Appendix 2

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Contents

Appendix A

Differential Equations

A.1 Fundamentals

A.1.1 Introduction to Ordinary Differential Equations (ODEs)

Here we introduce intuition for Ordinary Differential Equations (ODEs) and introductory solving methods.

The simplest type of differential equation looks like:

$$\frac{dy}{dx} = f(x)$$

which can be solved by the antiderivative $y = \int f(x)\,dx$.

Intuition
Now we consider a more interesting example:

$$\frac{dy}{dx} + xy = 0$$

This equation can be solved by *separation of variables*:

$$\begin{aligned} \frac{dy}{dx} + xy &= 0 \\ \frac{dy}{dx} &= -xy \\ \frac{dy}{y} &= -x\,dx \end{aligned}$$

(next page)

Since the problem is now set up in terms of differentials rather than ratios of differentials, we can integrate both sides.

$$\int \frac{dy}{y} = -\int x\,dx$$
$$\ln y + c_1 = -\frac{x^2}{2} + c_2 \quad \text{(assume } y > 0)$$

We can combine the constants and simplify:

$$\ln y = -\frac{x^2}{2} + c$$
$$e^{\ln y} = e^{-x^2/2+c}$$
$$y = e^c e^{-x^2/2}$$
$$y = Ae^{-x^2/2}, \quad \text{(where } A = e^c)$$

(The more apt $\ln|y|$ simplifies to $\pm Ae^{-x^2/2}$, which doesn't matter since A is some unspecific constant)

It turns out that our solution,

$$y = Ae^{-x^2/2}, \quad \text{(where } A = e^c)$$

Works for any constant multiple A. We can check this solution:

$$y = ae^{-x^2/2}$$
$$\frac{dy}{dx} = \frac{d}{dx}ae^{-x^2/2}$$
$$= a \cdot (-x)e^{-x^2/2}$$
$$= -x \cdot ae^{-x^2/2}$$
$$\frac{dy}{dx} = -xy$$

A is determined by an initial condition; for instance if $y(0) = 1$, $A = 1$.

A.1.2 Separation of Variables

Here we describe a rudimentary method for solving some differential equations—Separation of Variables.

In general, this method applies to differential equations of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Where we then *separate* the variables and integrate:

$$\begin{aligned}
\frac{dy}{dx} &= f(x)g(y) \\
\frac{dy}{g(y)} &= f(x)\,dx \\
h(y)\,dy &= f(x)\,dx \quad \text{where } h(y) = \frac{1}{g(y)} \\
\int h(y)\,dy &= \int f(x)\,dx
\end{aligned}$$

Antidifferentiating both sides:

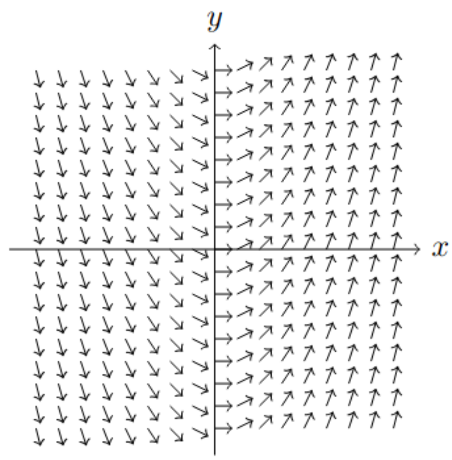
$$H(y) = \int h(y)\,dy; \quad F(x) = \int f(x)\,dx$$

we now have

$$\begin{aligned}
H(y) + c_1 &= F(x) + c_2 \\
H(y) &= F(x) + c
\end{aligned}$$

A.1.3 Direction fields, Isoclines, and Integral curves

Direction fields
Given an equation $y' = f(x, y)$, we can construct a *direction field*; imagine through each point (x, y), we draw a line segment whose slope is $f(x, y)$—consider $y'(x) = 2x$:



(note that in this case $f(x, y)$ does not depend on y (because of the equation)—it is invariant under vertical translation)

Plotting direction fields—Isoclines
In practice, computers are used to plot direction fields following the procedure:

1. Pick point (x, y)
2. Compute $y' = f(x, y)$
3. Plot line segment of slope at that point

Notice how a new slope has to be computed for each specified point; when plotting direction fields by hand, it is much more practical to utilise *isoclines*, which are, given the equation $y' = f(x, y)$, a one-parameter family of curves given by the equations

$$f(x, y) = m, \quad m \text{ constant}$$

Along a given isocline, all line segments have the same slope m.
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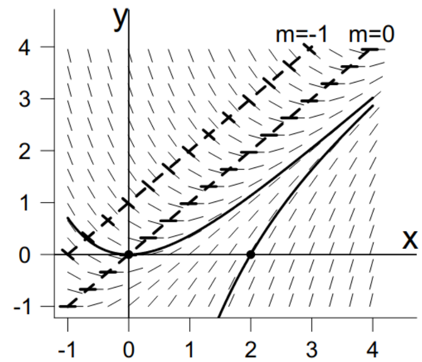
Example
Consider plotting the direction field for the equation $y' = x - y$; the isoclines are correspondingly the lines $x - y = m$ (shown in dashed lines):



The $m = 0$ isocline marks the points where the slope of the solution is 0; it is therefore of special interest and is called the *nullcline*.

Integral curves
As also shown in the figure above, once the direction field has been sketched, curves which are *at each point tangent to the line segment at that point* can be drawn; such curves are called *integral curves* or *solution curves* for the direction field. Their significance (this should be obvious) is that

The integral curves are the graphs of the solutions to $y' = f(x, y)$

Two integral curves have been drawn above (in solid lines).

Intersection Principle
Intuitively, see that at any point in the direction field it can only have one direction; therefore it is fairly obvious that integral curves cannot cross at an angle.

Consider the existence and uniqueness theorem for ODEs:

> *For any* (a, b) *in the region where* f *is defined,* $y' = f(x, y)$ *has exactly one solution such that* $y(a) = b$.

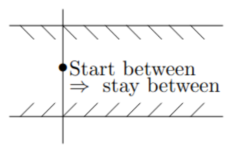
by the existence part of the theorem, there is an integral curve through any point where $f(x, y)$ is defined. Now supposing two integral curves through the same point, by the uniqueness part of the theorem they must agree.

As a result, *integral curves cannot intersect*; every point lies on exactly one integral curve.

A.1.4 Long term Behaviour: Fences, Funnels, and Separatrices

Fences
A *lower fence* for the equation $y' = f(x, y)$ is a curve that 'blocks' an integral curve from crossing from *above*; intuitively it is the curve whose direction field elements along the curve point up from it. Technically it can be described as a curve $y = L(x)$ such that $L'(x) < f(x, L(x))$ (the slope of the curve is always less than the slope of the direction field at that point).

Likewise an *upper fence* is a curve that 'blocks' integral curves from crossing from *above*. Illustrated:



(The upper curve is the upper fence and the lower curve is the lower fence). Solutions will be 'squeezed' between upper and lower fences.

Note that

- Note that fences aren't necessarily defined for all x; they could be defined only on an interval like $x \geq c$ for some constant c.
- Since integral curves can't cross an integral curve itself it is both an upper and lower fence.

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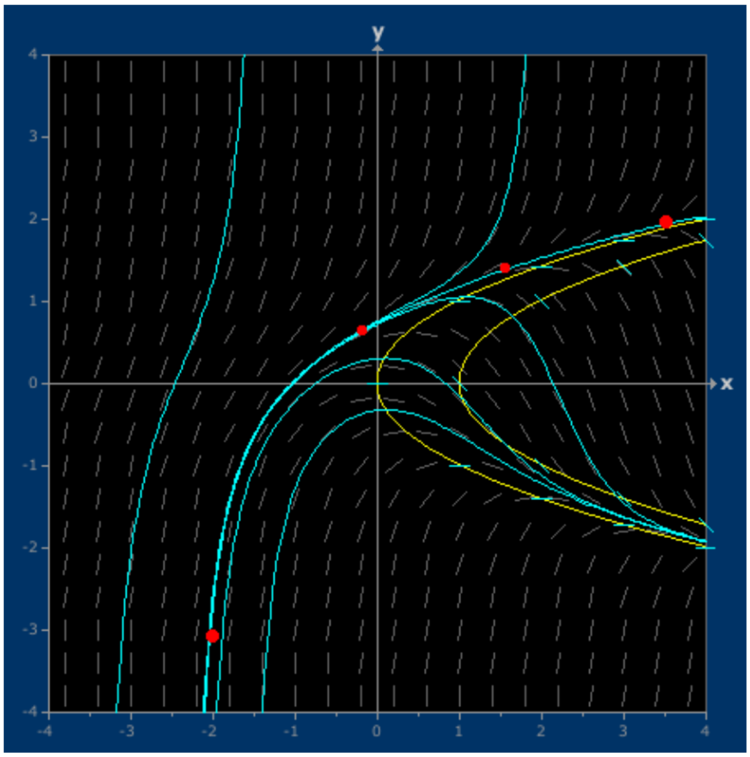
Example
Consider the direction field for the equation

$$y' = y^2 - x$$

The isoclines for $m = 0$ and $m = -1$ are plotted in yellow, with integral curves in blue:



Notice that the bottom hald of the isocline $m = 0$ is a lower fence and for x large enough the bottom half of the isocline $m = -1$ is an upper fence. (notice that the $m = -1$ isocline becomes an upper fence only for x large enough)

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Funnels
One use of fences is to construct funnels. A *funnel* for the equation $y' = f(x, y)$ consists of a pair of fences; one lower fence $L(x)$ and one upper fence $U(x)$ with the properties

1. For x large the lower fence is below the upper fence; $L(x) < U(x)$
2. The two fences come together asymptotically; $U(x) - L(x)$ is small for large x

For instance, in the above example the bottom parts of the two isoclines $m = 0$ and $m = -1$ act as a funnel once x is large enough. Given the equations of each isocline we have highly accurate estimates for solutions between them as

$$\underbrace{-\sqrt{x}}_{m=0} < y(x) < \underbrace{-\sqrt{x-1}}_{m=-1}$$

which is valid for large x.

Note that not all pairs of upper/lower fences form a funnel—they have to come together asymptotically as x gets large.

Separatrices
A *separatrix* is an integral curve such that the integral curves above it behave entirely differently from integral curves below it as $x \to \infty$.

A.1.5 Runge-Kutta 2 (Numerical methods)

General approach and Euler's method
Euler's method (for numerical estimation) follows a more general procedure for stepping from (x_n, y_n) to (x_{n+1}, y_{n+1}):

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + m_n h$$

Where h is the stepsize in the x direction and m is the slope of the line we step along. In Euler's method h is fixed ahead of time and $m_n = f(x_n, y_n)$.

Runge-Kutta 2
Naturally Euler's method is a fairly flawed method of numerical estimation. Other methods use other (and better) ways of choosing h and m. Here I describe the *Runge-Kutta 2* (RK2) method, which is a *fixed stepsize* method; meaning h is fixed and the added complexity comes from finding m.

Given an initial value problem $y' = f(x, y), y(x_0) = x_0$ and a step size h, one step of the RK2 method is as follows:

1. Compute the slope k_1 at (x_0, y_0): $k_1 = f(x_0, y_0)$
2. 'Take' an Euler step from (x_0, y_0) to $(a, b) : a = x_0 + h$, $b = y_0 + k_1 h$
3. Compute the slope k_2 at $(a, b) : k_2 = f(a, b)$
4. Average k_1 and k_2 to get m: $m = (k_1 + k_2)/2$
5. Now we use this averaged slope to take a step from (x_n, y_n) to (x_{n+1}, y_{n+1}):

$$x_1 = x_0 + h, \quad y_1 = y_0 + mh; \quad m = \frac{(k_1 + k_2)}{2}$$

Other methods such as RK4 or *variable stepsize methods* may (probably) work better. Though one might want to consider computational efficiency at the expense of accuracy.

A.1.6 First order Linear Differential Equations

Definition
The general *First order linear ODE* in the unknown function $x = x(t)$ has the form

$$A(t)\frac{dx}{dt} + B(t)x(t) = C(t)$$

If $A(t) \neq 0$ we can simplify the equation by dividing by $A(t)$:

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

This is called the *standard form* for a first order linear ODE. Should the *coefficients* $A(t), B(t)$ be constants (not dependent on t) we say the equation is a *constant coefficient* DE.

If $C(t) = 0$:

$$A(t)\frac{dx}{dt} + B(t)x(t) = 0$$

The DE is called *homogeneous* (notice that conversion to standard form doesn't change this fact); otherwise the equation is *inhomogeneous*.

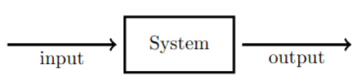
Signals and Systems—Terminology
Given a differential equation

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

Notice that the right-hand side does not depend on x. The left-hand side represents the *system* (think of it as defining the behaviour of a system); the right-hand side represents an outside influence on the system, which we can call the *input*.

In general, a signal is a function of t. The system *responds* to the input signal and yields the function $x(t)$, which we call the *output signal* or *system response*. (these terms should just be seen as convenient convention when describing an ODE)

Block diagrams can be used to visually represent systems:



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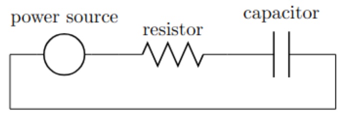
Example—RC circuits
Suppose we have an electrical circuit as shown



"Kirchhoff's Voltage Law" states that the total voltage change around the loop is 0, meaning

$$V(t) = V_R(t) + V_C(t)$$

The relationship between voltage drop and current are described as follows:

Resistor: $V_R(t) = RI(t)$ for a constant R, the "resistance"

Capacitor: $V_C'(t) = \dfrac{1}{C} I(t)$ for a constant C, the "capacitance"

the voltage drop from the capacitance can be seen from the equation defining capacitance

$$\begin{aligned} q &= CV \quad \text{(charge per unit voltage)} \\ I(t) = \frac{dq}{dt} &= \frac{d}{dt}(CV) \\ I(t) &= CV' \quad (C \text{ constant}) \\ V_C'(t) &= \frac{1}{C} I(t) \end{aligned}$$

The voltage drop across the capacitor is proportional to the *integral* of the current; it results from a buildup of charge on two plates of the capacitor.
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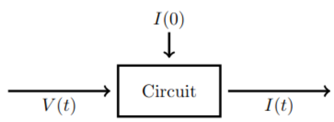
We can differentiate Kirchhoff's Voltage Law

$$\begin{aligned} V'(t) &= V_R'(t) + V_C'(t) \\ &= RI'(t) + \frac{1}{C} I(t) \end{aligned}$$

to obtain a first order linear differential equation

$$RI'(t) + \frac{1}{C} I(t) = V'(t)$$

In this circuit we consider the voltage $V(t)$ to be the input signal, and the circuit with resistance R and capacitance C to be the system. The current I is the output signal/system response:



$I(0)$ represents the initial condition.

A.1.7 Superposition (First order ODEs)

Considering the following the first order linear equation:

$$\dot{y} + p(t)y = q(t)$$

If a given input $q(t)$ has the output $y(t)$ we write

$$q \rightsquigarrow y$$

Here we show that if

$$q_1 \rightsquigarrow y_1 \text{ and } q_2 \rightsquigarrow y_2 \quad \text{then} \quad c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2$$

Proof
First see that (since differentiation does't change the constant coefficient)

$$\begin{aligned} \frac{dy}{dt} + py &= q \\ c\frac{dy}{dt} + cpy &= cq \\ = \frac{d(cy)}{dt} + p(cy) &= cq; \quad cq \rightsquigarrow cy \end{aligned}$$

Now see that

$$\begin{aligned} \frac{d(c_1y_1 + c_2y_2)}{dt} + p(c_1y_1 + c_2y_2) &= \underbrace{c_1\dot{y}_1 + pc_1y_1}_{=c_1q_1} + \underbrace{c_2\dot{y}_2 + pc_2y_2}_{=c_2q_2} \\ &= c_1q_1 + c_2q_2 \end{aligned}$$

Essentially, any linear combination of solutions is also a solution.

A.1.8 Solution by Integrating Factor (inhomogenous first order ODEs)

Here we prove the general solution to the inhomogeneous first order linear ODE

$$\dot{x} + p(t)x = q(t)$$

is

$$x(t) = \frac{1}{u(t)} \left(\int u(t)q(t)dt + C \right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

the function u is called an *integrating factor*.

Proof
We start with the product rule for differentiation:

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$$

Consider multiplying both sides of our inhomogenous first order ODE by some function $u(t)$:

$$u\dot{x} + upx = uq$$

We want to choose a function $u(t)$ such that we can apply the product rule to the sum on the left hand side of the equation. There may be many functions u that could work, but in this case we only need one. See that

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x \iff u\dot{x} + upx = u\dot{x} + \dot{u}x \iff \dot{u} = up$$

so now by separation of equations

$$\begin{aligned} \frac{du}{u} &= p(t)dt \\ \ln|u| &= \int p(t)dt \\ u &= e^{\int p\,dt} \end{aligned}$$

By using u to satisfy the product rule:

$$\begin{aligned} u\dot{x} + upx = \frac{d}{dt}(ux) &= uq \\ u(t)x(t) &= \int u(t)q(t)dt + c \\ x(t) &= \frac{1}{u(t)} \left(\int u(t)q(t)dt + c \right) \end{aligned}$$

which was what we wanted.
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Integrating factor and homogeneous equations
Given the homogeneous first order ODE

$$\dot{x} + p(t)x = 0$$

Solving by separation of variables gives

$$x_h(t) = Ae^{-\int p(t)dt}$$

Comparing this to the formula for the integrating factor

$$u(t) = e^{\int p(t)dt}$$

see that

$$x_h(t) = \frac{A}{u(t)}$$

A.1.9 General, Particular and Homogeneous solutions

Solving by method of Integrating factors allows us to come up with a solution for inhomogeneous first order linear ODEs

$$\dot{x} + p(t)x = q(t)$$

Which have the form

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt + C\right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

Notice that the presence of the constant C implies a family of solutions; by setting $C = 0$ we get a *particular solution* x_p, which is simply one specific solution—we could have chosen any other:

$$\begin{aligned} x_p &= \frac{1}{u(t)}\left(\int u(t)q(t)dt + 0\right) \quad \text{is a solution} \\ x_p &= \frac{1}{u(t)}\left(\int u(t)q(t)dt + 999\right) \quad \text{is also a solution} \end{aligned}$$

The method of integrating factors naturally leaves us with a constant. But say we were to find a solution by *inspection*—how would we know that the constant of integration exists in the form $\frac{C}{u(t)}$? (as is in this case)

General solution
See that since

$$x_h(t) = \frac{1}{u(t)}$$

We can write the solution by integrating factor as

$$\begin{aligned} x(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{C}{u(t)} \\ &= x_p + Cx_h \end{aligned}$$

One way to fully solve the inhomogeneous equation is by first solving the *homogeneous* equation, and then finding any *one* solution, a *particular solution*, to the inhomogeneous equation x_p. (We can use any method to find x_p since we the homogeneous solution handles the constant of integration):

$$\text{General solution} = \text{Particular solution} + \text{Homogeneous solution}$$

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Intuition
Given an inhomogeneous first order linear ODE and its associated homogeneous equation

$$\begin{aligned} \dot{x} + p(t)x &= q(t) \quad \text{(inhomogeneous)} \\ \dot{x} + p(t)x &= 0 \quad \text{(homogeneous)} \end{aligned}$$

Solving both equations by method of integrating factors gives

$$x_p(t) = \frac{1}{u(t)} \left(\int u(t)q(t)dt \right) + \frac{A}{u(t)}, \qquad x_h(t) = \frac{B}{u(t)}$$

(where A is any chosen constant, each constant giving a particular solution, and B the constant of integration) Now see that by adding the solutions together the constant for the inhomogeneous solution A gets absorbed into the homogeneous solution:

$$\begin{aligned} x_p(t) + x_h(t) &= \frac{1}{u(t)} \left(\int u(t)q(t)dt \right) + \frac{A+B}{u(t)} \\ &= \frac{1}{u(t)} \left(\int u(t)q(t)dt \right) + \frac{C}{u(t)} \end{aligned}$$

We can obtain the 'ambiguous part' of the general solution by simply solving the homogeneous equation; this means that when obtaining a particular solution we don't have to worry about the constant of integration.

Superposition
See that this also makes sense with respect to superposition of solutions, where since

$$\underbrace{q(t) \rightsquigarrow x_p(t)}_{\text{inhomogeneous}} \quad \text{and} \quad \underbrace{0 \rightsquigarrow x_h(t)}_{\text{homogeneous}}$$

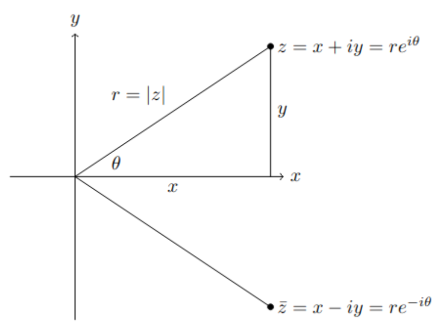
we can say

$$q(t) + 0 = q(t) \rightsquigarrow x_p(t) + x_h(t)$$

A.1.10 Polar form and Euler Identity

The Complex Plane, Polar Form

Complex numbers can be represented geometrically by points in a plane, where the number $a + ib$ is represented by the point (a, b); when points in a plane are thought of as representing complex numbers this way, the plane is known as a *Complex Plane*:



See that the magnitude of the coordinates of a complex number $x + iy$ can be represented by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

where r is the absolute value of the number:

$$r = |x + iy| = \sqrt{x^2 + y^2}$$

(its just the pythagorean theorem) thus the entire number can be written as

$$x + iy = r(\cos(\theta) + i\sin(\theta))$$

This is called the *Polar Form* of a non-zero complex number. We call θ the *angle* or *argument* of $x + iy$:

$$\theta = \arg(x + iy)$$

Notice that the angle can be increased by any integer multiple of 2π and will still represent the same thing. To simplify this one can specify the *principal value* of the angle:

$$0 \le \theta < 2\pi$$

this can be indicated by Arg(...); for instance

$$\text{Arg}(-1) = \pi, \quad \arg(-1) = \pm\pi, \pm3\pi, \pm5\pi$$

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Euler's Formula
Complex numbers have another *exponential* form called *Euler's formula*:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This should be regarded as a definition for the exponential of an imaginary power.

A good justification for Euler's formula can be found from its Taylor approximation:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos(\theta) + i\sin(\theta) \end{aligned}$$

Note that the argument above is not a proof; rather it just shows that Euler's formula is formally compatible with the series expansions for the exponential, sine, and cosine functions.

Polar form again
We can now write

$$x + iy = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

Polar representation in exponential form allows for much simpler multiplication of complex numbers. Since once can show that (using angle addition formulas)

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ &\quad + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1+\theta_2)} \end{aligned}$$

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Complex Exponential properties
We had

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

This property can be extrapolated to further justify Euler's formula—the complex exponential follows the same exponential addition rules as any typical exponential. See that we can now conclude:

Multiplication rule:

$$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$

also see that since

$$\frac{1}{r}e^{-i\theta} \cdot re^{i\theta} = 1$$

Reciprocal Rule:

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

DeMoivre's Formula
Since

$$(x+iy)^n = r^ne^{in\theta}$$

we can show *DeMoivre's formula:*

$$(\cos(\theta) + i\sin(\theta))^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Combining pure oscillations of the same frequency
We can also show that

$$a\cos(\lambda t) + b\sin(\lambda t) = A\cos(\lambda t - \phi)$$

where

$$A = \sqrt{a^2+b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

See that

$$\begin{aligned} a\cos(\lambda t) + b\sin(\lambda t) &= \mathrm{Re}((a-bi)(\cos(\lambda t + i\sin(\lambda t)) \\ &= \mathrm{Re}(Ae^{-i\phi} \cdot e^{i\lambda t}) \\ &= \mathrm{Re}(Ae^{i(\lambda t-\phi)}) \\ &= A\cos(\lambda t - \phi) \end{aligned}$$

A.1.11 More on Complex Exponentials

Notable properties
We know that (as proven)

$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$$

So see that

$$\mathrm{Re}(e^{a+ib}) = e^a\cos(b), \quad \mathrm{Im}(e^{a+ib}) = e^a\sin(b)$$

this can be extrapolated further to show

$$\begin{aligned} \cos(x) &= \mathrm{Re}(e^{ix}), & \sin(x) &= \mathrm{Im}(e^{ix} \\ \cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix}), & \sin(x) &= \frac{1}{2i}(e^{ix} - e^{-ix}) \end{aligned}$$

A.1.12 Superposition (Second order ODEs)

The Principle of Superposition for Second Order Differential Equations; if

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

is a second order linear differential equation and $y = y_1(t)$ and $y = y_2(t)$ are both solutions to this differential equation, then for C and D as constants,

$$y = Cy_1(t) + Dy_2(t) \quad \text{is also a solution}$$

Essentially, any linear combination of solutions is also a solution.

Proof: Consider $y = y_1$ and $y = y_2$ are solutions to the second order linear differential equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$. Then we have that:

$$\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 = 0 \quad \text{and} \quad \frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2 = 0$$

If C and D are constants, plugging in $y = Cy_1(t) + Dy_2(t)$:

$$\begin{aligned}
&\frac{d^2}{dt^2}(Cy_1(t) + Dy_2(t)) + p(t)\frac{d}{dt}(Cy_1(t) + Dy_2(t)) + q(t)(Cy_1(t) + Dy_2(t)) \\
&= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\
&= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=0} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=0} \\
&= 0
\end{aligned}$$

Therefore, $y = Cy_1(t) + Dy_2(t)$ is also a solution. Note that the superposition principle **does not** work for nonlinear differential equations.
(next page)

In context of inhomogenous differential equations
In addition, if y_1 is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_1(t)$$

and y_2 is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_2(t)$$

then for constants C and D, $Cy_1 + Dy_2$ is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = Cf_1(t) + Df_2(t)$$

Proof: Plugging in $y = Cy_1 + Dy_2$:

$$\begin{aligned}
&\frac{d^2}{dt^2}(Cy_1 + Dy_2) + p(t)\frac{d}{dt}(Cy_1 + Dy_2) + q(t)(Cy_1 + Dy_2) \\
&= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\
&= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=f_1(t)} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=f_2(t)} \\
&= Cf_1(t) + Df_2(t)
\end{aligned}$$

Superposition is therefore *not* limited to homogenous equations.

A.1.13 General solution for inhomogenous linear ODEs

Therefore, to get the general solution $y(t)$ to an inhomogenous linear ODE:

$$\text{inhomogenous:} \quad \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t)$$

1. Find the general solution y_h to the associated **homogenous** equation:

$$\text{homogenous:} \quad \frac{d^2y_h}{dt^2} + p(t)\frac{dy_h}{dt} + q(t)y_h = 0$$

2. Find (in some way) any **one particular solution** y_p to the **inhomogenous** ODE.
3. Add y_p to y_h to get the general solution to the inhomogenous ODE:

$$\underbrace{y}_{\text{general } \textbf{inhomogenous} \text{ solution}} = \underbrace{y_p}_{\textbf{any particular solution}} + \underbrace{y_h}_{\text{general } \textbf{homogenous} \text{ solution}}$$

Note that the superposition principle **does not** work for nonlinear differential equations.

A.1.14 Existence and uniqueness

Solving a first-order linear ODE leads to a 1-parameter family of solutions (a general solution). To derive a specific solution, we need an initial condition, such as $y(0)$. One may wonder if there are other solutions. Here is a general result which says that there aren't and confirms that our methods find all solutions:

Existence and uniqueness theorem for a linear ODE:
Let $p(t)$ and $q(t)$ be continuous functions on an open interval I. Let $a \in I$, and let b be a given number. Then there **exists** a **unique** solution defined on the entire interval I to the first order linear ODE

$$\dot{y} + p(t)y = q(t)$$

satisfying the initial condition

$$y(a) = b$$

Existence means there is **at least one** solution.
Uniqueness means that there is **only one** solution.

A.1.15 Exponential response formula

The exponential response formula gives us a quick method for finding the particular solution to any linear, constant coefficient, differential equations whose input can be expressed in terms of an exponential function.

The Exponential Response Formula(ERF):

A.2 Fourier Series

A.2.1 Fourier Series

If the input function $f(t)$ is periodic (of period 2π) ,we can express the function (where it is continuous) as an infinite sum of sines and cosines. This series representation is called a Fourier Series:

$$f(t) = c_0 + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], \qquad c_0, a_n, b_n \text{ real constants}$$