

Signals And Systems by Alan V. Oppenheim: Notes

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0.1 Introduction

0.1.1 Signal Energy and Power

Motivation and definition

In many but not all, applications, the signals considered directly related to physical quantities capturing power and energy in a physical system. (for instance v^2/R for the power across a resistor)

As such it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal, denoted $x(t)$, or any discrete-time signal $x[n]$. In this case, the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x ; see that the time-averaged signal can be obtained by dividing by $(t_2 - t_1)$. Similarly for a discrete signal $x[n]$ over the interval $n_1 \leq n \leq n_2$ the total energy is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

with the average power calculated by dividing by $(n_2 - n_1 + 1)$.

It is important to remember that the terms ‘power’ and ‘energy’ are used here *independently* of their relation to physical energy (they clearly don’t correlate since their units or scalings would differ). Nevertheless we will find it convenient to use these terms in a general fashion.

Power and energy over infinite intervals

Considering signals over an infinite time interval, meaning for $-\infty < t < +\infty$ or $-\infty < n < +\infty$. Here we define the total energy as the limits of the aforementioned equations increase without bound; in continuous time,

$$E_\infty \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Note that these expressions may not converge; for instance say $x(t)$ or $x[n]$ equal some nonzero constant for all time: such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

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Analagously, we can define the time-averaged power over an infinite interval as

$$P_{\infty} \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

and

$$P_{\infty} \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

In continuous and discrete time respectively.

Intuition

See that with these definitions, we can identify three classes of signals: first those with finite total energy, meaning $E_{\infty} < \infty$. See that such a signal would have zero average power:

$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{E_{\infty}}{2T} = 0$$

Second would be signals with finite average power P_{∞} ; see from the above expression that for $P_{\infty} > 0$, this requires that $E_{\infty} = \infty$.

Last would be signals for which neither P_{∞} nor E_{∞} are finite. An example of this might be $x(t) = t$.

Note on discrete signals

It is important to note that the discrete-time signal $x[n]$ is defined *only* for *integer* values of the independent variable.

0.1.2 Even and Odd signals

Definition

A continuous-time signal is *even* if

$$x(-t) = x(t)$$

while a discrete-time signal is *even* if

$$x[-n] = x[n]$$

These signals are referred to as *odd* if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n]$$

Note that an odd signal must be 0 at $t = 0$ or $n = 0$ since the equations require that $x(0) = -x(0)$ and $x[0] = -x[0]$.

Decomposition

An important fact is that any signal can be broken into a sum of two signals, where one is even and the other odd. To see this, consider

$$\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

See that $x(t)$ is the sum of the two. Exactly analogous definitions hold in the discrete time case.



Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

0.1.3 Differences between continuous and discrete periodic complex exponentials

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers. An important class of complex exponentials is obtained by constraining a to be purely imaginary:

$$x(t) = e^{i\omega_0 t}$$

Periodicity and harmonic relations (purely imaginary power)

An important property of this signal is that it is periodic; recall that $x(t)$ will be periodic with period T if

$$e^{i\omega_0 t} = e^{i\omega_0(t+T)}$$

this means

$$e^{i\omega_0(t+T)} = e^{i\omega_0 t} e^{i\omega_0 T} \implies e^{i\omega_0 T} = 1$$

If $\omega_0 = 0$ then this is satisfied for any T . If $\omega_0 \neq 0$, see that the *fundamental period* T_0 of $x(t)$ —that is, the smallest positive value of T for which this holds—is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

(the signals $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ have the same fundamental period) Naturally, there is a set of exponentials periodic to a common period T_0 . These are said to be *harmonically related* complex exponentials; the necessary condition they satisfy is

$$e^{i\omega T_0} = 1$$

which implies that

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

if we define

$$\omega_0 = \frac{2\pi}{T_0}$$

this means that the harmonic frequencies ω must be integer multiples of ω_0 :

$$\phi_k(t) = e^{ik\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \dots$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k , $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Note that the k th harmonic $\phi_k(t)$ is still periodic with T_0 ; it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0 . (the term ‘harmonic’ is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency)

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Discrete case

As in continuous time, an important signal in discrete time is the *complex exponential signal*, defined as

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. See that this could also be expressed as

$$x[n] = Ce^{\beta n}$$

where $\alpha = e^\beta$. See that we can constrain β to be purely imaginary:

$$x[n] = e^{i\omega_0 n}$$

Periodicity properties of Discrete-time complex exponentials

While there are many similarities between continuous and discrete-time signals, there are a number of important differences. For the continuous time signal $e^{i\omega_0 t}$, we know that

- The larger the magnitude of ω_0 , the higher the rate of oscillation of the signal
- $e^{i\omega_0 t}$ is periodic for any value of ω_0

These properties are different in the discrete-time case.

Given the first property, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{i(\omega_0+2\pi)n} = e^{i2\pi n}e^{i\omega_0 n} = e^{i\omega_0 n}$$

(see that this is a direct result of the fact that we iterate through discrete time as integers) The exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0 . This is unlike the continuous-time case where each distinct ω_0 represents a distinct signal.

In discrete time, the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$, and so on. Therefore when considering discrete time complex exponentials, see that we need only consider a frequency interval of length 2π in which to choose ω_0 , such as $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$.

Also see that because of this the discrete exponential $e^{i\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 increases in magnitude; the signals will oscillate faster until we reach $\omega_0 = \pi$, after which the rate of oscillation decreases until we reach $\omega_0 = 2\pi$, at which the same constant sequence as $\omega_0 = 0$ is produced.

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