

## Appendix 2

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# Appendix A

## Differential Equations

### A.1 Fundamentals

#### A.1.1 Introduction to Ordinary Differential Equations (ODEs)

Here we introduce intuition for Ordinary Differential Equations (ODEs) and introductory solving methods.

The simplest type of differential equation looks like:

$$\frac{dy}{dx} = f(x)$$

which can be solved by the antiderivative  $y = \int f(x) dx$ .

#### Intuition

Now we consider a more interesting example:

$$\frac{dy}{dx} + xy = 0$$

This equation can be solved by *separation of variables*:

$$\begin{aligned}\frac{dy}{dx} + xy &= 0 \\ \frac{dy}{dx} &= -xy \\ \frac{dy}{y} &= -x dx\end{aligned}$$

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Since the problem is now set up in terms of differentials rather than ratios of differentials, we can integrate both sides.

$$\int \frac{dy}{y} = - \int x \, dx$$

$$\ln y + c_1 = -\frac{x^2}{2} + c_2 \quad (\text{assume } y > 0)$$

We can combine the constants and simplify:

$$\ln y = -\frac{x^2}{2} + c$$

$$e^{\ln y} = e^{-x^2/2+c}$$

$$y = e^c e^{-x^2/2}$$

$$y = A e^{-x^2/2}, \quad (\text{where } A = e^c)$$

(The more apt  $\ln |y|$  simplifies to  $\pm A e^{-x^2/2}$ , which doesn't matter since  $A$  is some unspecific constant)

It turns out that our solution,

$$y = A e^{-x^2/2}, \quad (\text{where } A = e^c)$$

Works for any constant multiple  $A$ . We can check this solution:

$$y = a e^{-x^2/2}$$

$$\frac{dy}{dx} = \frac{d}{dx} a e^{-x^2/2}$$

$$= a \cdot (-x) e^{-x^2/2}$$

$$= -x \cdot a e^{-x^2/2}$$

$$\frac{dy}{dx} = -xy$$

$A$  is determined by an initial condition; for instance if  $y(0) = 1$ ,  $A = 1$ .

### A.1.2 Separation of Variables

Here we describe a rudimentary method for solving some differential equations—Separation of Variables.

In general, this method applies to differential equations of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Where we then *separate* the variables and integrate:

$$\begin{aligned}\frac{dy}{dx} &= f(x)g(y) \\ \frac{dy}{g(y)} &= f(x) dx \\ h(y) dy &= f(x) dx \quad \text{where } h(y) = \frac{1}{g(y)} \\ \int h(y) dy &= \int f(x) dx\end{aligned}$$

Antidifferentiating both sides:

$$H(y) = \int h(y) dy; \quad F(x) = \int f(x) dx$$

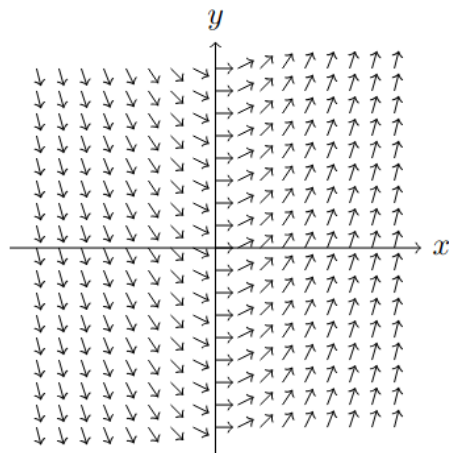
we now have

$$\begin{aligned}H(y) + c_1 &= F(x) + c_2 \\ H(y) &= F(x) + c\end{aligned}$$

### A.1.3 Direction fields, Isoclines, and Integral curves

#### Direction fields

Given an equation  $y' = f(x, y)$ , we can construct a *direction field*; imagine through each point  $(x, y)$ , we draw a line segment whose slope is  $f(x, y)$ —consider  $y'(x) = 2x$ :



(note that in this case  $f(x, y)$  does not depend on  $y$  (because of the equation)—it is invariant under vertical translation)

#### Plotting direction fields—Isoclines

In practice, computers are used to plot direction fields following the procedure:

1. Pick point  $(x, y)$
2. Compute  $y' = f(x, y)$
3. Plot line segment of slope at that point

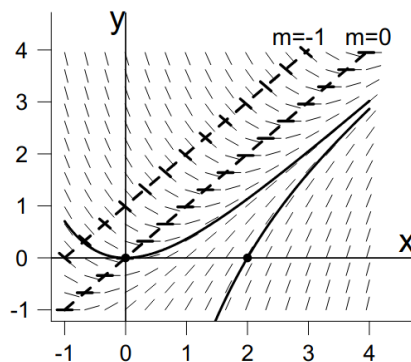
Notice how a new slope has to be computed for each specified point; when plotting direction fields by hand, it is much more practical to utilise *isoclines*, which are, given the equation  $y' = f(x, y)$ , a one-parameter family of curves given by the equations

$$f(x, y) = m, \quad m \text{ constant}$$

Along a given isocline, all line segments have the same slope  $m$ .  
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**Example**

Consider plotting the direction field for the equation  $y' = x - y$ ; the isoclines are correspondingly the lines  $x - y = m$  (shown in dashed lines):



The  $m = 0$  isocline marks the points where the slope of the solution is 0; it is therefore of special interest and is called the *nullcline*.

**Integral curves**

As also shown in the figure above, once the direction field has been sketched, curves which are *at each point tangent to the line segment at that point* can be drawn; such curves are called *integral curves* or *solution curves* for the direction field. Their significance (this should be obvious) is that

*The integral curves are the graphs of the solutions to  $y' = f(x, y)$*

Two integral curves have been drawn above (in solid lines).

**Intersection Principle**

Intuitively, see that at any point in the direction field it can only have one direction; therefore it is fairly obvious that integral curves cannot cross at an angle.

Consider the existence and uniqueness theorem for ODEs:

*For any  $(a, b)$  in the region where  $f$  is defined,  $y' = f(x, y)$  has exactly one solution such that  $y(a) = b$ .*

by the existence part of the theorem, there is an integral curve through any point where  $f(x, y)$  is defined. Now supposing two integral curves through the same point, by the uniqueness part of the theorem they must agree.

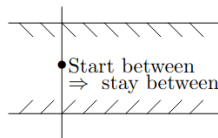
As a result, *integral curves cannot intersect*; every point lies on exactly one integral curve.

### A.1.4 Long term Behaviour: Fences, Funnels, and Separatrices

#### Fences

A *lower fence* for the equation  $y' = f(x, y)$  is a curve that ‘blocks’ an integral curve from crossing from *above*; intuitively it is the curve whose direction field elements along the curve point up from it. Technically it can be described as a curve  $y = L(x)$  such that  $L'(x) < f(x, L(x))$  (the slope of the curve is always less than the slope of the direction field at that point).

Likewise an *upper fence* is a curve that ‘blocks’ integral curves from crossing from *above*. Illustrated:



(The upper curve is the upper fence and the lower curve is the lower fence). Solutions will be ‘squeezed’ between upper and lower fences.

Note that

- Note that fences aren’t necessarily defined for all  $x$ ; they could be defined only on an interval like  $x \geq c$  for some constant  $c$ .
- Since integral curves can’t cross an integral curve itself it is both an upper and lower fence.

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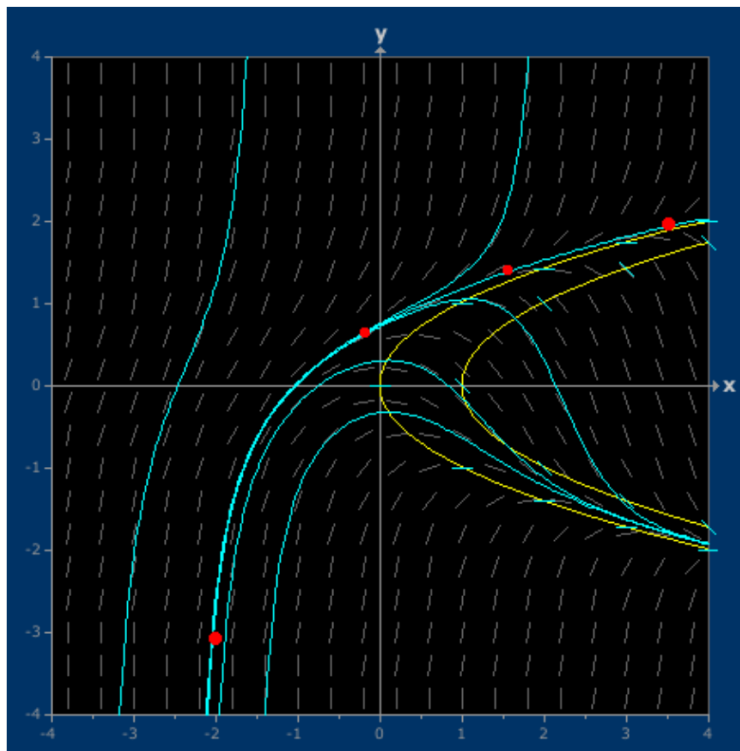


**Example**

Consider the direction field for the equation

$$y' = y^2 - x$$

The isoclines for  $m = 0$  and  $m = -1$  are plotted in yellow, with integral curves in blue:



Notice that the bottom half of the isocline  $m = 0$  is a lower fence and for  $x$  large enough the bottom half of the isocline  $m = -1$  is an upper fence. (notice that the  $m = -1$  isocline becomes an upper fence only for  $x$  large enough)

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### Funnels

One use of fences is to construct funnels. A *funnel* for the equation  $y' = f(x, y)$  consists of a pair of fences; one lower fence  $L(x)$  and one upper fence  $U(x)$  with the properties

1. For  $x$  large the lower fence is below the upper fence;  $L(x) < U(x)$
2. The two fences come together asymptotically;  $U(x) - L(x)$  is small for large  $x$

For instance, in the above example the bottom parts of the two isoclines  $m = 0$  and  $m = -1$  act as a funnel once  $x$  is large enough. Given the equations of each isocline we have highly accurate estimates for solutions between them as

$$\underbrace{-\sqrt{x}}_{m=0} < y(x) < \underbrace{-\sqrt{x-1}}_{m=-1}$$

which is valid for large  $x$ .

Note that not all pairs of upper/lower fences form a funnel—they have to come together asymptotically as  $x$  gets large.

### Separatrices

A *separatrix* is an integral curve such that the integral curves above it behave entirely differently from integral curves below it as  $x \rightarrow \infty$ .

### A.1.5 Runge-Kutta 2 (Numerical methods)

#### General approach and Euler's method

Euler's method (for numerical estimation) follows a more general procedure for stepping from  $(x_n, y_n)$  to  $(x_{n+1}, y_{n+1})$ :

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + m_n h$$

Where  $h$  is the stepsize in the  $x$  direction and  $m$  is the slope of the line we step along. In Euler's method  $h$  is fixed ahead of time and  $m_n = f(x_n, y_n)$ .

#### Runge-Kutta 2

Naturally Euler's method is a fairly flawed method of numerical estimation. Other methods use other (and better) ways of choosing  $h$  and  $m$ . Here I describe the *Runge-Kutta 2* (RK2) method, which is a *fixed stepsize* method; meaning  $h$  is fixed and the added complexity comes from finding  $m$ .

Given an initial value problem  $y' = f(x, y), y(x_0) = x_0$  and a step size  $h$ , one step of the RK2 method is as follows:

1. Compute the slope  $k_1$  at  $(x_0, y_0)$ :  $k_1 = f(x_0, y_0)$
2. 'Take' an Euler step from  $(x_0, y_0)$  to  $(a, b)$ :  $a = x_0 + h, b = y_0 + k_1 h$
3. Compute the slope  $k_2$  at  $(a, b)$ :  $k_2 = f(a, b)$
4. Average  $k_1$  and  $k_2$  to get  $m$ :  $m = (k_1 + k_2)/2$
5. Now we use this averaged slope to take a step from  $(x_n, y_n)$  to  $(x_{n+1}, y_{n+1})$ :

$$x_1 = x_0 + h, \quad y_1 = y_0 + mh; \quad m = \frac{(k_1 + k_2)}{2}$$

Other methods such as RK4 or *variable stepsize methods* may (probably) work better. Though one might want to consider computational efficiency at the expense of accuracy.

### A.1.6 First order Linear Differential Equations

#### Definition

The general *First order linear ODE* in the unknown function  $x = x(t)$  has the form

$$A(t) \frac{dx}{dt} + B(t)x(t) = C(t)$$

If  $A(t) \neq 0$  we can simplify the equation by dividing by  $A(t)$ :

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

This is called the *standard form* for a first order linear ODE. Should the *coefficients*  $A(t), B(t)$  be constants (not dependent on  $t$ ) we say the equation is a *constant coefficient* DE.

If  $C(t) = 0$ :

$$A(t) \frac{dx}{dt} + B(t)x(t) = 0$$

The DE is called *homogeneous* (notice that conversion to standard form doesn't change this fact); otherwise the equation is *inhomogeneous*.

#### Signals and Systems—Terminology

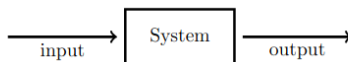
Given a differential equation

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

Notice that the right-hand side does not depend on  $x$ . The left-hand side represents the *system* (think of it as defining the behaviour of a system); the right-hand side represents an outside influence on the system, which we can call the *input*.

In general, a signal is a function of  $t$ . The system *responds* to the input signal and yields the function  $x(t)$ , which we call the *output signal* or *system response*. (these terms should just be seen as convenient convention when describing an ODE)

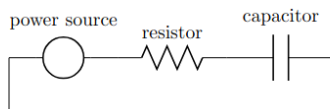
*Block diagrams* can be used to visually represent systems:



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### Example—RC circuits

Suppose we have an electrical circuit as shown



“Kirchhoff’s Voltage Law” states that the total voltage change around the loop is 0, meaning

$$V(t) = V_R(t) + V_C(t)$$

The relationship between voltage drop and current are described as follows:

Resistor:  $V_R(t) = RI(t)$  for a constant  $R$ , the “resistance”

Capacitor:  $V'_C(t) = \frac{1}{C}I(t)$  for a constant  $C$ , the “capacitance”

the voltage drop from the capacitance can be seen from the equation defining capacitance

$$q = CV \quad (\text{charge per unit voltage})$$

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}(CV)$$

$$I(t) = CV' \quad (C \text{ constant})$$

$$V'_C(t) = \frac{1}{C}I(t)$$

The voltage drop across the capacitor is proportional to the *integral* of the current; it results from a buildup of charge on two plates of the capacitor.

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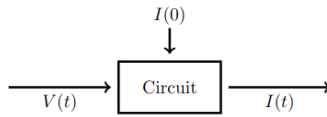
We can differentiate Kirchhoff's Voltage Law

$$\begin{aligned} V'(t) &= V_R'(t) + V_C'(t) \\ &= RI'(t) + \frac{1}{C}I(t) \end{aligned}$$

to obtain a first order linear differential equation

$$RI'(t) + \frac{1}{C}I(t) = V'(t)$$

In this circuit we consider the voltage  $V(t)$  to be the input signal, and the circuit with resistance  $R$  and capacitance  $C$  to be the system. The current  $I$  is the output signal/system response:



$I(0)$  represents the initial condition.

### A.1.7 Superposition (First order ODEs)

Considering the following the first order linear equation:

$$\dot{y} + p(t)y = q(t)$$

If a given input  $q(t)$  has the output  $y(t)$  we write

$$q \rightsquigarrow y$$

Here we show that if

$$q_1 \rightsquigarrow y_1 \text{ and } q_2 \rightsquigarrow y_2 \quad \text{then} \quad c_1 q_1 + c_2 q_2 \rightsquigarrow c_1 y_1 + c_2 y_2$$

#### **Proof**

First see that (since differentiation doesn't change the constant coefficient)

$$\begin{aligned} \frac{dy}{dt} + py &= q \\ c \frac{dy}{dt} + cpy &= cq \\ &= \frac{d(cy)}{dt} + p(cy) = cq; \quad cq \rightsquigarrow cy \end{aligned}$$

Now see that

$$\begin{aligned} \frac{d(c_1 y_1 + c_2 y_2)}{dt} + p(c_1 y_1 + c_2 y_2) &= \underbrace{c_1 \dot{y}_1 + p c_1 y_1}_{=c_1 q_1} + \underbrace{c_2 \dot{y}_2 + p c_2 y_2}_{=c_2 q_2} \\ &= c_1 q_1 + c_2 q_2 \end{aligned}$$

Essentially, any linear combination of solutions is also a solution.

### A.1.8 Solution by Integrating Factor (inhomogenous first order ODEs)

Here we prove the general solution to the inhomogeneous first order linear ODE

$$\dot{x} + p(t)x = q(t)$$

is

$$x(t) = \frac{1}{u(t)} \left( \int u(t)q(t)dt + C \right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

the function  $u$  is called an *integrating factor*.

#### Proof

We start with the product rule for differentiation:

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$$

Consider multiplying both sides of our inhomogenous first order ODE by some function  $u(t)$ :

$$u\dot{x} + upx = uq$$

We want to choose a function  $u(t)$  such that we can apply the product rule to the sum on the left hand side of the equation. There may be many functions  $u$  that could work, but in this case we only need one. See that

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x \iff u\dot{x} + upx = u\dot{x} + \dot{u}x \iff \dot{u} = up$$

so now by separation of equations

$$\begin{aligned} \frac{du}{u} &= p(t)dt \\ \ln |u| &= \int p(t)dt \\ u &= e^{\int p dt} \end{aligned}$$

By using  $u$  to satisfy the product rule:

$$\begin{aligned} u\dot{x} + upx &= \frac{d}{dt}(ux) = uq \\ u(t)x(t) &= \int u(t)q(t)dt + c \\ x(t) &= \frac{1}{u(t)} \left( \int u(t)q(t)dt + c \right) \end{aligned}$$

which was what we wanted.  
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**Integrating factor and homogeneous equations**

Given the homogeneous first order ODE

$$\dot{x} + p(t)x = 0$$

Solving by separation of variables gives

$$x_h(t) = Ae^{-\int p(t)dt}$$

Comparing this to the formula for the integrating factor

$$u(t) = e^{\int p(t)dt}$$

see that

$$x_h(t) = \frac{A}{u(t)}$$

### A.1.9 General, Particular and Homogeneous solutions

Solving by method of Integrating factors allows us to come up with a solution for inhomogeneous first order linear ODEs

$$\dot{x} + p(t)x = q(t)$$

Which have the form

$$x(t) = \frac{1}{u(t)} \left( \int u(t)q(t)dt + C \right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

Notice that the presence of the constant  $C$  implies a family of solutions; by setting  $C = 0$  we get a *particular solution*  $x_p$ , which is simply one specific solution—we could have chosen any other:

$$\begin{aligned} x_p &= \frac{1}{u(t)} \left( \int u(t)q(t)dt + 0 \right) \quad \text{is a solution} \\ x_p &= \frac{1}{u(t)} \left( \int u(t)q(t)dt + 999 \right) \quad \text{is also a solution} \end{aligned}$$

The method of integrating factors naturally leaves us with a constant. But say we were to find a solution by *inspection*—how would we know that the constant of integration exists in the form  $\frac{C}{u(t)}$ ? (as is in this case)

#### General solution

See that since

$$x_h(t) = \frac{1}{u(t)}$$

We can write the solution by integrating factor as

$$\begin{aligned} x(t) &= \frac{1}{u(t)} \left( \int u(t)q(t)dt \right) + \frac{C}{u(t)} \\ &= x_p + Cx_h \end{aligned}$$

One way to fully solve the inhomogeneous equation is by first solving the *homogeneous* equation, and then finding any *one* solution, a *particular solution*, to the inhomogeneous equation  $x_p$ . (We can use any method to find  $x_p$  since we the homogeneous solution handles the constant of integration):

$$\text{General solution} = \text{Particular solution} + \text{Homogeneous solution}$$

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**Intuition**

Given an inhomogeneous first order linear ODE and its associated homogeneous equation

$$\begin{aligned}\dot{x} + p(t)x &= q(t) & (\text{inhomogeneous}) \\ \dot{x} + p(t)x &= 0 & (\text{homogeneous})\end{aligned}$$

Solving both equations by method of integrating factors gives

$$x_p(t) = \frac{1}{u(t)} \left( \int u(t)q(t)dt \right) + \frac{A}{u(t)}, \quad x_h(t) = \frac{B}{u(t)}$$

(where  $A$  is any chosen constant, each constant giving a particular solution, and  $B$  the constant of integration) Now see that by adding the solutions together the constant for the inhomogeneous solution  $A$  gets absorbed into the homogeneous solution:

$$\begin{aligned}x_p(t) + x_h(t) &= \frac{1}{u(t)} \left( \int u(t)q(t)dt \right) + \frac{A+B}{u(t)} \\ &= \frac{1}{u(t)} \left( \int u(t)q(t)dt \right) + \frac{C}{u(t)}\end{aligned}$$

We can obtain the ‘ambiguous part’ of the general solution by simply solving the homogeneous equation; this means that when obtaining a particular solution we don’t have to worry about the constant of integration.

**Superposition**

See that this also makes sense with respect to superposition of solutions, where since

$$\underbrace{q(t) \rightsquigarrow x_p(t)}_{\text{inhomogeneous}} \quad \text{and} \quad \underbrace{0 \rightsquigarrow x_h(t)}_{\text{homogeneous}}$$

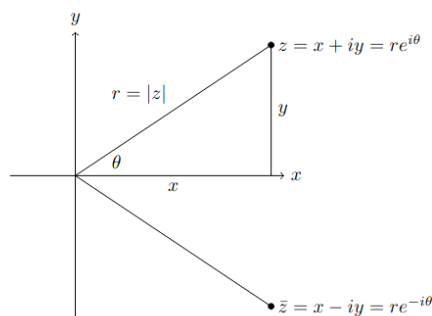
we can say

$$q(t) + 0 = q(t) \rightsquigarrow x_p(t) + x_h(t)$$

### A.1.10 Polar form and Euler Identity

#### The Complex Plane, Polar Form

Complex numbers can be represented geometrically by points in a plane, where the number  $a + ib$  is represented by the point  $(a, b)$ ; when points in a plane are thought of as representing complex numbers this way, the plane is known as a *Complex Plane*:



See that the magnitude of the coordinates of a complex number  $x + iy$  can be represented by

$$x = r \cos(\theta), \quad y = r \sin(\theta)$$

where  $r$  is the absolute value of the number:

$$r = |x + iy| = \sqrt{x^2 + y^2}$$

(its just the pythagorean theorem) thus the entire number can be written as

$$x + iy = r(\cos(\theta) + i \sin(\theta))$$

This is called the *Polar Form* of a non-zero complex number. We call  $\theta$  the *angle* or *argument* of  $x + iy$ :

$$\theta = \arg(x + iy)$$

Notice that the angle can be increased by any integer multiple of  $2\pi$  and will still represent the same thing. To simplify this one can specify the *principal value* of the angle:

$$0 \leq \theta < 2\pi$$

this can be indicated by  $\text{Arg}(\dots)$ ; for instance

$$\text{Arg}(-1) = \pi, \quad \arg(-1) = \pm\pi, \pm3\pi, \pm5\pi$$

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### Euler's Formula

Complex numbers have another *exponential* form called *Euler's formula*:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

This should be regarded as a definition for the exponential of an imaginary power.

A good justification for Euler's formula can be found from its Taylor approximation:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos(\theta) + i \sin(\theta) \end{aligned}$$

Note that the argument above is not a proof; rather it just shows that Euler's formula is formally compatible with the series expansions for the exponential, sine, and cosine functions.

### Polar form again

We can now write

$$x + iy = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}$$

Polar representation in exponential form allows for much simpler multiplication of complex numbers. Since one can show that (using angle addition formulas)

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos(\theta_1) + i \sin(\theta_1))(\cos(\theta_2) + i \sin(\theta_2)) \\ &= \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2) \\ &\quad + i(\sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1 + \theta_2)} \end{aligned}$$

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### Complex Exponential properties

We had

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

This property can be extrapolated to further justify Euler's formula—the complex exponential follows the same exponential addition rules as any typical exponential. See that we can now conclude:

*Multiplication rule:*

$$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$

also see that since

$$\frac{1}{r}e^{-i\theta} \cdot re^{i\theta} = 1$$

*Reciprocal Rule:*

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

### DeMoivre's Formula

Since

$$(x + iy)^n = r^n e^{in\theta}$$

we can show *DeMoivre's formula*:

$$(\cos(\theta) + i \sin(\theta))^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

### Combining pure oscillations of the same frequency

We can also show that

$$a \cos(\lambda t) + b \sin(\lambda t) = A \cos(\lambda t - \phi)$$

where

$$A = \sqrt{a^2 + b^2}, \quad \phi = \tan^{-1} \left( \frac{b}{a} \right)$$

See that

$$\begin{aligned} a \cos(\lambda t) + b \sin(\lambda t) &= \operatorname{Re}((a - bi)(\cos(\lambda t) + i \sin(\lambda t))) \\ &= \operatorname{Re}(Ae^{-i\phi} \cdot e^{i\lambda t}) \\ &= \operatorname{Re}(Ae^{i(\lambda t - \phi)}) \\ &= A \cos(\lambda t - \phi) \end{aligned}$$

### A.1.11 More on Complex Exponentials

#### Notable properties

We know that (as proven)

$$e^{a+ib} = e^a e^{ib} = e^a (\cos(b) + i \sin(b))$$

So see that

$$\operatorname{Re}(e^{a+ib}) = e^a \cos(b), \quad \operatorname{Im}(e^{a+ib}) = e^a \sin(b)$$

this can be extrapolated further to show

$$\begin{aligned} \cos(x) &= \operatorname{Re}(e^{ix}), & \sin(x) &= \operatorname{Im}(e^{ix}) \\ \cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix}), & \sin(x) &= \frac{1}{2i}(e^{ix} - e^{-ix}) \end{aligned}$$

#### Derivatives and integrals

Note that a function like

$$e^{ix} = \cos(x) + i \sin(x)$$

is a *complex-valued function of the real variable  $x$* . Such a function may be written as

$$u(x) + iv(x), \quad u, v \text{ real-valued}$$

with its derivative and integral with respect to  $x$  defined to be

$$\text{a) } D(u + iv) = Du + iDv, \quad \text{b) } \int (u + iv)dx = \int udx + i \int vdx$$

It follows easily that

$$D(e^{(a+ib)x}) = (a + ib)e^{(a+ib)x}$$

since

$$\begin{aligned} D(e^{(a+ib)x}) &= D(e^{ax} \cos(bx) + ie^{ax} \sin(bx)) \\ &= ae^{ax} \cos(bx) - be^{ax} \sin(bx) + i(ae^{ax} \sin(bx) + be^{ax} \cos(bx)) \\ &= e^{ax} \cos(bx)(a + ib) + e^{ax} \sin(bx)(ia + i^2b) \\ &= (a + ib)e^{ax}(\cos(bx) + i \sin(bx)) \\ &= (a + ib)e^{(a+ib)x} \end{aligned}$$

Therefore we can also write the down the integral as

$$\int e^{(a+ib)x} dx = \frac{1}{a + ib} e^{(a+ib)x}$$

### A.1.12 Finding $n$ -th roots

To solve linear DEs with constant coefficients, we need to be able to find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha$$

where  $\alpha$  is a complex number—finding the  $n$ -th roots of  $\alpha$ .

#### $n$ -th roots of unity

Consider first a special case; we want the solutions to

$$z^n = 1$$

We use polar representation for both sides, setting  $z = re^{i\theta}$  on the left. See that

$$\underbrace{r^n e^{in\theta}}_{(re^{i\theta})^n} = \underbrace{1 \cdot e^{(2k\pi i)}}_{=1}, \quad k = 0, \pm 1, \pm 2, \dots$$

Equating the absolute values and the arguments of each side:

$$r^n = 1, \quad n\theta = 2k\pi, \quad k = 0, \pm 1, \pm 2, \dots$$

(notice the arguments for  $k = a$  and  $k = -a$ , where  $a$  is an integer, are the same) we can conclude that

$$r = 1, \quad \theta = \frac{2k\pi}{n}, \quad k = 0, 1, \dots, n-1$$

we don't need any integer values of  $k$  other than  $0, \dots, n-1$ —they would not produce a complex number that isn't already among the above  $n$  numbers. See that if we add  $an$ , an integer multiple of  $n$ , to any  $k$  we get the same complex number:

$$\theta' = \frac{2(k+an)\pi}{n} = \theta + 2a\pi$$

(this is the same as having  $k = n, n+1, n+2, \dots$ ) so

$$e^{i\theta'} = e^{i\theta} e^{2a\pi i} = e^{i\theta}$$

We can conclude therefore that *the  $n$ -th roots of 1 are the numbers*

$$e^{2k\pi i/n}, \quad k = 0, \dots, n-1$$

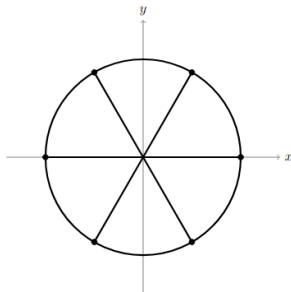
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### Roots of unity visualised

There are  $n$  complex  $n$ -th roots of unity. Since they all have absolute value 1 ( $r = 1$ ) they all lie on the unit circle in the complex plane. They are evenly spaced around the unit circle; the angle between two consecutive roots is  $2\pi/n$ .

Illustrated here is the case for  $n = 6$ :



The six solutions to  $z^6 = 1$  lie on the unit circle in the complex plane. See that we can express the roots of unity in a different notation:

*the  $n$ -th roots of 1 are  $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$ , where  $\zeta = e^{2\pi i/n}$*

### General case

Now we generalise to find the  $n$ -th roots of an arbitrary complex number  $w$ . We start by writing  $w$  in polar form:

$$w = re^{i\theta}; \quad \theta = \text{Arg}(w), 0 \leq \theta < 2\pi$$

Here  $\theta$  is the principal value of the polar angle of  $w$ . Following the same reasoning as before, see that

$$z^n = re^{i(\theta+2\pi k)}; \quad k = 0, \pm 1, \pm 2, \dots$$

where removing the redundant  $k$  (this can be shown using the same methods as above) and solving gives us

$$z = \sqrt[n]{r}e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, \dots, n-1$$

See that these  $n$  roots can be expressed with the roots of unity as

$$\sqrt[n]{w} = z_0, z_0\zeta, z_0\zeta^2, \dots, z_0\zeta^{n-1}, \quad \text{where } z_0 = \sqrt[n]{r}e^{i\theta/n}$$

( $z_0$  is just the case where  $k = 0$ ) See that all of the  $n$  roots satisfy  $z^n = w$ .

**A.1.13**

### A.1.14 Superposition (Second order ODEs)

The Principle of Superposition for Second Order Differential Equations; if

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

is a second order linear differential equation and  $y = y_1(t)$  and  $y = y_2(t)$  are both solutions to this differential equation, then for  $C$  and  $D$  as constants,

$$y = Cy_1(t) + Dy_2(t) \quad \text{is also a solution}$$

Essentially, any linear combination of solutions is also a solution.

*Proof:* Consider  $y = y_1$  and  $y = y_2$  are solutions to the second order linear differential equation  $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$ . Then we have that:

$$\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 = 0 \quad \text{and} \quad \frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2 = 0$$

If  $C$  and  $D$  are constants, plugging in  $y = Cy_1(t) + Dy_2(t)$ :

$$\begin{aligned} & \frac{d^2}{dt^2}(Cy_1(t) + Dy_2(t)) + p(t)\frac{d}{dt}(Cy_1(t) + Dy_2(t)) + q(t)(Cy_1(t) + Dy_2(t)) \\ &= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\ &= C \underbrace{\left[ \frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 \right]}_{=0} + D \underbrace{\left[ \frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2 \right]}_{=0} \\ &= 0 \end{aligned}$$

Therefore,  $y = Cy_1(t) + Dy_2(t)$  is also a solution. Note that the superposition principle **does not** work for nonlinear differential equations.  
(next page)

### In context of inhomogenous differential equations

In addition, if  $y_1$  is a solution to:

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = f_1(t)$$

and  $y_2$  is a solution to:

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = f_2(t)$$

then for constants  $C$  and  $D$ ,  $Cy_1 + Dy_2$  is a solution to:

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = Cf_1(t) + Df_2(t)$$

*Proof:* Plugging in  $y = Cy_1 + Dy_2$ :

$$\begin{aligned} & \frac{d^2}{dt^2}(Cy_1 + Dy_2) + p(t) \frac{d}{dt}(Cy_1 + Dy_2) + q(t)(Cy_1 + Dy_2) \\ &= C \frac{d^2 y_1}{dt^2} + D \frac{d^2 y_2}{dt^2} + p(t)C \frac{dy_1}{dt} + p(t)D \frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\ &= C \underbrace{\left[ \frac{d^2 y_1}{dt^2} + p(t) \frac{dy_1}{dt} + q(t)y_1 \right]}_{=f_1(t)} + D \underbrace{\left[ \frac{d^2 y_2}{dt^2} + p(t) \frac{dy_2}{dt} + q(t)y_2 \right]}_{=f_2(t)} \\ &= Cf_1(t) + Df_2(t) \end{aligned}$$

Superposition is therefore *not* limited to homogenous equations.

### A.1.15 General solution for inhomogenous linear ODEs

Therefore, to get the general solution  $y(t)$  to an inhomogenous linear ODE:

$$\text{inhomogenous: } \frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = f(t)$$

1. Find the general solution  $y_h$  to the associated **homogenous** equation:

$$\text{homogenous: } \frac{d^2 y_h}{dt^2} + p(t) \frac{dy_h}{dt} + q(t)y_h = 0$$

2. Find (in some way) any **one particular solution**  $y_p$  to the **inhomogenous** ODE.
3. Add  $y_p$  to  $y_h$  to get the general solution to the inhomogenous ODE:

$$\underbrace{y}_{\text{general inhomogenous solution}} = \underbrace{y_p}_{\text{any particular solution}} + \underbrace{y_h}_{\text{general homogenous solution}}$$

Note that the superposition principle **does not** work for nonlinear differential equations.

### A.1.16 Existence and uniqueness

Solving a first-order linear ODE leads to a 1-parameter family of solutions (a general solution). To derive a specific solution, we need an initial condition, such as  $y(0)$ . One may wonder if there are other solutions. Here is a general result which says that there aren't and confirms that our methods find all solutions:

#### Existence and uniqueness theorem for a linear ODE:

Let  $p(t)$  and  $q(t)$  be continuous functions on an open interval  $I$ . Let  $a \in I$ , and let  $b$  be a given number. Then there **exists** a **unique** solution defined on the entire interval  $I$  to the first order linear ODE

$$\dot{y} + p(t)y = q(t)$$

satisfying the initial condition

$$y(a) = b$$

**Existence** means there is **at least one** solution.

**Uniqueness** means that there is **only one** solution.

### A.1.17 Exponential response formula

The exponential response formula gives us a quick method for finding the particular solution to any linear, constant coefficient, differential equations whose input can be expressed in terms of an exponential function.

The Exponential Response Formula(ERF):

## A.2 Fourier Series

### A.2.1 Fourier Series

If the input function  $f(t)$  is periodic (of period  $2\pi$ ), we can express the function (where it is continuous) as an infinite sum of sines and cosines. This series representation is called a Fourier Series:

$$f(t) = c_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], \quad c_0, a_n, b_n \text{ real constants}$$