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Chapter 1

Logic

1.0.1 Logic Factsheet

De Morgan's laws

$$\neg(P \wedge Q) \text{ is equivalent to } \neg P \vee \neg Q$$
$$\neg(P \vee Q) \text{ is equivalent to } \neg P \wedge \neg Q$$

Commutative laws

$$P \wedge Q \text{ is equivalent to } Q \wedge P$$
$$P \vee Q \text{ is equivalent to } Q \vee P$$

Associative laws

$$P \wedge (Q \wedge R) \text{ is equivalent to } (P \wedge Q) \wedge R$$
$$P \vee (Q \vee R) \text{ is equivalent to } (P \vee Q) \vee R$$

Indempotent laws

$$P \wedge P \text{ is equivalent to } P$$
$$P \vee P \text{ is equivalent to } P$$

Distributive laws

$$P \wedge (Q \vee R) \text{ is equivalent to } (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \text{ is equivalent to } (P \vee Q) \wedge (P \vee R)$$

Absorption laws

$$P \vee (P \wedge Q) \text{ is equivalent to } P$$
$$P \wedge (P \vee Q) \text{ is equivalent to } P$$

Double Negation law

$$\neg\neg P \text{ is equivalent to } P$$

1.0.2 Set operation definitions

The *intersection* of two sets A and B is the set $A \cap B$ defined as follows:

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$$

The *union* of A and B is the set $A \cup B$ defined as follows:

$$A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$$

The *difference* of A and B is the set $A \setminus B$ defined as follows:

$$A \setminus B = \{x \,|\, x \in A \text{ and } x \notin B\}$$

See that

$$x \in A \cap B = x \in \{y \,|\, y \in A \text{ and } y \in B\}$$

where y is a dummy variable. So we can also write that

$$x \in A \cap B = x \in A \wedge x \in B$$

The same can be shown for the union and difference.

1.0.3 Distributivity of set operations

We show

$$x \in A \cap (B \cup C) \text{ is equivalent to } x \in (A \cap B) \cup (A \cap C)$$

By analysing their logical forms:

$$\begin{aligned}
&x \in A \cap (B \cup C) \\
&= x \in A \wedge x \in (B \cup C) \\
&= x \in A \wedge (x \in B \vee x \in C)
\end{aligned}$$

and

$$\begin{aligned}
&x \in (A \cap B) \cup (A \cap C) \\
&= x \in (A \cap B) \vee x \in (A \cap C) \\
&= (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\
&= [(x \in A \wedge x \in B) \vee x \in A)] \wedge [(x \in A \wedge x \in B) \vee x \in C)] \\
&= x \in A \wedge [(x \in A \vee x \in C) \wedge (x \in B \vee x \in C)] \\
&= [x \in A \wedge (x \in A \vee x \in C)] \wedge (x \in B \vee x \in C) \\
&= x \in A \wedge (x \in B \vee x \in C)
\end{aligned}$$

We can also show, in a similar manner, that

$$x \in A \cup (B \cap C) \text{ is equivalent to } x \in (A \cup B) \cap (A \cup C)$$

1.0.4 $x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$

We can also show

$$x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$$

See that

$$\begin{aligned}
&x \in A \setminus (B \cap C) \\
&= x \in A \wedge \neg(x \in B \cap C) && \text{(Definition of } \setminus) \\
&= x \in A \wedge \neg(x \in B \wedge x \in C) && \text{(Definition of } \cap) \\
&= x \in A \wedge (x \notin B \vee x \notin C) && \text{(De Morgan's)} \\
&= (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) && \text{(Distributivity)} \\
&= (x \in A \setminus B) \vee (x \in A \setminus C) && \text{(Definition of } \setminus) \\
&= x \in (A \setminus B) \cup (A \setminus C) && \text{(Definition of } \cup)
\end{aligned}$$

1.0.5 $x \in (A \cup B) \setminus (A \cap B) = x \in (A \setminus B) \cup (B \setminus A)$

$$\begin{aligned}
&x \in (A \cup B) \setminus (A \cap B) \\
&= (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B) && \text{(By definition)} \\
&= (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) && \text{(De Morgan's)} \\
&= [(x \in A \vee x \in B) \wedge (x \notin A)] \\
&\qquad \vee [(x \in A \vee x \in B) \wedge (x \notin B)] && \text{(Distributivity)} \\
&= [(x \notin A \wedge x \in A) \vee (x \notin A \wedge x \in B)] \\
&\qquad \vee [(x \notin B \wedge x \in A) \vee (x \notin B \wedge x \in B)] && \text{(Distributivity)} \\
&= (x \notin A \wedge x \in B) \vee (x \notin B \wedge x \in A) \\
&= (x \in A \wedge x \notin B) \wedge (x \in B \wedge x \notin A) && \text{(Commutativity)} \\
&= x \in (A \setminus B) \cup (B \setminus A) && \text{(By definition)}
\end{aligned}$$

1.0.6 $(A \cap B) \cap (A \setminus B) = \emptyset$

See that

$$\begin{aligned}
&x \in (A \cap B) \cap (A \setminus B) \\
&= (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin B) && \text{(Definition)} \\
&= x \in A \wedge \underbrace{(x \in B \wedge x \notin B)}_{\text{Contradiction}} && \text{(Associativity + Commutativity)}
\end{aligned}$$

The last statement is a contradiction, so the statement $x \in (A \cap B) \cap (A \setminus B)$ will always be false, no matter what x is. In other words, nothing can be an element of $(A \cap B) \cap (A \setminus B)$, so it must be the case that $(A \cap B) \cap (A \setminus B) = \emptyset$; $A \cap B$ and $A \setminus B$ are disjoint.

1.0.7 Conditional and Contrapositive laws

Conditional Law

$$P \to Q \text{ is equivalent to } \neg(P \wedge \neg Q)$$

by De Morgan's law we can also say that

$$P \to Q \text{ is equivalent to } \neg P \vee Q$$

Contrapositive law

$$P \to Q \text{ is equivalent to } \neg Q \to \neg P$$

This can be justified using

$$P \to Q = \neg(P \wedge \neg Q) = \neg(\neg Q \wedge P) = \neg Q \to \neg P$$

Intuition
Intuitive ways to think of $P \to Q$ (and equivalently $\neg Q \to \neg P$) include:

- P implies Q.
- Q, if P.
- P only if Q.
- P is a sufficient condition for Q.
- Q is a necessary condition for P.

1.0.8 Biconditional statements

We write

$$P \leftrightarrow Q = (P \to Q) \wedge (Q \to P)$$

Note that by the contrapositive law, this is also equivalent to

$$(P \to Q) \wedge (\neg P \to \neg Q)$$

Intuition
$Q \to P$ can be written as 'P if Q' and $P \to Q$ can be written as 'P only if Q' (since this means $\neg Q \to \neg P$ which is $P \to Q$).

Combining the two as $(P \to Q) \wedge (Q \to P) = P \leftrightarrow Q$ therefore corresponds to the statement 'P if and only if Q'.

$P \leftrightarrow Q$ means 'P iff Q', or 'P is a necessary and sufficient condition for Q'.

1.1 Quantificational logic

1.1.1 Quantifier negation laws

We have

$$\neg\exists x P(x) \text{ is equivalent to } \forall x \neg P(x)$$
$$\neg\forall x P(x) \text{ is equivalent to } \exists x \neg P(x)$$

Intuition
No matter what $P(x)$ stands for, the formula $\neg\exists x P(x)$ means that there is no value of x for which $P(x)$ is true; this is ths same as saying that for every value of x in the universe of discourse, $P(x)$ is false—meaning $\forall x \neg P(x)$.

Similarly, to say that $\neg\forall x P(x)$ means that it is not the case that for all values of x, $P(x)$ is true. This is equivalent to saying that that there is at least one value of x for which $P(x)$ is false—so $\exists x \neg P(x)$.

1.1.2 Notation

'Exactly one' notation
We write

$$\exists! x P(x) = \exists x(P(x) \wedge \neg\exists y(P(y) \wedge y \neq x))$$

As a shorthand way to write 'there is exactly one value of x such that $P(x)$ is true', or 'there is a unique x such that $P(x)$'.

Specifying quantifiers
We write

$$\forall x \in A\, P(x)$$

to mean that *for every value of x in the set A, $P(x)$ is true*. Similarly,

$$\exists x \in A\, P(x)$$

means *there is at least one value of x in the set A such that $P(x)$ is true*.

Formulas containing bounded quantifiers can also be thought of as abbreviations for more complicated formulas containing only normal, unbounded quantifiers. See that

$$\forall x \in A\, P(x) = \forall x(x \in A \rightarrow P(x))$$

and

$$\exists x \in A\, P(x) = \exists x(x \in A \wedge P(x))$$

1.1.3 Negation law for bounded quantifiers

We can show

$$\neg\forall x \in A\, P(x) = \exists x \in A\, \neg P(x)$$

See that

$$\begin{aligned}
&\neg\forall x \in A\, P(x) \\
&= \neg\forall x(x \in A \to P(x)) && \text{(as defined)} \\
&= \exists x\neg(x \in A \to P(x)) && \text{(negation law)} \\
&= \exists x\neg\neg(x \in A \land \neg P(x)) && \text{(conditional law)} \\
&= \exists x(x \in A \land \neg P(x)) \\
&= \exists x \in A\, \neg P(x) && \text{(as defined)}
\end{aligned}$$

Similarly we can show

$$\neg\exists x \in A\, P(x) = \forall x \in A\, \neg P(x)$$

See that

$$\begin{aligned}
&\neg\exists x \in A\, P(x) \\
&= \neg\exists x(x \in A \land P(x)) && \text{(as defined)} \\
&= \forall x\neg(x \in A \land P(x)) && \text{(negation law)} \\
&= \forall x(x \in A \to \neg P(x)) && \text{(conditional law)} \\
&= \forall x \in A\, \neg P(x) && \text{(as defined)}
\end{aligned}$$

1.1.4 Vacuously true

It is clear that if $A = \emptyset$ then $\exists x \in A\, P(x)$ will be false regardless of $P(x)$, since there is nothing in A that makes $P(x)$ come true (since there is nothing in A to being with).

Now consider $\forall x \in A\, P(x)$. We can reason that

$$\forall x \in A\, P(x) = \neg \exists x \in A\, \neg P(x) \quad \text{(quantifier negation)}$$

See that if $A = \emptyset$ then this formula will be true, no matter what $P(x)$ is. In this case we say that the statement is *vacuously true*.

Another way to see this is to rewrite

$$\forall x \in A\, P(x) = \forall x (x \in A \to P(x))$$

The only way this can be false is if there is some value of x such that $x \in A$ is true but $P(x)$ false; but there is no such value of x. Intuitively, because the condition cannot be met, it is impossible to provide a counterexample to prove something wrong.

An analogy would be me claiming 'i've never lost a race to Usain Bolt'. This is true, but vacuously so.

1.1.5 Alternate definition for indexed families

Say we are looking for the set $\{p_1, p_2, \ldots, p_{100}\}$; another way of describing this set would be to say that it consists of all numbers p_i, for i an element of the set $I = \{1, 2, 3, \ldots, 100\} = \{i \in \mathbb{N} | 1 \leq i \leq 100\}$. We can write

$$P = \{p_i | i \in I\}$$

Each element p_i in this set is identified by $i \in I$, called the *index* of each element. A set defined this way is called an *indexed family*, and I the *index set*. Although the indices for an indexed family are often numbers, they need not be.

In general, see that any indexed family

$$A = \{x_i | i \in I\}$$

Can also be defined as

$$A = \{x | \exists i \in I \, (x = x_i)\}$$

It follows that the statement

$$x \in \{x_i | i \in I\}$$

means the same thing as

$$\exists i \in I \, (x = x_i)$$

1.1.6 Power set

Suppose A is as set. The *power set* of A, denoted $\mathscr{P}(A)$, is the set whose elements are all subsets of A. In other words,

$$\mathscr{P}(A) = \{x | x \subseteq A\}$$

For instance, the set $A = \{7, 12\}$ has four subsets $\emptyset, \{7\}, \{12\}$, and $\{7, 12\}$; thus, $\mathscr{P}(A) = \{\emptyset, \{7\}, \{12\}, \{7, 12\}\}$.

1.1.7 Intersection and union of a family of sets

Suppose $\mathcal{F}$ is a family of sets. The *intersection* and *union* of $\mathcal{F}$ are the sets $\bigcap \mathcal{F}$ and $\bigcup \mathcal{F}$ are defined as follows:

$$\bigcap \mathcal{F} = \{x | \forall A \in \mathcal{F}(x \in A)\} = \{x | \forall A (A \in \mathcal{F} \to x \in A)\}$$
$$\bigcup \mathcal{F} = \{x | \exists A \in \mathcal{F}(x \in A)\} = \{x | \exists A (A \in \mathcal{F} \land x \in A)\}$$

Notice that if A and B are any two sets and $\mathcal{F} = \{A, B\}$, then $\bigcap \mathcal{F} = A \cap B$ and $\bigcup \mathcal{F} = A \cup B$; the definitions of intersection and union of a family of sets are generalisations of our old definitions of the intersection and union of two sets.