Velleman

Malcolm

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Contents

Chapter 1

Logic

1.0.1 Logic Factsheet

De Morgan's laws

$$\neg (P \wedge Q) \text{ is equivalent to } \neg P \vee \neg Q$$
$$\neg (P \vee Q) \text{ is equivalent to } \neg P \wedge \neg Q$$

Commutative laws

$$P \wedge Q \text{ is equivalent to } Q \wedge P$$
$$P \vee Q \text{ is equivalent to } Q \vee P$$

Associative laws

$$P \wedge (Q \wedge R) \text{ is equivalent to } (P \wedge Q) \wedge R$$
$$P \vee (Q \vee R) \text{ is equivalent to } (P \vee Q) \vee R$$

Indempotent laws

$$P \wedge P \text{ is equivalent to } P$$
$$P \vee P \text{ is equivalent to } P$$

Distributive laws

$$P \wedge (Q \vee R) \text{ is equivalent to } (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \text{ is equivalent to } (P \vee Q) \wedge (P \vee R)$$

Absorption laws

$$P \vee (P \wedge Q) \text{ is equivalent to } P$$
$$P \wedge (P \vee Q) \text{ is equivalent to } P$$

Double Negation law

$$\neg\neg P \text{ is equivalent to } P$$

1.0.2 Set operation definitions

The *intersection* of two sets A and B is the set $A \cap B$ defined as follows:

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$$

The *union* of A and B is the set $A \cup B$ defined as follows:

$$A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$$

The *difference* of A and B is the set $A \setminus B$ defined as follows:

$$A \setminus B = \{x \,|\, x \in A \text{ and } x \notin B\}$$

See that

$$x \in A \cap B = x \in \{y \,|\, y \in A \text{ and } y \in B\}$$

where y is a dummy variable. So we can also write that

$$x \in A \cap B = x \in A \land x \in B$$

The same can be shown for the union and difference.

1.0.3 Distributivity of set operations

We show

$$x \in A \cap (B \cup C) \text{ is equivalent to } x \in (A \cap B) \cup (A \cap C)$$

By analysing their logical forms:

$$\begin{aligned}
&x \in A \cap (B \cup C) \\
&= x \in A \land x \in (B \cup C) \\
&= x \in A \land (x \in B \lor x \in C)
\end{aligned}$$

and

$$\begin{aligned}
&x \in (A \cap B) \cup (A \cap C) \\
&= x \in (A \cap B) \lor x \in (A \cap C) \\
&= (x \in A \land x \in B) \lor (x \in A \land x \in C) \\
&= [(x \in A \land x \in B) \lor x \in A)] \land [(x \in A \land x \in B) \lor x \in C)] \\
&= x \in A \land [(x \in A \lor x \in C) \land (x \in B \lor x \in C)] \\
&= [x \in A \land (x \in A \lor x \in C)] \land (x \in B \lor x \in C) \\
&= x \in A \land (x \in B \lor x \in C)
\end{aligned}$$

We can also show, in a similar manner, that

$$x \in A \cup (B \cap C) \text{ is equivalent to } x \in (A \cup B) \cap (A \cup C)$$

1.0.4 $x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$

We can also show

$$x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$$

See that

$$\begin{aligned}
& x \in A \setminus (B \cap C) \\
& = x \in A \wedge \neg (x \in B \cap C) && \text{(Definition of } \setminus\text{)} \\
& = x \in A \wedge \neg (x \in B \wedge x \in C) && \text{(Definition of } \cap\text{)} \\
& = x \in A \wedge (x \notin B \vee x \notin C) && \text{(De Morgan's)} \\
& = (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) && \text{(Distributivity)} \\
& = (x \in A \setminus B) \vee (x \in A \setminus C) && \text{(Definition of } \setminus\text{)} \\
& = x \in (A \setminus B) \cup (A \setminus C) && \text{(Definition of } \cup\text{)}
\end{aligned}$$

1.0.5 $x \in (A \cup B) \setminus (A \cap B) = x \in (A \setminus B) \cup (B \setminus A)$

$$\begin{aligned}
& x \in (A \cup B) \setminus (A \cap B) \\
& = (x \in A \vee x \in B) \wedge \neg (x \in A \wedge x \in B) && \text{(By definition)} \\
& = (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) && \text{(De Morgan's)} \\
& = [(x \in A \vee x \in B) \wedge (x \notin A)] \\
& \qquad \vee [(x \in A \vee x \in B) \wedge (x \notin B)] && \text{(Distributivity)} \\
& = [(x \notin A \wedge x \in A) \vee (x \notin A \wedge x \in B)] \\
& \qquad \vee [(x \notin B \wedge x \in A) \vee (x \notin B \wedge x \in B)] && \text{(Distributivity)} \\
& = (x \notin A \wedge x \in B) \vee (x \notin B \wedge x \in A) \\
& = (x \in A \wedge x \notin B) \wedge (x \in B \wedge x \notin A) && \text{(Commutativity)} \\
& = x \in (A \setminus B) \cup (B \setminus A) && \text{(By definition)}
\end{aligned}$$

1.0.6 $(A \cap B) \cap (A \setminus B) = \emptyset$

See that

$$\begin{aligned}
& x \in (A \cap B) \cap (A \setminus B) \\
& = (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin B) && \text{(Definition)} \\
& = x \in A \wedge \underbrace{(x \in B \wedge x \notin B)}_{\text{Contradiction}} && \text{(Associativity + Commutativity)}
\end{aligned}$$

The last statement is a contradiction, so the statement $x \in (A \cap B) \cap (A \setminus B)$ will always be false, no matter what x is. In other words, nothing can be an element of $(A \cap B) \cap (A \setminus B)$, so it must be the case that $(A \cap B) \cap (A \setminus B) = \emptyset$; $A \cap B$ and $A \setminus B$ are disjoint.

1.0.7 Conditional and Contrapositive laws

Conditional Laws

$$P \to Q \text{ is equivalent to } \neg(P \wedge \neg Q)$$