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# Contents

<b>1</b>	<b>Logic</b>	<b>2</b>
1.0.1	Logic Factsheet . . . . .	2
1.0.2	Set operation definitions . . . . .	3
1.0.3	Distributivity of set operations . . . . .	3
1.0.4	$x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$ . . . . .	4
1.0.5	$x \in (A \cup B) \setminus (A \cap B) = x \in (A \setminus B) \cup (B \setminus A)$ . . . . .	4
1.0.6	$(A \cap B) \cap (A \setminus B) = \emptyset$ . . . . .	4
1.0.7	Conditional and Contrapositive laws . . . . .	5
1.0.8	Biconditional statements . . . . .	6
1.1	Quantifier Negation Laws . . . . .	7

# Chapter 1

## Logic

### 1.0.1 Logic Factsheet

#### De Morgan's laws

$\neg(P \wedge Q)$  is equivalent to  $\neg P \vee \neg Q$

$\neg(P \vee Q)$  is equivalent to  $\neg P \wedge \neg Q$

#### Commutative laws

$P \wedge Q$  is equivalent to  $Q \wedge P$

$P \vee Q$  is equivalent to  $Q \vee P$

#### Associative laws

$P \wedge (Q \wedge R)$  is equivalent to  $(P \wedge Q) \wedge R$

$P \vee (Q \vee R)$  is equivalent to  $(P \vee Q) \vee R$

#### Idempotent laws

$P \wedge P$  is equivalent to  $P$

$P \vee P$  is equivalent to  $P$

#### Distributive laws

$P \wedge (Q \vee R)$  is equivalent to  $(P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R)$  is equivalent to  $(P \vee Q) \wedge (P \vee R)$

#### Absorption laws

$P \vee (P \wedge Q)$  is equivalent to  $P$

$P \wedge (P \vee Q)$  is equivalent to  $P$

#### Double Negation law

$\neg\neg P$  is equivalent to  $P$

### 1.0.2 Set operation definitions

The *intersection* of two sets  $A$  and  $B$  is the set  $A \cap B$  defined as follows:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The *union* of  $A$  and  $B$  is the set  $A \cup B$  defined as follows:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The *difference* of  $A$  and  $B$  is the set  $A \setminus B$  defined as follows:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

See that

$$x \in A \cap B = x \in \{y \mid y \in A \text{ and } y \in B\}$$

where  $y$  is a dummy variable. So we can also write that

$$x \in A \cap B = x \in A \wedge x \in B$$

The same can be shown for the union and difference.

### 1.0.3 Distributivity of set operations

We show

$$x \in A \cap (B \cup C) \text{ is equivalent to } x \in (A \cap B) \cup (A \cap C)$$

By analysing their logical forms:

$$\begin{aligned} x \in A \cap (B \cup C) \\ &= x \in A \wedge x \in (B \cup C) \\ &= x \in A \wedge (x \in B \vee x \in C) \end{aligned}$$

and

$$\begin{aligned} x \in (A \cap B) \cup (A \cap C) \\ &= x \in (A \cap B) \vee x \in (A \cap C) \\ &= (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\ &= [(x \in A \wedge x \in B) \vee x \in A] \wedge [(x \in A \wedge x \in B) \vee x \in C] \\ &= x \in A \wedge [(x \in A \vee x \in C) \wedge (x \in B \vee x \in C)] \\ &= [x \in A \wedge (x \in A \vee x \in C)] \wedge (x \in B \vee x \in C) \\ &= x \in A \wedge (x \in B \vee x \in C) \end{aligned}$$

We can also show, in a similar manner, that

$$x \in A \cup (B \cap C) \text{ is equivalent to } x \in (A \cup B) \cap (A \cup C)$$

$$\mathbf{1.0.4} \quad x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$$

We can also show

$$x \in A \setminus (B \cap C) = x \in (A \setminus B) \cup (A \setminus C)$$

See that

$$\begin{aligned} x \in A \setminus (B \cap C) & \\ &= x \in A \wedge \neg(x \in B \cap C) && \text{(Definition of } \setminus \text{)} \\ &= x \in A \wedge \neg(x \in B \wedge x \in C) && \text{(Definition of } \cap \text{)} \\ &= x \in A \wedge (x \notin B \vee x \notin C) && \text{(De Morgan's)} \\ &= (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) && \text{(Distributivity)} \\ &= (x \in A \setminus B) \vee (x \in A \setminus C) && \text{(Definition of } \setminus \text{)} \\ &= x \in (A \setminus B) \cup (A \setminus C) && \text{(Definition of } \cup \text{)} \end{aligned}$$

$$\mathbf{1.0.5} \quad x \in (A \cup B) \setminus (A \cap B) = x \in (A \setminus B) \cup (B \setminus A)$$

$$\begin{aligned} x \in (A \cup B) \setminus (A \cap B) & \\ &= (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B) && \text{(By definition)} \\ &= (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) && \text{(De Morgan's)} \\ &= [(x \in A \vee x \in B) \wedge (x \notin A)] && \\ &\quad \vee [(x \in A \vee x \in B) \wedge (x \notin B)] && \text{(Distributivity)} \\ &= [(x \notin A \wedge x \in A) \vee (x \notin A \wedge x \in B)] && \\ &\quad \vee [(x \notin B \wedge x \in A) \vee (x \notin B \wedge x \in B)] && \text{(Distributivity)} \\ &= (x \notin A \wedge x \in B) \vee (x \notin B \wedge x \in A) && \\ &= (x \in A \wedge x \notin B) \wedge (x \in B \wedge x \notin A) && \text{(Commutativity)} \\ &= x \in (A \setminus B) \cup (B \setminus A) && \text{(By definition)} \end{aligned}$$

$$\mathbf{1.0.6} \quad (A \cap B) \cap (A \setminus B) = \emptyset$$

See that

$$\begin{aligned} x \in (A \cap B) \cap (A \setminus B) & \\ &= (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin B) && \text{(Definition)} \\ &= x \in A \wedge \underbrace{(x \in B \wedge x \notin B)}_{\text{Contradiction}} && \text{(Associativity + Commutativity)} \end{aligned}$$

The last statement is a contradiction, so the statement  $x \in (A \cap B) \cap (A \setminus B)$  will always be false, no matter what  $x$  is. In other words, nothing can be an element of  $(A \cap B) \cap (A \setminus B)$ , so it must be the case that  $(A \cap B) \cap (A \setminus B) = \emptyset$ ;  $A \cap B$  and  $A \setminus B$  are disjoint.

### 1.0.7 Conditional and Contrapositive laws

#### Conditional Law

$$P \rightarrow Q \text{ is equivalent to } \neg(P \wedge \neg Q)$$

by De Morgan's law we can also say that

$$P \rightarrow Q \text{ is equivalent to } \neg P \vee Q$$

#### Contrapositive law

$$P \rightarrow Q \text{ is equivalent to } \neg Q \rightarrow \neg P$$

This can be justified using

$$P \rightarrow Q = \neg(P \wedge \neg Q) = \neg(\neg Q \wedge P) = \neg Q \rightarrow \neg P$$

#### Intuition

Intuitive ways to think of  $P \rightarrow Q$  (and equivalently  $\neg Q \rightarrow \neg P$ ) include:

- $P$  implies  $Q$ .
- $Q$ , if  $P$ .
- $P$  only if  $Q$ .
- $P$  is a sufficient condition for  $Q$ .
- $Q$  is a necessary condition for  $P$ .

### 1.0.8 Biconditional statements

We write

$$P \leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$$

Note that by the contrapositive law, this is also equivalent to

$$(P \rightarrow Q) \wedge (\neg P \rightarrow \neg Q)$$

#### **Intuition**

$Q \rightarrow P$  can be written as ‘ $P$  if  $Q$ ’ and  $P \rightarrow Q$  can be written as ‘ $P$  only if  $Q$ ’ (since this means  $\neg Q \rightarrow \neg P$  which is  $P \rightarrow Q$ ).

Combining the two as  $(P \rightarrow Q) \wedge (Q \rightarrow P) = P \leftrightarrow Q$  therefore corresponds to the statement ‘ $P$  if and only if  $Q$ ’.

$P \leftrightarrow Q$  means ‘ $P$  iff  $Q$ ’, or ‘ $P$  is a necessary and sufficient condition for  $Q$ ’.

## 1.1 Quantifier Negation Laws

We have

$\neg\exists xP(x)$  is equivalent to  $\forall x\neg P(x)$

$\neg\forall xP(x)$  is equivalent to  $\exists x\neg P(x)$

### Intuition

No matter what  $P(x)$  stands for, the formula  $\neg\exists xP(x)$  means that there is no value of  $x$  for which  $P(x)$  is true; this is the same as saying that for every value of  $x$  in the universe of discourse,  $P(x)$  is false—meaning  $\forall x\neg P(x)$ .

Similarly, to say that  $\neg\forall xP(x)$  means that it is not the case that for all values of  $x$ ,  $P(x)$  is true. This is equivalent to saying that there is at least one value of  $x$  for which  $P(x)$  is false—so  $\exists x\neg P(x)$ .