Signals And Systems by Alan V. Oppenheim: Notes

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Contents

0.1 Introduction

0.1.1 Signal Energy and Power

Motivation and definition
In many but not all, applications, the signals considered directly related to physical quantities capturing power and energy in a physical system. (for instance v^2/R for the power across a resistor)

As such it is a common and worthwhile convention to use similar terminology for power and energy for *any* continuous-time signal, denoted $x(t)$, or any discrete-time signal $x[n]$. In this case, the total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x; see that the time-averaged signal can be obtained by dividing by $(t_2 - t_1)$. Similarly for a discrete signal $x[n]$ over the interval $n_1 \leq n \leq n_2$ the total energy is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

with the average power calculated by dividing by $(n_2 - n_1 + 1)$.

It is important to remember that the terms 'power' and 'energy' are used here *independently* of their relation to physical energy (they clearly don't correlate since their units or scalings would differ). Nevertheless we will find it convenient to use these terms in a general fashion.

Power and energy over infinite intervals
Considering signals over an infinite time interval, meaning for $-\infty < t < +\infty$ or $-\infty < n < +\infty$. Here we define the total energy as the limits of the aforementioned equations increase without bound; in continuous time,

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and in discrete time,

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Note that these expressions may not converge; for instance say $x(t)$ or $x[n]$ equal some nonzero constant for all time: such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.
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Analagously, we can define the time-averaged power over an infinite interval as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

and

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2$$

In continuous and discrete time respectively.

Intuition
See that with these definitions, we can identify three classes of signals: first those with finite total energy, meaning $E_\infty < \infty$. See that such a signal would have zero average power:

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

Second would be signals with finite average power P_∞; see from the above expression that for $P_\infty > 0$, this requires that $E_\infty = \infty$.

Last would be signals for which neither P_∞ nor E_∞ are finite. An example of this might be $x(t) = t$.

Note on discrete signals
It is omportant to note that the discrete-time signal $x[n]$ is defined *only* for *integer* values of the independent variable.

0.1.2 Even and Odd signals

Definition
A continuous-time signal is *even* if

$$x(-t) = x(t)$$

while a discrete-time signal is *even* if

$$x[-n] = x[n]$$

These signals are referred to as *odd* if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

Note that an odd signal must be 0 at $t = 0$ or $n = 0$ since the equations require that $x(0) = -x(0)$ and $x[0] = -x[0]$.

Decomposition
An important fact is that any signal can be broken into a sum of two signals, where one is even and the other odd. To see this, consider

$$\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

See that $x(t)$ is the sum of the two. Exactly analagous definitions hold in the discrete time case.

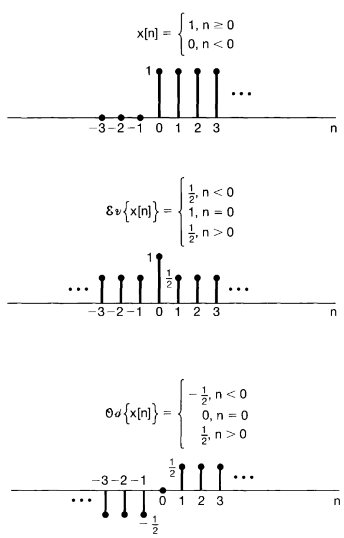

Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

0.1.3 Complex exponential and sinusoidal signals

0.1.4 Differences between continuous and discrete periodic complex exponentials

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers. An important class of complex exponentials is obtained by constraining a to be purely imaginary:

$$x(t) = e^{i\omega t}$$

Periodicity and harmonic relations (purely imaginary power)
An important property of this signal is that it is periodic; recall that $x(t)$ will be periodic with period T if

$$e^{i\omega t} = e^{i\omega(t+T)}$$

this means

$$e^{i\omega(t+T)} = e^{i\omega t}e^{i\omega T} \implies e^{i\omega T} = 1$$

If $\omega = 0$ then this is satisfied for any T. If $\omega \neq 0$, see that the *fundamental period* T_0 of $x(t)$—that is, the smallest positive value of T for which this holds—is

$$T_0 = \frac{2\pi}{|\omega|}$$

(the signals $e^{i\omega 0t}$ and $e^{-i\omega t}$ have the same fundamental period) Naturally, there is a set of exponentials periodic to a common period T_0. These are said to be *harmonically related* complex exponentials; the necessary condition they satisfy is

$$e^{i\omega T_0} = 1$$

which implies that

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

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Cont.
We had

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

if we define

$$\omega_0 = \frac{2\pi}{T_0}$$

this means that the harmonic frequencies ω must be integer multiples of ω_0:

$$\phi_k(t) = e^{ik\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Each $\phi_k(t)$ itself defines a fundamental frequency and a corresponding fundamental period. (see that $|k|\omega_0 \cdot T_0/|k| = 2\pi$, so this scaled down period is the corresponding period for this scaled up frequency. Each frequency is unique, point here is that they are also periodic with T_0, but with fundamental periods getting proportionally smaller.)

Note that the kth harmonic $\phi_k(t)$ is still periodic with T_0; it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0. (the term 'harmonic' is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency)
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Discrete case

As in continuous time, an important signal in discrete time is the *complex exponential signal*, defined as

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. See that this could also be expressed as

$$x[n] = Ce^{\beta n}$$

where $\alpha = e^{\beta}$. See that we can constrain β to be purely imaginary:

$$x[n] = e^{i\omega_0 n}$$

Periodicity properties of Discrete-time complex exponentials

While there are many similarities between continuous and discrete-time signals, there are a number of important differences. For the continuous time signal $e^{i\omega_0 t}$, we know that

- The larger the magnitude of ω_0, the higher the rate of oscillation of the signal
- $e^{i\omega_0 t}$ is periodic for any value of ω_0

These properties are different in the discrete-time case.

Given the first property, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{i(\omega_0+2\pi)n} = e^{i2\pi n}e^{i\omega_0 n} = e^{i\omega_0 n}$$

(see that this is a direct result of the fact that we iterate through discrete time as integers) The exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. This is unlike the continuous-time case where each distinct ω_0 represents a distinct signal.

In discrete time, the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi, \omega_0 \pm 4\pi$, and so on. Therefore when considering discrete time complex exponentials, see that we need only consider a frequency interval of length 2π in which to choose ω_0, such as $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$.

Also see that because of this the discrete exponential $e^{i\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 increases in magnitude; the signals will oscillate faster until we reach $\omega_0 = \pi$, after which the rate of oscillation decreases until we reach $\omega_0 = 2\pi$, at which the same constant sequence as $\omega_0 = 0$ is produced.

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The second property we wish to consider concerns the periodicity of the discrete time complex exponential. In order for the signal $e^{i\omega_0 n}$ to be periodic with period $N > 0$ we must have

$$e^{i\omega_0(n+N)} = e^{i\omega_0 n}$$

or equivalently

$$e^{i\omega_0 N} = 1$$

For this to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m$$

or equivalently

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

The signal $e^{i\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise.

Fundamental period
Recall the idea of a *fundamental period*; in this case it would mean the smallest N such that $\omega_0 N = 2\pi m$ holds (this is unlike the continuous case where there is always some T where $\omega_0 T = 2\pi$); see that this occurs when m and N do not have any factors in common.

See that from this we can derive a *fundamental frequency* as

$$\frac{2\pi}{N} = \frac{\omega_0}{m}$$

(see that this frequency is always equal or lower—intuitively, to have a different wave that completes one oscillation in N time, its frequency will either be equal or lower)

To summarize

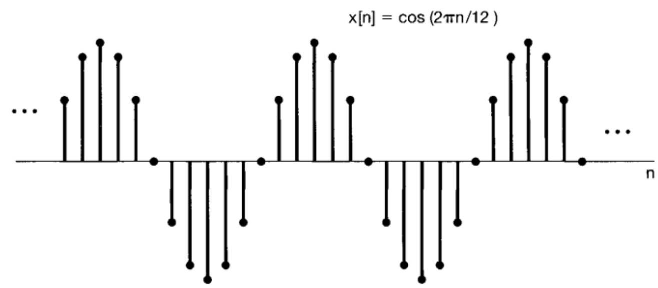
TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

0.1.5 Intuition for discrete-time periodicity

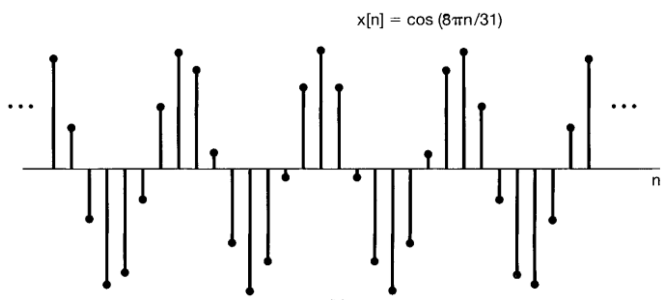
Consider the sequence $x[n] = \cos(2\pi n/12)$:



(a)

we can think of this as a set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, see that both $x(t)$ and $x[n]$ are periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

Now consider the signal $x[n] = \cos(8\pi n/31)$:



(b)

This can also be viewed as a set of samples of $x(t) = \cos(8\pi t/31)$ at integer points in time. But now see that in this case $x(t)$ is periodic with fundamental period 31/4, while $x[n]$ is periodic with fundamental period 31.

This difference stems from the fact that the discrete-time signal is defined only for integer values of the independent variable—there is no sample at time $t = 31/4$, when $x(t)$ completes one period, or at $t = 2 \cdot 31/4$ or $t = 3 \cdot 31/4$, when $x(t)$ has completed two or three periods. Only at sample $t = 4 \cdot 31/4 = 31$, when $x(t)$ has completed *four* periods is the discrete sequence defined.

This manifests as the pattern of $x[n]$ not repeating with each cycle of positive and negative values, but rather only after four of such cycles, specifically 31 points.
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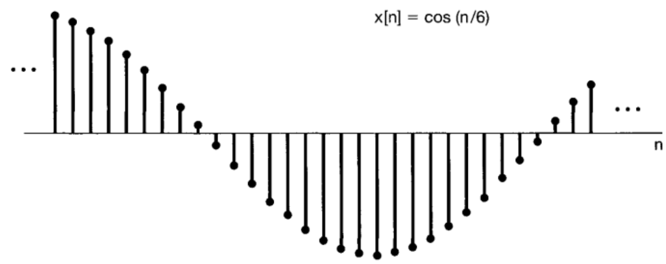
Finally consider the signal $x[n] = \cos(n/6)$:



(c)

In this case, the values of $x(t)$ at integer sample points *never repeat*, as these sample points never span an interval that is an exact multiple of the period, 12π, of $x(t)$.

Thus, $x[n]$ *is not periodic*, although the eye visually interpolates between the sample points, suggesting *the envelope* $x(t)$ which is periodic.

0.1.6 Difference in harmonic relations in discrete and continuous periodic exponentials

As in continuous time, it is also of considerable value in discrete-time to consider sets of harmonically related periodic exponentials—that is, *periodic exponentials with a common period N.*

We know that these are precisely the signals which are at frequencies which are multiples of $2\pi/N$; that is

$$\phi_k[n] = e^{ik(2\pi/N)n}, \quad k = 0, \pm 1, \ldots$$

In the continuous-time case, all the harmonically related complex exponentials $e^{ik(2\pi/T_0)t}$, $k = 0, \pm 1, \pm 2, \ldots$ are distinct. However, recall that for discrete signals we have

$$e^{i(\omega_0 + 2\pi)n} = e^{i2\pi n} e^{i\omega_0 n} = e^{i\omega_0 n}$$

(this is a direct result of the fact that we iterate through discrete time as integers) As such the harmonically related complex exponentials *are not all unique in discrete time*; specifically,

$$\begin{aligned} \phi_{k+N}[n] &= e^{i(k+N)(2\pi/N)n} \\ &= e^{ik(2\pi/N)n} e^{i2\pi n} = \phi_k[n] \end{aligned}$$

See that this implies that there are only N distinct periodic exponentials in the set of $\phi_k[n]$; meaning

$$\phi_0[n] = 1,\ \phi_1[n] = e^{i(2\pi/N)n},\ \phi_2[n] = e^{i2(2\pi/N)n}, \ldots,\ \phi_{N-1}[n] = e^{i(N-1)(2\pi/N)n}$$

are all distinct, but any other $\phi_k[n]$ would just be identical to one of them. (for instance $\phi_N[n] = \phi_0[n]$ or $\phi_{-1}[n] = \phi_{N-1}[n]$.)