

# Signals And Systems by Alan V. Oppenheim: Notes

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## 0.1 Introduction

### 0.1.1 Signal Energy and Power

#### Motivation and definition

In many but not all, applications, the signals considered directly related to physical quantities capturing power and energy in a physical system. (for instance  $v^2/R$  for the power across a resistor)

As such it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal, denoted  $x(t)$ , or any discrete-time signal  $x[n]$ . In this case, the total energy over the time interval  $t_1 \leq t \leq t_2$  in a continuous signal  $x(t)$  is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where  $|x|$  denotes the magnitude of the (possibly complex) number  $x$ ; see that the time-averaged signal can be obtained by dividing by  $(t_2 - t_1)$ . Similarly for a discrete signal  $x[n]$  over the interval  $n_1 \leq n \leq n_2$  the total energy is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

with the average power calculated by dividing by  $(n_2 - n_1 + 1)$ .

It is important to remember that the terms ‘power’ and ‘energy’ are used here *independently* of their relation to physical energy (they clearly don’t correlate since their units or scalings would differ). Nevertheless we will find it convenient to use these terms in a general fashion.

#### Power and energy over infinite intervals

Considering signals over an infinite time interval, meaning for  $-\infty < t < +\infty$  or  $-\infty < n < +\infty$ . Here we define the total energy as the limits of the aforementioned equations increase without bound; in continuous time,

$$E_\infty \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Note that these expressions may not converge; for instance say  $x(t)$  or  $x[n]$  equal some nonzero constant for all time: such signals have infinite energy, while signals with  $E_\infty < \infty$  have finite energy.

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Analagously, we can define the time-averaged power over an infinite interval as

$$P_{\infty} \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

and

$$P_{\infty} \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

In continuous and discrete time respectively.

**Intuition**

See that with these definitions, we can identify three classes of signals: first those with finite total energy, meaning  $E_{\infty} < \infty$ . See that such a signal would have zero average power:

$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{E_{\infty}}{2T} = 0$$

Second would be signals with finite average power  $P_{\infty}$ ; see from the above expression that for  $P_{\infty} > 0$ , this requires that  $E_{\infty} = \infty$ .

Last would be signals for which neither  $P_{\infty}$  nor  $E_{\infty}$  are finite. An example of this might be  $x(t) = t$ .

**Note on discrete signals**

It is important to note that the discrete-time signal  $x[n]$  is defined *only* for *integer* values of the independent variable.

### 0.1.2 Even and Odd signals

#### Definition

A continuous-time signal is *even* if

$$x(-t) = x(t)$$

while a discrete-time signal is *even* if

$$x[-n] = x[n]$$

These signals are referred to as *odd* if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n]$$

Note that an odd signal must be 0 at  $t = 0$  or  $n = 0$  since the equations require that  $x(0) = -x(0)$  and  $x[0] = -x[0]$ .

#### Decomposition

An important fact is that any signal can be broken into a sum of two signals, where one is even and the other odd. To see this, consider

$$\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the *even part* of  $x(t)$ . Similarly, the *odd part* of  $x(t)$  is given by

$$\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

See that  $x(t)$  is the sum of the two. Exactly analogous definitions hold in the discrete time case.



**Figure 1.18** Example of the even-odd decomposition of a discrete-time signal.

### 0.1.3 Differences between continuous and discrete periodic complex exponentials

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where  $C$  and  $a$  are, in general, complex numbers. An important class of complex exponentials is obtained by constraining  $a$  to be purely imaginary:

$$x(t) = e^{i\omega t}$$

#### Periodicity and harmonic relations (purely imaginary power)

An important property of this signal is that it is periodic; recall that  $x(t)$  will be periodic with period  $T$  if

$$e^{i\omega t} = e^{i\omega(t+T)}$$

this means

$$e^{i\omega(t+T)} = e^{i\omega t} e^{i\omega T} \implies e^{i\omega T} = 1$$

If  $\omega = 0$  then this is satisfied for any  $T$ . If  $\omega \neq 0$ , see that the *fundamental period*  $T_0$  of  $x(t)$ —that is, the smallest positive value of  $T$  for which this holds—is

$$T_0 = \frac{2\pi}{|\omega|}$$

(the signals  $e^{i\omega_0 t}$  and  $e^{-i\omega_0 t}$  have the same fundamental period) Naturally, there is a set of exponentials periodic to a common period  $T_0$ . These are said to be *harmonically related* complex exponentials; the necessary condition they satisfy is

$$e^{i\omega T_0} = 1$$

which implies that

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

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We had

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

if we define

$$\omega_0 = \frac{2\pi}{T_0}$$

this means that the harmonic frequencies  $\omega$  must be integer multiples of  $\omega_0$ :

$$\phi_k(t) = e^{ik\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \dots$$

For  $k = 0$ ,  $\phi_k(t)$  is a constant, while for any other value of  $k$ ,  $\phi_k(t)$  is periodic with fundamental frequency  $|k|\omega_0$  and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Each  $\phi_k(t)$  itself defines a fundamental frequency and a corresponding fundamental period. (see that  $|k|\omega_0 \cdot T_0/|k| = 2\pi$ , so this scaled down period is the corresponding period for this scaled up frequency. Each frequency is unique, point here is that they are also periodic with  $T_0$ )

Note that the  $k$ th harmonic  $\phi_k(t)$  is still periodic with  $T_0$ ; it goes through exactly  $|k|$  of its fundamental periods during any time interval of length  $T_0$ . (the term ‘harmonic’ is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency)

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### Discrete case

As in continuous time, an important signal in discrete time is the *complex exponential signal*, defined as

$$x[n] = C\alpha^n$$

where  $C$  and  $\alpha$  are, in general, complex numbers. See that this could also be expressed as

$$x[n] = Ce^{\beta n}$$

where  $\alpha = e^\beta$ . See that we can constrain  $\beta$  to be purely imaginary:

$$x[n] = e^{i\omega_0 n}$$

### Periodicity properties of Discrete-time complex exponentials

While there are many similarities between continuous and discrete-time signals, there are a number of important differences. For the continuous time signal  $e^{i\omega_0 t}$ , we know that

- The larger the magnitude of  $\omega_0$ , the higher the rate of oscillation of the signal
- $e^{i\omega_0 t}$  is periodic for any value of  $\omega_0$

These properties are different in the discrete-time case.

Given the first property, consider the discrete-time complex exponential with frequency  $\omega_0 + 2\pi$ :

$$e^{i(\omega_0+2\pi)n} = e^{i2\pi n} e^{i\omega_0 n} = e^{i\omega_0 n}$$

(see that this is a direct result of the fact that we iterate through discrete time as integers) The exponential at frequency  $\omega_0 + 2\pi$  is the *same* as that at frequency  $\omega_0$ . This is unlike the continuous-time case where each distinct  $\omega_0$  represents a distinct signal.

In discrete time, the signal with frequency  $\omega_0$  is identical to the signals with frequencies  $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$ , and so on. Therefore when considering discrete time complex exponentials, see that we need only consider a frequency interval of length  $2\pi$  in which to choose  $\omega_0$ , such as  $0 \leq \omega_0 < 2\pi$  or  $-\pi \leq \omega_0 < \pi$ .

Also see that because of this the discrete exponential  $e^{i\omega_0 n}$  does *not* have a continually increasing rate of oscillation as  $\omega_0$  increases in magnitude; the signals will oscillate faster until we reach  $\omega_0 = \pi$ , after which the rate of oscillation decreases until we reach  $\omega_0 = 2\pi$ , at which the same constant sequence as  $\omega_0 = 0$  is produced.

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The second property we wish to consider concerns the periodicity of the discrete time complex exponential. In order for the signal  $e^{i\omega_0 n}$  to be periodic with period  $N > 0$  we must have

$$e^{i\omega_0(n+N)} = e^{i\omega_0 n}$$

or equivalently

$$e^{i\omega_0 N} = 1$$

For this to hold,  $\omega_0 N$  must be a multiple of  $2\pi$ . That is, there must be an integer  $m$  such that

$$\omega_0 N = 2\pi m$$

or equivalently

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

The signal  $e^{i\omega_0 n}$  is periodic if  $\omega_0/2\pi$  is a rational number and is not periodic otherwise.

**Fundamental period**

Recall the idea of a *fundamental period*; in this case it would mean the smallest  $N$  such that  $\omega_0 N = 2\pi m$  holds (this is unlike the continuous case where there is always some  $T$  where  $\omega_0 T = 2\pi$ ); see that this occurs when  $m$  and  $N$  do not have any factors in common.

See that from this we can derive a *fundamental frequency* as

$$\frac{2\pi}{N} = \frac{\omega_0}{m}$$

(see that this frequency is always equal or lower—intuitively, to have a different wave that completes one oscillation in  $N$  time, its frequency will either be equal or lower)

To summarize

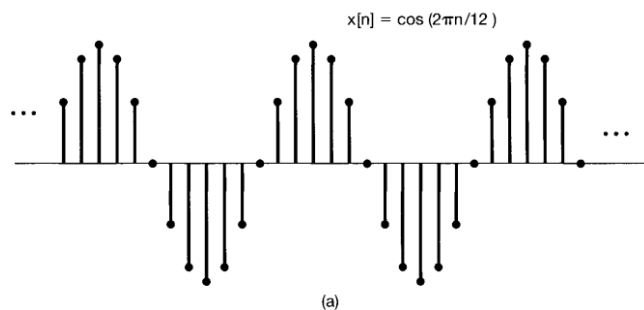
**TABLE 1.1** Comparison of the signals  $e^{i\omega_0 t}$  and  $e^{i\omega_0 n}$ .

$e^{i\omega_0 t}$	$e^{i\omega_0 n}$
Distinct signals for distinct values of $\omega_0$	Identical signals for values of $\omega_0$ separated by multiples of $2\pi$
Periodic for any choice of $\omega_0$	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and $m$ .
Fundamental frequency $\omega_0$	Fundamental frequency* $\omega_0/m$
Fundamental period $\omega_0 = 0$ : undefined $\omega_0 \neq 0$ : $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$ : undefined $\omega_0 \neq 0$ : $m \left( \frac{2\pi}{\omega_0} \right)$

\*Assumes that  $m$  and  $N$  do not have any factors in common.

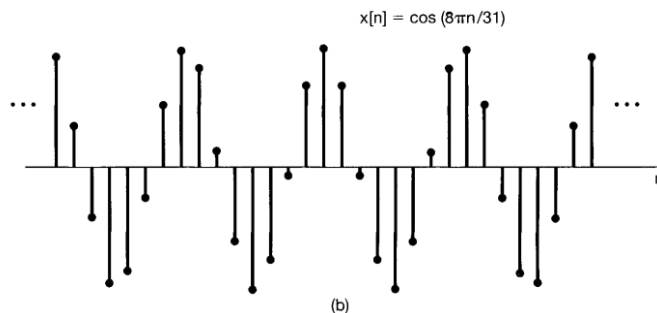
### 0.1.4 Intuition for discrete-time periodicity

Consider the sequence  $x[n] = \cos(2\pi n/12)$ :



we can think of this as a set of samples of the continuous-time sinusoid  $x(t) = \cos(2\pi t/12)$  at integer time points. In this case, see that both  $x(t)$  and  $x[n]$  are periodic with fundamental period 12. That is, the values of  $x[n]$  repeat every 12 points, exactly in step with the fundamental period of  $x(t)$ .

Now consider the signal  $x[n] = \cos(8\pi n/31)$ :



This can also be viewed as a set of samples of  $x(t) = \cos(8\pi t/31)$  at integer points in time. But now see that in this case  $x(t)$  is periodic with fundamental period  $31/4$ , while  $x[n]$  is periodic with fundamental period 31.

This difference stems from the fact that the discrete-time signal is defined only for integer values of the independent variable—there is no sample at time  $t = 31/4$ , when  $x(t)$  completes one period, or at  $t = 2 \cdot 31/4$  or  $t = 3 \cdot 31/4$ , when  $x(t)$  has completed two or three periods. Only at sample  $t = 4 \cdot 31/4 = 31$ , when  $x(t)$  has completed *four* periods is the discrete sequence defined.

This manifests as the pattern of  $x[n]$  not repeating with each cycle of positive and negative values, but rather only after four of such cycles, specifically 31 points. (next page)

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Finally consider

**0.1.5 Difference in harmonic relations in discrete and continuous periodic exponentials**