Signals And Systems by Alan V. Oppenheim: Notes

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0.1 Introduction

0.1.1 Signal Energy and Power

Motivation and definition
In many but not all, applications, the signals considered directly related to physical quantities capturing power and energy in a physical system. (for instance v^2/R for the power across a resistor)

As such it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal, denoted $x(t)$, or any discrete-time signal $x[n]$. In this case, the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x; see that the time-averaged signal can be obtained by dividing by $(t_2 - t_1)$. Similarly for a discrete signal $x[n]$ over the interval $n_1 \leq n \leq n_2$ the total energy is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

with the average power calculated by dividing by $(n_2 - n_1 + 1)$.

It is important to remember that the terms 'power' and 'energy' are used here *independently* of their relation to physical energy (they clearly don't correlate since their units or scalings would differ). Nevertheless we will find it convenient to use these terms in a general fashion.

Power and energy over infinite intervals
Considering signals over an infinite time interval, meaning for $-\infty < t < +\infty$ or $-\infty < n < +\infty$. Here we define the total energy as the limits of the aforementioned equations increase without bound; in continuous time,

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N \to \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Note that these expressions may not converge; for instance say $x(t)$ or $x[n]$ equal some nonzero constant for all time: such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.
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Cont.
Analagously, we can define the time-averaged power over an infinite interval as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

and

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2$$

In continuous and discrete time respectively.

Intuition
See that with these definitions, we can identify three classes of signals: first those with finite total energy, meaning $E_\infty < \infty$. See that such a signal would have zero average power:

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

Second would be signals with finite average power P_∞; see from the above expression that for $P_\infty > 0$, this requires that $E_\infty = \infty$.

Last would be signals for which neither P_∞ nor E_∞ are finite. An example of this might be $x(t) = t$.

Note on discrete signals
It is important to note that the discrete-time signal $x[n]$ is defined *only* for *integer* values of the independent variable.

0.1.2 Even and Odd signals

Definition
A continuous-time signal is *even* if

$$x(-t) = x(t)$$

while a discrete-time signal is *even* if

$$x[-n] = x[n]$$

These signals are referred to as *odd* if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

Note that an odd signal must be 0 at $t = 0$ or $n = 0$ since the equations require that $x(0) = -x(0)$ and $x[0] = -x[0]$.

Decomposition
An important fact is that any signal can be broken into a sum of two signals, where one is even and the other odd. To see this, consider

$$\mathrm{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\mathrm{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

See that $x(t)$ is the sum of the two. Exactly analagous definitions hold in the discrete time case.

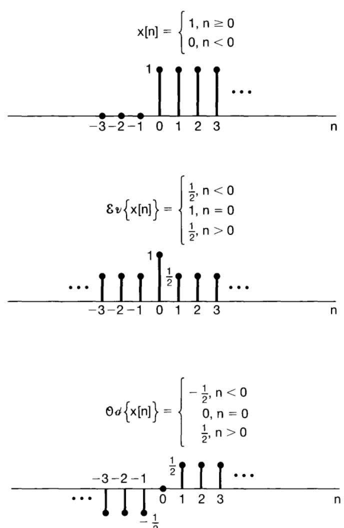

Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

0.1.3 Differences between continuous and discrete periodic complex exponentials

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers. An important class of complex exponentials is obtained by constraining a to be purely imaginary:

$$x(t) = e^{i\omega t}$$

Periodicity and harmonic relations (purely imaginary power)
An important property of this signal is that it is periodic; recall that $x(t)$ will be periodic with period T if

$$e^{i\omega t} = e^{i\omega(t+T)}$$

this means

$$e^{i\omega(t+T)} = e^{i\omega t}e^{i\omega T} \implies e^{i\omega T} = 1$$

If $\omega = 0$ then this is satisfied for any T. If $\omega \neq 0$, see that the *fundamental period* T_0 of $x(t)$—that is, the smallest positive value of T for which this holds—is

$$T_0 = \frac{2\pi}{|\omega|}$$

(the signals $e^{i\omega 0t}$ and $e^{-i\omega t}$ have the same fundamental period) Naturally, there is a set of exponentials periodic to a common period T_0. These are said to be *harmonically related* complex exponentials; the necessary condition they satisfy is

$$e^{i\omega T_0} = 1$$

which implies that

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

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Cont.
We had

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

if we define

$$\omega_0 = \frac{2\pi}{T_0}$$

this means that the harmonic frequencies ω must be integer multiples of ω_0:

$$\phi_k(t) = e^{ik\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Each $\phi_k(t)$ itself defines a fundamental frequency and a corresponding fundamental period. (see that $|k|\omega_0 \cdot T_0/|k| = 2\pi$, so this scaled down period is the corresponding period for this scaled up frequency. Each frequency is unique, point here is that they are also periodic with T_0, but with fundamental periods getting proportionally smaller.)

Note that the kth harmonic $\phi_k(t)$ is still periodic with T_0; it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0. (the term 'harmonic' is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency)
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Discrete case
As in continuous time, an important signal in discrete time is the *complex exponential signal*, defined as
$$x[n] = C\alpha^n$$
where C and α are, in general, complex numbers. See that this could also be expressed as
$$x[n] = Ce^{\beta n}$$
where $\alpha = e^{\beta}$. See that we can constrain β to be purely imaginary:
$$x[n] = e^{i\omega_0 n}$$

Periodicity properties of Discrete-time complex exponentials
While there are many similarities between continuous and discrete-time signals, there are a number of important differences. For the continuous time signal $e^{i\omega_0 t}$, we know that

- The larger the magnitude of ω_0, the higher the rate of oscillation of the signal
- $e^{i\omega_0 t}$ is periodic for any value of ω_0

These properties are different in the discrete-time case.

Given the first property, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:
$$e^{i(\omega_0+2\pi)n} = e^{i2\pi n}e^{i\omega_0 n} = e^{i\omega_0 n}$$
(see that this is a direct result of the fact that we iterate through discrete time as integers) The exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. This is unlike the continuous-time case where each distinct ω_0 represents a distinct signal.

In discrete time, the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$, and so on. Therefore when considering discrete time complex exponentials, see that we need only consider a frequency interval of length 2π in which to choose ω_0, such as $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$.

Also see that because of this the discrete exponential $e^{i\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 increases in magnitude; the signals will oscillate faster until we reach $\omega_0 = \pi$, after which the rate of oscillation decreases until we reach $\omega_0 = 2\pi$, at which the same constant sequence as $\omega_0 = 0$ is produced.
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Cont.
The second property we wish to consider concerns the periodicity of the discrete time complex exponential. In order for the signal $e^{i\omega_0 n}$ to be periodic with period $N > 0$ we must have

$$e^{i\omega_0(n+N)} = e^{i\omega_0 n}$$

or equivalently

$$e^{i\omega_0 N} = 1$$

For this to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m$$

or equivalently

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

The signal $e^{i\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise.

Fundamental period
Recall the idea of a *fundamental period*; in this case it would mean the smallest N such that $\omega_0 N = 2\pi m$ holds (this is unlike the continuous case where there is always some T where $\omega_0 T = 2\pi$); see that this occurs when m and N do not have any factors in common.

See that from this we can derive a *fundamental frequency* as

$$\frac{2\pi}{N} = \frac{\omega_0}{m}$$

(see that this frequency is always equal or lower—intuitively, to have a different wave that completes one oscillation in N time, its frequency will either be equal or lower)

To summarize

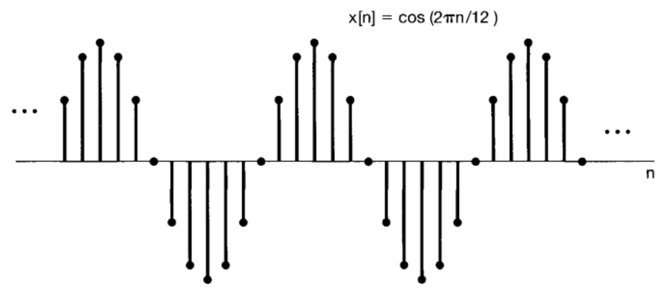
TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

0.1.4 Intuition for discrete-time periodicity

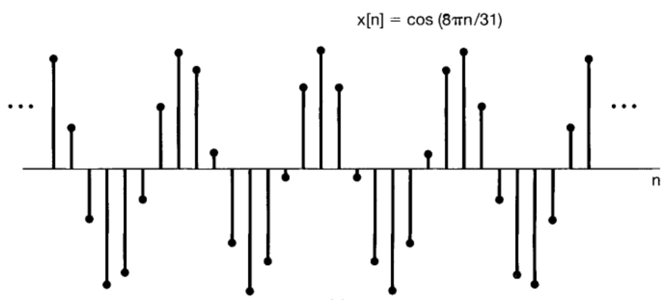
Consider the sequence $x[n] = \cos(2\pi n/12)$:



(a)

we can think of this as a set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, see that both $x(t)$ and $x[n]$ are periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

Now consider the signal $x[n] = \cos(8\pi n/31)$:



(b)

This can also be viewed as a set of samples of $x(t) = \cos(8\pi t/31)$ at integer points in time. But now see that in this case $x(t)$ is periodic with fundamental period 31/4, while $x[n]$ is periodic with fundamental period 31.

This difference stems from the fact that the discrete-time signal is defined only for integer values of the independent variable—there is no sample at time $t = 31/4$, when $x(t)$ completes one period, or at $t = 2 \cdot 31/4$ or $t = 3 \cdot 31/4$, when $x(t)$ has completed two or three periods. Only at sample $t = 4 \cdot 31/4 = 31$, when $x(t)$ has completed *four* periods is the discrete sequence defined.

This manifests as the pattern of $x[n]$ not repeating with each cycle of positive and negative values, but rather only after four of such cycles, specifically 31 points.
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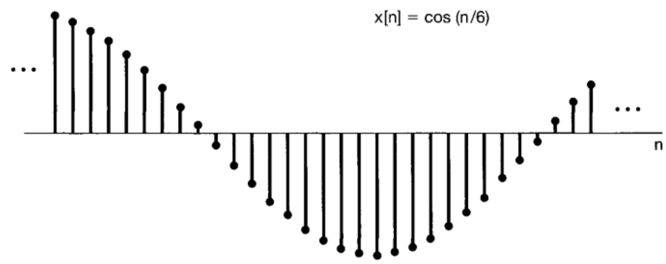
Cont.
Finally consider the signal $x[n] = \cos(n/6)$:



(c)

In this case, the values of $x(t)$ at integer sample points *never repeat*, as these sample points never span an interval that is an exact multiple of the period, 12π, of $x(t)$.

Thus, $x[n]$ *is not periodic*, although the eye visually interpolates between the sample points, suggesting *the envelope* $x(t)$ which is periodic.

0.1.5 Difference in harmonic relations in discrete and continuous periodic exponentials

As in continuous time, it is also of considerable value in discrete-time to consider sets of harmonically related periodic exponentials—that is, *periodic exponentials with a common period* N.

We know that these are precisely the signals which are at frequencies which are multiples of $2\pi/N$; that is

$$\phi_k[n] = e^{ik(2\pi/N)n}, \quad k = 0, \pm 1, \ldots$$

In the continuous-time case, all the harmonically related complex exponentials $e^{ik(2\pi/T_0)t}$, $k = 0, \pm 1, \pm 2, \ldots$ are distinct. However, recall that for discrete signals we have

$$e^{i(\omega_0 + 2\pi)n} = e^{i2\pi n} e^{i\omega_0 n} = e^{i\omega_0 n}$$

(this is a direct result of the fact that we iterate through discrete time as integers) As such the harmonically related complex exponentials *are not all unique in discrete time*; specifically,

$$\begin{aligned}\phi_{k+N}[n] &= e^{i(k+N)(2\pi/N)n} \\ &= e^{ik(2\pi/N)n} e^{i2\pi n} = \phi_k[n]\end{aligned}$$

See that this implies that there are only N distinct periodic exponentials in the set of $\phi_k[n]$; meaning

$$\phi_0[n] = 1,\ \phi_1[n] = e^{i(2\pi/N)n},\ \phi_2[n] = e^{i2(2\pi/N)n}, \ldots,\ \phi_{N-1}[n] = e^{i(N-1)(2\pi/N)n}$$

are all distinct, but any other $\phi_k[n]$ would just be identical to one of them. (for instance $\phi_N[n] = \phi_0[n]$ or $\phi_{-1}[n] = \phi_{N-1}[n]$.)

0.1.6 More on complex exponential and sinusoidal signals

Continuous case
A continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers.

Euler identity and 'combined' sinusoidal form
Recall euler's identity:

$$e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t)$$

See that the scaled and phase-delayed sinusoid can be written in terms of these periodic complex exponentials with the same fundamental period:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{i\phi}e^{i\omega_0 t} + \frac{A}{2}e^{-i\phi}e^{-i\omega_0 t}$$

We can also express

$$A\cos(\omega_0 t + \phi) = A\mathrm{Re}\{e^{i(\omega_0 t+\phi)}\}$$

and

$$A\sin(\omega_0 t + \phi) = A\mathrm{Im}\{e^{i(\omega_0 t+\phi)}\}$$

Energy and power
Periodic signals—and in particular, the complex periodic exponential signal—are examples of signals with infinite total energy but finite average power. Calculating the total energy and of the periodic exponential signal over one period:

$$\begin{aligned} E_{\text{period}} &= \int_0^{T_0} |e^{i\omega_0 t}|^2 dt \\ &= \int_0^{T_0} 1\, dt = T_0 \end{aligned}$$

(The absolute value of a complex number is its magnitude. Think of the absolute value as the (possibly multidimensional) distance from zero.) Calculating the average power:

$$P_{\text{period}} = \frac{1}{T_0}E_{\text{period}} = 1$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. However, since the average power over each period is 1, averaging over multiple periods always yields an average power of 1. That is, the complex periodic exponential signal has finite average power equal to

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |e^{i\omega_0 t}|^2 dt = 1$$

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General continuous complex exponential signals
In the most general case Ce^{at} where both C and a are complex, see that since C and a can are just

$$C = |C|e^{i\theta}, \quad a = r + i\omega_0$$

we can express the general complex signal as

$$Ce^{at} = |C|e^{i\theta}e^{(r+i\omega_0)t} = |C|e^{rt}e^{i(\omega_0 t+\theta)}$$

we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + i|C|e^{rt}\sin(\omega_0 t + \theta)$$

Discrete case
As in continuous time, the discrete time *complex exponential signal* is defined by

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}$$

where $\alpha = e^{\beta}$.

Euler identity and 'combined' sinusoidal form
As with the continuous case, constraining β to be purely imaginary, we have euler's identity

$$e^{i\omega_0 n} = \cos\omega_0 n + i\sin\omega_0 n$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{i\phi}e^{i\omega_0 n} + \frac{A}{2}e^{-i\phi}e^{-i\omega_0 n}$$

General discrete complex exponential signals
As with the continuous case, for complex C and α, we have

$$C = |C|e^{i\theta}, \quad \alpha = |\alpha|e^{i\omega_0}$$

so the general complex exponential signal can be expressed as

$$C\alpha^n = |C||\alpha|^n\cos(\omega_0 n + \theta) + i|C||\alpha|^n\sin(\omega_0 n + \theta)$$

0.1.7 Unit impulse and Unit step functions

Discrete-Time

One of the simplest discrete-time signals is the *unit impulse/unit sample*, defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

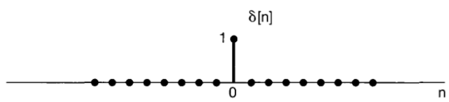

Figure 1.28 Discrete-time unit impulse (sample).

Another basic discrete-time signal is the discrete-time *unit step*, denoted by $u[n]$ and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

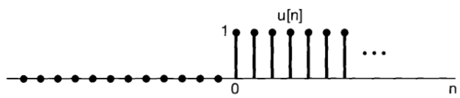

Figure 1.29 Discrete-time unit step sequence.

See that the discrete-time unit impulse is the *first difference* of the discrete-time step:

$$\delta[n] = u[n] - u[n-1]$$

Conversely, the discrete-time unit step is the *running sum* of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

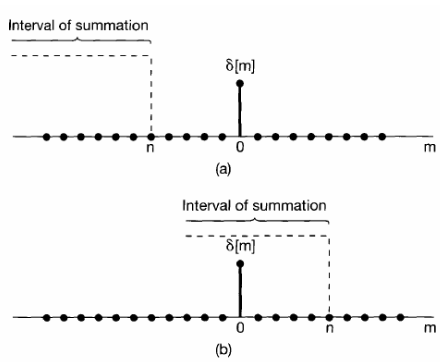

Figure 1.30 Running sum of eq. (1.66): (a) $n < 0$; (b) $n > 0$.

Since the only nonzero value of the unit sample is at 0, the running sum is 0 for $n < 0$ and 1 for $n \geq 0$.
(next page)

Cont.
we had the discrete-time unit step as the running sum of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

See that by changing the variable of summation from m to $k = n - m$, we can rewrite this as

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

and equivalently

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

An interpretation of this is a superposition of delayed impulses; we can view the unit step as the sum of unit impulses $\delta[n]$ (nonzero at $n = 0$), $\delta[n-1]$ (nonzero at $n = 1$), and all other $\delta[n-k]$ for integer k extending to infinity.

See that the unit impulse can also be used to sample the value of a signal at $n = 0$; since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n]$$

More generally, if we consider a unit impulse $\delta[n - n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

(next page)

Continuous-Time

The continuous-time *unit step function* $u(t)$ is defined in a similar manner to its discrete-time counterpart:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

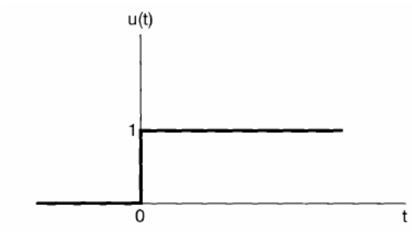

Figure 1.32 Continuous-time unit step function.

Note that the unit step is *discontinuous* at $t = 0$. The continuous-time *unit inpulse* $\delta(t)$ is related to the unit step in a manner analagous to that of their discrete counterparts; in particular, the continuous-time unit step is the *running integral* of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

It also follows that the continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step:

$$\delta(t) = \frac{du(t)}{dt}$$