

# Signals And Systems by Alan V. Oppenheim: Notes

Malcolm

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## 0.1 Introduction

### 0.1.1 Signal Energy and Power

#### Motivation and definition

In many but not all, applications, the signals considered directly related to physical quantities capturing power and energy in a physical system. (for instance  $v^2/R$  for the power across a resistor)

As such it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal, denoted  $x(t)$ , or any discrete-time signal  $x[n]$ . In this case, the total energy over the time interval  $t_1 \leq t \leq t_2$  in a continuous signal  $x(t)$  is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where  $|x|$  denotes the magnitude of the (possibly complex) number  $x$ ; see that the time-averaged signal can be obtained by dividing by  $(t_2 - t_1)$ . Similarly for a discrete signal  $x[n]$  over the interval  $n_1 \leq n \leq n_2$  the total energy is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

with the average power calculated by dividing by  $(n_2 - n_1 + 1)$ .

It is important to remember that the terms ‘power’ and ‘energy’ are used here *independently* of their relation to physical energy (they clearly don’t correlate since their units or scalings would differ). Nevertheless we will find it convenient to use these terms in a general fashion.

#### Power and energy over infinite intervals

Considering signals over an infinite time interval, meaning for  $-\infty < t < +\infty$  or  $-\infty < n < +\infty$ . Here we define the total energy as the limits of the aforementioned equations increase without bound; in continuous time,

$$E_\infty \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Note that these expressions may not converge; for instance say  $x(t)$  or  $x[n]$  equal some nonzero constant for all time: such signals have infinite energy, while signals with  $E_\infty < \infty$  have finite energy.

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Analagously, we can define the time-averaged power over an infinite interval as

$$P_{\infty} \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

and

$$P_{\infty} \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

In continuous and discrete time respectively.

**Intuition**

See that with these definitions, we can identify three classes of signals: first those with finite total energy, meaning  $E_{\infty} < \infty$ . See that such a signal would have zero average power:

$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{E_{\infty}}{2T} = 0$$

Second would be signals with finite average power  $P_{\infty}$ ; see from the above expression that for  $P_{\infty} > 0$ , this requires that  $E_{\infty} = \infty$ .

Last would be signals for which neither  $P_{\infty}$  nor  $E_{\infty}$  are finite. An example of this might be  $x(t) = t$ .

**Note on discrete signals**

It is important to note that the discrete-time signal  $x[n]$  is defined *only* for *integer* values of the independent variable.

### 0.1.2 Even and Odd signals

#### Definition

A continuous-time signal is *even* if

$$x(-t) = x(t)$$

while a discrete-time signal is *even* if

$$x[-n] = x[n]$$

These signals are referred to as *odd* if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n]$$

Note that an odd signal must be 0 at  $t = 0$  or  $n = 0$  since the equations require that  $x(0) = -x(0)$  and  $x[0] = -x[0]$ .

#### Decomposition

An important fact is that any signal can be broken into a sum of two signals, where one is even and the other odd. To see this, consider

$$\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the *even part* of  $x(t)$ . Similarly, the *odd part* of  $x(t)$  is given by

$$\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

See that  $x(t)$  is the sum of the two. Exactly analogous definitions hold in the discrete time case.



**Figure 1.18** Example of the even-odd decomposition of a discrete-time signal.

### 0.1.3 More on Complex Exponential and Sinusoidal Signals

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where  $C$  and  $a$  are, in general, complex numbers. An important class of complex exponentials is obtained by constraining  $a$  to be purely imaginary:

$$x(t) = e^{i\omega_0 t}$$

#### Periodicity and harmonic relations (purely imaginary power)

An important property of this signal is that it is periodic; recall that  $x(t)$  will be periodic with period  $T$  if

$$e^{i\omega_0 t} = e^{i\omega_0(t+T)}$$

this means

$$e^{i\omega_0(t+T)} = e^{i\omega_0 t} e^{i\omega_0 T} \implies e^{i\omega_0 T} = 1$$

If  $\omega_0 = 0$  then this is satisfied for any  $T$ . If  $\omega_0 \neq 0$ , see that the *fundamental period*  $T_0$  of  $x(t)$ —that is, the smallest positive value of  $T$  for which this holds—is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

(the signals  $e^{i\omega_0 t}$  and  $e^{-i\omega_0 t}$  have the same fundamental period) Naturally, there is a set of exponentials periodic to a common period  $T_0$ . These are said to be *harmonically related* complex exponentials; the necessary condition they satisfy is

$$e^{i\omega T_0} = 1$$

which implies that

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

if we define

$$\omega_0 = \frac{2\pi}{T_0}$$

this means that the harmonic frequencies  $\omega$  must be integer multiples of  $\omega_0$ :

$$\phi_k(t) = e^{ik\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \dots$$

For  $k = 0$ ,  $\phi_k(t)$  is a constant, while for any other value of  $k$ ,  $\phi_k(t)$  is periodic with fundamental frequency  $|k|\omega_0$  and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Note that the  $k$ th harmonic  $\phi_k(t)$  is still periodic with  $T_0$ ; it goes through exactly  $|k|$  of its fundamental periods during any time interval of length  $T_0$ . (the term ‘harmonic’ is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency)

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**Discrete case**

As in continuous time, an important signal in discrete time is the *complex exponential signal*, defined as

$$x[n] = C\alpha^n$$

where  $C$  and  $\alpha$  are, in general, complex numbers. See that this could also be expressed as

$$x[n] = Ce^{\beta n}$$

where  $\alpha = e^\beta$ . See that we can constrain  $\beta$  to be purely imaginary:

$$x[n] = e^{i\omega_0 n}$$

**Periodicity properties of Discrete-time complex exponentials**

While