Appendix 2

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Started 24 May 2024

Contents

Appendix A

Differential Equations

A.1 First Order Differential Equations

A.1.1 Introduction to Ordinary Differential Equations (ODEs)

Here we introduce intuition for Ordinary Differential Equations (ODEs) and introductory solving methods.

The simplest type of differential equation looks like:

$$\frac{dy}{dx} = f(x)$$

which can be solved by the antiderivative $y = \int f(x)\, dx$.

Intuition
Now we consider a more interesting example:

$$\frac{dy}{dx} + xy = 0$$

This equation can be solved by *separation of variables*:

$$\begin{aligned}\frac{dy}{dx} + xy &= 0\\ \frac{dy}{dx} &= -xy\\ \frac{dy}{y} &= -x\, dx\end{aligned}$$

(next page)

Since the problem is now set up in terms of differentials rather than ratios of differentials, we can integrate both sides.

$$\begin{aligned} \int \frac{dy}{y} &= -\int x\,dx \\ \ln y + c_1 &= -\frac{x^2}{2} + c_2 \quad \text{(assume } y > 0\text{)} \end{aligned}$$

We can combine the constants and simplify:

$$\begin{aligned} \ln y &= -\frac{x^2}{2} + c \\ e^{\ln y} &= e^{-x^2/2+c} \\ y &= e^c e^{-x^2/2} \\ y &= Ae^{-x^2/2}, \quad \text{(where } A = e^c\text{)} \end{aligned}$$

(The more apt $\ln|y|$ simplifies to $\pm Ae^{-x^2/2}$, which doesn't matter since A is some unspecific constant)

It turns out that our solution,

$$y = Ae^{-x^2/2}, \quad \text{(where } A = e^c\text{)}$$

Works for any constant multiple A. We can check this solution:

$$\begin{aligned} y &= ae^{-x^2/2} \\ \frac{dy}{dx} &= \frac{d}{dx}ae^{-x^2/2} \\ &= a \cdot (-x)e^{-x^2/2} \\ &= -x \cdot ae^{-x^2/2} \\ \frac{dy}{dx} &= -xy \end{aligned}$$

A is determined by an initial condition; for instance if $y(0) = 1$, $A = 1$.

A.1.2 Separation of Variables

Here we describe a rudimentary method for solving some differential equations—Separation of Variables.

In general, this method applies to differential equations of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Where we then *separate* the variables and integrate:

$$\begin{aligned}
\frac{dy}{dx} &= f(x)g(y) \\
\frac{dy}{g(y)} &= f(x)\,dx \\
h(y)\,dy &= f(x)\,dx \quad \text{where } h(y) = \frac{1}{g(y)} \\
\int h(y)\,dy &= \int f(x)\,dx
\end{aligned}$$

Antidifferentiating both sides:

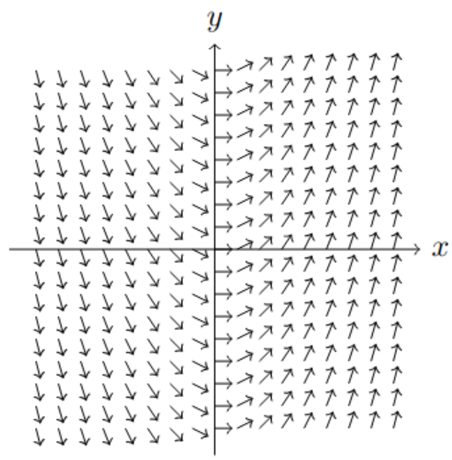
$$H(y) = \int h(y)\,dy; \quad F(x) = \int f(x)\,dx$$

we now have

$$\begin{aligned}
H(y) + c_1 &= F(x) + c_2 \\
H(y) &= F(x) + c
\end{aligned}$$

A.1.3 Direction fields, Isoclines, and Integral curves

Direction fields
Given an equation $y' = f(x, y)$, we can construct a *direction field*; imagine through each point (x, y), we draw a line segment whose slope is $f(x, y)$—consider $y'(x) = 2x$:



(note that in this case $f(x, y)$ does not depend on y (because of the equation)—it is invariant under vertical translation)

Plotting direction fields—Isoclines
In practice, computers are used to plot direction fields following the procedure:

1. Pick point (x, y)
2. Compute $y' = f(x, y)$
3. Plot line segment of slope at that point

Notice how a new slope has to be computed for each specified point; when plotting direction fields by hand, it is much more practical to utilise *isoclines*, which are, given the equation $y' = f(x, y)$, a one-parameter family of curves given by the equations

$$f(x, y) = m, \quad m \text{ constant}$$

Along a given isocline, all line segments have the same slope m.
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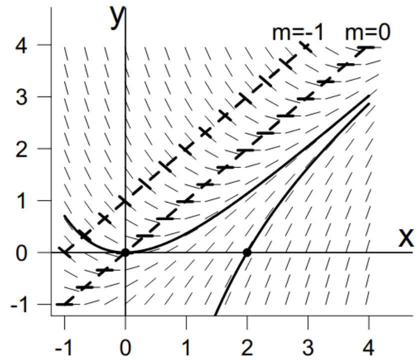
Example
Consider plotting the direction field for the equation $y' = x - y$; the isoclines are correspondingly the lines $x - y = m$ (shown in dashed lines):



The $m = 0$ isocline marks the points where the slope of the solution is 0; it is therefore of special interest and is called the *nullcline*.

Integral curves
As also shown in the figure above, once the direction field has been sketched, curves which are *at each point tangent to the line segment at that point* can be drawn; such curves are called *integral curves* or *solution curves* for the direction field. Their significance (this should be obvious) is that

The integral curves are the graphs of the solutions to $y' = f(x, y)$

Two integral curves have been drawn above (in solid lines).

Intersection Principle
Intuitively, see that at any point in the direction field it can only have one direction; therefore it is fairly obvious that integral curves cannot cross at an angle.

Consider the existence and uniqueness theorem for ODEs:

> *For any* (a, b) *in the region where* f *is defined,* $y' = f(x, y)$ *has exactly one solution such that* $y(a) = b$.

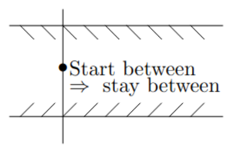
by the existence part of the theorem, there is an integral curve through any point where $f(x, y)$ is defined. Now supposing two integral curves through the same point, by the uniqueness part of the theorem they must agree.

As a result, *integral curves cannot intersect*; every point lies on exactly one integral curve.

A.1.4 Long term Behaviour: Fences, Funnels, and Separatrices

Fences
A *lower fence* for the equation $y' = f(x,y)$ is a curve that 'blocks' an integral curve from crossing from *above*; intuitively it is the curve whose direction field elements along the curve point up from it. Technically it can be described as a curve $y = L(x)$ such that $L'(x) < f(x, L(x))$ (the slope of the curve is always less than the slope of the direction field at that point).

Likewise an *upper fence* is a curve that 'blocks' integral curves from crossing from *above*. Illustrated:



(The upper curve is the upper fence and the lower curve is the lower fence). Solutions will be 'squeezed' between upper and lower fences.

Note that

- Note that fences aren't necessarily defined for all x; they could be defined only on an interval like $x \geq c$ for some constant c.
- Since integral curves can't cross an integral curve itself it is both an upper and lower fence.

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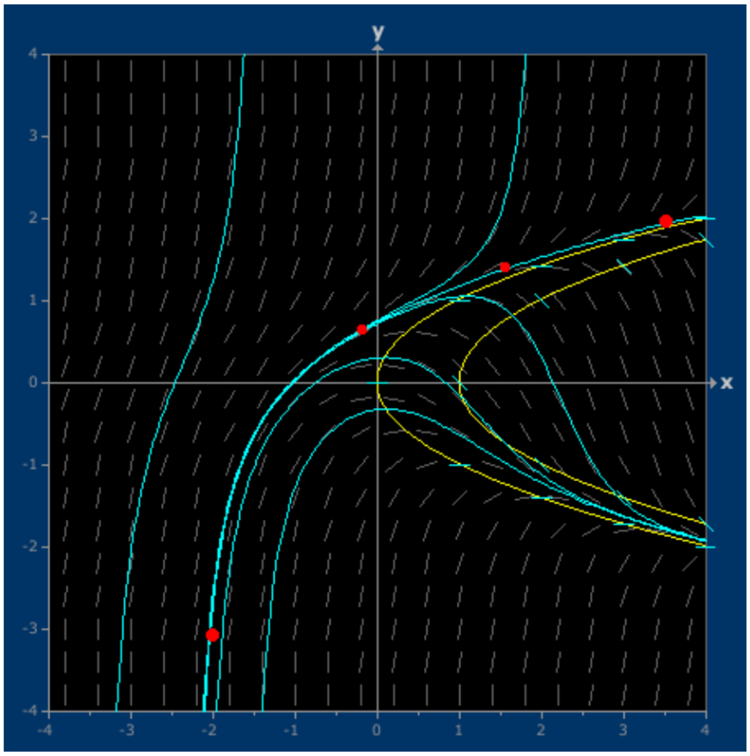
Example
Consider the direction field for the equation

$$y' = y^2 - x$$

The isoclines for $m = 0$ and $m = -1$ are plotted in yellow, with integral curves in blue:



Notice that the bottom hald of the isocline $m = 0$ is a lower fence and for x large enough the bottom half of the isocline $m = -1$ is an upper fence. (notice that the $m = -1$ isocline becomes an upper fence only for x large enough)

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Funnels
One use of fences is to construct funnels. A *funnel* for the equation $y' = f(x, y)$ consists of a pair of fences; one lower fence $L(x)$ and one upper fence $U(x)$ with the properties

1. For x large the lower fence is below the upper fence; $L(x) < U(x)$
2. The two fences come together asymptotically; $U(x) - L(x)$ is small for large x

For instance, in the above example the bottom parts of the two isoclines $m = 0$ and $m = -1$ act as a funnel once x is large enough. Given the equations of each isocline we have highly accurate estimates for solutions between them as

$$\underbrace{-\sqrt{x}}_{m=0} < y(x) < \underbrace{-\sqrt{x-1}}_{m=-1}$$

which is valid for large x.

Note that not all pairs of upper/lower fences form a funnel—they have to come together asymptotically as x gets large.

Separatrices
A *separatrix* is an integral curve such that the integral curves above it behave entirely differently from integral curves below it as $x \to \infty$.

A.1.5 Runge-Kutta 2 (Numerical methods)

General approach and Euler's method
Euler's method (for numerical estimation) follows a more general procedure for stepping from (x_n, y_n) to (x_{n+1}, y_{n+1}):

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + m_n h$$

Where h is the stepsize in the x direction and m is the slope of the line we step along. In Euler's method h is fixed ahead of time and $m_n = f(x_n, y_n)$.

Runge-Kutta 2
Naturally Euler's method is a fairly flawed method of numerical estimation. Other methods use other (and better) ways of choosing h and m. Here I describe the *Runge-Kutta 2* (RK2) method, which is a *fixed stepsize* method; meaning h is fixed and the added complexity comes from finding m.

Given an initial value problem $y' = f(x, y), y(x_0) = x_0$ and a step size h, one step of the RK2 method is as follows:

1. Compute the slope k_1 at (x_0, y_0): $k_1 = f(x_0, y_0)$
2. 'Take' an Euler step from (x_0, y_0) to $(a, b) : a = x_0 + h$, $b = y_0 + k_1 h$
3. Compute the slope k_2 at $(a, b) : k_2 = f(a, b)$
4. Average k_1 and k_2 to get m: $m = (k_1 + k_2)/2$
5. Now we use this averaged slope to take a step from (x_n, y_n) to (x_{n+1}, y_{n+1}):

$$x_1 = x_0 + h, \quad y_1 = y_0 + mh; \quad m = \frac{(k_1 + k_2)}{2}$$

Other methods such as RK4 or *variable stepsize methods* may (probably) work better. Though one might want to consider computational efficiency at the expense of accuracy.

A.1.6 First order Linear Differential Equations

Definition
The general *First order linear ODE* in the unknown function $x = x(t)$ has the form

$$A(t)\frac{dx}{dt} + B(t)x(t) = C(t)$$

If $A(t) \neq 0$ we can simplify the equation by dividing by $A(t)$:

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

This is called the *standard form* for a first order linear ODE. Should the *coefficients* $A(t), B(t)$ be constants (not dependent on t) we say the equation is a *constant coefficient* DE.

If $C(t) = 0$:

$$A(t)\frac{dx}{dt} + B(t)x(t) = 0$$

The DE is called *homogeneous* (notice that conversion to standard form doesn't change this fact); otherwise the equation is *inhomogeneous*.

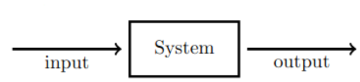
Signals and Systems—Terminology
Given a differential equation

$$\frac{dx}{dt} + p(t)x(t) = q(t)$$

Notice that the right-hand side does not depend on x. The left-hand side represents the *system* (think of it as defining the behaviour of a system); the right-hand side represents an outside influence on the system, which we can call the *input*.

In general, a signal is a function of t. The system *responds* to the input signal and yields the function $x(t)$, which we call the *output signal* or *system response*. (these terms should just be seen as convenient convention when describing an ODE)

Block diagrams can be used to visually represent systems:



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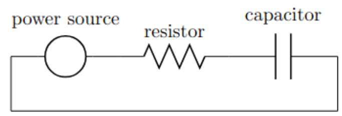
Example—RC circuits
Suppose we have an electrical circuit as shown



"Kirchhoff's Voltage Law" states that the total voltage change around the loop is 0, meaning

$$V(t) = V_R(t) + V_C(t)$$

The relationship between voltage drop and current are described as follows:

$$\text{Resistor: } V_R(t) = RI(t) \text{ for a constant } R, \text{ the "resistance"}$$
$$\text{Capacitor: } V_C'(t) = \frac{1}{C}I(t) \text{ for a constant } C, \text{ the "capacitance"}$$

the voltage drop from the capacitance can be seen from the equation defining capacitance

$$\begin{aligned} q &= CV \quad \text{(charge per unit voltage)} \\ I(t) = \frac{dq}{dt} &= \frac{d}{dt}(CV) \\ I(t) &= CV' \quad (C \text{ constant}) \\ V_C'(t) &= \frac{1}{C}I(t) \end{aligned}$$

The voltage drop across the capacitor is proportional to the *integral* of the current; it results from a buildup of charge on two plates of the capacitor.
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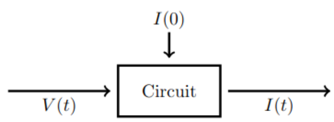
We can differentiate Kirchhoff's Voltage Law

$$\begin{aligned} V'(t) &= V_R'(t) + V_C'(t) \\ &= RI'(t) + \frac{1}{C} I(t) \end{aligned}$$

to obtain a first order linear differential equation

$$RI'(t) + \frac{1}{C} I(t) = V'(t)$$

In this circuit we consider the voltage $V(t)$ to be the input signal, and the circuit with resistance R and capacitance C to be the system. The current I is the output signal/system response:



$I(0)$ represents the initial condition.

A.1.7 Superposition (First order ODEs)

Considering the following the first order linear equation:

$$\dot{y} + p(t)y = q(t)$$

If a given input $q(t)$ has the output $y(t)$ we write

$$q \rightsquigarrow y$$

Here we show that if

$$q_1 \rightsquigarrow y_1 \text{ and } q_2 \rightsquigarrow y_2 \quad \text{then} \quad c_1 q_1 + c_2 q_2 \rightsquigarrow c_1 y_1 + c_2 y_2$$

Proof
First see that (since differentiation does't change the constant coefficient)

$$\begin{aligned} \frac{dy}{dt} + py = q \\ c\frac{dy}{dt} + cpy = cq \\ = \frac{d(cy)}{dt} + p(cy) = cq; \quad cq \rightsquigarrow cy \end{aligned}$$

Now see that

$$\begin{aligned} \frac{d(c_1 y_1 + c_2 y_2)}{dt} + p(c_1 y_1 + c_2 y_2) &= \underbrace{c_1 \dot{y}_1 + pc_1 y_1}_{=c_1 q_1} + \underbrace{c_2 \dot{y}_2 + pc_2 y_2}_{=c_2 q_2} \\ &= c_1 q_1 + c_2 q_2 \end{aligned}$$

Essentially, any linear combination of solutions is also a solution.

A.1.8 Solution by Integrating Factor (inhomogenous first order ODEs)

Here we prove the general solution to the inhomogeneous first order linear ODE

$$\dot{x} + p(t)x = q(t)$$

is

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt + C\right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

the function u is called an *integrating factor*.

Proof
We start with the product rule for differentiation:

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$$

Consider multiplying both sides of our inhomogenous first order ODE by some function $u(t)$:

$$u\dot{x} + upx = uq$$

We want to choose a function $u(t)$ such that we can apply the product rule to the sum on the left hand side of the equation. There may be many functions u that could work, but in this case we only need one. See that

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x \iff u\dot{x} + upx = u\dot{x} + \dot{u}x \iff \dot{u} = up$$

so now by separation of equations

$$\begin{aligned} \frac{du}{u} &= p(t)dt \\ \ln|u| &= \int p(t)dt \\ u &= e^{\int p\,dt} \end{aligned}$$

By using u to satisfy the product rule:

$$\begin{aligned} u\dot{x} + upx = \frac{d}{dt}(ux) &= uq \\ u(t)x(t) &= \int u(t)q(t)dt + c \\ x(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)dt + c\right) \end{aligned}$$

which was what we wanted.
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Integrating factor and homogeneous equations
Given the homogeneous first order ODE

$$\dot{x} + p(t)x = 0$$

Solving by separation of variables gives

$$x_h(t) = Ae^{-\int p(t)dt}$$

Comparing this to the formula for the integrating factor

$$u(t) = e^{\int p(t)dt}$$

see that

$$x_h(t) = \frac{A}{u(t)}$$

A.1.9 General, Particular and Homogeneous solutions

Solving by method of Integrating factors allows us to come up with a solution for inhomogeneous first order linear ODEs

$$\dot{x} + p(t)x = q(t)$$

Which have the form

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)dt + C\right), \quad \text{where } u(t) = e^{\int p(t)dt}$$

Notice that the presence of the constant C implies a family of solutions; by setting $C = 0$ we get a *particular solution* x_p, which is simply one specific solution—we could have chosen any other:

$$\begin{aligned} x_p &= \frac{1}{u(t)}\left(\int u(t)q(t)dt + 0\right) \quad \text{is a solution} \\ x_p &= \frac{1}{u(t)}\left(\int u(t)q(t)dt + 999\right) \quad \text{is also a solution} \end{aligned}$$

The method of integrating factors naturally leaves us with a constant. But say we were to find a solution by *inspection*—how would we know that the constant of integration exists in the form $\frac{C}{u(t)}$? (as is in this case)

General solution
See that since

$$x_h(t) = \frac{1}{u(t)}$$

We can write the solution by integrating factor as

$$\begin{aligned} x(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)dt\right) + \frac{C}{u(t)} \\ &= x_p + Cx_h \end{aligned}$$

One way to fully solve the inhomogeneous equation is by first solving the *homogeneous* equation, and then finding any *one* solution, a *particular solution*, to the inhomogeneous equation x_p. (We can use any method to find x_p since we the homogeneous solution handles the constant of integration):

$$\text{General solution} = \text{Particular solution} + \text{Homogeneous solution}$$

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Intuition
Given an inhomogeneous first order linear ODE and its associated homogeneous equation

$$\begin{aligned} \dot{x} + p(t)x &= q(t) \quad \text{(inhomogeneous)} \\ \dot{x} + p(t)x &= 0 \quad \text{(homogeneous)} \end{aligned}$$

Solving both equations by method of integrating factors gives

$$x_p(t) = \frac{1}{u(t)} \left(\int u(t)q(t)dt \right) + \frac{A}{u(t)}, \qquad x_h(t) = \frac{B}{u(t)}$$

(where A is any chosen constant, each constant giving a particular solution, and B the constant of integration) Now see that by adding the solutions together the constant for the inhomogeneous solution A gets absorbed into the homogeneous solution:

$$\begin{aligned} x_p(t) + x_h(t) &= \frac{1}{u(t)} \left(\int u(t)q(t)dt \right) + \frac{A+B}{u(t)} \\ &= \frac{1}{u(t)} \left(\int u(t)q(t)dt \right) + \frac{C}{u(t)} \end{aligned}$$

We can obtain the 'ambiguous part' of the general solution by simply solving the homogeneous equation; this means that when obtaining a particular solution we don't have to worry about the constant of integration.

Superposition
See that this also makes sense with respect to superposition of solutions, where since

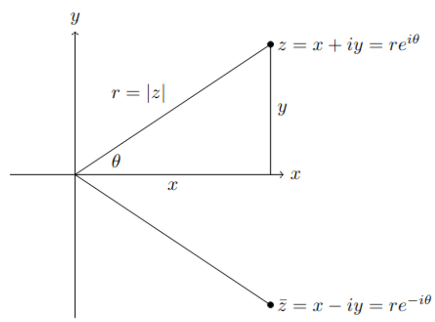
$$\underbrace{q(t) \rightsquigarrow x_p(t)}_{\text{inhomogeneous}} \quad \text{and} \quad \underbrace{0 \rightsquigarrow x_h(t)}_{\text{homogeneous}}$$

we can say

$$q(t) + 0 = q(t) \rightsquigarrow x_p(t) + x_h(t)$$

A.1.10 Polar form and Euler Identity

The Complex Plane, Polar Form
Complex numbers can be represented geometrically by points in a plane, where the number $a + ib$ is represented by the point (a, b); when points in a plane are thought of as representing complex numbers this way, the plane is known as a *Complex Plane*:



See that the magnitude of the coordinates of a complex number $x + iy$ can be represented by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

where r is the absolute value of the number:

$$r = |x + iy| = \sqrt{x^2 + y^2}$$

(its just the pythagorean theorem) thus the entire number can be written as

$$x + iy = r(\cos(\theta) + i\sin(\theta))$$

This is called the *Polar Form* of a non-zero complex number. We call θ the *angle* or *argument* of $x + iy$:

$$\theta = \arg(x + iy)$$

Notice that the angle can be increased by any integer multiple of 2π and will still represent the same thing. To simplify this one can specify the *principal value* of the angle:

$$0 \leq \theta < 2\pi$$

this can be indicated by Arg(...); for instance

$$\mathrm{Arg}(-1) = \pi, \quad \arg(-1) = \pm\pi, \pm 3\pi, \pm 5\pi$$

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Euler's Formula
Complex numbers have another *exponential* form called *Euler's formula*:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This should be regarded as a definition for the exponential of an imaginary power.

A good justification for Euler's formula can be found from its Taylor approximation:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos(\theta) + i\sin(\theta) \end{aligned}$$

Note that the argument above is not a proof; rather it just shows that Euler's formula is formally compatible with the series expansions for the exponential, sine, and cosine functions.

Polar form again
We can now write

$$x + iy = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

Polar representation in exponential form allows for much simpler multiplication of complex numbers. Since one can show that (using angle addition formulas)

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ &\quad + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1+\theta_2)} \end{aligned}$$

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Complex Exponential properties
We had

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

This property can be extrapolated to further justify Euler's formula—the complex exponential follows the same exponential addition rules as any typical exponential. See that we can now conclude:

Multiplication rule:

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

also see that since

$$\frac{1}{r} e^{-i\theta} \cdot r e^{i\theta} = 1$$

Reciprocal Rule:

$$\frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

DeMoivre's Formula
Since

$$(x+iy)^n = r^n e^{in\theta}$$

we can show *DeMoivre's formula:*

$$(\cos(\theta) + i\sin(\theta))^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Combining pure oscillations of the same frequency
We can also show that

$$a\cos(\lambda t) + b\sin(\lambda t) = A\cos(\lambda t - \phi)$$

where

$$A = \sqrt{a^2+b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

See that

$$\begin{aligned} a\cos(\lambda t) + b\sin(\lambda t) &= \mathrm{Re}((a-bi)(\cos(\lambda t) + i\sin(\lambda t)) \\ &= \mathrm{Re}(Ae^{-i\phi} \cdot e^{i\lambda t}) \\ &= \mathrm{Re}(Ae^{i(\lambda t - \phi)}) \\ &= A\cos(\lambda t - \phi) \end{aligned}$$

A.1.11 More on Complex Exponentials

Notable properties
We know that (as proven)

$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$$

So see that

$$\mathrm{Re}(e^{a+ib}) = e^a\cos(b), \quad \mathrm{Im}(e^{a+ib}) = e^a\sin(b)$$

this can be extrapolated further to show

$$\begin{aligned} \cos(x) &= \mathrm{Re}(e^{ix}), & \sin(x) &= \mathrm{Im}(e^{ix}) \\ \cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix}), & \sin(x) &= \frac{1}{2i}(e^{ix} - e^{-ix}) \end{aligned}$$

Derivatives and integrals
Note that a function like

$$e^{ix} = \cos(x) + i\sin(x)$$

is a *complex-valued function of the real variable* x. Such a function may be written as

$$u(x) + iv(x), \quad u, v \text{ real-valued}$$

with its derivative and integral with respect to x defined to be

$$\text{a) } D(u + iv) = Du + iDv, \quad \text{b) } \int (u + iv)dx = \int udx + i\int vdx$$

It follows easily that

$$D(e^{(a+ib)x}) = (a + ib)e^{(a+ib)x}$$

since

$$\begin{aligned} D(e^{(a+ib)x}) &= D(e^{ax}\cos(bx) + ie^{ax}\sin(bx)) \\ &= ae^{ax}\cos(bx) - be^{ax}\sin(bx) + i(ae^{ax}\sin(bx) + be^{ax}\cos(bx)) \\ &= e^{ax}\cos(bx)(a + ib) + e^{ax}\sin(bx)(ia + i^2 b) \\ &= (a + ib)e^{ax}(\cos(bx) + i\sin(bx)) \\ &= (a + ib)e^{(a+ib)x} \end{aligned}$$

Therefore we can also write the down the integral as

$$\int e^{(a+ib)x}dx = \frac{1}{a + ib}e^{(a+ib)x}$$

A.1.12 Finding n-th roots

To solve linear DEs with constant coefficients, we need to be able to find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha$$

where α is a complex number—finding the n-th roots of α.

n-th roots of unity
Consider first a special case; we want the solutions to

$$z^n = 1$$

We use polar representation for both sides, setting $z = re^{i\theta}$ on the left. See that

$$\underbrace{r^n e^{in\theta}}_{(re^{i\theta})^n} = \underbrace{1 \cdot e^{(2k\pi i)}, \quad k = 0, \pm 1, \pm 2, \ldots}_{=1}$$

Equating the absolute values and the arguments of each side:

$$r^n = 1, \quad n\theta = 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

(Notice the arguments for $k = a$ and $k = -a$, where a is an integer, are the same. Also see that r can only be 1 it is defined to be *real and non-negative* so it can't be anything else) we can conclude that

$$r = 1, \quad \theta = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1$$

we don't need any integer values of k other than $0, \ldots, n-1$—they would not produce a complex number that isn't already among the above n numbers. See that if we add an, an integer multiple of n, to any k we get the same complex number:

$$\theta' = \frac{2(k + an)\pi}{n} = \theta + 2a\pi$$

(this is the same as having $k = n, n+1, n+2 \ldots$) so

$$e^{i\theta'} = e^{i\theta} e^{2a\pi i} = e^{i\theta}$$

We can conclude therefore that *the n-th roots of 1 are the numbers*

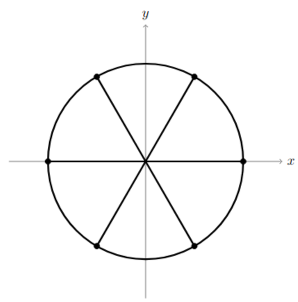
$$e^{2k\pi i/n}, \quad k = 0, \ldots, n-1$$

(next page)

Roots of unity visualised

There are n complex n-th roots of unity. Since they all have absolute value 1 ($r = 1$) they all lie on the unit circle in the complex plane. They are evenly spaced around the unit circle; the angle between two consecutive roots is $2\pi/n$.

Illustrated here is the case for $n = 6$:



The six solutions to $z^6 = 1$ lie on the unit circle in the complex plane. See that we can express the roots of unity in a different notation:

$$\textit{the } n\textit{-th roots of 1 are } 1, \zeta, \zeta^2, \dots, \zeta^{n-1}, \text{ where } \zeta = e^{2\pi i/n}$$

General case

Now we generalise to find the n-th roots of an arbitrary complex number w. We start by writing w in polar form:

$$w = re^{i\theta}; \quad \theta = \text{Arg}(w), 0 \leq \theta < 2\pi$$

Here θ is the principal value of the polar angle of w. Following the same reasoning as before, see that

$$z^n = re^{i(\theta+2\pi k)}; \quad k = 0, \pm 1, \pm 2, \dots$$

where removing the redundant k (this can be shown using the same methods as above) and solving gives us

$$z = \sqrt[n]{r}e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, \dots, n-1$$

See that these n roots can be expressed with the roots of unity as

$$\sqrt[n]{w} = z_0, z_0\zeta, z_0\zeta^2, \dots, z_0\zeta^{n-1}, \quad \text{where } z_0 = \sqrt[n]{r}e^{i\theta/n}$$

(z_0 is just the case where $k = 0$) See that all of the n roots satisfy $z^n = w$.

A.1.13 Sinusoidal functions

Definition and properties

A *sinusoidal function/oscillation/signal* is one that can be written in the from

$$f(t) = A\cos(\omega t - \phi)$$

The function $f(t)$ is a cosine function which has been *amplified* by A, *shifted* by ϕ/ω, and *compressed* by ω.

- $A > 0$ is its *amplitude*: how high the graph of $f(t)$ rises above the t-axis at its maximum values
- ϕ is its *phase lag*: the value of ωt for which the graph has its maximum (a positive phase lag shifts the sinusoid *forward*; consider a maximum at $\cos(a)$, without phase lag it is reached at $\omega t = a$, with phase lag its now $\omega t = a + \phi$.)
- $\tau = \phi/\omega$ is its *time delay/lag*: how far along the t-axis the graph of $\cos(wt)$ has been shifted due to phase lag. (τ and ϕ have the same sign; consider a maximum at $\cos(0)$, without phase lag it is reached at $\omega t = 0 \implies t = 0$, with phase lag its now $\omega t - \phi = 0 \implies t = \phi/\omega$.)
- ω is its *angular frequency*: the number of complete oscillations $f(t)$ makes per time interval of 2π; that is, the *number of radians per unit time* (1 radian in 1 second means 1 oscillation in 2π seconds—1 radian is the angle subtended at the centre of a circle by an arc equal in length to the radius).
- $v = \omega/2\pi$ is the *frequency* of $f(t)$: the number of complete oscillations made in a time interval of 1; that is, the number of cycles per unit time.
- $P = 2\pi/\omega = 1/v$ is its *period*: the t-interval required for one complete oscillation.

See that one can also write the sinusoidal function using the time lag $\tau = \phi/\omega$:

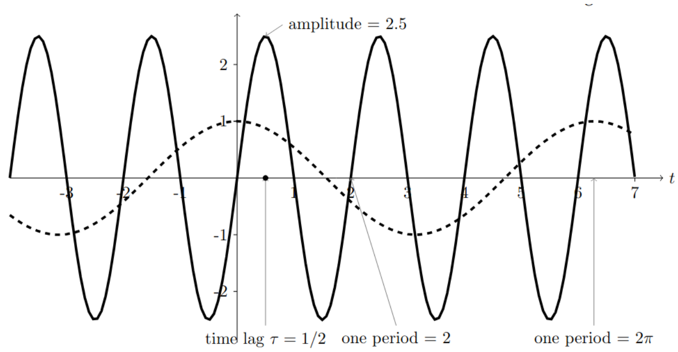
$$f(t) = A\cos(\omega(t - \tau))$$

(next page)

Example

In the figure below the dotted curve is $\cos(t)$ and the solid curve is $2.5\cos(\pi t - \pi/2)$. The solid curve has

$$A = 2.5, \quad \omega = \pi, \quad \phi = \pi/2, \quad \tau = 1/2$$



A.1.14 Solution to the Constant Coefficient First Order Equation

Solution
Considering the constant coefficient equation (constant coefficient meaning k is a constant)

$$\dot{y} + ky = q(t)$$

This is easily solvable by integrating factor:

$$\begin{aligned} y &= e^{-kt}\left(\int e^{kt}q(t)dt + c\right) \\ &= e^{-kt}\int e^{kt}q(t)dt + ce^{-kt} \end{aligned}$$

(integrating factor gives us a way of finding the particular solution, but see that it also gives us the homogeneous solution) We have the *particular* solution and *homogeneous* solution respectively

$$y_p(t) = e^{-kt}\int e^{kt}q(t)dt \quad \text{and} \quad y_h(t) = e^{-kt}$$

The general solution is then

$$y(t) = y_p(t) + cy_h(t)$$

Behaviour for $k > 0$:
For $k > 0$ the system models *exponential decay*. When the input is 0 the system response is $y(t) = ce^{-kt}$, which decays exponentially to 0 as t goes to ∞.

In the general solution we call ce^{-kt} the *transient* because it goes to 0. The other term $e^{-kt}\int e^{kt}q(t)dt$ is called the *steady-state/long-term* solution. That is, cy_h is the transient and y_p is the steady-state solution.

The value of c is determined by the initial value $y(0)$. See that this initial value only affects the transient and not the long-term behaviour of the solution—no matter what the initial condition, every solution goes asymptotically to the steady-state—all solution curves approach the steady-state as $t \to \infty$.

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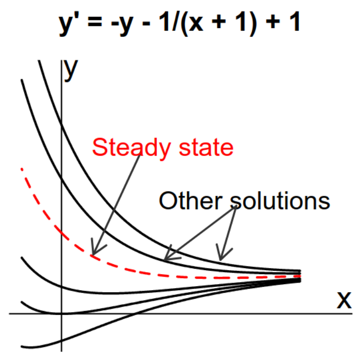
Behaviour for $k > 0$ illustrated
In the case $k > 0$ all solutions go asymptotically to the steady-state:



Since all the solutions approach each other, there is no precise way to choose the one to we call the steady-state—we can *choose any one* to be the steady-state solution. Generally we just choose the simplest looking solution.

The case $k \leq 0$:
When $k \leq 0$ the homogeneous solution $e^{-}kt$ does not decay asymptotically to 0—it is not transient. In this case it does not make sense to talk about the steady-state solution.

A.1.15 First Order Response to Sinusoidal/Exponential Input—complex replacement

Context
Consider solving the first order constant coefficient DE with sinusoidal input

$$\dot{x} + kx = B\cos(\omega t)$$

The idea here is to replace $\cos(\omega t)$ by the complex exponenial $e^{i\omega t}$; this is called *complex replacement*.

Complex Replacement
Consider introducing a new variable y with its own related ODE:

$$\dot{y} + ky = B\sin(\omega t)$$

Combining x and y to make a complex variable $z = x + iy$, see that we get

$$\dot{z} + kz = B(\cos(\omega t) + i\sin(\omega t)) = Be^{i\omega t}$$

where

$$\cos(\omega t) = \mathrm{Re}(e^{i\omega t}) \quad \text{and} \quad x = \mathrm{Re}(z)$$

Exponential input
Using complex replacement we now have the same problem but with exponential input

$$\dot{z} + kz = Be^{i\omega t}$$

This can be solved using integrating factors, but we present a simpler solution: consider a particular solution of the form $z_p(t) = Ae^{i\omega t}$ (this is a reasonable choice given that differentiation reproduces exponentials); this gives us

$$\dot{z}_p + kz_p = i\omega Ae^{i\omega t} + kAe^{i\omega t} = (k + i\omega)Ae^{i\omega t}$$

so we have

$$(k + i\omega)Ae^{i\omega t} = Be^{i\omega t} \implies A = B/(k + i\omega)$$

As such we have the particular solution

$$z_p(t) = Be^{i\omega t}/(k + i\omega)$$

simplifying with polar coordinates:

$$z_p(t) = \frac{Be^{i\omega t}}{\sqrt{k^2 + \omega^2}e^{i\phi}} = \frac{Be^{i(\omega t - \phi)}}{\sqrt{k^2 + \omega^2}}$$

since

$$k + i\omega = \sqrt{k^2 + \omega^2}e^{i\phi}, \quad \text{where } \phi = \tan^{-1}(\omega/k) \text{ in the first quadrant}$$

Since $\tan^{-1}$ is ambiguous ($\tan(\pi/4) = \tan(5\pi/4) = 1$), we clarify by saying which quadrant the complex number is in. In this case since $k, \omega > 0$ its the first quadrant.
(next page)

Solving for sinusoidal input
We have

$$z_p(t) = \frac{Be^{i(\omega t - \phi)}}{\sqrt{k^2 + \omega^2}}$$

we wanted x_p, where $x_p = \mathrm{Re}(z_p)$:

$$x_p(t) = \frac{B}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi)$$

To get the general solution we add the homogeneous solution:

$$x(t) = x_p(t) + Ce^{-kt} = \frac{B}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi) + Ce^{-kt}$$

A.1.16 Amplitude, Phase, Gain, and Bode Plots —Terminology and introduction

Terminology
We found that the ODE

$$\dot{x} + kx = kB\cos(\omega t)$$

has a particular solution

$$x(t) = \frac{kB}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi)$$

where $\phi = \tan^{-1}(\omega/k)$. If we consider the input to be $B\cos(\omega t)$ then the gain g (output amplitude/input amplitude) is $g = k/\sqrt{k^2+\omega^2}$:

$$x(t) = gB\cos(\omega t - \phi)$$

We define the terminology as follows:

- $B\cos(\omega t)$ is the input/input signal.
- B is the input amplitude and ω is the input angular/circular frequency.
- $x(t)$ is the output or response.
- $g = k/\sqrt{k^2+\omega^2}$ is called the gain/amplitude response. See that the input amplitude is scaled by the gain to give the output amplitude.
- ϕ is called the phase lag.

(next page)

Bode plots
Since g and ϕ vary with ω, we can regard them as functions of ω—$g(\omega)$ and $\phi(\omega)$. k is called the *coupling constant.* Consider the graphs of $g(\omega)$ and $-\phi(\omega)$ for the values of coupling constant $k = .25, .5, .75, 1, 1.25, 1.5$:

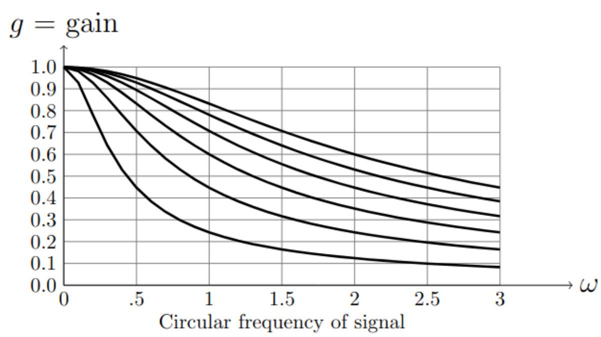


Fig. 1. First order amplitude response curves

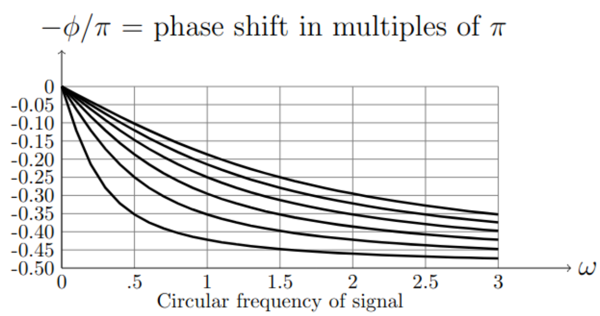


Fig. 2. First order phase response curves

These graphs are essentially *Bode plots.* (Bode plots display $\log g(\omega)$ and $-\phi(\omega)$ against $\log \omega$).

A.1.17 Autonomous equations, Logistic Model, Stable/Unstable equilibria

Here we consider *autonomous first order differential equations.* These are (in general) nonlinear equations of the form

$$\dot{x} = f(x)$$

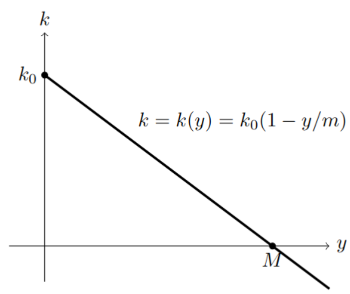
(compare this with the general first order ODE $\dot{x} = f(x,t)$.) The word autonomous means self governing—the rate of change of x is governed by x itself and is not dependent on time.

Example: Logistic Population Model
Suppose we have a model for a population y with variable growth rate $k(y)$ which depends on the current population but *not on time*:

$$\dot{y} = k(y) \cdot y$$

Say we model $k(y)$ as

$$k(y) = k_0 \left(1 - \frac{y}{M}\right)$$



The idea here is that population growth is positive until the population reaches some M, after which it becomes negative and declines until it becomes lower than M. In the simplest version of this we model $k(y)$ as a straight line as above. The final equation is known as the *logistic population model*:

$$\dot{y} = k_0(1 - (y/M))y = f(y)$$

The equation is nonlinear and autonomous. Autonomous equations are always separable; in this case partial fractions could be used to compute an integral, but here we consider a qualitative approach.
(next page)

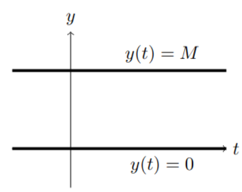
Qualitative perspective
We start by looking for *constant* solutions $y(t) = y_0$. We do this by considering $\dot{y} = 0$; see that this occurs in two situations, either

$$y(t) = 0, \quad \text{or} \quad y(t) = M$$

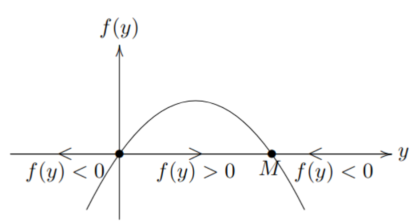
Because a system at equilibrium is unchanging, we call these solutions *equilibrium solutions*. Since equilibrium is achieved when y is 0 or M we call 0 and M the *critical points* of the DE. To summarise, these statements all mean the same thing:

1. $f(y_0) = 0$.
2. $y(t) = y_0$ is an equilibrium solution.
3. $y = y_0$ is a critical point.

Consider the direction field for these solutions; (recall each isocline represents $f(y) = c$) they correspond to the *nullclines*, where $f(y) = 0$:



(Note that the nullclines are also solution curves, if y starts at M it will never change since its derivative will be 0 forever.) For a clear picture of the other isoclines consider a graph of $f(y)$ against y:



See that

$$\begin{aligned} \text{for } y < 0 \quad & \dot{y} = f(y) \text{ is negative,} \\ \text{for } 0 < y < M \quad & \dot{y} = f(y) \text{ is positive,} \\ \text{for } M < y \quad & \dot{y} = f(y) \text{ is negative} \end{aligned}$$

These are indicated on the graph by the arrows on the horizontal axis.
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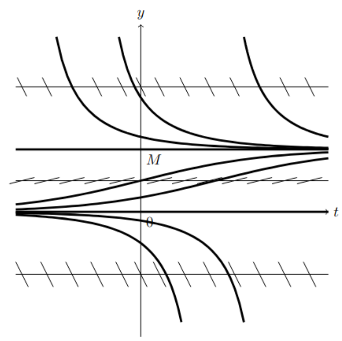
Direction field
We sketch the direction field and some solution curves:



See that

1. Since the isoclines are constant in the t direction any solution curve can be translated left or right and still be a solution—time invariance.

2. Since the lines $y = 0$ and $y = M$ are solutions the other curves can't cross them.

3. The solutions that start above $y = 0$ must increase, and since they can't cross $y = M$ they tend toward it asymptotically. These bounded solutions are called *logistic curves*. They represent small populations increasing to M.

4. If the population exceeds M, they tend back towards it. This represents overpopulation. M is called the *carrying capacity* of the environment.

5. Although it doesn't make sense to model a negative numbered population $y < 0$, mathematically the solution curves that start below $y = 0$ decrease without bound.

Stable and Unstable Equilibria
See that solution curves near the equilibrium $y = M$ tend asymptotically towards it; this is called a *stable equilibrium*. Solution curves near the other equilibrium $y = 0$ tend away from it; this is called an *unstable equilibrium*.

A.1.18 Phase lines, Semistable equilibria

Phase lines allows for the essential content of an autonomous DE:

$$\dot{y} = f(y)$$

to be conveyed more efficiently. A phase line is drawn using the following steps:

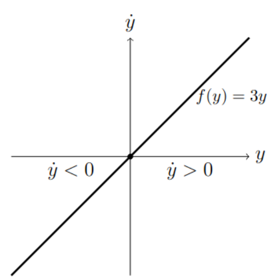
1. Draw the y-axis as a vertical line and mark on it the equilibria—where $f(y) = 0$.
2. In each of the intervals delimited by the equilibria draw an upward pointing arrow if $f(y) > 0$ and a downward arrow if $f(y) < 0$.

Phase lines tells us roughly how the system behaves, capturing the information of a qualitative sketch. Consider the following examples.

Example 1:
Consider the simple autonomous equation

$$\dot{y} = 3y$$

1) First we find the critical points; see that only one critical point exists: $y = 0$.
2) Next we plot the graph of $f(y)$ (in this case a straight line); see that $\dot{y} > 0$ for $y > 0$ and $\dot{y} < 0$ for $y < 0$:



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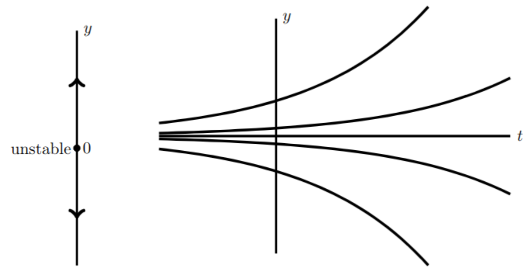
Example 1 (cont.)
3) With this we can draw the phase line as outlined above. Since the arrows on the phase line point away from the critical point, the equilibrium is *unstable*:



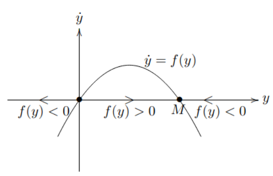
3. The phase line. 4. Qualitative sketch of solution curves.

4) See how the phase line conveys qualitiative information. The equilibrium solution corresponds to the critical point.

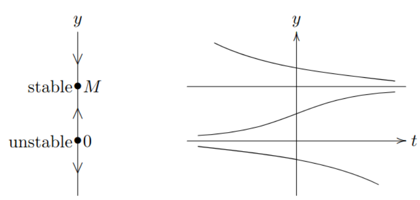
Example 2: Logistic equation
Now consider the same solution for the logistic equation:

$$\dot{y} = k_0(1 - y/M)y$$

With critical points $y = 0, y = M$:



2. Graph of $f(y)$



3. Phase line. 4. Sketch of solution curves.

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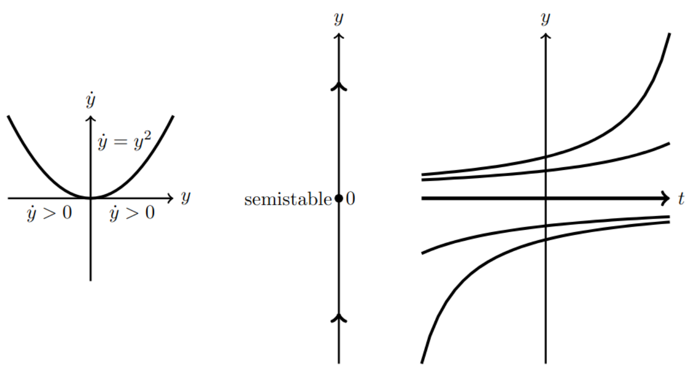
Semistable Equilibria

Some equilibria are stable on one side and unstable on the other. We call them *semistable*. Consider the DE

$$\dot{y} = y^2$$

With only one critical point $y = 0$:



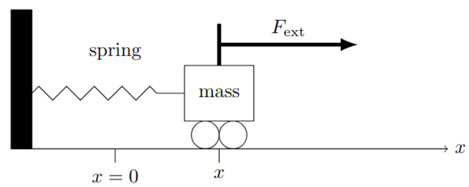
Graph of $f(y)$. Phase line. Sketch of solution curves.

A.2 Second Order Constant Coefficient Linear Equations

A.2.1 Second Order Physical systems—Spring-Mass-Dashpot

Spring and Mass

Here we model a second order differential equation. Consider a spring attached to a wall and a cart:



Consider the coordinate system set in a way that at $x = 0$ the spring doesn't exert any force—the equilibrium position. Now also consider an external force acting on the mass, the system can be modelled as

$$m\ddot{x} = F_{\text{spr}} + F_{\text{ext}}$$

The spring's behaviour can be characterised by the fact that it depends on the deviation from equilibrium position, meaning

$$\begin{aligned} &\text{if } x > 0, \quad F_{\text{spr}}(x) < 0 \\ &\text{if } x = 0, \quad F_{\text{spr}}(x) = 0 \\ &\text{if } x < 0, \quad F_{\text{spr}}(x) > 0 \end{aligned}$$

The simplest way to model the force exerted by the spring (which is valid in general for small x) is

$$F_{\text{spr}}(x) = -kx, \text{ where } k > 0$$

This is called *Hooke's law*, and k is called the *spring constant.*

Replacing F_{spr} by $-kx$ we get

$$m\ddot{x} + kx = F_{\text{ext}}$$

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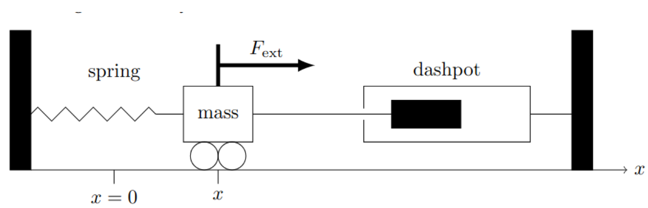
Dashpot
Any real mechanical system has friction, which can take many forms; it is characterised by the fact that it depends on the motion of the mass. We will suppose that it depends only on the velocity of the mass and not on its position.

Often dampening is controlled by a device called the *dashpot* (its a cylinder filled with oil that a piston moves through):



We write $F_{\mathrm{dash}}(\dot{x})$ for the force exerted by the dashpot. It opposes the velocity:

$$\begin{aligned} &\text{if } \dot{x} > 0, \quad F_{\mathrm{dash}}(\dot{x}) < 0 \\ &\text{if } \dot{x} = 0, \quad F_{\mathrm{dash}}(\dot{x}) = 0 \\ &\text{if } \dot{x} < 0, \quad F_{\mathrm{dash}}(\dot{x}) > 0 \end{aligned}$$

The simplest way to model this (which is also valid for small $\dot{x}$) is

$$F_{\mathrm{dash}}(\dot{x}) = -b\dot{x}, \text{ where } b > 0$$

This is called *linear damping*, and b is called the *damping constant*.

Putting this together

$$m\ddot{x} = F_{\mathrm{spr}} + F_{\mathrm{dash}} + F_{\mathrm{ext}}$$

we get the differential equation for the displacement x of the mass from equilibrium as

$$m\ddot{x} + b\dot{x} + kx = F_{\mathrm{ext}}$$

A.2.2 Linear DEs—Notation

A *linear differential equation* is of the following form:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 \dot{x} + a_0 x = q(t)$$

The a_k are the *coefficients*; they may depend on t. If a_n is not zero then the differential equation is said to be of order n.

If the a_k are constant then the equation is said to be a *constant coefficient linear equation*.

A.2.3 Second order homogeneous constant coefficient linear equations—Spring system, Simple harmonic oscillator

Consider the spring system in the case where $F_{\text{ext}} = 0$:

$$m\ddot{x} + b\dot{x} + kx = 0$$

With no external force the equation is *homogeneous*.

Undamped case: Simple harmonic oscillator
The special case where $b = 0$ (no dashpot) is called *undamped*. This is called the *simple harmonic oscillator*. We can write its ODE as

$$\ddot{x} + \frac{k}{m} x = 0$$

If we let $\omega = \sqrt{k/m}$ our equation becomes

$$\ddot{x} + \omega^2 x = 0$$

See that $x_1(t) = \cos(\omega t)$ and $x_2(t) = \sin(\omega t)$ are solutions to this equation. Since the equation is linear we can use superposition of solutions to get the solution

$$x(t) = a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi)$$

This is the *general solution*. We know it gives every solution because $x(0) = a$ and $\dot{x}(0) = \omega b$—by solving (uniquely) for a and b we can get any desired initial condition.

A.2.4 Characteristic polynomial

2nd Order Case
For m, b, k constant, consider the homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

Consider solutions of the form $x = e^{rt}$. We have

$$m\ddot{x} + b\dot{x} + kx = (mr^2 + br + k)e^{rt} = 0$$

Since an exponential is never zero, e^{rt} is therefore a solution exactly when r satisfies the *characteristic equation* (the left hand side is the *characteristic polynomial*):

$$mr^2 + br + k = 0$$

Example
Consider the DE

$$\ddot{x} + 8\dot{x} + 7x = 0$$

The characteristic polynomial here is $r^2 + 8r + 7$. Solving for r by factorisation we have $(r + 1)(r + 7)$ and the roots $r = -1$ and $r = -7$. Therefore the corresponding exponential solutions are $x_1(t) = e^{-t}$ and $x_2(t) = e^{-7t}$.

By superposition, the linear combination of independent solutions gives the general solution:

$$x(t) = c_1 e^{-t} + c_2 e^{-7t}$$

Where given initial conditions for x and $\dot{x}$ we can solve for c_1 and c_2.

General nth Order Case
See that using the same principle we can take the homogeneous constant coefficient linear equation of degree n:

$$a_n x^{(n)} + \cdots + a_1 \dot{x} + a_0 x = 0$$

and get its characteristic polynomial

$$p(r) = a_n r^n + \cdots + a_1 r + a_0$$

In which the exponential $x(t) = e^{rt}$ is a solution of the homogeneous DE if and only if r is a root of $p(r)$ (meaning $p(r) = 0$). By superposition, any linear combination of these exponentials is also a solution.

A.2.5 Modes and Roots (real and complex) (homogeneous constant coefficient linear equations)

Modes
A solution of the form $x(t) = ce^{rt}$ to the homogeneous constant coefficient linear equation:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 \dot{x} + a_0 x = 0$$

is called a *modal solution* and ce^{rt} the *mode* of the system. Recall that e^{rt} is a solution exactly when r is a root of the characteristic polynomial

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

(note this only works for *homogeneous constant coefficient linear equations*; it won't apply to non-constant coefficient or inhomogeneous or nonlinear equations.)

Real roots
The roots of these polynomials can be real or complex. Roots can also be repeated. First consider the real case for a second order homogeneous constant coefficient DE: if the characteristic polynomial has real roots r_1 and r_2 then the *modal* solutions are $x_1(t) = e^{r_1 t}$ and $x_2(t) = e^{r_2 t}$. The general solution can be found by superposition:

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example
Solving $\ddot{x} + 5\dot{x} + 4x = 0$: The characteristic equation is

$$s^2 + 5s + 4 = (s + 1)(s + 4) = 0$$

Which has roots -1 and -4. The modal solutions are $x_1(t) = e^{-t}$ and $x_2(t) = e^{-4t}$. Therefore the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-4t}$$

(next page)

Complex roots—illustrative example
Consider now the equation $\ddot{x} + 4\dot{x} + 5x = 0$. The characterisic polynomial is $s^2 + 4s + 5$. Using the quadratic formula the roots are

$$s = \frac{-4\sqrt{16-20}}{2} = -2 \pm \sqrt{-1} = -2 \pm i$$

So our exponential solutions are (using the letter z to indicate they are complex valued):

$$z_1(t) = e^{(-2+i)t} \quad \text{and} \quad z_2(t) = e^{(-2-i)t}$$

The DE has real coefficients, we expect *real solutions.* To get them, consider the following theorem:

Real Solution Theorem:
Theorem: If $z(t)$ is a complex-values solution to $m\ddot{z} + b\dot{z} + kz = 0$, where m, b, k are real, then the real and imaginary parts of z are also solutions.

Proof: Letting $u(t)$ be the real part of z and $v(t)$ the imaginary part, so that $z(t) = u(t) + iv(t)$, see that the DE can be written as

$$(m\ddot{u} + b\dot{u} + ku) + i(m\ddot{v} + b\dot{v} + kv) = 0$$

Both expressions in parentheses are real. The only way for the sum to be 0 is if both expressions are 0. That is, both u and v are solutions.

Illustrative example cont.
We had $z_1(t) = e^{(-2+i)t}$ and $z_2(t) = e^{(-2-i)t}$. Using Euler's formula:

$$z_1(t) = e^{(-2+i)t} = e^{-2t}\cos t + ie^{-2t}\sin t$$

Both the real part $e^{-2t}\cos t$ and imaginary part $e^{-2t}\sin t$ are solutions. We now have two *basic* solutions and can use superposition to obtain the general *real valued* solution

$$x(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$$

See that choosing the other exponential solution

$$z_2(t) = e^{(-2+i)t} = e^{-2t}\cos(-t) + ie^{-2t}\sin(-t)$$

would give the basic real solutions

$$e^{-2t}\cos(t) \quad \text{and} \quad -e^{-2t}\sin(t)$$

Which would give the same general solution.
(next page)

More on complex roots
We had the general solution

$$c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$$

See that the solution can be written in a different form:

$$x(t) = e^{-2t}(c_1\cos(t) + c_2\sin(t)) = Ae^{-2t}\cos(t-\phi)$$

This is a *damped sinusoid* with *circular pseudo-frequency* 1.

Example
Solving $\ddot{x}+\dot{x}+x=0$, the characteristic equation is $s^2+s+1=0$ and the roots

$$\frac{-1\pm\sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

We have the complex exponential solutions

$$z_1(t) = e^{(-1+i\sqrt{3})t/2}, \quad z_2(t) = e^{(-1-i\sqrt{3})t/2}$$

From this we obtain the basic real solutions

$$\mathrm{Re}(z_1(t)) = e^{-t/2}\cos(\sqrt{3}t/2), \quad \mathrm{Im}(z_1(t)) = e^{-t/2}\sin(\sqrt{3}t/2)$$

and therefore the general real solution

$$e^{-t/2}(c_1\cos(\sqrt{3}t/2) + c_2\sin(\sqrt{3}t/2)) = Ae^{-t/2}\cos(\sqrt{3}t/2-\phi)$$

In general
In general, supposing the equation $m\ddot{x}+b\dot{x}+kx=0$ has the characteristic roots $a\pm ib$, two real solutions are

$$e^{at}\cos(bt) \quad \text{and} \quad e^{at}\sin(bt)$$

and the general real solution is

$$c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt) = Ae^{at}\cos(bt-\phi)$$

A.2.6 Repeated roots (homogeneous constant coefficient linear equations)

Illustrative example
Consider $\ddot{x} + 4\dot{x} + 4x = 0$. In this case the characteristic equation:

$$P(s) = s^2 + 4s + 4 = (s+2)^2$$

has $r = -2$ as a repeated root. The only exponential solution is $e^{-2}t$. To get the second basic exponential solution, see that te^{-2t} is also a solution. Our general solution is therefore

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Deriving solution for second order repeated roots
Considering the DE $a\ddot{x} + b\dot{x} + cx = 0$, from the characteristic equation we know

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Should we only have one solution, as per the quadratic formula, we must have

$$b^2 - 4ac = 0, \quad \text{and} \quad r_{1,2} = -\frac{b}{2a}$$

We know one solution is $x_1 = e^{-b/(2a)t}$. Consider a second solution of the form

$$x_2 = v(t)x_1$$

we want to plug x_2 into the DE. For that we require its derivatives:

$$\begin{aligned}
x_2' &= v' e^{-b/(2a)t} - \frac{b}{2a} v e^{-b/(2a)t} \\
x_2'' &= v'' e^{-b/(2a)t} - \frac{b}{2a} v' e^{-b/(2a)t} - \frac{b}{2a} v' e^{-b/(2a)t} + \frac{b^2}{4a^2} v e^{-b/(2a)t} \\
&= v'' e^{-b/(2a)t} - \frac{b}{a} v' e^{-b/(2a)t} + \frac{b^2}{4a^2} v e^{-b/(2a)t}
\end{aligned}$$

Evaluating the DE with our proposed solution x_2:

$$a\left(v'' e^{-b/(2a)t} - \frac{b}{a} v' e^{-b/(2a)t} + \frac{b^2}{4a^2} v e^{-b/(2a)t}\right) + \\ b\left(v' e^{-b/(2a)t} - \frac{b}{2a} v e^{-b/(2a)t}\right) + c\left(v e^{-b/(2a)t}\right) = 0$$

Factoring out the exponential we get

$$\begin{aligned}
& e^{-b/(2a)t}\left(av'' - bv' + \frac{b^2}{4a} v + bv' - \frac{b^2}{2a} v + cv\right) \\
&= e^{-b/(2a)t}\left(av'' + \left(-\frac{b^2}{4a} + c\right) v\right) \\
&= e^{-b/(2a)t}\left(av'' - \frac{1}{4a}\left(b^2 - 4ac\right) v\right) = 0
\end{aligned}$$

(next page)

Derivation continued
Evaluating the DE with our proposed solution $x_2 = ve^{-b/(2a)t}$ gave us

$$e^{-b/(2a)t}\left(av'' - \frac{1}{4a}\left(b^2 - 4ac\right)v\right) = 0$$

We know that $b^2 - 4ac = 0$ (since the quadratic equation only has one solution as mentioned before). Thus since exponentials cannot be zero, we have

$$av'' = 0 \implies v'' = 0$$

(since $a \neq 0$.) We can then determine $v(t)$:

$$v' = \int v'' dt = k \implies v = \int v' dt = k_1 t + k_2$$

We therefore have our proposed basic solutions as

$$x_1 = e^{-b/(2a)t}, \quad x_2 = vx_1 = (k_1 t + k_2)e^{-b/(2a)t}$$

see that combining our solutions into a general solution via superposition gives

$$x(t) = c_1 e^{-b/(2a)t} + c_2(k_1 t + k_2)e^{-b/(2a)t}$$

See that this can be simplified since c_1, c_2, k_1, k_2 are all unknown constants

$$x(t) = (c_1 + c_2 k_2)e^{-b/(2a)t} + c_2 k_1 t e^{-b/(2a)t}$$

We have

$$x(t) = C_1 e^{-b/(2a)t} + C_2 t e^{-b/(2a)t}$$

See that since we are dealing with a homogeneous equation, $te^{-b/(2a)t}$ by itself satisfies the DE (because of superposition).

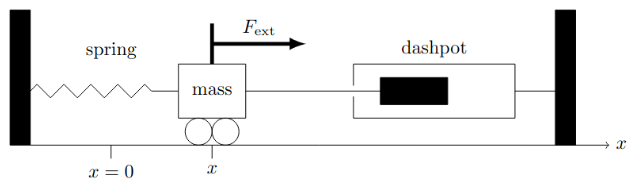
A.2.7 Damped harmonic oscillators

Spring-mass-dashpot

Recall the model spring-mass-dashpot system with the constant coefficient linear DE:

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$$

where m is the mass, b the damping constant, k the spring constant, and $x(t)$ the displacement of the mass from its equilibrium position.



Assuming zero external force ($F_{\text{ext}} = 0$), we have the homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

The algebra doesn't require any restrictions on m, b, k, except $m \neq 0$ (for the equation to be second order in the first place). But in this physical model we require $m > 0, b \geq 0$, and $k > 0$.

(next page)

Damped harmonic oscillator

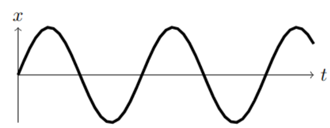
The *undamped* ($b = 0$) system has the equation

$$m\ddot{x} + kx = 0$$

We have its solution as

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(\omega t - \phi)$$

Here $\omega = \sqrt{k/m}$. The solution is always a sinusoid, thus we call this a *simple harmonic oscillator*:



When we add damping ($b > 0$) we then call the system a *damped harmonic oscillator*.

This emphasises an important fact about uding DEs to model physical systems: Any system modeled by the same equation will respond just like the spring-mass-dashpot (regardless of what m, d, k, x represent). That is, all damped harmonic oscillators exhibit similar behaviour.

A.2.8 Under, Over and Critical damping

Response to damping
As we saw, the unforced damped harmonic oscillator has equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

with $m > 0, b \geq 0$ and $k > 0$. It has characteristic equation

$$ms^2 + bs + k = 0$$

with characteristic roots

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

There are three cases depending on the sign of the expression under the square root:

1. $b^2 < 4mk$—*Underdamping*
2. $b^2 > 4mk$—*Overdamping*
3. $b^2 = 4mk$—*Critical damping*

First case: Underdamping
If $b^2 < 4mk$ the square root is negative and the characteristic roots are complex. See that the roots are given by

$$-\frac{b}{2m} \pm i\omega_d, \quad \text{where } \omega_d = \frac{\sqrt{|b^2 - 4mk|}}{2m}$$

With that we have the complex exponential solutions

$$e^{(-b/(2m)+i\omega_d)t}, \quad e^{(-b/(2m)-i\omega_d)t}$$

With the basic real solutions

$$e^{-bt/(2m)}\cos(\omega_d t), \quad e^{-bt/(2m)}\sin(\omega_d t)$$

The general real solution is found by taking linear combinations of two basic solutions:

$$x(t) = c_1 e^{-bt/(2m)}\cos(\omega_d t) + c_2 e^{-bt/(2m)}\sin(\omega_d t)$$

This can also be written as

$$x(t) = e^{-bt/(2m)}(c_1\cos(\omega_d t) + c_2\sin(\omega_d t)) = Ae^{-bt/(2m)}\cos(\omega_d t - \phi)$$

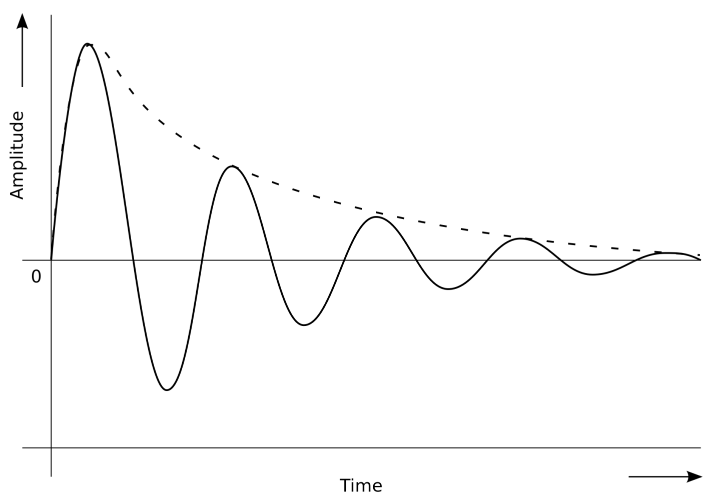
(next page)

Underdamping—Intuition
We had the behaviour of an underdamped system as

$$Ae^{-bt/(2m)}\cos(\omega_d t - \phi)$$

Intuitively, when $b = 0$ the response is a sinusoid. In terms of the spring-mass-dashpot setup this corresponds to zero damping/no friction. When the damping constant b is small (underdamped), we expect the system to still oscillate, but with decreasing amplitude as its energy is lost to friction.

This can be seen from the response of the underdamped system. The factor $\cos(\omega_d t - \phi)$ represents the oscillation while the exponential factor $e^{-bt/(2m)}$ has a negative exponent—thus representing a decaying amplitude. As $t \to \infty$, $x(t) \to 0$:



We call ω_d the *damped angular/circular frequency* of the system, or the *psuedo-frequency* of $x(t)$ ('psuedo' because $x(t)$ is not periodic and only periodic functions have a frequency).

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Underdamping—Example
Consider the system

$$\ddot{x} + \dot{x} + 3x = 0$$

(See that $b^2 < 4mk$) We have the characteristic equation $s^2 + s + 3 = 0$ and the characteristic roots $-1/2 \pm \sqrt{11}/2$. We have the basic real solutions as

$$e^{-t/2}\cos(\sqrt{11}t/2), \quad e^{-t/2}\sin(\sqrt{11}t/2)$$

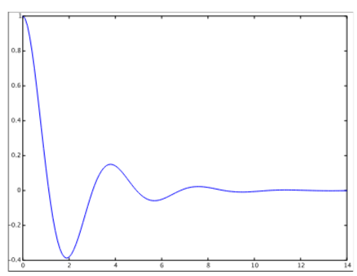
and the general solution

$$x(t) = e^{-t/2}(c_1\cos(\sqrt{11}t/2) + c_2\sin(\sqrt{11}t/2)) = Ae^{-t/2}\cos(\sqrt{11}t/2 - \phi)$$

The damped angular frequency, ω_d, in this case is $\sqrt{11}/2$. Say we have the initial conditions $x(0) = 1, \dot{x}(0) = 0$, this gives us constants $c_1 = 1, c_2 = 1/\sqrt{11}$. So

$$\begin{aligned} x(t) &= e^{-t/2}\left(\cos(\sqrt{11}t/2) + \frac{1}{\sqrt{11}}\sin(\sqrt{11}t/2)\right) \\ &= \frac{\sqrt{12}}{\sqrt{11}}e^{-t/2}\cos(\sqrt{11}t/2 - \phi) \end{aligned}$$

Where $\phi = \tan^{-1}(1/\sqrt{11})$. Plotted:



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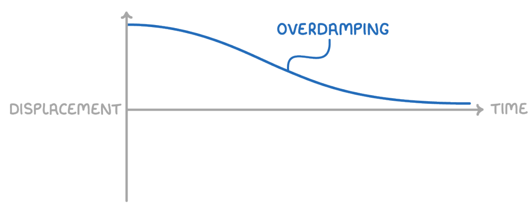
Overdamping
Overdamping corresponds to the case where $b^2 > 4mk$ (distinct real roots). In this case *both characteristic roots are negative*—see that it is always the case that $b^2 - 4mk < b^2$ and therefore that $-b + \sqrt{b^2 - 4mk} < 0$, while the other root is clearly negative. Thus we have the real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, r_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

(both r are *real and negative* and are *different*) We have the general solution

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Intuitively, this corresponds to the frictional force being so great that the system can't oscillate—the system just goes asymptotically to the equilibrium $x = 0$:

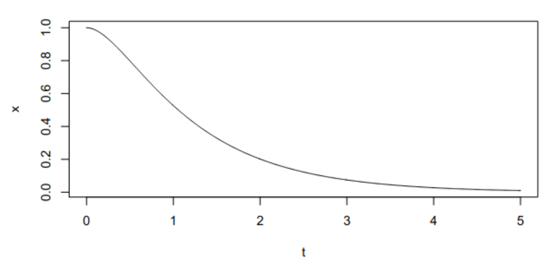


Example
For instance, consider the system $\ddot{x} + 4\dot{x} + 3x = 0$ with inital conditions $x(0) = 1, \dot{x}(0) = 0$. This has characteristic equation $s^2 + 4s + 3 = 0$ and characteristic roots $-1, -3$. Its general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$

Because the roots are real and different, this system is classified as overdamped. To satisfy the initial conditions $c_1 = 3/2, c_2 = -1/2$. So

$$x(t) = 3/2e^{-t} - 1/2e^{-3t}$$



Because e^{-t} goes to 0 *more slowly* than $e^{-3t/2}$ it controls the rate at which x goes to 0—the term that goes to zero slowest controls the rate.

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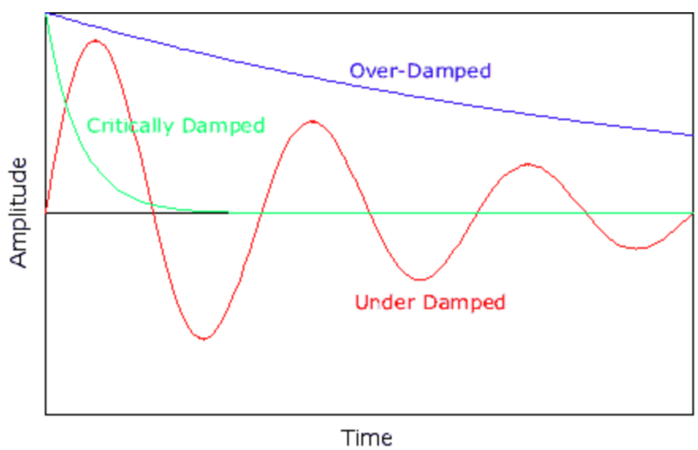
Critical damping
Critical damping corresponds to the case where $b^2 = 4mk$ (repeated real roots) —the characteristic polynomial has the roots $-b/2m, -b/2m$. With repeated roots we have the solutions

$$e^{-bt/(2m)}, \quad te^{-bt/(2m)}$$

and the general solution

$$e^{-bt/(2m)}(c_1 + c_2 t)$$

As in the overdamped case, this does not oscillate. It is worth noting that for a fixed m and k, choosing b to be the critically damping value gives the fastest return of the system to equilibrium:



Intuition for rate of decay
This can be seen from the roots. Considering fixed m, k and only varying the damping constant b, in the overdamped case notice that one characteristic root is less negative than that of the critically damped case; since the less negative root controls the decay rate this leads to the rate of decay always being smaller. See that $f = b$ has a higher derivative than $g = \sqrt{b^2 - 4mk}$:

$$\frac{df}{db} = 1 \quad \frac{dg}{db} = \frac{b}{\sqrt{b^2 - 4mk}} > \frac{b}{\sqrt{b}} = 1$$

For $m, k > 0$. So $-b + \sqrt{b^2 - 4mk}$ only becomes less negative (slower rate of decay).

For the underdamped case to exist we must have $b < 4mk$. See that this b is always smaller than in the critically damped case, which corresponds to $b = 4mk$.

A.2.9 Superposition (Second order ODEs)

The Principle of Superposition for Second Order Differential Equations; if

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

is a second order linear differential equation and $y = y_1(t)$ and $y = y_2(t)$ are both solutions to this differential equation, then for C and D as constants,

$$y = Cy_1(t) + Dy_2(t) \quad \text{is also a solution}$$

Essentially, any linear combination of solutions is also a solution.

Proof: Consider $y = y_1$ and $y = y_2$ are solutions to the second order linear differential equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$. Then we have that:

$$\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 = 0 \quad \text{and} \quad \frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2 = 0$$

If C and D are constants, plugging in $y = Cy_1(t) + Dy_2(t)$:

$$\begin{aligned}
&\frac{d^2}{dt^2}(Cy_1(t) + Dy_2(t)) + p(t)\frac{d}{dt}(Cy_1(t) + Dy_2(t)) + q(t)(Cy_1(t) + Dy_2(t)) \\
&= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\
&= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=0} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=0} \\
&= 0
\end{aligned}$$

Therefore, $y = Cy_1(t) + Dy_2(t)$ is also a solution. Note that the superposition principle **does not** work for nonlinear differential equations.
(next page)

In context of inhomogenous differential equations
In addition, if y_1 is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_1(t)$$

and y_2 is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f_2(t)$$

then for constants C and D, $Cy_1 + Dy_2$ is a solution to:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = Cf_1(t) + Df_2(t)$$

Proof: Plugging in $y = Cy_1 + Dy_2$:

$$\begin{aligned}
&\frac{d^2}{dt^2}(Cy_1 + Dy_2) + p(t)\frac{d}{dt}(Cy_1 + Dy_2) + q(t)(Cy_1 + Dy_2) \\
&= C\frac{d^2y_1}{dt^2} + D\frac{d^2y_2}{dt^2} + p(t)C\frac{dy_1}{dt} + p(t)D\frac{dy_2}{dt} + q(t)Cy_1 + q(t)Dy_2 \\
&= C\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{=f_1(t)} + D\underbrace{\left[\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right]}_{=f_2(t)} \\
&= Cf_1(t) + Df_2(t)
\end{aligned}$$

Superposition is therefore *not* limited to homogenous equations.

A.2.10 General solution for inhomogenous linear ODEs

Therefore, to get the general solution $y(t)$ to an inhomogenous linear ODE:

$$\text{inhomogenous:} \quad \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t)$$

1. Find the general solution y_h to the associated **homogenous** equation:

$$\text{homogenous:} \quad \frac{d^2y_h}{dt^2} + p(t)\frac{dy_h}{dt} + q(t)y_h = 0$$

2. Find (in some way) any **one particular solution** y_p to the **inhomogenous** ODE.
3. Add y_p to y_h to get the general solution to the inhomogenous ODE:

$$\underbrace{y}_{\text{general }\textbf{inhomogenous}\text{ solution}} = \underbrace{y_p}_{\textbf{any particular solution}} + \underbrace{y_h}_{\text{general }\textbf{homogenous}\text{ solution}}$$

Note that the superposition principle **does not** work for nonlinear differential equations.

A.2.11 Superposition of homogeneous equations only requires linearity

Consider the illustrative example. Given the ODE

$$t^2 y'' + ty' - 4y = 0$$

One can check by substitution that $y_1(t) = t^2$ and $y_2(t) = 1/t^2$ are both solutions. Thus

$$y(t) = c_1 t^2 + c_2/t^2$$

Is a solution for any c_1, c_2. Point here is that *we didn't need the differential equation to have constant coefficients*: linearity and homogeneity is sufficient.

A.2.12 Existence and uniqueness

Solving a first-order linear ODE leads to a 1-parameter family of solutions (a general solution). To derive a specific solution, we need an initial condition, such as $y(0)$. One may wonder if there are other solutions. Here is a general result which says that there aren't and confirms that our methods find all solutions:

Existence and uniqueness theorem for a linear ODE:
Let $p(t)$ and $q(t)$ be continuous functions on an open interval I. Let $a \in I$, and let b be a given number. Then there **exists** a **unique** solution defined on the entire interval I to the first order linear ODE

$$\dot{y} + p(t)y = q(t)$$

satisfying the initial condition

$$y(a) = b$$

Existence means there is **at least one** solution.
Uniqueness means that there is **only one** solution.

A.2.13 Exponential input (2nd order ODEs) —exponential response

Context
Consider a case where the driving function (the external force in a spring-mass-dashpot system) is an exponential Be^{at}:

$$mx'' + bx' + kx = Be^{at}$$

where B, a are constants. We obtain a *particular* solution by considering a solution of the form Ae^{at} (same as the first order case). Substituting gives us

$$\begin{aligned} mx'' + bx' + kx &= ma^2Ae^{at} + baAe^{at} + kAe^{at} \\ &= (ma^2 + ba + k)Ae^{at} \end{aligned}$$

Thus we get (by comparing coefficients)

$$A = \frac{B}{ma^2 + ba + k}$$

See that the denominator is just the characteristic polymonial, which we will denote as $p(a)$ (this is not suprising, since the characteristic polynomial is derived using the same principles).

This concept isn't new, the same idea was used in the context of exponential/sinusoidal input to first order ODEs.

Exponential response formula (ERF)
Considering the second order equation

$$mx'' + bx' + kx = Be^{at}$$

and letting $p(r) = mr^2 + br + k$ be its characteristic polynomial, then

$$x(t) = \frac{B}{p(a)}e^{at}$$

is a *particular solution*, as long as $p(a) \neq 0$.
(next page)

Example
Consider finding the *general* solution to

$$x'' + 8x' + 7x = 9e^{2t}$$

considering solutions of the form $x(t) = Ae^{2t}$, substituting gives us

$$x'' + 8x' + 7x = Ae^{2t}\underbrace{(4 + 16 + 7)}_{p(2)} = 27Ae^{2t}$$

Comparing coefficients gives us

$$A = \frac{\overbrace{9}^{B}}{\underbrace{27}_{p(2)}}, \quad x_p = \frac{1}{3}e^{2t}$$

This is our *particular* solution (see that the ERF is just a shortcut of this reasoning). To obtain a general solution we need to solve the homogeneous equation:

$$x'' + 8x' + 7x = 0$$

Where we get the homogeneous solution

$$x_h = c_1e^{-7t} + c_2e^{-t}$$

Thus the general solution to the original equation is

$$x = x_h + x_p = c_1e^{-7t} + c_2e^{-t} + \frac{1}{3}e^{2t}$$

A.2.14 Sinusoidal input—complex replacement

General case, Gain

The ERF can be applied using complex replacement to solve equations with sinusoidal driving. Consider the general case:

$$mx'' + bx' + kx = B\cos(\omega t)$$

can be *complexified* as

$$mz'' + bz' + kz = Be^{i\omega t}$$

where $z = x + iy$, y defined as

$$my'' + by' + ky = B\sin(\omega t)$$

Now we solve for z_p using the ERF:

$$z_p = \frac{B}{p(i\omega)}e^{i\omega t}$$

We get x_p by taking $\mathrm{Re}(z)$:

$$x_p = \frac{B}{|p(i\omega)|}\cos(\omega t - \phi)$$

where $|p(i\omega)|$ refers to the magnitude of $p(i\omega)$, and $\phi = \mathrm{Arg}(p(i\omega))$.

See that the output looks similar to the input—apart from the phase lag, the output is a multiple of the input by a factor of $1/|p(i\omega)|$. This factor is called the *gain* of the system:

$$\text{output amplitude} = \text{gain} \times \text{input amplitude}$$

(next page)

Example
Say we want the general solution to

$$x'' + 8x' + 7x = 9\cos(2t)$$

using complex replacement we instead solve

$$z'' + 8z' + 7z = 9e^{2it}$$

Now we can apply the ERF:

$$\begin{aligned} z_p(t) &= \frac{9}{p(2i)}e^{2it} \\ &= \frac{9}{(2i)^2 + 16i + 7}e^{2it} \\ &= \frac{9}{3 + 16i}e^{2it} \end{aligned}$$

Simplify by expressing $p(2i)$ in polar form:

$$\begin{aligned} \frac{9}{3 + 16i}e^{2it} &= 9\left(\frac{1}{\sqrt{265}}e^{-i\phi}\right)e^{2it} \\ &= \frac{9}{\sqrt{265}}e^{i(2t-\phi)} \end{aligned}$$

where $\phi = \text{Arg}(p(2i)) = \tan^{-1}(16/3)$. We can then obtain x_p as

$$x_p(t) = \text{Re}(z_p(t)) = \frac{9}{\sqrt{265}}\cos(2t - \phi)$$

To get the general solution we add the homogeneous solution:

$$x_h(t) = c_1e^{-7t} + c_2e^{-t}$$

To obtain

$$x(t) = x_p + x_h = \frac{9}{\sqrt{265}}\cos(2t - \phi) + c_1e^{-7t} + c_2e^{-t}$$

In this case the gain is

$$1/|p(2i)| = 1/\sqrt{265}$$

A.2.15 Simple harmonic oscillator—intuition for resonance

Natural frequency
We consider the undamped harmonic oscillator ($b = 0$)

$$mx'' + kx = F_{\text{ext}}(t)$$

we define the *natural frequency* ω_n as

$$\omega_n = \sqrt{k/m}$$

this comes from considering the system with no driving force $F_{\text{ext}}(t) = 0$, in which case the characteristic equation $p(r) = mr^2 + k$ has the roots $\pm i\sqrt{k/m}$. This leads to the general solution

$$x_h = c_1 \cos((\sqrt{k/m})t) + c_2 \sin((\sqrt{k/m})t)$$

With the natural frequency defined the unforced solution is

$$x_h = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t)$$

and the original equation looks like

$$m(x'' + \omega_n^2 x) = F_{\text{ext}}(t)$$

Intuition for resonance
Now consider adding sinusoidal input $F_{\text{ext}}(t) = B\cos(\omega t)$:

$$m(x'' + \omega_n^2 x) = B\cos(\omega t)$$

Using complex replacement, we seek to obtain the z_p in

$$m(x'' + \omega_n^2 x) = Be^{i\omega t}$$

Applying the ERF with $a = i\omega$ we get

$$z_p = \frac{B}{p(i\omega)} e^{i\omega t} = \frac{B}{m(\omega_n^2 - \omega^2)} e^{i\omega t}$$

(Notice the lack of a damping makes the denominator non-complex.) Taking the real part to obtain x_p:

$$x_p = \text{Re}(z_p) = \frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t)$$

With the homogeneous solution, our general solution is

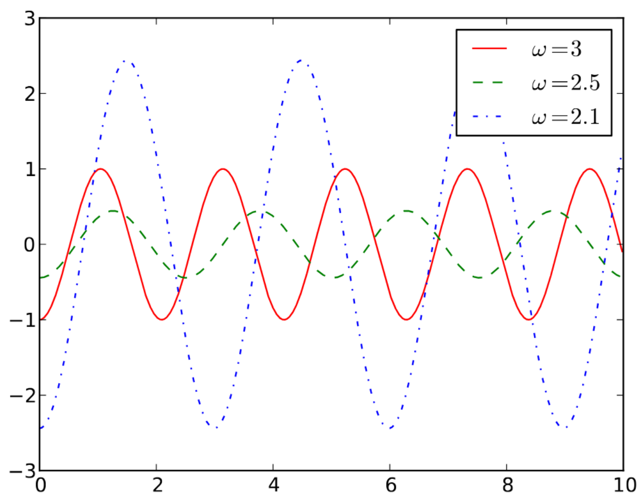
$$x_h = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t) + \frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t)$$

(next page)

Intuition for resonance (cont.)
We had the particular solution

$$\frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t)$$

See that the gain is $1/|p(a)| = 1/m(\omega_n^2 - \omega^2)$—the closer ω is to ω_n. Setting $m = 1, B = 1$ and $\omega_n = 2$ while varying ω (the frequency of the driving):



See that the solution breaks down when $\omega = \omega_n$. This is called **pure resonance**; notice it corresponds to the case $p(a) = 0$.
(next page)

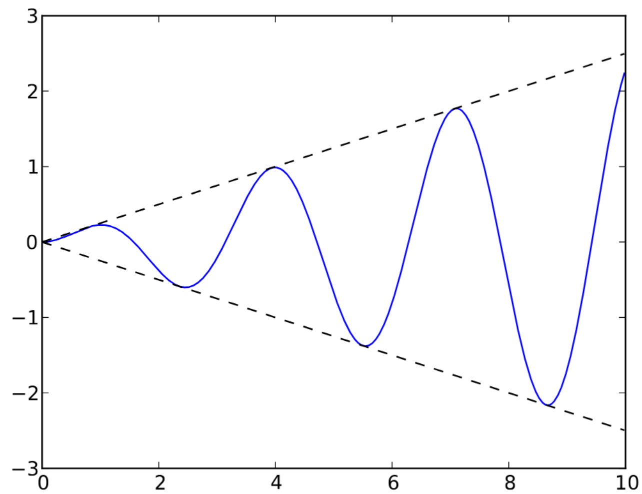
The workaround solution—resonant response formula
When $p(a) = 0$ the initial solution we found breaks down. It can be checked that the following is a solution *to the complexified equation*:

$$z_p(t) = \frac{B}{p'(a)} t e^{at}$$

where taking the real part gives a valid x_p (which can also be checked):

$$x_p(t) = \frac{B}{2m\omega} t \sin(\omega t)$$

See the extra factor of t before the sine term—the amplitude of the response in the case of pure resonance grows with time.



The following is a counterpart to the ERF in the pure resonance case, when $p(a) = 0$. It is called the *Resonant Response Formula (RRF)*; essentially, in the second order equation

$$mx'' + bx' + kx = Be^{at}$$

with characteristic polynomial p, if $p(a) = 0$ and $p'(a) \neq 0$, then

$$x(t) = \frac{B}{p'(a)} t e^{at}$$

is a *particular* solution.

A.2.16 Stability

The notion of Stability
A system is called *stable* if its long-term behaviour does not depend significantly on the initial conditions.

In this context
In terms of DEs, the simplest spring-mass system or RLC-circuit is represented as

$$a_0 y'' + a_1 y' + a_2 y = r(t), \quad a_i \text{ constants, } t = \text{time.}$$

By the theory of inhomogeneous equations, the general solution to this is of the form

$$y = c_1 y_1 + c_2 y_2 + y_p, \quad c_1, c_2 \text{ arbitrary constants}$$

where y_p is a *particular solution*, and $c_1 y_1 + c_2 y_2$ is the *complementary function* (the general solution to the associated homogeneous equation).

See that the initial conditions determine the exact values of c_1 and c_2. (the same initial conditions might lead to different constants for different particular solutions, but considering a fixed particular solution, they are controlled by the initial conditions) See that

$$\textit{the system is stable} \iff \forall c_1, c_2 \quad \lim_{t \to \infty} (c_1 y_1 + c_2 y_2) = 0$$

If the ODE is stable, the two parts of the solution can be classified as

$$y_p = \textit{steady-state solution}, \quad c_1 y_1 + c_2 y_2 = \textit{transient}$$

The effects of transient term, which depends on the initial conditions, disappear over time. The steady state term therefore more and more closely represents the response of the system as time goes to ∞, regardless of initial conditions.
(next page)

Conditions for stability (second order case)
First we consider the second order case. See that stability concerns just the behaviour of the solutions to the associated homogeneous equation

$$a_0 y'' + a_1 y' + a_2 y = 0$$

(the forcing term only determines the particular solution. what matters for stability is that the homogeneous solution decays to 0—so the response resembles the particular solution regardless of constants of integration) See that the conditions can be split into three cases, summarised as follows:

roots	(homogeneous) solution	condition for stability
$r_1 \neq r_2$	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1 < 0, r_2 < 0$
$r_1 = r_2$	$e^{r_1 t}(c_1 + c_2 t)$	$r_1 < 0$
$a \pm ib$	$e^{at}(c_1 \cos bt + c_2 \sin bt)$	$a < 0$

Root, coefficient form for second order case
The three cases can be summarised as (root form)

$$a_0 y'' + a_1 y' + a_2 y = r(t) \text{ is stable } \iff \begin{array}{l}\text{all roots of } a_0 r^2 + a_1 r + a_2 = 0 \\ \text{have negative real part.}\end{array}$$

it can be further shown that (coefficient form) *assuming* $a_0 > 0$*:*

$$a_0 y'' + a_1 y' + a_2 y = r(t) \text{ is stable } \iff a_0, a_1, a_2 > 0$$

See that the characteristic equation can be written as

$$r^2 + \frac{a_1}{a_0} r + \frac{a_2}{a_0} = r^2 + br + c = 0$$

by the quadratic formula

$$(-b \pm \sqrt{b^2 - 4c})/2$$

First considering the case where both roots are complex, then the *real* part of each of them is $-b/2$. The condition for stability is that the real part is negative, therefore for stability we must have $b > 0$. A similar principle applies for the case of repeated roots, where the real part is again given by $-b/2$, so $b > 0$ for stability.

Now consider the case of two real roots. See that $(-b + \sqrt{b^2 - 4c})/2 > 0$ if $b < \sqrt{b^2 - 4c}$, which only occurs when $c < 0$ (assuming the square root gives a real result; also the other root is always less than 0). Therefore to have both real roots be negative we must have $c > 0$.

Assuming $a_0 > 0$, we must have $a_1, a_2 > 0$.
(next page)

Stability of higher order ODEs (root form only)
The stability criterion in the root form can also be applied to higher-order ODEs with constant coefficients:

$$(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n) y = f(t)$$

where the characteristic equation looks like

$$a_0 r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0$$

The real and complex roots of the characteristic equation give rise to solutions to the associated homogeneous equation just as they do in the second order case (for a k-fold repeated root, one gets additional solutions by multiplying $1, t, t^2, \ldots, t^{k-1}$).

Thus the stability criterion for the second order case can be extrapolated to the higher order case:

$$n\text{-th order ODE is stable} \iff \text{all roots have negative real parts}$$

A.2.17 $p(D)$ Notation, Gain, Phase lag, Complex Gain

$p(D)$ Notation
Consider the typical linear constant coefficient DE

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 x' + a_0 x = q(t)$$

as usual, this has the characteristic polynomial

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

We write $D = \frac{d}{dt}$ to represent the operation of differentiation applied to functions of t, where for instance $Dx = \frac{dx}{dt}$. Similarly we have the representation $D^2 = \frac{d^2}{dt^2}$ for differentiation twice. See that this notation can be fit into the characteristic polynomial:

$$p(D) = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0$$

which allows us to represent the initial DE as

$$p(D)x = q$$

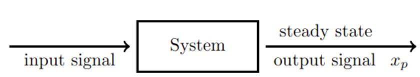
giving us an efficient representation.

Context for Gain, Phase lag
We have the *transience theorem*: All solutions $x = x(t)$ to the linear homogeneous constant coefficient DE

$$p(D)x = 0$$

decay to 0 as $t \to \infty$ exactly when the roots r of the characteristic polynomial $p(s)$ have negative real part.

In this case the homogeneous solutions are called *transients*; by superposition, should we fix a driving term, and therefore a particular solution, all solutions then converge to the *same* solution as t gets large, and we say that the DE is *stable*. Therefore if we have a system modelled by a stable equation, and are only interested in what happens after the transients have died down, we can ignore the initial condition:



The solutions converge to the particular solution x_p. When the input signal is sinusoidal, recall that the particular solution will also be sinusoidal. Here we review this.
(next page)

Review of Complex replacement
We find the particular solution using the ERF, where the DE

$$p(D)x = Be^{at}$$

has the particular solution

$$x_p = \frac{Be^{at}}{p(a)}$$

provided $p(a) \neq 0$ (this corresponds to the resonant case). Given sinusoidal input:

$$p(D)x = B\cos(\omega t)$$

We consider the complex equation since $B\cos(\omega t) = \mathrm{Re}(Be^{i\omega t})$:

$$p(D)z = Be^{i\omega t}, \quad x = \mathrm{Re}(z)$$

The ERF gives

$$z_p = \frac{B}{p(i\omega)}e^{i\omega t} \implies x_p = B\,\mathrm{Re}\left(\frac{e^{i\omega t}}{p(i\omega)}\right)$$

Thus

$$x_p = \frac{B}{|p(i\omega)|}\cos(\omega t - \phi)$$

where $\phi = \mathrm{Arg}(p(i\omega))$. This is the particular, and therefore steady-state (periodic) solution.

Gain, Phase lag
Compare the periodic input $q(t) = B\cos(\omega t)$ and its periodic output $x_p(t) = \frac{B}{|p(i\omega)|}\cos(\omega t - \phi)$. See that the amplitude is scaled by $1/|p(i\omega)|$ and the output sinusoid is shifted by an angle $\phi = \mathrm{Arg}(p(i\omega))$ relative to the input sinusoid. This motivates the following definitions: for a constant coefficient linear DE

$$P(D)x = q(t) \quad \text{with sinusoidal input } q(t)$$

1. The *gain* is defined to be the *ratio of the amplitude of the output sinusoid to the amplitude of the input sinusoid.*

2. The *phase lag* is defined to be the *angle by which the output sinusoid is shifted relative to the input sinusoid.*

Complex Gain
When solving by complex replacement we had $x_p = \mathrm{Re}(z_p)$ where $z_p(t)$ is the complex solution to $p(D)z = Be^{i\omega t}$:

$$z_p = \frac{B}{p(i\omega)}e^{i\omega t}$$

see that the complexified exponential input can also be compared to the complex output. We therefore define the *complex gain*, in this case it is $1/p(i\omega)$.

A.2.18 Polynomial Input: Undetermined Coefficients

Definition
A *polynomial* is a function of the form

$$q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

where the largest k for which $a_k \neq 0$ is the *degree* of $q(x)$ (0 is also a polynomial with no degree).

Theorem. (Undetermined coefficients): If $p(0) \neq 0$, and $q(x)$ is a polynomial of degree n, then

$$p(D)y = q(x)$$

has exactly one solution which is a polynomial, and is of degree n.

Instructive example
We now show this with an instructive example. Consider finding a particular solution y_p to

$$y'' + 3y' + 4y = 4x^2 - 2x$$

Consider a polynomial solution with the same degree as the input—of the form $y_p = Ax^2 + Bx + C$. See that calculating its derivatives gives us

$$\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}$$

Substituting into the DE gives us

$$\begin{aligned} &2A + 3(2Ax + B)a + 4(Ax^2 + Bx + C) \\ =&4Ax^2 + (4B + 6A)x + (4C + 3B + 2A) = 4x^2 - 2x \end{aligned}$$

Comparing coefficients we have $A = 1, B = -2, C = 1$. So we obtain the particular solution $y_p = x^2 - 2x + 1$.
(next page)

Another example
Consider
$$y'' + 5y' + 4y = 2x + 3$$
We have a *trial solution* of the form $y_p = Ax + B$ (same degree as input). Substituting gives us
$$0 + 5A + 4(Ax + B) = 2x + 3$$
Comparing coefficients and solving gives us $A = 1/2, B = 1/8$. We obtain the particular solution
$$y_p = \frac{1}{2}x + \frac{1}{8}$$
Should we want a general solution we add the homogeneous solution.

Case where $p(0) = 0$ or homogeneous DE has polynomial solutions
Should $p(0) = 0$, then coefficient of y in the equation is 0, this looks like
$$a_2 y'' + a_1 y' = q$$
See that in this case the homogeneous DE $p(D)$ has polynomial solutions (constants are polynomials, and any constant y_p would satisfy the homogeneous differential equation.

In this case, the polynomial solution to the inhomogeneous DE $p(D)y = q$ would be of *higher degree* than that of $q(x)$; consider the example
$$y'' + y' = x + 1$$
A particular solution of the form $y_p = Ax + B$ (same degree as input) doesn't work—substituting gives $0 + A = x + 1$. We fix this by considering a solution one degree higher $y_p = Ax^2 + Bx$, where substituting gives us
$$2Ax + (2A + B) = x + 1 \implies A = 1/2, B = 0 \implies y_p = \frac{1}{2}x^2$$
Another example
Consider
$$y''' + 3y'' = x^2 + x$$
Lowest order derivative is 2, so we try increasing the degree of the proposed solution by 2: $y_p = Ax^4 + Bx^3 + Cx^2$. Substituting, we have
$$(24Ax + 6B) + 3(12Ax^2 + 6Bx + 2C) = x^2 + x$$
so $A = 1/36, B = 1/27, C = -1/27$.

A.2.19 Linear Operators

Polynomial differential operator
The general linear ODE with constant coefficients of order n for a function $y = y(t)$:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = q(t)$$

can be written compactly using the differentiation operator $D = \frac{d}{dt}$:

$$p(D)y = q(t)$$

where

$$p(D) = D^n + a_1 D^{n-1} + \ldots + a_n$$

we call $p(D)$ a *polynomial differential operator with constant coefficients.*

Properties
We now state a few rules surrounding these operators. We will assume that the functions involved are sufficiently differentiable so that the operators can be applied.

Sum rule: If $p(D)$ and $q(D)$ are polynomial operators, then for any (sufficiently differentiable) function u,

$$[p(D) + q(D)]u = p(D)u + q(D)u$$

Linearity rule: If f, g are functions and c_1, c_2 constants,

$$p(D)(c_1 f + c_2 g) = c_1 p(D)f + c_2 p(D)g$$

Proof of linearity rule: This follows from the linearity of differentiaton, see that

$$D(c_1 f + c_2 g) = (c_1 f + c_2 g)' = c_1 f' + c_2 g' = c_1 Df + c_2 Dg$$

also see that taking the second or higher derivative also follows the linearity rule. That is,

$$D^n(c_1 f + c_2 g) = \frac{d^n}{dt}(c_1 f + c_2 g) = c_1 f^{(n)} + c_2 g^{(n)} = c_1 D^n f + c_2 D^n g$$

We can scale the linear operator D^n by a (a constant/function/or whatever independent variable) and it stays linear:

$$aD^n(c_1 f + c_2 g) = a\frac{d^n}{dt}(c_1 f + c_2 g) = c_1 a f^{(n)} + c_2 a g^{(n)} = c_1 a D^n f + c_2 a D^n g$$

Finally we can combine these operators into a polynomial operator

$$D^n + a_1 D^{n-1} + \ldots + a_n$$

which clearly still obeys the linearity rule (consider $c_1 f + c_2 g$ a single function u and apply sum rule to each term in the differential polynomial operator).
(next page)

Multiplication rule: If $p(D) = g(D)h(D)$ as polynomials in D, then

$$p(D)u = g(D)(h(D)u)$$

First consider the simple operator aD^k and see that

$$D^m(aD^k u) = aD^{m+k}u$$

(Note that in this case a must be a constant.) This then extends to general polynomial operators $h(D)$ by the linearity rule.

An important corollary of the multiplication property is that *polynomial operators with constant coefficients commute*:

$$g(D)(h(D)u) = h(D)(g(D)u)$$

This can be shown since $g(D)h(D) = h(D)g(D) = p(D)$, so both sides of the equation are equal to $p(D)u$.

Substitution rule

$$p(D)e^{at} = p(a)e^{at}$$

See that by repeated differentiation

$$De^{at} = ae^{at}, D^2e^{at} = a^2e^{at}, \ldots, D^ke^{at} = a^ke^{at}$$

therefore;

$$(D^n + c_1D^{n-1} + \ldots + c_n)e^{at} = (a^n + c_1a^{n-1} + \ldots + c_n)e^{at}$$

This is the substitution rule.
(next page)

Exponential-shift rule: This handles expressions such as $t^k e^{at}$ and $t^k \sin(at)$. Let $u = u(t)$. Then

$$p(D)e^{at}u = e^{at}p(D+a)u$$

Proof: First see that when $p(D) = D$, by product rule

$$De^{at}u(t) = e^{at}Du(t) + ae^{at}u(t) = e^{at}(D+a)u(t)$$

To show this rule is true for D^k, we apply the differentiation repeatedly:

$$\begin{aligned} D^2e^{at}u = D(De^{at}u) &= D(e^{at}(D+a)u) \\ &= e^{at}(D+a)((D+a)u) \\ &= e^{at}(D+a)^2u \end{aligned}$$

in the same way,

$$\begin{aligned} D^3e^{at}u = D(D^2e^{at}u) &= D(e^{at}(D+a)^2u) \\ &= e^{at}(D+a)((D+a)^2u) \\ &= e^{at}(D+a)^3u \end{aligned}$$

and so on; see therefore that

$$D^ke^{at}u = e^{at}(D+a)^ku$$

and

$$(D^n + c_1D^{n-1} + \ldots + c_n)e^{at}u(t) = ((D+a)^n + c_1(D+a)^{n-1} + \ldots + c_n)e^{at}u(t)$$

A.2.20 Time Invariance

Definition
In the case of *constant coefficient* operators $p(D)$, there is an important relationship between solutions of to $p(D)x = q(t)$ for input signals $q(t)$. The following result shows why these operators are called 'Linear Time Invariant' (or LTI).

Translation invariance: If $p(D)$ is a constant-coefficient differential operator and $p(D)x = q(t)$, then $p(D)y = q(t - c)$, where $y(t) = x(t - c)$.

This is the 'time invariance' of $p(D)$.

Example
Consider two systems:

$$\text{System A: } \frac{1}{t}y(t) = x(t) \implies y(t) = tx(t)$$
$$\text{System B: } \frac{1}{10}y(t) = x(t) \implies y(t) = 10x(t)$$

System A is not time-invariant, System B is. First consider system A with a delayed input $x_d(t) = x(t + \delta)$. We have the corresponding response y_1 as

$$y_1(t) = tx_d(t) = tx(t + \delta)$$

Now consider delaying the output by δ

$$y_2(t) = y(t + \delta) = (t + \delta)x(t + \delta)$$

Clearly $y_1(t) \neq y_2(t)$, therefore the system is not time-invariant. Now consider the same for system B; for the delayed input:

$$y_1(t) = 10x(t + \delta)$$

Now with the delayed output

$$y_2(t) = y(t + \delta) = 10x(t + \delta)$$

$y_1(t) = y_2(t)$, therefore the system is time-invariant.

Example 2
Suppose that we know that $x_p(t) = \sqrt{2}\sin(t/2 - \pi/4)$ is a solution to the DE

$$2\ddot{x} + \dot{x} + x = \sin(t/2)$$

Should we want to find a solution y_p to

$$2\ddot{x} + \dot{x} + x = \sin(t/2 - \pi/3)$$

by translation-invariance we immediately have

$$y_p = \sqrt{2}\sin(t/2 - \pi/4 - \pi/3) = \sqrt{2}\sin(t/2 - 7\pi/12)$$

Intuitively:
A linear map is a function between two vector spaces where addition and scalar multiplication are preserved; a function $T : V \to W$ is linear if

$$T(u + v) = T(u) + T(v)$$
$$T(av) = aT(v)$$

Linearity in the context of a system means that the relation between the input $x(t)$ and the output $y(t)$, both being regarded as functions, is a linear mapping: if a is a constant then the system output to $ax(t)$ is $ay(t)$ (so $T(ax) = aT(x) = ay$).

If $x'(t)$ is a further input with system output $y'(t)$ then the output of the system to an input $x(t) + x'(t)$ is $y(t) + y'(t)$ (so $T(u + v) = T(u) + T(v)$). (this is also superposition)

Time invariance means that whether we apply an input to the system now or T seconds from now, the output would be identical except for a time delay of T seconds. That is, if the output due to input $x(t)$ is $y(t)$, then the output due to $x(t - T)$ is $y(t - T)$. The system is time invariant because the output does not depend on the particular time the input is applied.

A.2.21 Generalised ERF with proof

We can solve a LTI DE $p(D)x = q(t)$ with exponential input $q(t) = Be^{at}$ even when $p(a) = 0$ using the *generalised exponential response* formula:

Definition
Generalised ERF: Let $p(D)$ be a polynomial operator with constant coefficients, and $p^{(s)}$ its s-th derivative. Then

$$p(D)x = Be^{at}, \quad \text{where } a \text{ is real or complex}$$

has the particular solution

$$x_p = \begin{cases} \frac{Be^{at}}{p(a)} & \text{if } p(a) \neq 0 \\ \frac{Bte^{at}}{p'(a)} & \text{if } p(a) = 0 \text{ and } p'(a) \neq 0 \\ \frac{Bt^2e^{at}}{p''(a)} & \text{if } p(a) = p'(a) = 0 \text{ and } p''(a) \neq 0 \\ & \vdots \\ \frac{Bt^se^{at}}{p^{(s)}(a)} & \text{if } a \text{ is an } s\text{-fold zero} \end{cases}$$

Proof
The first case can be shown fairly easily:

$$p(D)x_p = p(D)\frac{e^{at}}{p(a)} = \frac{1}{p(a)}p(D)e^{at} = \frac{p(a)e^{at}}{p(a)} = e^{at}$$

Now we prove the last general case (since the other cases are just instances of it). First understand that saying the polynomial $p(D)$ has the number a as an s-fold zero is the same as saying $p(D)$ has a factorization

$$p(D) = q(D)(D-a)^s$$

(both $p(D)$ and $q(D)$ are polynomials; this just says $(-a)$ can be factored out s times). We will first prove that the above implies

$$p^{(s)}(a) = q(a)s!$$

(next page)

Proof cont.
We show

$$p(D) = q(D)(D-a)^s$$

implies

$$p^{(s)}(a) = q(a)s!$$

Say that we factor out $(D-a)$ as many times as possible. We know that $q(D)$ is still a polynomial. Say of degree k:

$$C_0 + C_1 D + \ldots + C_k D^k$$

we know that $(D-a)$ can no longer be factored out. Now see that we can write $q(D)$ *in terms of* $(D-a)$:

$$q(D) = c_0 + c_1(D-a) + \ldots + c_k(D-a)^k$$

which is still a polynomial. See that $q(a) = c_0$ so

$$q(D) = q(a) + c_1(D-a) + \ldots + c_k(D-a)^k$$

We can then write $p(D)$ as

$$p(D) = q(a)(D-a)^s + c_1(D-a)^{s+1} + \ldots + c_k(D-a)^{s+k}$$

taking the derivative s times, we obtain

$$p^{(s)}(D) = q(a)s! + (\text{positive powers of } D-a)$$

where substituting a gives us

$$p^{(s)}(D) = q(a)s!$$

(next page)

Proof cont.
Using the result
$$p^{(s)}(D) = q(a)s!$$
we can use the exponential-shift rule to prove the resonant response formulas:
$$\begin{aligned} p(D)\frac{t^s e^{at}}{p^{(s)}(a)} &= \frac{e^{at}}{p^{(s)}(a)} p(D+a)t^s \\ &= \frac{e^{at}}{p^{(s)}(a)} q(D+a)D^s t^s \\ &= \frac{e^{at}}{q(a)s!} q(D+a)s! \\ &= \frac{e^{at}}{q(a)s!} q(a)s! = e^{at} \end{aligned}$$
where the last line follows from
$$q(D+a)s! = (q(a) + c_1 D + \ldots + c_k D^k)s! = q(a)s!$$
Note that by linearity we could have stated the formula with a factor of B in the input and a corresponding factor of B to the output. That is, the DE
$$p(D)x = Be^{at}$$
has a particular solution
$$x_p = \frac{Be^{at}}{p(a)}, \quad \text{if } p(a) \neq 0 \text{ etc.}$$

A.2.22 Resonant response formula, example

Recall the usual case of the ERF, where a solution to

$$p(D)x = Be^{at}$$

is given by

$$x_p = \frac{Be^{at}}{p(a)} \quad \text{provided that } p(a) \neq 0$$

We generalised this formula for the case when $p(a) = 0$. Say $p'(a) \neq 0$, a particular solution is given by

$$x_p = \frac{Bte^{at}}{p'(a)} \quad \text{if } p(a) = 0 \text{ and } p'(a) \neq 0$$

we call this the *Resonant Response Formula.*

Example
Consider $x'' + 4x = 2\cos 2t$. We use complex replacement and the ERF to get a complex particular solution, then take the real part. We have

$$z'' + 4z = 2e^{2it}, \quad x_p = \text{Re}(z_p)$$

and characteristic polynomial $p(s) = s^2 + 4$. See that $p(2i) = 0$ (resonant case). So we use the resonant response formula; we have $p'(2i) = 4i \neq 0$:

$$z_p = \frac{2te^{2it}}{4i}$$

Taking the real part gives us the desired particular solution

$$x_p = \frac{1}{2}t\sin 2t$$

A.2.23 Pure resonance

Here we look at the pure resonant case for a second-order LTI DE. We use the spring-mass model for a physical interpretation of the system, but obviously the same behaviour applies to other systems with the same DE:

$$x'' + w_0^2 x = F_0 \cos \omega t$$

We solve a system like this with complex replacement:

$$z'' + w_0^2 z = F_0 e^{i\omega t}, \quad x = \mathrm{Re}(z)$$

Characteristic polynomial:

$$p(r) = r^2 + \omega_0^2 \implies p(i\omega) = \omega_0^2 - \omega^2$$

ERF:

$$z_p = \begin{cases} \frac{F_0 e^{i\omega t}}{p(i\omega)} = \frac{F_0 e^{i\omega t}}{\omega_0^2 - \omega^2} & \text{if } \omega \neq \omega_0 \\ \frac{F_0 t e^{i\omega t}}{p'(i\omega)} = \frac{F_0 t e^{i\omega t}}{2i\omega} & \text{if } \omega = \omega_0 \end{cases}$$

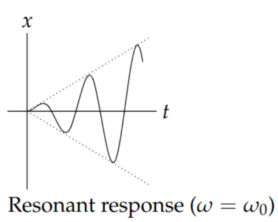
Taking real part

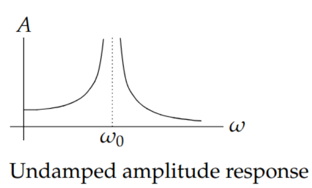
$$x_p = \begin{cases} \frac{F_0 \cos \omega t}{\omega_0^2 - \omega^2} & \text{if } \omega \neq \omega_0 \\ \frac{F_0 t \sin \omega_0 t}{2\omega_0} & \text{if } \omega = \omega_0 \end{cases}$$

See that the amplitude $A(\omega)$ is given by

$$A(\omega) = \left| \frac{F_0}{\omega_0^2 - \omega} \right|$$

See that as ω gets closer to ω_0 the amplitude increases (it is similar to the damped amplitude response except that the peak is infinitely high).





When $\omega = \omega_0$ we have $x_p = \frac{F_0 t \sin \omega_0 t}{2\omega_0}$. This is called *pure resonance*. The frequency ω_0 is called the *resonant* or *natural* frequency of the system. Notice that the response is *oscillatory but not periodic*—the amplitude keeps growing in time.

A.2.24 Recap and Practical resonance

Recap: Gain, Phase lag, Complex gain
Given a second order linear CC DE with a *sinusoidal* driving force $B\cos\omega t$:

$$mx'' + bx' + kx = B\cos\omega t$$

An equation like this is solved using complex replacement:

$$mz'' + bz' + kz = Be^{i\omega t}, \quad x = \mathrm{Re}(z)$$

characteristic equation:

$$p(s) = ms^2 + bs + k$$

ERF:

$$z_p = \frac{Be^{i\omega t}}{p(i\omega)} = \frac{Be^{i\omega t}}{k - m\omega^2 + ib\omega}$$

So

$$x_p = \mathrm{Re}(z_p) = \frac{B}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}\cos(\omega t - \phi)$$

where $\phi = \mathrm{Arg}(p(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k-m\omega^2}\right)$. See that the numerator of the inverse tangent is positive—ϕ must be between 0 and π (in first or second quadrants).

Recall that the *complex gain* is defined as the ratio between the output and input amplitudes in the *complexified* equation:

$$\tilde{g}(\omega) = \frac{1}{p(i\omega)} = \frac{1}{k - m\omega^2 + ib\omega}$$

The *gain* is defined as the ratio between the output and input amplitudes in the *real* equation:

$$g = g(\omega) = \frac{1}{|p(i\omega)|} = \frac{1}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$$

The *phase lag* is

$$\phi = \phi(\omega) = \mathrm{Arg}(p(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$$

from this we also have the *time lag* as ϕ/ω.

We call the gain $g(\omega)$ the *amplitude response* of the system, the phase lag $\phi(\omega)$ the *phase response* of the system, and both of them collectively as the *frequency response* of the system.
(next page)

Frequency response and Practical Resonance
We have a periodic solution to the equation

$$mx'' + bx' + kx = B\cos(\omega t)$$

given by $x_p = gB\cos(\omega t - \phi)$, where g is the gain

$$g = g(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

and ϕ is the phase lag

$$\phi = \phi(\omega) = \mathrm{Arg}(p(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$$

The gain/amplitude response is a function of ω. It tells us the size of the system's response to the given input frequency. If the amplitude has a peak at ω_r we call this the *practical resonance frequency*:

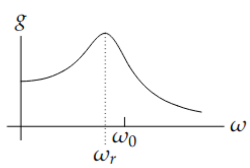


Fig 1a. Small b (has resonance).

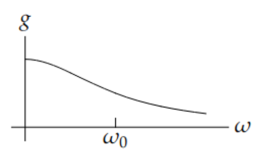


Fig 1b. Large b (no resonance)

If the damping b gets too large, there will be no peak and therefore no practical resonance. (note this is different from overdamping and critical damping, where the response is not sinusoidal and therefore does not have a measureable gain)

Finding the practical resonant frequency
Practical resonance occurs at the frequency ω_r where $g(\omega)$ has a maximum. Accordingly we look for the *minimum* of

$$f(\omega) = (k - m\omega^2)^2 + b^2\omega^2$$

Setting $f'(\omega) = 0$ and solving gives

$$\begin{aligned} f'(\omega) &= -4m\omega(k - m\omega^2) + 2b^2\omega \\ &= 2\omega(b^2 - 2mk - 2m^2\omega^2) = 0 \end{aligned}$$

so we have $\omega = 0$ or $(b^2 - 2mk - 2m^2\omega^2) = 0$:

$$\begin{aligned} m^2\omega^2 &= mk - b^2/2 \\ \omega_r &= \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \end{aligned}$$

See that there is a practical resonant frequency when $mk - b^2/2 > 0$.
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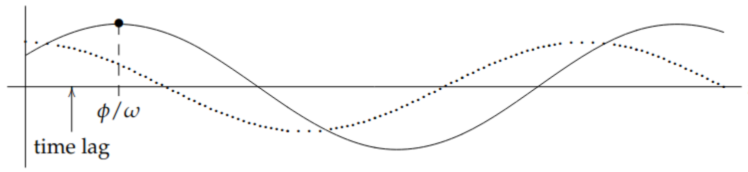
Phase lag visualised

The dotted line is the input and the solid line is the response:



The damping causes a lag between the input and output. In radians, the angle ϕ is called the *phase lag* and the time ϕ/ω the *time lag*.

A.3 Fourier series and Laplace Transform

A.3.1 Periodic functions—Definitions

Definition
A function $f(t)$ is *periodic* with *period* $P > 0$ if

$$f(t + P) = f(t) \quad \text{for all } t$$

Example
$f(t) = \sin(2t)$ is periodic with period $P = \pi$. This can be seen because for all t,

$$f(t + \pi) = \sin(2(t + \pi)) = \sin(2t + 2\pi) = \sin(2t) = f(t)$$

Notice however that in the this example $f(t) = \sin(2t)$ also has period $P = n\pi$ for any integer $n = 1, 2, 3 \ldots$.

Base period
Most periodic functions have a *minimal period*—which the smallest value that satisfies the definition of the period. This is also called the *base period*. An exception to this is the constant function, where every value of $P > 0$ is a period.

Windows
To fully describe a periodic function we only need to specify the period and the value of the function over one full period. We call an interval containing one full period a *window*. Typical choices are $[-P/2, P/2)$ and $[0, P)$, but any interval of length P would work.

A.3.2 Fourier series—Definition and Coefficients

If the $f(t)$ is periodic (for now considering period 2π) ,we can express the function (where it is continuous) as an infinite sum of sines and cosines which all have period 2π.

Theorem (Fourier):
Suppose $f(t)$ has period 2π then we have

$$
\begin{aligned}
f(t) \approx & \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + \dots \\
& + b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + \dots \\
= & \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nt) + b_n\sin(nt)]
\end{aligned}
$$

where the coefficients $a_0, a_1, \dots$ and $b_1, b_2, \dots$ are computed by

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt
\end{aligned}
$$

See that all the sinusoids have 2π as a period. The series is called a *Fourier series*; and the coefficients $a_0, a_1, \dots, b_1, b_2 \dots$ are called the *Fourier coefficients* of $f(t)$.

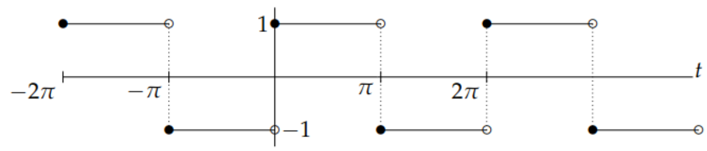
We used an approximation instead of an equality in the first statement because the two sides may differ for discontinuities in $f(t)$. We use an equal sign from now on.

A.3.3 Example: Square wave

Definition
Say we have $f(t)$ as the *square wave* with period 2π, defined as

$$f(t) = \begin{cases} -1 & \text{for } -\pi \leq t < 0 \\ 1 & \text{for } 0 \leq t < \pi \end{cases}$$

Plotted:



Fourier series
Consider computing its Fourier series. Recall

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)]$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

First see that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt = 0$$

now for the other coefficients, see that we can split the integral to get around the discontinuity in the square wave function

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt = \frac{1}{\pi} \int_{-\pi}^{0} (-1) \cos(nt)\, dt + \frac{1}{\pi} \int_{0}^{\pi} (1) \cos(nt)\, dt$$

and so

$$a_n = -\frac{\sin(nt)}{n\pi} \bigg|_{-\pi}^{0} + \frac{\sin(nt)}{n\pi} \bigg|_{0}^{\pi} = 0 \quad \text{for } n \neq 0$$

(or see visually that both integrals are 0)
(next page)

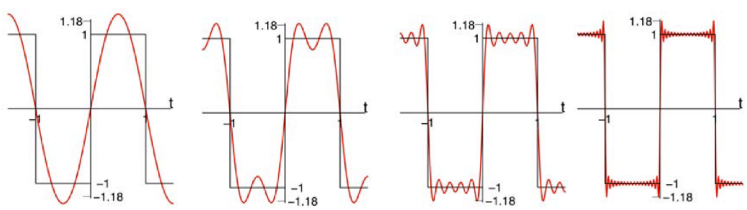
Fourier series cont.
Likewise

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin(nt)\,dt \\
&= \left.\frac{\cos(nt)}{n\pi}\right|_{-\pi}^{0} - \left.\frac{\cos(nt)}{n\pi}\right|_{0}^{\pi} = \frac{1-\cos(-n\pi)}{n\pi} - \frac{\cos(n\pi)-1}{n\pi} \\
&= \frac{2}{n\pi}(1-\cos(-n\pi)) = \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}
\end{aligned}
$$

The last step comes from the the simplification $\cos(n\pi) = (-1)^n$. With that we have the fourier series for $f(t)$ as

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right)$$

Illustrated convergence of Fourier series
Here we plot the sums of the first N terms of the series for $N = 1, 3, 9, 33$:



See that the more terms considered, the better the series approximates the candidate function.

Intuition for odd and even functions
It turns out that for *odd* $f(t)$, meaning $f(-t) = -f(t)$, such functions only have sines (which are also odd functions) in their Fourier series, it resembles

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

Similarly for even functions, where $f(-t) = f(t)$, their Fourier series consist of cosines:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$$

For instance see that in the case of the square wave (an odd function), the Fourier series consists of just sines.

A.3.4 Fourier series for functions with period 2L

Generalising
Suppose that we have a periodic function $f(t)$ with arbitrary period $P = 2L$; generalising the special case $P = 2\pi$ by a re-scaling of the interval $(-\pi, \pi)$ to $(-L, L)$ allows us to write down the general Fourier series and Fourier coefficient formulas:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\frac{\pi}{L}t\right) + b_n \sin\left(n\frac{\pi}{L}t\right) \right]$$

with Fourier coefficients given by

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(t)\, dt \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(n\frac{\pi}{L}t\right) dt \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(n\frac{\pi}{L}t\right) dt \end{aligned}$$

The number $L = P/2$ is called the *half-period*.

Example: Triangle wave
Let $f(t)$ be a period 2 function, defined on the window $[-1, 1)$ by $f(t) = |t|$; this function is called a *triangle wave* or *continuous sawtooth function*:

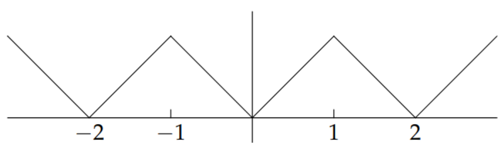


Figure 1: The period 2 triangle wave.

Consider computing its Fourier series. In this case the period is $P = 2$, and therefore the half-period is $L = 1$. Computing a_0:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t)\, dt = \int_{-1}^{1} |t|\, dt = 2 \int_{0}^{1} t\, dt = 1$$

Now for a_n for $n \neq 0$ (and integrating by parts):

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(n\frac{\pi}{L}t\right) dt \\ &= \frac{1}{1} \int_{-1}^{1} |t| \cos(n\pi t)\, dt = 2 \int_{0}^{1} t \cos(n\pi t)\, dt \\ &= 2 \left(\frac{t \sin(n\pi t)}{n\pi} + \frac{\cos(n\pi t)}{n^2\pi^2} \right) \Bigg|_0^1 = \frac{2}{n^2\pi^2}((-1)^n - 1) = \begin{cases} -\frac{4}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \end{aligned}$$

(next page)

Triangle wave (cont.)
Now for the sine coefficients.

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(n\frac{\pi}{L}t\right)dt = \frac{1}{1}\int_{-1}^{1}|t|\sin(n\pi t)\,dt \\
&= \int_{-1}^{0} -t\sin(n\pi t)\,dt + \int_{0}^{1} t\sin(n\pi t)\,dt \\
&= -\left(-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2}\right)\Bigg|_{-1}^{0} + \left(-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2}\right)\Bigg|_{0}^{1} \\
&= \frac{\cos(n\pi)}{n\pi} - \frac{\cos(n\pi)}{n\pi} = 0
\end{aligned}$$

Thus the Fourier series for $f(t)$ is

$$\begin{aligned}
f(t) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{\cos(3\pi t)}{3^2} + \frac{\cos(5\pi t)}{5^2} + \cdots\right) \\
&= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\text{ odd}} \frac{\cos(n\pi t)}{n^2}
\end{aligned}$$

A.3.5 Orthogonality Rules Justification

Here we justify a few integral formulas, called the orthogonality relations (note the integral can be viewed as a continuous dot product between two sinusoids, thus the link to the name). n, m are integers:

$$\frac{1}{L}\int_{-L}^{L} \cos\left(n\frac{\pi}{L}t\right)\cos\left(m\frac{\pi}{L}t\right)\,dt = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \\ 2 & n = m = 0 \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L} \cos\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = 0$$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \\ 0 & n = m = 0 \end{cases}$$

First relation
First see that, denoting $a = n\pi/L$, $b = m\pi/L$, n, m integers:

$$\begin{aligned} \int_{-L}^{L} e^{i(a+b)t} + e^{-i(a+b)t}\,dt &= \left[\frac{e^{i(a+b)t}}{i(a+b)} - \frac{e^{-i(a+b)t}}{i(a+b)}\right]_{-L}^{L} \\ &= \left.\frac{2\sin((a+b)t)}{i(a+b)}\right|_{-L}^{L} \\ &= \left.\frac{2L\sin((n\pi+m\pi)t/L)}{i(n\pi+m\pi)}\right|_{-L}^{L} \\ &= \frac{4L\sin(n\pi+m\pi)}{i(n\pi+m\pi)} = \begin{cases} 0 & n = m \neq 0 \\ 0 & n \neq m \end{cases} \end{aligned}$$

And for $n = m = 0$:

$$\int_{-L}^{L} e^{i(a+b)t} + e^{-i(a+b)t}\,dt = \int_{-L}^{L} e^0 + e^0\,dt = \int_{-L}^{L} 2\,dt = 4L$$

so

$$\int_{-L}^{L} e^{i(a+b)t} + e^{-i(a+b)t}\,dt = \begin{cases} 0 & n = m \neq 0 \\ 0 & n \neq m \\ 4L & n = m = 0 \end{cases}$$

(next page)

First relation (cont.)
Then see that

$$\begin{aligned}
\int_{-L}^{L} e^{i(a-b)t} + e^{-i(a-b)t}\, dt &= \left[\frac{e^{i(a-b)t}}{i(a-b)} - \frac{e^{-i(a-b)t}}{i(a-b)}\right]_{-L}^{L} \\
&= \left.\frac{2\sin((a-b)t)}{i(a-b)}\right|_{-L}^{L} \\
&= \left.\frac{2L\sin((n\pi - m\pi)t/L)}{i(n\pi - m\pi)}\right|_{-L}^{L} \\
&= \frac{4L\sin(n\pi - m\pi)}{i(n\pi - m\pi)} = 0 \quad \text{for } n \neq m
\end{aligned}$$

and for $a = b \neq 0$ or $a = b = 0$

$$\int_{-L}^{L} e^{i(a-b)t} + e^{-i(a-b)t}\, dt = \int_{-L}^{L} e^{0} + e^{0}\, dt = \int_{-L}^{L} 2\, dt = 4L$$

so

$$\int_{-L}^{L} e^{i(a-b)t} + e^{-i(a-b)t}\, dt = \begin{cases} 4L & n = m \neq 0 \\ 0 & n \neq m \\ 4L & n = m = 0 \end{cases}$$

Finally we prove the first relation:

$$\frac{1}{L}\int_{-L}^{L} \cos\left(n\frac{\pi}{L}t\right)\cos\left(m\frac{\pi}{L}t\right)\, dt = \frac{1}{L}\int_{-L}^{L} \cos(at)\cos(bt)\, dt$$

using the fact that $\cos(x) = (e^{ix} + e^{-ix})/2$

$$\begin{aligned}
\frac{1}{L}\int_{-L}^{L} \cos(at)\cos(bt)\, dt &= \frac{1}{L}\int_{-L}^{L} \frac{e^{iat} + e^{-iat}}{2} \cdot \frac{e^{ibt} + e^{-ibt}}{2}\, dt \\
&= \frac{1}{4L}\int_{-L}^{L} e^{i(a+b)t} + e^{-i(a+b)t} + e^{i(a-b)t} + e^{-i(a-b)t}\, dt
\end{aligned}$$

$$= \underbrace{\frac{1}{4L}\int_{-L}^{L} e^{i(a+b)t} + e^{-i(a+b)t}\, dt}_{=\begin{cases} 0 & n = m \neq 0 \\ 0 & n \neq m \\ 1 & n = m = 0 \end{cases}} + \underbrace{\frac{1}{4L}\int_{-L}^{L} e^{i(a-b)t} + e^{-i(a-b)t}\, dt}_{=\begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \\ 1 & n = m = 0 \end{cases}}$$

so

$$\frac{1}{L}\int_{-L}^{L} \cos\left(n\frac{\pi}{L}t\right)\cos\left(m\frac{\pi}{L}t\right)\, dt = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \\ 2 & n = m = 0 \end{cases}$$

(next page)

Second relation

See that for $a = n\pi/L$, $b = m\pi/L$, n, m integers:

$$\begin{aligned}
\int_{-L}^{L} e^{i(a+b)t} - e^{-i(a+b)t}\, dt &= \left[\frac{e^{i(a+b)t}}{i(a+b)} + \frac{e^{-i(a+b)t}}{i(a+b)}\right]_{-L}^{L} \\
&= \left.\frac{2\cos((a+b)t)}{i(a+b)}\right|_{-L}^{L} \\
&= \frac{2\cos((a+b)L)}{i(a+b)} - \frac{2\cos(-(a+b)L)}{i(a+b)} \\
&= \begin{cases} 0 & n = m \neq 0 \\ 0 & n \neq m \end{cases}
\end{aligned}$$

(since $\cos(-x) = \cos(x)$) and for $n = m = 0$

$$\int_{-L}^{L} e^{i(a+b)t} - e^{-i(a+b)t}\, dt = \int_{-L}^{L} e^0 - e^0\, dt = \int_{-L}^{L} 0\, dt = 0$$

so

$$\int_{-L}^{L} e^{i(a+b)t} - e^{-i(a+b)t}\, dt = \begin{cases} 0 & n = m \neq 0 \\ 0 & n \neq m \\ 0 & n = m = 0 \end{cases}$$

see that by exactly the same line of reasoning:

$$\begin{aligned}
\int_{-L}^{L} e^{i(a-b)t} - e^{-i(a-b)t}\, dt &= \left[\frac{e^{i(a-b)t}}{i(a-b)} + \frac{e^{-i(a-b)t}}{i(a-b)}\right]_{-L}^{L} \\
&= \left.\frac{2\cos((a-b)t)}{i(a-b)}\right|_{-L}^{L} \\
&= \frac{2\cos((a-b)L)}{i(a-b)} - \frac{2\cos(-(a-b)L)}{i(a-b)} \\
&= 0 \quad \text{for } n \neq m
\end{aligned}$$

for $n = m \neq 0$ and $n = m = 0$

$$\int_{-L}^{L} e^{i(a-b)t} - e^{-i(a-b)t}\, dt = \int_{-L}^{L} e^0 - e^0\, dt = \int_{-L}^{L} 0\, dt = 0$$

so

$$\int_{-L}^{L} e^{i(a-b)t} - e^{-i(a-b)t}\, dt = \begin{cases} 0 & n = m \neq 0 \\ 0 & n \neq m \\ 0 & n = m = 0 \end{cases}$$

(next page)

Second relation (cont.)
Finally see that

$$\frac{1}{L}\int_{-L}^{L}\cos\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = \frac{1}{L}\int_{-L}^{L}\cos(at)\sin(bt)\,dt$$

using $\cos(x) = (e^{ix}+e^{-ix})/2$ and $\sin(x) = (e^{ix}-e^{-ix})/(2i)$:

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L}\cos(at)\sin(bt)\,dt &= \frac{1}{L}\int_{-L}^{L}\frac{e^{iat}+e^{-iat}}{2}\cdot\frac{e^{ibt}-e^{-ibt}}{2i}\,dt \\ &= \frac{1}{4Li}\int_{-L}^{L}e^{i(a+b)t}-e^{-i(a+b)t}-e^{i(a-b)t}+e^{-i(a-b)t}\,dt\end{aligned}$$

as derived above,

$$= \underbrace{\frac{1}{4Li}\int_{-L}^{L}e^{i(a+b)t}-e^{-i(a+b)t}\,dt}_{\begin{cases}0 & n=m\neq 0\\ 0 & n\neq m\\ 0 & n=m=0\end{cases}} - \underbrace{\frac{1}{4Li}\int_{-L}^{L}e^{i(a-b)t}-e^{-i(a-b)t}\,dt}_{\begin{cases}0 & n=m\neq 0\\ 0 & n\neq m\\ 0 & n=m=0\end{cases}}$$

so

$$\frac{1}{L}\int_{-L}^{L}\cos\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = 0$$

(next page)

Third relation
Lastly, see that

$$\frac{1}{L}\int_{-L}^{L} \sin\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = \frac{1}{L}\int_{-L}^{L} \sin(at)\sin(bt)\,dt$$

using $\sin(x) = (e^{ix} - e^{-ix})/(2i)$:

$$\begin{aligned}
\frac{1}{L}\int_{-L}^{L} \sin(at)\sin(bt)\,dt &= \frac{1}{L}\int_{-L}^{L} \frac{e^{iat} - e^{-iat}}{2i}\cdot\frac{e^{ibt} - e^{-ibt}}{2i}\,dt \\
&= \frac{1}{4Li^2}\int_{-L}^{L} e^{i(a+b)t} + e^{-i(a+b)t}\,dt - \frac{1}{4Li^2}\int_{-L}^{L} e^{i(a-b)t} + e^{-i(a-b)t}\,dt \\
&= \underbrace{\frac{1}{4L}\int_{-L}^{L} e^{i(a-b)t} + e^{-i(a-b)t}\,dt}_{=\begin{cases}1 & n=m\neq 0\\ 0 & n\neq m\\ 1 & n=m=0\end{cases}} - \underbrace{\frac{1}{4L}\int_{-L}^{L} e^{i(a+b)t} + e^{-i(a+b)t}\,dt}_{=\begin{cases}0 & n=m\neq 0\\ 0 & n\neq m\\ 1 & n=m=0\end{cases}}
\end{aligned}$$

so

$$\frac{1}{L}\int_{-L}^{L} \sin\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = \begin{cases}1 & n=m\neq 0\\ 0 & n\neq m\\ 0 & n=m=0\end{cases}$$

A.3.6 Orthogonality rules to justify coefficient formulas

We had the orthogonality rules as: for n, m integers:

$$\frac{1}{L}\int_{-L}^{L} \cos\left(n\frac{\pi}{L}t\right)\cos\left(m\frac{\pi}{L}t\right)\,dt = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \\ 2 & n = m = 0 \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L} \cos\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = 0$$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(n\frac{\pi}{L}t\right)\sin\left(m\frac{\pi}{L}t\right)\,dt = \begin{cases} 1 & n = m \neq 0 \\ 0 & n \neq m \\ 0 & n = m = 0 \end{cases}$$

Cosine coefficients
Given a function (with arbitrary period $2L$)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\frac{\pi}{L}t\right) + b_n \sin\left(n\frac{\pi}{L}t\right)\right]$$

First we consider a_n coefficients; say that we want to find a_k, where k is an integer and $k > 0$:

$$\begin{aligned} &\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(k\frac{\pi}{L}t\right)dt \\ =&\frac{1}{L}\int_{-L}^{L} \frac{a_0}{2}\cos\left(k\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty}\left[a_n\cos\left(n\frac{\pi}{L}t\right)\cos\left(k\frac{\pi}{L}t\right)\right] \\ &+\sum_{n=1}^{\infty}\left[b_n\sin\left(n\frac{\pi}{L}t\right)\cos\left(k\frac{\pi}{L}t\right)\right]dt \end{aligned}$$

Using the orthogonality relations see that all except one term in this expression are equal to 0

$$= 0 + \ldots + 0 + \frac{1}{L}\int_{-L}^{L} a_k\cos\left(k\frac{\pi}{L}t\right)\cos\left(k\frac{\pi}{L}t\right) + 0 + \ldots + 0$$

so

$$\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(k\frac{\pi}{L}t\right) = a_k \underbrace{\frac{1}{L}\int_{-L}^{L}\cos\left(k\frac{\pi}{L}t\right)\cos\left(k\frac{\pi}{L}t\right)}_{=1} = a_k$$

which gives us a formula for the cosine coefficients.
(next page)

Sine coefficients
The sine coefficients can be proven the same way; say we want to find b_k where again k is an integer and $k > 0$:

$$\begin{aligned}
&\frac{1}{L}\int_{-L}^{L} f(t)\sin\left(k\frac{\pi}{L}t\right)dt\\
=&\frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\sin\left(k\frac{\pi}{L}t\right)+\sum_{n=1}^{\infty}\left[a_n\cos\left(n\frac{\pi}{L}t\right)\sin\left(k\frac{\pi}{L}t\right)\right]\\
&+\sum_{n=1}^{\infty}\left[b_n\sin\left(n\frac{\pi}{L}t\right)\sin\left(k\frac{\pi}{L}t\right)\right]dt
\end{aligned}$$

Again see that by the orthogonality rules

$$=0+\ldots+0+\frac{1}{L}\int_{-L}^{L} b_k\sin\left(k\frac{\pi}{L}t\right)\sin\left(k\frac{\pi}{L}t\right)+0+\ldots+0$$

so

$$\frac{1}{L}\int_{-L}^{L} f(t)\sin\left(k\frac{\pi}{L}t\right)dt=b_k\underbrace{\frac{1}{L}\int_{-L}^{L}\sin\left(k\frac{\pi}{L}t\right)\sin\left(k\frac{\pi}{L}t\right)}_{=1}=b_k$$

which gives us a formula for the sine coefficients.

Justifying $a_0/2$
See that should we consider $k=0$, attempting to compute the cosine coefficients gives

$$\begin{aligned}
&\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(0\cdot\frac{\pi}{L}t\right)dt\\
=&\frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\cos\left(0\cdot\frac{\pi}{L}t\right)+\underbrace{\sum_{n=1}^{\infty}\left[a_n\cos\left(n\frac{\pi}{L}t\right)\cos\left(0\cdot\frac{\pi}{L}t\right)\right]}_{=0}\\
&+\underbrace{\sum_{n=1}^{\infty}\left[b_n\sin\left(n\frac{\pi}{L}t\right)\cos\left(0\cdot\frac{\pi}{L}t\right)\right]}_{=0}dt\\
=&\frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\,dt=a_0
\end{aligned}$$

See that

$$\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(0\cdot\frac{\pi}{L}t\right)dt=\frac{1}{L}\int_{-L}^{L} f(t)\,dt=a_0$$

which gives us the formula for a_0. Also see that this justifies the convention of scaling a_0 by a factor of $1/2$ in the fourier series formula. (Computing the sine coefficients for $k=0$ amounts to 0 since $\sin(0)=0$)

A.3.7 Even and Odd Functions

Definition
A function $f(t)$ is called *even* if $f(-t) = f(t)$ for all t. Examples include t^2, t^4, t^6 (even powers), $\cos(at)$ or a constant.

See that

$$\text{If } f(t) \text{ is even then} \quad \int_{-L}^{L} f(t)\,dt = 2\int_{0}^{L} f(t)\,dt$$

A function is $f(t)$ is called *odd* if $f(-t) = -f(t)$ for all t. Examples include t, t^3, t^5 (odd powers) and $\sin(at)$.

Now see that

$$\text{If } f(t) \text{ is odd then} \quad \int_{-L}^{L} f(t)\,dt = 0$$

Illustrated:

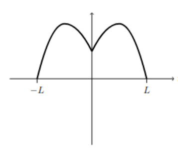


Fig. 1: Even functions:
(total area = twice area of right half)

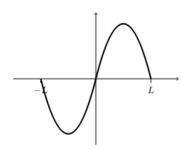


Fig. 2: Odd functions:
(total (signed) area is 0)

Even and odd Fourier coefficient rules
Consider multiplying even and odd functions; intuitively (consider for instance multiplying odd and even powers of t):

1. even$\times$even = even
2. odd$\times$odd = even
3. odd$\times$even = odd

We can use this to show that, assuming periodic $f(t)$:

$$\text{If } f(t) \text{ is } \textit{even} \text{ then we have } b_n = 0, \text{ and } a_n = \frac{2}{L}\int_0^L f(t)\cos\left(n\frac{\pi}{L}\right)\,dt$$

and

$$\text{If } f(t) \text{ is } \textit{odd} \text{ then we have } a_n = 0, \text{ and } b_n = \frac{2}{L}\int_0^L f(t)\sin\left(n\frac{\pi}{L}\right)\,dt$$

(next page)

Justification for Even and odd Fourier coefficient rules
Starting with the *Even* case, assuming even $f(t)$, see that attempting to obtain the fourier coefficients using their respective formulas:

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(n\frac{\pi}{l}t\right)}_{\text{is even}} dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\sin\left(n\frac{\pi}{l}t\right)}_{\text{is odd}} dt$$

we can thus rewrite the integrals as

$$a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(n\frac{\pi}{L}\right) dt, \quad b_n = 0$$

The odd case can be shown the same way; if $f(t)$ is *odd*:

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(n\frac{\pi}{l}t\right)}_{\text{is odd}} dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\sin\left(n\frac{\pi}{l}t\right)}_{\text{is even}} dt$$

so

$$a_n = 0, \quad b_n = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(n\frac{\pi}{L}\right) dt$$

A.3.8 Example: Simple Harmonic Oscillator with square wave input

Let $f(t)$ be an odd square wave of period 2π with $f(t) = 1$ for $0 < t < \pi$. Given the DE

$$\ddot{x} + 9.1x = f(t)$$

We can use the already derived fourier series for $f(t)$:

$$f(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right) = \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin(nt)}{n}$$

So the DE becomes

$$\ddot{x} + 9.1x = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right)$$

See that we can solve for each term separately and combine all the outputs using superposition. The general case would be

$$\ddot{x} + 9.1x = \frac{\sin(nt)}{n}$$

Complex replacement and ERF
This is solved by complex replacement and exponential response. To recapitulate:

$$\ddot{z} + 9.1z = \frac{1}{n}e^{int} \quad \text{where } x_p = \mathrm{Im}(z_p)$$

we consider a particular solution of form $z_p = Ae^{int}$ and solve for A (basis for exponential response)

$$(-n^2 + 9.1)Ae^{int} = \frac{1}{n}e^{int}$$

comparing coefficients and taking the imaginary part we have

$$A = \frac{1}{n(9.1 - n^2)}, \quad \text{so } z_p = \frac{1}{n(9.1 - n^2)}e^{int}$$

$$x_{n,p} = \mathrm{Im}(z_{n,p}) = \frac{\sin(nt)}{n(9.1 - n^2)}$$

so by superposition we get the *steady periodic solution*:

$$\begin{aligned}
\ddot{x} + 9.1x &= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right) = \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin(nt)}{n} \\
x_{sp}(t) &= \frac{4}{\pi}(x_{1,p}(t) + x_{3,p}(t) + x_{5,p}(t)) = \frac{4}{\pi}\sum_{n\text{ odd}} x_{n,p}(t) \\
&= \frac{4}{\pi}\left(\frac{\sin(nt)}{n(9.1 - 1^2)} + \frac{\sin(nt)}{n(9.1 - 3^2)} + \frac{\sin(nt)}{n(9.1 - 5^2)} + \ldots\right) \\
&= \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin(nt)}{n(9.1 - n^2)}
\end{aligned}$$

(next page)

Resonant frequencies
We had

$$x_{sp}(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n(9.1 - n^2)}$$

The amplitudes for the first few terms are

$$\frac{4}{\pi}\left(\frac{1}{9.1 - 1^2}\right) \approx 0.157, \quad \frac{4}{\pi}\left(\frac{1}{3(9.1 - 3^2)}\right) \approx 4.244,$$
$$\frac{4}{\pi}\left(\frac{1}{5(9.1 - 5^2)}\right) \approx -0.016, \quad \ldots$$

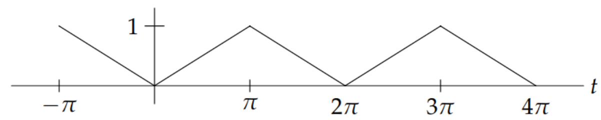
for $n = 1, 3, 5$ respectively. Then for $n > 5$ the amplitudes are much smaller. See that the $n = 3$ term has by far the largest amplitude.

We can explain this by noticing that the natural frequency of this system is $\sqrt{9.1} \approx 3$ and so the system has a resonant-type response to the "embedded third harmonic" $\frac{\sin(3t)}{3}$ in the input signal.

The input signal has a base frequency 1—the presence of this third harmonic is not apprent to the eye—yet the driven oscillator picked it out in its reponse in the form of a dominant frequency *three times* the fundamental frequency of the input.

A.3.9 Example: Damped Harmonic Oscillator with triangle wave input

Say $f(t)$ is a triangle wave as shown



Computing its fourier series would yield

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \dots\right)$$

Now consider the DE

$$\ddot{x} + 2\dot{x} + 9x = f(t)$$

Solving
We solve for the individual component

$$\ddot{x} + 2\dot{x} + 9x = \cos nt$$

First (for the constant), when $n = 0$, a particular solution is $x_{n,p} = 1/9$. Now for $n \geq 1$ we have by complex replacement

$$\ddot{z}_n + 2\dot{z}_n + 9z_n = e^{int}, \quad x_n = \mathrm{Re}(z_n)$$

By the ERF we have

$$z_{n,p} = \frac{e^{int}}{9 - n^2 + 2in}$$

We can write the complex number in polar form $9 - n^2 + 2in = R_n e^{i\phi_n}$ where

$$R_n = \sqrt{(9-n^2)^2 + 4n^2} \text{ and } \phi_n = \mathrm{Arg}(9 - n^2 + 2in) = \tan^{-1}\frac{2n}{9-n^2}$$

So we have

$$z_{n,p} = \frac{1}{R_n}e^{i(nt-\phi_n)}, \text{ implying } x_{n,p} = \frac{1}{R_n}\cos(nt - \phi_n)$$

By superposition we have our solution

$$x_{sp}(t) = \frac{1}{18} - \frac{4}{\pi^2}\left(\frac{\cos(t-\phi_1)}{R_1} + \frac{\cos(3t-\phi_3)}{3^2R_3} + \frac{\cos(5t-\phi_5)}{5^2R_5} + \dots\right)$$

where R_n and ϕ_n are defined as above.

A.3.10 General fourier series input to damped harmonic oscillator

Now a generalisation of the previous example. Consider solving

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

for the steady-periodic response $x_{sp}(t)$, where

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right)$$

Solving
For $n = 0$ we have $x_{0,p} = 1/k$. For $n \geq 1$ we use complex replacement

$$m\ddot{z}_n + b\dot{z}_n + kz_n = e^{in\frac{\pi}{L}t}, \quad x_n = \mathrm{Re}(z_n)$$

and exponential response

$$z_{n,p}(t) = \frac{e^{in\frac{\pi}{L}t}}{p\left(in\frac{\pi}{L}\right)}$$

We have $p\left(in\frac{\pi}{L}\right)$ (where $p(s) = ms^2 + bs + k$)

$$p\left(in\frac{\pi}{L}\right) = \left(k - m\left(n\frac{\pi}{L}\right)^2\right) + ibn\frac{\pi}{L}$$

written in polar coordinates

$$= \left|p\left(in\frac{\pi}{L}\right)\right| e^{i\phi_n}$$

where

$$\left|p\left(in\frac{\pi}{L}\right)\right| = \sqrt{\left(k - m\left(n\frac{\pi}{L}\right)^2\right)^2 + \left(bn\frac{\pi}{L}\right)^2}$$

and

$$\phi_n = \mathrm{Arg}\left(p\left(in\frac{\pi}{L}\right)\right) = \tan^{-1}\left(\frac{bn\frac{\pi}{L}}{k - m\left(n\frac{\pi}{L}\right)^2}\right)$$

So we have

$$z_{n,p} = g_n e^{i\left(n\frac{\pi}{L}t - \phi_n\right)}, \quad \text{where } g_n = \frac{1}{\left|p\left(in\frac{\pi}{L}\right)\right|}$$

taking the real part we have

$$x_{n,p} = g_n \cos\left(n\frac{\pi}{L}t - \phi_n\right)$$

Using superposition we finally get

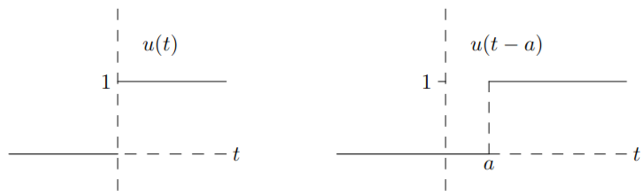
$$x_{sp}(t) = \frac{a_0}{2}x_{0,p} + \sum_{n=1}^{\infty} a_n x_{n,p}(t) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} a_n g_n \cos\left(n\frac{\pi}{L}t - \phi_n\right)$$

This is the general formula for the steady state periodic response of a second order LTI DE to an even periodic driver $f(t)$.

A.3.11 Step and Box functions

Step functions
The *unit step function* (also called the *heaviside function*) is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$



See that $u(t-a)$ is just $u(t)$ shifted to the right

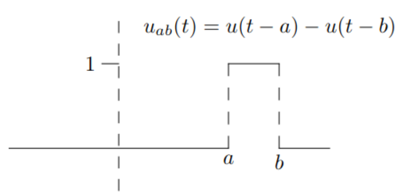
$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

Note that

- $u(t)$ is not defined when $t = 0$. It has a *jump discontinuity* at that point.
- The graph shows that $u(0^-) = 0$ and $u(0^+) = 1$. Where $u(0^-)$ indicates the left-hand limit and $u(0^+)$ the right-hand limit.

Box Functions
We define a *box function* as

$$u_{ab}(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } a < t < b \\ 0 & \text{for } t > b \end{cases}$$



See that this same function can be written as

$$u_{ab}(t) = u(t-a) - u(t-b)$$

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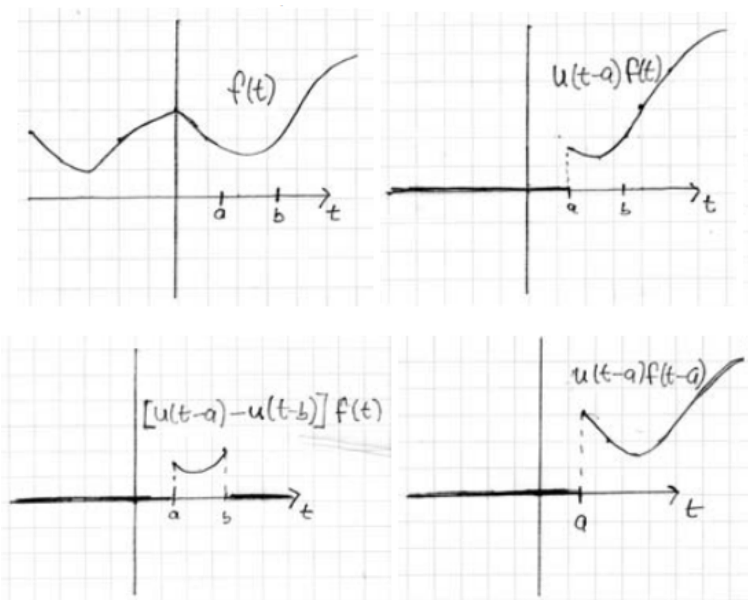
Switches

By multiplying a function $f(t)$ with a step or box function, see that they act as switches to turn $f(t)$ on and off.



- *Top left:* $f(t)$
- *Top right:* $u(t-a)f(t)$ is 0 for $t < a$ and $f(t)$ for $t > a$
- *Bottom left:* $[u(t-a) - u(t-b)]f(t)$ is $f(t)$ within the window $a < t < b$ and 0 otherwise.
- *Bottom right:* $u(t-a)f(t-a)$ translates $f(t)$ to the right a units, and the result is switched on at time a.

(next page)

Multiple functions
See that we can change formulas for different intervals of t. The function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ f_1(t) & \text{for } 0 < t < 1 \\ f_2(t) & \text{for } 1 < t < 2 \\ f_3(t) & \text{for } 2 < t \end{cases}$$

can be written as

$$f(t) = [u(t) - u(t-1)]f_1(t) + [u(t-1) - u(t-2)]f_2(t) + u(t-2)f_3(t)$$

We can also add on new functions at specific timings; see that the function

$$f(t) = u(t)f_1(t) + u(t-2)f_2(t) + u(t-4)f_3(t)$$

looks like

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ f_1(t) & \text{for } 0 < t < 2 \\ f_1(t) + f_2(t) & \text{for } 2 < t < 4 \\ f_1(t) + f_2(t) + f_3(t) & \text{for } 4 < t \end{cases}$$

With no 'off' switch the number of terms in each successive case grows.

A.3.12 Delta function

The discontinuity in the unit step function $u(t)$ is an idealised model of a quantity that goes from 0 to 1 very quickly. We assume it jumps across these values in no time.

This modelled 'jump' is called the *delta function* or *Dirac delta function* or *unit impulse*. It is important to note that the delta function isn't really a function, but rather a *generalised function*.

Intuition
Consider a substance being added to a container at a rate of $q(t)$. The total amount added to the container from time 0 to time t is

$$Q(t) = \int_0^t q(u)du$$

Equivalently

$$\dot{Q}(t) = q(t)$$

Assume that $q(t)$ is only nonzero for a short period of time and that the total amount of substance added over that period is 1(unit mass). Consider two of the many possible ways this could happen:

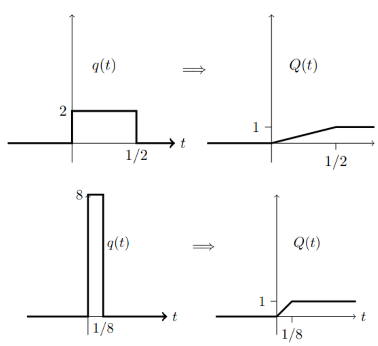


Figure 1: two possible graphs of $q(t)$ and $Q(t)$, both with total input = 1.

See that the area under each box has total area equal to 1, and that the graphs for $Q(t)$ rise linearly to 1 and remain constant at that value thereafter.
(next page)

Definition
Now consider a generalisation of the example; let $q_h(t)$ be a box of width h and height $1/h$. As $h \to 0$, the width of the box becomes 0, but the *area of the box remains at 1*:

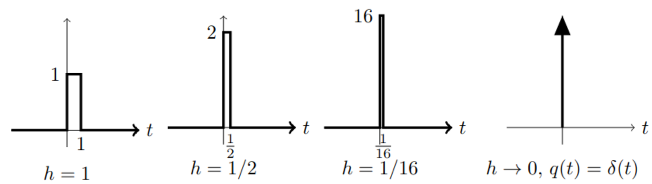


Figure 2: Box functions $q_h(t)$ becoming the delta function as $h \to 0$.

We define the *delta function* to be the formal limit

$$\delta(t) = \lim_{h \to 0} q_h(t)$$

Graphically $\delta(t)$ is represented as a spike or harpoon at $t = 0$. It is an infinitely tall spike of infinitesimal width enclosing a total area of 1.

As an input function $\delta(t)$ represents the ideal case where 1 unit of input is applied at once at time $t = 0$.

Properties
We have

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}$$

Since $\delta(t)$ is the limit of graphs of area 1, the area under its graph is 1, meaning

$$\int_c^d \delta(t) dt = \begin{cases} 1 & \text{if } c < 0 < d \\ 0 & \text{otherwise} \end{cases}$$

Also see that for any continuous function we have

$$f(t)\delta(t) = f(0)\delta(t) \quad \text{and} \quad \int_c^d f(t)\delta(t) dt = \begin{cases} f(0) & \text{if } c < 0 < d \\ 0 & \text{otherwise} \end{cases}$$

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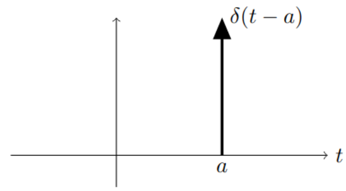
Properties cont.
As with the unit function, we can translate the delta function; where $\delta(t-a)$ is 0 everywhere except $t = a$. Its total area remains as 1:



where then

$$f(t)\delta(t-a) = f(a)\delta(t-a)$$

and

$$\int_c^d f(t)\delta(t-a)dt = \begin{cases} f(a) & \text{if } c < a < d \\ 0 & \text{otherwise} \end{cases}$$

More intuition
Note that any sequence of functions with unit area become an $\delta(t)$ in the limit; practically, $\delta(t)$ should be thought of as *any function* of unit area, concentrated very near $t = 0$.

In a sense this justifies the idea of $\delta(t)$ not really being a function, but rather a *generalised function*.

Limits
The left and right-handed limits of the impulse are

$$\delta(0^-) = 0, \quad \delta(0^+) = 0, \quad \delta(0) = \infty$$

So 0 is in the interval $[0^-, \infty)$ but not $[0^+, \infty)$, and

$$\int_{0^-}^{\infty} \delta(t)dt = 1 \quad \text{and} \quad \int_{0^+}^{\infty} \delta(t)dt = 0$$

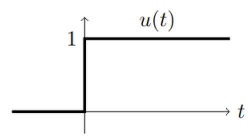
further

$$\int_{0^-}^{0^+} \delta(t)dt = 1$$

(next page)

Generalised Derivative—Intuition
We say that
$$\delta(t) = u'(t)$$
where $u(t)$ is the unit step function. Given the discontinuous nature of $u(t)$ this isn't like a usual derivative, but rather a *generalised derivative*. Here we justify this.

First see that the slope of the unit function $u(t)$ is 0 everywhere except at $t = 0$ and that its slope is ∞ at $t = 0$.



that is, its derivative is
$$u'(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}$$
this derivative matches the properties of the delta function. It doesn't exist in a calculus sense since discontinuities aren't differentiable. The function $u(t)$ isn't even defined at 0. So we call this derivative a *generalised derivative*.

Further, consider the anti-derivative of $\delta(t)$; let
$$f(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$
The fundamental theorem of calculus would lead us to say $f'(t) = \delta(t)$. (this is only in a generalised sense since technically the FTC requires continuity) See however that we can state
$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$
That is, $f(t) = u(t)$, so $u(t)$ is the antiderivative of $\delta(t)$.
(next page)

Scaling
See that the antiderivative of a scaled impulse is also a scaled unit function, where a constant

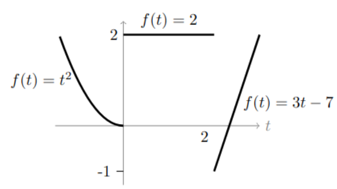
$$f(t) = \int_{-\infty}^{t} a\delta(\tau)d\tau$$

yields

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ a & \text{if } t > 0 \end{cases}$$

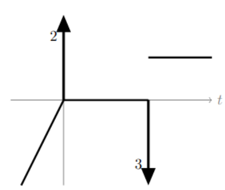
which is just $au(t)$.

Deriving jump discontinuities
In that sense, a jump discontinuity contributes a delta function to the generalised derivative. For instance suppose $f(t)$ has the following graph



The corresponding derivative would look like

$$f'(t) = 2\delta(t) - 3\delta(t-2) + \begin{cases} 2t & \text{if } t < 0 \\ 0 & \text{if } 0 < t < 2 \\ 3 & \text{if } 2 < t \end{cases}$$



$f'(t)$ is a generalised function. Referring to different parts of a generalised function, we call the delta function pieces the *singular part* and the remainder the *regular part*. If the singular part contains a multiple of $\delta(t-a)$, we say the function *contains* $\delta(t-a)$.

A.3.13 Initial conditions

Recall that we had the left and right-handed limits of the unit function as

$$u(0^-) = 0, \quad u(0^+) = 1, \quad u(0), \text{ is undefined}$$

and that of the delta function as

$$\delta(0^-) = 0, \quad \delta(0^+) = 0, \quad \delta(0) = \infty$$

Generally, see that at a discontinuity we may need to distinguish between 0^- and 0^+. Assuming x is the output, we do this by calling $x(0^-), \dot{x}(0^-), \ldots$ the *pre-initial conditions* and $x(0^+), \dot{x}(0^+), \ldots$ the *post-initial conditions*.

We define the *rest initial conditions* as the pre-initial conditions

$$x(0^-) = 0, \dot{x}(0^-) = 0, \ldots, x^{(n-1)}(0^-) = 0$$

A.3.14 First order unit step response

Consider

$$\dot{x} + kx = ru(t), \quad x(0^-) = 0, \quad k, r \text{ constants}$$

This can be rewritten as

$$\dot{x} + kx = \begin{cases} 0 & \text{for } t < 0 \\ r & \text{for } t > 0 \end{cases} \quad x(0^-) = 0$$

Solving each case gives

$$x(t) = \begin{cases} c_1 e^{-kt} & \text{for } t < 0 \\ \frac{r}{k} + c_2 e^{-kt} & \text{for } t > 0 \end{cases}$$

See that $x(0^-) = c_1$ and $x(0^+) = r/k + c_2$. If the two differ then there will be a jump of magnitude

$$x(0^+) - x(0^-) = r/k + c_2 - c_1$$

The initial condition $x(0^-) = 0$ implies $c_1 = 0$ so

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{r}{k} + c_2 e^{-kt} & \text{for } t > 0 \end{cases}$$

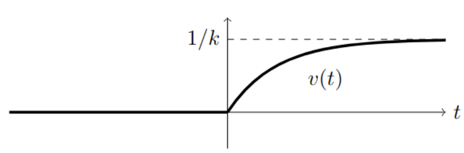
To find c_2 we substitute this back into the differential equation (we use a generalised derivative for the jump at $t = 0$ from 0^- to 0^+)

$$\begin{aligned} \dot{x} + kx &= (r/k + c_2)\delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ -kc_2 e^{-kt} + r + kc_2 e^{-kt} & \text{for } t > 0 \end{cases} \\ &= (r/k + c_2)\delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ r & \text{for } t > 0 \end{cases} \end{aligned}$$

Comparing this result with the second statement we see that $r/k + c_2 = 0$ or $c_2 = -r/k$. so

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{r}{k}(1 - e^{-kt}) & \text{for } t > 0 \end{cases} \quad = \quad \frac{r}{k}(1 - e^{-kt})u(t)$$

With $r = 1$, this is the *unit step response*, sometimes written as $v(t)$, where $v(t) = (1/k)(1 - e^{-kt})u(t)$:



Notice that it starts at 0 and goes asymptotically up to $1/k$.

A.3.15 Unit impulse response

(Note that the unit impulse and the unit step are very different—the magnitude of the instantaneous impulse extends to infinity, while the discontinuity of the step function only extends to a fixed value).

Consider the initial value problem

$$\dot{x} + kx = \delta(t), \quad x(0^-) = 0, \quad k, r \text{ constants}$$

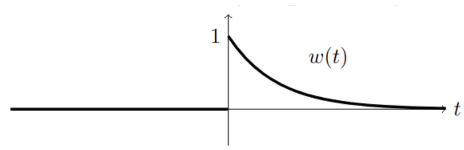
Due to the rest initial conditions we have $x(t) = 0$ for $t < 0$. The idea here is that $x(t)$ jumps from 0 to 1 at $t = 0$. That is, $x(0^+) = 1$ (not $\delta(0^+)$); for $t > 0$ the input $\delta(t) = 0$ and therefore for $t > 0$ we solve

$$\dot{x} + kx = 0, \quad x(0) = 1$$

So we have

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-kt} & \text{for } t > 0 \end{cases}$$

This is called the *Unit impulse response*, denoted by $w(t)$:



At $t = 0$ it jumps to $x = 1$ and then decays exponentially to 0.
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$\delta(t)$ as a limit of box functions
Now we compute the *unit impulse response* as the *limit of the responses to box functions*. Defining the box function as

$$u_h(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1/h & \text{for } 0 < t < h \\ 0 & \text{for } h < t \end{cases}$$

It has a total area of 1 for all $h > 0$ and the graph of $u_h(t)$ becomes a delta function as $h \to 0$:

$$\lim_{h\to 0} u_h(t) = \delta(t)$$

Given the equation

$$\dot{x} + kx = u_h(t), \quad x(0^-) = 0$$

the solution (worked out piecewise) is found to be

$$x(t) = \begin{cases} c_1 e^{-kt} & \text{for } t < 0 \\ \frac{1}{hk} + c_2 e^{-kt} & \text{for } 0 < t < h \\ c_3 e^{-kt} & \text{for } h < t \end{cases}$$

Finding the constants, the initial condition $x(0^-) = 0$ we find that $c_1 = 0$; substitution into the original equation (like in the a step response) gives us $c_2 = -1/hk$.

For the third case the principle follows that of the impulse response, setting the initial condition $x(h) = 1/hk(1 - e^{-hk})$ (since we found c_2). So

$$c_3 e^{-hk} = \frac{1}{hk}(1 - e^{-hk})$$
$$c_3 = \frac{1}{hk}(e^{hk} - 1)$$

This gives the solution

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{hk}(1 - e^{-kt}) & \text{for } 0 < t < h \\ \frac{1}{hk}(e^{hk} - 1)e^{-kt} & \text{for } h < t \end{cases}$$

(next page)

Cont.
We had

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{hk}(1 - e^{-kt}) & \text{for } 0 < t < h \\ \frac{1}{hk}(e^{hk} - 1)e^{-kt} & \text{for } h < t \end{cases}$$

Now consider the limit $h \to 0$. The second case becomes irrelevant (its range becomes 0), and since by l'hospital's rule we can say

$$\lim_{h \to 0} \frac{e^{hk} - 1}{hk} = 1$$

Our solution tends to

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-kt} & \text{for } 0 < t \end{cases}$$

This limit is *exactly the unit impulse response* found previously:

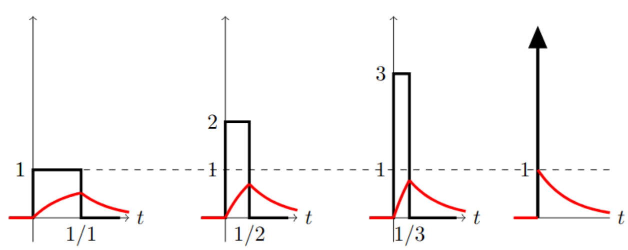

Figure 2. Responses for $h = 1$, $h = .5$, $h = .333$, and $h \to 0$.

(input in black and output in red) See that the output rises faster and gets closer to 1 as $h \to 0$.