Appendix 3

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Appendix A

Probability

A.1 Fundamental concepts

A.1.1 Expectation and Variance

Expectation
We define the *expected value* of a random variable X with a PMF p_X by

$$\boxed{\mathbb{E}[X] = \sum_x x p_X(x)}$$

Variance and Standard Deviation
We define the *variance* associated with a random variable X as

$$\boxed{\mathrm{var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \sum_x (X - \mathbb{E}[X])^2 p_X(x)}$$

(See that the because of the square the variance is always nonnegative). The variance provides a measure of dispersion of X around the mean. Another measure of dispersion is the *Standard deviation* of X, which is defined as the square root of the variance and is denoted by σ_X:

$$\boxed{\sigma_X = \sqrt{\mathrm{var}(X)}}$$

The standard deviation is often easier to interpret because it has the same units as X.

A.1.2 Expected value of a function of a RV

Expectation of a function
Let X be a RV with PMF p_X, and let $g(X)$ be a function of X. Then the expected value of the random variable $g(X)$ is given by

$$\boxed{\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)}$$

This can be shown, since

$$p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x)$$

we have

$$\begin{aligned}
\mathbb{E}[g(X)] &= \mathbb{E}[Y] \\
&= \sum_y y p_Y(y) \\
&= \sum_y y \sum_{\{x|g(x)=y\}} p_X(x) \\
&= \sum_y \sum_{\{x|g(x)=y\}} y p_X(x) \\
&= \sum_y \sum_{\{x|g(x)=y\}} g(x) p_X(x) \\
&= \sum_x g(x) p_X(x)
\end{aligned}$$

Variance
Using this we can write the variance of X as

$$\operatorname{var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \sum_x (X - \mathbb{E}[X])^2 p_X(x)$$

A.1.3 Expectation and variance of linear functions

We show for a random variable X, and letting $Y = aX + b$:

$$\boxed{\mathbb{E}[Y] = a\mathbb{E}[X] + b, \quad \text{var}(Y) = a^2\text{var}(X)}$$

Linearity of Expectations:

$$\mathbb{E}[Y] = \sum_x (ax+b)p_X(x) = a\underbrace{\sum_x xp_x(x)}_{=\mathbb{E}[X]} + b\underbrace{\sum_x p_x(x)}_{=1} = a\mathbb{E}[X] + b$$

Variance:

$$\begin{aligned}
\text{var}(Y) &= \sum_x (ax + b - \mathbb{E}[aX+b])^2 p_X(x) \\
&= \sum_x (ax + b - a\mathbb{E}[X] + b)^2 p_X(x) \\
&= a^2 \sum_x (x - \mathbb{E}[X])^2 p_X(x) \\
&= a^2 \text{var}(X)
\end{aligned}$$

Note that unless $g(X)$ is a linear function, it is not generally true that $\mathbb{E}[g(X)]$ is equal to $g(\mathbb{E}[X])$.

A.1.4 Variance in terms of Moments Expression

We show

$$\boxed{\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

see that

$$\begin{aligned}
\text{var}(X) &= \sum_x (x - \mathbb{E}[X])^2 p_X(x) \\
&= \sum_x (x^2 - 2x\mathbb{E}[X] + (\mathbb{E}[X])^2) p_X(x) \\
&= \sum_x x^2 p_X(x) - 2\mathbb{E}[X] \sum_x x p_X(x) + (\mathbb{E}[X])^2 \sum_x p_X(x) \\
&= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{aligned}$$

A.1.5 Expectation and Variance of Bernoulli

Consider a Bernoulli RV X with PMF

$$p_X(k) = \begin{cases} p, & \text{if } k = 1. \\ 1-p, & \text{if } k = 0. \end{cases}$$

The mean, second moment, and variance of X are as follows:

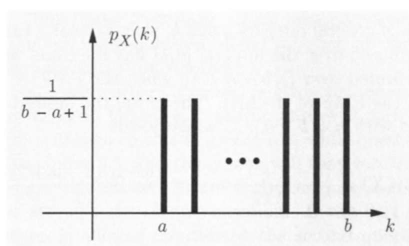
$$\begin{aligned} \mathbb{E}[X] &= 1 \cdot p + 0 \cdot (1-p) = p \\ \mathbb{E}[X^2] &= 1^2 \cdot p + 0 \cdot (1-p) = p \\ \text{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p(1-p) \end{aligned}$$

A.1.6 Expectation of Discrete Uniform

Consider a Discrete Uniform RV X with PMF, for $k \in [a, b]$:

$$p_X(k) = \begin{cases} \frac{1}{b-a+1}, & \text{if } k = a, a+1 \ldots, b \\ 0, & \text{otherwise.} \end{cases}$$

An illustration is useful here:



Expectation
Upon inspection one might suppose that the expectation is

$$\mathbb{E}[X] = \frac{a+b}{2}$$

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Expectation (cont.)
The formula can be elucidated from the definition of the expectation. First see that a sequence $\sum_{k=a}^{b} k$ can be written as

$$\begin{aligned}\sum_{k=a}^{b} k = \sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k \\ &= \frac{(b)(b+1)}{2} - \frac{(a-1)(a)}{2} \quad \text{(see B.1)} \\ &= \frac{b^2+b-a^2+a}{2} = \frac{(b-a+1)(a+b)}{2}\end{aligned}$$

The last step isn't easy to factor, but working back from our 'hypothesis' for the expectation it coincides.

so now we have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=a}^{b} k \left(\frac{1}{b-a+1}\right) \\ &= \frac{1}{b-a+1} \sum_{k=a}^{b} k \\ &= \frac{1}{b-a+1} \cdot \frac{(b-a+1)(a+b)}{2} \\ &= \frac{(a+b)}{2}\end{aligned}$$

A.1.7 Variance of Discrete Uniform

Appendix B

Supplementary Notes

B.1 The sum of the first n natural numbers is $n(n+1)/2$

We have that

$$\sum_{i=1}^{i} i = 1 + 2 + \cdots + n$$

Now consider $2\sum_{i=1}^{n} i$:

$$\begin{aligned}
2\sum_{i=1}^{n} i &= 2(1 + 2 + \cdots + (n-1) + n) \\
&= (1 + 2 + \cdots + (n-1) + n) + (n + (n-1) + \cdots + 2 + 1) \\
&= (1 + n) + (2 + (n-1)) + \cdots + ((n-1) + 2) + (n + 1) \\
&= (n+1)_1 + (n+1)_2 + \cdots + (n+1)_n \\
&= n(n+1)
\end{aligned}$$

so

$$\begin{aligned}
2\sum_{i=1}^{n} i &= n(n+1) \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}
\end{aligned}$$