Counting With Symmetries

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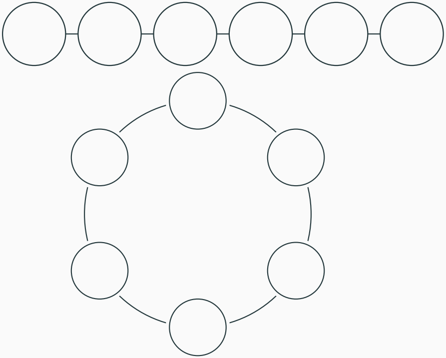
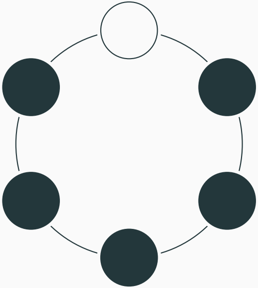
UC Riverside

Introduction



- We have black and white beads, and want to make a 6 bead bracelet.
- Well, we have 6 beads and 2 choices per bead so 2^6 right?
- This is only true so long as you don't care about symmetry.

Symmetry?



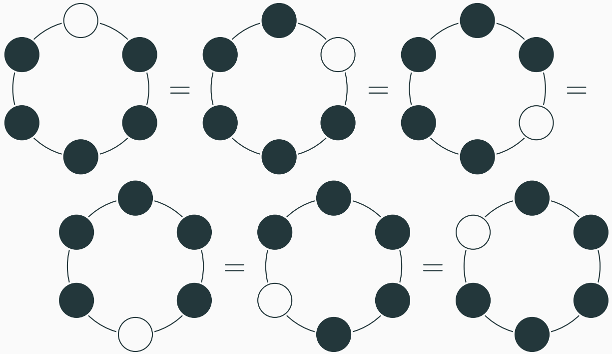
- Our initial count considers these different
- But one is just the other one rotated
- We would like to consider these as the same when we count

The hard way

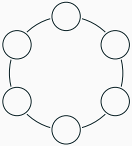
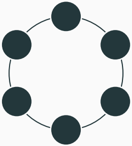
We can count this way by making a table of bracelets and counting how many different bracelets can be made via rotations of that bracelet.

The hard way

Specifically, we want to count colorings with rotations by picking combinations of beads and treating them as equal when they're rotations of each other:

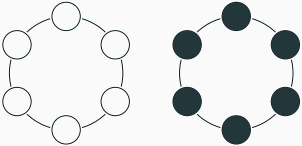


The hard way

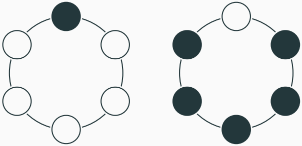



$2 \to 2$

The hard way

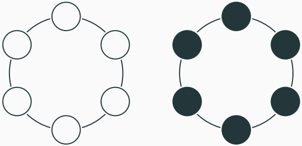


2→2

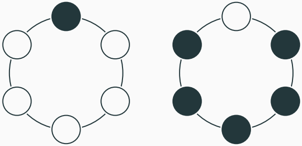


12→2

The hard way



2→2



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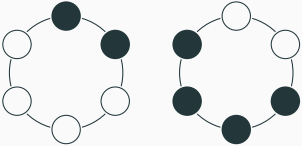

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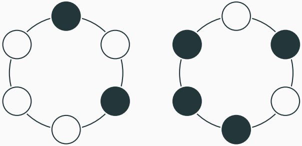

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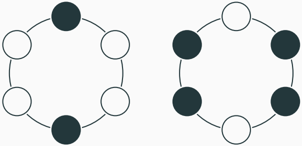
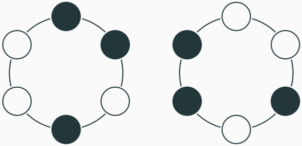
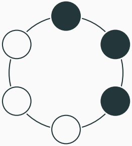


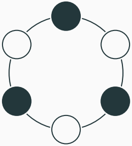
6→2



12→2



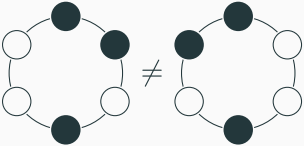
6→1



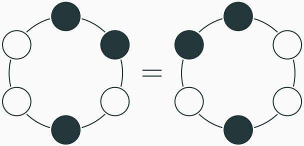
2→1

Flips and Colorings

- However, the count is different if we allow for more actions than just rotations.
- Another action we could add is flipping the bracelet over.
- This results in different unique colorings.



when no flips are allowed.

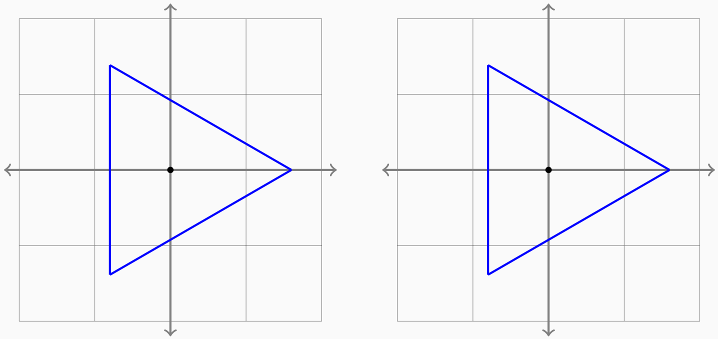


when flips are allowed.

Dihedral Group and Group Actions

- We seek to generalize the types of things we can do to sets with the right symmetries.
- To get an idea of these, we will look at the symmetries of an equilateral triangle.
- We will let Δ denote the set of points that make up an equilateral triangle.
- We want to describe the symmetries of this triangle.
- We then want to think about the actions that make up these symmetries more generally.

Dihedral Group and Group Actions



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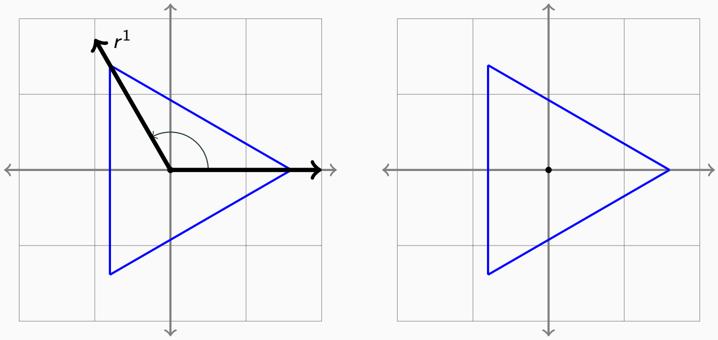

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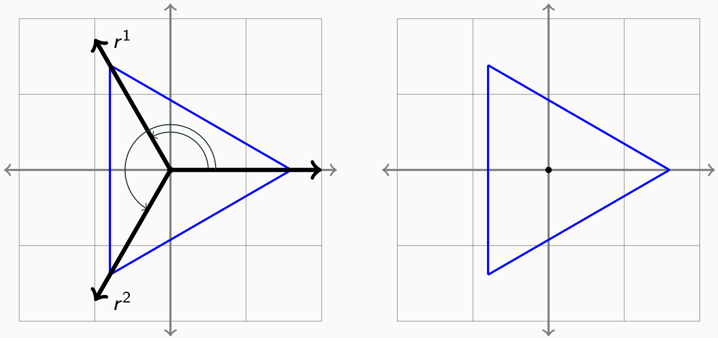

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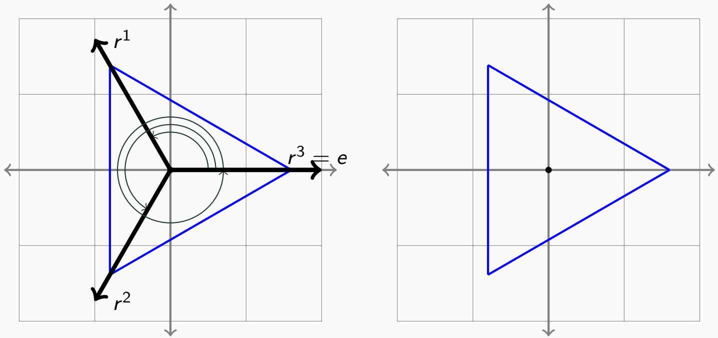

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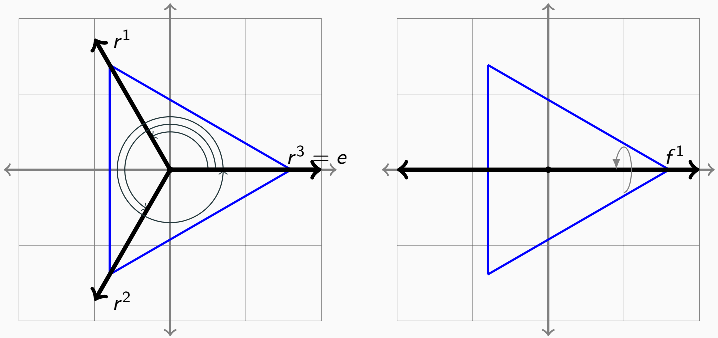

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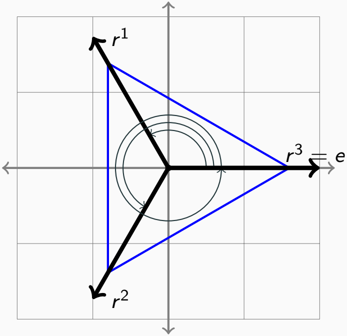

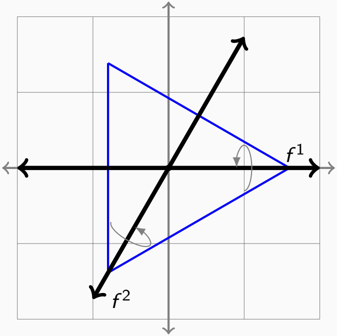

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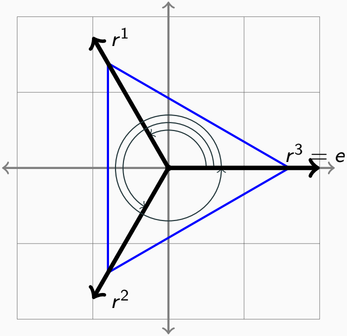

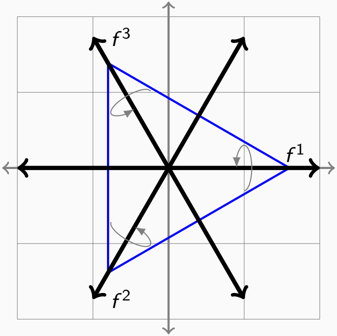

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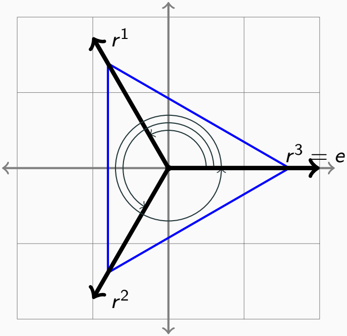
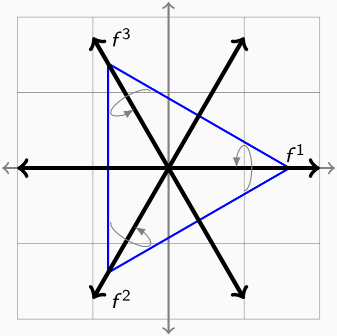
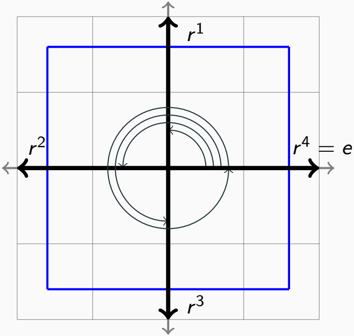




$D_3 = \{e, r^1, r^2, f^1, f^2, f^3\}$ is the dihedral group on 3 elements.

We say D_3 acts on Δ, or $D_3 \circlearrowright \Delta$.

D_4



$D_4 = \{e, r^1, r^2, r^3, f^1, f^2, f^3, f^4\}$ is the dihedral group on 4 elements.

This generalizes to D_n, the dihedral group on n elements: it's a group of $2n$ elements with n rotations and n flips.

The Goal

The goal of this talk is to develop a framework for answering coloring questions like these where symmetry is crucial

Polya Build up - Ethan

Permutations

- A permutation is a bijection from a set S into itself
- Can be viewed as changing the order of elements
- Consider the set $S = \{1, 2, 3, 4\}$ And let $f : S \to S$ be defined

$$f(1) = 2$$
$$f(2) = 1$$
$$f(3) = 4$$
$$f(4) = 3$$

Permutations

To specify permutations we use cycle notation.

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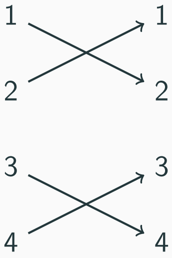

$$(12)(34)$$

Permutations

To specify permutations we use cycle notation.

$$\begin{array}{ccc} 1 & \quad & 1 \\ 2 & & 2 \\ 3 & & 3 \\ 4 & & 4 \end{array}$$

$$(123)(4)$$

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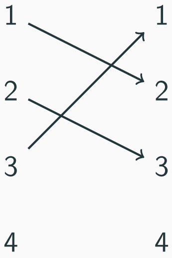

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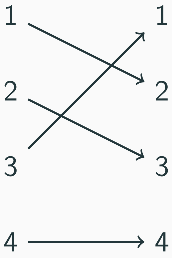

$$(123)(4)$$

Permutations

To specify permutations we use cycle notation.

$$1 \longrightarrow 1$$

$$2 \longrightarrow 2$$

$$3 \longrightarrow 3$$

$$4 \longrightarrow 4$$

$$(1)(2)(3)(4) = e$$

Colorings

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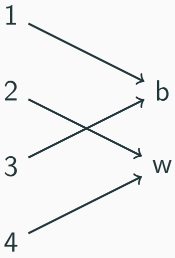

$(bwbw)$

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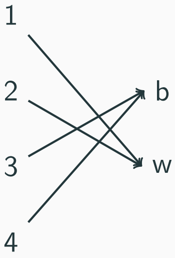

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Permuting Colorings

As a coloring is a map from our set of objects to our set of colors, if we compose a coloring c with a permutation we get a new coloring.

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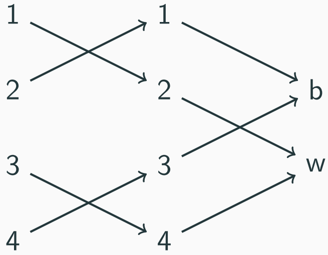

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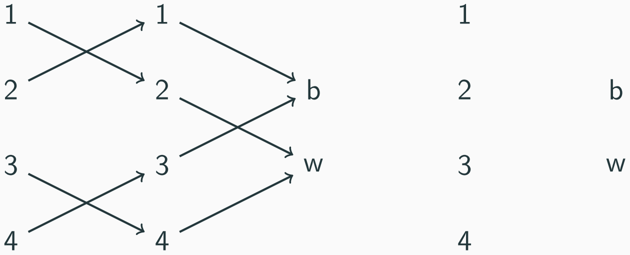

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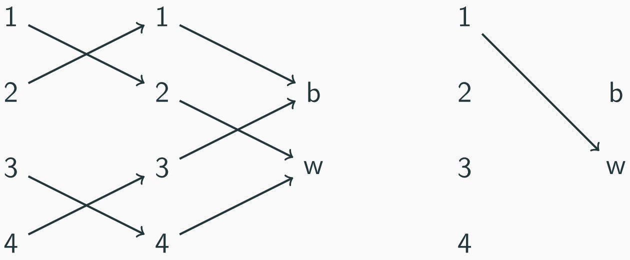

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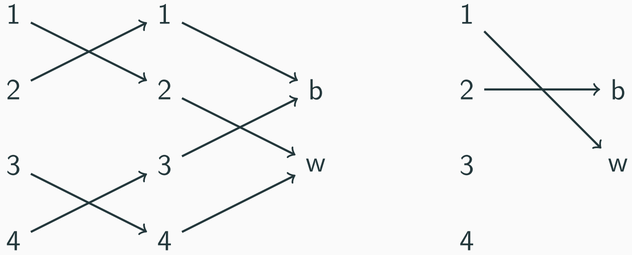

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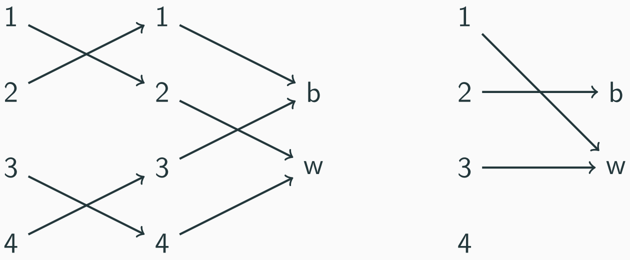

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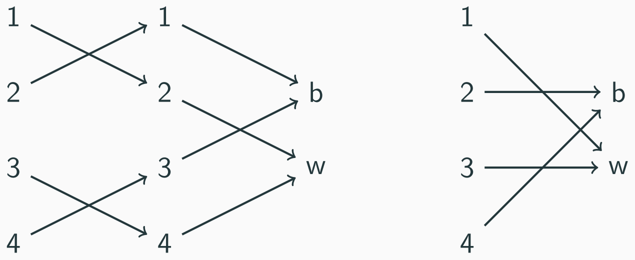

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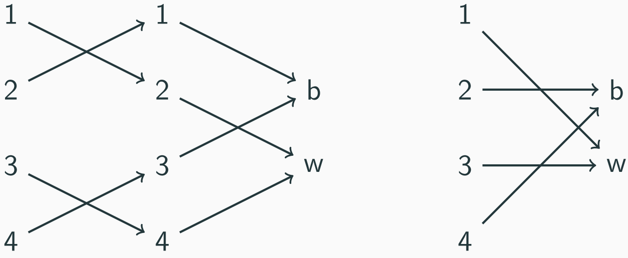
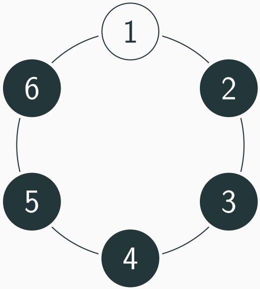
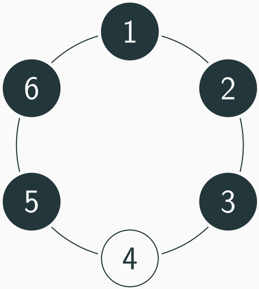


$(12)(34) \circ (bwbw) \qquad (wbwb)$

Symmetry

This gives us a language to discuss symmetry now. Consider the two bracelets from earlier:





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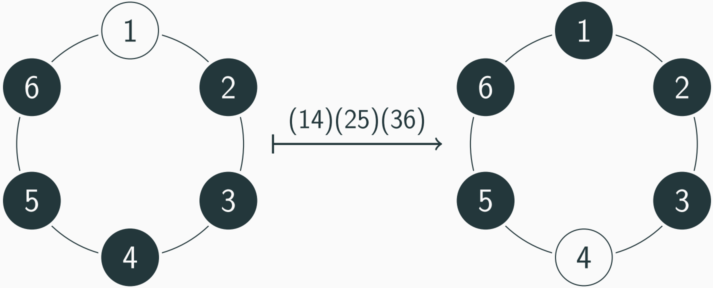

Symmetry

Thus we can describe the symmetry of an object by defining the group of permutations we allow on it.

Taking our bracelet example from earlier we see that our set of permutations is:

$$\{e, (123456), (135)(246), (14)(25)(36), (153)(264), (165432)\}$$

Abstracting Polya from the cycle index

- We can use the six bead example and definition of permutations to generalize properties from equations and definitions.
- Definition: Let G be a group whose elements are the permutations on S and $|S| = m$. Next we let m variables $x_1, x_2, ..., x_m$ with nonnegative coefficients form the product $\beta = x_1^{\alpha_1}, x_2^{\alpha_2}, ..., x_m^{\alpha_m}$ for every permutation in G.
- Also let α_i represents the the number of disjoint cycles of length i in the given permutation.

- We can obtain the cyle index of G
  $P_G(x_1, x_2, ..., x_m) = \frac{1}{|G|} \sum_{\pi \in G} x_1^{\alpha_1}, x_2^{\alpha_2}, ..., x_m^{\alpha_m}$.

- For example, referring back to the 6 beaded example, we get the cycle index $P_{D_6}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{9}(x_1^6 + x_6^1 + x_3^2 + x_2^3 + x_3^2 + x_6^1 + x_2^2 x_1^2 + x_2^2 x_1^2 + x_2^2 x_1^2)$.
- Every term represent a permutation and the superscript represents the number of disjoint cycles in the permutation while the superscript represents the length of each cycle.
- $7th$ term in the polynomial represents the permutation $\pi = (1)(26)(35)(4)$. Therefore two cycles of length two and two cycles of length one correspond to $x = x_2^2 x_1^2$.

- The purpose of the cycle index is to determine the number of distinct colorings acted on by the group of symmetries G.
- For example, Let m equal the number of any distinct colors, we obtain the polynomial,
  $P_{D_6}(m, m, m, m) = \frac{1}{9}(m_1^6 + 2m_6^1 + 2m_3^2 + m_2^3 + 3m_2^2 m_1^2)$.
- Specifically, when $m = 2$ we obtain $P_{D_6}(2, 2, 2, 2) =$.

- While the cycle index tells us the number of distinct objects we seek, we can abstract even further to obtain not only the number of distinct objects but also an idea of the appearance of what each object should look like.
- Polya's Enumeration Formula: Let S be a set of elements and G a group of permutations on S, where each permutation induces an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using colors $c_1, c_2, ..., c_m$ is the function $P_G(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, ..., \sum_{j=1}^{m} c_j^k)$.
- As an observation, the k in $P_G(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, ..., \sum_{j=1}^{m} c_j^k)$ refers to the largest cycle length.

- Resorting back to our bead example, if we let b and w represent our two colors, then we obtain $P_{D_4}((b+w),(b^2+w^2),(b^3+w^3),(b^4+w^4)) = b^6 + b^5w + 3b^4w^2 + 4b^3w^3 + 3b^2w^4 + bw^5 + w^6$.
- The coefficient, of each term, $A_i b^k w^j$ determines the number of possible distinct colorings on the six beaded bracelet using the colors b and w, i and j times for each bead.
- for example, the third term $3b^4w^2$ corresponds to 3 possible colorings, coloring 4 bead's black and 2 bead's white.

Abstracting Polya from the cycle index

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- Definition: Let G be a group whose elements are the permutations on S and $|S| = m$. Next we let m variables $x_1, x_2, ..., x_m$ with nonnegative coefficients form the product $\beta = x_1^{\alpha_1}, x_2^{\alpha_2}, ..., x_m^{\alpha_m}$ for every permutation in G.
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- Polya's Enumeration Formula: Let S be a set of elements and G a group of permutations on S, where each permutation induces an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using colors $c_1, c_2, ..., c_m$ is the function $P_G(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, ..., \sum_{j=1}^{m} c_j^k)$.
- As an observation, the k in $P_G(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, ..., \sum_{j=1}^{m} c_j^k)$ refers to the largest cycle length.

- Resorting back to our bead example, if we let b and w represent our two colors, then we obtain $P_{D_4}((b+w),(b^2+w^2),(b^3+w^3),(b^4+w^4)) = b^6+b^5w+3b^4w^2+4b^3w^3+3b^2w^4+bw^5+w^6$.
- The coefficient, of each term, $A_ib^kw^j$ determines the number of possible distinct colorings on the six beaded bracelet using the colors b and w, i and j times for each bead.
- for example, the third term $3b^4w^2$ corresponds to 3 possible colorings, coloring 4 bead's black and 2 bead's white.

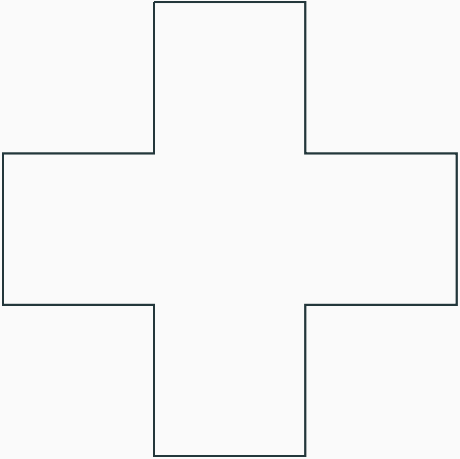
Using What We've Learned with an Example! - Amber

An Application in Industry

- Suppose a medical relief agency plans to design a symbol for their organization in the shape of a regular cross.
- They decide that the cross should be white in color, with each of the twelve line segments outlining the cross colored red, green, blue, or yellow.
- And should have an equal number of lines of each color.

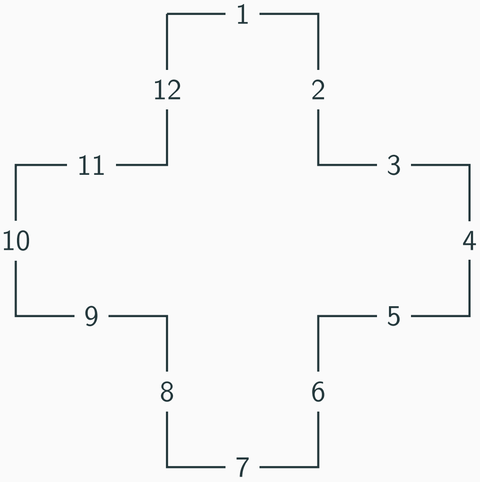
An Application in Industry

How many different ways are there to design the symbol, taking into account rotations and flips?

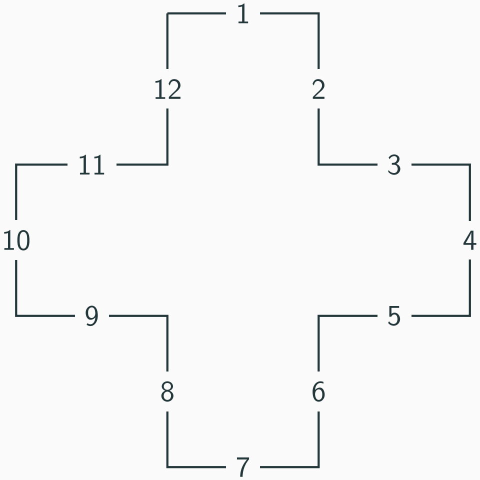


An Application in Industry

First, WLOG we can number the sides of the cross as so:

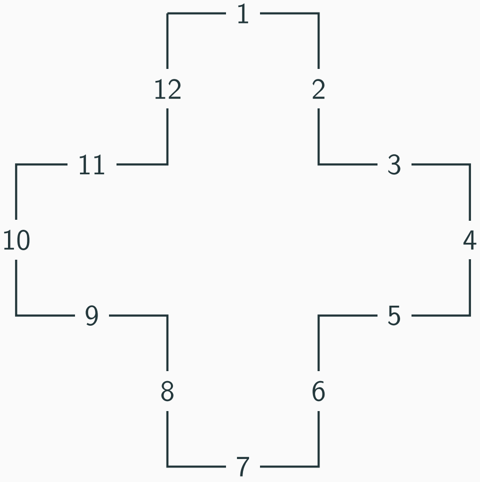


An Application in Industry



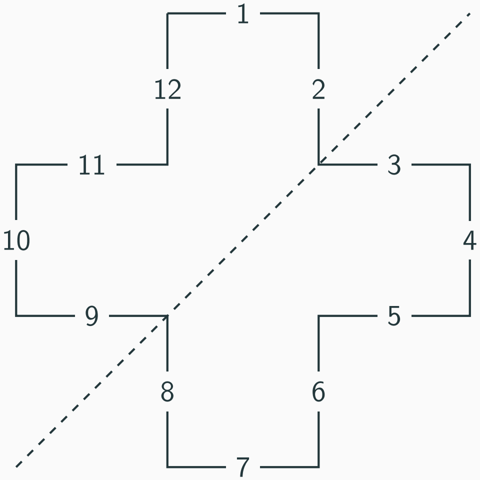
- Identity:

An Application in Industry



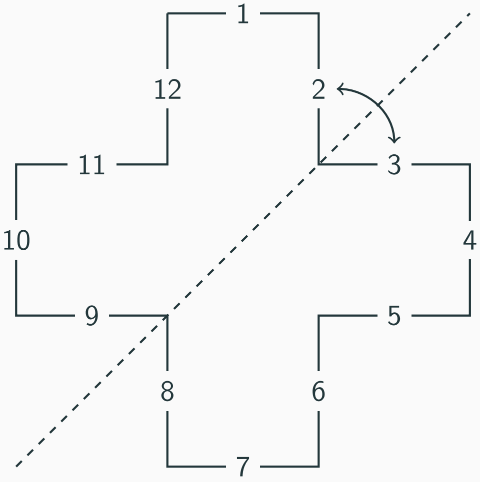
- Identity: x_1^{12}

An Application in Industry



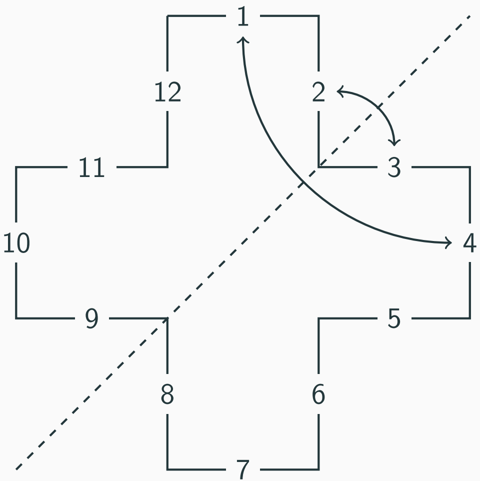
- Identity: x_1^{12}
- Diagonal:

An Application in Industry



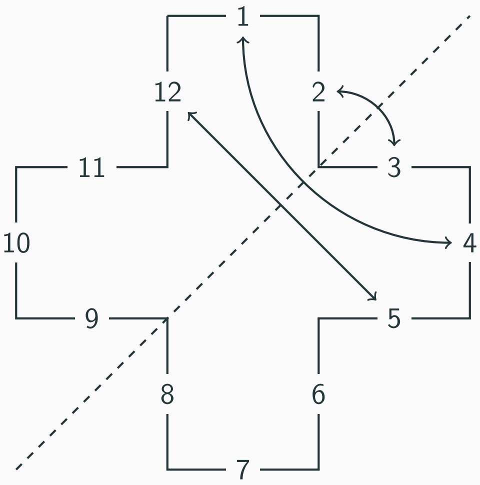
- Identity: x_1^{12}
- Diagonal:

An Application in Industry



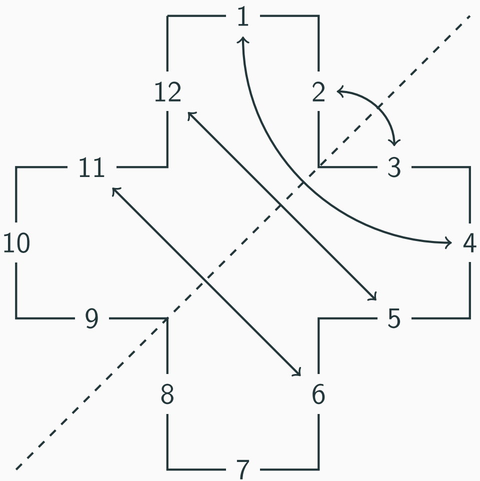
- Identity: x_1^{12}
- Diagonal:

An Application in Industry



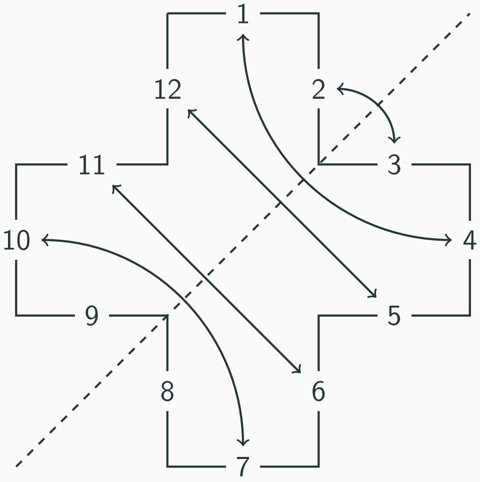
- Identity: x_1^{12}
- Diagonal:

An Application in Industry



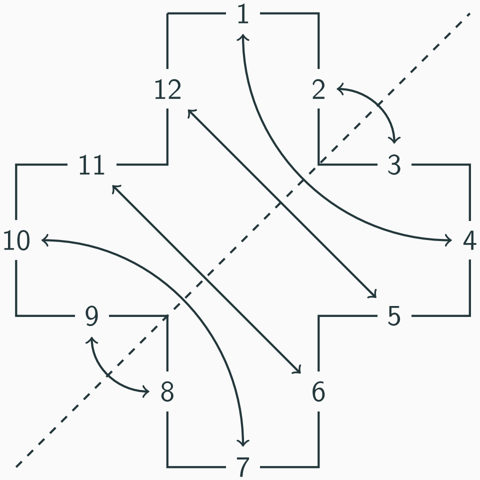
- Identity: x_1^{12}
- Diagonal:

An Application in Industry



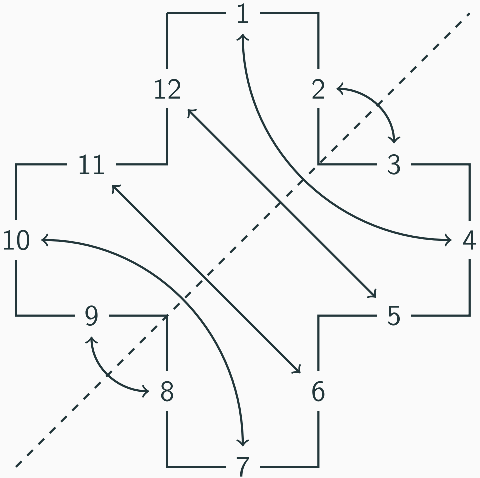
- Identity: x_1^{12}
- Diagonal:

An Application in Industry



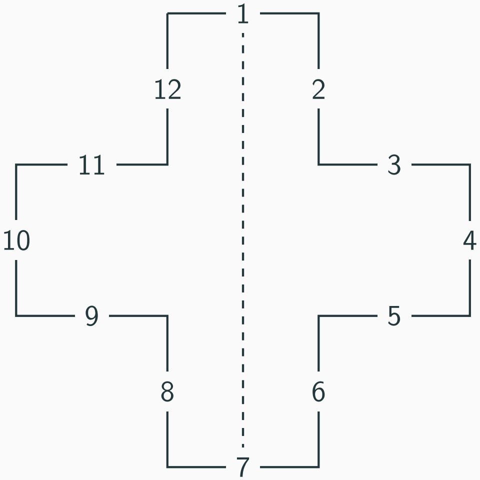
- Identity: x_1^{12}
- Diagonal:

An Application in Industry



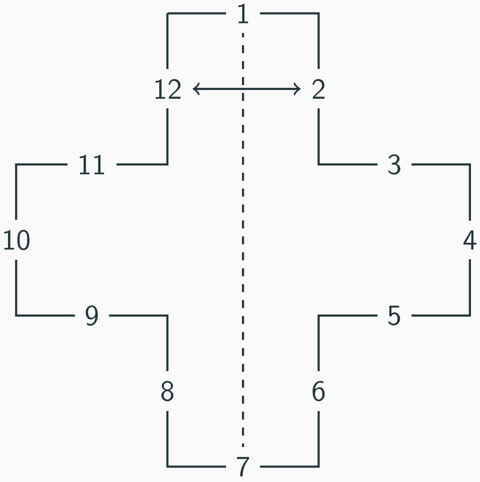
- Identity: x_1^{12}
- Diagonal: x_2^6

An Application in Industry



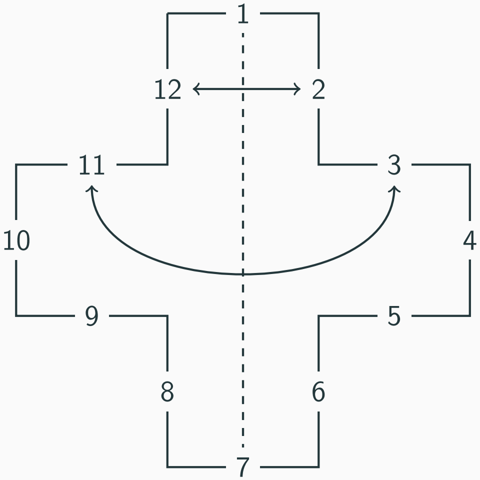
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical:

An Application in Industry



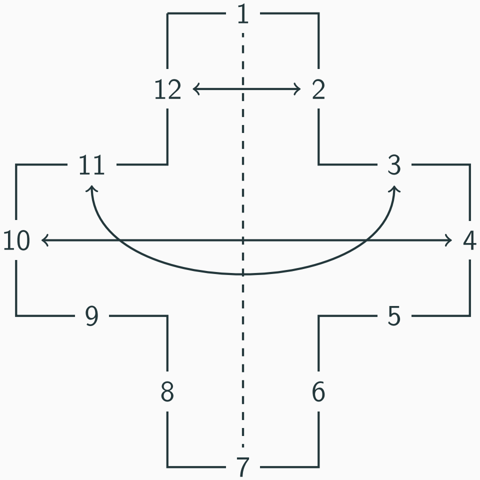
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical:

An Application in Industry



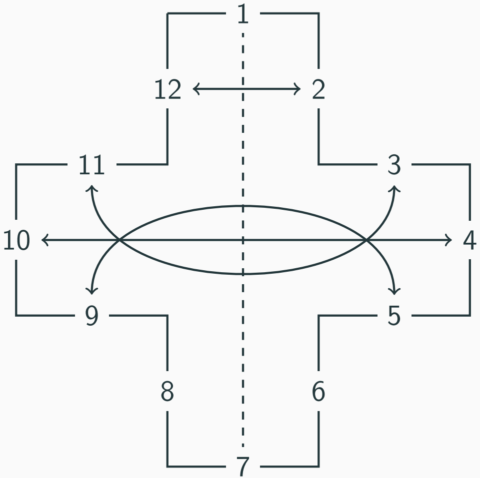
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical:

An Application in Industry



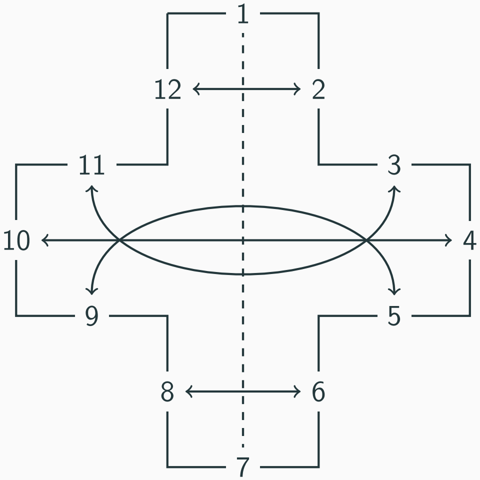
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical:

An Application in Industry



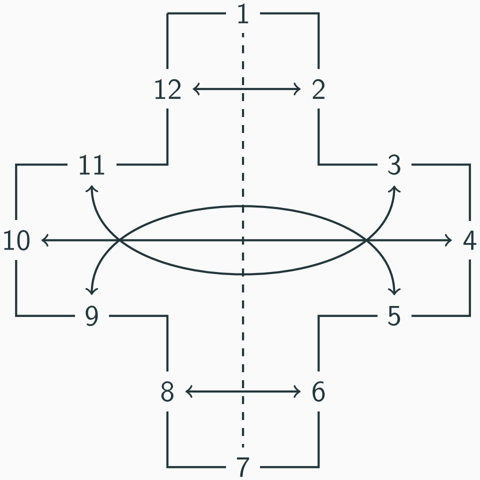
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical:

An Application in Industry



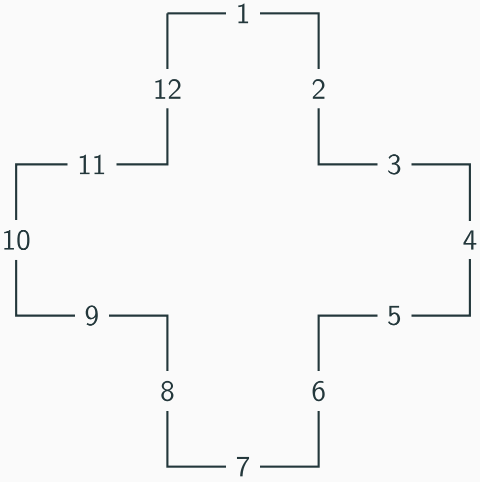
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical:

An Application in Industry



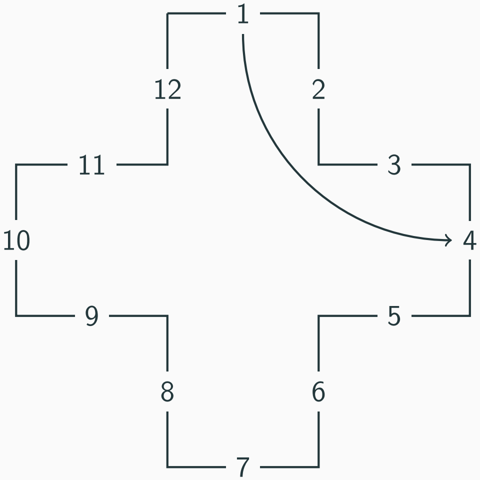
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$

An Application in Industry



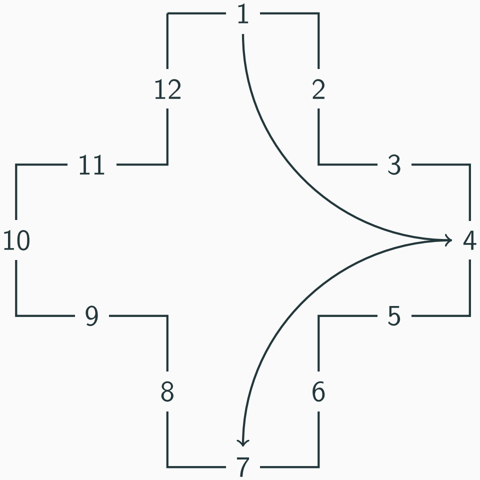
- Identity: x_1^{12}
- Diagonal: x_2^{6}
- Vertical: $x_1^2 x_2^5$
- 90°:

An Application in Industry



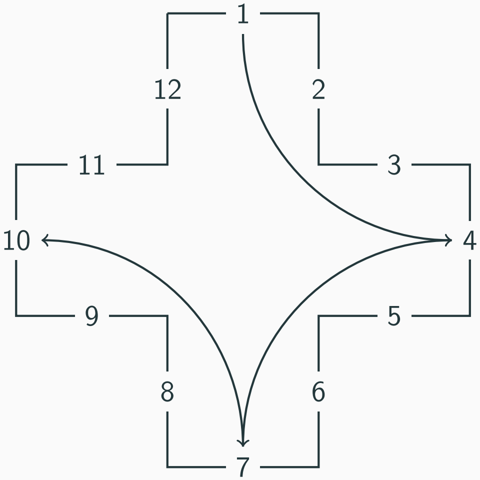
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°:

An Application in Industry



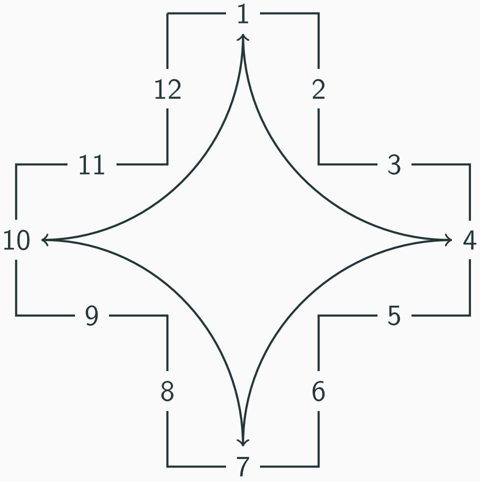
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°:

An Application in Industry



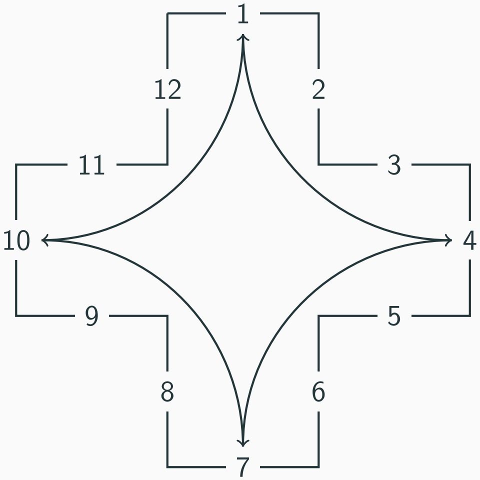
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°:

An Application in Industry



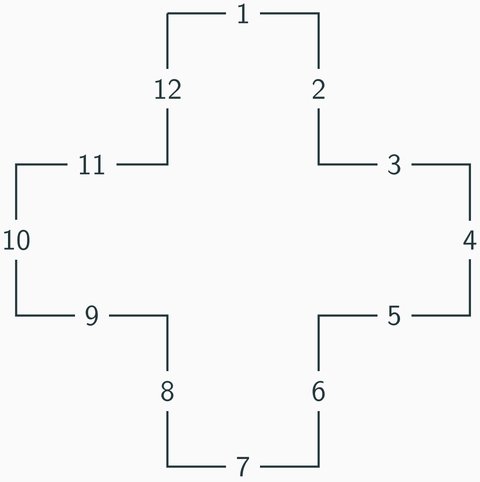
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°:

An Application in Industry



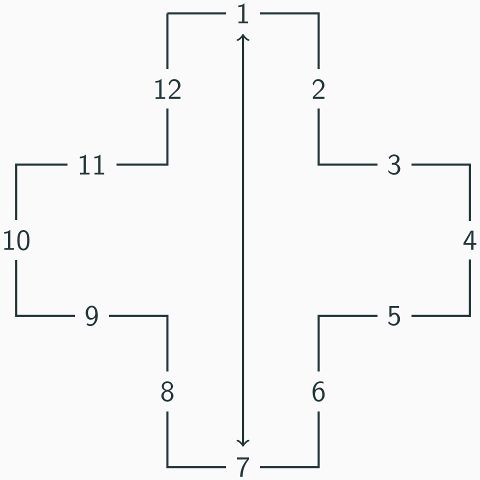
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3

An Application in Industry



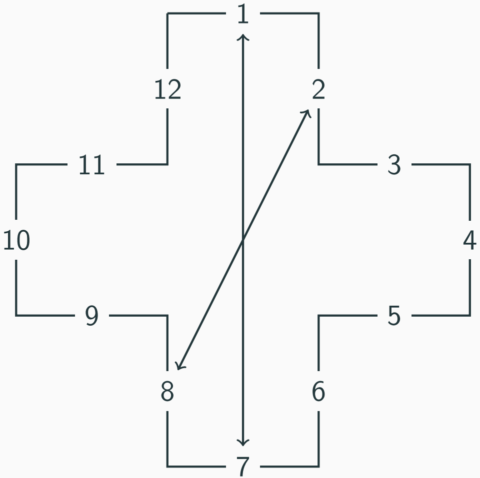
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



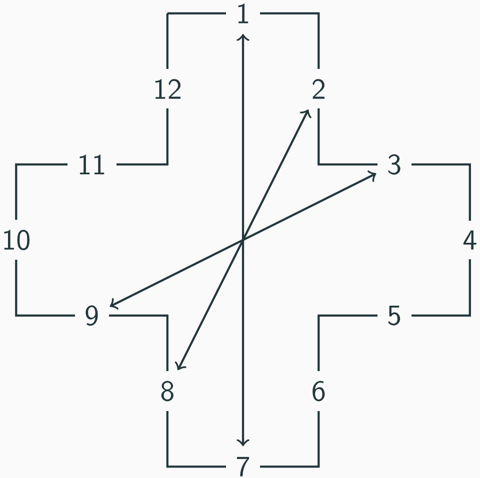
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



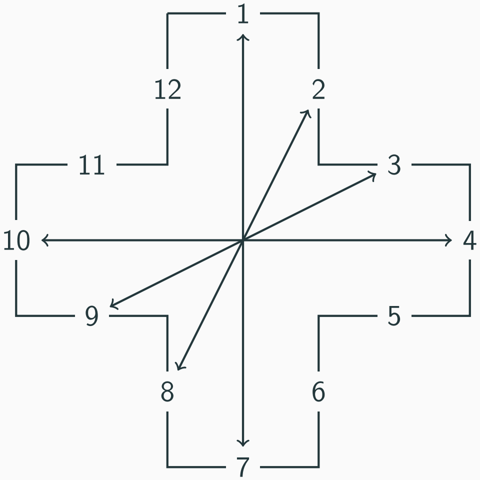
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



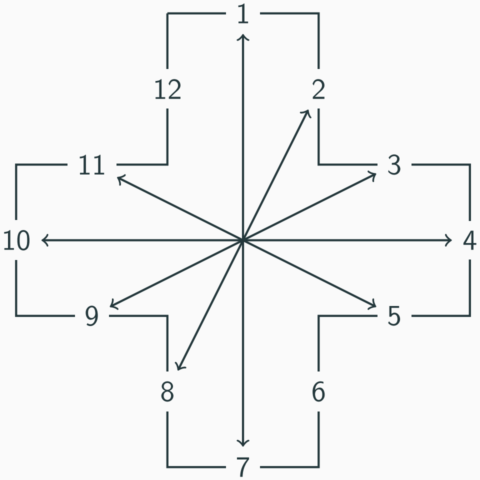
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



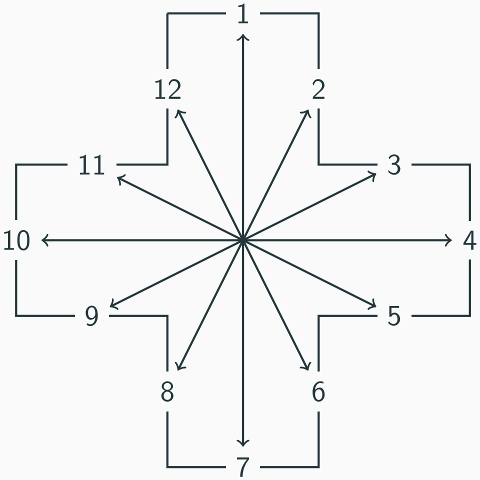
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



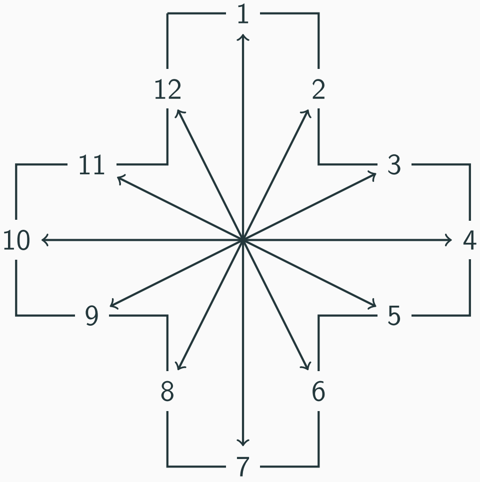
- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°:

An Application in Industry



- Identity: x_1^{12}
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$
- 90°: x_4^3
- 180°: x_2^6

An Application in Industry

From this, we get the cycle index:

$$P = \frac{1}{8}(x_1^{12} + 2x_4^3 + 3x_2^6 + 2x_1^2x_2^5)$$

We want the coefficient of $r^3g^3b^3y^3$
(where r = red, g = green, b = blue, y = yellow),
so we look at the terms x_1^{12}, x_4^3, x_2^6, $x_1^2x_2^5$ individually.

An Application in Industry

- $P = \frac{1}{8}(x_1^{12} + 2x_4^3 + 3x_2^6 + 2x_1^2 x_2^5)$

Using Pólya's Enumeration Theorem, we'll plug in the colors r, g, b, y:

$$x_k = r^k + g^k + b^k + y^k$$

So looking at each term individually:

$$\begin{aligned} x_1^{12} &= (r + g + b + y)^{12} \\ x_4^3 &= (r^4 + g^4 + b^4 + y^4)^3 \\ x_2^6 &= (r^2 + g^2 + b^2 + y^2)^6 \\ x_1^2 x_2^5 &= (r + g + b + y)^2 (r^2 + g^2 + b^2 + y^2)^5 \end{aligned}$$

An Application in Industry

- $P = \frac{1}{8}(x_1^{12} + 2x_4^3 + 3x_2^6 + 2x_1^2 x_2^5)$

So the coefficient for the term $r^3g^3b^3y^3$ from P will be...

$$\frac{1}{8}\binom{12}{3,3,3,3}$$

$= 46200$

Therefore, we have found that there are 46,200 different ways to design the symbol!

An Interesting Find

- While working on this example, we discovered that the group of symmetries for the edges of a regular cross is actually the dihedral group D_4, like that of a square.
- This is interesting! So we started looking into the effects of dihedral groups acting on polygons.

Results, reflection groups, pictures - Clarisse

Group Action

For the purposes of analyzing D_k dihedral groups acting on $nk - gons$, it is best to imagine the $nk - gon$ inscribed inside the $k - gon$, and being restricted to the rotations and flips of the $k - gon$.

Algorithm for constructing the polynomials which represent a dihedral group D_k acting on a $nk - gon$.

- construct the normal k-gon.
    - Rotations
    - Flips
- Account for the n-cases

Rotations

First approach the normal k-gon. In order to construct the terms which represent rotations of the normal $k - gon$ we will take the following steps:

- Suppose $a_1, a_2, ..., a_n$ are divisors of k.
- For each divisor a_i assemble the values which are coprime to it.
- Denote these coprime values b_{ij} for each respective a_i.
- Now the **rotation** terms in our polynomial for the normal $k - gon$ will be:

$$\sum |b_{ij}| x_{a_i}^{k/a_i}$$

So

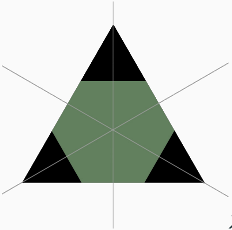
$$\sum |b_{ij}| x_{a_i}^{nk/a_i}$$

Flips

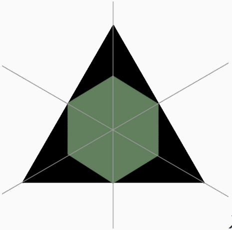
Next, in order to find the polynomial of $d_k \circlearrowright nk - gon$ we need to add in the terms which represent flips.
If k is prime, then we have three cases:

1. $kx_1^1 x_2^{(kn-1)/2}$ for n odd $\longmapsto$
2. $kx_2^{(kn)/2}$ for n even$\longleftrightarrow$
3. $kx_1^2 x_2^{(kn-2)/2}$ for n even $|--|$

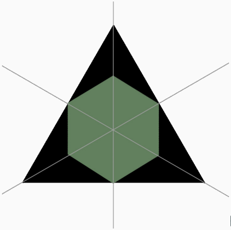
Understanding the Different Cases Concerning Flips



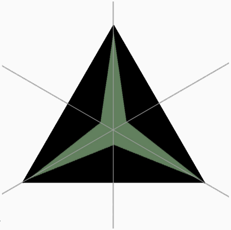
$x_1^6 + 2x_3^2 + 3x_1^2x_2^2$



$x_1^6 + 2x_3^2 + 3x_2^3$



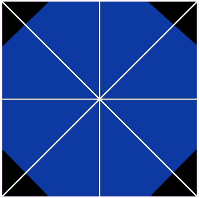
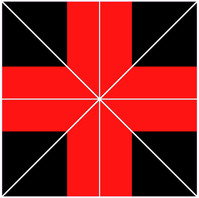
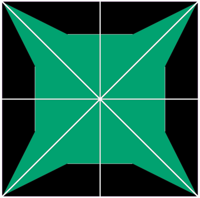
$\mapsto$

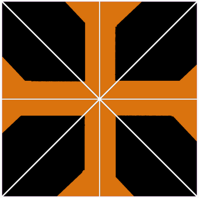
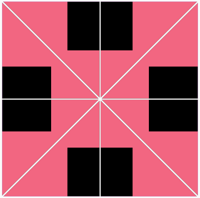
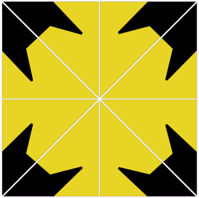
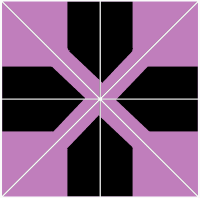


Cycle index of a triangle: $x_1^3 + 2x_3^1 + 3x_1^1 x_2^1$

n	Truncated Vertices	Subdivision of Edges
2	$x_1^6 + 2x_3^2 + 3x_1^2 x_2^2$	$x_1^6 + 2x_3^2 + 3x_2^3$
3	$x_1^9 + 2x_3^3 + 3x_1^1 x_2^4$	$x_1^9 + 2x_3^3 + 3x_1^1 x_2^4$
4	$x_1^1 2 + 2x_3^4 + 3x_1^2 x_2^5$	$x_1^1 2 + 2x_3^4 + 3x_2^6$
5	$x_1^1 5 + 2x_3^5 + 3x_1^1 x_2^7$	$x_1^1 5 + 2x_3^5 + 3x_1^1 x_2^7$

Table 1: $D_3 \circlearrowright 3n - gon$

n	Truncated Vertices	Subdivision of Edges
2		
3		

n	Truncated Vertices	Subdivision of Edges
5		
6		

Cycle index of a square: $x_1^4 + 2x_4^1 + x_2^2 + x_1^2x_2^1 + 2x_2^2$

$D_4 \circlearrowright 4n - gon$

n	Truncated Vertices	Subdivision of Edges
2	$x_1^8 + 2x_4^2 + x_2^4 + 4x_1^2x_2^3$	$x_1^8 + 2x_4^2 + x_2^4 + 4x_2^4$
3	$x_1^{12} + 2x_4^3 + x_2^6 + 2x_1^2x_2^5 + 2x_2^6$	$x_1^{12} + 2x_4^3 + x_2^6 + 2x_1^2x_2^5 + 2x_2^6$
4	$x_1^{16} + 2x_4^4 + x_2^8 + 4x_1^2x_2^7$	$x_1^{16} + 2x_4^4 + x_2^8 + 4x_2^8$
5	$x_1^{20} + 2x_4^5 + x_2^{10} + 2x_1^2x_2^4 + 2x_2^{10}$	$x_1^{20} + 2x_4^5 + x_2^{10} + 2x_1^2x_2^4 + 2x_2^{10}$
6	$x_1^{24} + 2x_4^6 + x_2^{12} + 4x_1^2x_2^{11}$	$x_1^24 + 2x_4^6 + x_2^{12} + 4x_2^{12}$

Table 2: $D_4 \circlearrowright 4n - gon$

So for any D_n acting on a $nk - gon$ we can produce the flips using cases:

k Even:

- $k/2x_1^2x_2^{(kn-2)/2} + k/2x_2^{kn/2}$ n odd ($\longleftrightarrow$ and $|\!-\!|$)
- $kx_1^2x_2^{(kn-2)/2}$ n even (uniformly $|--|$)
- $kx_2^{kn/2}$ n odd (uniformly $\longleftrightarrow$)

k Odd:

- $kx_1^1x_2^{nk/2}$ n odd ($\mapsto$)
- $kx_2^{nk/2}$ n even ($|--|$)
- $kx_1^2x_2^{(nk-2)/2}$ n even ($\longleftrightarrow$)