Appendix 2

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Contents

Appendix A

Differential Equations

A.1   First Order Differential Equations

A.1.1   Introduction to Ordinary Differential Equations (ODEs)

Here we introduce intuition for Ordinary Differential Equations (ODEs) and introductory solving methods.

The simplest type of differential equation looks like:

$$\frac{dy}{dx} = f(x)$$

which can be solved by the antiderivative $y = \int f(x)\, dx$.

Intuition
Now we consider a more interesting example:

$$\frac{dy}{dx} + xy = 0$$

This equation can be solved by *separation of variables*:

$$\frac{dy}{dx} + xy = 0$$
$$\frac{dy}{dx} = -xy$$
$$\frac{dy}{y} = -x\, dx$$

Since the problem is now set up in terms of differentials rather than ratios of differentials, we can integrate both sides.

$$\int \frac{dy}{y} = -\int x \, dx$$

$$\ln y + c_1 = -\frac{x^2}{2} + c_2 \quad \text{(assume } y > 0\text{)}$$

We can combine the constants and simplify:

$$\ln y = -\frac{x^2}{2} + c$$

$$e^{\ln y} = e^{-x^2/2 + c}$$

$$y = e^c e^{-x^2/2}$$

$$y = Ae^{-x^2/2}, \quad \text{(where } A = e^c\text{)}$$

(The more apt $\ln |y|$ simplifies to $\pm Ae^{-x^2/2}$, which doesn't matter since A is some unspecific constant)

It turns out that our solution,

$$y = Ae^{-x^2/2}, \quad \text{(where } A = e^c\text{)}$$

Works for any constant multiple A. We can check this solution:

$$y = ae^{-x^2/2}$$

$$\frac{dy}{dx} = \frac{d}{dx} ae^{-x^2/2}$$

$$= a \cdot (-x)e^{-x^2/2}$$

$$= -x \cdot ae^{-x^2/2}$$

$$\frac{dy}{dx} = -xy$$

A is determined by an initial condition; for instance if $y(0) = 1$, $A = 1$.

A.1.2 Separation of Variables

Here we describe a rudimentary method for solving some differential equations—
Separation of Variables.

In general, this method applies to differential equations of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Where we then *separate* the variables and integrate:

$$\frac{dy}{dx} = f(x)g(y)$$
$$\frac{dy}{g(y)} = f(x)\,dx$$
$$h(y)\,dy = f(x)\,dx \quad \text{where } h(y) = \frac{1}{g(y)}$$
$$\int h(y)\,dy = \int f(x)\,dx$$

Antidifferentiating both sides:

$$H(y) = \int h(y)\,dy; \quad F(x) = \int f(x)\,dx$$

we now have

$$H(y) + c_1 = F(x) + c_2$$
$$H(y) = F(x) + c$$

A.1.3   Direction fields, Isoclines, and Integral curves

Direction fields

Given an equation $y' = f(x, y)$, we can construct a *direction field*; imagine through each point (x, y), we draw a line segment whose slope is $f(x, y)$— consider $y'(x) = 2x$:

(note that in this case $f(x, y)$ does not depend on y (because of the equation)—it is invariant under vertical translation)

Plotting direction fields—Isoclines

In practice, computers are used to plot direction fields following the procedure:

1. Pick point (x, y)

2. Compute $y' = f(x, y)$

3. Plot line segment of slope at that point

Notice how a new slope has to be computed for each specified point; when plotting direction fields by hand, it is much more practical to utilise *isoclines*, which are, given the equation $y' = f(x, y)$, a one-parameter family of curves given by the equations

$$f(x, y) = m, \quad m \text{ constant}$$

Along a given isocline, all line segments have the same slope m.
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Example

Consider plotting the direction field for the equation $y' = x - y$; the isoclines are correspondingly the lines $x - y = m$ (shown in dashed lines):

The $m = 0$ isocline marks the points where the slope of the solution is 0; it is therefore of special interest and is called the *nullcline*.

Integral curves

As also shown in the figure above, once the direction field has been sketched, curves which are *at each point tangent to the line segment at that point* can be drawn; such curves are called *integral curves* or *solution curves* for the direction field. Their significance (this should be obvious) is that

The integral curves are the graphs of the solutions to $y' = f(x, y)$

Two integral curves have been drawn above (in solid lines).

Intersection Principle

Intuitively, see that at any point in the direction field it can only have one direction; therefore it is fairly obvious that integral curves cannot cross at an angle.

Consider the existence and uniqueness theorem for ODEs:

For any (a, b) in the region where f is defined, $y' = f(x, y)$ has exactly one solution such that $y(a) = b$.

by the existence part of the theorem, there is an integral curve through any point where $f(x, y)$ is defined. Now supposing two integral curves through the same point, by the uniqueness part of the theorem they must agree.

As a result, *integral curves cannot intersect*; every point lies on exactly one integral curve.

A.1.4 Long term Behaviour: Fences, Funnels, and Separatrices

Fences

A *lower fence* for the equation $y' = f(x, y)$ is a curve that 'blocks' an integral curve from crossing from *above*; intuitively it is the curve whose direction field elements along the curve point up from it. Technically it can be described as a curve $y = L(x)$ such that $L'(x) < f(x, L(x))$ (the slope of the curve is always less than the slope of the direction field at that point).

Likewise an *upper fence* is a curve that 'blocks' integral curves from crossing from *above*. Illustrated:

(The upper curve is the upper fence and the lower curve is the lower fence). Solutions will be 'squeezed' between upper and lower fences.

Note that

- Note that fences aren't necessarily defined for all x; they could be defined only on an interval like $x \geq c$ for some constant c.

- Since integral curves can't cross an integral curve itself it is both an upper and lower fence.

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Example

Consider the direction field for the equation

$$y' = y^2 - x$$

The isoclines for $m = 0$ and $m = -1$ are plotted in yellow, with integral curves in blue:

Notice that the bottom hald of the isocline $m = 0$ is a lower fence and for x large enough the bottom half of the isocline $m = -1$ is an upper fence. (notice that the $m = -1$ isocline becomes an upper fence only for x large enough)

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Funnels

One use of fences is to construct funnels. A *funnel* for the equation $y' = f(x, y)$ consists of a pair of fences; one lower fence $L(x)$ and one upper fence $U(x)$ with the properties

1. For x large the lower fence is below the upper fence; $L(x) < U(x)$

2. The two fences come together asymptotically; $U(x) - L(x)$ is small for large x

For instance, in the above example the bottom parts of the two isoclines $m = 0$ and $m = -1$ act as a funnel once x is large enough. Given the equations of each isocline we have highly accurate estimates for solutions between them as

$$\underbrace{-\sqrt{x}}_{m=0} < y(x) < \underbrace{-\sqrt{x-1}}_{m=-1}$$

which is valid for large x.

Note that not all pairs of upper/lower fences form a funnel—they have to come together asymptotically as x gets large.

Separatrices

A *separatrix* is an integral curve such that the integral curves above it behave entirely differently from integral curves below it as $x \to \infty$.

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A.1.5 Runge-Kutta 2 (Numerical methods)

General approach and Euler's method

Euler's method (for numerical estimation) follows a more general procedure for stepping from (x_n, y_n) to (x_{n+1}, y_{n+1}):

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + m_n h$$

Where h is the stepsize in the x direction and m is the slope of the line we step along. In Euler's method h is fixed ahead of time and $m_n = f(x_n, y_n)$.

Runge-Kutta 2

Naturally Euler's method is a fairly flawed method of numerical estimation. Other methods use other (and better) ways of choosing h and m. Here I describe the *Runge-Kutta 2* (RK2) method, which is a *fixed stepsize* method; meaning h is fixed and the added complexity comes from finding m.

Given an initial value problem $y' = f(x, y), y(x_0) = x_0$ and a step size h, one step of the RK2 method is as follows:

1. Compute the slope k_1 at (x_0, y_0): $k_1 = f(x_0, y_0)$

2. 'Take' an Euler step from (x_0, y_0) to (a, b) : $a = x_0 + h$, $b = y_0 + k_1 h$

3. Compute the slope k_2 at (a, b) : $k_2 = f(a, b)$

4. Average k_1 and k_2 to get m: $m = (k_1 + k_2)/2$

5. Now we use this averaged slope to take a step from (x_n, y_n) to (x_{n+1}, y_{n+1}):

$$x_1 = x_0 + h, \quad y_1 = y_0 + mh; \quad m = \frac{(k_1 + k_2)}{2}$$

Other methods such as RK4 or *variable stepsize methods* may (probably) work better. Though one might want to consider computational efficiency at the expense of accuracy.

A.1.6 First order Linear Differential Equations

Definition
The general *First order linear ODE* in the unknown function $x = x(t)$ has the form

$$A(t)\frac{dx}{dt} + B(t)x(t) = C(t)$$

If $A(t) \neq 0$ we can simplify the equation by dividing by $A(t)$:

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

This is called the *standard form* for a first order linear ODE. Should the *coefficients* $A(t), B(t)$ be constants (not dependent on t) we say the equation is a *constant coefficient* DE.

If $C(t) = 0$:

$$A(t)\frac{dx}{dt} + B(t)x(t) = 0$$

The DE is called *homogeneous* (notice that conversion to standard form doesn't change this fact); otherwise the equation is *inhomogeneous.*

Signals and Systems—Terminology
Given a differential equation

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

Notice that the right-hand side does not depend on x. The left-hand side represents the *system* (think of it as defining the behaviour of a system); the right-hand side represents an outside influence on the system, which we can call the *input.*

In general, a signal is a function of t. The system *responds* to the input signal and yields the function $x(t)$, which we call the *output signal* or *system response.* (these terms should just be seen as convenient convention when describing an ODE)

Block diagrams can be used to visually represent systems:

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Example—RC circuits

Suppose we have an electrical circuit as shown

"Kirchhoff's Voltage Law" states that the total voltage change around the loop is 0, meaning

$$V(t) = V_R(t) + V_C(t)$$

The relationship between voltage drop and current are described as follows:

Resistor: $V_R(t) = RI(t)$ for a constant R, the "resistance"

Capacitor: $V_C'(t) = \dfrac{1}{C}I(t)$ for a constant C, the "capacitance"

the voltage drop from the capacitance can be seen from the equation defining capacitance

$$q = CV \quad \text{(charge per unit voltage)}$$

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}(CV)$$

$$I(t) = CV' \quad (C \text{ constant})$$

$$V_C'(t) = \frac{1}{C}I(t)$$

The voltage drop across the capacitor is proportional to the *integral* of the current; it results from a buildup of charge on two plates of the capacitor.
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We can differentiate Kirchhoff's Voltage Law

$$V'(t) = V_R'(t) + V_C'(t)$$
$$= RI'(t) + \frac{1}{C}I(t)$$

to obtain a first order linear differential equation

$$RI'(t) + \frac{1}{C}I(t) = V'(t)$$

In this circuit we consider the voltage $V(t)$ to be the input signal, and the circuit with resistance R and capacitance C to be the system. The current I is the output signal/system response:
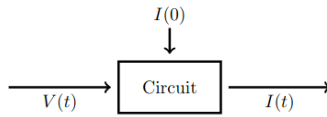
$I(0)$ represents the initial condition.

A.1.7   Superposition (First order ODEs)

Considering the following the first order linear equation:

$$\dot{y} + p(t)y = q(t)$$

If a given input $q(t)$ has the output $y(t)$ we write

$$q \rightsquigarrow y$$

Here we show that if

$$q_1 \rightsquigarrow y_1 \text{ and } q_2 \rightsquigarrow y_2 \quad \text{then} \quad c_1 q_1 + c_2 q_2 \rightsquigarrow c_1 y_1 + c_2 y_2$$

Proof
First see that (since differentiation does't change the constant coefficient)

$$\frac{dy}{dt} + py = q$$
$$c\frac{dy}{dt} + cpy = cq$$
$$= \frac{d(cy)}{dt} + p(cy) = cq; \quad cq \rightsquigarrow cy$$

Now see that

$$\frac{d(c_1 y_1 + c_2 y_2)}{dt} + p(c_1 y_1 + c_2 y_2) = \underbrace{c_1 \dot{y}_1 + pc_1 y_1}_{=c_1 q_1} + \underbrace{c_2 \dot{y}_2 + pc_2 y_2}_{=c_2 q_2}$$
$$= c_1 q_1 + c_2 q_2$$

Essentially, any linear combination of solutions is also a solution.

A.1.8 Solution by Integrating Factor (inhomogenous first order ODEs)

Here we prove the general solution to the inhomogeneous first order linear ODE

$$\dot{x} + p(t)x = q(t)$$

is

$$x(t) = \frac{1}{u(t)} \left(\int u(t)q(t)dt + C \right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

the function u is called an *integrating factor*.

Proof
We start with the product rule for differentiation:

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$$

Consider multiplying both sides of our inhomogenous first order ODE by some function $u(t)$:

$$u\dot{x} + upx = uq$$

We want to choose a function $u(t)$ such that we can apply the product rule to the sum on the left hand side of the equation. There may be many functions u that could work, but in this case we only need one. See that

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x \iff u\dot{x} + upx = u\dot{x} + \dot{u}x \iff \dot{u} = up$$

so now by separation of equations

$$\frac{du}{u} = p(t)dt$$

$$\ln|u| = \int p(t)dt$$

$$u = e^{\int p\,dt}$$

By using u to satisfy the product rule:

$$u\dot{x} + upx = \frac{d}{dt}(ux) = uq$$

$$u(t)x(t) = \int u(t)q(t)dt + c$$

$$x(t) = \frac{1}{u(t)} \left(\int u(t)q(t)dt + c \right)$$

which was what we wanted.
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Integrating factor and homogeneous equations

Given the homogeneous first order ODE

$$\dot{x} + p(t)x = 0$$

Solving by separation of variables gives

$$x_h(t) = Ae^{-\int p(t)dt}$$

Comparing this to the formula for the integrating factor

$$u(t) = e^{\int p(t)dt}$$

see that

$$x_h(t) = \frac{A}{u(t)}$$

A.1.9   General, Particular and Homogeneous solutions

Solving by method of Integrating factors allows us to come up with a solution for inhomogeneous first order linear ODEs

$$\dot{x} + p(t)x = q(t)$$

Which have the form

$$x(t) = \frac{1}{u(t)} \left(\int u(t)q(t)dt + C \right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

Notice that the presence of the constant C implies a family of solutions; by setting $C = 0$ we get a *particular solution* x_p, which is simply one specific solution—we could have chosen any other:

$$x_p = \frac{1}{u(t)} \left(\int u(t)q(t)dt + 0 \right) \quad \text{is a solution}$$

$$x_p = \frac{1}{u(t)} \left(\int u(t)q(t)dt + 999 \right) \quad \text{is also a solution}$$

The method of integrating factors naturally leaves us with a constant. But say we were to find a solution by *inspection*—how would we know that the constant of integration exists in the form $\frac{C}{u(t)}$? (as is in this case)

General solution
See that since

$$x_h(t) = \frac{1}{u(t)}$$

We can write the solution by integrating factor as

$$x(t) = \frac{1}{u(t)} \left(\int u(t)q(t)dt \right) + \frac{C}{u(t)}$$
$$= x_p + Cx_h$$

One way to fully solve the inhomogeneous equation is by first solving the *homogeneous* equation, and then finding any *one* solution, a *particular solution*, to the inhomogeneous equation x_p. (We can use any method to find x_p since we the homogeneous solution handles the constant of integration):

$$\text{General solution} = \text{Particular solution} + \text{Homogeneous solution}$$

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Intuition

Given an inhomogeneous first order linear ODE and its associated homogeneous equation

$$\dot{x} + p(t)x = q(t) \quad \text{(inhomogeneous)}$$
$$\dot{x} + p(t)x = 0 \quad \text{(homogeneous)}$$

Solving both equations by method of integrating factors gives

$$x_p(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{A}{u(t)}, \qquad x_h(t) = \frac{B}{u(t)}$$

(where A is any chosen constant, each constant giving a particular solution, and B the constant of integration) Now see that by adding the solutions together the constant for the inhomogeneous solution A gets absorbed into the homogeneous solution:

$$x_p(t) + x_h(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{A+B}{u(t)}$$
$$= \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{C}{u(t)}$$

We can obtain the 'ambiguous part' of the general solution by simply solving the homogeneous equation; this means that when obtaining a particular solution we don't have to worry about the constant of integration.

Superposition

See that this also makes sense with respect to superposition of solutions, where since

$$\underbrace{q(t) \rightsquigarrow x_p(t)}_{\text{inhomogeneous}} \quad \text{and} \quad \underbrace{0 \rightsquigarrow x_h(t)}_{\text{homogeneous}}$$

we can say

$$q(t) + 0 = q(t) \rightsquigarrow x_p(t) + x_h(t)$$

A.1.10 Polar form and Euler Identity

The Complex Plane, Polar Form

Complex numbers can be represented geometrically by points in a plane, where the number $a + ib$ is represented by the point (a, b); when points in a plane are thought of as representing complex numbers this way, the plane is known as a *Complex Plane*:

See that the magnitude of the coordinates of a complex number $x + iy$ can be represented by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

where r is the absolute value of the number:

$$r = |x + iy| = \sqrt{x^2 + y^2}$$

(its just the pythagorean theorem) thus the entire number can be written as

$$x + iy = r(\cos(\theta) + i\sin(\theta))$$

This is called the *Polar Form* of a non-zero complex number. We call θ the *angle* or *argument* of $x + iy$:

$$\theta = \arg(x + iy)$$

Notice that the angle can be increased by any integer multiple of 2π and will still represent the same thing. To simplify this one can specify the *principal value* of the angle:

$$0 \le \theta < 2\pi$$

this can be indicated by Arg(...); for instance

$$\text{Arg}(-1) = \pi, \quad \arg(-1) = \pm\pi, \pm3\pi, \pm5\pi$$

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Euler's Formula
Complex numbers have another *exponential* form called *Euler's formula*:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This should be regarded as a definition for the exponential of an imaginary power.

A good justification for Euler's formula can be found from its Taylor approximation:

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}
$$

Note that the argument above is not a proof; rather it just shows that Euler's formula is formally compatible with the series expansions for the exponential, sine, and cosine functions.

Polar form again
We can now write

$$x + iy = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

Polar representation in exponential form allows for much simpler multiplication of complex numbers. Since one can show that (using angle addition formulas)

$$
\begin{aligned}
e^{i\theta_1}e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\
&= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\
&\quad + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \\
&= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\
&= e^{i(\theta_1 + \theta_2)}
\end{aligned}
$$

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Complex Exponential properties
We had
$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

This property can be extrapolated to further justify Euler's formula—the complex exponential follows the same exponential addition rules as any typical exponential. See that we can now conclude:

Multiplication rule:
$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

also see that since
$$\frac{1}{r} e^{-i\theta} \cdot r e^{i\theta} = 1$$

Reciprocal Rule:
$$\frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

DeMoivre's Formula
Since
$$(x + iy)^n = r^n e^{in\theta}$$

we can show *DeMoivre's formula:*

$$(\cos(\theta) + i\sin(\theta))^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Combining pure oscillations of the same frequency
We can also show that

$$a\cos(\lambda t) + b\sin(\lambda t) = A\cos(\lambda t - \phi)$$

where
$$A = \sqrt{a^2 + b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

See that

$$
\begin{aligned}
a\cos(\lambda t) + b\sin(\lambda t) &= \mathrm{Re}((a - bi)(\cos(\lambda t) + i\sin(\lambda t))) \\
&= \mathrm{Re}(Ae^{-i\phi} \cdot e^{i\lambda t}) \\
&= \mathrm{Re}(Ae^{i(\lambda t - \phi)}) \\
&= A\cos(\lambda t - \phi)
\end{aligned}
$$

A.1.11   More on Complex Exponentials

Notable properties
We know that (as proven)

$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$$

So see that

$$\text{Re}(e^{a+ib}) = e^a \cos(b), \quad \text{Im}(e^{a+ib}) = e^a \sin(b)$$

this can be extrapolated further to show

$$\cos(x) = \text{Re}(e^{ix}), \qquad\qquad \sin(x) = \text{Im}(e^{ix})$$

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad\qquad \sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$$

Derivatives and integrals
Note that a function like

$$e^{ix} = \cos(x) + i\sin(x)$$

is a *complex-valued function of the real variable* x. Such a function may be written as

$$u(x) + iv(x), \quad u, v \text{ real-valued}$$

with its derivative and integral with respect to x defined to be

$$\text{a) } D(u+iv) = Du + iDv, \quad \text{b) } \int(u+iv)dx = \int u\,dx + i\int v\,dx$$

It follows easily that

$$D(e^{(a+ib)x}) = (a+ib)e^{(a+ib)x}$$

since

$$\begin{aligned}
D(e^{(a+ib)x}) &= D(e^{ax}\cos(bx) + ie^{ax}\sin(bx)) \\
&= ae^{ax}\cos(bx) - be^{ax}\sin(bx) + i(ae^{ax}\sin(bx) + be^{ax}\cos(bx)) \\
&= e^{ax}\cos(bx)(a+ib) + e^{ax}\sin(bx)(ia + i^2 b) \\
&= (a+ib)e^{ax}(\cos(bx) + i\sin(bx)) \\
&= (a+ib)e^{(a+ib)x}
\end{aligned}$$

Therefore we can also write the down the integral as

$$\int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x}$$

A.1.12 Finding n-th roots

To solve linear DEs with constant coefficients, we need to be able to find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha$$

where α is a complex number—finding the n-th roots of α.

n-th roots of unity
Consider first a special case; we want the solutions to

$$z^n = 1$$

We use polar representation for both sides, setting $z = re^{i\theta}$ on the left. See that

$$\underbrace{r^n e^{in\theta}}_{(re^{i\theta})^n} = \underbrace{1 \cdot e^{(2k\pi i)}, \quad k = 0, \pm 1, \pm 2, \ldots}_{=1}$$

Equating the absolute values and the arguments of each side:

$$r^n = 1, \quad n\theta = 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

(Notice the arguments for $k = a$ and $k = -a$, where a is an integer, are the same. Also see that r can only be 1 it is defined to be *real and non-negative* so it can't be anything else) we can conclude that

$$r = 1, \quad \theta = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n - 1$$

we don't need any integer values of k other than $0, \ldots, n - 1$—they would not produce a complex number that isn't already among the above n numbers. See that if we add an, an integer multiple of n, to any k we get the same complex number:

$$\theta' = \frac{2(k + an)\pi}{n} = \theta + 2a\pi$$

(this is the same as having $k = n, n + 1, n + 2 \ldots$) so

$$e^{i\theta'} = e^{i\theta} e^{2a\pi i} = e^{i\theta}$$

We can conclude therefore that *the n-th roots of 1 are the numbers*

$$e^{2k\pi i/n}, \quad k = 0, \ldots, n - 1$$

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Roots of unity visualised

There are n complex n-th roots of unity. Since they all have absolute value 1 ($r = 1$) they all lie on the unit circle in the complex plane. They are evenly spaced around the unit circle; the angle between two consecuitive roots is $2\pi/n$.

Illustrated here is the case for $n = 6$:

The six solutions to $z^6 = 1$ lie on the unit circle in the complex plane. See that we can express the roots of unity in a different notation:

the n-th roots of 1 are $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$, *where* $\zeta = e^{2\pi i/n}$

General case

Now we generalise to find the n-th roots of an arbitrary complex number w. We start by writing w in polar form:

$$w = re^{i\theta}; \quad \theta = \mathrm{Arg}(w), 0 \le \theta < 2\pi$$

Here θ is the principal value of the polar angle of w. Following the same reasoning as before, see that

$$z^n = re^{i(\theta + 2\pi k)}; \quad k = 0, \pm 1, \pm 2, \ldots$$

where removing the redundant k (this can be shown using the same methods as above) and solving gives us

$$z = \sqrt[n]{r}e^{i(\theta + 2\pi k)/n}, \quad k = 0, 1, \ldots, n - 1$$

See that these n roots can be expressed with the roots of unity as

$$\sqrt[n]{w} = z_0, z_0\zeta, z_0\zeta^2, \ldots, z_0\zeta^{n-1}, \quad \text{where } z_0 = \sqrt[n]{r}e^{i\theta/n}$$

(z_0 is just the case where $k = 0$) See that all of the n roots satisfy $z^n = w$.

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A.1.13 Sinusoidal functions

Definition and properties

A *sinusoidal function/oscillation/signal* is one that can be written in the from

$$f(t) = A\cos(\omega t - \phi)$$

The function $f(t)$ is a cosine function which has been *amplified* by A, *shifted* by ϕ/ω, and *compressed* by ω.

- $A > 0$ is its *amplitude*: how high the graph of $f(t)$ rises above the t-axis at its maximum values

- ϕ is its *phase lag*: the value of ωt for which the graph has its maximum (a positive phase lag shifts the sinusoid *forward*; consider a maximum at $\cos(a)$, without phase lag it is reached at $\omega t = a$, with phase lag its now $\omega t = a + \phi$.)

- $\tau = \phi/\omega$ is its *time delay/lag*: how far along the t-axis the graph of $\cos(wt)$ has been shifted due to phase lag. (τ and ϕ have the same sign; consider a maximum at $\cos(0)$, without phase lag it is reached at $\omega t = 0 \implies t = 0$, with phase lag its now $\omega t - \phi = 0 \implies t = \phi/\omega$.)

- ω is its *angular frequency*: the number of complete oscillations $f(t)$ makes per time interval of 2π; that is, the *number of radians per unit time* (1 radian in 1 second means 1 oscillation in 2π seconds—1 radian is the angle subtended at the centre of a circle by an arc equal in length to the radius).

- $v = \omega/2\pi$ is the *frequency* of $f(t)$: the number of complete oscillations made in a time interval of 1; that is, the number of cycles per unit time.

- $P = 2\pi/\omega = 1/v$ is its *period*: the t-interval required for one complete oscillation.

See that one can also write the sinusoidal function using the time lag $\tau = \phi/\omega$:

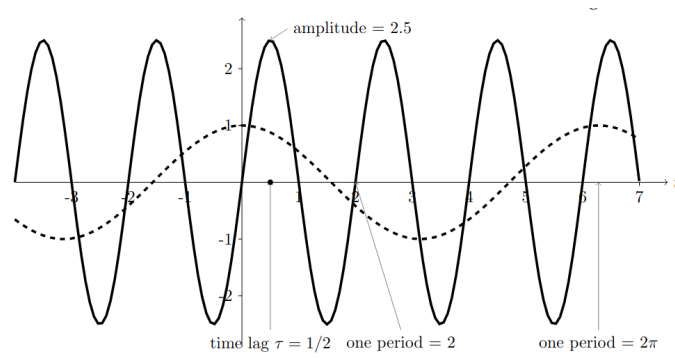$$f(t) = A\cos(\omega(t - \tau))$$

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Example

In the figure below the dotted curve is $\cos(t)$ and the solid curve is $2.5\cos(\pi t - \pi/2)$. The solid curve has

$$A = 2.5, \quad \omega = \pi, \quad \phi = \pi/2, \quad \tau = 1/2$$

A.1.14 Solution to the Constant Coefficient First Order Equation

Solution

Considering the constant coefficient equation (constant coefficient meaning k is a constant)

$$\dot{y} + ky = q(t)$$

This is easily solvable by integrating factor:

$$y = e^{-kt}\left(\int e^{kt}q(t)dt + c\right)$$

$$= e^{-kt}\int e^{kt}q(t)dt + ce^{-kt}$$

(integrating factor gives us a way of finding the particular solution, but see that it also gives us the homogeneous solution) We have the *particular* solution and *homogeneous* solution respectively

$$y_p(t) = e^{-kt}\int e^{kt}q(t)dt \quad \text{and} \quad y_h(t) = e^{-kt}$$

The general solution is then

$$y(t) = y_p(t) + cy_h(t)$$

Behaviour for $k > 0$:

For $k > 0$ the system models *exponential decay*. When the input is 0 the system response is $y(t) = ce^{-kt}$, which decays exponentially to 0 as t goes to ∞.

In the general solution we call ce^{-kt} the *transient* because it goes to 0. The other term $e^{-kt}\int e^{kt}q(t)dt$ is called the *steady-state/long-term* solution. That is, cy_h is the transient and y_p is the steady-state solution.

The value of c is determined by the initial value $y(0)$. See that this initial value only affects the transient and not the long-term behaviour of the solution—no matter what the initial condition, every solution goes asymptotically to the steady-state—all solution curves approach the steady-state as $t \to \infty$.

(next page)

Behaviour for $k > 0$ illustrated

In the case $k > 0$ all solutions go asymptotically to the steady-state:

y' = -y - 1/(x + 1) + 1

Since all the solutions approach each other, there is no precise way to choose the one to we call the steady-state—we can *choose any one* to be the steady-state solution. Generally we just choose the simplest looking solution.

The case $k \leq 0$:

When $k \leq 0$ the homogeneous solution $e^{-}kt$ does not decay asymptotically to 0—it is not transient. In this case it does not make sense to talk about the steady-state solution.

A.1.15   First Order Response to Sinusoidal/Exponential Input—complex replacement

Context

Consider solving the first order constant coefficient DE with sinusoidal input

$$\dot{x} + kx = B\cos(\omega t)$$

The idea here is to replace $\cos(\omega t)$ by the complex exponenial $e^{i\omega t}$; this is called *complex replacement*.

Complex Replacement

Consider introducing a new variable y with its own related ODE:

$$\dot{y} + ky = B\sin(\omega t)$$

Combining x and y to make a complex variable $z = x + iy$, see that we get

$$\dot{z} + kz = B(\cos(\omega t) + i\sin(\omega t)) = Be^{i\omega t}$$

where

$$\cos(\omega t) = \mathrm{Re}(e^{i\omega t}) \quad \text{and} \quad x = \mathrm{Re}(z)$$

Exponential input

Using complex replacement we now have the same problem but with exponential input

$$\dot{z} + kz = Be^{i\omega t}$$

This can be solved using integrating factors, but we present a simpler solution: consider a particular solution of the form $z_p(t) = Ae^{i\omega t}$ (this is a reasonable choice given that differentiation reproduces exponentials); this gives us

$$\dot{z}_p + kz_p = i\omega Ae^{i\omega t} + kAe^{i\omega t} = (k + i\omega)Ae^{i\omega t}$$

so we have

$$(k + i\omega)Ae^{i\omega t} = Be^{i\omega t} \implies A = B/(k + i\omega)$$

As such we have the particular solution

$$z_p(t) = Be^{i\omega t}/(k + i\omega)$$

simplifying with polar coordinates:

$$z_p(t) = \frac{Be^{i\omega t}}{\sqrt{k^2 + \omega^2}e^{i\phi}} = \frac{Be^{i(\omega t - \phi)}}{\sqrt{k^2 + \omega^2}}$$

since

$$k + i\omega = \sqrt{k^2 + \omega^2}e^{i\phi}, \quad \text{where } \phi = \tan^{-1}(\omega/k) \text{ in the first quadrant}$$

Since $\tan^{-1}$ is ambiguous ($\tan(\pi/4) = \tan(5\pi/4) = 1$), we clarify by saying which quadrant the complex number is in. In this case since $k, \omega > 0$ its the first quadrant.

(next page)

Solving for sinusoidal input

We have

$$z_p(t) = \frac{Be^{i(\omega t - \phi)}}{\sqrt{k^2 + \omega^2}}$$

we wanted x_p, where $x_p = \mathrm{Re}(z_p)$:

$$x_p(t) = \frac{B}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi)$$

To get the general solution we add the homogeneous solution:

$$x(t) = x_p(t) + Ce^{-kt} = \frac{B}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi) + Ce^{-kt}$$

A.1.16   Amplitude, Phase, Gain, and Bode Plots
—Terminology and introduction

Terminology
We found that the ODE

$$\dot{x} + kx = kB\cos(\omega t)$$

has a particular solution

$$x(t) = \frac{kB}{\sqrt{k^2 + \omega^2}}\cos(\omega t - \phi)$$

where $\phi = \tan^{-1}(\omega/k)$. If we consider the input to be $B\cos(\omega t)$ then the gain g (output amplitude/input amplitude) is $g = k/\sqrt{k^2 + \omega^2}$:

$$x(t) = gB\cos(\omega t - \phi)$$

We define the terminology as follows:

- $B\cos(\omega t)$ is the input/input signal.

- B is the input amplitude and ω is the input angular/circular frequency.

- $x(t)$ is the output or response.

- $g = k/\sqrt{k^2 + \omega^2}$ is called the gain/amplitude response. See that the input amplitude is scaled by the gain to give the output amplitude.

- ϕ is called the phase lag.

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Bode plots

Since g and ϕ vary with ω, we can regard them as functions of ω—$g(\omega)$ and $\phi(\omega)$. k is called the *coupling constant*. Consider the graphs of $g(\omega)$ and $-\phi(\omega)$ for the values of coupling constant $k = .25, .5, .75, 1, 1.25, 1.5$:

$g = \text{gain}$

Fig. 1. First order amplitude response curves

$-\phi/\pi = \text{phase shift in multiples of } \pi$

Fig. 2. First order phase response curves

These graphs are essentially *Bode plots*. (Bode plots display $\log g(\omega)$ and $-\phi(\omega)$ against $\log \omega$).

A.1.17 Autonomous equations, Logistic Model, Stable/Unstable equilibria

Here we consider *autonomous first order differential equations.* These are (in general) nonlinear equations of the form

$$\dot{x} = f(x)$$

(compare this with the general first order ODE $\dot{x} = f(x,t)$.) The word autonomous means self governing—the rate of change of x is governed by x itself and is not dependent on time.

Example: Logistic Population Model
Suppose we have a model for a population y with variable growth rate $k(y)$ which depends on the current population but *not on time*:

$$\dot{y} = k(y) \cdot y$$

Say we model $k(y)$ as

$$k(y) = k_0 \left(1 - \frac{y}{M}\right)$$

The idea here is that population growth is positive until the population reaches some M, after which it becomes negative and declines until it becomes lower than M. In the simplest version of this we model $k(y)$ as a straight line as above. The final equation is known as the *logistic population model*:

$$\dot{y} = k_0(1 - (y/M))y = f(y)$$

The equation is nonlinear and autonomous. Autonomous equations are always separable; in this case partial fractions could be used to compute an integral, but here we consider a qualitative approach.
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Qualitative perspective

We start by looking for *constant* solutions $y(t) = y_0$. We do this by considering $\dot{y} = 0$; see that this occurs in two situations, either

$$y(t) = 0, \quad \text{or} \quad y(t) = M$$

Because a system at equilibrium is unchanging, we call these solutions *equilibrium solutions*. Since equilibrium is achieved when y is 0 or M we call 0 and M the *critical points* of the DE. To summarise, these statements all mean the same thing:

1. $f(y_0) = 0$.

2. $y(t) = y_0$ is an equilibrium solution.

3. $y = y_0$ is a critical point.

Consider the direction field for these solutions; (recall each isocline represents $f(y) = c$) they correspond to the *nullclines*, where $f(y) = 0$:
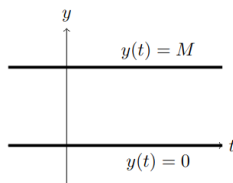
(Note that the nullclines are also solution curves, if y starts at M it will never change since its derivative will be 0 forever.) For a clear picture of the other isoclines consider a graph of $f(y)$ against y:
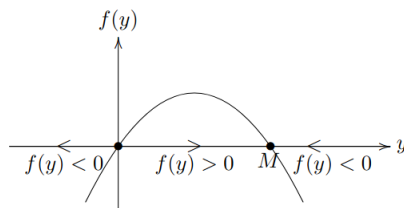
See that

$$\text{for } y < 0 \quad \dot{y} = f(y) \text{ is negative,}$$
$$\text{for } 0 < y < M \quad \dot{y} = f(y) \text{ is positive,}$$
$$\text{for } M < y \quad \dot{y} = f(y) \text{ is negative}$$

These are indicated on the graph by the arrows on the horizontal axis.
(next page)

Direction field
We sketch the direction field and some solution curves:

See that

1. Since the isoclines are constant in the t direction any solution curve can be translated left or right and still be a solution—time invariance.

2. Since the lines $y = 0$ and $y = M$ are solutions the other curves can't cross them.

3. The solutions that start above $y = 0$ must increase, and since they can't cross $y = M$ they tend toward it asymptotically. These bounded solutions are called *logistic curves*. They represent small populations increasing to M.

4. If the population exceeds M, they tend back towards it. This represents overpopulation. M is called the *carrying capacity* of the environment.

5. Although it doesn't make sense to model a negative numbered population $y < 0$, mathematically the solution curves that start below $y = 0$ decrease without bound.

Stable and Unstable Equilibria
See that solution curves near the equilibrium $y = M$ tend asymptotically towards it; this is called a *stable equilibrium*. Solution curves near the other equilibrium $y = 0$ tend away from it; this is called an *unstable equilibrium*.

A.1.18 Phase lines, Semistable equilibria

Phase lines allows for the essential content of an autonomous DE:

$$\dot{y} = f(y)$$

to be conveyed more efficiently. A phase line is drawn using the following steps:

1. Draw the y-axis as a vertical line and mark on it the equilibria—where $f(y) = 0$.

2. In each of the intervals delimited by the equilibria draw an upward pointing arrow if $f(y) > 0$ and a downward arrow if $f(y) < 0$.

Phase lines tells us roughly how the system behaves, capturing the information of a qualitative sketch. Consider the following examples.

Example 1:
Consider the simple autonomous equation

$$\dot{y} = 3y$$

1) First we find the critical points; see that only one critical point exists: $y = 0$.
2) Next we plot the graph of $f(y)$ (in this case a straight line); see that $\dot{y} > 0$ for $y > 0$ and $\dot{y} < 0$ for $y < 0$:

(next page)

Example 1 (cont.)
3) With this we can draw the phase line as outlined above. Since the arrows on the phase line point away from the critical point, the equilibrium is *unstable*:

unstable • 0

3. The phase line.   4. Qualitative sketch of solution curves.

4) See how the phase line conveys qualitiative information.   The equilibrium solution corresponds to the critical point.

Example 2: Logistic equation
Now consider the same solution for the logistic equation:

$$\dot{y} = k_0(1 - y/M)y$$

With critical points $y = 0, y = M$:

$\dot{y} = f(y)$

$f(y) < 0$ $f(y) > 0$ M $f(y) < 0$
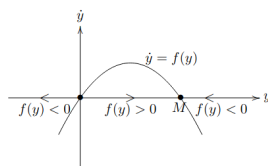
2. Graph of $f(y)$

stable • M

unstable • 0

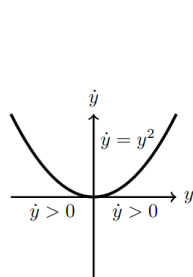3. Phase line.   4. Sketch of solution curves.

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Semistable Equilibria

Some equilibria are stable on one side and unstable on the other. We call them *semistable*. Consider the DE

$$\dot{y} = y^2$$

With only one critical point $y = 0$:
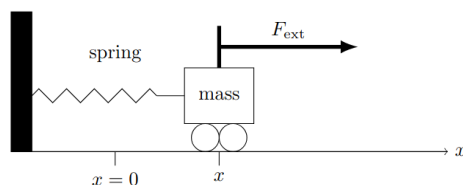

Graph of $f(y)$.   Phase line.   Sketch of solution curves.

A.2 Second Order Constant Coefficient Linear Equations

A.2.1 Second Order Physical systems— Spring-Mass-Dashpot

Spring and Mass

Here we model a second order differential equation. Consider a spring attached to a wall and a cart:

Consider the coordinate system set in a way that at $x = 0$ the spring doesn't exert any force—the equilibrium position. Now also consider an external force acting on the mass, the system can be modelled as

$$m\ddot{x} = F_{\text{spr}} + F_{\text{ext}}$$

The spring's behaviour can be characterised by the fact that it depends on the deviation from equilibrium position, meaning

$$\text{if } x > 0, \quad F_{\text{spr}}(x) < 0$$
$$\text{if } x = 0, \quad F_{\text{spr}}(x) = 0$$
$$\text{if } x < 0, \quad F_{\text{spr}}(x) > 0$$

The simplest way to model the force exerted by the spring (which is valid in general for small x) is

$$F_{\text{spr}}(x) = -kx, \text{ where } k > 0$$

This is called *Hooke's law*, and k is called the *spring constant*.

Replacing F_{spr} by $-kx$ we get

$$m\ddot{x} + kx = F_{\text{ext}}$$

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Dashpot

Any real mechanical system has friction, which can take many forms; it is characterised by the fact that it depends on the motion of the mass. We will suppose that it depends only on the velocity of the mass and not on its position.

Often dampening is controlled by a device called the *dashpot* (its a cylinder filled with oil that a piston moves through):
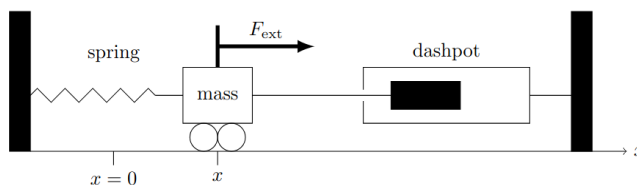
We write $F_{\mathrm{dash}}(\dot{x})$ for the force exerted by the dashpot. It opposes the velocity:

$$\begin{aligned} \text{if } \dot{x} > 0, \quad & F_{\mathrm{dash}}(\dot{x}) < 0 \\ \text{if } \dot{x} = 0, \quad & F_{\mathrm{dash}}(\dot{x}) = 0 \\ \text{if } \dot{x} < 0, \quad & F_{\mathrm{dash}}(\dot{x}) > 0 \end{aligned}$$

The simplest way to model this (which is also valid for small $\dot{x}$) is

$$F_{\mathrm{dash}}(\dot{x}) = -b\dot{x}, \text{ where } b > 0$$

This is called *linear damping*, and b is called the *damping constant*.

Putting this together

$$m\ddot{x} = F_{\mathrm{spr}} + F_{\mathrm{dash}} + F_{\mathrm{ext}}$$

we get the differential equation for the displacement x of the mass from equilibrium as

$$m\ddot{x} + b\dot{x} + kx = F_{\mathrm{ext}}$$

A.2.2   Linear DEs—Notation

A *linear differential equation* is of the following form:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 \dot{x} + a_0 x = q(t)$$

The a_k are the *coefficients*; they may depend on t. If a_n is not zero then the differential equation is said to be of order n.

If the a_k are constant then the equation is said to be a *constant coefficient linear equation*.

A.2.3   Second order homogeneous constant coefficient linear equations—Spring system, Simple harmonic oscillator

Consider the spring system in the case where $F_{\text{ext}} = 0$:

$$m\ddot{x} + b\dot{x} + kx = 0$$

With no external force the equation is *homogeneous*.

Undamped case: Simple harmonic oscillator
The special case where $b = 0$ (no dashpot) is called *undamped*. This is called the *simple harmonic oscillator*. We can write its ODE as

$$\ddot{x} + \frac{k}{m}x = 0$$

If we let $\omega = \sqrt{k/m}$ our equation becomes

$$\ddot{x} + \omega^2 x = 0$$

See that $x_1(t) = \cos(\omega t)$ and $x_2(t) = \sin(\omega t)$ are solutions to this equation. Since the equation is linear we can use superposition of solutions to get the solution

$$x(t) = a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi)$$

This is the *general solution*. We know it gives every solution because $x(0) = a$ and $\dot{x}(0) = \omega b$—by solving (uniquely) for a and b we can get any desired initial condition.

A.2.4 Characteristic polynomial

2nd Order Case

For m, b, k constant, consider the homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

Consider solutions of the form $x = e^{rt}$. We have

$$m\ddot{x} + b\dot{x} + kx = (mr^2 + br + k)e^{rt} = 0$$

Since an exponential is never zero, e^{rt} is therefore a solution exactly when r satisfies the *characteristic equation* (the left hand side is the *characteristic polynomial*):

$$mr^2 + br + k = 0$$

Example

Consider the DE

$$\ddot{x} + 8\dot{x} + 7x = 0$$

The characteristic polynomial here is $r^2 + 8r + 7$. Solving for r by factorisation we have $(r + 1)(r + 7)$ and the roots $r = -1$ and $r = -7$. Therefore the corresponding exponential solutions are $x_1(t) = e^{-t}$ and $x_2(t) = e^{-7t}$.

By superposition, the linear combination of independent solutions gives the general solution:

$$x(t) = c_1 e^{-t} + c_2 e^{-7t}$$

Where given initial conditions for x and $\dot{x}$ we can solve for c_1 and c_2.

General nth Order Case

See that using the same principle we can take the homogeneous constant coefficient linear equation of degree n:

$$a_n x^{(n)} + \cdots + a_1 \dot{x} + a_0 x = 0$$

and get its characteristic polynomial

$$p(r) = a_n r^n + \cdots + a_1 r + a_0$$

In which the exponential $x(t) = e^{rt}$ is a solution of the homogeneous DE if and only if r is a root of $p(r)$ (meaning $p(r) = 0$). By superposition, any linear combination of these exponentials is also a solution.

A.2.5 Modes and Roots (real and complex) (homogeneous constant coefficient linear equations)

Modes

A solution of the form $x(t) = ce^{rt}$ to the homogeneous constant coefficient linear equation:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 \dot{x} + a_0 x = 0$$

is called a *modal solution* and ce^{rt} the *mode* of the system. Recall that e^{rt} is a solution exactly when r is a root of the characteristic polynomial

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

(note this only works for *homogeneous constant coefficient linear equations*; it won't apply to non-constant coefficient or inhomogeneous or nonlinear equations.)

Real roots

The roots of these polynomials can be real or complex. Roots can also be repeated. First consider the real case for a second order homogeneous constant coefficient DE: if the characteristic polynomial has real roots r_1 and r_2 then the *modal* solutions are $x_1(t) = e^{r_1 t}$ and $x_2(t) = e^{r_2 t}$. The general solution can be found by superposition:

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example

Solving $\ddot{x} + 5\dot{x} + 4x = 0$: The characteristic equation is

$$s^2 + 5s + 4 = (s+1)(s+4) = 0$$

Which has roots -1 and -4. The modal solutions are $x_1(t) = e^{-t}$ and $x_2(t) = e^{-4t}$. Therefore the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-4t}$$

(next page)

Complex roots—illustrative example

Consider now the equation $\ddot{x} + 4\dot{x} + 5x = 0$. The characterisic polynomial is $s^2 + 4s + 5$. Using the quadratic formula the roots are

$$s = \frac{-4\sqrt{16-20}}{2} = -2 \pm \sqrt{-1} = -2 \pm i$$

So our exponential solutions are (using the letter z to indicate they are complex valued):

$$z_1(t) = e^{(-2+i)t} \quad \text{and} \quad z_2(t) = e^{(-2-i)t}$$

The DE has real coefficients, we expect *real solutions*. To get them, consider the following theorem:

Real Solution Theorem:

Theorem: If $z(t)$ is a complex-values solution to $m\ddot{z} + b\dot{z} + kz = 0$, where m, b, k are real, then the real and imaginary parts of z are also solutions.

Proof: Letting $u(t)$ be the real part of z and $v(t)$ the imaginary part, so that $z(t) = u(t) + iv(t)$, see that the DE can be written as

$$(m\ddot{u} + b\dot{u} + ku) + i(m\ddot{v} + b\dot{v} + kv) = 0$$

Both expressions in parentheses are real. The only way for the sum to be 0 is if both expressions are 0. That is, both u and v are solutions.

Illustrative example cont.

We had $z_1(t) = e^{(-2+i)t}$ and $z_2(t) = e^{(-2-i)t}$. Using Euler's formula:

$$z_1(t) = e^{(-2+i)t} = e^{-2t}\cos t + ie^{-2t}\sin t$$

Both the real part $e^{-2t}\cos t$ and imaginary part $e^{-2t}\sin t$ are solutions. We now have two *basic* solutions and can use superposition to obtain the general *real valued* solution

$$x(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$$

See that choosing the other exponential solution

$$z_2(t) = e^{(-2+i)t} = e^{-2t}\cos(-t) + ie^{-2t}\sin(-t)$$

would give the basic real solutions

$$e^{-2t}\cos(t) \quad \text{and} \quad -e^{-2t}\sin(t)$$

Which would give the same general solution.
(next page)

More on complex roots

We had the general solution

$$c_1 e^{-2t} \cos(t) + c_2 e^{-2t} \sin(t)$$

See that the solution can be written in a different form:

$$x(t) = e^{-2t}(c_1 \cos(t) + c_2 \sin(t)) = A e^{-2t} \cos(t - \phi)$$

This is a *damped sinusoid* with *circular pseudo-frequency* 1.

Example

Solving $\ddot{x} + \dot{x} + x = 0$, the characteristic equation is $s^2 + s + 1 = 0$ and the roots

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i \frac{\sqrt{3}}{2}$$

We have the complex exponential solutions

$$z_1(t) = e^{(-1+i\sqrt{3})t/2}, \quad z_2(t) = e^{(-1-i\sqrt{3})t/2}$$

From this we obtain the basic real solutions

$$\mathrm{Re}(z_1(t)) = e^{-t/2} \cos(\sqrt{3}t/2), \quad \mathrm{Im}(z_1(t)) = e^{-t/2} \sin(\sqrt{3}t/2)$$

and therefore the general real solution

$$e^{-t/2}(c_1 \cos(\sqrt{3}t/2) + c_2 \sin(\sqrt{3}t/2)) = A e^{-t/2} \cos(\sqrt{3}t/2 - \phi)$$

In general

In general, supposing the equation $m\ddot{x} + b\dot{x} + kx = 0$ has the characteristic roots $a \pm ib$, two real solutions are

$$e^{at} \cos(bt) \quad \text{and} \quad e^{at} \sin(bt)$$

and the general real solution is

$$c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt) = A e^{at} \cos(bt - \phi)$$

A.2.6 Repeated roots
(homogeneous constant coefficient linear equations)

Illustrative example

Consider $\ddot{x} + 4\dot{x} + 4x = 0$. In this case the characteristic equation:

$$P(s) = s^2 + 4s + 4 = (s+2)^2$$

has $r = -2$ as a repeated root. The only exponential solution is $e^{-2}t$. To get the second basic exponential solution, see that te^{-2t} is also a solution. Our general solution is therefore

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Deriving solution for second order repeated roots

Considering the DE $a\ddot{x} + b\dot{x} + cx = 0$, from the characteristic equation we know

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Should we only have one solution, as per the quadratic formula, we must have

$$b^2 - 4ac = 0, \quad \text{and} \quad r_{1,2} = -\frac{b}{2a}$$

We know one solution is $x_1 = e^{-b/(2a)t}$. Consider a second solution of the form

$$x_2 = v(t)x_1$$

we want to plug x_2 into the DE. For that we require its derivatives:

$$x_2' = v'e^{-b/(2a)t} - \frac{b}{2a}ve^{-b/(2a)t}$$

$$x_2'' = v''e^{-b/(2a)t} - \frac{b}{2a}v'e^{-b/(2a)t} - \frac{b}{2a}v'e^{-b/(2a)t} + \frac{b^2}{4a^2}ve^{-b/(2a)t}$$

$$= v''e^{-b/(2a)t} - \frac{b}{a}v'e^{-b/(2a)t} + \frac{b^2}{4a^2}ve^{-b/(2a)t}$$

Evaluating the DE with our proposed solution x_2:

$$a\left(v''e^{-b/(2a)t} - \frac{b}{a}v'e^{-b/(2a)t} + \frac{b^2}{4a^2}ve^{-b/(2a)t}\right) +$$

$$b\left(v'e^{-b/(2a)t} - \frac{b}{2a}ve^{-b/(2a)t}\right) + c\left(ve^{-b/(2a)t}\right) = 0$$

Factoring out the exponential we get

$$e^{-b/(2a)t}\left(av'' - bv' + \frac{b^2}{4a}v + bv' - \frac{b^2}{2a}v + cv\right)$$

$$= e^{-b/(2a)t}\left(av'' + \left(-\frac{b^2}{4a} + c\right)v\right)$$

$$= e^{-b/(2a)t}\left(av'' - \frac{1}{4a}\left(b^2 - 4ac\right)v\right) = 0$$

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Derivation continued

Evaluating the DE with our proposed solution $x_2 = ve^{-b/(2a)t}$ gave us

$$e^{-b/(2a)t} \left(av'' - \frac{1}{4a} \left(b^2 - 4ac \right) v \right) = 0$$

We know that $b^2 - 4ac = 0$ (since the quadratic equation only has one solution as mentioned before). Thus since exponentials cannot be zero, we have

$$av'' = 0 \implies v'' = 0$$

(since $a \neq 0$.) We can then determine $v(t)$:

$$v' = \int v'' dt = k \implies v = \int v' dt = k_1 t + k_2$$

We therefore have our proposed basic solutions as

$$x_1 = e^{-b/(2a)t}, \quad x_2 = vx_1 = (k_1 t + k_2)e^{-b/(2a)t}$$

see that combining our solutions into a general solution via superposition gives

$$x(t) = c_1 e^{-b/(2a)t} + c_2 (k_1 t + k_2)e^{-b/(2a)t}$$

See that this can be simplified since c_1, c_2, k_1, k_2 are all unknown constants

$$x(t) = (c_1 + c_2 k_2)e^{-b/(2a)t} + c_2 k_1 t e^{-b/(2a)t}$$

We have

$$x(t) = C_1 e^{-b/(2a)t} + C_2 t e^{-b/(2a)t}$$

See that since we are dealing with a homogeneous equation, $te^{-b/(2a)t}$ by itself satisfies the DE (because of superposition).

A.2.7 Damped harmonic oscillators

Spring-mass-dashpot

Recall the model spring-mass-dashpot system with the constant coefficient linear DE:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$$

where m is the mass, b the damping constant, k the spring constant, and $x(t)$ the displacement of the mass from its equilibrium position.

Assuming zero external force ($F_{\text{ext}} = 0$), we have the homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

The algebra doesn't require any restrictions on m, b, k, except $m \neq 0$ (for the equation to be second order in the first place). But in this physical model we require $m > 0, b \geq 0$, and $k > 0$.

(next page)

Damped harmonic oscillator

The *undamped* ($b = 0$) system has the equation

$$m\ddot{x} + kx = 0$$

We have its solution as

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \cos(\omega t - \phi)$$

Here $\omega = \sqrt{k/m}$. The solution is always a sinusoid, thus we call this a *simple harmonic oscillator*:

When we add damping ($b > 0$) we then call the system a *damped harmonic oscillator*.

This emphasises an important fact about uding DEs to model physical systems: Any system modeled by the same equation will respond just like the spring-mass-dashpot (regardless of what m, d, k, x represent). That is, all damped harmonic oscillators exhibit similar behaviour.

A.2.8   Under, Over and Critical damping

Response to damping

As we saw, the unforced damped harmonic oscillator has equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

with $m > 0$, $b \geq 0$ and $k > 0$. It has characteristic equation

$$ms^2 + bs + k = 0$$

with characteristic roots

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

There are three cases depending on the sign of the expression under the square root:

1. $b^2 < 4mk$—*Underdamping*

2. $b^2 > 4mk$—*Overdamping*

3. $b^2 = 4mk$—*Critical damping*

First case: Underdamping

If $b^2 < 4mk$ the square root is negative and the characteristic roots are complex. See that the roots are given by

$$-\frac{b}{2m} \pm i\omega_d, \quad \text{where } \omega_d = \frac{\sqrt{|b^2 - 4mk|}}{2m}$$

With that we have the complex exponential solutions

$$e^{(-b/(2m)+i\omega_d)t}, \quad e^{(-b/(2m)-i\omega_d)t}$$

With the basic real solutions

$$e^{-bt/(2m)}\cos(\omega_d t), \quad e^{-bt/(2m)}\sin(\omega_d t)$$

The general real solution is found by taking linear combinations of two basic solutions:

$$x(t) = c_1 e^{-bt/(2m)}\cos(\omega_d t) + c_2 e^{-bt/(2m)}\sin(\omega_d t)$$

This can also be written as

$$x(t) = e^{-bt/(2m)}(c_1\cos(\omega_d t) + c_2\sin(\omega_d t)) = Ae^{-bt/(2m)}\cos(\omega_d t - \phi)$$
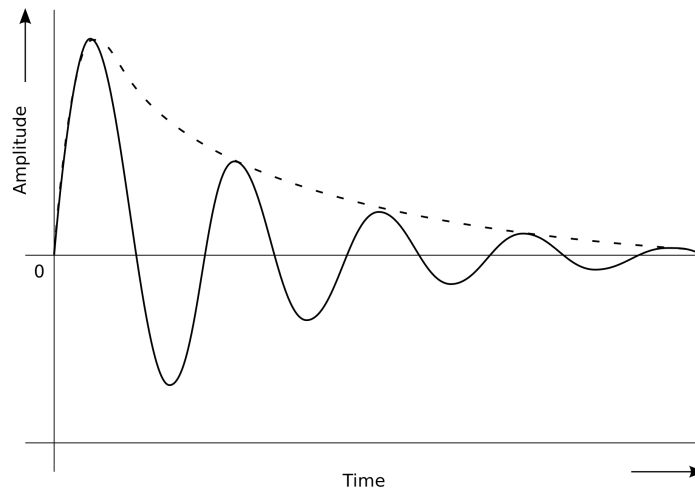
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Underdamping—Intuition

We had the behaviour of an underdamped system as

$$Ae^{-bt/(2m)}\cos(\omega_d t - \phi)$$

Intuitively, when $b = 0$ the response is a sinusoid. In terms of the spring-mass-dashpot setup this corresponds to zero damping/no friction. When the damping constant b is small (underdamped), we expect the system to still oscillate, but with decreasing amplitude as its energy is lost to friction.

This can be seen from the response of the underdamped system. The factor $\cos(\omega_d t - \phi)$ represents the oscillation while the exponential factor $e^{-bt/(2m)}$ has a negative exponent—thus representing a decaying amplitude. As $t \to \infty$, $x(t) \to 0$:

We call ω_d the *damped angular/circular frequency* of the system, or the *psuedo-frequency* of $x(t)$ ('psuedo' because $x(t)$ is not periodic and only periodic functions have a frequency).

(next page)

Underdamping—Example

Consider the system

$$\ddot{x} + \dot{x} + 3x = 0$$

(See that $b^2 < 4mk$) We have the characteristic equation $s^2 + s + 3 = 0$ and the characteristic roots $-1/2 \pm \sqrt{11}/2$. We have the basic real solutions as

$$e^{-t/2}\cos(\sqrt{11}t/2), \quad e^{-t/2}\sin(\sqrt{11}t/2)$$

and the general solution

$$x(t) = e^{-t/2}(c_1\cos(\sqrt{11}t/2) + c_2\sin(\sqrt{11}t/2)) = Ae^{-t/2}\cos(\sqrt{11}t/2 - \phi)$$

The damped angular frequency, ω_d, in this case is $\sqrt{11}/2$. Say we have the initial conditions $x(0) = 1, \dot{x}(0) = 0$, this gives us constants $c_1 = 1, c_2 = 1/\sqrt{11}$. So

$$x(t) = e^{-t/2}\left(\cos(\sqrt{11}t/2) + \frac{1}{\sqrt{11}}\sin(\sqrt{11}t/2)\right)$$

$$= \frac{\sqrt{12}}{\sqrt{11}}e^{-t/2}\cos(\sqrt{11}t/2 - \phi)$$

Where $\phi = \tan^{-1}(1/\sqrt{11})$. Plotted:

(next page)

Overdamping

Overdamping corresponds to the case where $b^2 > 4mk$ (distinct real roots). In this case *both characteristic roots are negative*—see that it is always the case that $b^2 - 4mk < b^2$ and therefore that $-b + \sqrt{b^2 - 4mk} < 0$, while the other root is clearly negative. Thus we have the real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, r_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

(both r are *real and negative* and are *different*) We have the general solution

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Intuitively, this corresponds to the frictional force being so great that the system can't oscillate—the system just goes asymptotically to the equilibrium $x = 0$:

Example

For instance, consider the system $\ddot{x} + 4\dot{x} + 3x = 0$ with inital conditions $x(0) = 1, \dot{x}(0) = 0$. This has characteristic equation $s^2 + 4s + 3 = 0$ and characteristic roots $-1, -3$. Its general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$

Because the roots are real and different, this system is classified as overdamped. To satisfy the initial conditions $c_1 = 3/2, c_2 = -1/2$. So

$$x(t) = 3/2 e^{-t} - 1/2 e^{-3t}$$

Because e^{-t} goes to 0 *more slowly* than $e^{-3t/2}$ it controls the rate at which x goes to 0—the term that goes to zero slowest controls the rate.

(next page)

Critical damping

Critical damping corresponds to the case where $b^2 = 4mk$ (repeated real roots) —the characteristic polynomial has the roots $-b/2m, -b/2m$. With repeated roots we have the solutions

$$e^{-bt/(2m)}, \quad te^{-bt/(2m)}$$

and the general solution

$$e^{-bt/(2m)}(c_1 + c_2 t)$$

As in the overdamped case, this does not oscillate. It is worth noting that for a fixed m and k, choosing b to be the critically damping value gives the fastest return of the system to equilibrium:
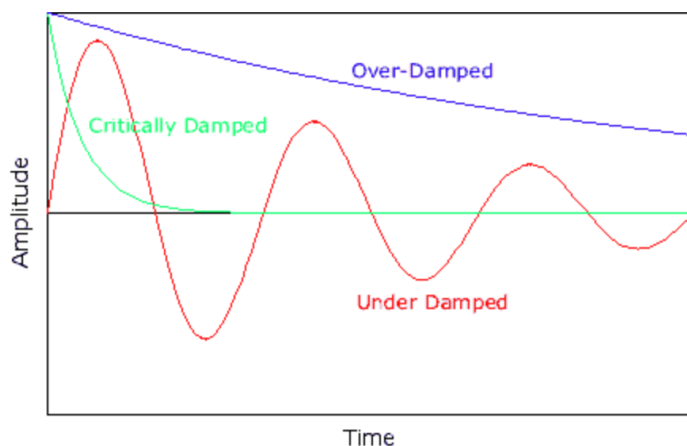
Intuition for rate of decay

This can be seen from the roots. Considering fixed m, k and only varying the damping constant b, in the overdamped case notice that one characteristic root is less negative than that of the critically damped case; since the less negative root controls the decay rate this leads to the rate of decay always being smaller. See that $f = b$ has a higher derivative than $g = \sqrt{b^2 - 4mk}$:

$$\frac{df}{db} = 1 \quad \frac{dg}{db} = \frac{b}{\sqrt{b^2 - 4mk}} > \frac{b}{\sqrt{b}} = 1$$

For $m, k > 0$. So $-b + \sqrt{b^2 - 4mk}$ only becomes less negative (slower rate of decay).

For the underdamped case to exist we must have $b < 4mk$. See that this b is always smaller than in the critically damped case, which corresponds to $b = 4mk$.

A.2.9 Superposition (Second order ODEs)

The Principle of Superposition for Second Order Differential Equations; if

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

is a second order linear differential equation and $y = y_1(t)$ and $y = y_2(t)$ are both solutions to this differential equation, then for C and D as constants,

$$y = Cy_1(t) + Dy_2(t) \quad \text{is also a solution}$$

Essentially, any linear combination of solutions is also a solution.

Proof: Consider $y = y_1$ and $y = y_2$ are solutions to the second order linear differential equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$. Then we have that:

$$\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 = 0 \quad \text{and} \quad \frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2 = 0$$

If C and D are constants, plugging in $y = Cy_1(t) + Dy_2(t)$:

$$\frac{d^2}{dt^2}(Cy_1(t) + Dy_2(t)) + p(t)\frac{d}{dt}(Cy_1(t) + Dy_2(t)) + q(t)(Cy_1(t) + Dy_2(t))$$

$$= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2$$

$$= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=0} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=0}$$

$$= 0$$

Therefore, $y = Cy_1(t) + Dy_2(t)$ is also a solution. Note that the superposition principle **does not** work for nonlinear differential equations.
(next page)

In context of inhomogenous differential equations

In addition, if y_1 is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_1(t)$$

and y_2 is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_2(t)$$

then for constants C and D, $Cy_1 + Dy_2$ is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = Cf_1(t) + Df_2(t)$$

Proof: Plugging in $y = Cy_1 + Dy_2$:

$$\frac{d^2}{dt^2}(Cy_1 + Dy_2) + p(t)\frac{d}{dt}(Cy_1 + Dy_2) + q(t)(Cy_1 + Dy_2)$$

$$= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2$$

$$= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=f_1(t)} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=f_2(t)}$$

$$= Cf_1(t) + Df_2(t)$$

Superposition is therefore *not* limited to homogenous equations.

A.2.10 General solution for inhomogenous linear ODEs

Therefore, to get the general solution $y(t)$ to an inhomogenous linear ODE:

$$\text{inhomogenous:} \quad \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t)$$

1. Find the general solution y_h to the associated **homogenous** equation:

$$\text{homogenous:} \quad \frac{d^2y_h}{dt^2} + p(t)\frac{dy_h}{dt} + q(t)y_h = 0$$

2. Find (in some way) any **one particular solution y_p** to the **inhomogenous** ODE.

3. Add y_p to y_h to get the general solution to the inhomogenous ODE:

$$\underbrace{y}_{\text{general inhomogenous solution}} = \underbrace{y_p}_{\text{any particular solution}} + \underbrace{y_h}_{\text{general homogenous solution}}$$

Note that the superposition principle **does not** work for nonlinear differential equations.

A.2.11 Superposition of homogeneous equations only requires linearity

Consider the illustrative example. Given the ODE

$$t^2 y'' + ty' - 4y = 0$$

One can check by substitution that $y_1(t) = t^2$ and $y_2(t) = 1/t^2$ are both solutions. Thus

$$y(t) = c_1 t^2 + c_2/t^2$$

Is a solution for any c_1, c_2. Point here is that *we didn't need the differential equation to have constant coefficients*: linearity and homogeneity is sufficient.

A.2.12 Existence and uniqueness

Solving a first-order linear ODE leads to a 1-parameter family of solutions (a general solution). To derive a specific solution, we need an initial condition, such as $y(0)$. One may wonder if there are other solutions. Here is a general result which says that there aren't and confirms that our methods find all solutions:

Existence and uniqueness theorem for a linear ODE:
Let $p(t)$ and $q(t)$ be continuous functions on an open interval I. Let $a \in I$, and let b be a given number. Then there **exists** a **unique** solution defined on the entire interval I to the first order linear ODE

$$\dot{y} + p(t)y = q(t)$$

satisfying the initial condition

$$y(a) = b$$

Existence means there is **at least one** solution.
Uniqueness means that there is **only one** solution.

A.2.13 Exponential input (2nd order ODEs) —exponential response

Context

Consider a case where the driving function (the external force in a spring-mass-dashpot system) is an exponential Be^{at}:

$$mx'' + bx' + kx = Be^{at}$$

where B, a are constants. We obtain a *particular* solution by considering a solution of the form Ae^{at} (same as the first order case). Substituting gives us

$$mx'' + bx' + kx = ma^2 Ae^{at} + baAe^{at} + kAe^{at}$$
$$= (ma^2 + ba + k)Ae^{at}$$

Thus we get (by comparing coefficients)

$$A = \frac{B}{ma^2 + ba + k}$$

See that the denominator is just the characteristic polymonial, which we will denote as $p(a)$ (this is not suprising, since the characteristic polynomial is derived using the same principles).

This concept isn't new, the same idea was used in the context of exponential/sinusoidal input to first order ODEs.

Exponential response formula (ERF)

Considering the second order equation

$$mx'' + bx' + kx = Be^{at}$$

and letting $p(r) = mr^2 + br + k$ be its characteristic polynomial, then

$$x(t) = \frac{B}{p(a)} e^{at}$$

is a *particular solution*, as long as $p(a) \neq 0$.
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Example

Consider finding the *general* solution to

$$x'' + 8x' + 7x = 9e^{2t}$$

considering solutions of the form $x(t) = Ae^{2t}$, substituting gives us

$$x'' + 8x' + 7x = Ae^{2t} \underbrace{(4 + 16 + 7)}_{p(2)} = 27Ae^{2t}$$

Comparing coefficients gives us

$$A = \frac{\overbrace{9}^{B}}{\underbrace{27}_{p(2)}}, \quad x_p = \frac{1}{3}e^{2t}$$

This is our *particular* solution (see that the ERF is just a shortcut of this reasoning). To obtain a general solution we need to solve the homogeneous equation:

$$x'' + 8x' + 7x = 0$$

Where we get the homogeneous solution

$$x_h = c_1 e^{-7t} + c_2 e^{-t}$$

Thus the general solution to the original equation is

$$x = x_h + x_p = c_1 e^{-7t} + c_2 e^{-t} + \frac{1}{3}e^{2t}$$

A.2.14 Sinusoidal input—complex replacement

General case, Gain

The ERF can be applied using complex replacement to solve equations with sinusoidal driving. Consider the general case:

$$mx'' + bx' + kx = B\cos(\omega t)$$

can be *complexified* as

$$mz'' + bz' + kz = Be^{i\omega t}$$

where $z = x + iy$, y defined as

$$my'' + by' + ky = B\sin(\omega t)$$

Now we solve for z_p using the ERF:

$$z_p = \frac{B}{p(i\omega)}e^{i\omega t}$$

We get x_p by taking Re(z):

$$x_p = \frac{B}{|p(i\omega)|}\cos(\omega t - \phi)$$

where $|p(i\omega)|$ refers to the magnitude of $p(i\omega)$, and $\phi = \text{Arg}(p(i\omega))$.

See that the output looks similar to the input—apart from the phase lag, the output is a multiple of the input by a factor of $1/|p(i\omega)|$. This factor is called the *gain* of the system:

$$\text{output amplitude} = \text{gain} \times \text{input amplitude}$$

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Example

Say we want the general solution to

$$x'' + 8x' + 7x = 9\cos(2t)$$

using complex replacement we instead solve

$$z'' + 8z' + 7z = 9e^{2it}$$

Now we can apply the ERF:

$$z_p(t) = \frac{9}{p(2i)}e^{2it}$$

$$= \frac{9}{(2i)^2 + 16i + 7}e^{2it}$$

$$= \frac{9}{3 + 16i}e^{2it}$$

Simplify by expressing $p(2i)$ in polar form:

$$\frac{9}{3 + 16i}e^{2it} = 9\left(\frac{1}{\sqrt{265}}e^{-i\phi}\right)e^{2it}$$

$$= \frac{9}{\sqrt{265}}e^{i(2t-\phi)}$$

where $\phi = \mathrm{Arg}(p(2i)) = \tan^{-1}(16/3)$. We can then obtain x_p as

$$x_p(t) = \mathrm{Re}(z_p(t)) = \frac{9}{\sqrt{265}}\cos(2t - \phi)$$

To get the general solution we add the homogeneous solution:

$$x_h(t) = c_1 e^{-7t} + c_2 e^{-t}$$

To obtain

$$x(t) = x_p + x_h = \frac{9}{\sqrt{265}}\cos(2t - \phi) + c_1 e^{-7t} + c_2 e^{-t}$$

In this case the gain is

$$1/|p(2i)| = 1/\sqrt{265}$$

A.2.15 Simple harmonic oscillator—
intuition for resonance

Natural frequency

We consider the undamped harmonic oscillator ($b = 0$)

$$mx'' + kx = F_{\text{ext}}(t)$$

we define the *natural frequency* ω_n as

$$\omega_n = \sqrt{k/m}$$

this comes from considering the system with no driving force $F_{\text{ext}}(t) = 0$, in which case the characteristic equation $p(r) = mr^2 + k$ has the roots $\pm i\sqrt{k/m}$. This leads to the general solution

$$x_h = c_1 \cos((\sqrt{k/m})t) + c_2 \sin((\sqrt{k/m})t)$$

With the natural frequency defined the unforced solution is

$$x_h = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t)$$

and the original equation looks like

$$m(x'' + \omega_n^2 x) = F_{\text{ext}}(t)$$

Intuition for resonance

Now consider adding sinusoidal input $F_{\text{ext}}(t) = B\cos(\omega t)$:

$$m(x'' + \omega_n^2 x) = B\cos(\omega t)$$

Using complex replacement, we seek to obtain the z_p in

$$m(x'' + \omega_n^2 x) = Be^{i\omega t}$$

Applying the ERF with $a = i\omega$ we get

$$z_p = \frac{B}{p(i\omega)}e^{i\omega t} = \frac{B}{m(\omega_n^2 - \omega^2)}e^{i\omega t}$$

(Notice the lack of a damping makes the denominator non-complex.) Taking the real part to obtain x_p:

$$x_p = \text{Re}(z_p) = \frac{B}{m(\omega_n^2 - \omega^2)}\cos(\omega t)$$

With the homogeneous solution, our general solution is

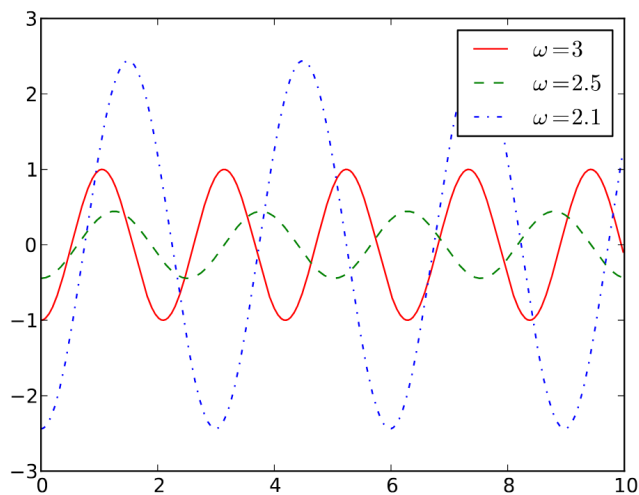$$x_h = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t) + \frac{B}{m(\omega_n^2 - \omega^2)}\cos(\omega t)$$

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Intuition for resonance (cont.)
We had the particular solution

$$\frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t)$$

See that the gain is $1/|p(a)| = 1/m(\omega_n^2 - \omega^2)$—the closer ω is to ω_n. Setting $m = 1, B = 1$ and $\omega_n = 2$ while varying ω (the frequency of the driving):

See that the solution breaks down when $\omega = \omega_n$. This is called **pure resonance**; notice it corresponds to the case $p(a) = 0$.
(next page)

The workaround solution—resonant response formula

When $p(a) = 0$ the initial solution we found breaks down. It can be checked that the following is a solution *to the complexified equation*:

$$z_p(t) = \frac{B}{p'(a)} t e^{at}$$

where taking the real part gives a valid x_p (which can also be checked):

$$x_p(t) = \frac{B}{2m\omega} t \sin(\omega t)$$

See the extra factor of t before the sine term—the amplitude of the response in the case of pure resonance grows with time.

The following is a counterpart to the ERF in the pure resonance case, when $p(a) = 0$. It is called the *Resonant Response Formula (RRF)*; essentially, in the second order equation

$$mx'' + bx' + kx = Be^{at}$$

with characteristic polynomial p, if $p(a) = 0$ and $p'(a) \neq 0$, then

$$x(t) = \frac{B}{p'(a)} t e^{at}$$

is a *particular* solution.

A.2.16   Generalised ERF

A.2.17 Exponential response formula

The exponential response formula gives us a quick method for finding the particular solution to any linear, constant coefficient, differential equations whose input can be expressed in terms of an exponential function.

The Exponential Response Formula(ERF):

A.3 Fourier Series

A.3.1 Fourier Series

If the input function $f(t)$ is periodic (of period 2π) ,we can express the function (where it is continuous) as an infinite sum of sines and cosines. This series representation is called a Fourier Series:

$$f(t) = c_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], \qquad c_0, a_n, b_n \text{ real constants}$$