Appendix 4

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0.1 Real Numbers

0.1.1 Dedekind Cuts

Motivation
Recall the definitions:

- The set $\mathbb{N}$ of natural numbers, 1, 2, 3, 4,...
- The set $\mathbb{Z}$ of integers, 0, 1, -1, -2, 2,...
- The set $\mathbb{Q}$ of rational numbers p/q where p, q are integers, $q \neq 0$.

It is clear that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$. Intuitively $\mathbb{Z}$ improves $\mathbb{N}$ because it contains negatives and $\mathbb{Q}$ improves $\mathbb{Z}$ because it contains reciprocals. However notice that $\mathbb{Q}$ is incomplete—doesn't admit irrational roots such as $\sqrt{2}$ or numbers like π. We solve this with $\mathbb{R}$ such that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

As an example of the fact that $\mathbb{Q}$ is incomplete:

Theorem *No number r in $\mathbb{Q}$ has the square equal to 2; that is, $\sqrt{2} \notin \mathbb{Q}$.*

Proof To prove that every $r = p/q$ has $r^2 \neq 2$ we show that $p^2 \neq 2q^2$. (since $r = p/q \neq 2 \implies r^2 = p^2/q^2 \neq 2$) We also assume that p and q have no common factors since they would have been canceled out beforehand.

Case 1: p is odd. Then p^2 is odd while $2q^2$ is not. Therefore $p^2 \neq 2q^2$.

(an even number n would also have even n^2: because $n = 2a$ for some a would have a square $n^2 = 4a^2$ which is also divisible by 2. An odd number squared is odd: the expression $(n+1)$ for even n is odd; its square $(n+1)^2 = n^2 + 2n + 1$ is also odd, since $n^2 + 2n$ is even.)

Case 2: p is even. Since p and q have no common factors, q is odd. Then p^2 is divisible by 4 while $2q^2$ is not (since q^2 is odd). Therefore $p^2 \neq 2q^2$.

Since $p^2 \neq 2q^2$ for all integers p, there is no rational number $r = p/q$ whose square is 2. $\square$
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Dedekind cuts

The set $\mathbb{Q}$ of rational numbers is incomplete with 'gaps' of negligible width. An elegant method to fill in these gaps are with *Dedekind cuts* in which one visualises real numbers as places on a number line that can be 'cut':

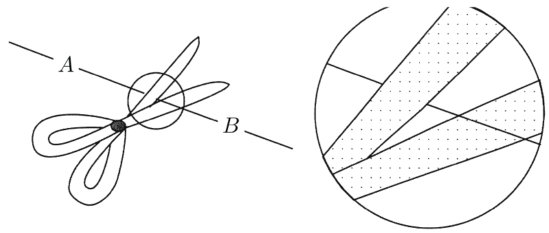

Figure 3 A Dedekind cut

Definition A *cut* in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that

- $A \cup B = \mathbb{Q}$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$
- If $a \in A$ and $b \in B$ then $a < b$
- A contains no largest element

A is the left-hand part of the cut and B the right. We denote the cut as $x = A|B$. Now we define:

Definition A *real number* is a cut in $\mathbb{Q}$

$\mathbb{R}$ is the class (of set-pairs) of all real numbers $x = A|B$. Here are two examples of cuts:

1. $A|B = \{r \in \mathbb{Q} : r < 1\}|\{r \in \mathbb{Q} : r \geq 1\}$
2. $A|B = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\}|\{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \geq 2\}$

The first cut is a *rational cut*, where for some fixed rational number c, A is the set of all rationals $< c$ while B is the rest of $\mathbb{Q}$. We write c^* for the rational cut at c.