Appendix 3

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Contents

Appendix A

Probability

A.1 Fundamental concepts

A.1.1 Permutations and Combinations, Binomial Coefficient

k-permutations
Starting with n distinct objects, and letting k be some positive integer where $k \leq n$, consider counting the number of different ways that we ca pick k out of these n objects and arrange them into a sequence—the number of distinct k-object sequences.

We first have n choices for the first object. Having chosen the first, there are only $n-1$ possible choices for the second, $n-2$ for the third, and so on. This continues until we have chosen $k-1$ objects, leaving us with $n-(k-1)$ choices for the last one. The number of possible sequences, called *k-permutations*, can be written as

$$n(n-1)\cdots(n-k+1)$$

This can be rewritten, giving us

$$\begin{aligned} n(n-1)\cdots(n-k+1) &= \frac{n(n-1)\cdots(n-k+1)(n-k)\cdots 2\cdot 1}{(n-k)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

See that in the special case where $k=n$ we have

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

(This can also be seen from substituting $k=n$ into the formula and recalling the convention $0! = 1$.)
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Reordering a set
Starting with k objects, consider trying to find how many ways can we order them in a set of k elements. This follows a fairly similar principle to permutation; think of having k 'slots' to order k elements in: the first 'slot' has k possible inputs, the second $k-1$ and so on. See that this just gives us $k!$.

Combinations
Combinations can be viewed as counting the number of k-element subsets of a given n-element set. Combinations are different from permutations in that *there is no ordering of selected elements.* For instance, where the 2-permutations of the letters A, B, C, and D are

$$\text{AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC}$$

the *combinations* of two out of these four letters are

$$\text{AB, AC, AD, BC, BD, CD}$$

See that the 'duplicates' are grouped together; for instance AB and BA are not viewed as distinct.

This reasoning can be generalised: each combination is associated with $k!$ 'duplicate' k-permutations—all 'duplicate' permutations of any given combination is just that permutation reordered for the maximum number of times:

$$(\text{any single combination of length } k) \cdot k! = (\text{permutations of that combination})$$

The number $n!/(n-k)!$ of k-permutations is equal to the number of combinations times $k!$. Hence the number of possible combinations is equal to

$$\frac{n!}{k!\,(n-k)!}$$

Binomial Coefficient
Consider a bernoulli process with probability p. We want the probility of k 'successes' in n trials. See that the probability of one *specific* sequence of n trials yielding k 'successes' would be

$$p^k(1-p)^{n-k}$$

We obtain the desired probability by multiplying this by the number of *combinations* of k 'successes' we can obtain in n trials:

$$\binom{n}{k}p^k(1-p)^{n-k}$$

(think tossing a coin three times and obtaining two heads—the heads might occur on the first and third tosses, or other *combinations* of trials).

A.1.2 Expectation and Variance

Expectation
We define the *expected value* of a random variable X with a PMF p_X by

$$\boxed{\mathbb{E}[X] = \sum_x x p_X(x)}$$

Variance and Standard Deviation
We define the *variance* associated with a random variable X as

$$\boxed{\mathrm{var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \sum_x (X - \mathbb{E}[X])^2 p_X(x)}$$

(See that the because of the square the variance is always nonnegative). The variance provides a measure of dispersion of X around the mean. Another measure of dispersion is the *Standard deviation* of X, which is defined as the square root of the variance and is denoted by σ_X:

$$\boxed{\sigma_X = \sqrt{\mathrm{var}(X)}}$$

The standard deviation is often easier to interpret because it has the same units as X.

A.1.3 Expected value of a function of a RV

Expectation of a function
Let X be a RV with PMF p_X, and let $g(X)$ be a function of X. Then the expected value of the random variable $g(X)$ is given by

$$\boxed{\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)}$$

This can be shown, since

$$p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x)$$

we have

$$\begin{aligned}
\mathbb{E}[g(X)] &= \mathbb{E}[Y] \\
&= \sum_y y p_Y(y) \\
&= \sum_y y \sum_{\{x|g(x)=y\}} p_X(x) \\
&= \sum_y \sum_{\{x|g(x)=y\}} y p_X(x) \\
&= \sum_y \sum_{\{x|g(x)=y\}} g(x) p_X(x) \\
&= \sum_x g(x) p_X(x)
\end{aligned}$$

Variance
Using this we can write the variance of X as

$$\operatorname{var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \sum_x (X - \mathbb{E}[X])^2 p_X(x)$$

A.1.4 Expectation and variance of linear functions

We show for a random variable X, and letting $Y = aX + b$:

$$\boxed{\mathbb{E}[Y] = a\mathbb{E}[X] + b, \quad \text{var}(Y) = a^2\text{var}(X)}$$

Linearity of Expectations:

$$\mathbb{E}[Y] = \sum_x (ax+b)p_X(x) = a\underbrace{\sum_x xp_x(x)}_{=\mathbb{E}[X]} + b\underbrace{\sum_x p_x(x)}_{=1} = a\mathbb{E}[X] + b$$

Variance:

$$\begin{aligned}
\text{var}(Y) &= \sum_x (ax + b - \mathbb{E}[aX+b])^2 p_X(x) \\
&= \sum_x (ax + b - a\mathbb{E}[X] + b)^2 p_X(x) \\
&= a^2 \sum_x (x - \mathbb{E}[X])^2 p_X(x) \\
&= a^2 \text{var}(X)
\end{aligned}$$

Note that unless $g(X)$ is a linear function, it is not generally true that $\mathbb{E}[g(X)]$ is equal to $g(\mathbb{E}[X])$.

A.1.5 Variance in terms of Moments Expression

We show

$$\boxed{\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

see that

$$\begin{aligned}
\text{var}(X) &= \sum_x (x - \mathbb{E}[X])^2 p_X(x) \\
&= \sum_x (x^2 - 2x\mathbb{E}[X] + (\mathbb{E}[X])^2) p_X(x) \\
&= \sum_x x^2 p_X(x) - 2\mathbb{E}[X]\sum_x xp_X(x) + (\mathbb{E}[X])^2 \sum_x p_X(x) \\
&= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{aligned}$$

A.1.6 Expectation and Variance of Bernoulli

Consider a Bernoulli RV X with PMF

$$p_X(k) = \begin{cases} p, & \text{if } k = 1. \\ 1-p, & \text{if } k = 0. \end{cases}$$

The mean, second moment, and variance of X are as follows:

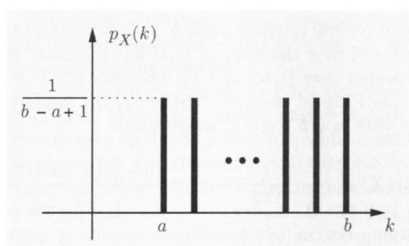
$$\begin{aligned} \mathbb{E}[X] &= 1 \cdot p + 0 \cdot (1-p) = p \\ \mathbb{E}[X^2] &= 1^2 \cdot p + 0 \cdot (1-p) = p \\ \mathrm{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p(1-p) \end{aligned}$$

A.1.7 Expectation of Discrete Uniform

Consider a Discrete Uniform RV X with PMF, for $k \in [a, b]$:

$$p_X(k) = \begin{cases} \frac{1}{b-a+1}, & \text{if } k = a, a+1 \ldots, b \\ 0, & \text{otherwise.} \end{cases}$$

An illustration is useful here:



Expectation
Upon inspection one might suppose that the expectation is

$$\mathbb{E}[X] = \frac{a+b}{2}$$

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Expectation (cont.)
The formula can be elucidated from the definition of the expectation. First see that a sequence $\sum_{k=a}^{b} k$ can be written as

$$\begin{aligned}
\sum_{k=a}^{b} k &= \sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k \\
&= \frac{(b)(b+1)}{2} - \frac{(a-1)(a)}{2} \quad \text{(see B.1)} \\
&= \frac{b^2 + b - a^2 + a}{2} = \frac{(b-a+1)(a+b)}{2}
\end{aligned}$$

The last step isn't easy to factor, but working back from our 'hypothesis' for the expectation it coincides.

so now we have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=a}^{b} k \left(\frac{1}{b-a+1}\right) \\
&= \frac{1}{b-a+1} \sum_{k=a}^{b} k \\
&= \frac{1}{b-a+1} \cdot \frac{(b-a+1)(a+b)}{2} \\
\mathbb{E}[X] &= \frac{(a+b)}{2}
\end{aligned}$$

A.1.8 Variance of Discrete Uniform

Case for $k \in [1, n]$:
We can obtain the second moment for a discrete uniform distributed over $k \in [1, n]$ as

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=1}^{n} k^2 \left(\frac{1}{n}\right) \\
&= \frac{1}{n} \sum_{k=1}^{n} k^2 \\
&= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} \qquad \text{(see B.4)} \\
&= \frac{(n+1)(2n+1)}{6}
\end{aligned}$$

We then use the formula for variance in terms of moments expression:

$$\begin{aligned}
\operatorname{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{12}(n+1)(4n+2-3n-3) \\
&= \frac{n^2-1}{12}
\end{aligned}$$

General case $k \in [a, b]$:
For the general case, note that a RV uniformly distributed over an interval $[a, b]$ has the *same variance* as one which is uniformly distributed over $[1, b-a+1]$—the PMF of the second is just a shifted version of the PMF of the first.

Therefore, the desired variance is given by the first case, but instead with $n = b - a + 1$, yielding

$$\boxed{\operatorname{var}(X) = \frac{(b-a+1)^2 - 1}{12} = \frac{(b-a)(b-a+2)}{12}}$$

A.2 Limit Theorems

A.2.1 Sample mean

Definition
Here we discuss asymptomatic behaviour of sequences of random variables. The principal context involves a sequence $X_1, X_2, \ldots$ of independent identically distributed random variables with expectation μ and variance σ^2. We denote

$$S_n = X_1 + \cdots + X_n$$

to be the sum of the first n of them. Since they are independent we also have

$$\operatorname{var}(S_n) = \operatorname{var}(X_1) + \ldots + \operatorname{var}(X_n) = n\sigma^2$$

See that the distribution of S_n spreads out (it's variance increases) as n increases and doesn't have a meaningful limit. Consider instead the *sample mean*

$$M_n = \frac{X_1 + \cdots + X_n}{n} = \frac{S_n}{n}$$

Expectation and Variance
We have the expectation as

$$\begin{aligned}\mathbb{E}[M_n] &= \frac{\mathbb{E}[X_1 + \ldots + X_n]}{n} \\ &= \frac{\mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]}{n} \\ &= \frac{n\mu}{n} = \mu\end{aligned}$$

and the variance as

$$\operatorname{var}(M_n) = \frac{1}{n^2}\operatorname{var}(S_n) = \frac{\sigma^2}{n}$$

See that the variance of M_n decreases to 0 as n increases.

With this consider a new random variable, that we modify based off M_n and S_n:

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

This has the properties

$$\mathbb{E}[Z_n] = 0, \quad \operatorname{var}(Z_n) = \frac{\operatorname{var}(S_n - n\mu)}{\sigma^2 n} = 1$$

A.2.2 Markov Inequality

Appendix B

Supplementary Notes

B.1 The sum of the first n natural numbers is $n(n+1)/2$

We have that

$$\sum_{i=1}^{i} i = 1 + 2 + \cdots + n$$

Now consider $2\sum_{i=1}^{n} i$:

$$\begin{aligned}
2\sum_{i=1}^{n} i &= 2(1 + 2 + \cdots + (n-1) + n) \\
&= (1 + 2 + \cdots + (n-1) + n) + (n + (n-1) + \cdots + 2 + 1) \\
&= (1+n) + (2 + (n-1)) + \cdots + ((n-1) + 2) + (n+1) \\
&= (n+1)_1 + (n+1)_2 + \cdots + (n+1)_n \\
&= n(n+1)
\end{aligned}$$

so

$$\begin{aligned}
2\sum_{i=1}^{n} i &= n(n+1) \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}
\end{aligned}$$

B.2 Telescoping series

Let $\langle b_n \rangle$ be a sequence in $\mathbb{R}$. Let $\langle a_n \rangle$ be a sequence defined as

$$a_k = b_k - b_{k-1}$$

we show

$$\boxed{\sum_{k=m}^{n} a_k = b_n - b_{m-1}}$$

See that

$$\begin{aligned}
\sum_{k=m}^{n} a_k &= \sum_{k=m}^{n} (b_k - b_{k-1}) \\
&= \sum_{k=m}^{n} b_k - \sum_{k=m}^{n} b_{k-1} \\
&= \sum_{k=m}^{n} b_k - \sum_{k=m-1}^{n-1} b_k \\
&= \left(\sum_{k=m}^{n-1} b_k + b_n \right) - \left(b_{m-1} + \sum_{k=m}^{n-1} b_k \right) \\
&= b_n - b_{m-1}
\end{aligned}$$

B.3 Sum of series of products of consecutive integers

We show

$$\boxed{\sum_{j=1}^{n} j(j+1) = 1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}}$$

See that

$$\begin{aligned} 3i(i+1) &= i(i+1)(i+2) - i(i+1)(i-1) \\ &= (i+1)((i+1)+1)((i+1)-1) - i(i+1)(i-1) \end{aligned}$$

Thus we have the basis of a telescoping series (see (B.2)):

$$3i(i+1) = b(i+1) - b(i)$$

where

$$b(i) = i(i+1)(i-1)$$

So we have

$$\begin{aligned} \sum_{j=1}^{n} 3j(j+1) &= \sum_{j=1}^{n} (j+1)((j+1)+1)((j+1)-1) - j(j+1)(j-1) \\ &= n(n+1)(n+2) - 0(0+1)(0-1) \\ &= n(n+1)(n+2) \end{aligned}$$

Thus

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

B.4 Sum of sequence of squares

We show

$$\boxed{\forall n \in \mathbb{N} : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

See that this follows from (B.3):

$$\begin{aligned}
\sum_{i=1}^{n} 3i(i+1) &= n(n+1)(n+2) \\
\sum_{i=1}^{n} 3i^2 + \sum_{i=1}^{n} 3i &= n(n+1)(n+2) \\
\sum_{i=1}^{n} 3i^2 &= n(n+1)(n+2) - 3\frac{n(n+1)}{2} \quad \text{see (B.1))} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\
&= \frac{2n(n+1)(n+2) - 3n(n+1)}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$