Signals And Systems by Alan V. Oppenheim: Notes

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Started 17 April 2025

Contents

Chapter 1

Introduction

1.1 Signal Energy and Power

Motivation and definition
In many but not all, applications, the signals considered directly related to physical quantities capturing power and energy in a physical system. (for instance v^2/R for the power across a resistor)

As such it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal, denoted $x(t)$, or any discrete-time signal $x[n]$. In this case, the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x; see that the time-averaged signal can be obtained by dividing by $(t_2 - t_1)$. Similarly for a discrete signal $x[n]$ over the interval $n_1 \leq n \leq n_2$ the total energy is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

with the average power calculated by dividing by $(n_2 - n_1 + 1)$.

It is important to remember that the terms 'power' and 'energy' are used here *independently* of their relation to physical energy (they clearly don't correlate since their units or scalings would differ). Nevertheless we will find it convenient to use these terms in a general fashion.

Power and energy over infinite intervals

Considering signals over an infinite time interval, meaning for $-\infty < t < +\infty$ or $-\infty < n < +\infty$. Here we define the total energy as the limits of the aforementioned equations increase without bound; in continuous time,

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Note that these expressions may not converge; for instance say $x(t)$ or $x[n]$ equal some nonzero constant for all time: such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.
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Cont.
Analagously, we can define the time-averaged power over an infinite interval as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

and

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2$$

In continuous and discrete time respectively.

Intuition
See that with these definitions, we can identify three classes of signals: first those with finite total energy, meaning $E_\infty < \infty$. See that such a signal would have zero average power:

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

Second would be signals with finite average power P_∞; see from the above expression that for $P_\infty > 0$, this requires that $E_\infty = \infty$.

Last would be signals for which neither P_∞ nor E_∞ are finite. An example of this might be $x(t) = t$.

Note on discrete signals
It is important to note that the discrete-time signal $x[n]$ is defined *only* for *integer* values of the independent variable.

1.2 Even and Odd signals

Definition
A continuous-time signal is *even* if

$$x(-t) = x(t)$$

while a discrete-time signal is *even* if

$$x[-n] = x[n]$$

These signals are referred to as *odd* if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

Note that an odd signal must be 0 at $t = 0$ or $n = 0$ since the equations require that $x(0) = -x(0)$ and $x[0] = -x[0]$.

Decomposition
An important fact is that any signal can be broken into a sum of two signals, where one is even and the other odd. To see this, consider

$$\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

See that $x(t)$ is the sum of the two. Exactly analagous definitions hold in the discrete time case.

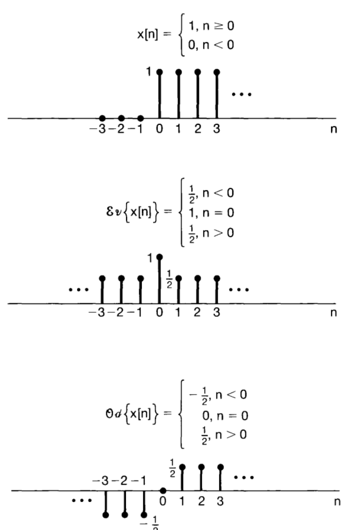

Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

1.3 Differences between continuous and discrete periodic complex exponentials

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers. An important class of complex exponentials is obtained by constraining a to be purely imaginary:

$$x(t) = e^{i\omega t}$$

Periodicity and harmonic relations (purely imaginary power)
An important property of this signal is that it is periodic; recall that $x(t)$ will be periodic with period T if

$$e^{i\omega t} = e^{i\omega(t+T)}$$

this means

$$e^{i\omega(t+T)} = e^{i\omega t}e^{i\omega T} \implies e^{i\omega T} = 1$$

If $\omega = 0$ then this is satisfied for any T. If $\omega \neq 0$, see that the *fundamental period* T_0 of $x(t)$—that is, the smallest positive value of T for which this holds—is

$$T_0 = \frac{2\pi}{|\omega|}$$

(the signals $e^{i\omega 0t}$ and $e^{-i\omega t}$ have the same fundamental period) Naturally, there is a set of exponentials periodic to a common period T_0. These are said to be *harmonically related* complex exponentials; the necessary condition they satisfy is

$$e^{i\omega T_0} = 1$$

which implies that

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

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Cont.
We had

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

if we define

$$\omega_0 = \frac{2\pi}{T_0}$$

this means that the harmonic frequencies ω must be integer multiples of ω_0:

$$\phi_k(t) = e^{ik\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Each $\phi_k(t)$ itself defines a fundamental frequency and a corresponding fundamental period. (see that $|k|\omega_0 \cdot T_0/|k| = 2\pi$, so this scaled down period is the corresponding period for this scaled up frequency. Each frequency is unique, point here is that they are also periodic with T_0, but with fundamental periods getting proportionally smaller.)

Note that the kth harmonic $\phi_k(t)$ is still periodic with T_0; it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0. (the term 'harmonic' is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency)
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Discrete case

As in continuous time, an important signal in discrete time is the *complex exponential signal*, defined as

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. See that this could also be expressed as

$$x[n] = Ce^{\beta n}$$

where $\alpha = e^{\beta}$. See that we can constrain β to be purely imaginary:

$$x[n] = e^{i\omega_0 n}$$

Periodicity properties of Discrete-time complex exponentials

While there are many similarities between continuous and discrete-time signals, there are a number of important differences. For the continuous time signal $e^{i\omega_0 t}$, we know that

- The larger the magnitude of ω_0, the higher the rate of oscillation of the signal
- $e^{i\omega_0 t}$ is periodic for any value of ω_0

These properties are different in the discrete-time case.

Given the first property, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{i(\omega_0+2\pi)n} = e^{i2\pi n}e^{i\omega_0 n} = e^{i\omega_0 n}$$

(see that this is a direct result of the fact that we iterate through discrete time as integers) The exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. This is unlike the continuous-time case where each distinct ω_0 represents a distinct signal.

In discrete time, the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$, and so on. Therefore when considering discrete time complex exponentials, see that we need only consider a frequency interval of length 2π in which to choose ω_0, such as $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$.

Also see that because of this the discrete exponential $e^{i\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 increases in magnitude; the signals will oscillate faster until we reach $\omega_0 = \pi$, after which the rate of oscillation decreases until we reach $\omega_0 = 2\pi$, at which the same constant sequence as $\omega_0 = 0$ is produced.

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Cont.
The second property we wish to consider concerns the periodicity of the discrete time complex exponential. In order for the signal $e^{i\omega_0 n}$ to be periodic with period $N > 0$ we must have

$$e^{i\omega_0(n+N)} = e^{i\omega_0 n}$$

or equivalently

$$e^{i\omega_0 N} = 1$$

For this to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m$$

or equivalently

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

The signal $e^{i\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise.

Fundamental period
Recall the idea of a *fundamental period*; in this case it would mean the smallest N such that $\omega_0 N = 2\pi m$ holds (this is unlike the continuous case where there is always some T where $\omega_0 T = 2\pi$); see that this occurs when m and N do not have any factors in common.

See that from this we can derive a *fundamental frequency* as

$$\frac{2\pi}{N} = \frac{\omega_0}{m}$$

(see that this frequency is always equal or lower—intuitively, to have a different wave that completes one oscillation in N time, its frequency will either be equal or lower)

To summarize

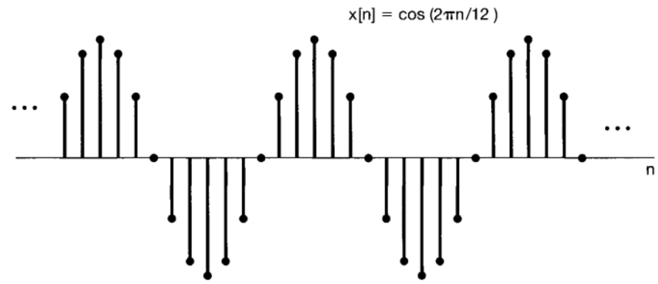
TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

1.4 Intuition for discrete-time periodicity

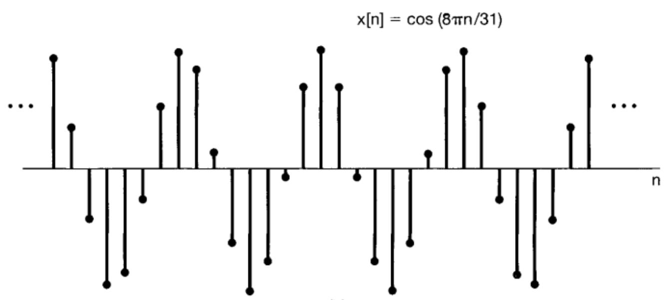
Consider the sequence $x[n] = \cos(2\pi n/12)$:



(a)

we can think of this as a set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, see that both $x(t)$ and $x[n]$ are periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

Now consider the signal $x[n] = \cos(8\pi n/31)$:



(b)

This can also be viewed as a set of samples of $x(t) = \cos(8\pi t/31)$ at integer points in time. But now see that in this case $x(t)$ is periodic with fundamental period 31/4, while $x[n]$ is periodic with fundamental period 31.

This difference stems from the fact that the discrete-time signal is defined only for integer values of the independent variable—there is no sample at time $t = 31/4$, when $x(t)$ completes one period, or at $t = 2 \cdot 31/4$ or $t = 3 \cdot 31/4$, when $x(t)$ has completed two or three periods. Only at sample $t = 4 \cdot 31/4 = 31$, when $x(t)$ has completed *four* periods is the discrete sequence defined.

This manifests as the pattern of $x[n]$ not repeating with each cycle of positive and negative values, but rather only after four of such cycles, specifically 31 points.
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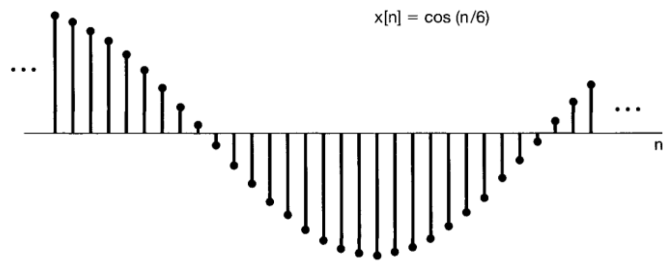
Cont.
Finally consider the signal $x[n] = \cos(n/6)$:



(c)

In this case, the values of $x(t)$ at integer sample points *never repeat*, as these sample points never span an interval that is an exact multiple of the period, 12π, of $x(t)$.

Thus, $x[n]$ *is not periodic*, although the eye visually interpolates between the sample points, suggesting *the envelope* $x(t)$ which is periodic.

1.5 Difference in harmonic relations in discrete and continuous periodic exponentials

As in continuous time, it is also of considerable value in discrete-time to consider sets of harmonically related periodic exponentials—that is, *periodic exponentials with a common period* N.

We know that these are precisely the signals which are at frequencies which are multiples of $2\pi/N$; that is

$$\phi_k[n] = e^{ik(2\pi/N)n}, \quad k = 0, \pm 1, \dots$$

In the continuous-time case, all the harmonically related complex exponentials $e^{ik(2\pi/T_0)t}$, $k = 0, \pm 1, \pm 2, \dots$ are distinct. However, recall that for discrete signals we have

$$e^{i(\omega_0 + 2\pi)n} = e^{i2\pi n} e^{i\omega_0 n} = e^{i\omega_0 n}$$

(this is a direct result of the fact that we iterate through discrete time as integers) As such the harmonically related complex exponentials *are not all unique in discrete time*; specifically,

$$\begin{aligned} \phi_{k+N}[n] &= e^{i(k+N)(2\pi/N)n} \\ &= e^{ik(2\pi/N)n} e^{i2\pi n} = \phi_k[n] \end{aligned}$$

See that this implies that there are only N distinct periodic exponentials in the set of $\phi_k[n]$; meaning

$$\phi_0[n] = 1,\ \phi_1[n] = e^{i(2\pi/N)n},\ \phi_2[n] = e^{i2(2\pi/N)n}, \dots,\ \phi_{N-1}[n] = e^{i(N-1)(2\pi/N)n}$$

are all distinct, but any other $\phi_k[n]$ would just be identical to one of them. (for instance $\phi_N[n] = \phi_0[n]$ or $\phi_{-1}[n] = \phi_{N-1}[n]$.)

1.6 More on complex exponential and sinusoidal signals

Continuous case
A continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers.

Euler identity and 'combined' sinusoidal form
Recall euler's identity:

$$e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t)$$

See that the scaled and phase-delayed sinusoid can be written in terms of these periodic complex exponentials with the same fundamental period:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{i\phi}e^{i\omega_0 t} + \frac{A}{2}e^{-i\phi}e^{-i\omega_0 t}$$

We can also express

$$A\cos(\omega_0 t + \phi) = A\mathrm{Re}\{e^{i(\omega_0 t+\phi)}\}$$

and

$$A\sin(\omega_0 t + \phi) = A\mathrm{Im}\{e^{i(\omega_0 t+\phi)}\}$$

Energy and power
Periodic signals—and in particular, the complex periodic exponential signal—are examples of signals with infinite total energy but finite average power. Calculating the total energy and of the periodic exponential signal over one period:

$$\begin{aligned} E_{\text{period}} &= \int_0^{T_0} |e^{i\omega_0 t}|^2 dt \\ &= \int_0^{T_0} 1\, dt = T_0 \end{aligned}$$

(The absolute value of a complex number is its magnitude. Think of the absolute value as the (possibly multidimensional) distance from zero.) Calculating the average power:

$$P_{\text{period}} = \frac{1}{T_0}E_{\text{period}} = 1$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. However, since the average power over each period is 1, averaging over multiple periods always yields an average power of 1. That is, the complex periodic exponential signal has finite average power equal to

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |e^{i\omega_0 t}|^2 dt = 1$$

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General continuous complex exponential signals
In the most general case Ce^{at} where both C and a are complex, see that since C and a can are just

$$C = |C|e^{i\theta}, \quad a = r + i\omega_0$$

we can express the general complex signal as

$$Ce^{at} = |C|e^{i\theta}e^{(r+i\omega_0)t} = |C|e^{rt}e^{i(\omega_0 t+\theta)}$$

we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + i|C|e^{rt}\sin(\omega_0 t + \theta)$$

Discrete case
As in continuous time, the discrete time *complex exponential signal* is defined by

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}$$

where $\alpha = e^{\beta}$.

Euler identity and 'combined' sinusoidal form
As with the continuous case, constraining β to be purely imaginary, we have euler's identity

$$e^{i\omega_0 n} = \cos\omega_0 n + i\sin\omega_0 n$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{i\phi}e^{i\omega_0 n} + \frac{A}{2}e^{-i\phi}e^{-i\omega_0 n}$$

General discrete complex exponential signals
As with the continuous case, for complex C and α, we have

$$C = |C|e^{i\theta}, \quad \alpha = |\alpha|e^{i\omega_0}$$

so the general complex exponential signal can be expressed as

$$C\alpha^n = |C||\alpha|^n\cos(\omega_0 n + \theta) + i|C||\alpha|^n\sin(\omega_0 n + \theta)$$

1.7 Unit impulse and Unit step functions

Discrete-Time

One of the simplest discrete-time signals is the *unit impulse/unit sample*, defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

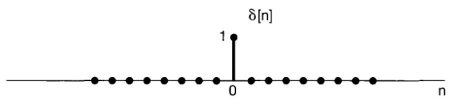

Figure 1.28 Discrete-time unit impulse (sample).

Another basic discrete-time signal is the discrete-time *unit step*, denoted by $u[n]$ and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

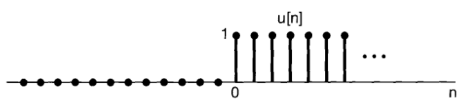

Figure 1.29 Discrete-time unit step sequence.

See that the discrete-time unit impulse is the *first difference* of the discrete-time step:

$$\delta[n] = u[n] - u[n-1]$$

Conversely, the discrete-time unit step is the *running sum* of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

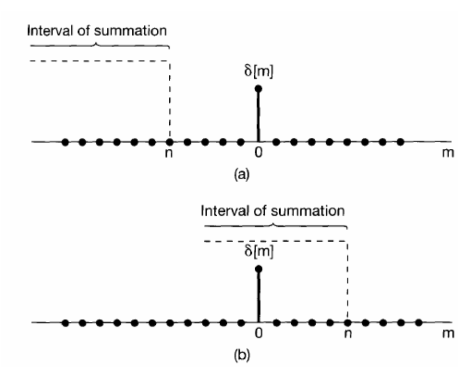

Figure 1.30 Running sum of eq. (1.66): (a) $n < 0$; (b) $n > 0$.

Since the only nonzero value of the unit sample is at 0, the running sum is 0 for $n < 0$ and 1 for $n \geq 0$.

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Alternative form
we had the discrete-time unit step as the running sum of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

See that by changing the variable of summation from m to $k = n - m$, we can rewrite this as

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

and equivalently

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

An interpretation of this is a superposition of delayed impulses; we can view the unit step as the sum of unit impulses $\delta[n]$ (nonzero at $n = 0$), $\delta[n-1]$ (nonzero at $n = 1$), and all other $\delta[n-k]$ for integer k extending to infinity.

Sampling property
See that the unit impulse can also be used to sample the value of a signal at $n = 0$; since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n]$$

More generally, if we consider a unit impulse $\delta[n-n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

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Continuous-Time

The continuous-time *unit step function* $u(t)$ is defined in a similar manner to its discrete-time counterpart:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

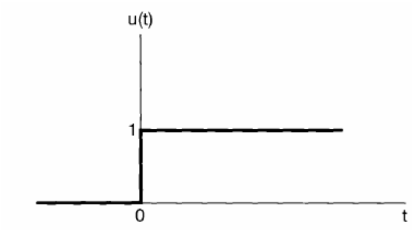

Figure 1.32 Continuous-time unit step function.

Note that the unit step is *discontinuous* at $t = 0$. The continuous-time *unit inpulse* $\delta(t)$ is related to the unit step in a manner analagous to that of their discrete counterparts; in particular, the continuous-time unit step is the *running integral* of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

It also follows that the continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step:

$$\delta(t) = \frac{du(t)}{dt}$$

In contrast to discrete-time, there is some formal difficulty with this equation as a representation of the unit impulse—since $u(t)$ is discontinuous at $t = 0$ and consequently is not formally differentiable.

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We get around this by considering an approximation to the unit step $u_\Delta(t)$, rising from the value 0 to 1 in a short time interval of length Δ:

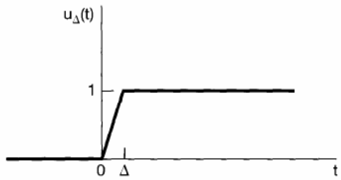


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

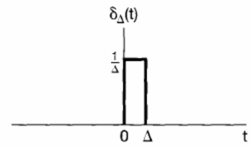


Figure 1.34 Derivative of $u_\Delta(t)$.

Also considering the derivative:

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

The unit step changes from value 0 to 1 instantaneously and can be thought of an idealisation of $u_\Delta(t)$ for Δ so short that its duration doesn't matter for any practical purpose. Formally, $u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \to 0$.

Consider the derivative again; $\delta_\Delta(t)$ is a short pulse, of duration Δ and unit area for any value of Δ. As $\Delta \to 0$, $\delta_\Delta(t)$ becomes narrower and higher while mantaining its unit area. Its limiting form

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

can then be thought of as an idealisation of the short pulse $\delta_\Delta(t)$ as the duration Δ becomes insignificant:

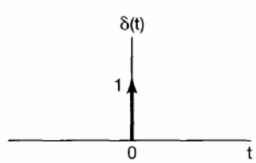


Figure 1.35 Continuous-time unit impulse.

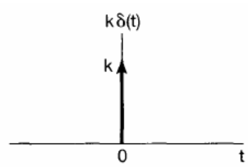


Figure 1.36 Scaled impulse.

$\delta(t)$ has, in effect, no duration but unit area. The arrow at $t = 0$ indicates the area of the pulse is concentrated at $t = 0$ and the height of the arrow and the '1' next to the arrow is used to represent the *area* of the impulse. More generally, a scaled impulse $k\delta(t)$ will have an area k, and thus

$$ku(t) = \int_{-\infty}^{t} k\delta(\tau)d\tau$$

A scaled impulse, where the height of the arrow is chosen to be proportional to the area of the impulse.
(next page)

Alternative form

As with discrete time, the relationship between the continuous time unit step and impulse can be rewritten in a different form, by changing the variable of integration from τ to $\sigma = t - \tau$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \int_{\infty}^{0} \delta(t - \sigma)(-d\sigma)$$

or equivalently

$$u(t) = \int_{0}^{\infty} \delta(t - \sigma)\, d\sigma$$

(This negation after inverstion of the limits of the integral can be derived from the first fundamental theorem. For a more intuitive understanding, consult the definition of the Riemann sum:

$$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$$

Considering the $d\tau$, or respectively $d\sigma$, as the width of the step between two argumentsof the sum, if we change the direction in which we integrate, the step also changes its sign. In the discrete sum, the summands are not multiplied by this step.)

Sampling property

As with discrete-time, the continuous-time impulse has a very important sampling property; consider

$$x_1(t) = x(t)\delta_\Delta(t)$$

By construction, $x_1(t)$ is zero outside the interval $0 \le t \le \Delta$. See that for Δ sufficiently small so that $x(t)$ is approximately constant over this integral:

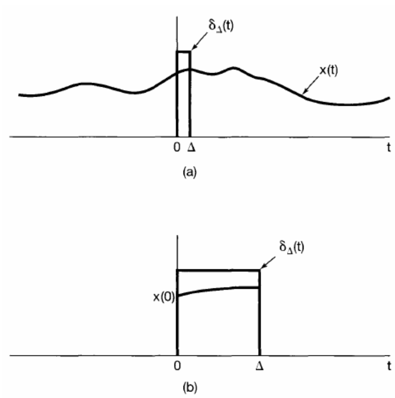


Figure 1.39 The product $x(t)\delta_\Delta(t)$: (a) graphs of both functions; (b) enlarged view of the nonzero portion of their product.

so we have

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

(next page)

Cont.
We had, for small Δ,

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

since $\delta(t)$ is the limit as $\Delta \to 0$ of $\delta_\Delta(t)$, it follows that

$$x(t)\delta(t) = x(0)\delta(t)$$

By the same argument, we have an analagous expression for an impulse concentrated at an arbitrary point, say t_0. That is,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

1.8 Basic system properties 1

General form
A general formula for a continuous first-order linear differential equation is

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

where $x(t)$ is the input, $y(t)$ the output, and a, b constants. An example of this might be

$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

Discrete cases have general first-order linear difference equations of the form

$$y[n] + ay[n-1] = bx[n]$$

Considering the earlier example, if we let $v[n] = v(n\Delta)$ and $f[n] = f(n\Delta)$ and approximate $dv(t)/dt$ at $t = n\Delta$ by the first backward difference:

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}$$

we can obtain

$$\begin{aligned}
\frac{v[n] - v[n-1]}{\Delta} + \frac{\rho}{m}v[n] &= \frac{1}{m}f[n] \\
v[n] - v[n-1] + \frac{\rho\Delta}{m}v[n] &= \frac{\Delta}{m}f[n] \\
v[n]\left(1 + \frac{\rho\Delta}{m}\right) - v[n-1] &= \frac{\Delta}{m}f[n] \\
v[n]\frac{m + \rho\Delta}{m} - v[n-1] &= \frac{\Delta}{m}f[n] \\
v[n] - \frac{m}{m + \rho\Delta}v[n-1] &= \frac{\Delta}{m + \rho\Delta}f[n]
\end{aligned}$$

which is in the general form as described above.
(next page)

Basic system properties

Memory
A system is said to be *memoryless* if its output for each value of the independent variable is dependent only on the *input at that same time.*

For instance, the system

$$y[n] = (2x[n] - x^2[n])^2$$

is memoryless, since the value of $y[n]$ at any particular time n_0 depends only on the value of $x[n]$ at that time. Other examples of memoryless systems include the input-output relationship of a resistor, with input $x(t)$ taken to be current and output $y(t)$ voltage:

$$y(t) = Rx(t)$$

where R is the resistance. The *identity system*, whose output is identical to its input, is also memoryless

$$y(t) = x(t)$$

Written in discrete time:

$$y[n] = x[n]$$

An example of a discrete-time system *with memory* is an *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

see that the accumulator must store or 'remember' information about the past inputs up to current time. Another example of a system with memory is a *delay*

$$y[n] = x[n-1]$$

While the concept of memory in a system would typically suggest storing *past* input and output values, our formal definition also leads to our referring to a system as having memory if the current output is dependent on *future* values of the input and output (noncausal systems).
(next page)

Causality
A system is *causal* if the output at any time depends only on values of the input at the present time and in the past. Such a system is often referred to as being *nonanticipative*, as the system output does not 'anticipate' future values of the input.

Consequently, if two inputs to a causal system are identical up to some point in time t_0 or n_0, the corresponding outputs must also be equal up to this same time. For instance the capacitor:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

is a causal system, with input taken to be current, output voltage, and C capacitance. The capacitor voltage responds only to the present and past values of the input.

Examples of *noncausal* systems might be

$$y[n] = x[n] - x[n+1]$$

or

$$y(t) = x(t+1)$$

See that all memoryless systems are causal, since the output responds only to the current value of the input in such systems.

An example of a noncausal system might be when averaging data over an interval to smooth out fluctuations and keep only the trend. Such an averaging system might look like

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k]$$

More examples
Consider

$$y[n] = x[-n]$$

see that for $n < 0$ the output depends on a future value of the input, and hence the system is not causal. Another example: consider

$$y(t) = x(t)\cos(t+1)$$

This can be rewritten as

$$y(t) = x(t)g(t)$$

Thus only the current value of the input $x(t)$ influences the current value of the output $y(t)$. This system is causal (and, in fact, memoryless).
(next page)

Invertibility and inverse systems
A system is said to be *invertible* if distinct inputs lead to distinct outputs. If a system is invertible, then an *inverse* system exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ in the first system:

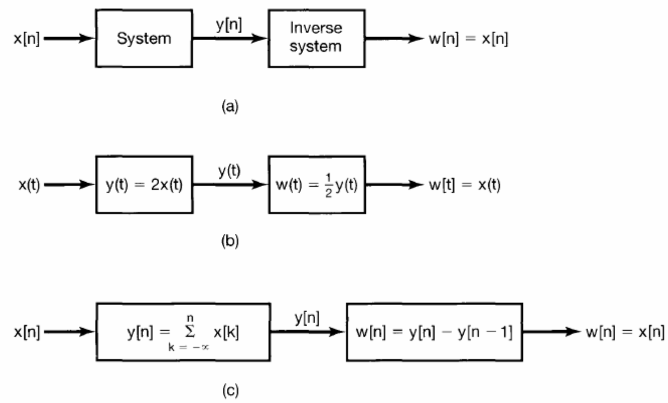


Figure 1.45 Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

See that the series interconnection between the invertible system and its inverse system has an overall input-output relationship same as that for the identity system. As illustrated in the second graph, an example of an invertible continuous-time system is

$$y(t) = 2x(t)$$

for which the inverse system is

$$w(t) = \frac{1}{2}y(t)$$

Another example is illustrated in the third block diagram, see that the accumulator is an invertible system:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

with inverse system

$$w[n] = y[n] - y[n-1]$$

Examples of noninvertible systems include

$$y[n] = 0$$

or

$$y(t) = x^2(t)$$

where a single output can correspond to different inputs.
(next page)

Stability
Another important system property is *stability*. Informally, a stable system is one in which small inputs lead to responses that do not diverge (grow uncontrollably).

More formally, if the input to a stable system is bounded (if its magnitude of the input does not grow without bound), then the output must also be bounded and therefore cannot diverge. (this is the definition to take note of)

For instance consider the averaging system brought up earlier

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]$$

Suppose that the input $x[n]$ is bounded in magnitude by some number, say B for all values of n. Then the largest possible magnitude for $y[n]$ is also B since it is the average. Therefore, $y[n]$ is bounded and the system is stable.

On the other hand, the accumulator sums all of the past values of the input, so the output will grow continually even if $x[n]$ is bounded—it is an unstable system.

A useful strategy to verify that a system is unstable is to look for a specific bounded input that leads to an unbounded output. For instance consider

$$y(t) = tx(t)$$

See that a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded. Since for any finite constant bound, $|y(t)|$ will exceed that bound at some t, this system is unstable.

Now consider a different system

$$y(t) = e^{x(t)}$$

See that for an arbitrary positive number B, if $x(t)$ is bound by B, that is

$$|x(t)| < B$$

or

$$-B < x(t) < B$$

for all t, then $y(t)$ must satisfy

$$e^{-B} < |y(t)| < e^{B}$$

Any input to this system bounded by an arbitrary positive number B has a output guaranteed to be bounded by e^B—this system is stable.

1.9 Basic system properties 2

Time invariance
A system it *time invariant* if a time shift in the input signal results in an identical time shift in the output signal.

That is, if $y[n]$ is the output of a discrete-time, time-invariant system with input $x[n]$, then $y[n - n_0]$ is the output when $x[n - n_0]$ is applied. In continuous time with $y(t)$ the output corresponding to the input $x(t)$, a time-invariant system will have $y(t - t_0)$ as the output when $x(t - t_0)$ is the input.

Examples
Consider the continuous-time system

$$y(t) = \sin[x(t)]$$

To check that this system is time invariant, we must determine whether the time-invariance property holds for *any* input and *any* time shift t_0. Thus, letting $x_1(t)$ be an arbitrary input to the system, with corresponding output

$$y_1(t) = \sin[x_1(t)]$$

Now consider a second input x_2 obtained by shifting $x_1(t)$ in time

$$x_2(t) = x_1(t - t_0)$$

The output corresponding to this input is

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)]$$

Now consider translating the output y_1, see that

$$y_1(t - t_0) = \sin[x_1(t - t_0)]$$

Since y_2, the output of the shifted input, is the same as $y_1(t - t_0)$, which is if we had shifted the output instead, this system is time invariant.
(next page)

More examples
As a second example, consider

$$y[n] = nx[n]$$

See that time shifting the input doesn't correspond with an equivalent shift in the output—this is a time-varying system. This system represents one with a time-varying gain, even if we know the input value, we cannot determine the output value without knowing the current time.

For a counterexample, consider having input $x_1[n] = \delta[n]$, which yields an output $y_1[n] = 0$ (since $n\delta[n] = 0$). However, the input $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1] = \delta[n-1]$—while $x_2[n]$ is a shifted version of $x_1[n]$, $y_2[n]$ is *not* a shifted version of $y_1[n]$.

For a final example, consider the system

$$y(t) = x(2t)$$

This system represents a time scaling. That is, $y(t)$ is a time-compressed (by a factor of 2) version of $x(t)$. Intuitively then, any time shift in the input will also be compressed by a factor of 2, and it is for this reason that the system is not time invariant. Consider a counterexample:

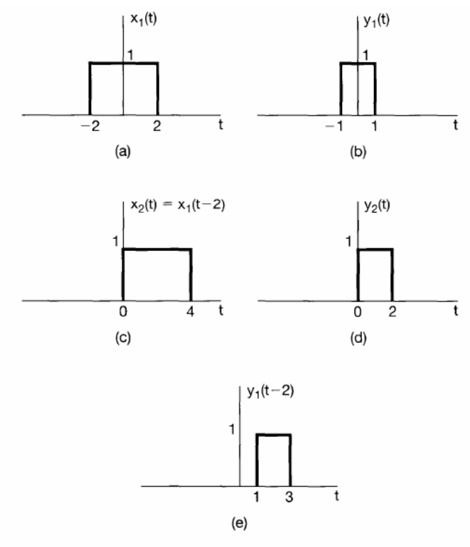


Figure 1.47 (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

see that if we shift the input signal by 2, the resulting output is not the same as if we had shifted the output signal by 2.
(next page)

Linearity
A *linear system*, in continuous or discrete time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition—that is, the weighted sum—of the responses of the system to each of those signals.

More precisely, letting $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and $y_2(t)$ the response to $x_2(t)$, the system is linear if:

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. The response to $ax_1(t)$ is $ay(1)$, where a is any complex constant.

The first property is known as the *additivity* propery, while the second the *scaling* or *homogeneity* property; the same definition holds for continuous time. Note that a system can be linear without being time invariant, and it can be time invariant without being linear.

The two properties defining a linear system can be combined into a single statement:

$$\begin{aligned} \text{continuous time: } & ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t) \\ \text{discrete time: } & ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n] \end{aligned}$$

where a and b are any complex constants.

From this see that for a set of inputs $x_k[n], k = 1, 2, 3, \ldots$ to a discrete-time linear system with corresponding outputs $y_k[n], k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots$$

is

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \ldots$$

This is known as the *superposition property*, which holds for linear systems in both continuous and discrete time.

See that a direct consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time; if $x[n] \rightarrow y[n]$, then by the homogeneity property

$$0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0$$

(next page)

Examples
Consider the system

$$y(t) = tx(t)$$

To determine whether or not it is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$.

$$\begin{aligned} x_1(t) &\to y_1(t) = tx_1(t) \\ x_2(t) &\to y_2(t) = tx_2(t) \end{aligned}$$

Now consider $x_3(t)$, a linear combination of $x_1(t)$ and $x_2(t)$:

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. See that for input $x_3(t)$, we have the output

$$\begin{aligned} y_3(t) &= tx_3(t) \\ &= t(ax_1(t) + bx_2(t)) \\ &= atx_1(t) + btx_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

So we conclude that the system is linear (also see that it is not time invariant).

For another example, consider the system

$$y(t) = x^2(t)$$

as defining $x_1(t), x_2(t)$, and $x_3(t)$ as in the previous example, we have

$$\begin{aligned} x_1(t) &\to y_1(t) = x_1^2(t) \\ x_2(t) &\to y_2(t) = x_2^2(t) \end{aligned}$$

and

$$\begin{aligned} x_3(t) \to y_3(t) &= x_3^2(t) \\ &= (ax_1(t) + bx_2(t))^2 \\ &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \end{aligned}$$

which isn't a superposition of the inputs, thus the system is not linear.
(next page)

More examples
It is important to remember that the scaling constants of the superposition criteria are allowed to be complex. Consider the system

$$y[n] = \mathrm{Re}\{x[n]\}$$

This system is additive, but does not satisfy the homogeneity property, consider input

$$x_1[n] = r[n] + is[n]$$

the corresponding output is

$$y_1[n] = r[n]$$

Now consider scaling $x_1[n]$ by a complex number, for instance $a = i$; defined as x_2:

$$\begin{aligned} x_2[n] = ix_1[n] &= i(r[n] + is[n]) \\ &= -s[n] + ir[n] \end{aligned}$$

The corresponding output then is

$$y_2[n] = \mathrm{Re}\{x_2[n]\} = -s[n]$$

which is not equal to the scaled version of $y_1[n]$:

$$ay_1[n] = ir[n]$$

thus the system violates the homogeneity property and is not linear.
(next page)

Example—incrementally linear system
Consider the system

$$y[n] = 2x[n] + 3$$

This system is not linear, see that it violates the additivity property:

$$x_1[n] \to y_1[n] = 2x_1[n] + 3$$
$$x_2[n] \to y_2[n] = 2x_2[n] + 3$$

where the response to $x_3[n] = x_1[n] + x_2[n]$ is

$$y_3[n] = 2(x_1[n] + x_2[n]) + 3$$

while

$$y_1[n] + y_2[n] = 2(x_1[n] + x_2[n]) + 6$$

See that this system violates the property of linear systems where zero input yields zero output.

It may be surprising that the system here is nonlinear since it describes a linear equation. Intuitively, see that the output of this system can be represented as the sum of the output of a linear system

$$x[n] \to 2x[n]$$

and another signal equal to the *zero-input response* of the system (when the input is zero),

$$y_0[n] = 3$$

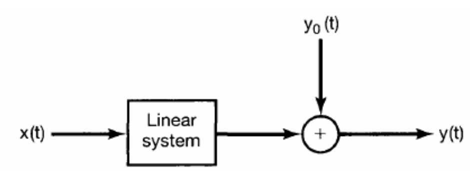


Figure 1.48 Structure of an incrementally linear system. Here, $y_0[n]$ is the zero-input response of the system.

There are large classes of systems that can be represented like this, for which the overall system output consists of the superposition of the response of a linear system with a zero-input response; these systems correspond to the class of *incrementally linear systems*.

The *difference* between the resposnes to any two inputs to an incrementally system is a linear (additive and homogeneous) function of the *difference* between the two inputs. For example, for inputs $x_1[n]$, $x_2[n]$ and corresponding outputs $y_1[n]$, $y_2[n]$ we have

$$y_1[n] - y_2[n] = 2x_1[n] + 3 - (2x_2[n] + 3) = 2[x_1[n] - x_2[n]]$$

Chapter 2

Linear Time-Invariant Systems

2.1 Discrete-time convolution sum

Representing a discrete signal in terms of impulses
The key idea in visualising how the discrete-time unit impulse can be used to construct any discrete-time signal is to think of a discrete-time signal as a sequence of individual impulses:

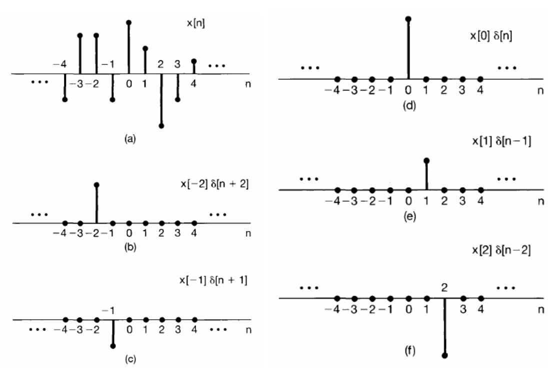


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

See that we can express express each value of $x[n]$ as an individual scaled, shifted impulse; for instance

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases}$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}$$

More generally, see that we can write

$$\begin{aligned} x[n] = \ldots &+ x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ &+ x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots \end{aligned}$$

For any value of n, only one of the terms on the right-hand side of the equation is nonzero, and the scaling associated with that term is precisely $x[n]$. Writing this summation in a more compact form, we have

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

This equation is called the *sifting property* of the discrete-time unit impulse; the summation 'sifts' through $x[k]$ and preserves only the value corresponding to $k = n$.

(next page)

Convolution-sum representation of LTI systems
The fact that $x[n]$ can be represented as a superposition of scaled versions of (time shifted) impulses, means that the response of a linear system to $x[n]$ will be a *superposition of the scaled responses of the system* to each of these shifted impulses.

Moreover, the property ot time invariance tells us that that the *responses of a time-invariant system to the time-shifted unit impulses are simply time-shifted versions of one another.* The convolution-sum representation for discrete-time systems that are both linear and time invariant results from putting these two basic facts together.

Linear, not necessarily time invariant response
Consider the response of a linear (but possibly time-varying) system to an arbitrary input $x[n]$. We can represent $x[n]$ as a linear combination of shifted impulses.

Letting $h_k[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n-k]$, see that the superposition property of linear systems means that the response $y[n]$ of a linear system to the input $x[n]$ is simply the weighted linear combination of these basic responses:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

The response value at a specific time n of a linear system is the superposition of the 'contributions' to that output point from each input (from all points in time).
(next page)

Illustrated

For instance, given the input signal $x[n]$ to a linear (non-time invariant) system with the responses $h_{-1}[n], h_0[n]$, and $h_1[n]$ to the signals $\delta[n+1], \delta[n]$, and $\delta[n-1]$:

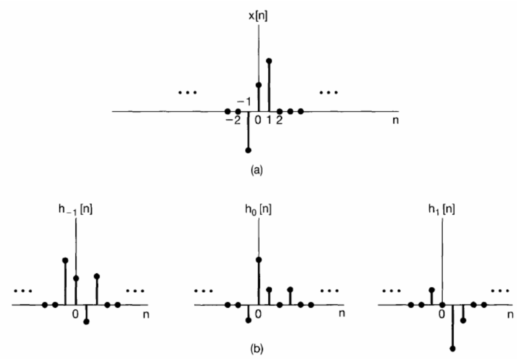


Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).

Since $x[n]$ can be written as a linear combination of the shifted impulses. The system *responses* to these time shifted impulses, h, scaled accordingly by their respective input value, sum to produce $y[n]$.

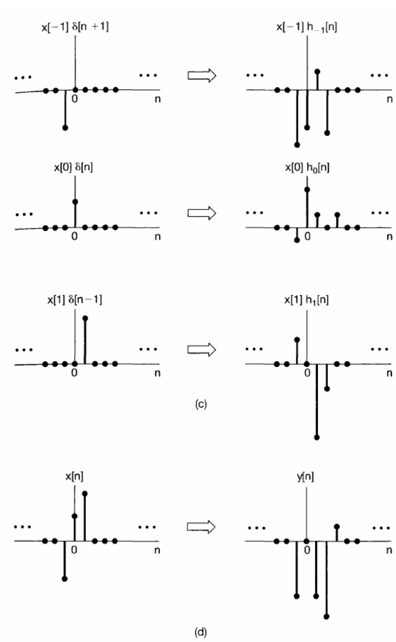


Figure 2.2 *Continued*

(This illustration only considers three points, but see that the summation on the previous page accounts for the possibility that every input could contribute to a particular output point in time, thus the limits $-\infty$ to ∞)

(next page)

LTI response—convolution sum

In the previous case, the responses $h_k[n]$ need not be related to each other for different values of k. However, if the linear system is also *time invariant*, then these responses to time-shifted unit impulses are all *time-shifted versions of each other*, meaning

$$h_k[n] = h_0[n-k]$$

For notational convenience, we dr the subscript on $h_0[n]$ and define the *unit impulse (sample) response* as

$$h[n] = h_0[n]$$

That is, when $h[n]$ is the output of the LTI system when $\delta[n]$ is the input, we have

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

This result is referred to as the *convolution sum* or *superposition sum*, and the operation on the right-hand side is known as the *convolution* of the sequences $x[n]$ and $h[n]$, represented symbolically as

$$y[n] = x[n] * h[n]$$

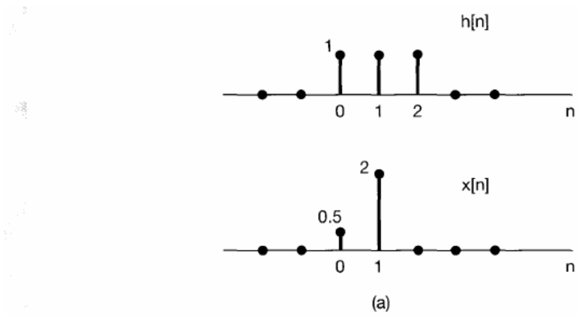
See that an LTI system is completely characterised by its response to a single signal, the unit impulse. As before, the actual output is a superposition of the responses, but due to time invariance this time every response is just a time shift of each other.

2.2 Discrete-time convolution sum—examples

These examples are crucial for a deeper understanding of the discrete convolution. For reference, the definition of the convolution sum: When $h[n]$ is the output of the LTI system when $\delta[n]$ is the input, we have output $y[n]$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Example 1
Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated



In this case, since only $x[0]$ and $x[1]$ are nonzero, the output (via convolution), simplifies to the expression

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$$

The sequences $0.5h[n]$ and $2h[n-1]$ are two 'echoes' of the impulse response needed for the superposition involved in generating $y[n]$, obtained by summing up each echo for each value of n.
(next page)

Example 1 illustrated

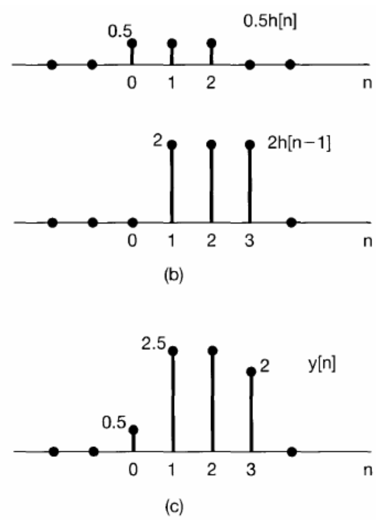


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

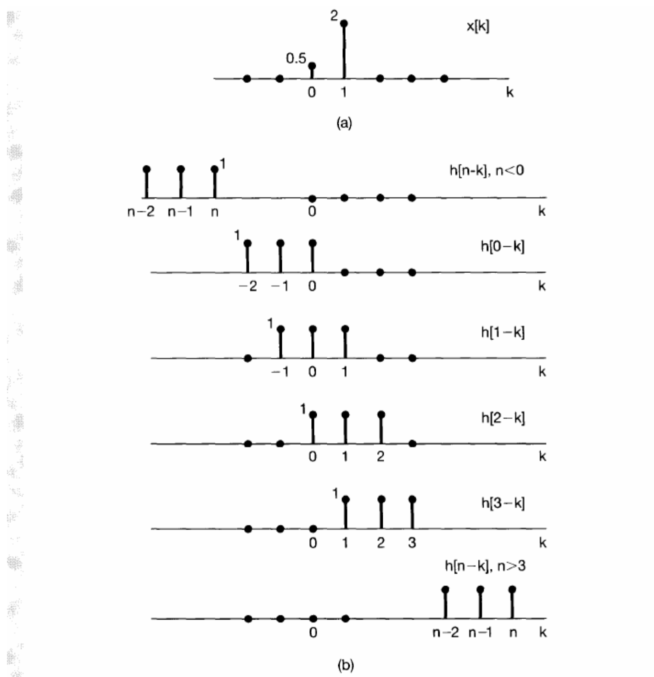
The output for point 0 depends on only on the first point of the impulse response at point 0, $x[0]h[0]$ (in this case there aren't any inputs before this point to produce other impulse responses to affect this). In contrast, the output for point 1 depends on both the *second* point of the impulse response at point 0, $x[0]h[1-0]$ and the *first* point of the impulse response at point 1, $x[1]h[1-1]$. (see how this relates to the structure of the convolution sum)

Alternative view of the convolution
Consider the evaluation of the output value at a specific time n, a particularly convenient way of displaying this calculation graphically begins with the two signals $x[k]$ and $h[n-k]$ viewed as functions of k.

Multiplying these two functions, we obtain a sequence $g[k] = x[k]h[n-k]$, which, at each time k, can be seen as representing the contribution of $x[k]$ to the output at time n. Summing all the samples in the sequence of $g[k]$ yields the output value at the selected time n.

Thus, to calculate $y[n]$ for all values of n requires repeating this procedure for each value of n. Fortunately, changing the value of n has a very simple graphical interpretation for the two signals $x[k]$ and $h[n-k]$, viewed as functions of k. The following examples illustrate this.
(next page)

Example 2—view as a function of k
Let us consider again the convolution problem encountered in the previous example. Now we have the sequences $x[k]$ and $h[n-k]$, for n fixed and viewed as a function of k; see that the graph of $h[n-k]$ is a time reversal and shift—to the *right* if n is positive and *left* if n is negative:



Having sketched $x[k]$ and $h[n-k]$ for a given n, we multiply the two signal and sum over all values of k (like a dot product) to get $y[n]$. (We multiply and sum up the points where the graphs 'overlap')

(Intuit that the view in the first example is the same as 'reversing' the impulse response and moving it along the input like a sliding window, which is essentially what is happening here.)

For $n < 0$, neither of the graphs overlap at points where both are nonzero, so $y[n] = 0$ for $n < 0$. For $n = 0$, since the product of the sequence $x[k]$ and the sequence $h[0-k]$ has only one nonzero sample (at $k = 0$), we have

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = x[0]h[0] = 0.5$$

(next page)

cont. from example 2
The other examples have more ‘overlapping’ points:

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 + 2.0 = 2.5$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 + 2.0 = 2.5$$

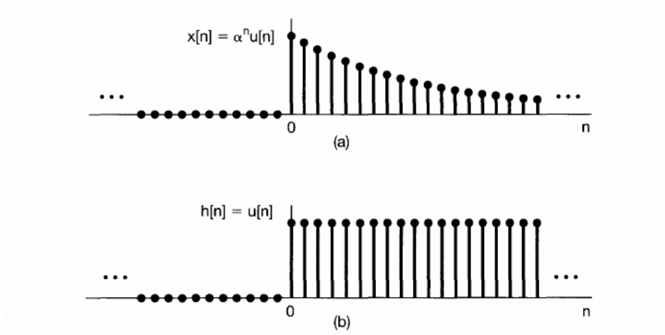
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2.0$$

(next page)

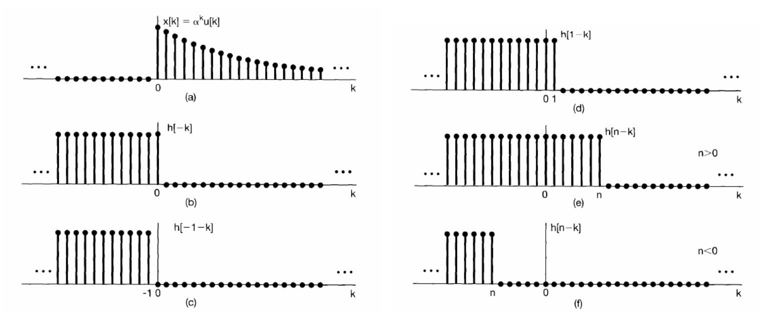
Example 3
Consider input $x[n]$ and unit impulse $h[n]$ given by

$$x[n] = \alpha^n u[n]$$
$$h[n] = u[n]$$

where $0 < a < 1$:



As before, consider $x[k]$ and $h[n-k]$ for different n:



For $n < 0$ there is no overlap between nonzero points in $x[k]$ and $h[n-k]$, so $y[n] = 0$ for $n < 0$. See that for $n \geq 0$,

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

Thus for $n \geq 0$, the usual infinite limits are superfluous, and we can write

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

this is a geometric series, we can write

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad \text{for } n \geq 0$$

(see appendix for sum of geometric series formula)
(next page)

cont. example 3
We had

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad \text{for } n \geq 0$$

which can be rewritten as

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$

Illustrated:

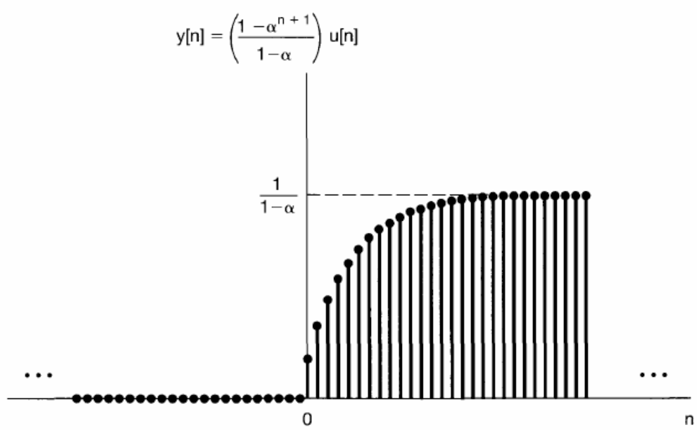


Figure 2.7 Output for Example 2.3.

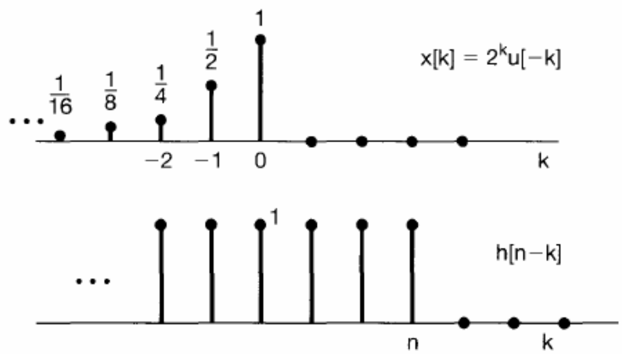
See that the convolution operation can be described as a 'sliding' of the sequence $h[n-k]$ past $x[k]$ (as we increase n the $h[n-k]$ graph just gets shifted to the right).

For example, suppose we have evaluated $y[n]$ for some particular value of n, say $n = n_0$, meaning we have sketched $h[n_0 - k]$, multiplied it by $x[k]$, and summed the results over all k. To evaluate $y[n]$ at the next value of n, so $n = n_0 + 1$, we sketch $h[(n_0+1)-k]$; however, we can do this by simply translating $h[n_0 - k]$ to the right by one point.

Repeating this for each successive n demonstrates this 'sliding' idea.
(next page)

Example 4
Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as

$$x[n] = 2^n u[-n]$$
$$h[n] = u[n]$$



(a)

See that $x[k]$ is zero for $k > 0$ and $h[n-k]$ is zero for $k > n$. Also see that regardless of n, the sequence $x[k]h[n-k]$ always has nonzero samples along the k-axis, and when $n \geq 0$, $x[k]h[n-k]$ has nonzero samples in the interval $k \leq 0$. As such we can write, for $n \geq 0$

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[n-k] = \sum_{k=-\infty}^{0} 2^k$$

and for $n < 0$, $x[k]h[n-k]$ has nonzero samples for $k \leq n$. It follows that, for $n < 0$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=-\infty}^{n} 2^k$$

(next page)

cont. example 4

In both cases we can simplify using the the formula for summing geometric series (see appendix)

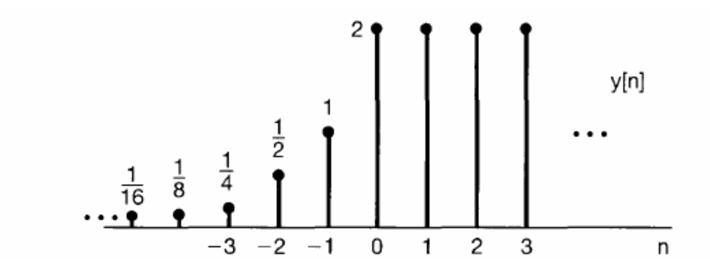
$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$

Where for the $n \geq 0$ case, by performing a change of variable $r = -k$,

$$y[n] = \sum_{k=-\infty}^{0} 2^k = \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1-(1/2)} = 2$$

For the $n < 0$ case, by performing a change of variable $l = -k$, and then $m = l + n$,

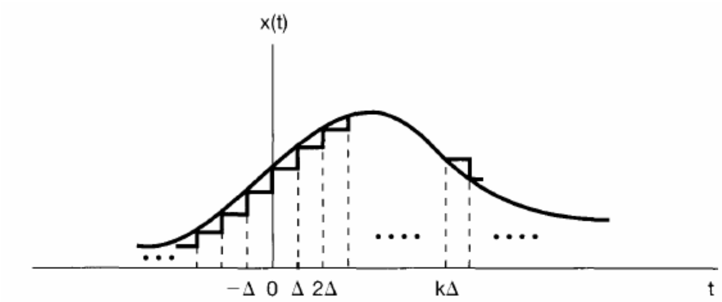
$$\begin{aligned} y[n] = \sum_{k=-\infty}^{n} 2^k = \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l &= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n} \\ &= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1} \end{aligned}$$



(b)

2.3 Continuous-time—the convolution integral

Representating a continuous signal in terms of impulses
We develop a continuous-time conterpart of the discrete-time 'sifting' property. Consider a riemann-sum like approximation, $\hat{x}(t)$, to a continuous-time signal $x(t)$:



(a)

This approximation can be expressed as a linear combination of delayed rectangular pulses. If we define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

then since $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

(δ_Δ 'turns on' a specific rectangle, x specifies its height, and Δ the base length) See that for any t, only one term in the summation on the right is nonzero.

As we let Δ approach 0, the approximation $\hat{x}(t)$ becomes better, and in the limit equals $x(t)$, therefore we write

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

Also, as $\Delta \to 0$, the summation approaches an integral; see that since the limit of $\delta \to 0$ of $\delta_\Delta(t)$ is the unit impulse function $\delta(t)$, we have

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau$$

This should be viewed as an idealisation in the sense that, for any Δ 'small enough', the approximation by this summation is essentially exact for any practical purpose. The integral then represents an idealisation of the summation by taking Δ to be vanishingly small.
(next page)

Viewed as a function of τ
See that when the signal $\delta(t-\tau)$ is viewed as a function of τ with t fixed, it is still a unit impulse located at $\tau = t$:

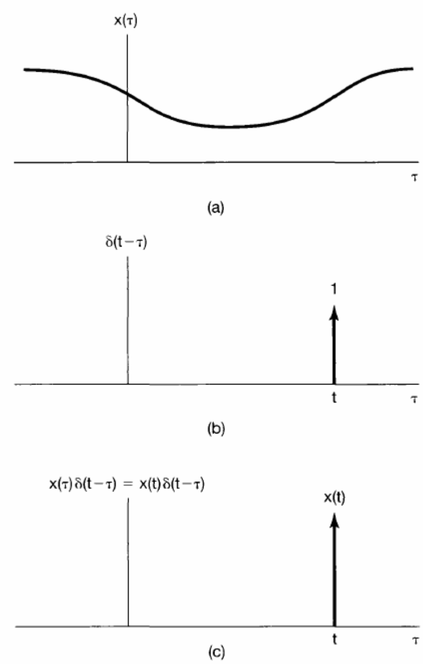


Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a function of τ with t fixed; (c) product of these two signals.

As such, we have

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$$

which is just a scaled impulse at $\tau = t$ with area equal to $x(t)$. See that this can be used to show the statement on the previous page

$$\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{+\infty} \delta(t-\tau)d\tau = x(t)$$

Emphasising the interpretation of the $x(t)$ being represented as a sum (or more precisely, an integral) of weighted, shifted impulses.
(next page)

Continuous-time impulse response—the convolution integral
We could represent $\hat{x}(t)$ as a sum of scaled and shifted versions of the basic pulse signal $\delta_\Delta(t)$.

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

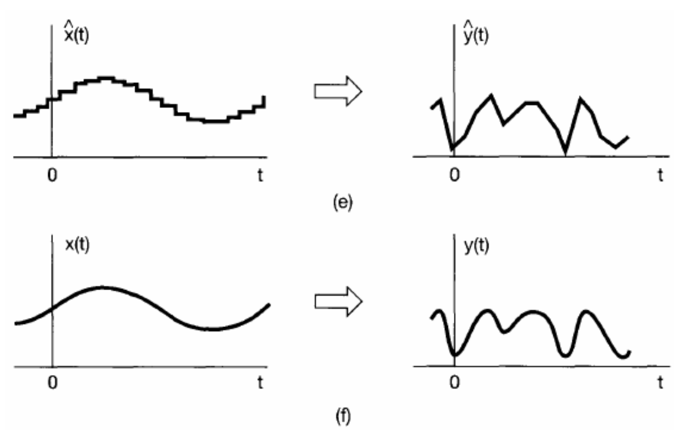
Consequently, the response $\hat{y}(t)$ of a linear system to this signal will be the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$.

Specifically, let us define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$, then by superposition we have

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

Now consider what happens as Δ becomes vanishingly small, meaning $\Delta \to 0$. $\hat{x}(t)$ becomes an increasingly good approximation to $x(t)$, and in fact, the two coincide as $\Delta \to 0$.

Consequently, the response to $\hat{x}(t)$, namely, $\hat{y}(t)$, must converge to $y(t)$, the response to the actual input $x(t)$. Either way, we say that for Δ 'small enough', the duration of the pulse $\delta_\Delta(t - k\Delta)$ is of no significance, in that, as far as the system is concerned, the response to this pulse is essentially the same as the response to a unit impulse as $\Delta \to 0$:



Since the pulse $\delta_\Delta(t - k\Delta)$ corresponds to a shifted unit impulse as $\Delta \to 0$, the response $\hat{h}_{k\Delta}(t)$ to this input also becomes the impulse response in the limit. As such we have

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

(next page)

cont.
We had

Appendix A

Other proofs

A.1 Sum of geometric series

Here we show

$$S_n = \sum_{i=0}^{n-1} \alpha^i = \frac{\alpha^n - 1}{\alpha - 1}$$

This can be seen from

$$\begin{aligned}
(\alpha - 1)S_n &= \alpha \sum_{i=0}^{n-1} \alpha^i - \sum_{i=0}^{n-1} \alpha^i \\
&= \sum_{i=1}^{n} \alpha^i - \sum_{i=0}^{n-1} \alpha^i \\
&= \alpha^n + \sum_{i=1}^{n-1} \alpha^i - (1 + \sum_{i=1}^{n-1} \alpha^i) \\
&= \alpha^n - 1
\end{aligned}$$

and so

$$S_n = \frac{\alpha^n - 1}{\alpha - 1}$$