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Contents

Chapter 1

The real numbers

1.1 Definitions and the necessity for real numbers

We begin with the *natural numbers*

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

If we restrict our attention to the natural numbers $\mathbb{N}$, then we can perform addition perfectly well, but if we want to have an additive identity (zero) and the additive inverses necessary to define subtraction, we must extend our system to the *integers*

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The next issue is multiplication and division. The number 1 acts as the multiplicative identity, but in order to define division we need to have multiplicative inverses. Thus we extend our system again to the *rational numbers*

$$\mathbb{Q} = \left\{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0\right\}$$

The properties of $\mathbb{Q}$ essentially make up the definition of what is called a *field.* More formally stated, a field is any set where addition and multiplication are well-defined operations that are commutative, associative, and distributive $a(b+c) = ab + ac$. There must be an additive identity, and every element must have an additive inverse. There must be a multiplicative identity, and multiplicative inverses must exist for all nonzero elements of the field.
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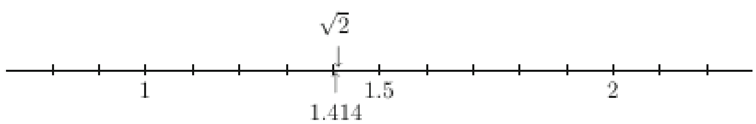
Cont.
The set $\mathbb{Q}$ has a natural *order* defined on it. Given any two rational numbers r and s, exactly one of the following is true:

$$r < s, \quad r = s, \quad \text{or} \quad r > s$$

The ordering is transitive in the sense that if $r < s$ and $s < t$ then $r < t$, so we are led to the mental picture of the rational numbers as being laid out from left to right along a number line. Unlike $\mathbb{Z}$, there are no intervals of empty space; given any two rational numbers $r < s$, the rational number $(r+s)/2$ sits halfway in between, implying that the rational numbers are densely nestled together.

With the field properties of $\mathbb{Q}$ allowing us to safely carry out addition, subtraction, multiplication, and division, we want to consider what $\mathbb{Q}$ is lacking. As an example it can be proven that $\sqrt{2}$ is irrational; using rational numbers it is possible to *approximate* $\sqrt{2}$ quite well:



For instance, $1.414^2 = 1.999396$, and by adding more decimal places to the approximation we can get even closer to the value for $\sqrt{2}$, but even so, it is clear that there is a 'hole' in the rational number line where $\sqrt{2}$ ought to be. $\sqrt{3}$ and $\sqrt{5}$ are also examples of this. If we want every length along the number line to correspond to an actual number, then another extension to our number system is in order. Thus, to the chain $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ we append the *real numbers* $\mathbb{R}$.

Intuition for real numbers
The question of how to construct $\mathbb{R}$ from $\mathbb{Q}$ is complicated; it is discussed later on. For now it is not too inaccurate to say that $\mathbb{R}$ is obtained by filling in the gaps in $\mathbb{Q}$. Wherever there is a hole, a new *irrational* number is defined and placed into the ordering that already exists on $\mathbb{Q}$. The real numbers are the union of these irrational numbers together with the more familiar rational ones.

1.2 The axiom of completeness

What exactly is a real number? We got as far as saying that the set $\mathbb{R}$ of real numbers is an extension of the rational numbers $\mathbb{Q}$ in which there are no holes or gaps, where every length along the number line—such as $\sqrt{2}$—corresponds to a real number.

One will want to improve on this definition, and we could; continuing to use this unprecise definition would mean that whatever precise statements we formulate will rest on unproven assumptions and undefined terms. However, at some point we must draw a line and confess that this is what we have decidd to accept as a reasonable place to start. Naturally there is some debate as to where this line should be drawn. The majority of the material covered in these notes is attributable to many prominent mathematicians; the interesting point is that nearly all of this work was done using intuitive assumptions about the nature of $\mathbb{R}$ quite similar to our own informal understanding at this point. It was these very theorems that motivated a rigorous construction of $\mathbb{R}$; not the other way around.

We will follow this historical model: our own rigorous construction of $\mathbb{R}$ from $\mathbb{Q}$ will be postponed till much later, when the need for such a construction will be more justified and easier to appreciate.