Counting With Symmetries

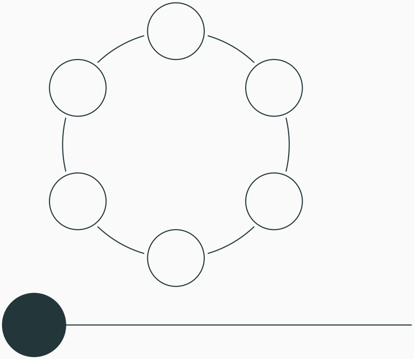
Amber McKeough, Carlos Lira, Clarisse Bonnand, Ethan Kowalenko, Ethan Rooke, Reid Booth, Xavier Ramos

Friday, June 2nd

UC Riverside

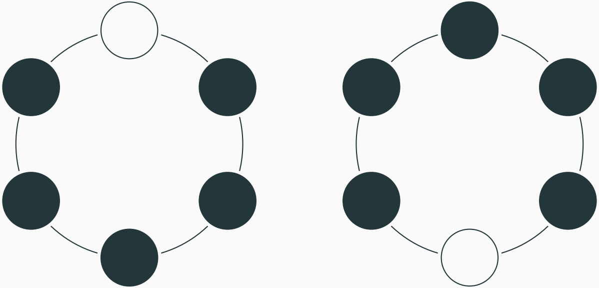
Introduction

Introduction - Reid



- We have black and white beads, and want to make a 6 bead bracelet.
- Well, we have 6 beads and 2 choices per bead so 2^6 right?
- This is only true so long as you don't care about symmetry.

Symmetry?

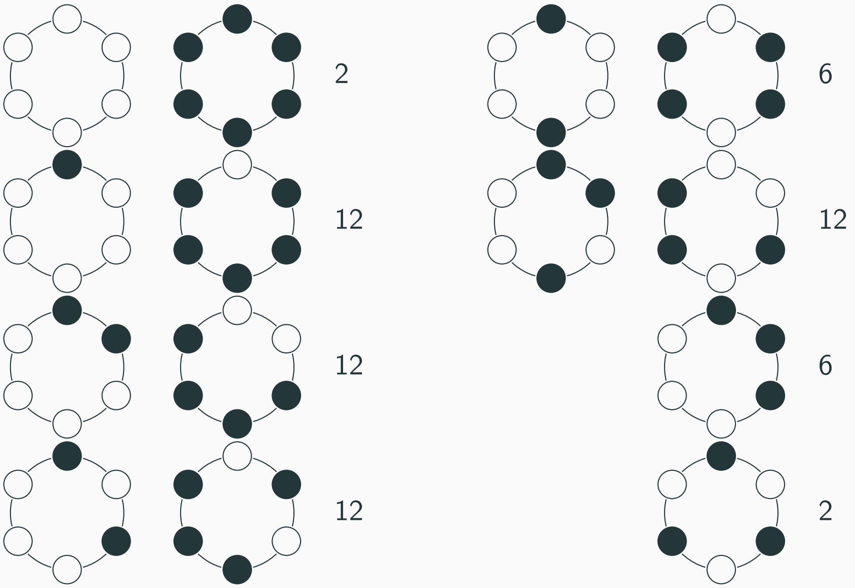


- Our initial count considers these different
- But one is just the other one rotated
- We would like to consider these as the same when we count

The hard way

We can count this way by making a table of bracelets and counting how many different bracelets can be made via rotations of that bracelet.

The hard way



The Goal

The goal of this talk is to develop a framework for answering coloring questions like these where symmetry is crucial

Polya Build up - Ethan

Permutations

- A permutation is a bijection from a set S into itself
- Can be viewed as changing the order of elements
- Consider the set $S = \{1, 2, 3, 4\}$ And let $f : S \to S$ be defined

$$f(1) = 2$$
$$f(2) = 1$$
$$f(3) = 4$$
$$f(4) = 3$$

Permutations

To specify permutations we use cycle notation.

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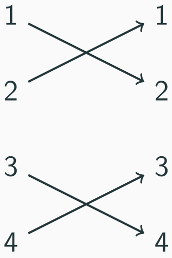

$$(12)(34)$$

Permutations

To specify permutations we use cycle notation.

1	1
2	2
3	3
4	4

$$(123)(4)$$

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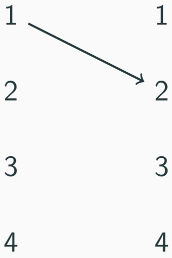

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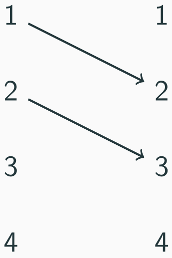

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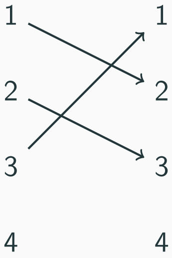

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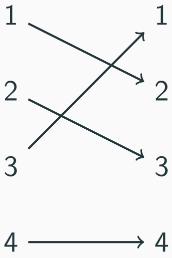

$$(123)(4)$$

Permutations

To specify permutations we use cycle notation.

$$1 \longrightarrow 1$$

$$2 \longrightarrow 2$$

$$3 \longrightarrow 3$$

$$4 \longrightarrow 4$$

$$(1)(2)(3)(4) = e$$

Colorings

A coloring c is a map from our set of objects into our set of colors.

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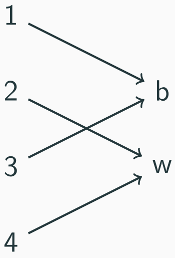

$(bwbw)$

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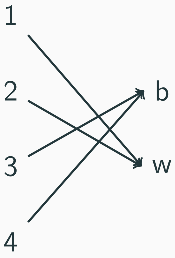

$(wwbb)$

Permuting Colorings

As a coloring is a map from our set of objects to our set of colors, if we compose a coloring c with a permutation we get a new coloring.

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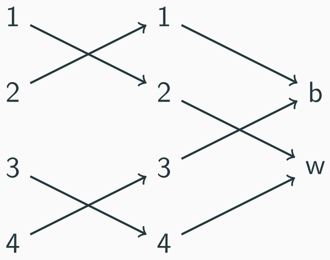

$$(12)(34) \ \circ \ (bwbw)$$

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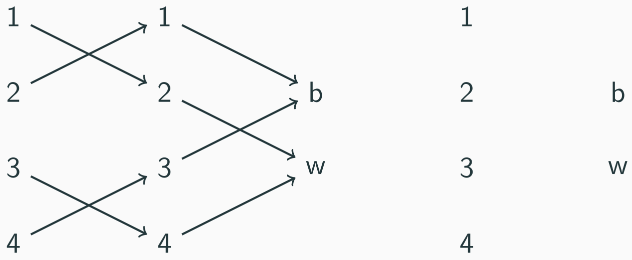

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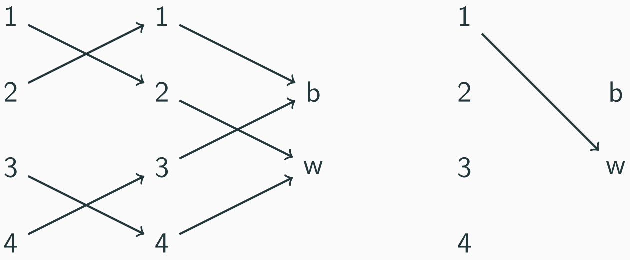

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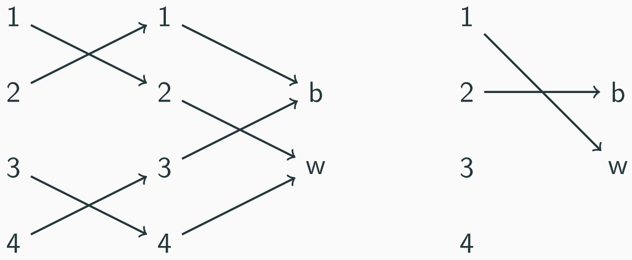

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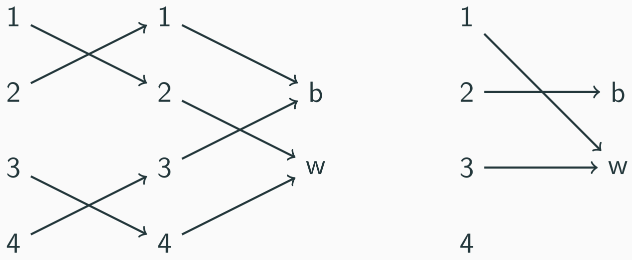

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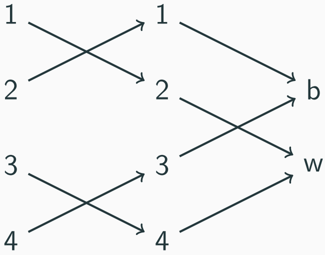

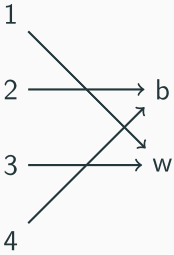

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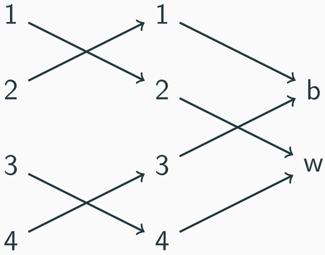

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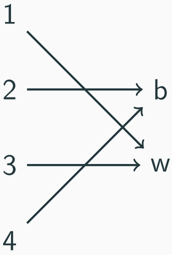
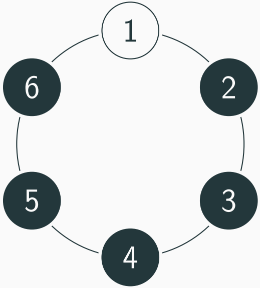
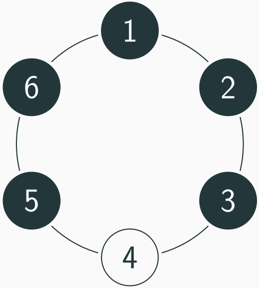


$(wbwb)$

This gives us a language to discuss symmetry now. Consider the two bracelets from earlier:





Symmetry

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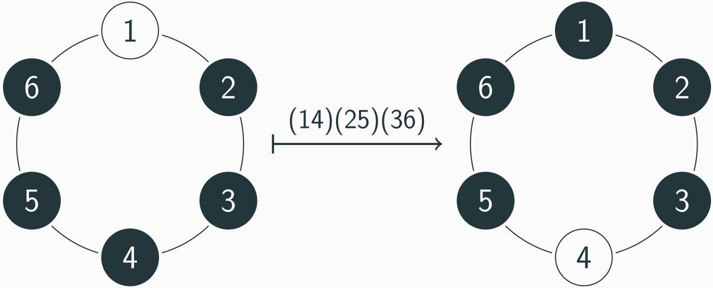


Symmetry

Thus we can describe the symmetry of an object by defining the group of permutations we allow on it.

Taking our bracelet example from earlier we see that our set of permutations is:

$$\{e, (123456), (135)(246), (14)(25)(36), (153)(264), (165432)\}$$

How to use Polya - Carlos

Invariant Set C_π

- Given a set of objects S, A set of colorings C (i.e. functions assigning colors to these objects) and a group G, we can denote the set of colors not changing when acted upon any permutation π in G, as C_π.
- For example, given a permutation (12)(3456) in G and letting w = white and b = black, a possible coloring in C_π would be $c = (bbwwww)$ because the coloring remains unchanged under the permutation. Yet, a coloring (bwwwww) would not belong in the invariant set because the permutation would cause the color b to become w and the w in the first position to become b.

Burnsides Lemma

- Let the number N correspond to the number of equivalence classes of the set of colorings C. Each equivalence class is formed by an equivalence relaion , where two colorings in C are related if one coloring can be transformed to the other via a permutation in G.
- For example, let a permutation in G be defined by (165432). Then, this permutation will simply rotate any coloring, meaning that the initial coloring and the coloring after the permutation acts on, will be the same.
- Therefore we define Burnsides Lemma: $N = \frac{1}{|G|}\sum_{n \in G} |C_\pi|$

Burnsides Lemma

- The problem of using Burnsides Lemma is that the size of the invariant set $|C_\pi|$ must be computed with any permutation, π , in the group G
- A better way is to observe that a coloring invariant under the action of a permutation in G, implies that every object permuted by one cycle of π must have the same color

Cycle Index

- As a generalization if π has k disjoint cycles, then $|C_\pi|$ equals m^k.
- Therefore we obtain monomial $M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{l_i}$, where permutation π is a product of k disjoint cycles, each x_{l_i} represents the length of each cycle, and the i'th cycle has length l_i.
- For example, given $M_{(12)(34)}$ we obtain x_2^2.
- Therefore we define the cycle index as

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n)$$

.

Polya

- Let the set $\{y_1, y_2\}$ be a set of colors where k_i is any color.
- Using the group $\{(1)(2)(3)(4), (12)(34)\}$ and the cycle index $P(x_1, x_2) = \frac{1}{2}(x_1^4 + x_2^2)$ we obtain the polynomial $P(y_1 + y_2, y_1^2 + y_2^2) = y_1^4 + y_2^4 + 2y_1^3 y_2 + 2y_1^2 y_2^3 + 4y_1^2 y_2^2$.

Polya

- Choosing the term $4y_1^2y_2^2$ as an example and using the permutation (12)(34) we can see that the coefficient 4 represents four ways to color a set of object using the color y_1 twice for two objects in the permutation and y_2 twice for two objects in the permutation.

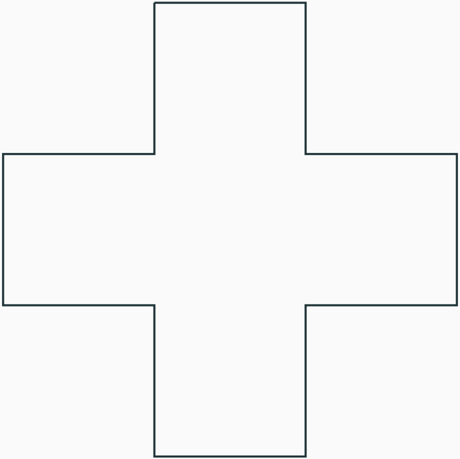
Using What We've Learned with an Example! - Amber

An Application in Industry

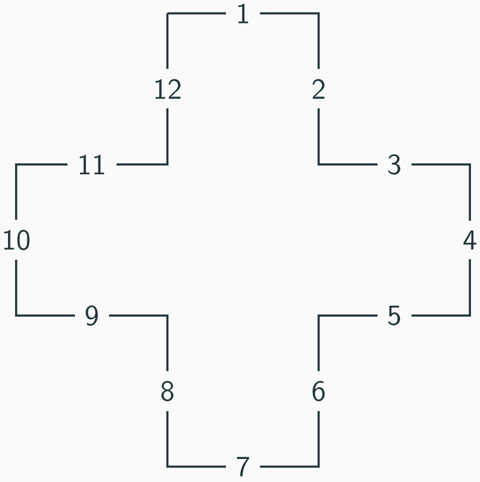
Suppose a medical relief agency plans to design a symbol for their organization in the shape of a regular cross. To symbolize the purpose of the organization and emphasize its international constituency, its board of directors decides that the cross should be white in color, with each of the twelve line segments outlining the cross colored red, green, blue, or yellow, with an equal number of lines of each color.

An Application in Industry

How many different ways are there to design the symbol, taking into account rotations and flips?

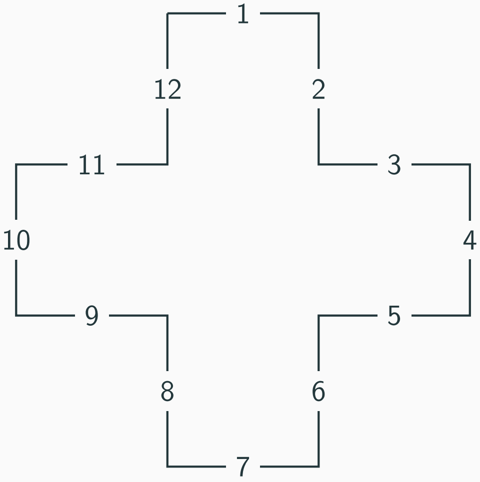


An Application in Industry



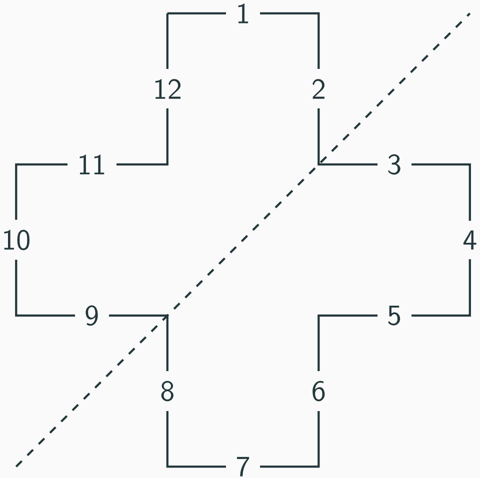
- Identity:

An Application in Industry



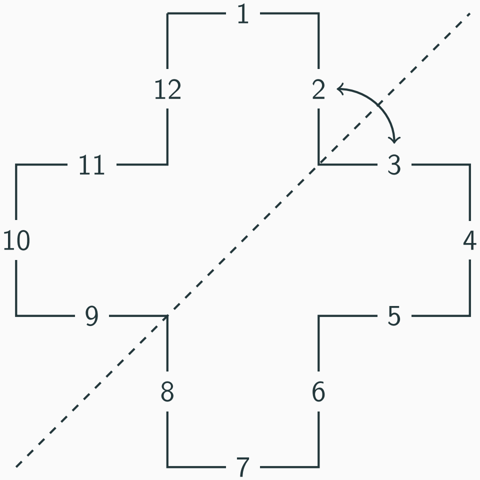
- Identity: x_1^{12}

An Application in Industry



- Identity: x_1^{12}
- Diagonal:

An Application in Industry



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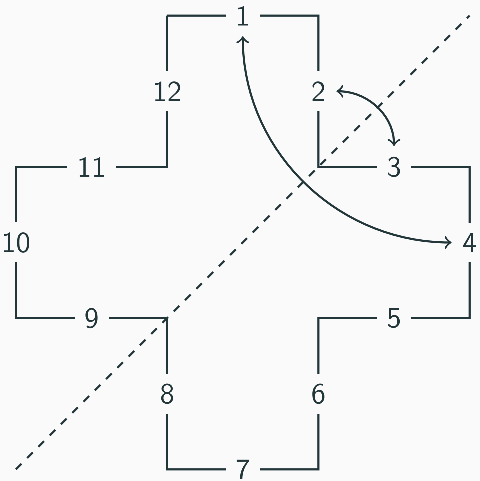

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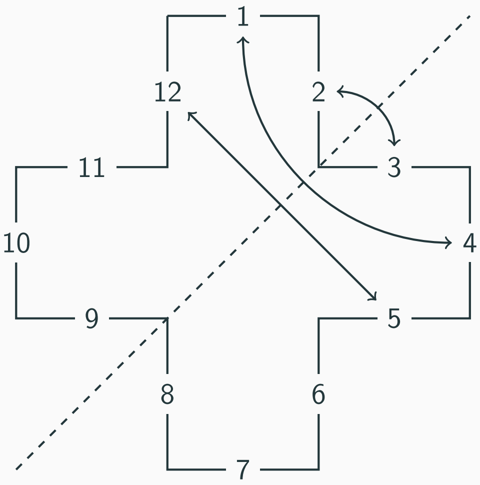

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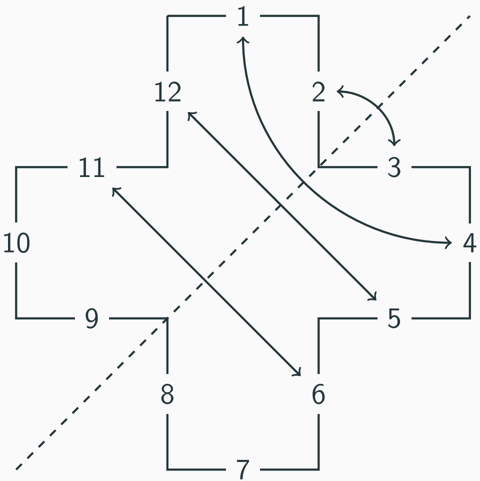

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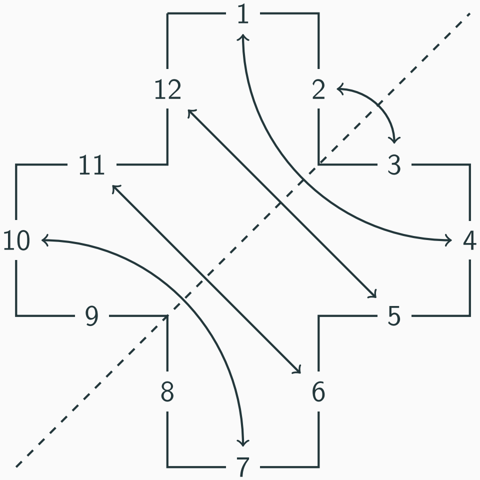

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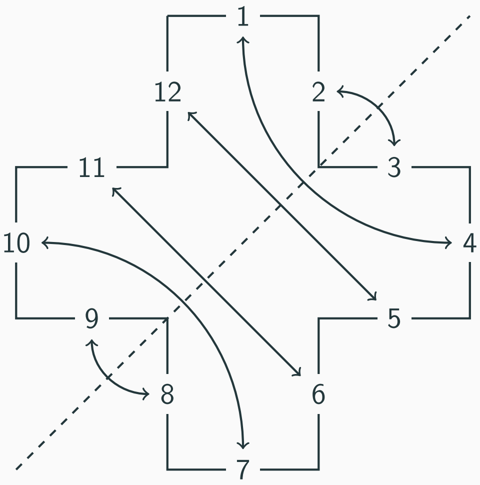

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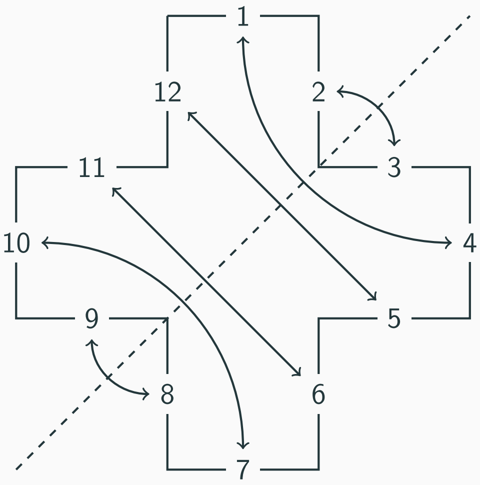

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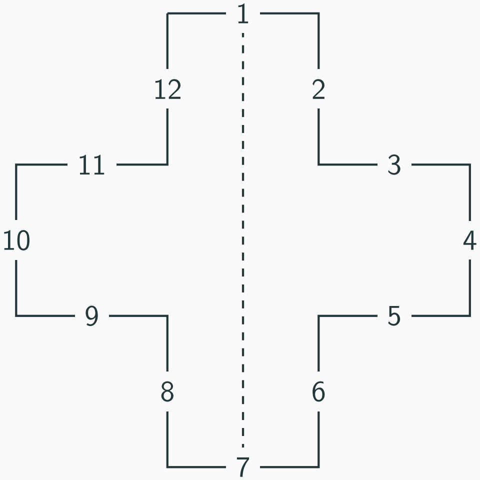
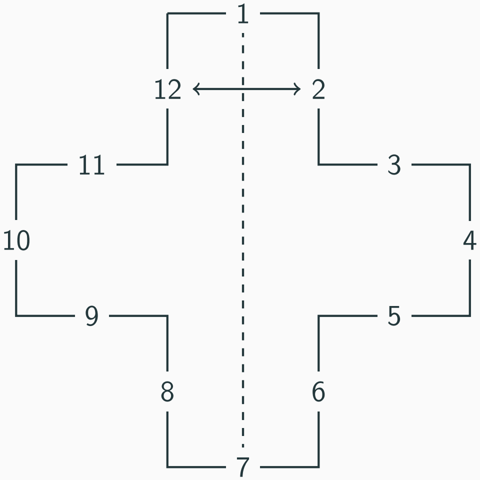
An Application in Industry



- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical:

An Application in Industry



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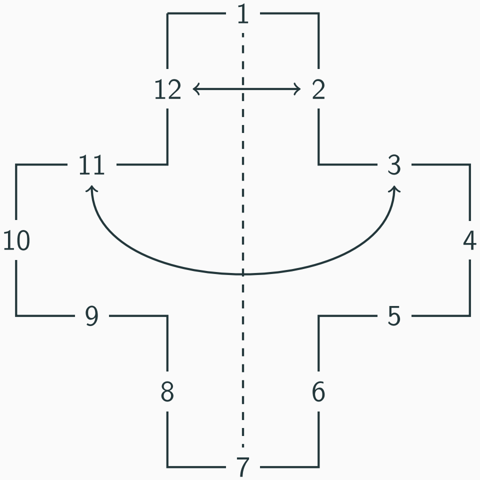

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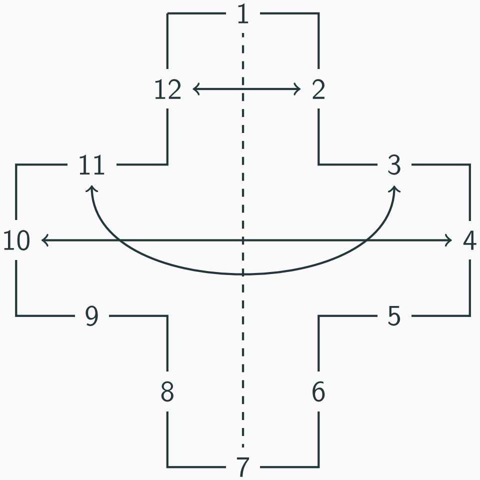

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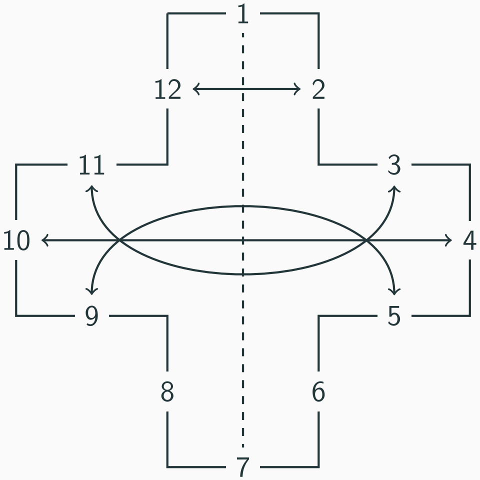

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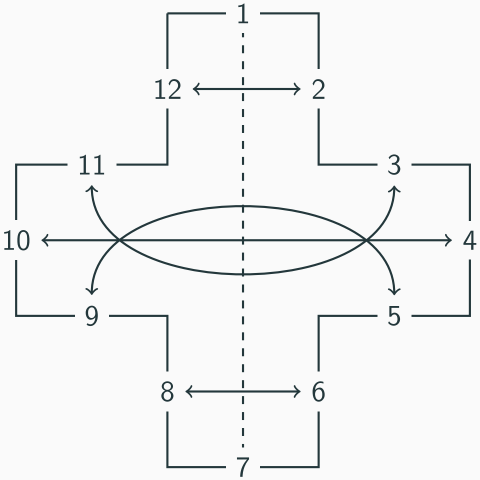

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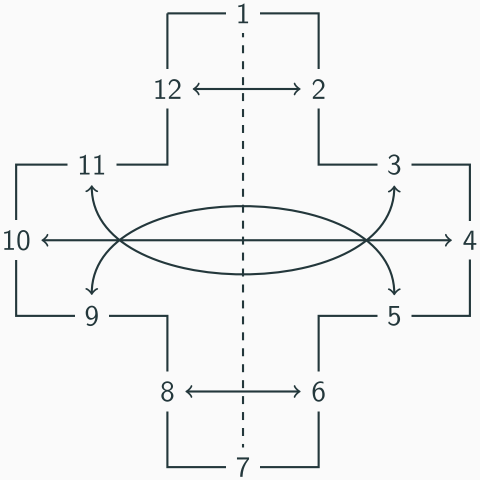
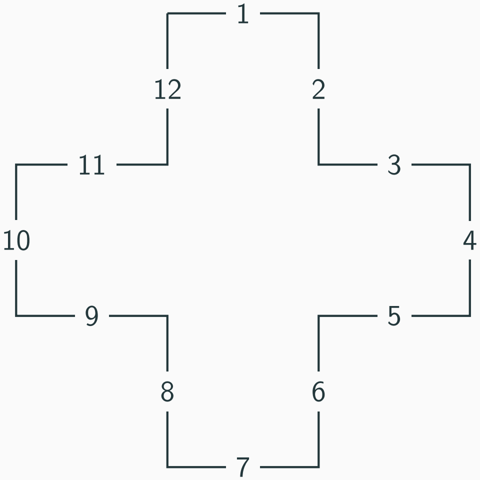


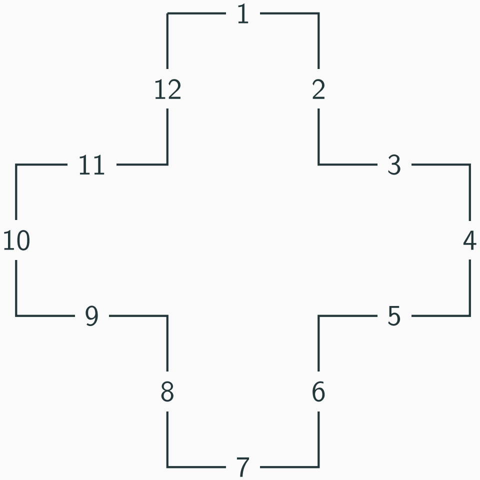
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°:

An Application in Industry



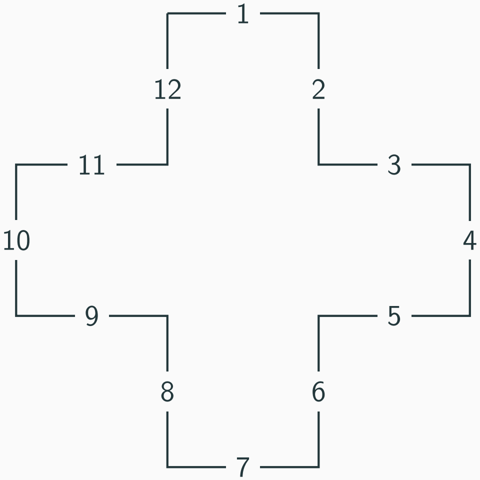
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3

An Application in Industry



- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°: x_2^6

An Application in Industry

From this, we get the cycle index:

$$P = \frac{1}{8}(x_1^{12} + 2x_4^3 + 3x_2^6 + 2x_1^2x_2^5)$$

We want the coefficient of $r^3g^3b^3y^3$, so we look at $x_1^{12}, x_4^3, x_2^6, x_1^2x_2^5$ individually.
(where r = red, g = green, b = blue, y = yellow)

An Application in Industry

Using Pólya's Enumeration Theorem, plugging in the colors r, g, b, y:

$$x_1 = r + g + b + y$$
$$x_2 = r^2 + g^2 + b^2 + y^2$$
$$x_4 = r^4 + g^4 + b^4 + y^4$$

So looking at each term individually:

$$x_1^{12} = (r + g + b + y)^{12}$$
$$x_4^3 = (r^4 + g^4 + b^4 + y^4)^3$$
$$x_2^6 = (r^2 + g^2 + b^2 + y^2)^6$$
$$x_1^2 x_2^5 = (r + g + b + y)^2 (r^2 + g^2 + b^2 + y^2)^5$$

An Application in Industry

- $P = \frac{1}{8}(x_1^{12} + 2x_4^3 + 3x_2^6 + 2x_1^2 x_2^5)$

So the ways to get the coefficient for $r^3 g^3 b^3 y^3$ from P are...

$$\frac{1}{8}\binom{12}{3,3,3,3} + \frac{1}{4}(0) + \frac{3}{8}(0) + \frac{1}{4}(0)$$

$= 46200$

Therefore, we have found that there are 46,200 different ways to design the symbol.

An Interesting Find

- While working on this example, we discovered that the group of symmetries for the edges of a regular cross is actually the dihedral group D_4, like that of a square.
- This is interesting! So we started looking into the effects of dihedral groups acting on polygons.

Results, reflection groups, pictures - Clarisse

Group Action

For the purposes of analyzing D_k dihedral groups acting on $nk - gons$, it is best to imagine the $nk - gon$ inscribed inside the $k - gon$, and being restricted to the rotations and flips of the $k - gon$.

Algorithm for constructing the polynomials which represent a dihedral group D_k acting on a $nk - gon$.

- construct the normal k-gon.
  - Rotations
  - Flips
- Account for the n-cases

Rotations

First approach the normal k-gon. In order to construct the terms which represent rotations of the normal $k - gon$ we will take the following steps:

- Suppose $a_1, a_2, ..., a_n$ are divisors of k.
- For each divisor a_i assemble the values which are coprime to it.
- Denote these coprime values b_{ij} for each respective a_i.
- Now the **rotation** terms in our polynomial for the normal $k - gon$ will be:

$$\sum |b_{ij}| x_{a_i}^{k/a_i}$$

So

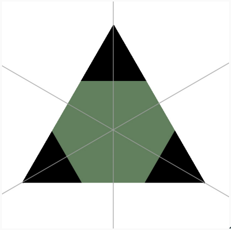
$$\sum |b_{ij}| x_{a_i}^{nk/a_i}$$

Flips

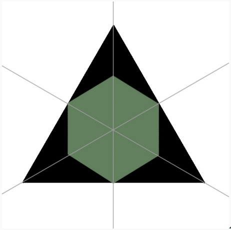
Next, in order to find the polynomial of $d_k \circlearrowright nk - gon$ we need to add in the terms which represent flips.
If k is prime, then we have three cases:

1. $kx_1^1 x_2^{(kn-1)/2}$ for n odd $\longmapsto$
2. $kx_2^{(kn)/2}$ for n even$\longleftrightarrow$
3. $kx_1^2 x_2^{(kn-2)/2}$ for n even $|--|$

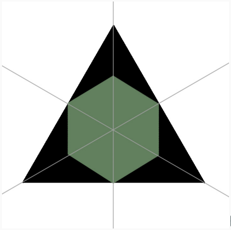
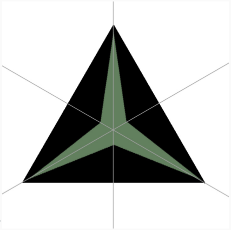
Understanding the Different Cases Concerning Flips



$x_1^6 + 2x_3^2 + 3x_1^2x_2^2$



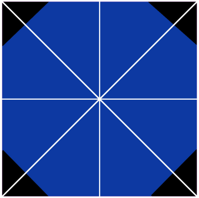
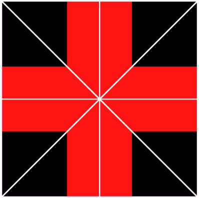
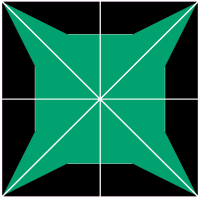
$x_1^6 + 2x_3^2 + 3x_2^3$

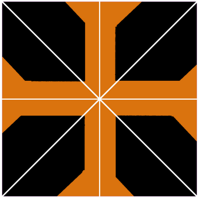
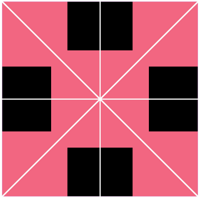
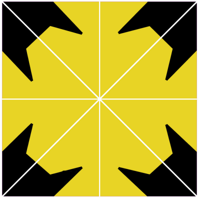
 $\mapsto$ 

Cycle index of a triangle: $x_1^3 + 2x_3^1 + 3x_1^1 x_2^1$

n	Truncated Vertices	Subdivision of Edges
2	$x_1^6 + 2x_3^2 + 3x_1^2 x_2^2$	$x_1^6 + 2x_3^2 + 3x_2^3$
3	$x_1^9 + 2x_3^3 + 3x_1^1 x_2^4$	$x_1^9 + 2x_3^3 + 3x_1^1 x_2^4$
4	$x_1^1 2 + 2x_3^4 + 3x_1^2 x_2^5$	$x_1^1 2 + 2x_3^4 + 3x_2^6$
5	$x_1^1 5 + 2x_3^5 + 3x_1^1 x_2^7$	$x_1^1 5 + 2x_3^5 + 3x_1^1 x_2^7$

Table 1: $D_3 \circlearrowright 3n - gon$

n	Truncated Vertices	Subdivision of Edges
2		
3		

n	Truncated Vertices	Subdivision of Edges
5		
6		

Cycle index of a square: $x_1^4 + 2x_4^1 + x_2^2 + x_1^2x_2^1 + 2x_2^2$

$D_4 \circlearrowright 4n - gon$

n	Truncated Vertices	Subdivision of Edges
2	$x_1^8 + 2x_4^2 + x_2^4 +$ $4x_1^2x_2^3$	$x_1^8 + 2x_4^2 + x_2^4 +$ $4x_2^4$
3	$x_1^{12} + 2x_4^3 + x_2^6 +$ $2x_1^2x_2^5$ $+$ $2x_2^6$	$x_1^{12} + 2x_4^3 + x_2^6 +$ $2x_1^2x_2^5$ $+$ $2x_2^6$
4	$x_1^{16} + 2x_4^4 + x_2^8 +$ $4x_1^2x_2^7$	$x_1^{16} + 2x_4^4 + x_2^8 +$ $4x_2^8$
5	$x_1^{20} + 2x_4^5 + x_2^{10} +$ $2x_1^2x_2^4$ $+$ $2x_2^{10}$	$x_1^{20} + 2x_4^5 + x_2^{10} +$ $2x_1^2x_2^4$ $+$ $2x_2^{10}$
6	$x_1^{24} + 2x_4^6 + x_2^{12} +$ $4x_1^2x_2^{11}$	$x_1^24 + 2x_4^6 + x_2^{12} +$ $4x_2^{12}$

Table 2: $D_4 \circlearrowright 4n - gon$

So for any D_n acting on a $nk - gon$ we can produce the flips using cases:

k Even:

- $k/2x_1^2x_2^{(kn-2)/2} + k/2x_2^{kn/2}$ n odd ($\longleftrightarrow$ and $|\!-\!|$)
- $kx_1^2x_2^{(kn-2)/2}$ n even (uniformly $| - -|$)
- $kx_2^{kn/2}$ n odd (uniformly $\longleftrightarrow$)

k Odd:

- $kx_1^1x_2^{nk/2}$ n odd ($\mapsto$)
- $kx_2^{nk/2}$ n even ($| - -|$)
- $kx_1^2x_2^{(nk-2)/2}$ n even ($\longleftrightarrow$)