# Appendix 2

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# Contents

# Appendix A

# Differential Equations

## A.1 Fundamentals

### A.1.1 Introduction to Ordinary Differential Equations (ODEs)

Here we introduce intuition for Ordinary Differential Equations (ODEs) and introductory solving methods.

The simplest type of differential equation looks like:

$$\frac{dy}{dx} = f(x)$$

which can be solved by the antiderivative $y = \int f(x)\,dx$.

**Intuition**
Now we consider a more interesting example:

$$\frac{dy}{dx} + xy = 0$$

This equation can be solved by *separation of variables*:

$$\begin{aligned}\frac{dy}{dx} + xy &= 0 \\ \frac{dy}{dx} &= -xy \\ \frac{dy}{y} &= -x\,dx\end{aligned}$$

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Since the problem is now set up in terms of differentials rather than ratios of differentials, we can integrate both sides.

$$\int \frac{dy}{y} = -\int x\,dx$$
$$\ln y + c_1 = -\frac{x^2}{2} + c_2 \quad \text{(assume } y > 0\text{)}$$

We can combine the constants and simplify:

$$\ln y = -\frac{x^2}{2} + c$$
$$e^{\ln y} = e^{-x^2/2+c}$$
$$y = e^c e^{-x^2/2}$$
$$y = Ae^{-x^2/2}, \quad \text{(where } A = e^c\text{)}$$

(The more apt $\ln|y|$ simplifies to $\pm Ae^{-x^2/2}$, which doesn't matter since $A$ is some unspecific constant)

It turns out that our solution,

$$y = Ae^{-x^2/2}, \quad \text{(where } A = e^c\text{)}$$

Works for any constant multiple $A$. We can check this solution:

$$y = ae^{-x^2/2}$$
$$\frac{dy}{dx} = \frac{d}{dx} ae^{-x^2/2}$$
$$= a \cdot (-x)e^{-x^2/2}$$
$$= -x \cdot ae^{-x^2/2}$$
$$\frac{dy}{dx} = -xy$$

$A$ is determined by an initial condition; for instance if $y(0) = 1$, $A = 1$.

### A.1.2 Separation of Variables

Here we describe a rudimentary method for solving some differential equations—Separation of Variables.

In general, this method applies to differential equations of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Where we then *separate* the variables and integrate:

$$\begin{aligned}
\frac{dy}{dx} &= f(x)g(y) \\
\frac{dy}{g(y)} &= f(x)\,dx \\
h(y)\,dy &= f(x)\,dx \quad \text{where } h(y) = \frac{1}{g(y)} \\
\int h(y)\,dy &= \int f(x)\,dx
\end{aligned}$$

Antidifferentiating both sides:

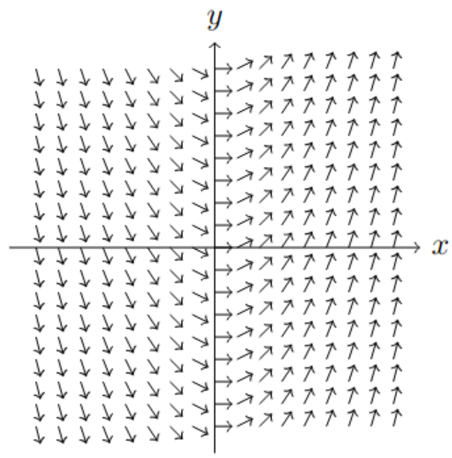
$$H(y) = \int h(y)\,dy; \quad F(x) = \int f(x)\,dx$$

we now have

$$\begin{aligned}
H(y) + c_1 &= F(x) + c_2 \\
H(y) &= F(x) + c
\end{aligned}$$

### A.1.3 Direction fields, Isoclines, and Integral curves

**Direction fields**
Given an equation $y' = f(x, y)$, we can construct a *direction field*; imagine through each point $(x, y)$, we draw a line segment whose slope is $f(x, y)$—consider $y'(x) = 2x$:




(note that in this case $f(x, y)$ does not depend on $y$ (because of the equation)—it is invariant under vertical translation)

**Plotting direction fields—Isoclines**
In practice, computers are used to plot direction fields following the procedure:

1. Pick point $(x, y)$
2. Compute $y' = f(x, y)$
3. Plot line segment of slope at that point

Notice how a new slope has to be computed for each specified point; when plotting direction fields by hand, it is much more practical to utilise *isoclines*, which are, given the equation $y' = f(x, y)$, a one-parameter family of curves given by the equations

$$f(x, y) = m, \quad m \text{ constant}$$

Along a given isocline, all line segments have the same slope $m$.
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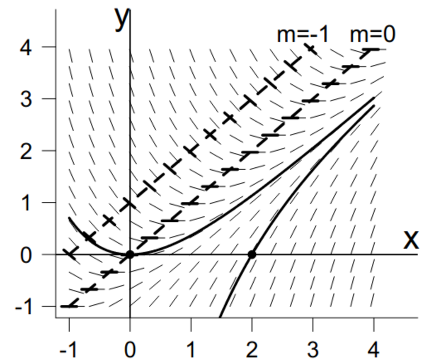
**Example**
Consider plotting the direction field for the equation $y' = x - y$; the isoclines are correspondingly the lines $x - y = m$ (shown in dashed lines):




The $m = 0$ isocline marks the points where the slope of the solution is 0; it is therefore of special interest and is called the *nullcline*.

**Integral curves**
As also shown in the figure above, once the direction field has been sketched, curves which are *at each point tangent to the line segment at that point* can be drawn; such curves are called *integral curves* or *solution curves* for the direction field. Their significance (this should be obvious) is that

*The integral curves are the graphs of the solutions to* $y' = f(x, y)$

Two integral curves have been drawn above (in solid lines).

**Intersection Principle**
Intuitively, see that at any point in the direction field it can only have one direction; therefore it is fairly obvious that integral curves cannot cross at an angle.

Consider the existence and uniqueness theorem for ODEs:

> *For any* $(a, b)$ *in the region where* $f$ *is defined,* $y' = f(x, y)$ *has exactly one solution such that* $y(a) = b$*.*

by the existence part of the theorem, there is an integral curve through any point where $f(x, y)$ is defined. Now supposing two integral curves through the same point, by the uniqueness part of the theorem they must agree.

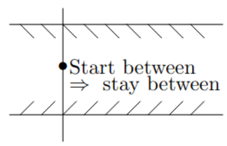
As a result, *integral curves cannot intersect*; every point lies on exactly one integral curve.

### A.1.4 Long term Behaviour: Fences, Funnels, and Separatrices

**Fences**

A *lower fence* for the equation $y' = f(x, y)$ is a curve that 'blocks' an integral curve from crossing from *above*; intuitively it is the curve whose direction field elements along the curve point up from it. Technically it can be described as a curve $y = L(x)$ such that $L'(x) < f(x, L(x))$ (the slope of the curve is always less than the slope of the direction field at that point).

Likewise an *upper fence* is a curve that 'blocks' integral curves from crossing from *above*. Illustrated:




(The upper curve is the upper fence and the lower curve is the lower fence). Solutions will be 'squeezed' between upper and lower fences.

Note that

- Note that fences aren't necessarily defined for all $x$; they could be defined only on an interval like $x \geq c$ for some constant $c$.
- Since integral curves can't cross an integral curve itself it is both an upper and lower fence.

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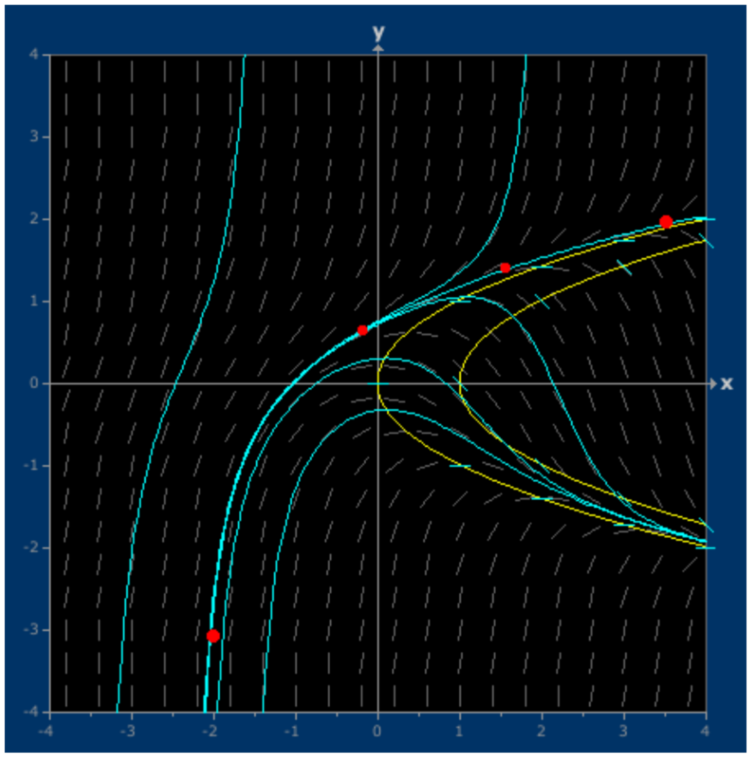
**Example**
Consider the direction field for the equation

$$y' = y^2 - x$$

The isoclines for $m = 0$ and $m = -1$ are plotted in yellow, with integral curves in blue:




Notice that the bottom hald of the isocline $m = 0$ is a lower fence and for $x$ large enough the bottom half of the isocline $m = -1$ is an upper fence. (notice that the $m = -1$ isocline becomes an upper fence only for $x$ large enough)

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**Funnels**

One use of fences is to construct funnels. A *funnel* for the equation $y' = f(x, y)$ consists of a pair of fences; one lower fence $L(x)$ and one upper fence $U(x)$ with the properties

1. For $x$ large the lower fence is below the upper fence; $L(x) < U(x)$
2. The two fences come together asymptotically; $U(x) - L(x)$ is small for large $x$

For instance, in the above example the bottom parts of the two isoclines $m = 0$ and $m = -1$ act as a funnel once $x$ is large enough. Given the equations of each isocline we have highly accurate estimates for solutions between them as

$$\underbrace{-\sqrt{x}}_{m=0} < y(x) < \underbrace{-\sqrt{x-1}}_{m=-1}$$

which is valid for large $x$.

Note that not all pairs of upper/lower fences form a funnel—they have to come together asymptotically as $x$ gets large.

**Separatrices**

A *separatrix* is an integral curve such that the integral curves above it behave entirely differently from integral curves below it as $x \to \infty$.

### A.1.5 Runge-Kutta 2 (Numerical methods)

**General approach and Euler's method**
Euler's method (for numerical estimation) follows a more general procedure for stepping from $(x_n, y_n)$ to $(x_{n+1}, y_{n+1})$:

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + m_n h$$

Where $h$ is the stepsize in the $x$ direction and $m$ is the slope of the line we step along. In Euler's method $h$ is fixed ahead of time and $m_n = f(x_n, y_n)$.

**Runge-Kutta 2**
Naturally Euler's method is a fairly flawed method of numerical estimation. Other methods use other (and better) ways of choosing $h$ and $m$. Here I describe the *Runge-Kutta 2* (RK2) method, which is a *fixed stepsize* method; meaning $h$ is fixed and the added complexity comes from finding $m$.

Given an initial value problem $y' = f(x, y), y(x_0) = x_0$ and a step size $h$, one step of the RK2 method is as follows:

1. Compute the slope $k_1$ at $(x_0, y_0)$: $k_1 = f(x_0, y_0)$
2. 'Take' an Euler step from $(x_0, y_0)$ to $(a, b) : a = x_0 + h$, $b = y_0 + k_1 h$
3. Compute the slope $k_2$ at $(a, b) : k_2 = f(a, b)$
4. Average $k_1$ and $k_2$ to get $m$: $m = (k_1 + k_2)/2$
5. Now we use this averaged slope to take a step from $(x_n, y_n)$ to $(x_{n+1}, y_{n+1})$:

$$x_1 = x_0 + h, \quad y_1 = y_0 + mh; \quad m = \frac{(k_1 + k_2)}{2}$$

Other methods such as RK4 or *variable stepsize methods* may (probably) work better. Though one might want to consider computational efficiency at the expense of accuracy.

### A.1.6 First order Linear Differential Equations

**Definition**
The general *First order linear ODE* in the unknown function $x = x(t)$ has the form

$$A(t)\frac{dx}{dt} + B(t)x(t) = C(t)$$

If $A(t) \neq 0$ we can simplify the equation by dividing by $A(t)$:

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

This is called the *standard form* for a first order linear ODE. Should the *coefficients* $A(t), B(t)$ be constants (not dependent on $t$) we say the equation is a *constant coefficient* DE.

If $C(t) = 0$:

$$A(t)\frac{dx}{dt} + B(t)x(t) = 0$$

The DE is called *homogeneous* (notice that conversion to standard form doesn't change this fact); otherwise the equation is *inhomogeneous*.

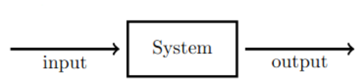
**Signals and Systems—Terminology**
Given a differential equation

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

Notice that the right-hand side does not depend on $x$. The left-hand side represents the *system* (think of it as defining the behaviour of a system); the right-hand side represents an outside influence on the system, which we can call the *input*.

In general, a signal is a function of $t$. The system *responds* to the input signal and yields the function $x(t)$, which we call the *output signal* or *system response*. (these terms should just be seen as convenient convention when describing an ODE)

*Block diagrams* can be used to visually represent systems:

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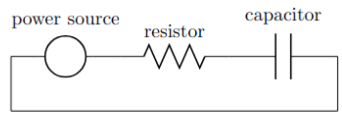
**Example—RC circuits**
Suppose we have an electrical circuit as shown




"Kirchhoff's Voltage Law" states that the total voltage change around the loop is 0, meaning

$$V(t) = V_R(t) + V_C(t)$$

The relationship between voltage drop and current are described as follows:

$$\text{Resistor: } V_R(t) = RI(t) \text{ for a constant } R, \text{ the "resistance"}$$
$$\text{Capacitor: } V_C'(t) = \frac{1}{C}I(t) \text{ for a constant } C, \text{ the "capacitance"}$$

the voltage drop from the capacitance can be seen from the equation defining capacitance

$$\begin{aligned} q &= CV \quad \text{(charge per unit voltage)} \\ I(t) = \frac{dq}{dt} &= \frac{d}{dt}(CV) \\ I(t) &= CV' \quad (C \text{ constant}) \\ V_C'(t) &= \frac{1}{C}I(t) \end{aligned}$$

The voltage drop across the capacitor is proportional to the *integral* of the current; it results from a buildup of charge on two plates of the capacitor.
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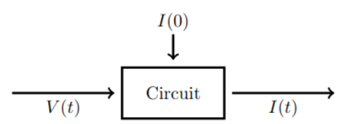
We can differentiate Kirchhoff's Voltage Law

$$\begin{aligned} V'(t) &= V_R'(t) + V_C'(t) \\ &= RI'(t) + \frac{1}{C} I(t) \end{aligned}$$

to obtain a first order linear differential equation

$$RI'(t) + \frac{1}{C} I(t) = V'(t)$$

In this circuit we consider the voltage $V(t)$ to be the input signal, and the circuit with resistance $R$ and capacitance $C$ to be the system. The current $I$ is the output signal/system response:




$I(0)$ represents the initial condition.

### A.1.7 Superposition (First order ODEs)

Considering the following the first order linear equation:

$$\dot{y} + p(t)y = q(t)$$

If a given input $q(t)$ has the output $y(t)$ we write

$$q \rightsquigarrow y$$

Here we show that if

$$q_1 \rightsquigarrow y_1 \text{ and } q_2 \rightsquigarrow y_2 \quad \text{then} \quad c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2$$

**Proof**
First see that (since differentiation does't change the constant coefficient)

$$\begin{aligned} \frac{dy}{dt} + py &= q \\ c\frac{dy}{dt} + cpy &= cq \\ = \frac{d(cy)}{dt} + p(cy) &= cq; \quad cq \rightsquigarrow cy \end{aligned}$$

Now see that

$$\begin{aligned} \frac{d(c_1y_1 + c_2y_2)}{dt} + p(c_1y_1 + c_2y_2) &= \underbrace{c_1\dot{y}_1 + pc_1y_1}_{=c_1q_1} + \underbrace{c_2\dot{y}_2 + pc_2y_2}_{=c_2q_2} \\ &= c_1q_1 + c_2q_2 \end{aligned}$$

Essentially, any linear combination of solutions is also a solution.

### A.1.8 Solution by Integrating Factor (inhomogenous first order ODEs)

Here we prove the general solution to the inhomogeneous first order linear ODE

$$\dot{x} + p(t)x = q(t)$$

is

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt + C\right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

the function $u$ is called an *integrating factor*.

**Proof**
We start with the product rule for differentiation:

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$$

Consider multiplying both sides of our inhomogenous first order ODE by some function $u(t)$:

$$u\dot{x} + upx = uq$$

We want to choose a function $u(t)$ such that we can apply the product rule to the sum on the left hand side of the equation. There may be many functions $u$ that could work, but in this case we only need one. See that

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x \iff u\dot{x} + upx = u\dot{x} + \dot{u}x \iff \dot{u} = up$$

so now by separation of equations

$$\begin{aligned} \frac{du}{u} &= p(t)dt \\ \ln|u| &= \int p(t)dt \\ u &= e^{\int p\,dt} \end{aligned}$$

By using $u$ to satisfy the product rule:

$$\begin{aligned} u\dot{x} + upx = \frac{d}{dt}(ux) &= uq \\ u(t)x(t) &= \int u(t)q(t)dt + c \\ x(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)dt + c\right) \end{aligned}$$

which was what we wanted.
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**Integrating factor and homogeneous equations**
Given the homogeneous first order ODE

$$\dot{x} + p(t)x = 0$$

Solving by separation of variables gives

$$x_h(t) = Ae^{-\int p(t)dt}$$

Comparing this to the formula for the integrating factor

$$u(t) = e^{\int p(t)dt}$$

see that

$$x_h(t) = \frac{A}{u(t)}$$

### A.1.9 General, Particular and Homogeneous solutions

Solving by method of Integrating factors allows us to come up with a solution for inhomogeneous first order linear ODEs

$$\dot{x} + p(t)x = q(t)$$

Which have the form

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt + C\right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

Notice that the presence of the constant $C$ implies a family of solutions; by setting $C = 0$ we get a *particular solution* $x_p$, which is simply one specific solution—we could have chosen any other:

$$x_p = \frac{1}{u(t)}\left(\int u(t)q(t)dt + 0\right) \quad \text{is a solution}$$
$$x_p = \frac{1}{u(t)}\left(\int u(t)q(t)dt + 999\right) \quad \text{is also a solution}$$

The method of integrating factors naturally leaves us with a constant. But say we were to find a solution by *inspection*—how would we know that the constant of integration exists in the form $\frac{C}{u(t)}$? (as is in this case)

**General solution**
See that since

$$x_h(t) = \frac{1}{u(t)}$$

We can write the solution by integrating factor as

$$\begin{aligned} x(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{C}{u(t)} \\ &= x_p + Cx_h \end{aligned}$$

One way to fully solve the inhomogeneous equation is by first solving the *homogeneous* equation, and then finding any *one* solution, a *particular solution*, to the inhomogeneous equation $x_p$. (We can use any method to find $x_p$ since we the homogeneous solution handles the constant of integration):

$$\text{General solution} = \text{Particular solution} + \text{Homogeneous solution}$$

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**Intuition**
Given an inhomogeneous first order linear ODE and its associated homogeneous equation

$$\begin{aligned} \dot{x} + p(t)x &= q(t) \quad \text{(inhomogeneous)} \\ \dot{x} + p(t)x &= 0 \quad \text{(homogeneous)} \end{aligned}$$

Solving both equations by method of integrating factors gives

$$x_p(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{A}{u(t)}, \qquad x_h(t) = \frac{B}{u(t)}$$

(where $A$ is any chosen constant, each constant giving a particular solution, and $B$ the constant of integration) Now see that by adding the solutions together the constant for the inhomogeneous solution $A$ gets absorbed into the homogeneous solution:

$$\begin{aligned} x_p(t) + x_h(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{A+B}{u(t)} \\ &= \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{C}{u(t)} \end{aligned}$$

We can obtain the 'ambiguous part' of the general solution by simply solving the homogeneous equation; this means that when obtaining a particular solution we don't have to worry about the constant of integration.

**Superposition**
See that this also makes sense with respect to superposition of solutions, where since

$$\underbrace{q(t) \rightsquigarrow x_p(t)}_{\text{inhomogeneous}} \quad \text{and} \quad \underbrace{0 \rightsquigarrow x_h(t)}_{\text{homogeneous}}$$

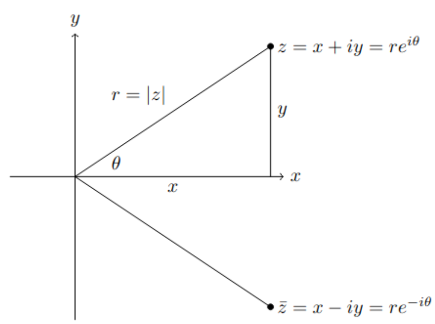
we can say

$$q(t) + 0 = q(t) \rightsquigarrow x_p(t) + x_h(t)$$

### A.1.10 Polar form and Euler Identity

**The Complex Plane, Polar Form**

Complex numbers can be represented geometrically by points in a plane, where the number $a + ib$ is represented by the point $(a, b)$; when points in a plane are thought of as representing complex numbers this way, the plane is known as a *Complex Plane*:




See that the magnitude of the coordinates of a complex number $x + iy$ can be represented by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

where $r$ is the absolute value of the number:

$$r = |x + iy| = \sqrt{x^2 + y^2}$$

(its just the pythagorean theorem) thus the entire number can be written as

$$x + iy = r(\cos(\theta) + i\sin(\theta))$$

This is called the *Polar Form* of a non-zero complex number. We call $\theta$ the *angle* or *argument* of $x + iy$:

$$\theta = \arg(x + iy)$$

Notice that the angle can be increased by any integer multiple of $2\pi$ and will still represent the same thing. To simplify this one can specify the *principal value* of the angle:

$$0 \leq \theta < 2\pi$$

this can be indicated by Arg(...); for instance

$$\mathrm{Arg}(-1) = \pi, \quad \arg(-1) = \pm\pi, \pm 3\pi, \pm 5\pi$$

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**Euler's Formula**
Complex numbers have another *exponential* form called *Euler's formula*:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This should be regarded as a definition for the exponential of an imaginary power.

A good justification for Euler's formula can be found from its Taylor approximation:

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}$$

Note that the argument above is not a proof; rather it just shows that Euler's formula is formally compatible with the series expansions for the exponential, sine, and cosine functions.

**Polar form again**
We can now write

$$x + iy = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

Polar representation in exponential form allows for much simpler multiplication of complex numbers. Since one can show that (using angle addition formulas)

$$\begin{aligned}
e^{i\theta_1}e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\
&= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\
&\quad + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \\
&= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\
&= e^{i(\theta_1 + \theta_2)}
\end{aligned}$$

(next page)

**Complex Exponential properties**
We had

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

This property can be extrapolated to further justify Euler's formula—the complex exponential follows the same exponential addition rules as any typical exponential. See that we can now conclude:

*Multiplication rule:*

$$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$

also see that since

$$\frac{1}{r}e^{-i\theta} \cdot re^{i\theta} = 1$$

*Reciprocal Rule:*

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

**DeMoivre's Formula**
Since

$$(x+iy)^n = r^ne^{in\theta}$$

we can show *DeMoivre's formula:*

$$(\cos(\theta) + i\sin(\theta))^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

**Combining pure oscillations of the same frequency**
We can also show that

$$a\cos(\lambda t) + b\sin(\lambda t) = A\cos(\lambda t - \phi)$$

where

$$A = \sqrt{a^2+b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

See that

$$\begin{aligned}
a\cos(\lambda t) + b\sin(\lambda t) &= \mathrm{Re}((a-bi)(\cos(\lambda t) + i\sin(\lambda t)) \\
&= \mathrm{Re}(Ae^{-i\phi} \cdot e^{i\lambda t}) \\
&= \mathrm{Re}(Ae^{i(\lambda t-\phi)}) \\
&= A\cos(\lambda t - \phi)
\end{aligned}$$

### A.1.11 More on Complex Exponentials

**Notable properties**
We know that (as proven)

$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$$

So see that

$$\mathrm{Re}(e^{a+ib}) = e^a\cos(b), \quad \mathrm{Im}(e^{a+ib}) = e^a\sin(b)$$

this can be extrapolated further to show

$$\begin{aligned} \cos(x) &= \mathrm{Re}(e^{ix}), & \sin(x) &= \mathrm{Im}(e^{ix}) \\ \cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix}), & \sin(x) &= \frac{1}{2i}(e^{ix} - e^{-ix}) \end{aligned}$$

**Derivatives and integrals**
Note that a function like

$$e^{ix} = \cos(x) + i\sin(x)$$

is a *complex-valued function of the real variable* $x$. Such a function may be written as

$$u(x) + iv(x), \quad u, v \text{ real-valued}$$

with its derivative and integral with respect to $x$ defined to be

$$\text{a) } D(u+iv) = Du + iDv, \quad \text{b) } \int (u+iv)dx = \int udx + i\int vdx$$

It follows easily that

$$D(e^{(a+ib)x}) = (a+ib)e^{(a+ib)x}$$

since

$$\begin{aligned} D(e^{(a+ib)x}) &= D(e^{ax}\cos(bx) + ie^{ax}\sin(bx)) \\ &= ae^{ax}\cos(bx) - be^{ax}\sin(bx) + i(ae^{ax}\sin(bx) + be^{ax}\cos(bx)) \\ &= e^{ax}\cos(bx)(a+ib) + e^{ax}\sin(bx)(ia + i^2 b) \\ &= (a+ib)e^{ax}(\cos(bx) + i\sin(bx)) \\ &= (a+ib)e^{(a+ib)x} \end{aligned}$$

Therefore we can also write the down the integral as

$$\int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x}$$

### A.1.12 Finding $n$-th roots

To solve linear DEs with constant coefficients, we need to be able to find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha$$

where $\alpha$ is a complex number—finding the $n$-th roots of $\alpha$.

**$n$-th roots of unity**

Consider first a special case; we want the solutions to

$$z^n = 1$$

We use polar representation for both sides, setting $z = re^{i\theta}$ on the left. See that

$$\underbrace{r^n e^{in\theta}}_{(re^{i\theta})^n} = \underbrace{1 \cdot e^{(2k\pi i)}, \quad k = 0, \pm 1, \pm 2, \ldots}_{=1}$$

Equating the absolute values and the arguments of each side:

$$r^n = 1, \quad n\theta = 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

(Notice the arguments for $k = a$ and $k = -a$, where $a$ is an integer, are the same. Also see that $r$ can only be 1 it is defined to be *real and non-negative* so it can't be anything else) we can conclude that

$$r = 1, \quad \theta = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1$$

we don't need any integer values of $k$ other than $0, \ldots, n-1$—they would not produce a complex number that isn't already among the above $n$ numbers. See that if we add $an$, an integer multiple of $n$, to any $k$ we get the same complex number:

$$\theta' = \frac{2(k + an)\pi}{n} = \theta + 2a\pi$$

(this is the same as having $k = n, n+1, n+2 \ldots$) so

$$e^{i\theta'} = e^{i\theta} e^{2a\pi i} = e^{i\theta}$$

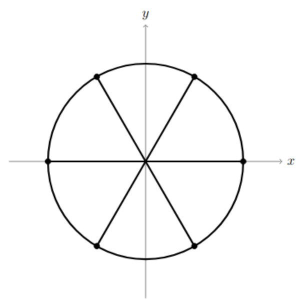
We can conclude therefore that *the $n$-th roots of 1 are the numbers*

$$e^{2k\pi i/n}, \quad k = 0, \ldots, n-1$$

(next page)

**Roots of unity visualised**
There are $n$ complex $n$-th roots of unity. Since they all have absolute value 1 ($r = 1$) they all lie on the unit circle in the complex plane. They are evenly spaced around the unit circle; the angle between two consecuitive roots is $2\pi/n$.

Illustrated here is the case for $n = 6$:




The six solutions to $z^6 = 1$ lie on the unit circle in the complex plane. See that we can express the roots of unity in a different notation:

$$\textit{the } n\textit{-th roots of 1 are } 1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, \text{ where } \zeta = e^{2\pi i/n}$$

**General case**
Now we generalise to find the $n$-th roots of an arbitrary complex number $w$. We start by writing $w$ in polar form:

$$w = re^{i\theta}; \quad \theta = \text{Arg}(w), 0 \leq \theta < 2\pi$$

Here $\theta$ is the principal value of the polar angle of $w$. Following the same reasoning as before, see that

$$z^n = re^{i(\theta+2\pi k)}; \quad k = 0, \pm 1, \pm 2, \ldots$$

where removing the redundant $k$ (this can be shown using the same methods as above) and solving gives us

$$z = \sqrt[n]{r} e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, \ldots, n-1$$

See that these $n$ roots can be expressed with the roots of unity as

$$\sqrt[n]{w} = z_0, z_0\zeta, z_0\zeta^2, \ldots, z_0\zeta^{n-1}, \quad \text{where } z_0 = \sqrt[n]{r} e^{i\theta/n}$$

($z_0$ is just the case where $k = 0$) See that all of the $n$ roots satisfy $z^n = w$.

### A.1.13 Sinusoidal functions

**Definition and properties**

A *sinusoidal function/oscillation/signal* is one that can be written in the from

$$f(t) = A\cos(\omega t - \phi)$$

The function $f(t)$ is a cosine function which has been *amplified* by $A$, *shifted* by $\phi/\omega$, and *compressed* by $\omega$.

- $A > 0$ is its *amplitude*: how high the graph of $f(t)$ rises above the $t$-axis at its maximum values

- $\phi$ is its *phase lag*: the value of $\omega t$ for which the graph has its maximum (a positive phase lag shifts the sinusoid *forward*; consider a maximum at $\cos(a)$, without phase lag it is reached at $\omega t = a$, with phase lag its now $\omega t = a + \phi$.)

- $\tau = \phi/\omega$ is its *time delay/lag*: how far along the $t$-axis the graph of $\cos(wt)$ has been shifted due to phase lag. ($\tau$ and $\phi$ have the same sign; consider a maximum at $\cos(0)$, without phase lag it is reached at $\omega t = 0 \implies t = 0$, with phase lag its now $\omega t - \phi = 0 \implies t = \phi/\omega$.)

- $\omega$ is its *angular frequency*: the number of complete oscillations $f(t)$ makes per time interval of $2\pi$; that is, the *number of radians per unit time* (1 radian in 1 second means 1 oscillation in $2\pi$ seconds—1 radian is the angle subtended at the centre of a circle by an arc equal in length to the radius).

- $v = \omega/2\pi$ is the *frequency* of $f(t)$: the number of complete oscillations made in a time interval of 1; that is, the number of cycles per unit time.

- $P = 2\pi/\omega = 1/v$ is its *period*: the $t$-interval required for one complete oscillation.

See that one can also write the sinusoidal function using the time lag $\tau = \phi/\omega$:

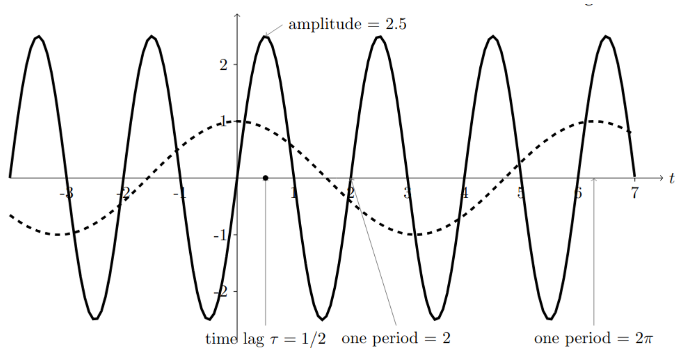
$$f(t) = A\cos(\omega(t - \tau))$$

(next page)

**Example**

In the figure below the dotted curve is $\cos(t)$ and the solid curve is $2.5\cos(\pi t - \pi/2)$. The solid curve has

$$A = 2.5, \quad \omega = \pi, \quad \phi = \pi/2, \quad \tau = 1/2$$



### A.1.14 Solution to the Constant Coefficient First Order Equation

**Solution**
Considering the constant coefficient equation (constant coefficient meaning $k$ is a constant)

$$\dot{y} + ky = q(t)$$

This is easily solvable by integrating factor:

$$\begin{aligned} y &= e^{-kt}\left(\int e^{kt}q(t)dt + c\right) \\ &= e^{-kt}\int e^{kt}q(t)dt + ce^{-kt} \end{aligned}$$

(integrating factor gives us a way of finding the particular solution, but see that it also gives us the homogeneous solution) We have the *particular* solution and *homogeneous* solution respectively

$$y_p(t) = e^{-kt}\int e^{kt}q(t)dt \quad \text{and} \quad y_h(t) = e^{-kt}$$

The general solution is then

$$y(t) = y_p(t) + cy_h(t)$$

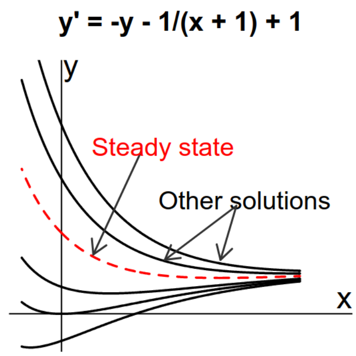
**Behaviour for $k > 0$:**
For $k > 0$ the system models *exponential decay*. When the input is 0 the system response is $y(t) = ce^{-kt}$, which decays exponentially to 0 as $t$ goes to $\infty$.

In the general solution we call $ce^{-kt}$ the *transient* because it goes to 0. The other term $e^{-kt}\int e^{kt}q(t)dt$ is called the *steady-state/long-term* solution. That is, $cy_h$ is the transient and $y_p$ is the steady-state solution.

The value of $c$ is determined by the initial value $y(0)$. See that this initial value only affects the transient and not the long-term behaviour of the solution—no matter what the initial condition, every solution goes asymptotically to the steady-state—all solution curves approach the steady-state as $t \to \infty$.

(next page)

**Behaviour for $k > 0$ illustrated**
In the case $k > 0$ all solutions go asymptotically to the steady-state:




Since all the solutions approach each other, there is no precise way to choose the one to we call the steady-state—we can *choose any one* to be the steady-state solution. Generally we just choose the simplest looking solution.

**The case $k \leq 0$:**
When $k \leq 0$ the homogeneous solution $e^{-}kt$ does not decay asymptotically to 0—it is not transient. In this case it does not make sense to talk about the steady-state solution.

### A.1.15

### A.1.16 Superposition (Second order ODEs)

The Principle of Superposition for Second Order Differential Equations; if

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

is a second order linear differential equation and $y = y_1(t)$ and $y = y_2(t)$ are both solutions to this differential equation, then for $C$ and $D$ as constants,

$$y = Cy_1(t) + Dy_2(t) \quad \text{is also a solution}$$

Essentially, any linear combination of solutions is also a solution.

*Proof*: Consider $y = y_1$ and $y = y_2$ are solutions to the second order linear differential equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$. Then we have that:

$$\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 = 0 \quad \text{and} \quad \frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2 = 0$$

If $C$ and $D$ are constants, plugging in $y = Cy_1(t) + Dy_2(t)$:

$$\begin{aligned}
&\frac{d^2}{dt^2}(Cy_1(t) + Dy_2(t)) + p(t)\frac{d}{dt}(Cy_1(t) + Dy_2(t)) + q(t)(Cy_1(t) + Dy_2(t)) \\
&= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\
&= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=0} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=0} \\
&= 0
\end{aligned}$$

Therefore, $y = Cy_1(t) + Dy_2(t)$ is also a solution. Note that the superposition principle **does not** work for nonlinear differential equations.
(next page)

**In context of inhomogenous differential equations**
In addition, if $y_1$ is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_1(t)$$

and $y_2$ is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_2(t)$$

then for constants $C$ and $D$, $Cy_1 + Dy_2$ is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = Cf_1(t) + Df_2(t)$$

*Proof*: Plugging in $y = Cy_1 + Dy_2$:

$$\begin{aligned}
&\frac{d^2}{dt^2}(Cy_1 + Dy_2) + p(t)\frac{d}{dt}(Cy_1 + Dy_2) + q(t)(Cy_1 + Dy_2) \\
&= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\
&= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=f_1(t)} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=f_2(t)} \\
&= Cf_1(t) + Df_2(t)
\end{aligned}$$

Superposition is therefore *not* limited to homogenous equations.

### A.1.17 General solution for inhomogenous linear ODEs

Therefore, to get the general solution $y(t)$ to an inhomogenous linear ODE:

$$\text{inhomogenous:} \quad \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t)$$

1. Find the general solution $y_h$ to the associated **homogenous** equation:

$$\text{homogenous:} \quad \frac{d^2y_h}{dt^2} + p(t)\frac{dy_h}{dt} + q(t)y_h = 0$$

2. Find (in some way) any **one particular solution** $y_p$ to the **inhomogenous** ODE.
3. Add $y_p$ to $y_h$ to get the general solution to the inhomogenous ODE:

$$\underbrace{y}_{\text{general } \textbf{inhomogenous} \text{ solution}} = \underbrace{y_p}_{\textbf{any particular solution}} + \underbrace{y_h}_{\text{general } \textbf{homogenous} \text{ solution}}$$

Note that the superposition principle **does not** work for nonlinear differential equations.

### A.1.18 Existence and uniqueness

Solving a first-order linear ODE leads to a 1-parameter family of solutions (a general solution). To derive a specific solution, we need an initial condition, such as $y(0)$. One may wonder if there are other solutions. Here is a general result which says that there aren't and confirms that our methods find all solutions:

**Existence and uniqueness theorem for a linear ODE:**
Let $p(t)$ and $q(t)$ be continuous functions on an open interval $I$. Let $a \in I$, and let $b$ be a given number. Then there **exists** a **unique** solution defined on the entire interval $I$ to the first order linear ODE

$$\dot{y} + p(t)y = q(t)$$

satisfying the initial condition

$$y(a) = b$$

**Existence** means there is **at least one** solution.
**Uniqueness** means that there is **only one** solution.

### A.1.19 Exponential response formula

The exponential response formula gives us a quick method for finding the particular solution to any linear, constant coefficient, differential equations whose input can be expressed in terms of an exponential function.

The Exponential Response Formula(ERF):

## A.2 Fourier Series

### A.2.1 Fourier Series

If the input function $f(t)$ is periodic (of period $2\pi$) ,we can express the function (where it is continuous) as an infinite sum of sines and cosines. This series representation is called a Fourier Series:

$$f(t) = c_0 + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)], \qquad c_0, a_n, b_n \text{ real constants}$$