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Contents

Chapter 1

Flows on a line

1.1 Introduction

We have the general system as

$$\begin{aligned}
\dot{x}_1 =& f_1(x_1, \ldots, x_n) \\
&\vdots \\
\dot{x}_n =& f_n(x_1, \ldots, x_n)
\end{aligned}$$

its solutions can be intuitively visualised as trajectories flowing through an n-dimensional phase space. In these early sections we start with the simple case where $n = 1$; then we get a single equation of the form

$$\dot{x} = f(x)$$

We call such systems one-dimensional or first-order systems.

We clarify the following:

1. The word *system* is being used here in the context of a dynamical system, not in the classical sense of a collection of two or more equations. Thus a single equation can be called a 'system'.

2. We do not allow f to depend explicitly on time. Time-dependent or 'nonautonomous' equations of the form $\dot{x} = f(x, t)$ are more complicated because they require two variables. Such a system in this case would be regarded as a two-dimensional system, to be discussed later.

1.2 Geometric intuition

1.2.1 Instructive example

Consider the following nonlinear differential equation:

$$\dot{x} = \sin x$$

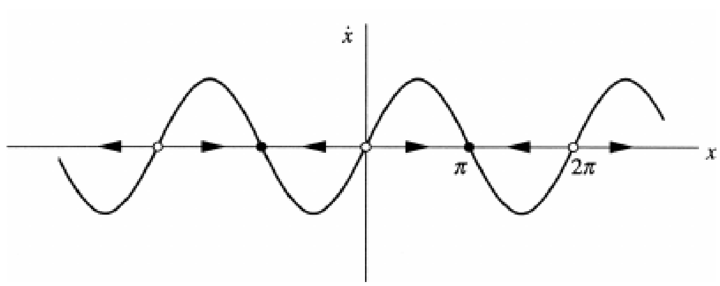
This particular equation can be solved in closed form; using separation of variables then integration we get

$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$

where $x = x_0$ at $t = 0$. Although the result is exact, it isn't really possible to interpret. For instance consider the following

1. Suppose $x_0 = \pi/4$. Describe the qualitative features of the solution $x(t)$ for all $t > 0$. What happens as $t \to \infty$?
2. For an arbitrary initial condition x_0, what is the behaviour of $x(t)$ as $t \to \infty$?

The formula above is not transparent. We consider instead a graphical analysis; the differential equation $\dot{x} = \sin x$ represents a *vector field* on the line: it dictates the velocity vector $\dot{x}$ at each x. To sketch this vector field, it is convenient to plot $\dot{x}$ vs x, drawing arrows on the x-axis to indicate the corresponding velocity vector at each x. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$:



The *flow* is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$; at points where $\dot{x} = 0$, there is no flow, and such points are therefore called *fixed points.* Solid dots represent *stable* fixed points (often called *attractors/sinks*, bscause the flow is toward them). Open dots represent *unstable* fixed points (also known as *repellers/sources*).
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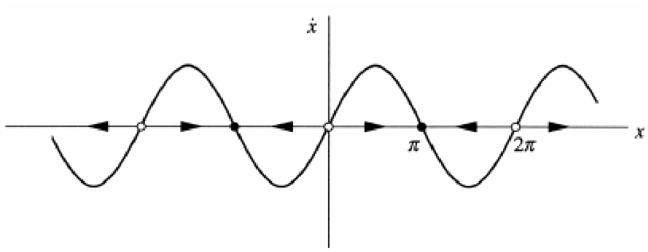
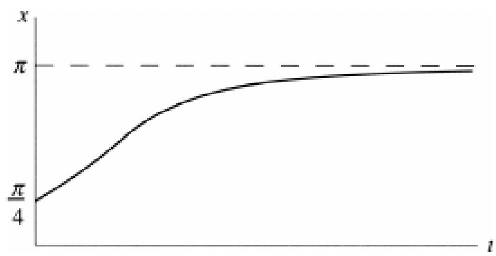
We had



We start at x_0 and watch how x is carried along by the flow. Consider now the questions proposed earlier.

1. Starting at $x_0 = \pi/4$, the flow moves to the right faster and faster until x crosses $\pi/2$ (where $\sin x$ reaches its maximum). Then the flow slows down and eventually approaches the stable fixed point $x = \pi$ from the left. As such the qualitative form of the solution is as follows

   See that the curve is concave up at first, then concave down; this change in concavity corresponds to the initial acceleration for $x < \pi/2$, followed by the deceleration toward $x = \pi$.
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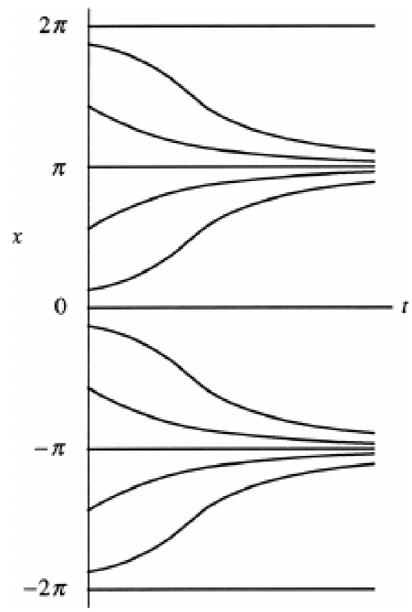
2. The same reasoning applies to any initial condition x_0. If $\dot{x} > 0$ initially, the flow is to the right and x asymptotically approaches the nearest stable point, and similarly to the left for the other case where $\dot{x} < 0$. If $\dot{x} = 0$, then x remains constant. The qualitative form of the solution for any initial condition can be sketched:



Note that the picture can't tell us certain *quantitative* things: for instance we don't know the time at which the speed $|\dot{x}|$ is greatest. But in many cases the *qualitative* information is what we care about. In this case these sketches are useful.

1.2.2 Fixed points and stability