

Velleman

Malcolm

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Chapter 1

Logic

1.0.1 Logic Factsheet

De Morgan's laws

$\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$

$\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$

Commutative laws

$P \wedge Q$ is equivalent to $Q \wedge P$

$P \vee Q$ is equivalent to $Q \vee P$

Associative laws

$P \wedge (Q \wedge R)$ is equivalent to $(P \wedge Q) \wedge R$

$P \vee (Q \vee R)$ is equivalent to $(P \vee Q) \vee R$

Idempotent laws

$P \wedge P$ is equivalent to P

$P \vee P$ is equivalent to P

Distributive laws

$P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$

Absorption laws

$P \vee (P \wedge Q)$ is equivalent to P

$P \wedge (P \vee Q)$ is equivalent to P

Double Negation law

$\neg\neg P$ is equivalent to P

1.0.2 Set operation definitions

The *intersection* of two sets A and B is the set $A \cap B$ defined as follows:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The *union* of A and B is the set $A \cup B$ defined as follows:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The *difference* of A and B is the set $A \setminus B$ defined as follows:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

See that

$$x \in A \cap B = x \in \{y \mid y \in A \text{ and } y \in B\}$$

where y is a dummy variable. So we can also write that

$$x \in A \cap B = x \in A \wedge x \in B$$

The same can be shown for the union and difference.

1.0.3 Distributivity of set operations

We show

$$x \in A \cap (B \cup C) \text{ is equivalent to } x \in (A \cap B) \cup (A \cap C)$$

By analysing their logical forms:

$$\begin{aligned} x \in A \cap (B \cup C) \\ &= x \in A \wedge x \in (B \cup C) \\ &= x \in A \wedge (x \in B \vee x \in C) \end{aligned}$$

and

$$\begin{aligned} x \in (A \cap B) \cup (A \cap C) \\ &= x \in (A \cap B) \vee x \in (A \cap C) \\ &= (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\ &= [(x \in A \wedge x \in B) \vee x \in A] \wedge [(x \in A \wedge x \in B) \vee x \in C] \\ &= x \in A \wedge [(x \in A \vee x \in C) \wedge (x \in B \vee x \in C)] \\ &= [x \in A \wedge (x \in A \vee x \in C)] \wedge (x \in B \vee x \in C) \\ &= x \in A \wedge (x \in B \vee x \in C) \end{aligned}$$

We can also show, in a similar manner, that

$$x \in A \cup (B \cap C) \text{ is equivalent to } x \in (A \cup B) \cap (A \cup C)$$

$$\mathbf{1.0.4} \quad x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$$

We can also show

$$x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$$

See that

$$\begin{aligned} x \in A \setminus (B \cap C) & \\ &= x \in A \wedge \neg(x \in B \cap C) && \text{(Definition of } \setminus \text{)} \\ &= x \in A \wedge \neg(x \in B \wedge x \in C) && \text{(Definition of } \cap \text{)} \\ &= x \in A \wedge (x \notin B \vee x \notin C) && \text{(De Morgan's)} \\ &= (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) && \text{(Distributivity)} \\ &= (x \in A \setminus B) \vee (x \in A \setminus C) && \text{(Definition of } \setminus \text{)} \\ &= x \in (A \setminus B) \cup (A \setminus C) && \text{(Definition of } \cup \text{)} \end{aligned}$$

$$\mathbf{1.0.5} \quad x \in (A \cup B) \setminus (A \cap B) = x \in (A \setminus B) \cup (B \setminus A)$$

$$\begin{aligned} x \in (A \cup B) \setminus (A \cap B) & \\ &= (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B) && \text{(By definition)} \\ &= (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) && \text{(De Morgan's)} \\ &= [(x \in A \vee x \in B) \wedge (x \notin A)] && \\ &\quad \vee [(x \in A \vee x \in B) \wedge (x \notin B)] && \text{(Distributivity)} \\ &= [(x \notin A \wedge x \in A) \vee (x \notin A \wedge x \in B)] && \\ &\quad \vee [(x \notin B \wedge x \in A) \vee (x \notin B \wedge x \in B)] && \text{(Distributivity)} \\ &= (x \notin A \wedge x \in B) \vee (x \notin B \wedge x \in A) && \\ &= (x \in A \wedge x \notin B) \wedge (x \in B \wedge x \notin A) && \text{(Commutativity)} \\ &= x \in (A \setminus B) \cup (B \setminus A) && \text{(By definition)} \end{aligned}$$

$$\mathbf{1.0.6} \quad (A \cap B) \cap (A \setminus B) = \emptyset$$

See that

$$\begin{aligned} x \in (A \cap B) \cap (A \setminus B) & \\ &= (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin B) && \text{(Definition)} \\ &= x \in A \wedge \underbrace{(x \in B \wedge x \notin B)}_{\text{Contradiction}} && \text{(Associativity + Commutativity)} \end{aligned}$$

The last statement is a contradiction, so the statement $x \in (A \cap B) \cap (A \setminus B)$ will always be false, no matter what x is. In other words, nothing can be an element of $(A \cap B) \cap (A \setminus B)$, so it must be the case that $(A \cap B) \cap (A \setminus B) = \emptyset$; $A \cap B$ and $A \setminus B$ are disjoint.

1.0.7 Conditional and Contrapositive laws

Conditional Law

$$P \rightarrow Q \text{ is equivalent to } \neg(P \wedge \neg Q)$$

by De Morgan's law we can also say that

$$P \rightarrow Q \text{ is equivalent to } \neg P \vee Q$$

Contrapositive law

$$P \rightarrow Q \text{ is equivalent to } \neg Q \rightarrow \neg P$$

This can be justified using

$$P \rightarrow Q = \neg(P \wedge \neg Q) = \neg(\neg Q \wedge P) = \neg Q \rightarrow \neg P$$

Intuition

Intuitive ways to think of $P \rightarrow Q$ (and equivalently $\neg Q \rightarrow \neg P$) include:

- P implies Q .
- Q , if P .
- P only if Q .
- P is a sufficient condition for Q .
- Q is a necessary condition for P .

1.0.8 Biconditional statements

We write

$$P \leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$$

Note that by the contrapositive law, this is also equivalent to

$$(P \rightarrow Q) \wedge (\neg P \rightarrow \neg Q)$$

Intuition

$Q \rightarrow P$ can be written as ‘ P if Q ’ and $P \rightarrow Q$ can be written as ‘ P only if Q ’ (since this means $\neg Q \rightarrow \neg P$ which is $P \rightarrow Q$).

Combining the two as $(P \rightarrow Q) \wedge (Q \rightarrow P) = P \leftrightarrow Q$ therefore corresponds to the statement ‘ P if and only if Q ’.

$P \leftrightarrow Q$ means ‘ P iff Q ’, or ‘ P is a necessary and sufficient condition for Q ’.

1.1 Quantificational logic

1.1.1 Quantifier negation laws

We have

$\neg\exists xP(x)$ is equivalent to $\forall x\neg P(x)$

$\neg\forall xP(x)$ is equivalent to $\exists x\neg P(x)$

Intuition

No matter what $P(x)$ stands for, the formula $\neg\exists xP(x)$ means that there is no value of x for which $P(x)$ is true; this is the same as saying that for every value of x in the universe of discourse, $P(x)$ is false—meaning $\forall x\neg P(x)$.

Similarly, to say that $\neg\forall xP(x)$ means that it is not the case that for all values of x , $P(x)$ is true. This is equivalent to saying that there is at least one value of x for which $P(x)$ is false—so $\exists x\neg P(x)$.

1.1.2 Notation

‘Exactly one’ notation

We write

$$\exists!xP(x) = \exists x(P(x) \wedge \neg\exists y(P(y) \wedge y \neq x))$$

As a shorthand way to write ‘there is exactly one value of x such that $P(x)$ is true’, or ‘there is a unique x such that $P(x)$ ’.

Specifying quantifiers

We write

$$\forall x \in A P(x)$$

to mean that *for every value of x in the set A , $P(x)$ is true*. Similarly,

$$\exists x \in A P(x)$$

means *there is at least one value of x in the set A such that $P(x)$ is true*.

Formulas containing bounded quantifiers can also be thought of as abbreviations for more complicated formulas containing only normal, unbounded quantifiers. See that

$$\forall x \in A P(x) = \forall x(x \in A \rightarrow P(x))$$

and

$$\exists x \in A P(x) = \exists x(x \in A \wedge P(x))$$

1.1.3 Negation law for bounded quantifiers

We can show

$$\neg \forall x \in A P(x) = \exists x \in A \neg P(x)$$

See that

$$\begin{aligned} & \neg \forall x \in A P(x) \\ &= \neg \forall x (x \in A \rightarrow P(x)) && \text{(as defined)} \\ &= \exists x \neg (x \in A \rightarrow P(x)) && \text{(negation law)} \\ &= \exists x \neg \neg (x \in A \wedge \neg P(x)) && \text{(conditional law)} \\ &= \exists x (x \in A \wedge \neg P(x)) \\ &= \exists x \in A \neg P(x) && \text{(as defined)} \end{aligned}$$

Similarly we can show

$$\neg \exists x \in A P(x) = \forall x \in A \neg P(x)$$

See that

$$\begin{aligned} & \neg \exists x \in A P(x) \\ &= \neg \exists x (x \in A \wedge P(x)) && \text{(as defined)} \\ &= \forall x \neg (x \in A \wedge P(x)) && \text{(negation law)} \\ &= \forall x (x \in A \rightarrow \neg P(x)) && \text{(conditional law)} \\ &= \forall x \in A \neg P(x) && \text{(as defined)} \end{aligned}$$

1.1.4 Vacuously true

It is clear that if $A = \emptyset$ then $\exists x \in A P(x)$ will be false regardless of $P(x)$, since there is nothing in A that makes $P(x)$ come true (since there is nothing in A to being with).

Now consider $\forall x \in A P(x)$. We can reason that

$$\forall x \in A P(x) = \neg \exists x \in A \neg P(x) \quad (\text{quantifier negation})$$

See that if $A = \emptyset$ then this formula will be true, no matter what $P(x)$ is. In this case we say that the statement is *vacuously true*.

Another way to see this is to rewrite

$$\forall x \in A P(x) = \forall x (x \in A \rightarrow P(x))$$

The only way this can be false is if there is some value of x such that $x \in A$ is true but $P(x)$ false; but there is no such value of x . Intuitively, because the condition cannot be met, it is impossible to provide a counterexample to prove something wrong.

An analogy would be me claiming ‘i’ve never lost a race to Usain Bolt’. This is true, but vacuously so.

1.1.5 Alternate definition for indexed families

Say we are looking for the set $\{p_1, p_2, \dots, p_{100}\}$; another way of describing this set would be to say that it consists of all numbers p_i , for i an element of the set $I = \{1, 2, 3, \dots, 100\} = \{i \in \mathbb{N} | 1 \leq i \leq 100\}$. We can write

$$P = \{p_i | i \in I\}$$

Each element p_i in this set is identified by $i \in I$, called the *index* of each element. A set defined this way is called an *indexed family*, and I the *index set*. Although the indices for an indexed family are often numbers, they need not be.

In general, see that any indexed family

$$A = \{x_i | i \in I\}$$

Can also be defined as

$$A = \{x | \exists i \in I (x = x_i)\}$$

It follows that the statement

$$x \in \{x_i | i \in I\}$$

means the same thing as

$$\exists i \in I (x = x_i)$$

1.1.6 Power set

Suppose A is a set. The *power set* of A , denoted $\mathcal{P}(A)$, is the set whose elements are all subsets of A . In other words,

$$\mathcal{P}(A) = \{x \mid x \subseteq A\}$$

For instance, the set $A = \{7, 12\}$ has four subsets \emptyset , $\{7\}$, $\{12\}$, and $\{7, 12\}$; thus, $\mathcal{P}(A) = \{\emptyset, \{7\}, \{12\}, \{7, 12\}\}$.