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Chapter 1

Logic

1.0.1 Logic Factsheet

De Morgan's laws

$$\neg (P \wedge Q) \text{ is equivalent to } \neg P \vee \neg Q$$
$$\neg (P \vee Q) \text{ is equivalent to } \neg P \wedge \neg Q$$

Commutative laws

$$P \wedge Q \text{ is equivalent to } Q \wedge P$$
$$P \vee Q \text{ is equivalent to } Q \vee P$$

Associative laws

$$P \wedge (Q \wedge R) \text{ is equivalent to } (P \wedge Q) \wedge R$$
$$P \vee (Q \vee R) \text{ is equivalent to } (P \vee Q) \vee R$$

Indempotent laws

$$P \wedge P \text{ is equivalent to } P$$
$$P \vee P \text{ is equivalent to } P$$

Distributive laws

$$P \wedge (Q \vee R) \text{ is equivalent to } (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \text{ is equivalent to } (P \vee Q) \wedge (P \vee R)$$

Absorption laws

$$P \vee (P \wedge Q) \text{ is equivalent to } P$$
$$P \wedge (P \vee Q) \text{ is equivalent to } P$$

Double Negation law

$$\neg\neg P \text{ is equivalent to } P$$

1.0.2 Set operation definitions

The *intersection* of two sets A and B is the set $A \cap B$ defined as follows:

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$$

The *union* of A and B is the set $A \cup B$ defined as follows:

$$A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$$

The *difference* of A and B is the set $A \setminus B$ defined as follows:

$$A \setminus B = \{x \,|\, x \in A \text{ and } x \notin B\}$$

See that

$$x \in A \cap B = x \in \{y \,|\, y \in A \text{ and } y \in B\}$$

where y is a dummy variable. So we can also write that

$$x \in A \cap B = x \in A \wedge x \in B$$

The same can be shown for the union and difference.

1.0.3 Distributivity of set operations

We show

$$x \in A \cap (B \cup C) \text{ is equivalent to } x \in (A \cap B) \cup (A \cap C)$$

By analysing their logical forms:

$$\begin{aligned}
&x \in A \cap (B \cup C) \\
&= x \in A \wedge x \in (B \cup C) \\
&= x \in A \wedge (x \in B \vee x \in C)
\end{aligned}$$

and

$$\begin{aligned}
&x \in (A \cap B) \cup (A \cap C) \\
&= x \in (A \cap B) \vee x \in (A \cap C) \\
&= (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\
&= [(x \in A \wedge x \in B) \vee x \in A)] \wedge [(x \in A \wedge x \in B) \vee x \in C)] \\
&= x \in A \wedge [(x \in A \vee x \in C) \wedge (x \in B \vee x \in C)] \\
&= [x \in A \wedge (x \in A \vee x \in C)] \wedge (x \in B \vee x \in C) \\
&= x \in A \wedge (x \in B \vee x \in C)
\end{aligned}$$

We can also show, in a similar manner, that

$$x \in A \cup (B \cap C) \text{ is equivalent to } x \in (A \cup B) \cap (A \cup C)$$

1.0.4 $x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$

We can also show

$$x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$$

See that

$$\begin{aligned}
&x \in A \setminus (B \cap C) \\
&= x \in A \wedge \neg(x \in B \cap C) && \text{(Definition of } \setminus) \\
&= x \in A \wedge \neg(x \in B \wedge x \in C) && \text{(Definition of } \cap) \\
&= x \in A \wedge (x \notin B \vee x \notin C) && \text{(De Morgan's)} \\
&= (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) && \text{(Distributivity)} \\
&= (x \in A \setminus B) \vee (x \in A \setminus C) && \text{(Definition of } \setminus) \\
&= x \in (A \setminus B) \cup (A \setminus C) && \text{(Definition of } \cup)
\end{aligned}$$

1.0.5 $x \in (A \cup B) \setminus (A \cap B) = x \in (A \setminus B) \cup (B \setminus A)$

$$\begin{aligned}
&x \in (A \cup B) \setminus (A \cap B) \\
&= (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B) && \text{(By definition)} \\
&= (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) && \text{(De Morgan's)} \\
&= [(x \in A \vee x \in B) \wedge (x \notin A)] \\
&\qquad \vee [(x \in A \vee x \in B) \wedge (x \notin B)] && \text{(Distributivity)} \\
&= [(x \notin A \wedge x \in A) \vee (x \notin A \wedge x \in B)] \\
&\qquad \vee [(x \notin B \wedge x \in A) \vee (x \notin B \wedge x \in B)] && \text{(Distributivity)} \\
&= (x \notin A \wedge x \in B) \vee (x \notin B \wedge x \in A) \\
&= (x \in A \wedge x \notin B) \wedge (x \in B \wedge x \notin A) && \text{(Commutativity)} \\
&= x \in (A \setminus B) \cup (B \setminus A) && \text{(By definition)}
\end{aligned}$$

1.0.6 $(A \cap B) \cap (A \setminus B) = \emptyset$

See that

$$\begin{aligned}
&x \in (A \cap B) \cap (A \setminus B) \\
&= (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin B) && \text{(Definition)} \\
&= x \in A \wedge \underbrace{(x \in B \wedge x \notin B)}_{\text{Contradiction}} && \text{(Associativity + Commutativity)}
\end{aligned}$$

The last statement is a contradiction, so the statement $x \in (A \cap B) \cap (A \setminus B)$ will always be false, no matter what x is. In other words, nothing can be an element of $(A \cap B) \cap (A \setminus B)$, so it must be the case that $(A \cap B) \cap (A \setminus B) = \emptyset$; $A \cap B$ and $A \setminus B$ are disjoint.

1.0.7 Conditional and Contrapositive laws

Conditional Law

$$P \to Q \text{ is equivalent to } \neg(P \wedge \neg Q)$$

by De Morgan's law we can also say that

$$P \to Q \text{ is equivalent to } \neg P \vee Q$$

Contrapositive law

$$P \to Q \text{ is equivalent to } \neg Q \to \neg P$$

This can be justified using

$$P \to Q = \neg(P \wedge \neg Q) = \neg(\neg Q \wedge P) = \neg Q \to \neg P$$

Intuition
Intuitive ways to think of $P \to Q$ (and equivalently $\neg Q \to \neg P$) include:

- P implies Q.
- Q, if P.
- P only if Q.
- P is a sufficient condition for Q.
- Q is a necessary condition for P.

1.0.8 Biconditional statements

We write

$$P \leftrightarrow Q = (P \to Q) \wedge (Q \to P)$$

Note that by the contrapositive law, this is also equivalent to

$$(P \to Q) \wedge (\neg P \to \neg Q)$$

Intuition
$Q \to P$ can be written as 'P if Q' and $P \to Q$ can be written as 'P only if Q' (since this means $\neg Q \to \neg P$ which is $P \to Q$).

Combining the two as $(P \to Q) \wedge (Q \to P) = P \leftrightarrow Q$ therefore corresponds to the statement 'P if and only if Q'.

$P \leftrightarrow Q$ means 'P iff Q', or 'P is a necessary and sufficient condition for Q'.

1.1 Quantificational logic

1.1.1 Quantifier negation laws

We have

$$\neg\exists x P(x) \text{ is equivalent to } \forall x \neg P(x)$$
$$\neg\forall x P(x) \text{ is equivalent to } \exists x \neg P(x)$$

Intuition
No matter what $P(x)$ stands for, the formula $\neg\exists x P(x)$ means that there is no value of x for which $P(x)$ is true; this is ths same as saying that for every value of x in the universe of discourse, $P(x)$ is false—meaning $\forall x \neg P(x)$.

Similarly, to say that $\neg\forall x P(x)$ means that it is not the case that for all values of x, $P(x)$ is true. This is equivalent to saying that that there is at least one value of x for which $P(x)$ is false—so $\exists x \neg P(x)$.

1.1.2 Notation

'Exactly one' notation
We write

$$\exists! x P(x) = \exists x (P(x) \wedge \neg\exists y (P(y) \wedge y \neq x))$$

As a shorthand way to write 'there is exactly one value of x such that $P(x)$ is true', or 'there is a unique x such that $P(x)$'.

Specifying quantifiers
We write

$$\forall x \in A\, P(x)$$

to mean that *for every value of x in the set A, $P(x)$ is true.* Similarly,

$$\exists x \in A\, P(x)$$

means *there is at least one value of x in the set A such that $P(x)$ is true.*

Formulas containing bounded quantifiers can also be thought of as abbreviations for more complicated formulas containing only normal, unbounded quantifiers. See that

$$\forall x \in A\, P(x) = \forall x (x \in A \rightarrow P(x))$$

and

$$\exists x \in A\, P(x) = \exists x (x \in A \wedge P(x))$$

1.1.3 Negation law for bounded quantifiers

We can show

$$\neg\forall x \in A\, P(x) = \exists x \in A\, \neg P(x)$$

See that

$$\begin{aligned}
&\neg\forall x \in A\, P(x) \\
&= \neg\forall x(x \in A \to P(x)) && \text{(as defined)} \\
&= \exists x\neg(x \in A \to P(x)) && \text{(negation law)} \\
&= \exists x\neg\neg(x \in A \land \neg P(x)) && \text{(conditional law)} \\
&= \exists x(x \in A \land \neg P(x)) \\
&= \exists x \in A\, \neg P(x) && \text{(as defined)}
\end{aligned}$$

Similarly we can show

$$\neg\exists x \in A\, P(x) = \forall x \in A\, \neg P(x)$$

See that

$$\begin{aligned}
&\neg\exists x \in A\, P(x) \\
&= \neg\exists x(x \in A \land P(x)) && \text{(as defined)} \\
&= \forall x\neg(x \in A \land P(x)) && \text{(negation law)} \\
&= \forall x(x \in A \to \neg P(x)) && \text{(conditional law)} \\
&= \forall x \in A\, \neg P(x) && \text{(as defined)}
\end{aligned}$$

1.1.4 Vacuously true

It is clear that if $A = \emptyset$ then $\exists x \in A\, P(x)$ will be false regardless of $P(x)$, since there is nothing in A that makes $P(x)$ come true (since there is nothing in A to being with).

Now consider $\forall x \in A\, P(x)$. We can reason that

$$\forall x \in A\, P(x) = \neg \exists x \in A\, \neg P(x) \quad \text{(quantifier negation)}$$

See that if $A = \emptyset$ then this formula will be true, no matter what $P(x)$ is. In this case we say that the statement is *vacuously true*.

Another way to see this is to rewrite

$$\forall x \in A\, P(x) = \forall x (x \in A \to P(x))$$

The only way this can be false is if there is some value of x such that $x \in A$ is true but $P(x)$ false; but there is no such value of x. Intuitively, because the condition cannot be met, it is impossible to provide a counterexample to prove something wrong.

An analogy would be me claiming 'i've never lost a race to Usain Bolt'. This is true, but vacuously so.

1.1.5 Alternate definition for indexed families

Say we are looking for the set $\{p_1, p_2, \ldots, p_{100}\}$; another way of describing this set would be to say that it consists of all numbers p_i, for i an element of the set $I = \{1, 2, 3, \ldots, 100\} = \{i \in \mathbb{N} | 1 \leq i \leq 100\}$. We can write

$$P = \{p_i | i \in I\}$$

Each element p_i in this set is identified by $i \in I$, called the *index* of each element. A set defined this way is called an *indexed family*, and I the *index set*. Although the indices for an indexed family are often numbers, they need not be.

In general, see that any indexed family

$$A = \{x_i | i \in I\}$$

Can also be defined as

$$A = \{x | \exists i \in I \, (x = x_i)\}$$

It follows that the statement

$$x \in \{x_i | i \in I\}$$

means the same thing as

$$\exists i \in I \, (x = x_i)$$

1.1.6 Power set

Suppose A is as set. The *power set* of A, denoted $\mathscr{P}(A)$, is the set whose elements are all subsets of A. In other words,

$$\mathscr{P}(A) = \{x | x \subseteq A\}$$

For instance, the set $A = \{7, 12\}$ has four subsets $\emptyset, \{7\}, \{12\}$, and $\{7, 12\}$; thus, $\mathscr{P}(A) = \{\emptyset, \{7\}, \{12\}, \{7, 12\}\}$.

1.1.7 Intersection and union of a family of sets

Suppose $\mathcal{F}$ is a family of sets. The *intersection* and *union* of $\mathcal{F}$ are the sets $\bigcap \mathcal{F}$ and $\bigcup \mathcal{F}$ are defined as follows:

$$\bigcap \mathcal{F} = \{x | \forall A \in \mathcal{F}(x \in A)\} = \{x | \forall A(A \in \mathcal{F} \to x \in A)\}$$
$$\bigcup \mathcal{F} = \{x | \exists A \in \mathcal{F}(x \in A)\} = \{x | \exists A(A \in \mathcal{F} \wedge x \in A)\}$$

Notice that if A and B are any two sets and $\mathcal{F} = \{A, B\}$, then $\bigcap \mathcal{F} = A \cap B$ and $\bigcup \mathcal{F} = A \cup B$; the definitions of intersection and union of a family of sets are generalisations of our old definitions of the intersection and union of two sets.

Alternative notation
An alternative notation is sometimes used for the union or intersection of an indexed family of sets. Suppose $\mathcal{F} = \{A_i | i \in I\}$, where each A_i is a set, then $\bigcap \mathcal{F}$ and $\bigcup \mathcal{F}$ could also be written as $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$; as such

$$\bigcap \mathcal{F} = \bigcap_{i \in I} A_i = \{x | \forall i \in I(x \in A_i)\}$$
$$\bigcup \mathcal{F} = \bigcup_{i \in I} A_i = \{x | \exists i \in I(x \in A_i)\}$$

1.1.8 More on set notation

One generally defines a set using the elementhood test notation

$$\{x|P(x)\}$$

Where the set consists of all x that satisfy the specified condition $P(x)$. Sometimes this notation can be modified to allow the x before the vertical line to be replaced with a more complex expression. For example, suppose we wanted to define S to be the set of all perfect squres, we could write

$$S = \{n^2|n \in \mathbb{N}\}$$

This is the same as

$$S = \{x|\exists n \in \mathbb{N}(x = n^2)\}$$

See therefore that

$$x \in \{n^2|n \in \mathbb{N}\} = \exists n \in \mathbb{N}(x = n^2)$$