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Chapter 1

Vectors and Matrices

1.1 Intuition for Dot product, Cosine formula, Schwarz and Triangle inequalities

Intuition for dot product

The unit vectors $\boldsymbol{v} = (\cos\alpha, \sin\alpha)$ and $\boldsymbol{w} = (\cos\beta, \sin\beta)$ are plotted as follows

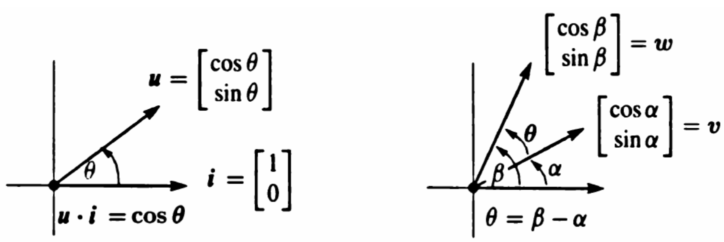

Figure 1.5: Unit vectors: $\boldsymbol{u} \cdot \boldsymbol{i} = \cos\theta$. The angle between the vectors is θ.

See first that when fixed in this form, the magnitude of both vectors is 1, with an angle $\beta - \alpha$ between them. These unit vectors have dot product

$$\boldsymbol{v} \cdot \boldsymbol{w} = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\beta - \alpha)$$

We have θ as the angle between the two vectors; see that the sign of $\boldsymbol{v} \cdot \boldsymbol{w}$ tells us whether θ is below or above a right angle (due to the cosine function being negative for its argument $> \pi/2$ and positive for $< \pi/2$):

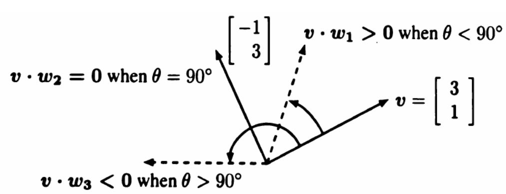

Figure 1.6: Small angle $\boldsymbol{v} \cdot \boldsymbol{w_1} > 0$. Right angle $\boldsymbol{v} \cdot \boldsymbol{w_2} = 0$. Large angle $\boldsymbol{v} \cdot \boldsymbol{w_3} < 0$.

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Cont.
The idea here is that the dot product reveals the exact angle θ; for unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$, the dot product $\boldsymbol{u} \cdot \boldsymbol{U}$ is the cosine of θ. Ths remains true in n dimensions (not shown).

See that any $\boldsymbol{u}$ and $\boldsymbol{v}$ can be fixed in the above form by normalising their lengths to get $\boldsymbol{u} = \boldsymbol{v}/||\boldsymbol{v}||$ and $\boldsymbol{U} = \boldsymbol{w}/||\boldsymbol{w}||$. After which their dot product would give $\cos\theta$. This leads us to the *cosine formula*:

$$\textbf{Cosine formula}: \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}||\,||\boldsymbol{w}||} = \cos\theta \quad \text{if } \boldsymbol{v} \text{ and } \boldsymbol{w} \text{ are nonzero vectors}$$

Perpendicular vectors
See that when the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is $90°$, its cosine is 0; this gives us a way to test this. Also see that for perpendicular vectors:

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = ||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2$$

because

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = (\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w}$$

where $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

Schwarz and Triangle inequalities
First, see from the cosine formula that the dot product of $\boldsymbol{v}/||\boldsymbol{v}||$ and $\boldsymbol{w}/||\boldsymbol{w}||$ never exceeds one (since $\cos\theta$ never exceeds one). This is the the *Schwarz inequality*:

$$\textbf{Schwarz inequality}: |\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}||\,||\boldsymbol{w}||$$

The *Triangle inequality* comes directly from the Schwarz inequality:

$$\textbf{Triangle inequality}: ||\boldsymbol{v} + \boldsymbol{w}|| \leq ||\boldsymbol{v}|| + ||\boldsymbol{w}||$$

This can be seen from

$$||\boldsymbol{v} + \boldsymbol{w}||^2 = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} \leq ||\boldsymbol{v}||^2 + 2||\boldsymbol{v}||\,||\boldsymbol{w}|| + ||\boldsymbol{w}||^2$$

The square root gives us the triangle equality (side 3 cannot exceed side 1 + side 2).

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