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Contents

Chapter 1

The Real Numbers

1.1 Definitions and the necessity for real numbers

We begin with the *natural numbers*

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

If we restrict our attention to the natural numbers $\mathbb{N}$, then we can perform addition perfectly well, but if we want to have an additive identity (zero) and the additive inverses necessary to define subtraction, we must extend our system to the *integers*

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The next issue is multiplication and division. The number 1 acts as the multiplicative identity, but in order to define division we need to have multiplicative inverses. Thus we extend our system again to the *rational numbers*

$$\mathbb{Q} = \left\{ \text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0 \right\}$$

The properties of $\mathbb{Q}$ essentially make up the definition of what is called a *field*. More formally stated, a field is any set where addition and multiplication are well-defined operations that are commutative, associative, and distributive $a(b+c) = ab + ac$. There must be an additive identity, and every element must have an additive inverse. There must be a multiplicative identity, and multiplicative inverses must exist for all nonzero elements of the field.
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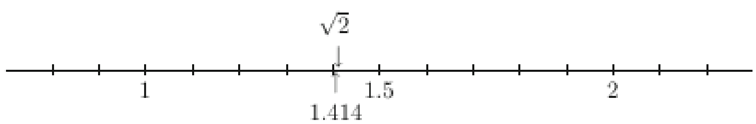
Cont.
The set $\mathbb{Q}$ has a natural *order* defined on it. Given any two rational numbers r and s, exactly one of the following is true:

$$r < s, \quad r = s, \quad \text{or} \quad r > s$$

The ordering is transitive in the sense that if $r < s$ and $s < t$ then $r < t$, so we are led to the mental picture of the rational numbers as being laid out from left to right along a number line. Unlike $\mathbb{Z}$, there are no intervals of empty space; given any two rational numbers $r < s$, the rational number $(r+s)/2$ sits halfway in between, implying that the rational numbers are densely nestled together.

With the field properties of $\mathbb{Q}$ allowing us to safely carry out addition, subtraction, multiplication, and division, we want to consider what $\mathbb{Q}$ is lacking. As an example it can be proven that $\sqrt{2}$ is irrational; using rational numbers it is possible to *approximate* $\sqrt{2}$ quite well:



For instance, $1.414^2 = 1.999396$, and by adding more decimal places to the approximation we can get even closer to the value for $\sqrt{2}$, but even so, it is clear that there is a 'hole' in the rational number line where $\sqrt{2}$ ought to be. $\sqrt{3}$ and $\sqrt{5}$ are also examples of this. If we want every length along the number line to correspond to an actual number, then another extension to our number system is in order. Thus, to the chain $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ we append the *real numbers* $\mathbb{R}$.

Intuition for real numbers
The question of how to construct $\mathbb{R}$ from $\mathbb{Q}$ is complicated; it is discussed later on. For now it is not too inaccurate to say that $\mathbb{R}$ is obtained by filling in the gaps in $\mathbb{Q}$. Wherever there is a hole, a new *irrational* number is defined and placed into the ordering that already exists on $\mathbb{Q}$. The real numbers are the union of these irrational numbers together with the more familiar rational ones.

1.2 The Axiom of Completeness

What exactly is a real number? We got as far as saying that the set $\mathbb{R}$ of real numbers is an extension of the rational numbers $\mathbb{Q}$ in which there are no holes or gaps, where every length along the number line—such as $\sqrt{2}$—corresponds to a real number.

One will want to improve on this definition, and we could; continuing to use this unprecise definition would mean that whatever precise statements we formulate will rest on unproven assumptions and undefined terms. However, at some point we must draw a line and confess that this is what we have decided to accept as a reasonable place to start. Naturally there is some debate as to where this line should be drawn. The majority of the material covered in these notes is attributable to many prominent mathematicians; the interesting point is that nearly all of this work was done using intuitive assumptions about the nature of $\mathbb{R}$ quite similar to our own informal understanding at this point. It was these very theorems that motivated a rigorous construction of $\mathbb{R}$; not the other way around.

We will follow this historical model: our own rigorous construction of $\mathbb{R}$ from $\mathbb{Q}$ will be postponed till much later, when the need for such a construction will be more justified and easier to appreciate.

1.2.1 An initial definition for $\mathbb{R}$

$\mathbb{R}$ is a set containing $\mathbb{Q}$. The operations of addition and multiplication on $\mathbb{Q}$ extend to all $\mathbb{R}$ in such a way that every element of $\mathbb{R}$ has an additive inverse and every nonzero element of $\mathbb{R}$ has a multiplicative inverse. We assume $\mathbb{R}$ is a *field*, meaning that addition and multiplication of real numbers are commutative, associative, and distributive. We also assume that the familiar properties of ordering on $\mathbb{Q}$ extend to all of $\mathbb{R}$ (for instance if $a < b$ and $c > 0$ then $ac < bc$). More formally, we assume that $\mathbb{R}$ is an *ordered field*, which contains $\mathbb{Q}$ as a subfield (rigorous definitions left to the end).

This brings us to the final, and most distinctive, assumption about the real number system. We must find some way to clearly articulate what we mean by insisting that $\mathbb{R}$ does not contain the gaps that permeate $\mathbb{Q}$. This is the defining difference between real numbers and rational numbers, and is referred to as the *Axiom of Completeness*:

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Explaining this statement is the focus of the rest of the section.

1.2.2 Least Upper Bounds and Greatest Lower Bounds

Definition 1. A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* for A.

Similarly, the set A is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 2. A real number s is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. s is an upper bound for A;
2. if b is any upper bound for A, then $s \leq b$.

The least upper bound is frequently called the *supremum* of the set A. Although the notation $s = \text{lub}A$ is sometimes used, we will always write $s = \sup A$ for the least upper bound.

The *greatest lower bound* or *infimum* for A is defined in a similar way. A real number l is the greatest lower bound for the set $A \subseteq \mathbb{R}$ if

1. l is a lower bound for A;
2. if b is any lower bound for A, then $b \leq l$.

Definition 3. A real number a_0 is the *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.

Axiom of Completeness. *Every nonempty set of real numbers that is bounded above has a least upper bound.*

The axiom of completeness asserts that every nonempty bounded set does have a least upper bound. We are not going to prove this. An *axiom* in mathematics is an accepted assumption, to be used without proof. Perferably, an axiom should be an elementary statement about the system in question so fundamental that it seems to need no justification.

1.2.3 Uniqueness of least upper bound

Theorem 1. *(Antisymmetry)* $\forall x \in \mathbb{R} \forall y \in \mathbb{R}((x \leq y) \wedge (y \leq x) \rightarrow y = x)$

Proof. (by me) Let x and y be arbitrary real numbers. Suppose $x \leq y$ and $y \leq x$. Then $x - y \leq 0$ and $0 \leq x - y$. Then $0 \leq x - y \leq 0$, so $x - y = 0$ and $x = y$. □

Theorem 2. *If $s \in \mathbb{R}$ is a least upper bound of $A \subseteq \mathbb{R}$, then s is the unique upper bound on A.*

Proof. (by me) Suppose $s \in \mathbb{R}$ is a least upper bound of $A \subseteq \mathbb{R}$. Now suppose $x \in \mathbb{R}$ is also a least upper bound of A. Then since $s \in \mathbb{R}$ and x is a least upper bound, $x \leq s$. But since $x \in \mathbb{R}$ and s is a least upper bound, $s \leq x$. By antisymmetry, we have $x = s$, showing that s is unique. □

1.2.4 sup(c+A)=c+sup A

Theorem 3. *Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Defining the set $c + A$ by*

$$c + A = \{c + a \mid a \in A\}$$

Then $\sup(c + A) = c + \sup(A)$

Proof. (by me) Denoting $\alpha = \sup(c+A)$ and $\beta = \sup(A)$, we will show $\alpha \leq c+\beta$ and $c + \beta \leq \alpha$.

Starting with $\alpha \leq c+\beta$, we first show that $c+\beta$ is an upper bound on $c+A$. Supposing arbitrary $a \in c + A$, we can declare some $x \in A$ such that $a = c + x$. Since $x \in A$ and $\beta = \sup(A)$, $x \leq \beta$, so $a = c+x \leq c+\beta$, and $c+\beta$ is an upper bound on $c + A$. Since $\alpha = \sup(c + A)$, $\alpha \leq c + \beta$.

Next we show $c + \beta \leq \alpha$. We will show that $\alpha - c$ is an upper bound on A. Supposing arbitrary $a \in A$, then $c + a \in c + A$. Since $\alpha = \sup(c + A)$, then $c + a \leq \alpha$ and $a \leq \alpha - c$, so $\alpha - c$ is an upper bound on A. Now, since $\beta = \sup(A)$, $\beta \leq \alpha - c$, so $c + \beta \leq \alpha$. By the antisymmetry of real numbers, $\alpha = c + \beta$, and $\sup(c + A) = c + \sup(A)$. □

Commentary

To prove that for real numbers x and y, $x = y$, one strategy is to prove $x \leq y$ and $y \leq x$. This is particularly relevant here because the definition of the supremum involves inequalities.

1.2.5 Alternative way of defining Supremum and Infimum

Recall the defintion of least upper bounds, which has two parts. Part 1 says that $\sup A$ must be an upper bound, and part 2 states that it must be the smallest one. The following lemma offers an alternative way to restate part 2.

Lemma 1. *Assume $s \in \mathbb{R}$ is an upper bound for the set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

Proof. ($\rightarrow$) Suppose $s = \sup A$. For arbitrary $\epsilon > 0$, $s - \epsilon < s$. Using the contrapositive of the second condition in the definition of the supremum, we can declare some $a \in A$ such that $s - \epsilon < a$.

($\leftarrow$) Conversely, suppose that for every $\epsilon > 0$ there exists $a \in A$ satisfying $s - \epsilon < a$. Let $b \in \mathbb{R}$ be some upper bound on A. Let $b \in \mathbb{R}$ be some upper bound on A. Suppose now that $b < s$, then $s - b > 0$. We would then be able to declare some $a \in A$ such that $s - (s - b) < a$, and $b < a$; but this contradicts the assumption that b is an upper bound on A, so $s \leq b$ and s is the least upper bound. $\square$

Commentary
To rephrase: given that s is an upper bound, s is the least upper bound if and only if any number smaller than s is not an upper bound.

Lemma 2. *Assume $l \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then $l = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $a < l + \epsilon$.*

Proof. ($\rightarrow$) Suppose $l = \inf A$. For arbitrary $\epsilon > 0$, $l + \epsilon > l$. Since $l = \inf A$, using the contrapositive of the second condition in the definition of the infimum, we can declare some $a \in A$ such that $a < l + \epsilon$.

($\leftarrow$) Now suppose that for every $\epsilon > 0$ there exists $a \in A$ satisfying $a < l + \epsilon$. Let $b \in \mathbb{R}$ be some lower bound on A and suppose $l < b$. Then $b - l > 0$ and we can declare some $a \in A$ such that $a < l + b - l$, so $a < b$; but this contradicts the fact that b is a lower bound, so $b \leq l$ and l is the greatest lower bound. $\square$

Commentary
Given that l is a lower bound, l is the greatest lower bound if and only if any number greater than l is not a lower bound.

1.2.6 Triangle inequality and Absolute value function

Definition 4. The *absolute value function* is so important that it merits the special notation $|x|$ in place of the usual $f(x)$ or $g(x)$. It is defined for every real number via the piecewise definition

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem 4. $\forall a \in \mathbb{R} \forall b \in \mathbb{R}(|ab| = |a||b|)$

Proof. Let a and b be arbitrary real numbers.

Case 1: $a \geq 0, b \geq 0$. Then $ab \geq 0$, and

$$|a||b| = ab = |ab|$$

Case 2: $a \geq 0, b < 0$.

Case 2a: $a = 0$. Then $ab = 0$, and

$$|a||b| = 0 = |ab|$$

Case 2b: $a > 0, b < 0$. Then $ab < 0$, and

$$|a||b| = (a)(-b) = -ab = |ab|$$

Case 3: $a < 0, b \geq 0$.

Case 3a: $b = 0$. Then $ab = 0$, and

$$|a||b| = 0 = |ab|$$

Case 3b: $b > 0$. Then $ab < 0$, and

$$|a||b| = (-a)b = -ab = |ab|$$

Case 4: $a < 0, b < 0$. Then $ab > 0$, and

$$|a||b| = (-a)(-b) = ab = |ab|$$

In all cases $|a||b| = |ab|$, so we are done. $\square$

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Lemma 3. $\forall x \in \mathbb{R}(x^2 = |x|^2)$

Proof. Case 1: $x \geq 0$. Then

$$|x|^2 = (x)^2 = x^2$$

Case 2: $x < 0$. Then

$$|x|^2 = (-x)^2 = x^2 \qquad \square$$

Lemma 4. $\forall x \in \mathbb{R}(|x| \geq x)$

Proof. Case 1: $x \geq 0$. Then $|x| = x$.
Case 2: $x < 0$. Then $|x| = -x > 0 > x$. $\square$

Lemma 5. $\forall x \in \mathbb{R}(|x| \geq 0)$

Proof. Case 1: $x \geq 0$. Then $|x| = x \geq 0$.
Case 2: $x < 0$. Then $|x| = -x > 0$. $\square$

Lemma 6. $\forall x \in \mathbb{R} \forall y \in \mathbb{R}[(x^2 \geq y^2 \wedge x \geq 0 \wedge y \geq 0) \to x \geq y]$

Proof. Let x and y be arbitrary real numbers. Suppose $x^2 \geq y^2$, $x \geq 0$, and $y \geq 0$. Now suppose $x < y$; we know $x \geq 0$ and $y \geq 0$, which leads to the following cases:

Case 1: $x = 0, y = 0$. This is immediately contradicted by $x < y$.

Case 2: $x > 0, y = 0$. Then $x > y$, which is immediately contradicted by $x < y$.

Case 3: $x = 0, y > 0$. Then $x^2 = 0 < y^2$, contradicting $x^2 \geq y^2$.

Case 4: $x > 0, y > 0$. Then since $x < y$, $x^2 < yx$ and $yx < y^2$. So $x^2 < y^2$, contradicting $x^2 \geq y^2$.

In all possible cases the assumption $x < y$ leads to a contradiction, so $x \geq y$. $\square$

Theorem 5. *(Triangle inequality)* $\forall a \in \mathbb{R} \forall b \in \mathbb{R}(|a + b| \leq |a| + |b|)$.

Proof. Let a and b be arbitrary real numbers. First we show $(a+b)^2 \leq (|a|+|b|)^2$. See that

$$\begin{aligned}
(|a| + |b|)^2 &= |a|^2 + |b|^2 + 2|a||b| \\
&= a^2 + b^2 + 2|ab| \\
&\geq a^2 + b^2 + 2ab = (a + b)^2
\end{aligned}$$

Now we show the inequality. See that

$$|a + b|^2 = (a + b)^2 \leq (|a| + |b|)^2 = ||a| + |b||^2$$

We've shown $\forall x \in \mathbb{R}(|x| \geq 0)$. So using the last lemma,

$$||a| + |b|| \geq |a + b|$$

and

$$|a| + |b| \geq |a + b| \qquad \square$$

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Theorem 6. $\forall a \in \mathbb{R} \forall b \in \mathbb{R} \forall c \in \mathbb{R}(|a-b| \leq |a-c| + |c-b|)$

Proof. Given three real numbers a, b and c, we certainly have

$$|a-b| = |(a-c) + (c-b)|$$

So by the triangle inequality,

$$|(a-c) + (c-b)| \leq |a-c| + |c-b|$$

and

$$|a-b| \leq |a-c| + |c-b| \qquad \square$$

The expression $|a-b|$ is equal to $|b-a|$ and is best understood as the *distance* between two points a and b on the number line. With this interpretation, this result makes the plausible statement that the distance from a to b is less than or equal to the distance from a to c plus the distance from c to b. Pretending for a moment that these are points on a plane instead of a line, it should be evident why this is referred to as the "triangle inequality".

1.2.7 Another way of showing equality between two real numbers

Theorem 7. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a-b| < \epsilon$.*

Proof. ($\rightarrow$) Suppose two real numbers a and b are equal. Then $a - b = 0$ and then $|a-b| = 0$. For arbitrary $\epsilon > 0$, $\epsilon > |a-b|$.

($\leftarrow$) Suppose that for every real $\epsilon > 0$ it follows that $|a-b| > \epsilon$. Now suppose $a \neq b$. Then $a-b$ is nonzero. Since it must be the case that $|a-b| \geq 0$, $|a-b| > 0$. Therefore $\epsilon_0 = |a-b| > 0$ contradicts the initial assumption, and it must be that $a = b$. $\square$

1.2.8

1.3 Consequences of Completeness

1.3.1 Nested interval property

Theorem 8. *(Nested Interval Property). For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Consider the set

$$A = \{a_n \mid n \in \mathbb{N}\}$$

First we will show that $\forall n \in \mathbb{N}(\forall a \in A(a \leq b_n))$, meaning that for every $n \in \mathbb{N}$, b_n serves as an upper bound for A. Suppose n is an arbitrary natural number. Let a be an arbitrary element of A. Then there exists $x \in \mathbb{N}$ such that $a = a_x$. We have two cases:

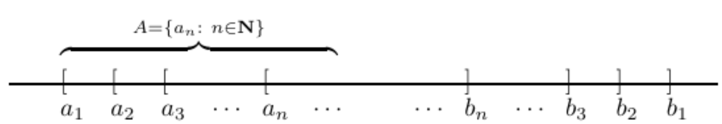
Case 1: $n \geq x$. Then $I_x \supseteq I_n$. We know $b_n \in I_n \subseteq I_x$, so $b_n \in I_x$ and $a = a_x \leq b_n$.

Case 2: $n < x$. Then $I_n \supseteq I_x$. We know $a_x \in I_x \subseteq I_n$, so $a_x \in I_n$ and $a = a_x \leq b_n$. Since a was an arbitrary element of A, and n was an arbitrary natural number, for every $n \in \mathbb{N}$, b_n is an upper bound of A.

Now that we've established $\forall n \in \mathbb{N}(\forall a \in A(a \leq b_n))$, by the Axiom of Completeness, since A is nonempty and has an upper bound, it has a supremum, which we will denote as $x = \sup A$. We want to show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, which can be rewritten as $\exists x \in \mathbb{R}(x \bigcap_{n=1}^{\infty} I_n)$, $\exists x \in \mathbb{R}(\forall n \in \mathbb{N}(x \in I_n))$, and finally $\exists x \in \mathbb{R}(\forall n \in \mathbb{N}(a_n \leq x \leq b_n))$.

We will show that $\forall n \in \mathbb{N}(a_n \leq x \leq b_n))$, where $x = \sup A$. Let n be an arbitrary natural number. Then $a_n \in A$, so $a_n \leq x$. Since x is the supremum of A and $\forall n \in \mathbb{N}(\forall a \in A(a \leq b_n))$, b_n is an upper bound of A, and $x \leq b_n$. So $a_n \leq x \leq b_n$ for arbitrary $n \in \mathbb{N}$. This means that x is in every I_n, and therefore their intersection is nonempty. □

Commentary
We showed that the supremum of the set $A = \{a_n \mid n \in \mathbb{N}\}$ was in every interval, proving that their intersection is nonempty.



1.3.2 Density of Q in R

Theorem 9. *(Archimedian property) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*

Proof. Assume, for contradiction, that $\mathbb{N}$ is bounded above. By the axiom of completeness, $\mathbb{N}$ should then have a least upper bound, and we can set $\alpha = \sup \mathbb{N}$. Then by lemma 1 if we consider $\alpha - 1$, we can declare some $n \in \mathbb{N}$ such that $\alpha - 1 < n$; this is equivalent to $\alpha < n + 1$. However, since $\mathbb{N}$ is closed under addition, $\alpha < n + 1 \in \mathbb{N}$, contradicting the fact that α is supposed to be an upper bound for $\mathbb{N}$. □

Theorem 10. *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

Proof. $1/y$ is a real number since $y > 0$. By theorem 9 we can declare some $n \in \mathbb{N}$ such that $n > 1/y$. Since $n > 0$ and $y > 0$, $1/n < y$. □

Theorem 11. *(Density of $\mathbb{Q}$ in $\mathbb{R}$) For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Proof. A rational number is a quotient of integers, so we must produce $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b$$

Using the archimedian property, since $a < b$ we can declare $n \in \mathbb{N}$ such that

$$\frac{1}{n} < b - a \tag{1.1}$$

The inequality we are trying to prove is equivalent to $na < m < nb$. The idea now is to choose m to be the smallest integer greater than na. In other words, pick $m \in \mathbb{Z}$ so that

$$m - 1 \leq na < m$$

First see that this immediately yields $a < m/n$. Inequality 1.1 is equivalent to $a < b - 1/n$, so we can write

$$\begin{aligned} m &\leq na + 1 \\ &< n\left(b - \frac{1}{n}\right) + 1 \\ &= nb \end{aligned}$$

where $m < nb$ implies $m/n < b$, so we have $a < m/n < b$, as desired. □

1.3.3