fact that we iterate through discrete time as integers) The exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. This is unlike the continuous-time case where each distinct ω_0 represents a distinct signal.

In discrete time, the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$, and so on. Therefore when considering discrete time complex exponentials, see that we need only consider a frequency interval of length 2π in which to choose ω_0, such as $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$.

Also see that because of this the discrete exponential $e^{i\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 increases in magnitude; the signals will oscillate faster until we reach $\omega_0 = \pi$, after which the rate of oscillation decreases until we reach $\omega_0 = 2\pi$, at which the same constant sequence as $\omega_0 = 0$ is produced.
(next page)

Cont.
The second property we wish to consider concerns the periodicity of the discrete time complex exponential. In order for the signal $e^{i\omega_0 n}$ to be periodic with period $N > 0$ we must have

$$e^{i\omega_0(n+N)} = e^{i\omega_0 n}$$

or equivalently

$$e^{i\omega_0 N} = 1$$

For this to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m$$

or equivalently

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

The signal $e^{i\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise.

Fundamental period
Recall the idea of a *fundamental period*; in this case it would mean the smallest N such that $\omega_0 N = 2\pi m$ holds (this is unlike the continuous case where there is always some T where $\omega_0 T = 2\pi$); see that this occurs when m and N do not have any factors in common.

See that from this we can derive a *fundamental frequency* as

$$\frac{2\pi}{N} = \frac{\omega_0}{m}$$

(see that this frequency is always equal or lower—intuitively, to have a different wave that completes one oscillation in N time, its frequency will either be equal or lower)

To summarize

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

1.4 A note on Discrete-time complex exponential frequencies

See that the discrete exponential $e^{i\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 increases in magnitude; the signals will oscillate faster until we reach $\omega_0 = \pi$, after which the rate of oscillation decreases until we reach $\omega_0 = 2\pi$, at which the same constant sequence as $\omega_0 = 0$ is produced.

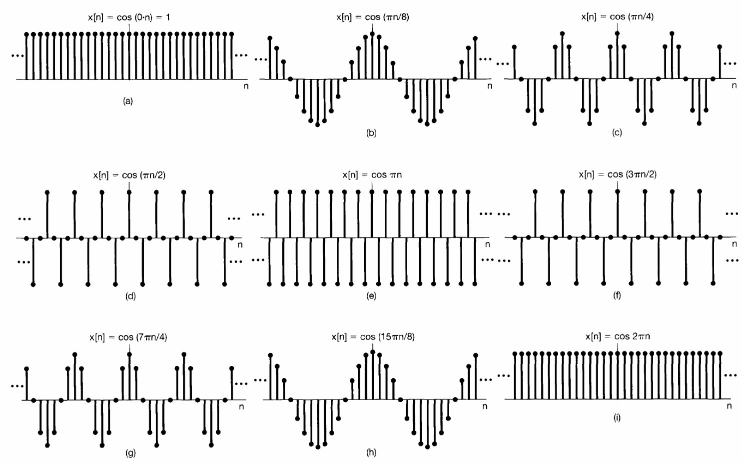


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

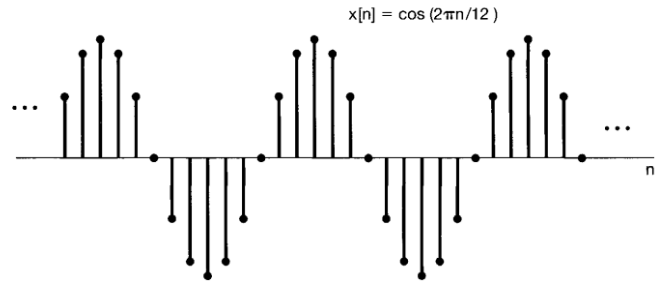
(See that integer multiples of $\omega_0 = \pi$ are the 'fastest' to reach some integer multple of $2/\pi$, while something else like $\omega_0 = 15\pi/8$ would take much longer. When plotted in continuous time, the higher ω_0 would correspond to a higher frequency, however the continuous graph only completes an oscillation at an *integer value of* x at $x = 16$.)

Even though ω_0 is higher, the fact that its integer multiples can only match a higher integer multiple of 2π means that its fundamental frequency is low (since N is higher).

As such, low frequencies for discrete time complex exponentials occur near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ and high frequencies near $\omega = \pm\pi, \pm 3\pi, \ldots$.

1.5 Intuition for discrete-time periodicity

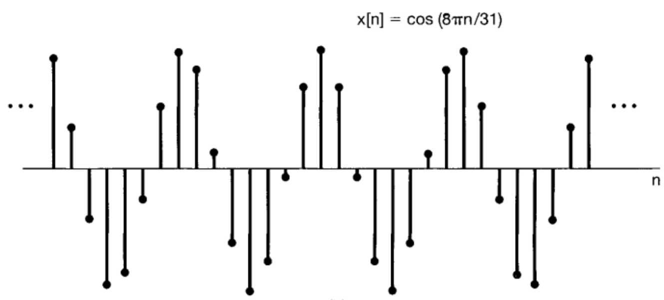
Consider the sequence $x[n] = \cos(2\pi n/12)$:



(a)

we can think of this as a set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, see that both $x(t)$ and $x[n]$ are periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

Now consider the signal $x[n] = \cos(8\pi n/31)$:



(b)

This can also be viewed as a set of samples of $x(t) = \cos(8\pi t/31)$ at integer points in time. But now see that in this case $x(t)$ is periodic with fundamental period $31/4$, while $x[n]$ is periodic with fundamental period 31.

This difference stems from the fact that the discrete-time signal is defined only for integer values of the independent variable—there is no sample at time $t = 31/4$, when $x(t)$ completes one period, or at $t = 2 \cdot 31/4$ or $t = 3 \cdot 31/4$, when $x(t)$ has completed two or three periods. Only at sample $t = 4 \cdot 31/4 = 31$, when $x(t)$ has completed *four* periods is the discrete sequence defined.

This manifests as the pattern of $x[n]$ not repeating with each cycle of positive and negative values, but rather only after four of such cycles, specifically 31 points.
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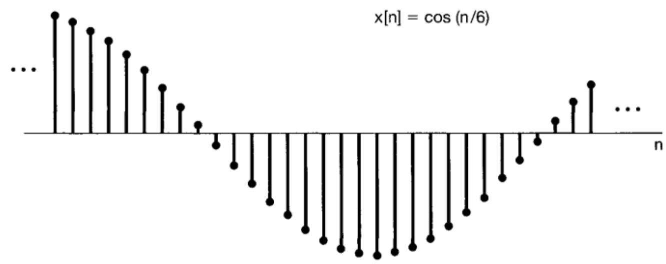
Cont.
Finally consider the signal $x[n] = \cos(n/6)$:



(c)

In this case, the values of $x(t)$ at integer sample points *never repeat*, as these sample points never span an interval that is an exact multiple of the period, 12π, of $x(t)$.

Thus, $x[n]$ *is not periodic*, although the eye visually interpolates between the sample points, suggesting *the envelope* $x(t)$ which is periodic.

1.6 Difference in harmonic relations in discrete and continuous periodic exponentials

As in continuous time, it is also of considerable value in discrete-time to consider sets of harmonically related periodic exponentials—that is, *periodic exponentials with a common period* N.

We know that these are precisely the signals which are at frequencies which are multiples of $2\pi/N$; that is

$$\phi_k[n] = e^{ik(2\pi/N)n}, \quad k = 0, \pm 1, \ldots$$

In the continuous-time case, all the harmonically related complex exponentials $e^{ik(2\pi/T_0)t}$, $k = 0, \pm 1, \pm 2, \ldots$ are distinct. However, recall that for discrete signals we have

$$e^{i(\omega_0 + 2\pi)n} = e^{i2\pi n} e^{i\omega_0 n} = e^{i\omega_0 n}$$

(this is a direct result of the fact that we iterate through discrete time as integers) As such the harmonically related complex exponentials *are not all unique in discrete time*; specifically,

$$\begin{aligned} \phi_{k+N}[n] &= e^{i(k+N)(2\pi/N)n} \\ &= e^{ik(2\pi/N)n} e^{i2\pi n} = \phi_k[n] \end{aligned}$$

See that this implies that there are only N distinct periodic exponentials in the set of $\phi_k[n]$; meaning

$$\phi_0[n] = 1,\ \phi_1[n] = e^{i(2\pi/N)n},\ \phi_2[n] = e^{i2(2\pi/N)n}, \ldots,\ \phi_{N-1}[n] = e^{i(N-1)(2\pi/N)n}$$

are all distinct, but any other $\phi_k[n]$ would just be identical to one of them. (for instance $\phi_N[n] = \phi_0[n]$ or $\phi_{-1}[n] = \phi_{N-1}[n]$.)

1.7 More on complex exponential and sinusoidal signals

Continuous case
A continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers.

Euler identity and 'combined' sinusoidal form
Recall euler's identity:

$$e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t)$$

See that the scaled and phase-delayed sinusoid can be written in terms of these periodic complex exponentials with the same fundamental period:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{i\phi}e^{i\omega_0 t} + \frac{A}{2}e^{-i\phi}e^{-i\omega_0 t}$$

We can also express

$$A\cos(\omega_0 t + \phi) = A\mathrm{Re}\{e^{i(\omega_0 t+\phi)}\}$$

and

$$A\sin(\omega_0 t + \phi) = A\mathrm{Im}\{e^{i(\omega_0 t+\phi)}\}$$

Energy and power
Periodic signals—and in particular, the complex periodic exponential signal—are examples of signals with infinite total energy but finite average power. Calculating the total energy and of the periodic exponential signal over one period:

$$\begin{aligned} E_{\text{period}} &= \int_0^{T_0} |e^{i\omega_0 t}|^2 dt \\ &= \int_0^{T_0} 1\, dt = T_0 \end{aligned}$$

(The absolute value of a complex number is its magnitude. Think of the absolute value as the (possibly multidimensional) distance from zero.) Calculating the average power:

$$P_{\text{period}} = \frac{1}{T_0}E_{\text{period}} = 1$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. However, since the average power over each period is 1, averaging over multiple periods always yields an average power of 1. That is, the complex periodic exponential signal has finite average power equal to

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |e^{i\omega_0 t}|^2 dt = 1$$

(next page)

General continuous complex exponential signals
In the most general case Ce^{at} where both C and a are complex, see that since C and a can are just

$$C = |C|e^{i\theta}, \quad a = r + i\omega_0$$

we can express the general complex signal as

$$Ce^{at} = |C|e^{i\theta}e^{(r+i\omega_0)t} = |C|e^{rt}e^{i(\omega_0 t+\theta)}$$

we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + i|C|e^{rt}\sin(\omega_0 t + \theta)$$

Discrete case
As in continuous time, the discrete time *complex exponential signal* is defined by

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}$$

where $\alpha = e^{\beta}$.

Euler identity and 'combined' sinusoidal form
As with the continuous case, constraining β to be purely imaginary, we have euler's identity

$$e^{i\omega_0 n} = \cos\omega_0 n + i\sin\omega_0 n$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{i\phi}e^{i\omega_0 n} + \frac{A}{2}e^{-i\phi}e^{-i\omega_0 n}$$

General discrete complex exponential signals
As with the continuous case, for complex C and α, we have

$$C = |C|e^{i\theta}, \quad \alpha = |\alpha|e^{i\omega_0}$$

so the general complex exponential signal can be expressed as

$$C\alpha^n = |C||\alpha|^n\cos(\omega_0 n + \theta) + i|C||\alpha|^n\sin(\omega_0 n + \theta)$$

1.8 Unit impulse and Unit step functions

Discrete-Time

One of the simplest discrete-time signals is the *unit impulse/unit sample*, defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

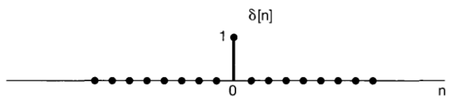


Figure 1.28 Discrete-time unit impulse (sample).

Another basic discrete-time signal is the discrete-time *unit step*, denoted by $u[n]$ and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

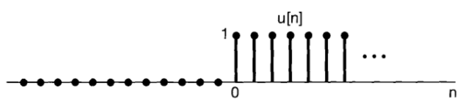


Figure 1.29 Discrete-time unit step sequence.

See that the discrete-time unit impulse is the *first difference* of the discrete-time step:

$$\delta[n] = u[n] - u[n-1]$$

Conversely, the discrete-time unit step is the *running sum* of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

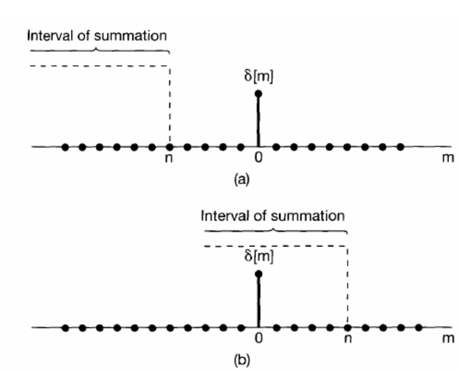


Figure 1.30 Running sum of eq. (1.66): (a) $n < 0$; (b) $n > 0$.

Since the only nonzero value of the unit sample is at 0, the running sum is 0 for $n < 0$ and 1 for $n \geq 0$.
(next page)

Alternative form
we had the discrete-time unit step as the running sum of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

See that by changing the variable of summation from m to $k = n - m$, we can rewrite this as

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

and equivalently

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

An interpretation of this is a superposition of delayed impulses; we can view the unit step as the sum of unit impulses $\delta[n]$ (nonzero at $n = 0$), $\delta[n-1]$ (nonzero at $n = 1$), and all other $\delta[n-k]$ for integer k extending to infinity.

Sampling property
See that the unit impulse can also be used to sample the value of a signal at $n = 0$; since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n]$$

More generally, if we consider a unit impulse $\delta[n-n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

(next page)

Continuous-Time

The continuous-time *unit step function* $u(t)$ is defined in a similar manner to its discrete-time counterpart:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

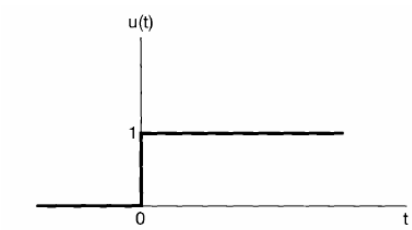


Figure 1.32 Continuous-time unit step function.

Note that the unit step is *discontinuous* at $t = 0$. The continuous-time *unit inpulse* $\delta(t)$ is related to the unit step in a manner analagous to that of their discrete counterparts; in particular, the continuous-time unit step is the *running integral* of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

It also follows that the continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step:

$$\delta(t) = \frac{du(t)}{dt}$$

In contrast to discrete-time, there is some formal difficulty with this equation as a representation of the unit impulse—since $u(t)$ is discontinuous at $t = 0$ and consequently is not formally differentiable.
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Cont.
We get around this by considering an approximation to the unit step $u_\Delta(t)$, rising from the value 0 to 1 in a short time interval of length Δ:

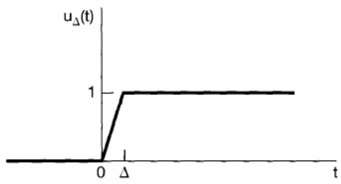


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

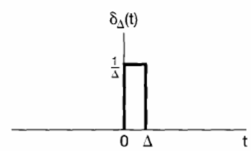


Figure 1.34 Derivative of $u_\Delta(t)$.

Also considering the derivative:

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

The unit step changes from value 0 to 1 instantaneously and can be thought of an idealisation of $u_\Delta(t)$ for Δ so short that its duration doesn't matter for any practical purpose. Formally, $u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \to 0$.

Consider the derivative again; $\delta_\Delta(t)$ is a short pulse, of duration Δ and unit area for any value of Δ. As $\Delta \to 0$, $\delta_\Delta(t)$ becomes narrower and higher while mantaining its unit area. Its limiting form

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

can then be thought of as an idealisation of the short pulse $\delta_\Delta(t)$ as the duration Δ becomes insignificant:

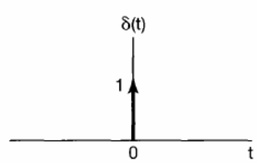


Figure 1.35 Continuous-time unit impulse.

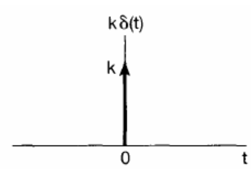


Figure 1.36 Scaled impulse.

$\delta(t)$ has, in effect, no duration but unit area. The arrow at $t = 0$ indicates the area of the pulse is concentrated at $t = 0$ and the height of the arrow and the '1' next to the arrow is used to represent the *area* of the impulse. More generally, a scaled impulse $k\delta(t)$ will have an area k, and thus

$$ku(t) = \int_{-\infty}^{t} k\delta(\tau)d\tau$$

A scaled impulse, where the height of the arrow is chosen to be proportional to the area of the impulse.
(next page)

Alternative form
As with discrete time, the relationship between the continuous time unit step and impulse can be rewritten in a different form, by changing the variable of integration from τ to $\sigma = t - \tau$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$

or equivalently

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$$

(This negation after inverstion of the limits of the integral can be derived from the first fundamental theorem. For a more intuitive understanding, consult the definition of the Riemann sum:

$$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$$

Considering the $d\tau$, or respectively $d\sigma$, as the width of the step between two argumentsof the sum, if we change the direction in which we integrate, the step also changes its sign. In the discrete sum, the summands are not multiplied by this step.)

Sampling property
As with discrete-time, the continuous-time impulse has a very important sampling property; consider

$$x_1(t) = x(t)\delta_\Delta(t)$$

By construction, $x_1(t)$ is zero outside the interval $0 \leq t \leq \Delta$. See that for Δ sufficiently small so that $x(t)$ is approximately constant over this integral:

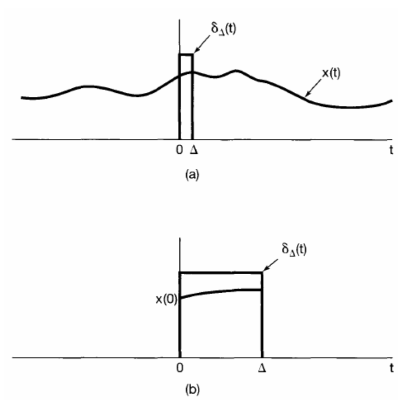


Figure 1.39 The product $x(t)\delta_\Delta(t)$: (a) graphs of both functions; (b) enlarged view of the nonzero portion of their product.

so we have

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

(next page)

Cont.
We had, for small Δ,

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

since $\delta(t)$ is the limit as $\Delta \to 0$ of $\delta_\Delta(t)$, it follows that

$$x(t)\delta(t) = x(0)\delta(t)$$

By the same argument, we have an analagous expression for an impulse concentrated at an arbitrary point, say t_0. That is,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

1.9 Basic system properties 1

General form
A general formula for a continuous first-order linear differential equation is

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

where $x(t)$ is the input, $y(t)$ the output, and a, b constants. An example of this might be

$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

Discrete cases have general first-order linear difference equations of the form

$$y[n] + ay[n-1] = bx[n]$$

Considering the earlier example, if we let $v[n] = v(n\Delta)$ and $f[n] = f(n\Delta)$ and approximate $dv(t)/dt$ at $t = n\Delta$ by the first backward difference:

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}$$

we can obtain

$$\begin{aligned}
\frac{v[n] - v[n-1]}{\Delta} + \frac{\rho}{m}v[n] &= \frac{1}{m}f[n] \\
v[n] - v[n-1] + \frac{\rho\Delta}{m}v[n] &= \frac{\Delta}{m}f[n] \\
v[n]\left(1 + \frac{\rho\Delta}{m}\right) - v[n-1] &= \frac{\Delta}{m}f[n] \\
v[n]\frac{m+\rho\Delta}{m} - v[n-1] &= \frac{\Delta}{m}f[n] \\
v[n] - \frac{m}{m+\rho\Delta}v[n-1] &= \frac{\Delta}{m+\rho\Delta}f[n]
\end{aligned}$$

which is in the general form as described above.
(next page)

Basic system properties

Memory
A system is said to be *memoryless* if its output for each value of the independent variable is dependent only on the *input at that same time*.

For instance, the system

$$y[n] = (2x[n] - x^2[n])^2$$

is memoryless, since the value of $y[n]$ at any particular time n_0 depends only on the value of $x[n]$ at that time. Other examples of memoryless systems include the input-output relationship of a resistor, with input $x(t)$ taken to be current and output $y(t)$ voltage:

$$y(t) = Rx(t)$$

where R is the resistance. The *identity system*, whose output is identical to its input, is also memoryless

$$y(t) = x(t)$$

Written in discrete time:

$$y[n] = x[n]$$

An example of a discrete-time system *with memory* is an *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

see that the accumulator must store or 'remember' information about the past inputs up to current time. Another example of a system with memory is a *delay*

$$y[n] = x[n-1]$$

While the concept of memory in a system would typically suggest storing *past* input and output values, our formal definition also leads to our referring to a system as having memory if the current output is dependent on *future* values of the input and output (noncausal systems).
(next page)

Causality
A system is *causal* if the output at any time depends only on values of the input at the present time and in the past. Such a system is often referred to as being *nonanticipative*, as the system output does not 'anticipate' future values of the input.

Consequently, if two inputs to a causal system are identical up to some point in time t_0 or n_0, the corresponding outputs must also be equal up to this same time. For instance the capacitor:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

is a causal system, with input taken to be current, output voltage, and C capacitance. The capacitor voltage responds only to the present and past values of the input.

Examples of *noncausal* systems might be

$$y[n] = x[n] - x[n+1]$$

or

$$y(t) = x(t+1)$$

See that all memoryless systems are causal, since the output responds only to the current value of the input in such systems.

An example of a noncausal system might be when averaging data over an interval to smooth out fluctuations and keep only the trend. Such an averaging system might look like

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k]$$

More examples
Consider

$$y[n] = x[-n]$$

see that for $n < 0$ the output depends on a future value of the input, and hence the system is not causal. Another example: consider

$$y(t) = x(t)\cos(t+1)$$

This can be rewritten as

$$y(t) = x(t)g(t)$$

Thus only the current value of the input $x(t)$ influences the current value of the output $y(t)$. This system is causal (and, in fact, memoryless).
(next page)

Invertibility and inverse systems
A system is said to be *invertible* if distinct inputs lead to distinct outputs. If a system is invertible, then an *inverse* system exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ in the first system:

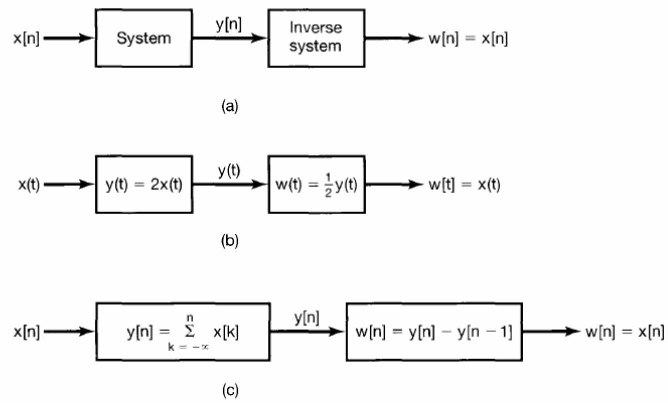


Figure 1.45 Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

See that the series interconnection between the invertible system and its inverse system has an overall input-output relationship same as that for the identity system. As illustrated in the second graph, an example of an invertible continuous-time system is

$$y(t) = 2x(t)$$

for which the inverse system is

$$w(t) = \frac{1}{2}y(t)$$

Another example is illustrated in the third block diagram, see that the accumulator is an invertible system:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

with inverse system

$$w[n] = y[n] - y[n-1]$$

Examples of noninvertible systems include

$$y[n] = 0$$

or

$$y(t) = x^2(t)$$

where a single output can correspond to different inputs.
(next page)

Stability
Another important system property is *stability.* Informally, a stable system is one in which small inputs lead to responses that do not diverge (grow uncontrollably).

More formally, if the input to a stable system is bounded (if its magnitude of the input does not grow without bound), then the output must also be bounded and therefore cannot diverge. (this is the definition to take note of)

For instance consider the averaging system brought up earlier

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k]$$

Suppose that the input $x[n]$ is bounded in magnitude by some number, say B for all values of n. Then the largest possible magnitude for $y[n]$ is also B since it is the average. Therefore, $y[n]$ is bounded and the system is stable.

On the other hand, the accumulator sums all of the past values of the input, so the output will grow continually even if $x[n]$ is bounded—it is an unstable system.

A useful strategy to verify that a system is unstable is to look for a specific bounded input that leads to an unbounded output. For instance consider

$$y(t) = tx(t)$$

See that a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded. Since for any finite constant bound, $|y(t)|$ will exceed that bound at some t, this system is unstable.

Now consider a different system

$$y(t) = e^{x(t)}$$

See that for an arbitrary positive number B, if $x(t)$ is bound by B, that is

$$|x(t)| < B$$

or

$$-B < x(t) < B$$

for all t, then $y(t)$ must satisfy

$$e^{-B} < |y(t)| < e^{B}$$

Any input to this system bounded by an arbitrary positive number B has a output guaranteed to be bounded by e^B—this system is stable.

1.10 Basic system properties 2

Time invariance
A system it *time invariant* if a time shift in the input signal results in an identical time shift in the output signal.

That is, if $y[n]$ is the output of a discrete-time, time-invariant system with input $x[n]$, then $y[n-n_0]$ is the output when $x[n-n_0]$ is applied. In continuous time with $y(t)$ the output corresponding to the input $x(t)$, a time-invariant system will have $y(t-t_0)$ as the output when $x(t-t_0)$ is the input.

Examples
Consider the continuous-time system

$$y(t) = \sin[x(t)]$$

To check that this system is time invariant, we must determine whether the time-invariance property holds for *any* input and *any* time shift t_0. Thus, letting $x_1(t)$ be an arbitrary input to the system, with corresponding output

$$y_1(t) = \sin[x_1(t)]$$

Now consider a second input x_2 obtained by shifting $x_1(t)$ in time

$$x_2(t) = x_1(t-t_0)$$

The output corresponding to this input is

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t-t_0)]$$

Now consider translating the output y_1, see that

$$y_1(t-t_0) = \sin[x_1(t-t_0)]$$

Since y_2, the output of the shifted input, is the same as $y_1(t-t_0)$, which is if we had shifted the output instead, this system is time invariant.
(next page)

More examples
As a second example, consider

$$y[n] = nx[n]$$

See that time shifting the input doesn't correspond with an equivalent shift in the output—this is a time-varying system. This system represents one with a time-varying gain, even if we know the input value, we cannot determine the output value without knowing the current time.

For a counterexample, consider having input $x_1[n] = \delta[n]$, which yields an output $y_1[n] = 0$ (since $n\delta[n] = 0$). However, the input $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1] = \delta[n-1]$—while $x_2[n]$ is a shifted version of $x_1[n]$, $y_2[n]$ is *not* a shifted version of $y_1[n]$.

For a final example, consider the system

$$y(t) = x(2t)$$

This system represents a time scaling. That is, $y(t)$ is a time-compressed (by a factor of 2) version of $x(t)$. Intuitively then, any time shift in the input will also be compressed by a factor of 2, and it is for this reason that the system is not time invariant. Consider a counterexample:

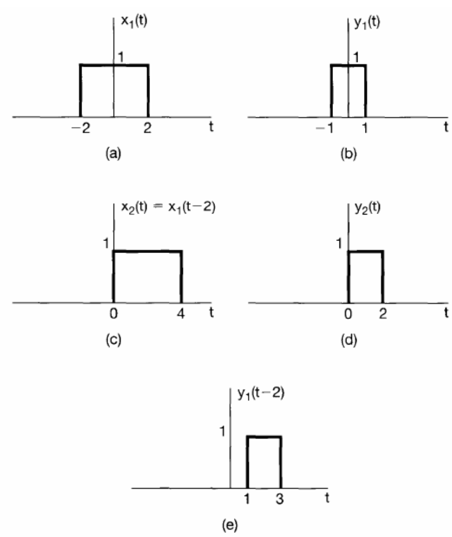


Figure 1.47 (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

see that if we shift the input signal by 2, the resulting output is not the same as if we had shifted the output signal by 2.
(next page)

Linearity
A *linear system*, in continuous or discrete time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition—that is, the weighted sum—of the responses of the system to each of those signals.

More precisely, letting $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and $y_2(t)$ the response to $x_2(t)$, the system is linear if:

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. The response to $ax_1(t)$ is $ay(1)$, where a is any complex constant.

The first property is known as the *additivity* propery, while the second the *scaling* or *homogeneity* property; the same definition holds for continuous time. Note that a system can be linear without being time invariant, and it can be time invariant without being linear.

The two properties defining a linear system can be combined into a single statement:

$$\begin{aligned} \text{continuous time: } & ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t) \\ \text{discrete time: } & ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n] \end{aligned}$$

where a and b are any complex constants.

From this see that for a set of inputs $x_k[n], k = 1, 2, 3, \ldots$ to a discrete-time linear system with corresponding outputs $y_k[n], k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots$$

is

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \ldots$$

This is known as the *superposition property*, which holds for linear systems in both continuous and discrete time.

See that a direct consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time; if $x[n] \to y[n]$, then by the homogeneity property

$$0 = 0 \cdot x[n] \to 0 \cdot y[n] = 0$$

(next page)

Examples
Consider the system

$$y(t) = tx(t)$$

To determine whether or not it is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \to y_1(t) = tx_1(t)$$
$$x_2(t) \to y_2(t) = tx_2(t)$$

Now consider $x_3(t)$, a linear combination of $x_1(t)$ and $x_2(t)$:

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. See that for input $x_3(t)$, we have the output

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

So we conclude that the system is linear (also see that it is not time invariant).

For another example, consider the system

$$y(t) = x^2(t)$$

as defining $x_1(t), x_2(t)$, and $x_3(t)$ as in the previous example, we have

$$x_1(t) \to y_1(t) = x_1^2(t)$$
$$x_2(t) \to y_2(t) = x_2^2(t)$$

and

$$\begin{aligned} x_3(t) \to y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

which isn't a superposition of the inputs, thus the system is not linear.
(next page)

More examples
It is important to remember that the scaling constants of the superposition criteria are allowed to be complex. Consider the system

$$y[n] = \mathrm{Re}\{x[n]\}$$

This system is additive, but does not satisfy the homogeneity property, consider input

$$x_1[n] = r[n] + is[n]$$

the corresponding output is

$$y_1[n] = r[n]$$

Now consider scaling $x_1[n]$ by a complex number, for instance $a = i$; defined as x_2:

$$\begin{aligned} x_2[n] &= ix_1[n] = i(r[n] + is[n]) \\ &= -s[n] + ir[n] \end{aligned}$$

The corresponding output then is

$$y_2[n] = \mathrm{Re}\{x_2[n]\} = -s[n]$$

which is not equal to the scaled version of $y_1[n]$:

$$ay_1[n] = ir[n]$$

thus the system violates the homogeneity property and is not linear.
(next page)

Example—incrementally linear system
Consider the system

$$y[n] = 2x[n] + 3$$

This system is not linear, see that it violates the additivity property:

$$x_1[n] \to y_1[n] = 2x_1[n] + 3$$
$$x_2[n] \to y_2[n] = 2x_2[n] + 3$$

where the response to $x_3[n] = x_1[n] + x_2[n]$ is

$$y_3[n] = 2(x_1[n] + x_2[n]) + 3$$

while

$$y_1[n] + y_2[n] = 2(x_1[n] + x_2[n]) + 6$$

See that this system violates the property of linear systems where zero input yields zero output.

It may be surprising that the system here is nonlinear since it describes a linear equation. Intuitively, see that the output of this system can be represented as the sum of the output of a linear system

$$x[n] \to 2x[n]$$

and another signal equal to the *zero-input response* of the system (when the input is zero),

$$y_0[n] = 3$$

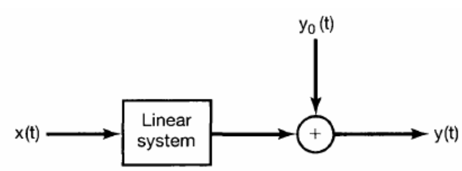


Figure 1.48 Structure of an incrementally linear system. Here, $y_0[n]$ is the zero-input response of the system.

There are large classes of systems that can be represented like this, for which the overall system output consists of the superposition of the response of a linear system with a zero-input response; these systems correspond to the class of *incrementally linear systems*.

The *difference* between the resposnes to any two inputs to an incrementally system is a linear (additive and homogeneous) function of the *difference* between the two inputs. For example, for inputs $x_1[n]$, $x_2[n]$ and corresponding outputs $y_1[n]$, $y_2[n]$ we have

$$y_1[n] - y_2[n] = 2x_1[n] + 3 - (2x_2[n] + 3) = 2[x_1[n] - x_2[n]]$$

Chapter 2

Linear Time-Invariant Systems

2.1 Discrete-time convolution sum

Representing a discrete signal in terms of impulses
The key idea in visualising how the discrete-time unit impulse can be used to construct any discrete-time signal is to think of a discrete-time signal as a sequence of individual impulses:

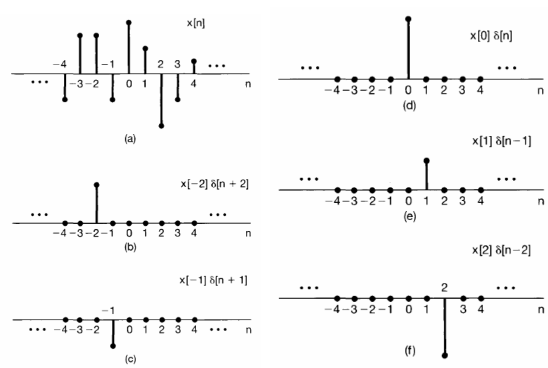


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

See that we can express express each value of $x[n]$ as an individual scaled, shifted impulse; for instance

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases}$$
$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases}$$
$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}$$

More generally, see that we can write

$$\begin{aligned} x[n] = \ldots &+ x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ &+ x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots \end{aligned}$$

For any value of n, only one of the terms on the right-hand side of the equation is nonzero, and the scaling associated with that term is precisely $x[n]$. Writing this summation in a more compact form, we have

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

This equation is called the *sifting property* of the discrete-time unit impulse; the summation 'sifts' through $x[k]$ and preserves only the value corresponding to $k = n$.
(next page)

Convolution-sum representation of LTI systems
The fact that $x[n]$ can be represented as a superposition of scaled versions of (time shifted) impulses, means that the response of a linear system to $x[n]$ will be a *superposition of the scaled responses of the system* to each of these shifted impulses.

Moreover, the property ot time invariance tells us that that the *responses of a time-invariant system to the time-shifted unit impulses are simply time-shifted versions of one another*. The convolution-sum representation for discrete-time systems that are both linear and time invariant results from putting these two basic facts together.

Linear, not necessarily time invariant response
Consider the response of a linear (but possibly time-varying) system to an arbitrary input $x[n]$. We can represent $x[n]$ as a linear combination of shifted impulses.

Letting $h_k[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n-k]$, see that the superposition property of linear systems means that the response $y[n]$ of a linear system to the input $x[n]$ is simply the weighted linear combination of these basic responses:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

The response value at a specific time n of a linear system is the superposition of the 'contributions' to that output point from each input (from all points in time).
(next page)

Illustrated
For instance, given the input signal $x[n]$ to a linear (non-time invariant) system with the responses $h_{-1}[n], h_0[n]$, and $h_1[n]$ to the signals $\delta[n+1], \delta[n]$, and $\delta[n-1]$:

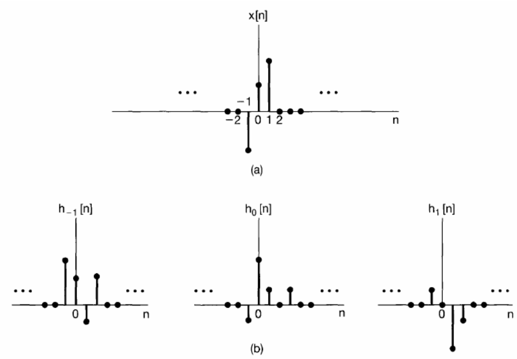


Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).

Since $x[n]$ can be written as a linear combination of the shifted impulses. The system *responses* to these time shifted impulses, h, scaled accordingly by their respective input value, sum to produce $y[n]$.

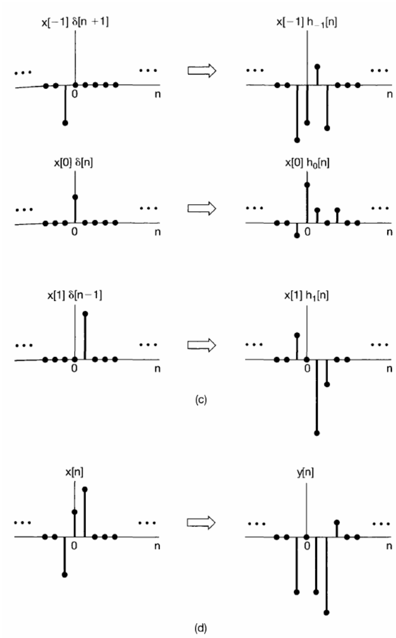


Figure 2.2 *Continued*

(This illustration only considers three points, but see that the summation on the previous page accounts for the possibility that every input could contribute to a particular output point in time, thus the limits $-\infty$ to ∞)
(next page)

LTI response—convolution sum

In the previous case, the responses $h_k[n]$ need not be related to each other for different values of k. However, if the linear system is also *time invariant*, then these responses to time-shifted unit impulses are all *time-shifted versions of each other*, meaning

$$h_k[n] = h_0[n-k]$$

For notational convenience, we dr the subscript on $h_0[n]$ and define the *unit impulse (sample) response* as

$$h[n] = h_0[n]$$

That is, when $h[n]$ is the output of the LTI system when $\delta[n]$ is the input, we have

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

This result is referred to as the *convolution sum* or *superposition sum*, and the operation on the right-hand side is known as the *convolution* of the sequences $x[n]$ and $h[n]$, represented symbolically as

$$y[n] = x[n] * h[n]$$

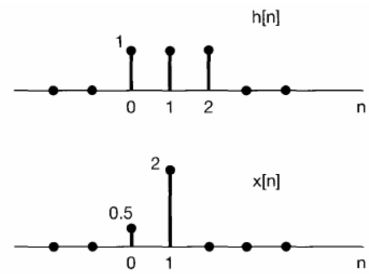
See that an LTI system is completely characterised by its response to a single signal, the unit impulse. As before, the actual output is a superposition of the responses, but due to time invariance this time every response is just a time shift of each other.

2.2 Discrete-time convolution sum—examples

These examples are crucial for a deeper understanding of the discrete convolution. For reference, the definition of the convolution sum: When $h[n]$ is the output of the LTI system when $\delta[n]$ is the input, we have output $y[n]$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Example 1
Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated



(a)

In this case, since only $x[0]$ and $x[1]$ are nonzero, the output (via convolution), simplifies to the expression

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$$

The sequences $0.5h[n]$ and $2h[n-1]$ are two 'echoes' of the impulse response needed for the superposition involved in generating $y[n]$, obtained by summing up each echo for each value of n.
(next page)

Example 1 illustrated

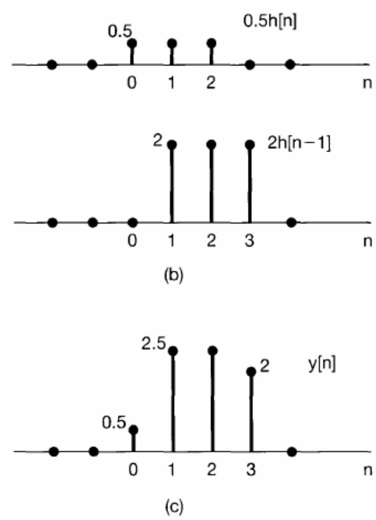


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

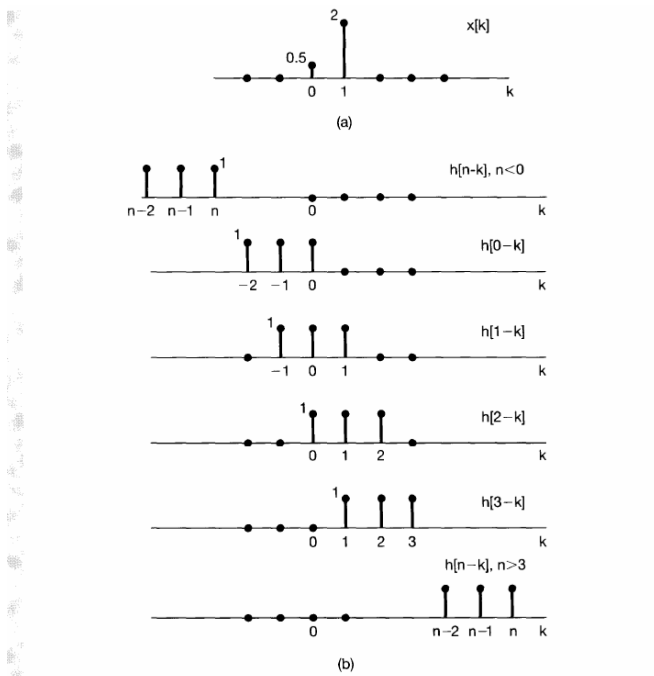
The output for point 0 depends on only on the first point of the impulse response at point 0, $x[0]h[0]$ (in this case there aren't any inputs before this point to produce other impulse responses to affect this). In contrast, the output for point 1 depends on both the *second* point of the impulse response at point 0, $x[0]h[1-0]$ and the *first* point of the impulse response at point 1, $x[1]h[1-1]$. (see how this relates to the structure of the convolution sum)

Alternative view of the convolution
Consider the evaluation of the output value at a specific time n, a particularly convenient way of displaying this calculation graphically begins with the two signals $x[k]$ and $h[n-k]$ viewed as functions of k.

Multiplying these two functions, we obtain a sequence $g[k] = x[k]h[n-k]$, which, at each time k, can be seen as representing the contribution of $x[k]$ to the output at time n. Summing all the samples in the sequence of $g[k]$ yields the output value at the selected time n.

Thus, to calculate $y[n]$ for all values of n requires repeating this procedure for each value of n. Fortunately, changing the value of n has a very simple graphical interpretation for the two signals $x[k]$ and $h[n-k]$, viewed as functions of k. The following examples illustrate this.
(next page)

Example 2—view as a function of k
Let us consider again the convolution problem encountered in the previous example. Now we have the sequences $x[k]$ and $h[n-k]$, for n fixed and viewed as a function of k; see that the graph of $h[n-k]$ is a time reversal and shift—to the *right* if n is positive and *left* if n is negative:



Having sketched $x[k]$ and $h[n-k]$ for a given n, we multiply the two signal and sum over all values of k (like a dot product) to get $y[n]$. (We multiply and sum up the points where the graphs 'overlap')

(Intuit that the view in the first example is the same as 'reversing' the impulse response and moving it along the input like a sliding window, which is essentially what is happening here.)

For $n < 0$, neither of the graphs overlap at points where both are nonzero, so $y[n] = 0$ for $n < 0$. For $n = 0$, since the product of the sequence $x[k]$ and the sequence $h[0-k]$ has only one nonzero sample (at $k = 0$), we have

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = x[0]h[0] = 0.5$$

(next page)

cont. from example 2
The other examples have more ‘overlapping’ points:

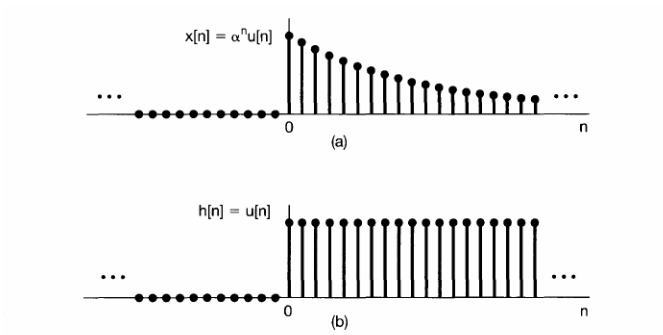
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5$$
$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5$$
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0$$

(next page)

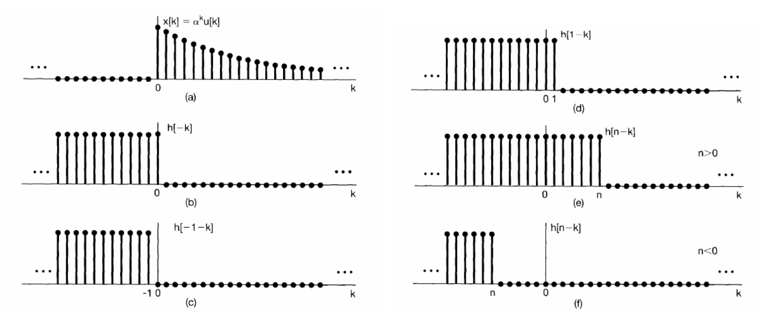
Example 3
Consider input $x[n]$ and unit impulse $h[n]$ given by

$$x[n] = \alpha^n u[n]$$
$$h[n] = u[n]$$

where $0 < a < 1$:



As before, consider $x[k]$ and $h[n-k]$ for different n:



For $n < 0$ there is no overlap between nonzero points in $x[k]$ and $h[n-k]$, so $y[n] = 0$ for $n < 0$. See that for $n \geq 0$,

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

Thus for $n \geq 0$, the usual infinite limits are superfluous, and we can write

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

this is a geometric series, we can write

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1 - \alpha^{n+1}}{1 - \alpha} \quad \text{for } n \geq 0$$

(see appendix for sum of geometric series formula)
(next page)

cont. example 3
We had

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad \text{for } n \geq 0$$

which can be rewritten as

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$

Illustrated:

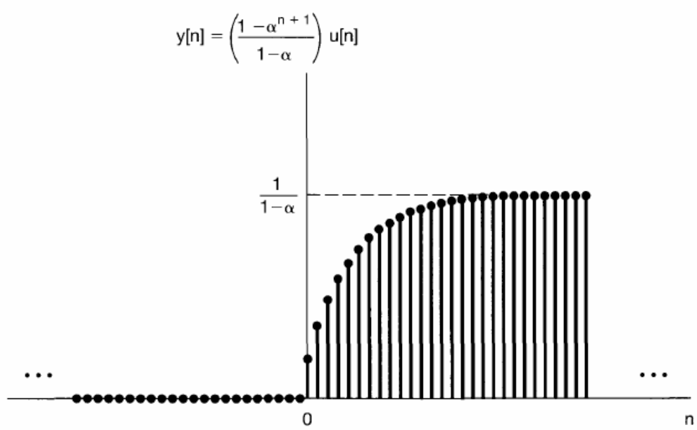


Figure 2.7 Output for Example 2.3.

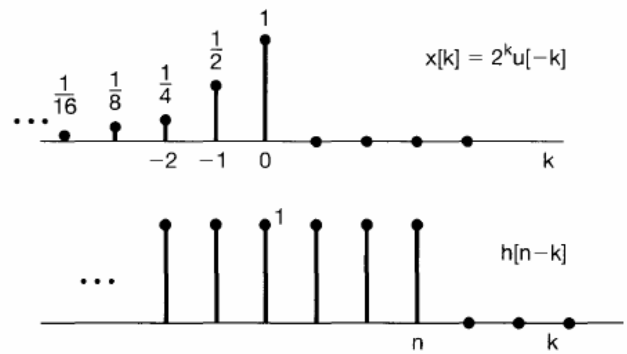
See that the convolution operation can be described as a 'sliding' of the sequence $h[n-k]$ past $x[k]$ (as we increase n the $h[n-k]$ graph just gets shifted to the right).

For example, suppose we have evaluated $y[n]$ for some particular value of n, say $n = n_0$, meaning we have sketched $h[n_0 - k]$, multiplied it by $x[k]$, and summed the results over all k. To evaluate $y[n]$ at the next value of n, so $n = n_0 + 1$, we sketch $h[(n_0 + 1) - k]$; however, we can do this by simply translating $h[n_0 - k]$ to the right by one point.

Repeating this for each successive n demonstrates this 'sliding' idea.
(next page)

Example 4
Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as

$$x[n] = 2^n u[-n]$$
$$h[n] = u[n]$$



(a)

See that $x[k]$ is zero for $k > 0$ and $h[n-k]$ is zero for $k > n$. Also see that regardless of n, the sequence $x[k]h[n-k]$ always has nonzero samples along the k-axis, and when $n \geq 0$, $x[k]h[n-k]$ has nonzero samples in the interval $k \leq 0$. As such we can write, for $n \geq 0$

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[n-k] = \sum_{k=-\infty}^{0} 2^k$$

and for $n < 0$, $x[k]h[n-k]$ has nonzero samples for $k \leq n$. It follows that, for $n < 0$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=-\infty}^{n} 2^k$$

(next page)

cont. example 4
In both cases we can simplify using the the formula for summing geometric series (see appendix)

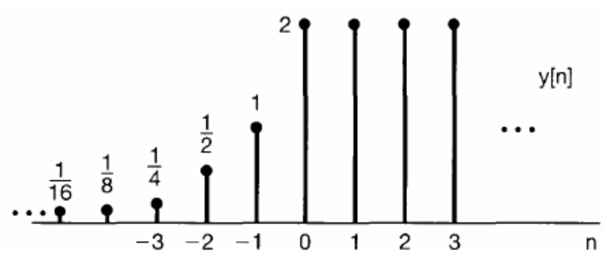
$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$

Where for the $n \geq 0$ case, by performing a change of variable $r = -k$,

$$y[n] = \sum_{k=-\infty}^{0} 2^k = \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1-(1/2)} = 2$$

For the $n < 0$ case, by performing a change of variable $l = -k$, and then $m = l + n$,

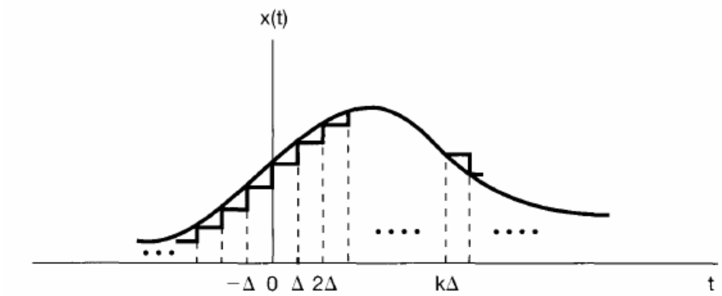
$$\begin{aligned} y[n] = \sum_{k=-\infty}^{n} 2^k &= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n} \\ &= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1} \end{aligned}$$



(b)

2.3 Continuous-time—the convolution integral

Representating a continuous signal in terms of impulses
We develop a continuous-time conterpart of the discrete-time 'sifting' property. Consider a riemann-sum like approximation, $\hat{x}(t)$, to a continuous-time signal $x(t)$:



(a)

This approximation can be expressed as a linear combination of delayed rectangular pulses. If we define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

then since $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

(δ_Δ 'turns on' a specific rectangle, x specifies its height, and Δ the base length) See that for any t, only one term in the summation on the right is nonzero.

As we let Δ approach 0, the approximation $\hat{x}(t)$ becomes better, and in the limit equals $x(t)$, therefore we write

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

Also, as $\Delta \to 0$, the summation approaches an integral; see that since the limit of $\delta \to 0$ of $\delta_\Delta(t)$ is the unit impulse function $\delta(t)$, we have

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau$$

This should be viewed as an idealisation in the sense that, for any Δ 'small enough', the approximation by this summation is essentially exact for any practical purpose. The integral then represents an idealisation of the summation by taking Δ to be vanishingly small.
(next page)

Viewed as a function of τ
See that when the signal $\delta(t-\tau)$ is viewed as a function of τ with t fixed, it is still a unit impulse located at $\tau = t$:

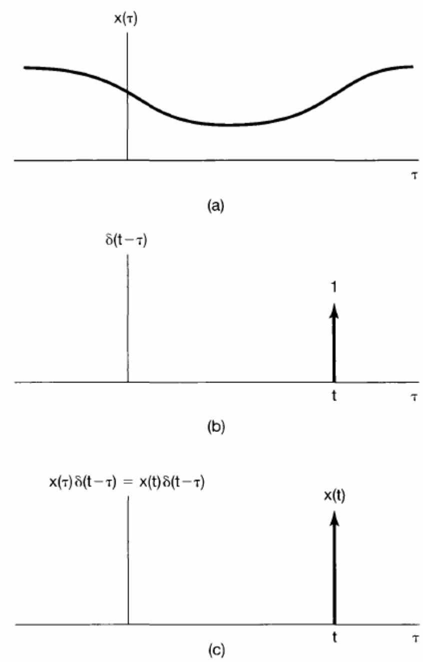


Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a function of τ with t fixed; (c) product of these two signals.

As such, we have

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$$

which is just a scaled impulse at $\tau = t$ with area equal to $x(t)$. See that this can be used to show the statement on the previous page

$$\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{+\infty} \delta(t-\tau)d\tau = x(t)$$

Emphasising the interpretation of the $x(t)$ being represented as a sum (or more precisely, an integral) of weighted, shifted impulses.
(next page)

Continuous-time impulse response
We could represent $\hat{x}(t)$ as a sum of scaled and shifted versions of the basic pulse signal $\delta_\Delta(t)$.

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

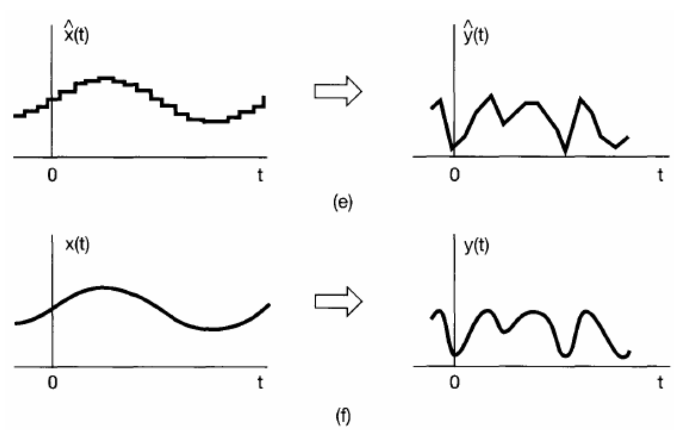
Consequently, the response $\hat{y}(t)$ of a linear system to this signal will be the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$.

Specifically, let us define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$, then by superposition we have

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

Now consider what happens as Δ becomes vanishingly small, meaning $\Delta \to 0$. $\hat{x}(t)$ becomes an increasingly good approximation to $x(t)$, and in fact, the two coincide as $\Delta \to 0$.

Consequently, the response to $\hat{x}(t)$, namely, $\hat{y}(t)$, must converge to $y(t)$, the response to the actual input $x(t)$. Either way, we say that for Δ 'small enough', the duration of the pulse $\delta_\Delta(t - k\Delta)$ is of no significance, in that, as far as the system is concerned, the response to this pulse is essentially the same as the response to a unit impulse as $\Delta \to 0$:



Since the pulse $\delta_\Delta(t - k\Delta)$ corresponds to a shifted unit impulse as $\Delta \to 0$, the response $\hat{h}_{k\Delta}(t)$ to this input also becomes the impulse response in the limit. As such we have

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

(next page)

cont.
We had

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

As $\Delta \to 0$, the summation on the right-hand side becomes an integral:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau$$

This interpretation is analagous to the one earlier, where any input $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

$x(t)$ is essentially represented as a 'sum', which, by linearity, leads to $y(t)$ being represented as a superposition of responses to each impulse in the 'sum'.

Convolution integral
$y(t)$ as decribed above represents the general form of the response of a linear system in continuous time. If, in addition to being linear, the system is also time invariant, then the response to the impulse $\delta(t-\tau)$ is just the response to $\delta(t)$ shifted by τ from the origin, meaning $h_\tau(t) = h_0(t-\tau)$.

For notational convenience, we can then drop the subscript and define the unit impulse response $h(t)$ (the response to $\delta(t)$) as

$$h(t) = h_0(t)$$

In this case, the expression for the $y(t)$ above becomes

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

This is referred to as the *convolution integral* or the *superposition integral*, the continuous-time counterpart of the convolution sum, representing a continuous-time LTI system in terms of its response to a unit impulse, written symbolically as

$$y(t) = x(t) * h(t)$$

See that a continuous-time LTI system is completely characterised by its impulse response—a single elementary signal.

Evaluation by visualisation in terms of τ
The procedure for evaluating the convolution integral is similar to the discrete-time counterpart: we first obtain the signal $h(t-\tau)$ regarded as a function of τ with t fixed; this is a reflection about the origin, shifted to the right by t if $t > 0$ or left if $t < 0$. Next multiply $x(\tau)$ and $h(t-\tau)$ and integrate the product over infinity. (see examples)

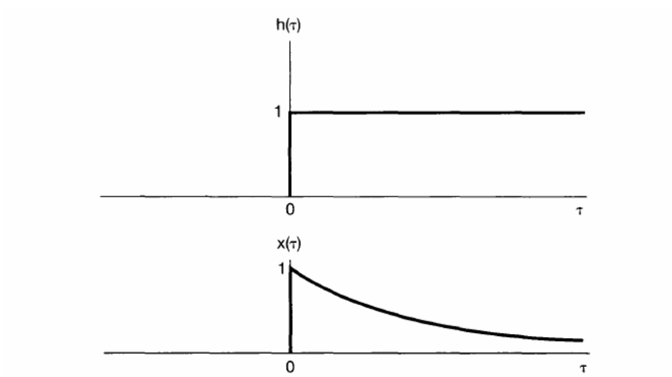
2.4 Convolution integral—examples

Example 1
Consider letting $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

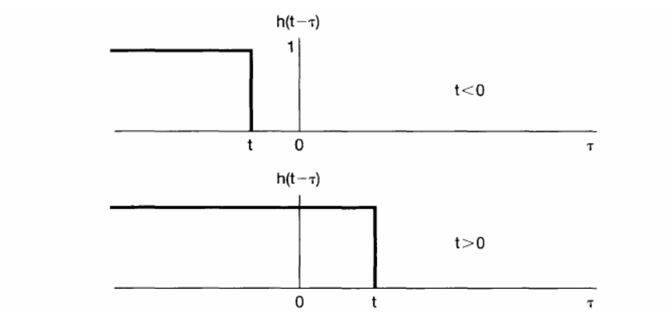
$$x(t) = e^{-at}u(t), \quad a > 0$$

and

$$h(t) = u(t)$$



See that plotting $h(t-\tau)$ with τ as the horizontal axis:



See that for $t < 0$, the product of $x(\tau)$ and $h(t-\tau)$ is zero, and consequently, $y(t)$ is zero. While for $t > 0$,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

As such, the convolution integral to compute $y(t)$ for $t > 0$ looks like

$$\begin{aligned} y(t) = \int_0^t e^{-a\tau} d\tau &= -\frac{1}{a}e^{-a\tau}\Big|_0^t \\ &= \frac{1}{a}(1 - e^{-at}) \end{aligned}$$

(next page)

cont. example 1
We had $y(t) = 0$ for $t < 0$, and for $t > 0$ we had

$$y(t) = \frac{1}{a}(1 - e^{-at})$$

for all t, we can write

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t)$$

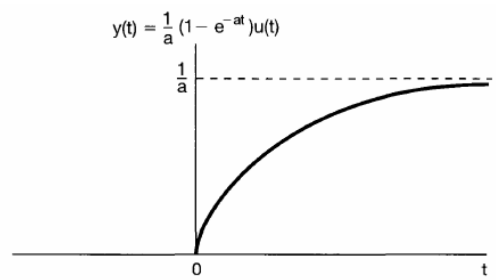


Figure 2.18 Response of the system in Example 2.6 with impulse response $h(t) = u(t)$ to the input $x(t) = e^{-at}u(t)$.

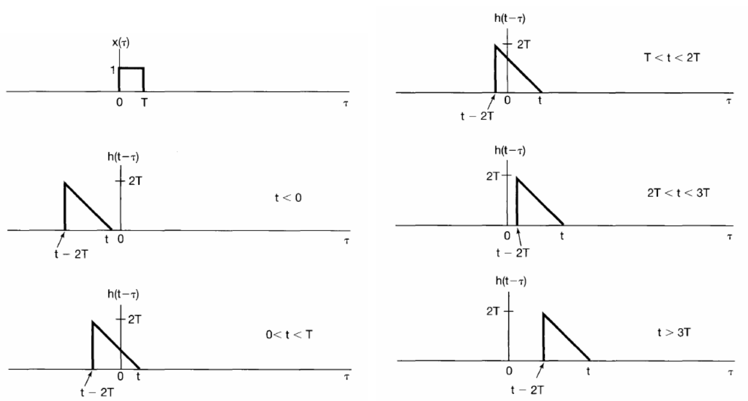
(next page)

Example 2
Consider the convolution of the following two signals

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

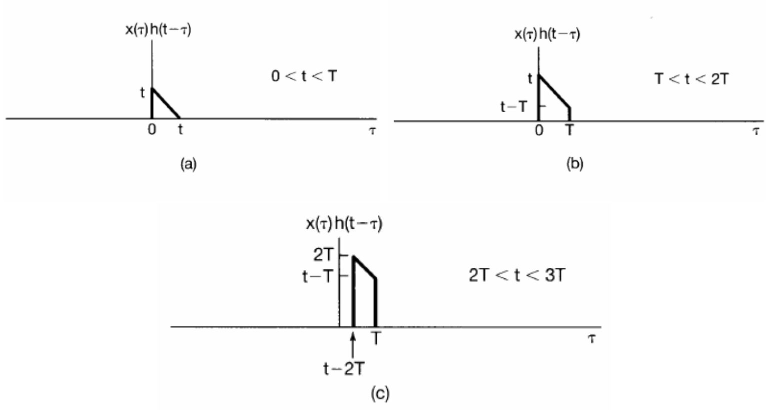
$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

Now consider the graphs of $x(\tau)$ and $h(t-\tau)$ plotted for different values of t:



See that the convolution integral is different depending on the value of t:

$$y(t) = \begin{cases} 0, & t < 0 \\ \int_0^t -\tau + t \, d\tau, & 0 < t < T \\ \int_0^T -\tau + t \, d\tau, & T < t < 2T \\ \int_{t-2T}^T -\tau + t \, d\tau, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$



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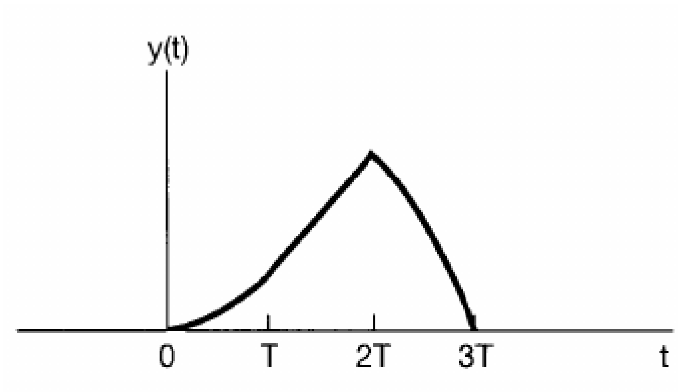
cont. example 2
Each convolution integral evaluates to

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$



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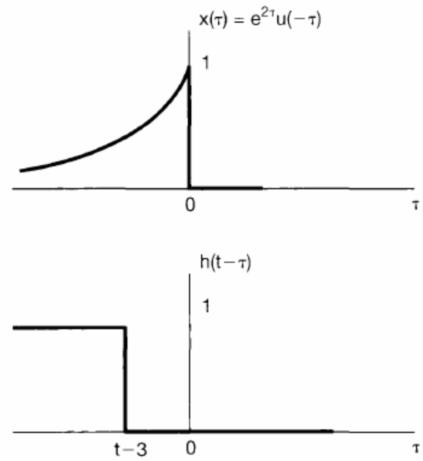
Example 3
Consider the convolution between the two signals

$$x(t) = e^{2t}u(-t)$$
$$h(t) = u(t-3)$$



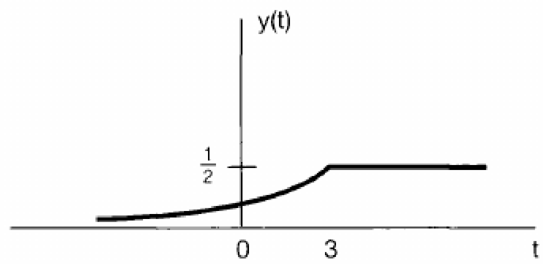
(a)

Observe that $x(\tau)$ and $h(t-\tau)$ have regions of nonzero overlap regardless of the value of t. When $t-3 \le 0$, the product of $x(\tau)$ and $h(t-\tau)$ is nonzero for $-\infty < \tau < t-3$; the convolution integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2}e^{2(t-3)}$$

For $t-3 \ge 0$, the product $x(\tau)h(t-\tau)$ is only nonzero for $-\infty < \tau < 0$, so the convolution integral is

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}$$



(b)

2.5 Properties of LTI systems 1

For reference, we can represent continuous and discrete-time LTI systems in terms of their unit impulse responses; in discrete-time this takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n] \\ y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \end{aligned}$$

It is important to emphasise that this property holds in general *only* for LTI systems (since the entire derivation is based on this).

Commutativity
Convolution in both continuous and discrete time is *commutative*. That is, in discrete time

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

and in continuous time

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

These expressions can verified by means of a substitution of variables. In the discrete-time case, substituting $r = n - k$:

$$\begin{aligned} x[n] * h[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=\infty}^{-\infty} x[n-r]h[r] \\ &= \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n] \end{aligned}$$

and in the continuous case, substituting $r = t - \tau$:

$$\begin{aligned} x(t) * h(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{+\infty}^{-\infty} -x(t-r)h(r)dr \\ &= \int_{-\infty}^{+\infty} x(t-r)h(r)dr = h(t) * x(t) \end{aligned}$$

In the context of computing the convolution, this means that the step where a signal is reflected and shifted according to n or t can be done with either x or h in both discrete and continuous time. One form may be easier to visualise, but both forms always result in the same answer.
(next page)

Distributivity
Convolution is also *distributive*. Specifically, convolution distributes over addition, meaning in discrete time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

and in continuous time

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

This can be verified by direct subsitution into the convolution sum/integral.

See this means that the system

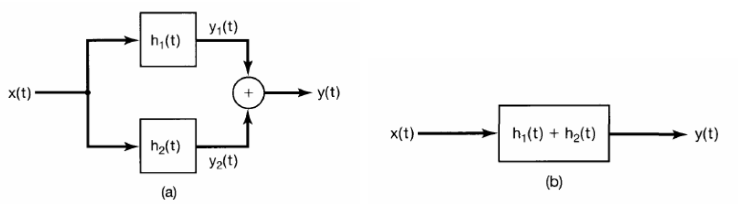
$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

and the system

$$y(t) = x(t) * [h_1(t) + h_2(t)]$$

are identical. There is an analagous interpretation in discrete time.

Illustrated using block diagrams, the two are identical:



See also that as a consequence of bothe the commutative and distributive properties, we have

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

and

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

this means that the response of an LTI system to the sum of two inputs must equal the sum of the responses to each input individually.
(next page)

Associativity
Convolution is *associative*; that is, in discrete time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

and in continuous time

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

This property can be proven by straigntforward manipulation of the summations and integrals. In discrete time (the infinite limit case obscures understanding, so assuming the convolution product is nonzero only for $0 < k < n$ (or t)),

$$\begin{aligned}
((f * g) * h)[n] &= \sum_{k=0}^{n} (f * g)[k]h[n-k] \\
&= \sum_{k=0}^{n} \left(\sum_{l=0}^{k} f[l]g[k-l] \right) h[n-k] \\
&= \sum_{0 \le l \le k \le n} f[l]g[k-l]h[n-k] \\
&= \sum_{l=0}^{n} \sum_{k=l}^{n} f[l]g[k-l]h[n-k] \\
&= \sum_{l=0}^{n} f[l] \left(\sum_{r=0}^{n-l} g[r]h[n-r-l] \right) \\
&= \sum_{l=0}^{n} f[l](g * h)[n-l] \\
&= (f * (g * h))[n]
\end{aligned}$$

In the third last step we subsitute $r = k - l$. The continuous case is analagously:

$$\begin{aligned}
((f * g) * h)(t) &= \int_0^t (f * g)(\tau)h(t-\tau)d\tau \\
&= \int_{\tau=0}^{t} \left(\int_{s=0}^{\tau} f(s)g(\tau - s)ds \right) h(t-\tau)d\tau \\
&= \iint_{0 \le s \le \tau \le t} f(s)g(\tau - s)h(t-\tau)ds\, d\tau \\
&= \int_{s=0}^{t} f(s) \left(\int_{\tau=s}^{t} g(\tau - s)h(t-\tau)d\tau \right) ds \\
&= \int_{s=0}^{t} f(s) \left(\int_{r=0}^{t-s} g(r)h(t-r-s)dr \right) ds \\
&= \int_{s=0}^{t} f(s)(g * h)(t-s)ds = (f * (g * h))(t)
\end{aligned}$$

(next page)

Associativity cont.

See that as a consequence of associativity, the expressions

$$y[n] = x[n] * h_1[n] * h_2[n]$$

and

$$y(t) = x(t) * h_1(t) * h_2(t)$$

are unambiguous, since it doesn't matter in what order we convolve the signals. It also follows that that the signals

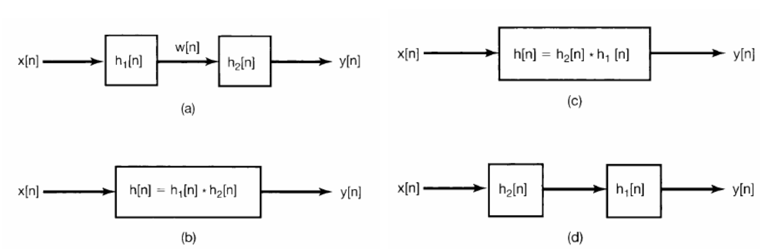
$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n] \end{aligned}$$

and

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (h_1[n] * h_2[n]) \end{aligned}$$

are equivalent. This can be generalised to an arbitrary number of LTI systems in cascade, and the analagous interpretation and conclusion also hold in continuous time.

Further, consider these block diagrams



Using the associative property, we can show (a)=(b); using the commutative property we know (b)=(c), and finally again the associative property shows (c)=(d).

Consequently, the unit impulse response of a cascade of an arbitrary number of LTI systems does not depend on the order in which they are cascaded. The same conclusions hold in continuous time.

2.6 Properties of LTI systems 2

Memory
Recall that a system is *memoryless* if its output at any time depends only on the value of the input at that same time.

See that the only way this can be true for a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$. The impulse response would be of the form

$$h[n] = K\delta[n]$$

Where $K = h[0]$ is a constant. (the response to each input only affects the input itself—the response at a single point is not influenced by any other points.)

See that the convolution sum reduces to the relation

$$y[n] = Kx[n]$$

If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory.

Similar properties hold for continuous-time LTI systems, which are memoryless if $h(t) = 0$ for $t \neq 0$. Such a memoryless LTI system has the form

$$y(t) = Kx(t)$$

for some constant K; it has the impulse response

$$h(t) = K\delta(t)$$

Note that if $K = 1$, then these systems become *identity systems*, with output equal to the input and with unit response equal to the unit impulse. In this case, the convolution sum and integral imply that

$$x[n] = x[n] * \delta[n]$$
$$x(t) = x(t) * \delta(t)$$

(next page)

Invertibility

A system is *invertible* only if an *inverse system* exists that, when cascaded in series with the original system, produces an output equal to the input to the first system. Consider the following block diagrams

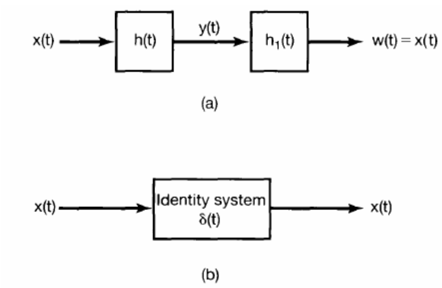


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

Given a system with impulse response $h(t)$, the inverse system, with impulse response $h_1(t)$, results in $w(t) = x(t)$—such that the series interconnection is essentially identical to the identity system.

See (by considering associativity) that the overall impulse response convolved with $x(t)$ is $(h(t) * h_1(t))$. As such we have a condition that $h_1(t)$ must satisfy for it to be the impulse response of the inverse system, namely,

$$h(t) * h_1(t) = \delta(t)$$

Similarly in discrete time, the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h_1[n] = \delta[n]$$

(next page)

Invertibility example
Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0)$$

Such a system is a *delay* if $t_0 > 0$ and an *advance* if $t_0 < 0$. If $t_0 = 0$, the system is the identity system and thus is memoryless; for any other value of t_0, this system has memory, (since it responds to an input value at a time other than the current time).

By taking input equal to the $\delta(t)$ we get the impulse response

$$h(t) = \delta(t - t_0)$$

See that as per the definition of the convolution (or by visualising the convolution)

$$y(t) = x(t - t_0) = x(t) * \delta(t - t_0)$$

See that to invert the system, all that is required is to shift the output back. If we consider another system

$$y(t) = x(t + t_0)$$

(see that this is just another time shift in the opposite direction) That would have the impulse response

$$h_1(t) = \delta(t + t_0)$$

then

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

Similarly in discrete time, a pure time shift with impulse response $\delta[n - n_0]$ would have an inverse consisting of a time shift in the opposite direction by the same amount—another LTI system with the impulse response $\delta[n - n_0]$.
(next page)

Causality
Recall the *causality* property: the output of a causal system depends only on the present and past values of the input to the system.

In the context of convolution, see that for a discrete-time LTI system to be causal, $y[n]$ must not depend on $x[k]$ for $k > n$, and that in order for this to be true we must have

$$h[n] = 0 \quad \text{for } n < 0$$

(try to visualise the convolution to see this) This means that the impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality.

More generally, causality for a linear system is equivalent to the condition of *initial rest*: if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time. It is important to emphasise that the equivalence of causality and the condition of initial rest applies only to linear systems.

(For example, the system $y[n] = 2x[n] + 3$ is not linear but is causal. However if $x[n] = 0$, $y[n] = 3 \neq 0$, so it does not satisfy the condition of initial rest.)

Once again visualising the convolution sum, see that for a causal discrete-time LTI system, the $h[n] = 0$ for $n < 0$ implies that it can be simplified to

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

and the alternative equivalent form (by commutativity) becomes (visualise as reflecting the input signal in time instead)

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

Similarly, a continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0$$

where the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

While causality is a property of systems, it is common terminology to refer to a signal as being causal if it is zero for $n < 0$ or $t < 0$. The motivation for this comes from the conditions for causality as defined above: Causality for an LTI system is equivalent to its impulse response being a ‘causal’ signal.
(next page)

Stability
Recall that a system is *stable* if every bounded input produces a bounded output. We want to derive a condition to determine whether a candidate LTI system is stable; consider an input $x[n]$ that is bounded in magnitude:

$$x[n] < B \quad \text{for all } n$$

Suppose we apply this input to an LTI system with unit impulse response $h[n]$. Then, using the convolution sum, we obtain an expression for the *magnitude* of the output as

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|$$

Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers, it follows that

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|$$

We know that $|x[n-k]| < B$ for all values of k and n. This implies that

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n$$

We conclude that the impulse response is *absolutely summable* if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$

See that if the impulse response is absolutely summable, $y[n]$ will be bounded in magnitude, and hence the system is stable. This means that the above is a sufficient condition to gaurantee the stability of a discrete-time LTI system. (see that this condition is also necessary, as if it were not satisfied then there would be unbounded outputs)
(next page)

Stability cont.
In continuous time we have an analagous characterisation of stabilityin terms of the impulse response of an LTI system; if $|x(t)| < B$ for all t, then it follows that

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right| \\ &\leq \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)|d\tau \\ &\leq B \int_{-\infty}^{+\infty} |h(\tau)|d\tau \end{aligned}$$

Therefore, the system is stable if it is *absolutely integrable*:

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$$

As with the discrete case this is actually a necessary condition for stability, as if it were unsatisfied, it would lead to unbounded outputs.

Stability example 1—stable
Consider a system that is a pure time shift in either continuous or discrete time. Then in discrete time

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |\delta[n-n_0]| = 1$$

while in continuous time

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau = \int_{-\infty}^{+\infty} |\delta(\tau - t_0)|d\tau = 1$$

and we can conclude both systems are stable.
(next page)

Stability example 2—unstable
Consider now the accumulator, with an impulse response equal to $u[n]$; see that the impulse response in this case is not absolutely summable:

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} u[n] = \infty$$

Similarly consider the integrator (the continuous-tiem counterpart of the accumulator):

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

This is also an unstable system; see that the impulse response for the integrator is:

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

and

$$\int_{-\infty}^{+\infty} |u(t)| d\tau = \int_{0}^{+\infty} |u(t)| d\tau = \infty$$

Since the impulse response is not absolutely integrable, the system is not stable.

2.7 The Unit Step Response of an LTI system

There is another signal that is also used quite often in describing the behaviour of LTI systems: the *unit step response* $s[n]$ or $s(t)$; as implied, it corresponds to the output when $x[n] = u[n]$ or $x(t) = u(t)$.

From the convolution-sum representation, see that the step response can be found by convolution of the unit step with the impulse response:

$$s[n] = u[n] * h[n]$$

Also see that by commutativity, $s[n] = h[n] * u[n]$, and therefore $s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit response $u[n]$. Since $u[n]$ is the unit impulse response of the *accumulator*, this means that an input $h[n]$ to an accumulator yields the unit step response:

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

The fact that we know the inverse system for the accumulator means we know then that $h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1]$$

That is, the step response of a discrete-time LTI system is the *running sum of its impulse response* (try to visualise this!). Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response.

Similarly in continuous time, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t) = h(t) * u(t)$. The *integrator* has $u(t)$ as its impulse response, so this is equivalent to inputting $h(t)$ into an integrator:

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

as such see that the unit impulse response can be recovered as the first derivative

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

The unit step response of a continuous-time LTI system is the running integral of its impulse response, and the unit impulse response is the first derivative of the unit step response.

In that sense, the unit step response can also be used to characterise an LTI system, since we can calculate the unit impulse response from it.

2.8 Causal LTI systems described by differential and difference equations

Linear constant-coefficient differential equations
An important point about differential equations is that they provide an *implicit* specification of a system. That is, they describe a relationship between the input and the output, rather than an *explicit* expression for the system output as a function of the input. (meaning they don't immediately tell you the output from a given input value, unlike say a function)

Auxiliary conditions—Initial rest
Recall that the formation of an explicit solution to a differential equation requires the specification of certain *auxiliary/initial conditions* in order to fully evaluate the *homogeneous solution* or the *natural response* of the system. Also recall that different choices of auxiliary conditions will lead to different relationships between the input and output.

In some cases this auxiliary contition might take the form of the condition of *initial rest*, where if $x(t) = 0$ for $t < t_0$, then $y(t)$ must also equal 0 for $t < t_0$. It is important to emphasise that a condition of initial rest does not specify a zero initial condition at a fixed point in time, but rather adjusts this point in time so that the *response is zero until the input becomes nonzero*. (thus, if $x(t) = 0$ for $t \leq t_0$, then we could use $y(t_0) = 0$ to solve for the output for $t > t_0$)
(next page)

General form
A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

The *order* refers to the highest derivative of the output $y(t)$ appearing in the equation. In the case where $N = 0$, see that this reduces to

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

In this case, $y(t)$ is an explicit function of the input $x(t)$ and its derivatives. For $N \geq 1$, the equation would specify an implicit relationship in terms of the input. In this case, the analysis of the equation would proceed in a way analagous to that of a first-order DE, with a solution consisting of two parts—a particular solution plus a solution to the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

As in the first-order case, the general form does not completely specify the output in terms of the input, and we need to identify auxiliary conditions to determine completely the input-output relationship for the system. Should we consider initial rest, we assume then that if $x(t) = 0$ for $t \leq t_0$, then $y(t) = 0$ for $t \leq t_0$, and therefore, the response for $t > t_0$ can be calculated with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1} y(t_0)}{dt^{N-1}} = 0$$

(For an Nth order DE we'll need N initial conditions. This comes from the fact that the characteristic polynomial will have N roots, so N exponentials where any linear combination of them would satisfy the homogeneous equation, thus warranting the need for the initial conditions to determine the coefficients of this linear combination.)

Under the condition of initial rest, the system described in the general form above is causal and LTI.
(next page)

Linear constant-coefficient difference equations—General form
The discrete-time counterpart of the previous general form is the Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

All equations of this type can be solved in a manner analagous to that for differential equations. As before, the solution $y[n]$ can be written as the sum of a particular solution to the above plus a solution to the homogeneous equation

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

As in the continuous-time case, the general form is not sufficient to completely specify an output, and requires specification of auxiliary conditions (in this case this might be $y[n-1], \ldots y[n-N]$). The condition of initial rest can also be applied here, where if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$ as well. With initial rest, the system would be LTI and causal.

Recursive equations
Although all of these properties can be developed following an approach that directly parallels those for differential equations, the discrete-time case offers an alternative path. See that the general form can be rearranged as

$$y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

See that the output at time n can be expressed in terms of previous values of the input and output. An equation of this form is called a *recursive equation*, since it specifies a recursive procedure for determining the output in terms of the input and previous outputs. In the special case where $N = 0$, this reduces to

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

Here $y[n]$ is an explicit function of the present and previous values of the input. For that reason this is often called a *nonrecursive equation* (since we do not recursively use previously computed values to compute the present output).

Although we do not require auxiliary conditions for the case of $N = 0$, such conditions are needed for the recursive case when $N \geq 1$.
(next page)

Difference equation example
Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

rewritten as

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

this highlights the idea that we need the previous value of the output, $y[n-1]$, to calculate the current value. Thus to begin the recursion we need an initial condition.

Suppose we impose the condition of initial rest and consider the input

$$x[n] = K\delta[n]$$

In this case, since $x[n] = 0$ for $n \leq -1$ (because of the delta function), the condition of initial rest implies that $y[n] = 0$ for $n \leq -1$, so we have as an initial condition $y[-1] = 0$. Starting from the initial condition, see that we can recursively solve for successive values of $y[n]$ for $n \geq 0$ as follows:

$$\begin{aligned}
y[0] &= x[0] + \frac{1}{2}y[-1] = K \\
y[1] &= x[1] + \frac{1}{2}y[0] = \frac{1}{2}K \\
y[2] &= x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 K \\
&\vdots \\
y[n] &= x[n] + \frac{1}{2}y[n-1] = \left(\frac{1}{2}\right)^n K
\end{aligned}$$

See additionally that by setting $K = 1$ we obtain the impulse response for the system as

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

(see that some initial conditions may lead to a non-LTI system, in which obtaining the impulse response be much less useful)

2.9 Block diagram representations of first-order systems described by differential and difference equations

Discrete-time

Consider the causal system described by the first-order difference equation

$$y[n] + ay[n-1] = bx[n]$$

Note that the evaluation of $y[n]$ requires three basic operations: addition, multiplication by a coefficient, and a delay (to capture the relationship between $y[n]$ and $y[n-1]$). We define these three basic network elements as follows:

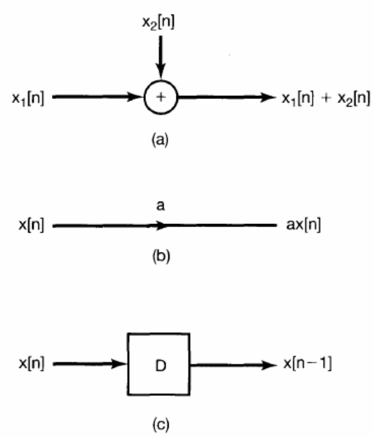


Figure 2.27 Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

Rewriting the equation in a form that directly suggests a recursive algorithm for computing successive values of $y[n]$:

$$y[n] = -ay[n-1] + bx[n]$$

see that we can represent this algorithm using the basic elements as

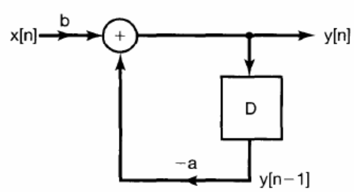


Figure 2.28 Block diagram representation for the causal discrete-time system described by eq. (2.126).

This is an example of a *feedback* system (since the output is fed back after a delay and multiplication as an addition to the next input). See that the presence of feedback is a direct consequence of the recursive nature of the system. The block diagram makes clear the required memory in this system and the consequent need for intial conditions (the initial memory element).
(next page)

Continuous-time

Consider next the causal continuous-time system described by a first-order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

which we can rewrite as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$

The three operations involved here are addition, multiplication by a coefficient, and differentiation. Therefore, if we define the three network elements as follows

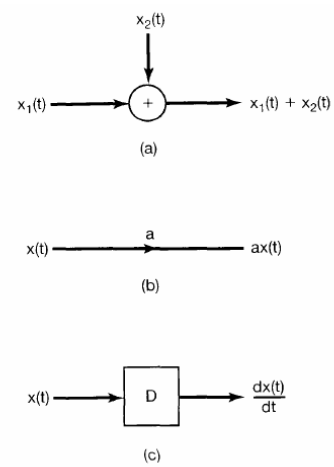


Figure 2.29 One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

we can consider representing the system as follows

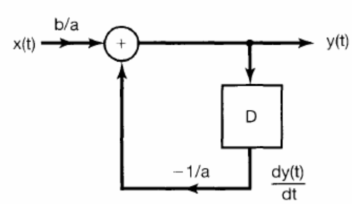


Figure 2.30 Block diagram representation for the system in eqs. (2.128) and (2.129), using adders, multiplications by coefficients, and differentiators.

While such a configuration is a valid representation of the causal system as described, it not the representation that is most frequently or practically used, since differentiators are both difficult to implement and extremely sensitive to errors and noise.

(next page)

Continuous-time cont.
The previously proposed representation of the system

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$

is valid but impractical, since differentiators are difficult to implement. An alternative implementation that is much more widely used can be obtained by first rewriting the system as

$$\frac{dy(t)}{dt} = bx(t) - ay(t)$$

and then *integrating* from $-\infty$ to t. Specifically, if we assume that the value of $y(-\infty)$ is zero, then the integral of $dy(t)/dt$ from $-\infty$ to t is precisely $y(t)$. Consequently, we obtain the equation

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$

In this form, see that this system can be implemented using the adder and coefficient multiplier as indicated earlier, together with an *integrator*:

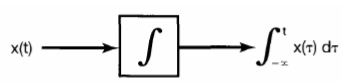


Figure 2.31 Pictorial representation of an integrator.

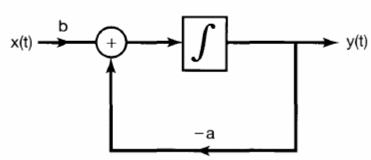


Figure 2.32 Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

Integrators are much more readily implemented. In the continuous-time case it is the integrator that represents the memory storage element of the system. This is perhaps more readily seen if we consider integrating the rearranged equation from a finite point in time t_0:

$$y(t) = y(t_0) + \int_{t_0}^{t} [bx(\tau) - ay(\tau)]d\tau$$

See that an initial condition $y(t_0)$ must be specified; this is the value that the integrator stores at time t_0.

2.10 The Unit impulse as an idealised short pulse

We know that the unit impulse $\delta(t)$ is the impulse response of the identity system. That is,

$$x(t) = x(t) * \delta(t)$$

for any signal $x(t)$. Therefore if we take $x(t) = \delta(t)$, we have

$$\delta(t) = \delta(t) * \delta(t)$$

This is a basic property of the unit impulse, and it also has a significant implication for our interpretation of the unit impulse as an idealised pulse. For instance, recall that we initially considered $\delta(t)$ to be the limiting form of a rectangular pulse $\delta_\Delta(t)$; now consider

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$$

Then $r_\Delta(t)$ looks like

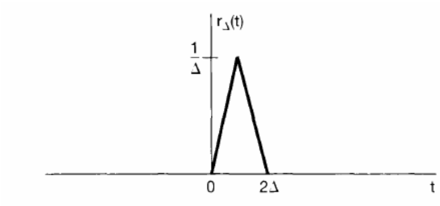


Figure 2.33 The signal $r_\Delta(t)$ defined in eq. (2.135).

See that it we wish to interpret $\delta(t)$ as the limit as $\Delta \to 0$ of $\delta_\Delta(t)$, then by the second statement, the limit as $\Delta \to 0$ of $r_\Delta(t)$ *must also be the unit impulse.*

Since we can argue that the limits as $\Delta \to 0$ for $r_\Delta(t)$ must also be a unit impulse, a similar argument can be made that as $\Delta \to 0$, $r_\Delta(t) * r_\Delta(t)$ or $r_\Delta(t) * \delta_\Delta(t)$ must also be unit impulses and so on.

Thus see that if we define the unit impulse as being the limiting form of some signal, there are an unlimited number of very dissimilar looking signals that all behave like an impulse in the limit.
(next page)

Cont.
We found that if we define the unit impulse as being the limiting form of some signal, there are an unlimited number of very dissimilar looking signals that all behave like an impulse in the limit.

The idea here is that they all 'behave like an impulse', meaning that the response an LTI system to all of these signals is essentially identical as long as the pulse is 'short enough', meaning Δ is 'small enough'. To illustrate this, consider an LTI system described by the first order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

assuming initial rest. The responses of this system to $\delta_\Delta(t), r_\Delta(t) * \delta_\Delta(t)$, and $r_\Delta(t) * r_\Delta(t)$ for several values are ilustrated here for several values of Δ:

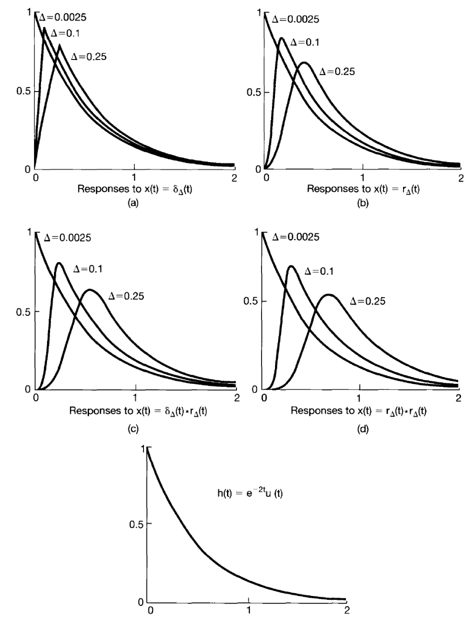


Figure 2.34 Interpretation of a unit impulse as the idealization of a pulse whose duration is "short enough" so that, as far as the response of an LTI system to this pulse is concerned, the pulse can be thought of as having been applied instantaneously: (a) responses of the causal LTI system described by eq. (2.136) to the input $\delta_\Delta(t)$ for $\Delta = 0.25$, 0.1, and 0.0025; (b) responses of the same system to $r_\Delta(t)$ for the same values of Δ; (c) responses to $\delta_\Delta(t) * r_\Delta(t)$; (d) responses to $r_\Delta(t) * r_\Delta(t)$; (e) the impulse response $h(t) = e^{-2t}u(t)$ for the system. Note that, for $\Delta = 0.25$, there are noticeable differences among the responses to these different signals; however, as Δ becomes smaller, the differences diminish, and all of the responses converge to the impulse response shown in (e).

See that for Δ large enough, the responses differ noticeably. However, for Δ sufficiently small, they all have a limiting form of $e^{-2t}u(t)$. The limit for each of these signals as $\Delta \to 0$ is the unit impulse, so we conclude that $e^{-2t}u(t)$ is the impulse response for this system.
(next page)

Cont.

One important point to be emphasised here is what we mean by 'Δ small enough' depends on the particular LTI system to which the preceding pulses are applied. For instance consider instead the causal LTI system

$$\frac{dy(t)}{dt} + 20y(t) = x(t)$$

plotted as per the previous instance for different values of Δ:

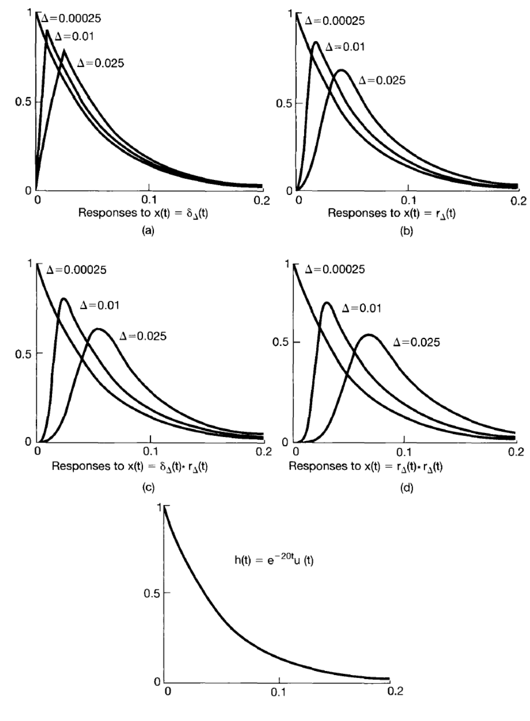


Figure 2.35 Finding a value of Δ that is "small enough" depends upon the system to which we are applying inputs: (a) responses of the causal LTI system described by eq. (2.137) to the input $\delta_\Delta(t)$ for $\Delta = 0.025$, 0.01, and 0.00025; (b) responses to $r_\Delta(t)$; (c) responses to $\delta_\Delta(t) * r_\Delta(t)$; (d) responses to $r_\Delta(t) * r_\Delta(t)$; (e) the impulse response $h(t) = e^{-20t}u(t)$ for the system. Comparing these responses to those in Figure 2.34, we see that we need to use a smaller value of Δ in this case before the duration and shape of the pulse are of no consequence.

In this case see that we need a smaller value of Δ for the responses to be indistinguishable from each other and from the impulse response $h(t) = e^{-20t}u(t)$ for the system.

As such see that what we mean by 'Δ small enough' differs depending on the system. In that sense the unit impulse is the idealisation for a short pulse whose duration is short enough for *all systems.*

2.11 Defining the unit impulse through convolution

As outlined previously, for Δ small enough, the signals $\delta_\Delta(t), r_\Delta(t), r_\Delta(t) * \delta_\Delta(t)$ and $r_\Delta(t) * r_\Delta(t)$, all act like impulses when applied to an LTI system (naturally there are an infinite number of signals for which this is true).

This suggests that we should think of the unit impulse as how an LTI system responds to it. While usually a function or signal is defined by what it is at each time point, the primary importance of the unit impulse is not that, but rather what it *does under convolution*. As such we may alternatively define the unit impulse as the signal which, when applied to an LTI system, yields the impulse response. That is, we define $\delta(t)$ as the signal for which

$$x(t) = x(t) * \delta(t)$$

for any $x(t)$. The signals that satisfy this property in the are all said to behave like a unit impulse.

All the previously mentioned properties of the unit impulse we need can be obtained from the *operational definition* given above. For instance, should we let $x(t) = 1$ for all t, then

$$\begin{aligned} 1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t) &= \int_{-\infty}^{+\infty} \delta(\tau) x(t-\tau) d\tau \\ &= \int_{-\infty}^{+\infty} \delta(\tau) d\tau \end{aligned}$$

showing that the unit impulse has unit area.
(next page)

Cont.

It is sometimes useful to use another completely equivalent operational definition of $\delta(t)$. To obtain this alternative form, consider taking an arbitrary signal $g(t)$, reversing it in time to obtain $g(-t)$, and then convolving this with $\delta(t)$:

$$g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{+\infty} g(\tau - t)\delta(\tau)d\tau$$

which, for $t = 0$, yields

$$g(0) = \int_{-\infty}^{+\infty} g(\tau)\delta(\tau)d\tau$$

This is another operational definition of the unit impulse: the unit impulse is the signal which, when multiplied by a signal $g(t)$ and then integrated from $-\infty$ to $+\infty$, produces the value $g(0)$.

See that the first operational definition defines this second one. On the other hand, this second definition also defines the first. To see this, let $x(t)$ be a given signal, fix a time t, and define

$$g(\tau) = x(t - \tau)$$

Then by the second definition we have

$$x(t) = g(0) = \int_{-\infty}^{+\infty} g(\tau)\delta(\tau)d\tau = \int_{-\infty}^{+\infty} x(t - \tau)\delta(\tau)d\tau$$

which leads us back to the first operational definition. Therefore the second operational definition above is an equivalent operational definition of the unit impulse.

The second operational definition won't be used as frequently as the first, but it does have some use in determining some other properties of the unit impulse. For instance, consider that for some signal $f(t)$:

$$\int_{-\infty}^{+\infty} g(\tau)f(\tau)\delta(\tau)d\tau = g(0)f(0)$$

On the other hand, also see that

$$\int_{-\infty}^{+\infty} g(\tau)f(0)\delta(\tau)d\tau = g(0)f(0)$$

Consequently, we can conclude that

$$f(t)\delta(t) = f(0)\delta(t)$$

which is a property we derived earlier.

2.12 Utility of singularity functions

Unit doublet as the impulse response for the differentiatior
The unit impulse is one of a class of signals known as *singularity functions*, each of which can be defined operationally in terms of its behaviour under convolution. Consider the LTI system for which the output is the derivative of the input:

$$y(t) = \frac{dx(t)}{dt}$$

The unit impulse response here is the derivative of the unit impulse, which is called the *unit doublet* $u_1(t)$. See that when represented as a convolution, we have

$$\frac{dx(t)}{dt} = x(t) * u_1(t)$$

for any signal $x(t)$. Just as we had an operational definition of $\delta(t)$, we take the above to be the operational definition of the unit doublet $u_1(t)$.

Similarly, we can define $u_2(t)$, the second derivative of $\delta(t)$, as the impulse response of an LTI system that takes the second derivative of the input:

$$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t)$$

Furthermore see that

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = x(t) * u_1(t) * u_1(t)$$

and therefore that

$$u_2(t) = u_1(t) * u_1(t)$$

See now that more generally, $u_k(t)$, $k > 0$, is the kth derivative of $\delta(t)$ and thus is the impulse response of a system that takes the kth derivative of the input. Since this system can be obtained as a cascade of k differentiators, we have

$$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{k \text{ times}}$$

(next page)

Properties of the unit doublet

As with the unit impulse, each of these singularity functions has properties that can be derived from their operational definition. For example, if we consider the constant signal $x(t) = 1$, we find

$$\begin{aligned} 0 = \frac{dx(t)}{dt} = x(t) * u_1(t) &= \int_{-\infty}^{+\infty} u_1(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} u_1(\tau)d\tau \end{aligned}$$

so that the unit doublet has zero area. Moreover, if we convolve the signal $g(-t)$ with $u_1(t)$, we obtain

$$\int_{-\infty}^{+\infty} g(\tau - t)u_1(\tau)d\tau = g(-t) * u(t) = \frac{dg(-t)}{dt} = -g'(-t)$$

which for $t = 0$ yields

$$-g'(0) = \int_{-\infty}^{+\infty} g(\tau)u_1(\tau)d\tau$$

In an analagous manner, we can derive related properties of $u_1(t)$ and higher order singularity functions.

(next page)

Intuition

As with the unit impulse, each of these singularity functions can be informally related to short pulses. For instance, since the unit doublet is formally the derivative of the unit impulse, we can think of it as the idealisation of the derivative of a short pulse with unit area. For instance, consider a short pulse $\delta_\Delta(t)$ as defined before; this pulse behaves like an impulse as $\Delta \to 0$. Consequently, we would expect its derivative to behave like a doublet as $\Delta \to 0$. Illustrated,

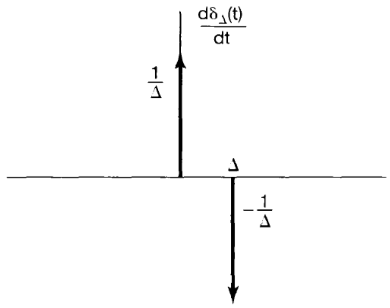


Figure 2.36 The derivative $d\delta_\Delta(t)/dt$ of the short rectangular pulse $\delta_\Delta(t)$ of Figure 1.34.

It consists of a unit impulse at $t = 0$ with area $+1/\Delta$, followed by another unit impulse of area $-1/\Delta$ at $t = \Delta$;

$$\frac{d\delta_\Delta(t)}{dt} = \frac{1}{\Delta}[\delta(t) - \delta(t - \Delta)]$$

Consequently, using the fact that $x(t) * \delta(t - t_0) = x(t - t_0)$, we have

$$x(t) * \frac{d\delta_\Delta(t)}{dt} = \frac{x(t) - x(t - \Delta)}{\Delta} \approxeq \frac{dx(t)}{dt}$$

where the approximation becomes increasingly accurate as $\Delta \to 0$. See therefore that $d\delta_\Delta(t)/dt$ does indeed behave like a unit doublet as $\Delta \to 0$.
(next page)

Unit step as the impulse response for the integrator
In addition to singularity functions that are derivatives of the unit impulse, we can also define signals that represent successive integrals of the unit impulse function. Recall that the unit step is the impulse response of an integrator, where we have:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

so

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

and we therefore have the operational definition of $u(t)$:

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

As with the differentiator, we can define a system that consists of a cascade of two integrators. Its impulse response is denoted by $u_{-2}(t)$, which is simply the convolution of $u(t)$ (the response of one integrator) with itself:

$$u_{-2} = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

Since $u(t)$ equals 0 for $t < 0$ and 1 for $t > 0$, it follows that

$$u_{-2} = tu(t)$$

This signal is referred to as the *unit ramp function*:

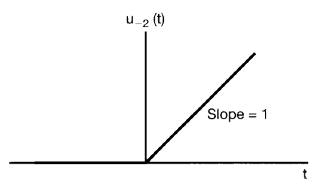


Figure 2.37 Unit ramp function.

We can obtain the operational definition for $u_{-2}(t)$ as

$$\begin{aligned} x(t) * u_{-2}(t) &= x(t) * u(t) * u(t) \\ &= \left(\int_{-\infty}^{t} x(\sigma)d\sigma \right) * u(t) \\ &= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma)d\sigma \right) d\tau \end{aligned}$$

(next page)

Cont.
In an analagous fashion, we can define higher order integrals of $\delta(t)$ as the impulse responses of cascades of integrators:

$$u_{-k}(t) = \underbrace{u(t) * \cdots * u(t)}_{k \text{ times}} = \int_{-\infty}^{t} u_{-(k-1)}(\tau) d\tau$$

The convolution of $x(t)$ with $u_{-3}(t), u_{-4}(t), \ldots$ generate correspondingly higher order integrals of $x(t)$. Also see that these functions can be evaluated directly to obtain

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t)$$

Unlike the derivatives of $\delta(t)$, the successive integrals of the unit impulse are functions that can be defined for each T, as well as their behaviour under convolution.

Combined
It is useful to consider an alternative notation for $\delta(t)$ and $u(t)$, namely

$$\delta(t) = u_0(t)$$
$$u(t) = u_{-1}(t)$$

With this notation, $u_k(t)$ for $k > 0$ denotes the impulse response of a cascade of k differentiators, $u_0(t)$ is the impulse response for the identity system; and for $k < 0$, $u_k(t)$ is the impulse response of a cascade of $|k|$ integrators. Furthermore, since a differentiator is the inverse system of an integrator,

$$u_{-1}(t) * u_1(t) = u_0(t)$$

More generally, see that for any integers k and r,

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

If k and r are both positive, it represents a cascade of k differentiatiors followed by r more differentiators, yielding an output that is the $(k+r)$th derivative of the input. Similarly, if k is negative and r is negative, we have a cascade of $|k|$ integrators followed by another r integrators.

If k is negative and r is positive, we have a cascade of $|k|$ integrators followed by r differentiators, and the overall system will be equivalent to a cascade of $|k+r|$ integrators if $k+r < 0$, a cascade of $k+r$ differentiators if $k+r > 0$, or the identity system if $k+r = 0$.

By defining singularity functions in terms of their behavior under convolution, we obtain a characerisation that allows us to manipulate them with relative ease and to interpret them directly in terms of their significance for LTI systems.

Chapter 3

Fourier series representation of periodic signals

3.1 The response of LTI systems to complex exponentials

Recall that complex exponential signals are represented as e^{st} in continuous time and z^n in discrete time, where s and z are complex numbers. One motivation for their use in the study of LTI systems stems from the idea that the response of an LTI system to a complex exponential input is the *same complex exponential with only a change in amplitude*; that is

$$\begin{aligned} \text{continuous time: } & e^{st} \to H(s)e^{st} \\ \text{discrete time: } & z^n \to H(z)z^n \end{aligned}$$

Where the complex amplitude factor $H(s)$ or $H(z)$ will in general be a function of the complex variable s or z.

A signal for which the system output is a (possibly complex) constant multiplied by the input is referred to as an *eigenfunction* of the system, and the amplitude factor is referred to as the systen's *eigenvalue*.
(next page)

Intuition
To show that complex exponentials are indeed eigenfunctions of LTI systems, let us consider a continuous-time LTI system with impulse response $h(t)$. For an input $x(t)$, we can determine the output using the convoltution integral, where with $x(t) = e^{st}$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \end{aligned}$$

See that this becomes

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Where the response to e^{st} is of the form

$$y(t) = H(s)e^{st}, \quad H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

See that $H(s)$ is a complex constant whose value depends on s. The constant $H(s)$ is the eigenvalue associated with the eigenfunction e^{st}.

In the same way, we can show that complex exponential sequences are eigenfunctions of discrete-time LTI systems. Supposing an LTI system with impulse response $h[n]$ with input

$$x[n] = z^n$$

where z is a complex number. Then the output of the system can be determined from the convolution sym as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \end{aligned}$$

Assuming that the summation on the right-hand side converges, the output is the same complex exponential multiplied by a constant that depends on z, that is

$$y[n] = H(z)z^n, \quad H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

The constant $H(z)$ for a specific value of z is the eigenvalue associated with the eigenfunction z^n.
(next page)

Utility

The usefulness of decomposing more general signals in terms of eigenfunctions can be seen from an example. Let $x(t)$ correspond to a linear combination of three complex exponentials; that is

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

Due to the fact that complex exponentials act as eigenfunctions, the response to each separately is

$$\begin{aligned} a_1 e^{s_1 t} &\to a_1 H(s_1) e^{s_1 t} \\ a_2 e^{s_2 t} &\to a_2 H(s_2) e^{s_2 t} \\ a_3 e^{s_3 t} &\to a_3 H(s_3) e^{s_3 t} \end{aligned}$$

and by superposition, see that the response is

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

More generally, the eigenfunction property together with superposition implies that the representation of signals as a linear combination of complex exponentials leads to a convenient expression for the response of an LTI system. See that an input to a continuous-time LTI system represented as a linear combination of complex exponentials:

$$x(t) = \sum_k a_k e^{s_k t}$$

will yield the output

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

Similarly for discrete-time LTI systems, the input

$$x[n] = \sum_k a_k z_k^n$$

will yield the output

$$y[n] = \sum_k a_k H(z_k) z_k^n$$

In both continuous and discrete time, if the input to an LTI system is represented as a linear combination of complex exponentials, then the output can also be represented as a linear combination of *the same exponential signals*; each coefficient in this representation of the output is obtained as the product of the corresponding coefficient a_k of the input and the system's eigenvalue $H(s_k)$ or $H(z_k)$ associated with the eigenfunction $e^{s_k t}$ or z_k^t respectively.

We will focus primarily on purely imaginary values of s, meaning $s = i\omega$, and therefore only complex exponentials of the form $e^{i\omega t}$. Similarly in discrete time we restrict z to be of unit magnitude, meaning $z = e^{i\omega}$, and therefore only complex exponentials of the form $e^{i\omega n}$.

3.2 Linear combinations of harmonically related complex exponentials

As defined earlier, a signal is periodic if, for some positive value of T,

$$x(t) = x(t+T) \quad \text{for all } t$$

Recall that the *fundamental period* of $x(t)$ is the minimum positive, nonzero value of T for which the above is satisfied, and therefore the value $\omega_0 = 2\pi/T$ is referred to as the *fundamental frequency.*

Recall the sinusoidal signal

$$x(t) = \cos\omega_0 t$$

and the periodic complex exponential

$$x(t) = e^{i\omega_0 t}$$

See that both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi/\omega_0$. Associated with the complex exponential is the set of *harmonically related* complex exponentials:

$$\phi_k(t) = e^{ik\omega_0 t} = e^{ik(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Each of these signals has a fundamental frequency that is a multiple of ω_0, and threfore, each is periodic with period T (see that for $|k| \geq 2|$, the fundamental period of $\phi_k(t)$ is a fraction of T).

A linear combination of harmonically related complex exponentials is of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{ik(2\pi/T)t}$$

See that this is *also periodic with period* T. The term for $k = 0$ is a constant.

The terms for $k = +1$ and $k = -1$ both have fundamental frequency equal to ω_0 and are collectively referred to as the *fundamental* or *first harmonic components.* The terms for $k = +2$ and $k = -2$ are periodic with half the period (or twice the frequency) of the fundamental components and are referred to as the *second harmonic components.* More generally, the components for $k = +N$ and $k = -N$ are referred to as the Nth harmonic components.

The representation of a periodic signal in this form is referred to as the *Fourier series* representation.

3.3 Fourier series representation of real periodic signals

Recall the *fourier series representation*

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{ik(2\pi/T)t}$$

Suppose that $x(t)$ is real and and can be represented in the form of the fourier series as above (note that a is complex by default). Then since $x^*(t) = x(t)$ (where x^* refers to the complex conjugate), we obtain

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-ik\omega_0 t}$$

(we can show that $\left((a+bi)e^{i\omega_0 t}\right)^* = (a+bi)^* e^{-i\omega_0 t}$ by just working the formulas out) Replacing k by $-k$ by summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{ik\omega_0 t}$$

which by comparison with the original fourier series representation, gives us the result $a_{-k}^* = a_k$, or equivalently, that

$$a_k^* = a_{-k}$$

(see that if we were to force the coefficients a to be real then we would just have $a_k = a_{-k}$, since the real parts are always equal)

In this case we can derive an alternate form of the fourier series; we can arrange the fourier series representation to

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{ik\omega_0 t} + a_{-k} e^{-ik\omega_0 t}]$$

and by substituting $a_k^* = a_{-k}$ we have

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{ik\omega_0 t} + a_k^* e^{-ik\omega_0 t}]$$

Since the two terms in the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left\{a_k e^{ik\omega_0 t}\right\}$$

(next page)

Cont.

We had an alternative representation for the fourier series representation of real periodic signals as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left\{a_k e^{ik\omega_0 t}\right\}$$

If a_k were to be expressed in polar form as

$$a_k = A_k e^{i\theta_k}$$

then we have

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left\{A_k e^{i(k\omega_0 t + \theta_k)}\right\}$$

That is

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

This is a commonly encountered form for fourier series of real periodic signals in continuous time. Another form can be obtained by writing a_k in rectangular form as

$$a_k = B_k + iC_k$$

where $B_k$and C_k are both real. This leads to the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)]$$

3.4 Determination of the fourier series representation of a continuous-time periodic signal

Assuming a signal can be represented as a fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{ik(2\pi/T)t}$$

We need a procedure for determining the coefficients a_k. Multiplying both sides by $e^{-in\omega_0 t}$, we obtain

$$x(t)e^{-in\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t} e^{-in\omega_0 t}$$

Integrating both sides from 0 to $T = 2\pi/\omega_0$, we have

$$\int_0^T x(t)e^{-in\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t} e^{-in\omega_0 t}dt$$

Interchanging the order of integration and summation (distributivity of the integral) yields

$$\int_0^T x(t)e^{-in\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{i(k-n)\omega_0 t}dt\right]$$

We want to evaluate the integral inside the square brackets. See that we have it set up so that T is the fundamental period of $x(t)$, and we are consequently integrating over one period. Rewriting the integral using Euler's formula we obtain

$$\int_0^T e^{i(k-n)\omega_0 t}dt = \int_0^T \cos((k-n)\omega_0 t)dt + i\int_0^T \sin((k-n)\omega_0 t)dt$$

See that for $k \neq n$, both $\cos((k-n)\omega_0 t)$ and $\sin((k-n)\omega_0 t)$ are periodic sinusoids with fundamental period $(T/|k-n|)$, and that since we integrate over an interval that is a period for all the sinusoids, for $k \neq n$ both of the integrals on the right hand side are zero. For $k = n$, the integral evaluates to 1, so we end up with

$$\int_0^T e^{i(k-n)\omega_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

(next page)

Cont.
We had
$$\int_0^T x(t)e^{-in\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{i(k-n)\omega_0 t}dt\right]$$
and
$$\int_0^T e^{i(k-n)\omega_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$
See that the first equation reduces to
$$\int_0^T x(t)e^{-in\omega_0 t}dt = a_n T$$
and so we have an equation for determining the coefficients as
$$a_n = \frac{1}{T}\int_0^T x(t)e^{-in\omega_0 t}dt$$
Note that the only fact we used concerning the interval of integration was that we were integrating over an interval of length T, which is an integral number of periods of $\cos((k-n)\omega_0 t)$ and $\sin((k-n)\omega_0 t)$. See that we would obtain the same result if we integrate over *any* interval of length T; that is, if denoted by $\int_T$, we have
$$\int_T e^{i(k-n)\omega_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$
and consequently
$$a_n = \frac{1}{T}\int_T x(t)e^{-in\omega_0 t}dt$$
Summary
To summarise, if $x(t)$ has a fourier series representation
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{ik(2\pi/T)t}$$
then the coefficients are given by
$$a_k = \frac{1}{T}\int_T x(t)e^{-ik\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-ik(2\pi/T)t}dt$$
Here we have written equivalent expressions for the fourier series in terms of the fundamental frequency ω_0 and the fundamental period T. The first equation is referred to as the *synthesis* equation and the second the *analysis* equation. The set of coefficients $\{a_k\}$ are often called the *fourier series coefficients* or *spectral coefficients* of $x(t)$, measuring the portion of the signal $x(t)$ that is at each harmonic of the fundamental component.

3.5 Convergence of fourier series

There are two somewhat different classes of conditions that a periodic signal can satisfy to gaurantee it can be represented by a fourier series. We will not attempte to provide a complete mathematical justification here.

One class of periodic signals that are representable through the fourier series are those signals which have finite energy over a single period, meaning

$$\int_T |x(t)|^2 dt < \infty$$

when this condition is satisfied, we can guarantee that the coefficients a_k obtained are finite. Furthermore, letting $x_N(t)$ denote the approximation to $x(t)$ obtained from using the coefficients for $|k| \leq N$,

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{ik\omega_0 t}$$

we can gaurantee that the energy E_N of the approximation error $e(t)$, defined as

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{ik\omega_0 t}$$

and

$$E_N = \int_T |e_N(t)|^2 dt$$

converges to 0 as $N \to \infty$. That is, if

$$e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t} \quad \text{then} \quad \int_T |e(t)|^2 dt = 0$$

Note that this does *not* imply that the signal $x(t)$ and its fourier series representation

$$\sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t}$$

are equal at every value of t, only that there is no energy in their difference. Because most periodic signals that one might consider do have finite energy over a single period, they have fourier series representations.
(next page)

Dirichlet conditions
An alternative set of conditions, the Dirichlet conditions, gaurantees that $x(t)$ *equals* its fourier series representation, except at isolated values of t, for which $x(t)$ is discontinuous. At these values, the infinite series converges to the *average* of the values on either side of the discontinuity. They are as follows:

Condition 1. Over any period, $x(t)$ must be *absolutely integrable*; that is,

$$\int_T |x(t)| dt < \infty$$

See that this gaurantees each coefficient a_k will be finite, since

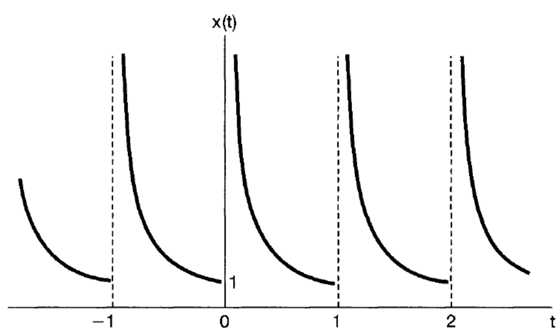
$$|a_k| \leq \frac{1}{T}\int_T |x(t)e^{-ik\omega_0 t}| dt = \frac{1}{T}\int_T |x(t)||e^{-ik\omega_0 t}| dt = \frac{1}{T}\int_T |x(t)| dt$$

so

$$|a_k| \leq \frac{1}{T}\int_T |x(t)| dt < \infty$$

An example of a periodic signal that violates this might be

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

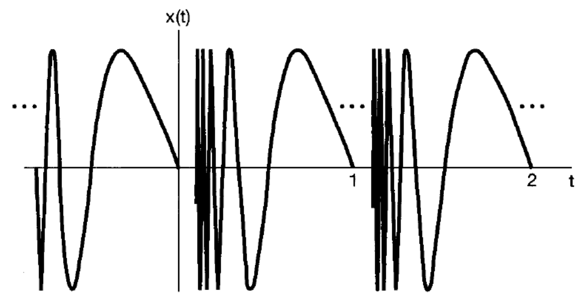


which is periodic wtih period 1, but has infinite energy over the period.
(next page)

Condition 2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

An example of a function that meets condition 1 but not 2 is

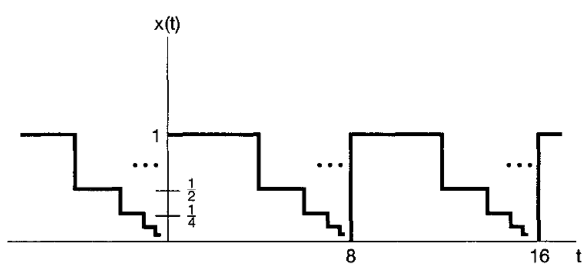
$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1$$



which has an infinite number of maxima and minima in a period.

Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

An example of this might be a signal that looks like



where the signal of period 8 is composed of an infinite number of secitons, each half the height and half the width of the previous section. Even though the area under one period is clearly less than 8, there are an infinite number of discontinuities in each period.

As one might notice, the signals that do not satisfy the Dirichlet conditions tend to be pathological in nature and consequently do not typically arise in practical contexts.
(next page)

Dirchlet conditions and discontinuities

For a periodic signal that has no discontinuities, the fourier series representation converges and equals the original signal at every value of t. For a periodic signal with a finite number of discontinuities in each period, the fourier series representation equals the signal everywhere except at the isolated points of discontinuity, at which the series converges to the average value of the signal on either side of the discontinuity.

In this case the difference between the original signal and its fourier series representation contains no energy, and consequently the two can be thought of as being the same for all practical purposes. Specifically, since the signals differ only at isolated points, the integrals of both signals over any interval *are* identical. For this reason, the two signals behave identically under convolution and are consequently identical in the context of LTI system analysis.

Consider the example of the square wave, where the partial sum $x_N(t)$ for different N are superimposed onto the original square wave.

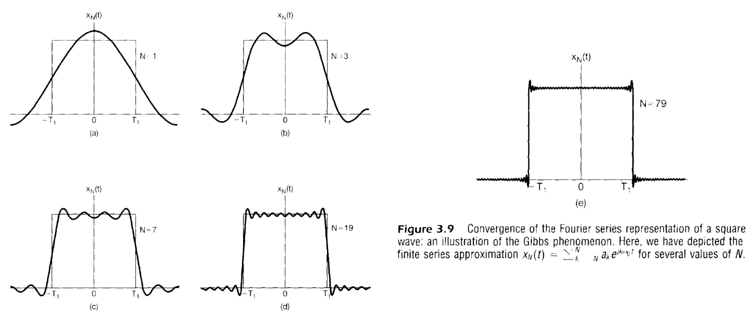


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

Since the square wave satisfies the Dirichlet conditions, the limit as $N \to \infty$ at of x_N at the discontinuities should be the the average value of the discontinuity. See that this is indeed the case, where for any N, x_N has exactly that value at the discontinuities. Furthermore, for any value of t, we are gauranteed that

$$\lim_{N\to\infty} x_N(t) = x(t)$$

However notice the behaviour of the partial sum in the vicinity of the discontinuity exhibits 'ripples' that don't appear to decrease with increasing N. One must be careful to interpret this correctly; for any *fixed* t, the partial sums will converge to the correct value, and at the discontinuity they converge to half the sum of the values of the signal on either side of the discontinuity. However, the closer t is chosen to the point of discontinuity, the larger N must be in order to reduce the error below a specified amount; as N increases, the ripples must in the partal sums become compressed toward the discontinuity, but for *any* finite value of of N, the peak amplitude of the ripples remains constant.
(next page)

Cont.

This behaviour is known as the *Gibbs phenomenon*; a *truncated* fourier series approximation $x_N(t)$ of a discontinuous signal $x(t)$ will in general exhibit high-frequency ripples and overshoot $x(t)$ near the discontinuities. If such an approximation is used in practice, a large enough value of N should be chosen so as to gaurantee that the total energy in these ripples is insignificant.

3.6 Properties of continuous-time fourier series 1

We will find it convenient to use a shorthand notation to indicate the relationship between a periodic signal and its fourier series coefficients. Specifically, suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the fourier series coefficients of $x(t)$ are denoted by a_k, we will use the notation

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$$

To signify a pairing of a periodic signal with its fourier series coefficients.

Linearity
Letting $x(t)$ and $y(t)$ denote two periodic signals of period T with fourier series coefficients denoted by a_k and b_k respectively:

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$$
$$y(t) \overset{\text{FS}}{\longleftrightarrow} b_k$$

Since $x(t)$ and $y(t)$ have the same period T, any linear combination of the signals will also be periodic with period T. Further, see that the fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination of the fourier series coefficients for $x(t)$ and $y(t)$:

$$z(t) = Ax(t) + By(t) \overset{\text{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$$

The proof of this follows directly from application of the fourier series coefficient formula. Note that this linearity property is easily extended to a linear combination of an arbitrary number of signals with period T.
(next page)

Time shifting
When a time shift is applied to a periodic signal $x(t)$, the period T of the signal is preserved. See that the fourier series coefficients B_k of the resulting signal $y(t) = x(t - t_0)$ may be expressed as

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-ik\omega_0 t}dt$$

Letting $\tau = t - t_0$ in the integral, and noting that the new variable τ will also range over an interval of duration T, we obtain

$$\begin{aligned}\frac{1}{T}\int_T x(\tau)e^{-ik\omega_0(\tau+t_0)}d\tau &= e^{-ik\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-ik\omega_0\tau}d\tau \\ &= e^{-ik\omega_0 t_0}a_k = e^{-ik(2\pi/T)t_0}a_k\end{aligned}$$

where a_k is the kth fourier series coefficient of $x(t)$. That is, if

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$$

then

$$x(t - t_0) \overset{\text{FS}}{\longleftrightarrow} e^{-ik\omega_0 t_0}a_k = e^{-ik(2\pi/T)t_0}a_k$$

See that time shifting a periodic signal doesn't alter the *magnitudes* of its fourier series coefficients, as $|b_k| = |a_k|$.

Time Reversal
The period T of a periodic $x(t)$ also remains unchanged when the signal undergoes a time reversal. To determine the fourier series of $y(t) = x(-t)$, consider the effect of the time reversal on the synthesis equation

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-ik2\pi t/T}$$

Making the substitution $k = -m$, we obtain

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m}e^{im2\pi t/T}$$

we observe that the right-hand side of this equation has the form of a fourier series synthesis equation, and therefore that the fourier series coefficients b_k are

$$b_k \overset{\text{FS}}{\longleftrightarrow} a_{-k}$$

that is, if

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$$

then

$$x(-t) = y(t) \overset{\text{FS}}{\longleftrightarrow} b_k = a_{-k}$$

(next page)

Cont.
we had

$$x(-t) = y(t) \overset{\text{FS}}{\longleftrightarrow} b_k = a_{-k}$$

In other words, time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of fourier series coefficients.

An interesting consequence of this is that if $x(t)$ is even, meaning $x(-t) = x(t)$, then its fourier series coefficients are also even, so $a_{-k} = a_k$. Similarly, if $x(t)$ is odd, meaning $x(-t) = -x(t)$, then its fourier series coefficients are also odd, so $a_{-k} = -a_k$.

Time Scaling
Time scaling, in general, changes the period of the underlying signal. Specifically, if $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

Since the time-scaling operation applies directly to each of he harmonic components of $x(t)$, we can conclude that the fourier coefficients remain the same, and so

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{ik(\alpha\omega_0)t}$$

is the fourier series representation $x(\alpha t)$. Note that although the fourier coefficients have not changed, the fourier series representation *has changed* because of the change in fundamental frequency.

3.7 Properties of continuous-time fourier series 2

Multiplication

Supposing that $x(t)$ and $y(t)$ are both periodic with period T and that

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$$
$$y(t) \overset{\text{FS}}{\longleftrightarrow} b_k$$

Since the product $x(t)y(t)$ is also periodic with period T, we can expand it in a fourier series with fourier coefficients expressed in terms of those of $x(t)$ and $y(t)$. The result is

$$x(t)y(t) \overset{\text{FS}}{\longleftrightarrow} h_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

We can show this. See that $x(t)(y(t)$ is given by

$$x(t)y(t) = \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_n e^{in\omega_0 t} b_l e^{il\omega_0 t} = \sum_n \sum_l a_n b_l e^{i(n+l)\omega_0 t}$$

see that the kth fourier coefficient of the product is given by

$$\begin{aligned} &\frac{1}{T}\int_T \sum_n \sum_l a_n b_l e^{i(n+l)\omega_0 t} e^{-ik\omega_0 t} dt \\ =&\frac{1}{T}\sum_n \sum_l a_n b_l \int_T e^{i(n+l-k)\omega_0 t} dt \end{aligned}$$

See that the integral part of this equation, over the range of a period, is easily solved as

$$\int_T e^{i(n+l-k)\omega_0 t} dt = \begin{cases} T, & \text{if } n+l-k=0 \\ 0, & \text{otherwise} \end{cases}$$

and so the expression for the kth fourier coefficient can be rewritten as

$$\begin{aligned} &\frac{1}{T}\sum_n \sum_l a_n b_l (T\delta(n+l-k)) \\ =&\sum_n \sum_l a_n b_l \delta(n+l-k) \end{aligned}$$

Consider fixing n, and see that for fixed n and k, there is only one l in the second summation that would yield a nonzero term. (only one specific value of l, $l = k - n$, would make the delta function argument zero and yield a nonzero term) In that sense, we can simplify the expression to only when the delta function is nonzero (so for any given n, l can just be replaced by $k - n$), giving us the kth term as

$$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

(next page)

Conjugation and Conjugate Symmetry

Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding fourier series coefficients. That is, if

$$x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$$

then

$$x^*(t) \overset{\text{FS}}{\longleftrightarrow} a^*_{-k}$$

To show this, see that if we have $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t}$$

Taking the complex conjugate on both sides we have

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-ik\omega_0 t}$$

(we can show that $\left((a+bi)e^{i\omega_0 t}\right)^* = (a+bi)^* e^{-i\omega_0 t}$ by just working the formulas out) Substituting k for $-k$, we then have

$$x^*(t) = \sum_{k=-\infty}^{\infty} a^*_{-k} e^{ik\omega_0 t}$$

which proves the property.

Recall that an interesting consequence of this property may be derived for real $x(t)$, since that implies $x(t) = x^*(t)$; the fourier series coefficients will be *conjugate symmetric*:

$$a_{-k} = a_k^*$$

which also implies

$$a_k = a^*_{-k}$$

(next page)

Parseval's Relation for Continuous-time Periodic Signals
Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

where the a_k are the fourier series coefficients of $x(t)$ and T is the period of the signal.

Intuition
This can be proved by finding the fourier coefficients of the product $x(t)x^*(t) = |x(t)|^2$, should we have the fourier series representations of the following as

$$x(t) = \sum_{l=-\infty}^{\infty} a_l e^{il\omega_0 t} dt$$
$$x^*(t) = \sum_{n=-\infty}^{\infty} a_n^* e^{-in\omega_0 t} dt$$

we have their product as

$$x(t)x^*(t) = \sum_l \sum_n a_l a_n^* e^{i(l-n)\omega_0 t}$$

We obtain the fourier coefficients of this product c_k in a similar manner to how we derived the multiplication property

$$\begin{aligned} c_k &= \frac{1}{T}\int_T \sum_l \sum_n a_l a_n^* e^{i(l-n)\omega_0 t} e^{-ik\omega_0 t} dt \\ &= \frac{1}{T}\sum_l \sum_n a_l a_n^* \underbrace{\int_T e^{i(l-n-k)\omega_0 t} dt}_{=\begin{cases} T, & l-n-k=0 \\ 0, & \text{otherwise} \end{cases}} \end{aligned}$$

so this simplifies to

$$\sum_l \sum_n a_l a_n^* \delta(l-n-k) = \sum_n a_{n+k} a_n^*$$

so we have

$$c_k = \frac{1}{T}\int_T |x(t)|^2 e^{-ik\omega_0 t} dt = \sum_n a_{n+k} a_n^*$$

and setting $k=0$ we have

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_n a_n a_n^* = \sum_n |a_n|^2$$

(next page)

Cont.
We have parseval's relation for continuous-time periodoc signals as

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Note that the left hand side of the equation is the average power (energy per unit time) in one period of the periodic signal $x(t)$. Also,

$$\frac{1}{T}\int_T \left|a_k e^{ik\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

see that $|a_k|^2$ is the average power of the kth harmonic component of $x(t)$. Thus, what Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Periodic convolution
We can show that for

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{in\omega_0 t}, \quad x(t) \overset{\mathrm{FS}}{\longleftrightarrow} a_k$$
$$y(t) = \sum_{l=-\infty}^{+\infty} b_l e^{il\omega_0 t}, \quad y(t) \overset{\mathrm{FS}}{\longleftrightarrow} b_k$$

that

$$z(t) = \int_T x(\tau)y(t-\tau)d\tau \overset{\mathrm{FS}}{\longleftrightarrow} Ta_k b_k$$

To show this, see that

$$\begin{aligned}
\int_T x(\tau)y(t-\tau)d\tau &= \int_T \sum_n a_n e^{in\omega_0\tau} \sum_l b_l e^{il\omega_0(t-\tau)} d\tau \\
&= \int_T \sum_n \sum_l a_n b_l e^{i(n-l)\omega_0\tau} e^{il\omega_0 t} d\tau \\
&= \sum_n \sum_l a_n b_l e^{il\omega_0 t} \int_T e^{i(n-l)\omega_0\tau} d\tau \\
&= \sum_n \sum_l a_n b_l e^{il\omega_0 t} (T\delta(n-l)) \\
&= \sum_n T a_n b_n e^{in\omega_0 t}
\end{aligned}$$

which proves our expression.
(next page)

Frequency shifting

We can show that for

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{in\omega_0 t}, \quad x(t) \overset{\text{fs}}{\longleftrightarrow} a_k$$

we have

$$x(t)e^{iM\omega_0 t} \overset{\text{fs}}{\longleftrightarrow} a_{k-M}$$

Showing this is simple:

$$\begin{aligned} x(t)e^{iM\omega_0 t} &= \sum_{n=-\infty}^{+\infty} a_n e^{in\omega_0 t} e^{iM\omega_0 t} \\ &= \sum_{n=-\infty}^{+\infty} a_n e^{i(n+M)\omega_0 t} \end{aligned}$$

substituting $k = n + M$ gives us

$$\sum_{k=-\infty}^{+\infty} a_{k-M} e^{ik\omega_0 t}$$

which proves our expression.

3.8 Summary of properties (continuous)

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

3.9 Linear combinations of harmonically related complex exponentials (discrete)

As defined before, a discrete-time signal $x[n]$ is periodic with period N if

$$x[n] = x[n+N]$$

Where the fundamental period is the smallest positive integer N for which the equation holds, and $\omega_0 = 2\pi/N$ is the fundamental frequency. The set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{ik\omega_0 n} = e^{ik(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

All of these signals have fundamental frequencies that are multiples of $2\pi/N$ and thus are harmonically related.

However, recall that there are only N distinct signals in the ϕ, since discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. For instance $\phi_0[n] = \phi_N[n]$ and $\phi_1[n] = \phi_{N+1}[n]$; more generally

$$\phi_k[n] = \phi_{k+rN}[n]$$

That is, when k is changed by any integer multiple of N, an identical sequence is generated. This differs from the situation in continuous time in which the higher harmonics are all different from one another.

We want to consider the representation of more general periodic sequences in terms of linear combinations of the sequences $\phi_k[n]$. Such a linear combination has the form

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{ik\omega_0 n} = \sum_k a_k e^{ik(2\pi/N)n}$$

See that since the harmonics $\phi_k[n]$ are distinct only over a range of N sucessive values of k, the summation need only include terms over this range. Thus k varies over a range of N successive integers, beginning with any k. This is indicated by expressing the limits of the summation as $k = \langle N \rangle$. That is

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{ik\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{ik(2\pi/N)n}$$

(for example, k could range from $0, 1, \ldots, N-1$, or $3, 4, \ldots, N+2$—as long as there are N successive integers for N distinct harmonics—this is sufficient to incorporate all the harmonics into the summation.) This equation is referred to as the *discrete-time Fourier series* and the coefficients a_k as the *Fourier series coefficients*.

3.10 Determination of the fourier series representation of a discrete-time periodic signal

Suppose that we are given a sequence $x[n]$ that is periodic with fundamental period N. We want to determine the values of the fourier coefficients in the discrete-time fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{ik\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{ik(2\pi/N)n}$$

By following steps parallel to those used in continuous time, it is possible to obtain a closed-form expression for the coefficients a_k in terms of the values of the sequence $x[n]$. First see that

$$\sum_{n=\langle N\rangle} e^{ik(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Which states that the sum over one period of the values of a periodic constant exponential is zero, unless that complex exponential is a constant.

Consider the fourier series representation, multiplying both sides by $e^{-ir(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-ir(2\pi/N)n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{ik(2\pi/N)n} e^{-ir(2\pi/N)n}$$

which simplifies to

$$\sum_{n=\langle N\rangle} x[n]e^{-ir(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{i(k-r)(2\pi/N)n}$$

Consider the innermost sum over n on the right-hand side. It is zero unless $k-r$ is zero or an integer multiple of N. Therefore, if we choose values for r over the same range as that over which k varies in the outer summation, the innermost sum on the right-hand side equals N if $k = r$ and 0 for $k \neq r$. See that this reduces to

$$\sum_{n=\langle N\rangle} x[n]e^{-ir(2\pi/N)n} = a_r N$$

and so

$$a_r = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]e^{-ir(2\pi/N)n}$$

This provides a closed-from expression for the coefficients.
(next page)

Summary
To summarise,

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{ik\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{ik(2\pi/N)n}$$
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-ik\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-ik(2\pi/N)n}$$

Tne first equation is the *synthesis* equation and second the *analysis* equation. As in continuous time, the discrete-time fourier coefficients a_k are often referred to as the *spectral coefficients* of $x[n]$, specifying a decomposition of $x[n]$ into a sum of N harmonically related complex exponentials.

Fourier coefficients repeat periodically
See that if we take k in the range from 0 to $N-1$ we have

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \ldots + a_{N-1}\phi_{N-1}[n]$$

Similarly, if k ranges from 1 to N, we obtain

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \ldots + a_N\phi_N[n]$$

We know that $\phi_0[n] = \phi_N[n]$, and therefore we can conclude that $a_0 = a_N$. Similarly. by letting k range over any set of N consecutive integers, we can conclude that

$$a_k = a_{k+N}$$

If we consider more than N sequential values of k, the values a_k repeat periodically with period N. It is important to interpret this carefully; since there are only N distinct complex exponentials that are periodic with period N, the discrete-time fourier series representation is a finite series with N terms.

If we fix the N consecutive values of k over which we define the fourier series, we will obtain a set of exactly N fourier coefficients. However, at times one might find it convenient to use different sets of N values of k; as such it is sometimes convenient to think of a_k as a sequence defined for all k, *but where only N successive elements in the sequence will be used in the fourier series representation.*
(next page)

An alternate method

Another way of solving for the fourier coefficients may be by finding a solution for a set of linear equations. Specifically, if we evaluate the fourier series representation

$$x(t) = \sum_{k=\langle N \rangle} a_k e^{ik(2\pi/N)n}$$

for N successive values of n corresponding to one period of $x[n]$, we obtain

$$\begin{aligned}
x[0] &= \sum_{k=\langle N \rangle} a_k \\
x[1] &= \sum_{k=\langle N \rangle} a_k e^{ik(2\pi/N)} \\
&\vdots \\
x[N-1] &= \sum_{k=\langle N \rangle} a_k e^{ik(2\pi/N)(N-1)}
\end{aligned}$$

Then we have a set of N lienar equations for N unknown coefficients a_k as k ranges over a set of N successive integers. It can be shown that this set of equations is linearly independent and consequently can be solved to obtain the coefficients a_k in terms of the given values of $x[n]$.

3.11 Convergence in discrete-time representation

In discussing the convergence of the continuous-time fourier series before, we considere the example of a symmetric square wave and observed how the finite sum converged to the square wave as the number of terms approached infinity.

In particular, we observed the Gibbs phenomenon in the discontinuity, where as the number of terms increased, the ripples in the partial sum became compressed toward the discontinuity, with the peak amplitude of the rupples remaining constant independently of the number of terms in the partIal sum.

Let us consider the analagous sequence of partial sums for the discrete-time square wavel, where for convenience, we will assume has an odd period N; we have depicted the signals

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{ik(2\pi/N)n}$$

for several values of M (so partial sums):

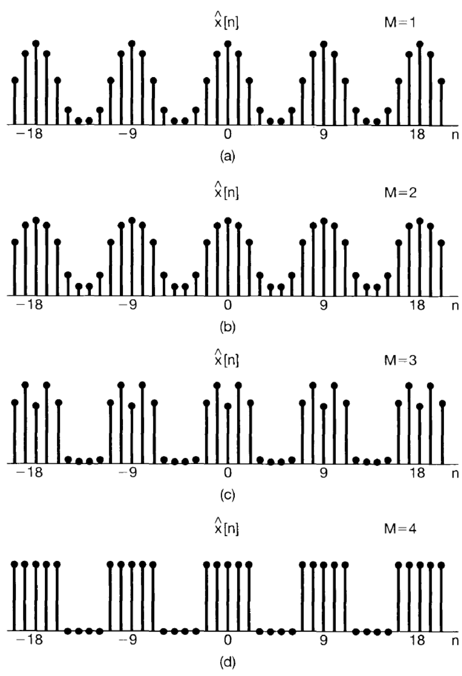


Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

(next page)

Cont.
See as per the graphs on the previous page, there is in contrast to the continuous-time case, there are no convergence issues and there is no Gibbs phenomenon.

Fact is that there are no convergence issues with the discrete-time fourier series in general. This stems from the fact that any discrete-time periodic sequence $x[n]$ is completely specified by a *finite* number of parameters, namely, the values of the sequence over one period.

The fourier series analysis equation simply transforms this set of N parameters into an equivalent set—the values of N fourier coefficients—and the synthesis equation tells us how to recover the original sequence in terms of a *finite* series.

If N is odd and we take $M = (N-1)/2$ as per

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{ik(2\pi/N)n}$$

the sum includes exactly N terms and we will have $\hat{x}[n] = x[n]$. Similarly if N is even and we let

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{ik(2\pi/N)n}$$

then with $M = N/2$ this sum consists of N terms and we can conclude that $\hat{x}[n] = x[n]$.

The main idea here is that a continuous-time periodic signal takes on a continuum of values over a single period, and thus an infinite number of fourier coefficients are required to represent it. As such *none* of the finite partial continuous fourier series representations yield exactly $x(t)$, leading to convergence issues.

3.12 Properties of discrete-time fourier series

Recall we had the synthesis and analysis equations of the discrete-time fourier series as

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{ik\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{ik(2\pi/N)n}$$
$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-ik\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-ik(2\pi/N)n}$$

The derivations of many properties are similar to those of the continuous-time counterpart. We use a shorthand notation to indicate the relationship between a periodic signal and its fourier series coefficients; specifically, if $x[n]$ is a periodic signal with period N and fourier series coefficients denoted by a_k, we write

$$x[n] \overset{\mathrm{FS}}{\longleftrightarrow} a_k$$

Multiplication
The multiplication property differs slightly in this case. For

$$x[n] \overset{\mathrm{FS}}{\longleftrightarrow} a_k$$
$$y[n] \overset{\mathrm{FS}}{\longleftrightarrow} b_k$$

both periodic with period N, the product $x[n]y[n]$ is also periodic with period N, with fourier coefficients c_k given by

$$x[n]y[n] \overset{\mathrm{FS}}{\longleftrightarrow} c_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

which is analagous to the continuous time case except that the summation is now restricted to an interval of N consecuitive samples. Due to the periodic nature of discrete-time fourier coefficients this can be taken over any set of N consecutive values of l.
(next page)

Cont.
Showing this is fairly simple; see that (first considering an N length interval from 0 to $N-1$)

$$x[n]y[n] = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_k b_l e^{i(k+l)\omega_0 n}$$

substituting $l' = k + l$

$$x[n]y[n] = \sum_{k=0}^{N-1} \sum_{l'=k}^{k+N-1} a_k b_{l'-k} e^{il'\omega_0 n}$$

Since both $b_{l'-k}$ and $e^{il'\omega_0 n}$ are periodic with period N we can rewrite this as

$$= \sum_{k=0}^{N-1} \sum_{l'=0}^{N-1} a_k b_{l'-k} e^{il'\omega_0 n} = \sum_{l=0}^{N-1} \left[\sum_{k=0}^{N-1} a_k b_{l-k} \right] e^{il\omega_0 n}$$

therefore

$$c_l = \sum_{k=0}^{N-1} a_k b_{l-k}$$

finally, by noting that both a_k and b_k are periodic with period N, we have

$$c_l = \sum_{k=\langle N \rangle} a_k b_{l-k}$$

First Difference
The discrete time parallel of the differentiation property of the continuous case involved the use of the first-difference operation. defined as $x[n] - x[n-1]$. Where for

$$x[n] \overset{\text{FS}}{\longleftrightarrow} a_k$$

then

$$x[n] - x[n-1] \overset{\text{FS}}{\longleftrightarrow} (1 - e^{-ik\omega_0}) a_k$$

This can be easily derived by applying the time-shifting and linearity properties, which are proved in a manner analagous to their continuous-time counterparts. (next page)

Parseval's relation for Discrete-time periodic signals
Parseval's relation for discrete-time periodic signals is given by

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 = \sum_{k=\langle N\rangle}|a_k|^2$$

where the a_k are the fourier series coefficients of $x[n]$ and N is the period. As in the continuous-time case, this states that the average power in a periodic signal equals the sum of the average powers in all its harmonic components. In discrete time there are only N distinct harmonic components, and since the a_k are periodic with period N, the sum of the right-hand side can be taken over any N consecutive values of k.

Showing this comes from finding the fourier coefficients of $x[n]x^*[n] = |x[n]|^2$. Using the analysis equation, the kth fourier coefficient c_k is given by

$$c_k = \frac{1}{N}\sum_{n=\langle N\rangle}x[n]x^*[n]e^{-ik\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 e^{-ik\omega_0 n}$$

for $k = 0$ we have

$$= \frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2$$

See that we have

$$x[n] \overset{\text{FS}}{\longleftrightarrow} a_k$$
$$x^*[n] \overset{\text{FS}}{\longleftrightarrow} a^*_{-k}$$

Now from the multiplication property, where

$$x[n]y[n] \overset{\text{FS}}{\longleftrightarrow} \sum_{l=\langle N\rangle}a_l b_{k-l}$$

we have

$$x[n]x^*[n] \overset{\text{FS}}{\longleftrightarrow} \sum_{l=\langle N\rangle}a_l a^*_{l-k}$$

(think of the fourier coefficient of the conjugate signal that scaled the $k-l$th harmonic) where for $k = 0$ we have

$$x[n]x^*[n] \overset{\text{FS}}{\longleftrightarrow} \sum_{l=\langle N\rangle}a_l a^*_l = \sum_{k=\langle N\rangle}|a_k|^2$$

and so this proves the expression

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 = \sum_{k=\langle N\rangle}|a_k|^2$$

3.13 Summary of properties (discrete)

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$$

3.14 Fourier series and LTI systems

Recall that the response of an LTI system to a linear combination of complex exponentials takes a particularly simple form. In continuous time, if $x(t) = e^{st}$ (where s is complex) is the input to a continuous-time LTI system, then the output is given by $y(t) = H(s)e^{st}$, where

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau$$

in which $h(t)$ is the impulse response of the LTI system. Similarly, if $x[n] = z^n$ (where z is complex) is the input to a discrete-time LTI system, then the output is given by $y[n] = H(z)z^n$, where

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

where $h[n]$ is the impulse response of the LTI system.

As a result of the superposition property of LTI systems, supposing a periodic input $x(t)$ given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{ik\omega_0 t}$$

is applied to an LTI system with impulse response $h(t)$, then since each complex exponential is an eigenfunction of the system it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(ik\omega_0)e^{ik\omega_0 t}$$

Thus $y(t)$ is also periodic with the same fundamental frequency as $x(t)$. Furthermore, see that if $\{a_k\}$ is the set of fourier coefficients for the input $x(t)$, then $\{a_k H(ik\omega_0)\}$ is the set of coefficients for the output $y(t)$. All the LTI system does is modify each coefficient.

3.15

Appendix A

Other proofs

A.1 Sum of geometric series

Here we show

$$S_n = \sum_{i=0}^{n-1} \alpha^i = \frac{\alpha^n - 1}{\alpha - 1}$$

This can be seen from

$$\begin{aligned}
(\alpha - 1)S_n &= \alpha \sum_{i=0}^{n-1} \alpha^i - \sum_{i=0}^{n-1} \alpha^i \\
&= \sum_{i=1}^{n} \alpha^i - \sum_{i=0}^{n-1} \alpha^i \\
&= \alpha^n + \sum_{i=1}^{n-1} \alpha^i - (1 + \sum_{i=1}^{n-1} \alpha^i) \\
&= \alpha^n - 1
\end{aligned}$$

and so

$$S_n = \frac{\alpha^n - 1}{\alpha - 1}$$