Counting With Symmetries

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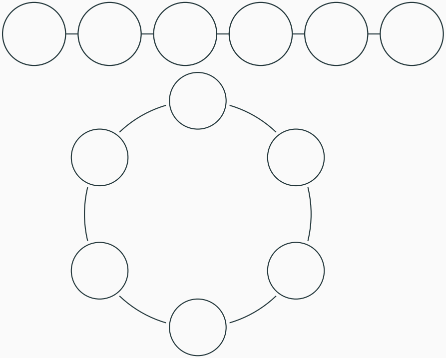
Thursday, June 1st

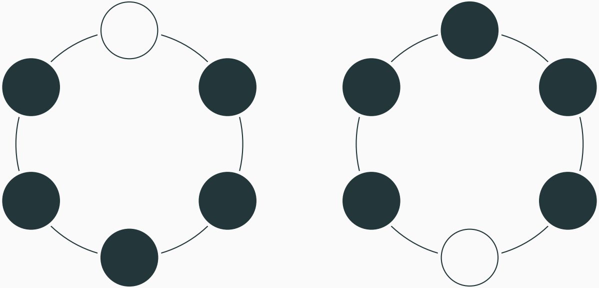
UC Riverside

Introduction



- We have black and white beads, and want to make a 6 bead bracelet.
- Well, we have 6 beads and 2 choices per bead so 2^6 right?
- This is only true so long as you don't care about symmetry.

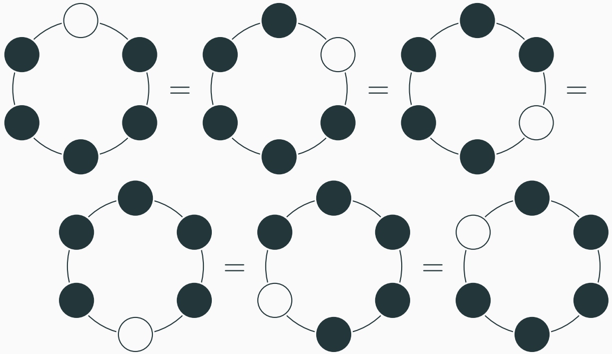
Symmetry?



- Our initial count considers these different
- But one is just the other one rotated
- We would like to consider these as the same when we count

The hard way

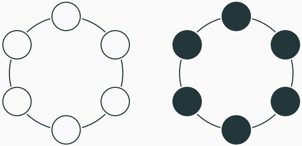
Specifically, we want to count colorings with rotations by picking combinations of beads and treating them as equal when they're rotations of each other:



The hard way

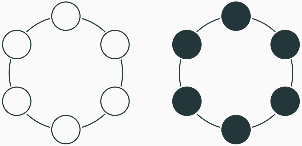
We can count this way by making a table of bracelets and counting how many different bracelets can be made via rotations of that bracelet.

The hard way



$2 \to 2$

The hard way



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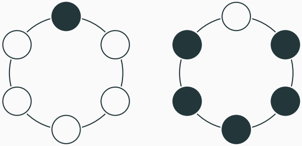
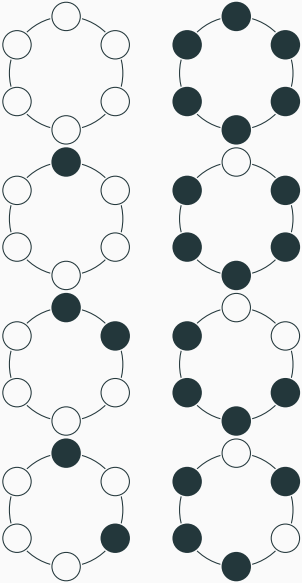


$12 \to 2$

The hard way

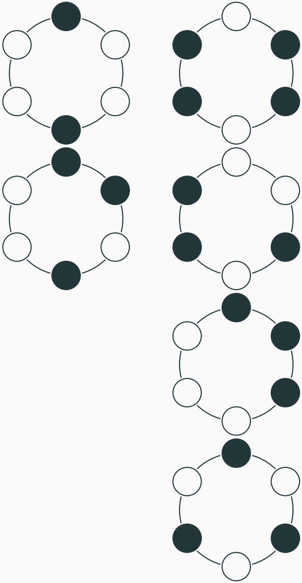


2→2

12→2

12→2

12→2



6→2

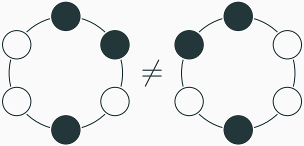
12→2

6→1

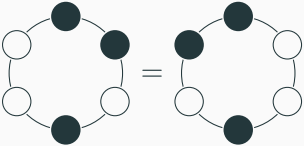
2→1

Flips and Colorings

- However, the count is different if we allow for more actions than just rotations.
- Another action we could add is flipping the bracelet over.
- This results in different unique colorings.



when no flips are allowed.

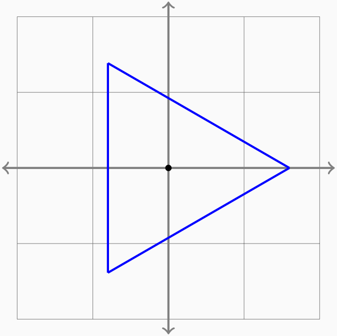


when flips are allowed.

Dihedral Group and Group Actions

- We seek to generalize the types of things we can do to sets with the right symmetries.
- To get an idea of these, we will look at the symmetries of an equilateral triangle.
- We will let Δ denote the set of points that make up an equilateral triangle.
- We want to describe the symmetries of this triangle.
- We then want to think about the actions that make up these symmetries more generally.

Dihedral Group and Group Actions



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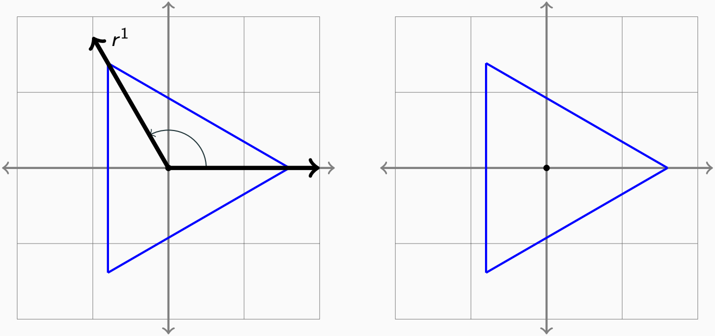

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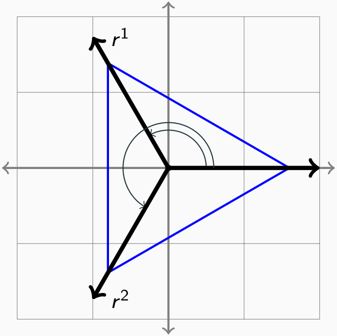

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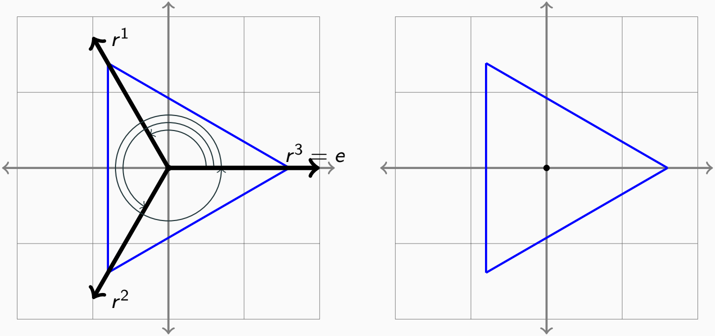

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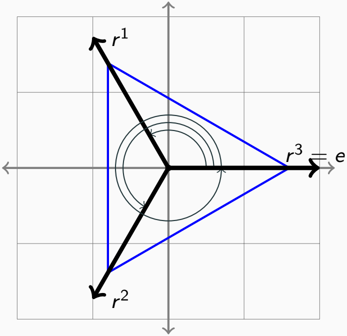

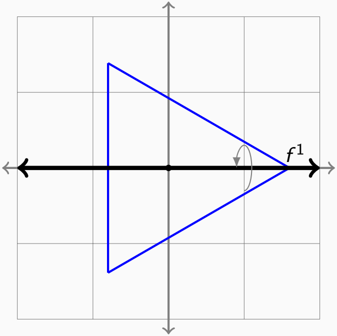

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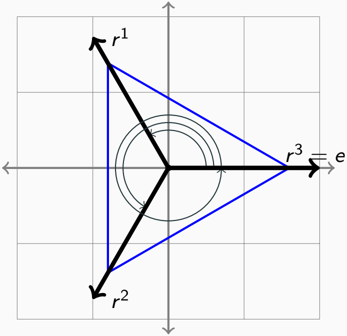

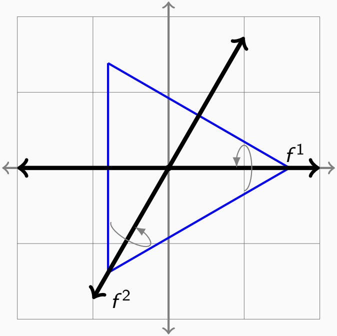

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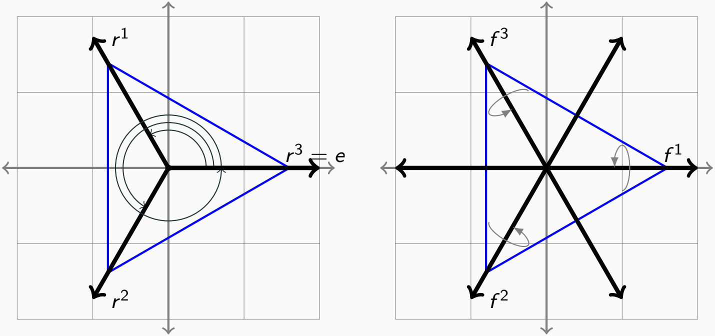

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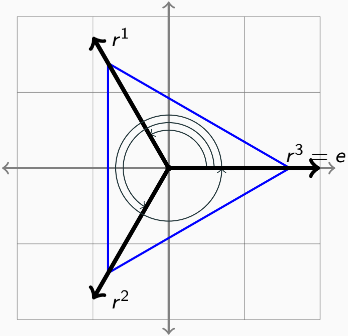
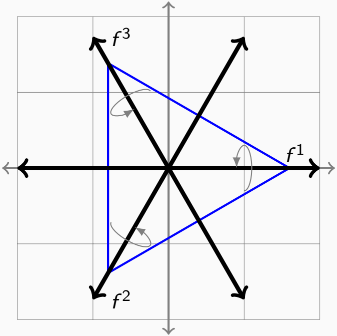
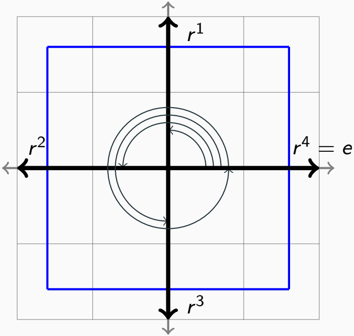


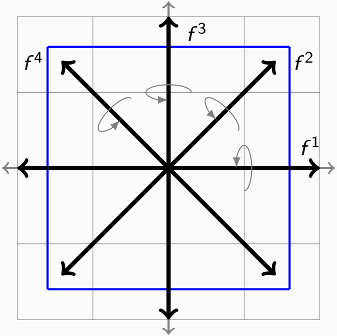


$D_3 = \{e, r^1, r^2, f^1, f^2, f^3\}$ is the dihedral group on 3 elements.

We say D_3 acts on Δ, or $D_3 \circlearrowright \Delta$.

D_4





$D_4 = \{e, r^1, r^2, r^3, f^1, f^2, f^3, f^4\}$ is the dihedral group on 4 elements.

This generalizes to D_n, the dihedral group on n elements: it's a group of $2n$ elements with n rotations and n flips.

The Goal

The goal of this talk is to develop a framework for answering coloring questions like these where symmetry is crucial

Polya Build up

Permutations

- A permutation is a bijection from a set S into itself
- Can be viewed as changing the order of elements
- Consider the set $S = \{1, 2, 3, 4\}$ And let $f : S \to S$ be defined

$$f(1) = 2$$
$$f(2) = 1$$
$$f(3) = 4$$
$$f(4) = 3$$

Permutations

To specify permutations we use cycle notation.

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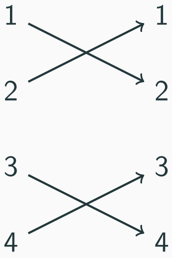

$$(12)(34)$$

Permutations

To specify permutations we use cycle notation.

1	1
2	2
3	3
4	4

$$(123)(4)$$

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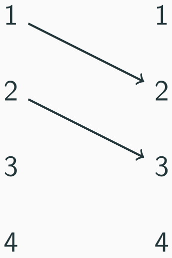

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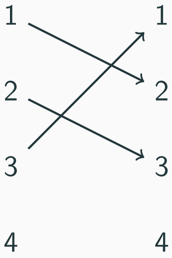

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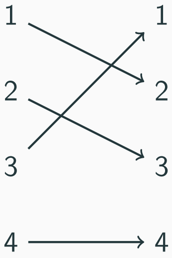

$$(123)(4)$$

Permutations

To specify permutations we use cycle notation.

$$1 \longrightarrow 1$$

$$2 \longrightarrow 2$$

$$3 \longrightarrow 3$$

$$4 \longrightarrow 4$$

$$(1)(2)(3)(4) = e$$

Colorings

A coloring c is a map from our set of objects into our set of colors.

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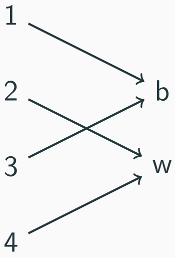

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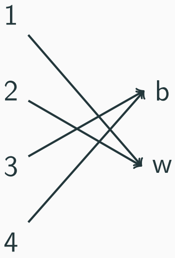

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As a coloring is a map from our set of objects to our set of colors, if we compose a coloring c with a permutation we get a new coloring.

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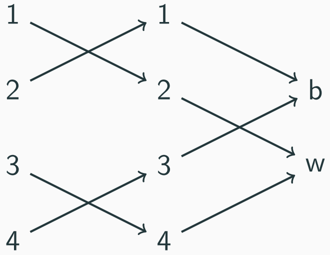

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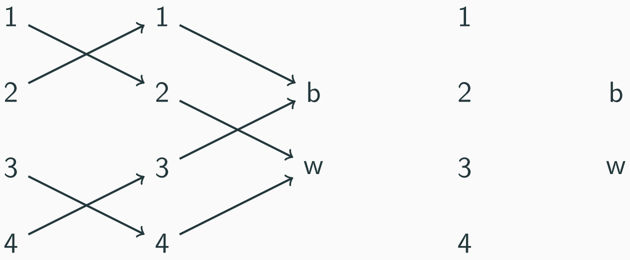

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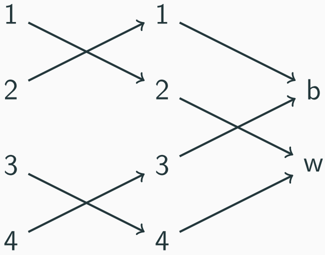

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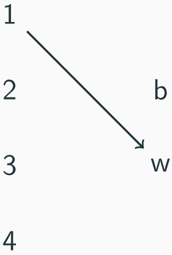

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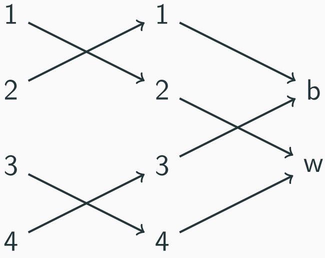

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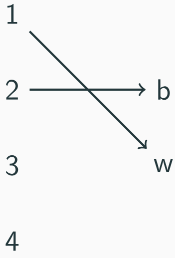

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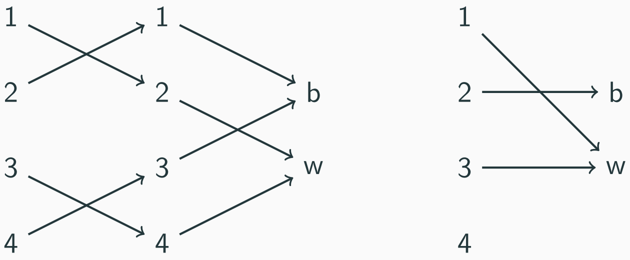

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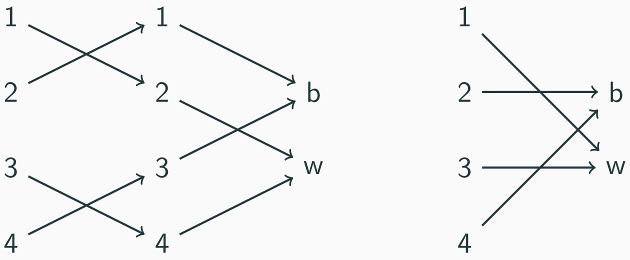
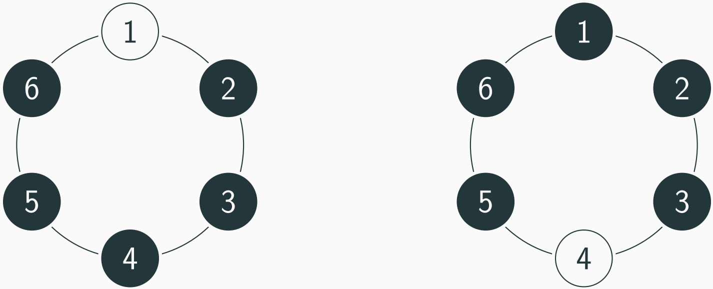


(12)(34)

Symmetry

This gives us a language to discuss symmetry now. Consider the two bracelets from earlier:



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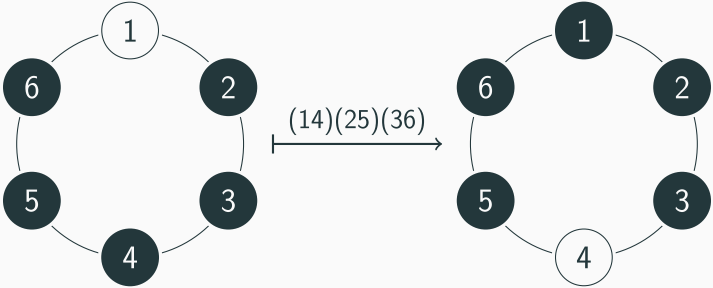

Symmetry

Thus we can describe the symmetry of an object by defining the group of permutations we allow on it.

Taking our bracelet example from earlier we see that our set of permutations is:

$$\{e, (123456), (135)(246), (14)(25)(36), (153)(264), (165432)\}$$

Polya Theorem

- Definition: Let G be a group of symmetries on set S where $|S| = n$.

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- Next define an arbitrary variable x_i.
- Let α_i represent the number of disjoint cycles of length i corresponding to a given permutation in G.
- For any π, we define $\beta_\pi = \prod_{i=1}^{n} x_i^{\alpha_i}$. We Observe, β_π encodes all the info about the number of cycles and lengths of cycles of π.

- Define cycle index of G as $P_G(x_1, x_2, ..., x_n) = \frac{1}{|G|}\sum_{\pi \in G} \beta_\pi$.

- Bracelet with 6 beads,
  $P_{C_6}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{6}(x_1^6 + x_6^1 + x_3^2 + x_2^3 + x_3^2 + x_6^1)$.

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- Every term represents a permutation.
- The 3*rd* term x_3^2 corresponding to the permutation $\pi = (153)(264)$ has two cycles of length three.

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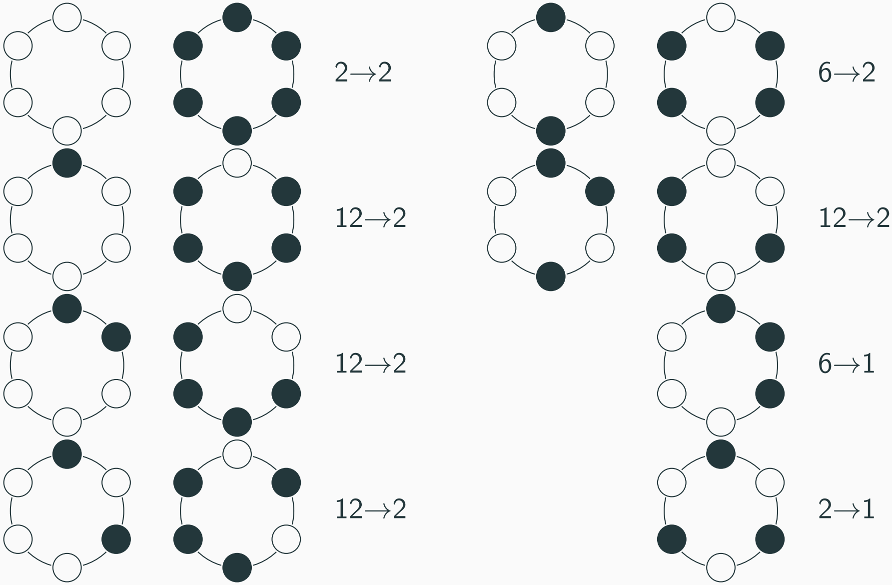
- $P_{C_6}(k,k,k,k,k,k) = \frac{1}{6}(k^6 + 2k + 2k^2 + k^3)$.
- $k = 2$, implies $P_{C_6}(2,2,2,2,2,2) = 14$.



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- What Polya's theorem asks us to consider is the new polynomial $F_G = P_G(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, ..., \sum_{j=1}^{m} c_j^n)$.

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- What we get is, given any term $Ac_1^{\gamma_1} c_2^{\gamma_2} ... c_m^{\gamma_m}$ of polya'a enumeration formula, the coefficient A equals the number of colorings with exactly γ^i elements of color c_i.

- Example using the 6 beaded bracelet. If we let b and w represent our two colors, then we obtain $P_{C_6}((b+w),(b^2+w^2),(b^3+w^3),(b^4+w^4),(b^5+w^5),(b^6+w^6)) = b^6+b^5w+3b^4w^2+4b^3w^3+3b^2w^4+bw^5+w^6$.

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- The coefficient, of each term, $A_i b^k w^j$ determines the number of possible distinct colorings on the six beaded bracelet using the colors b and w, i and j times for each bead.
- for example, the third term $3b^4w^2$ corresponds to 3 possible colorings, coloring 4 bead's black and 2 bead's white.

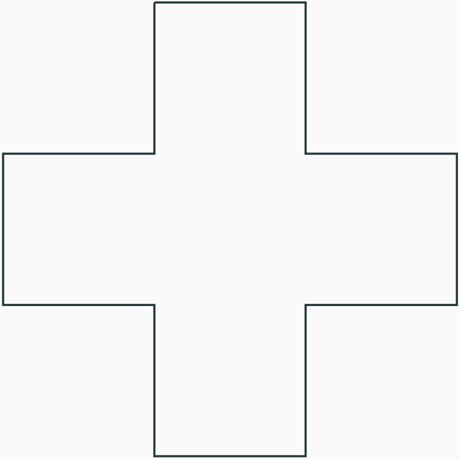
Using What We've Learned with an Example!

An Application in Industry

- A medical agency plans to design a symbol for their organization in the shape of a regular cross.
- They decide that the cross should be white in color, with each of the 12 line segments outlining the cross colored red, green, blue, or yellow.
- And should have an equal number of lines of each color.

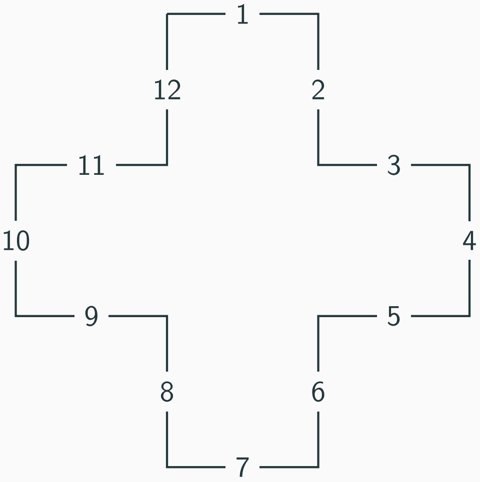
An Application in Industry

How many different ways are there to design the symbol, taking into account rotations and flips?



An Application in Industry

First, WLOG we can number the sides of the cross as so:



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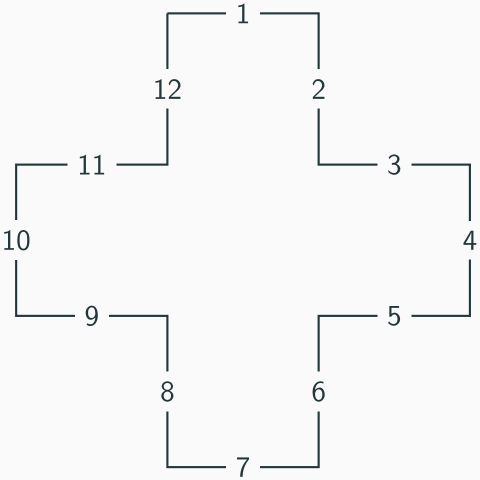
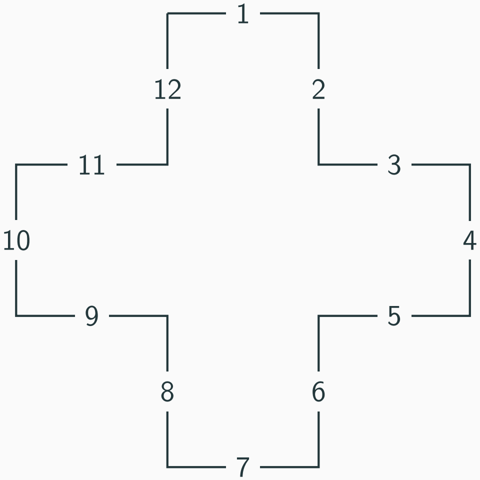


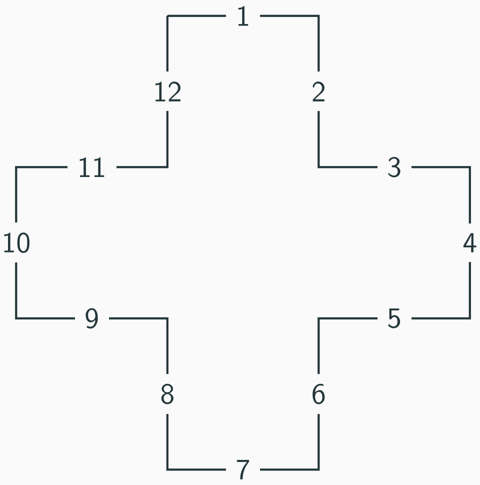
- Identity:

An Application in Industry



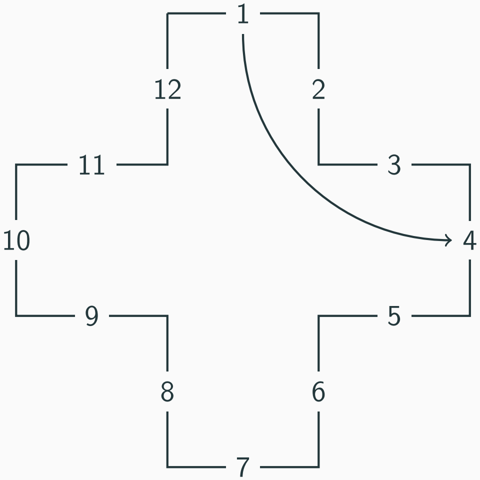
- Identity: x_1^{12}

An Application in Industry



- Identity: x_1^{12}
- 90°:

An Application in Industry



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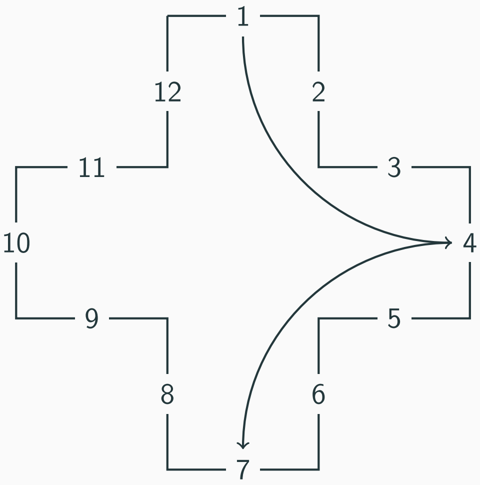

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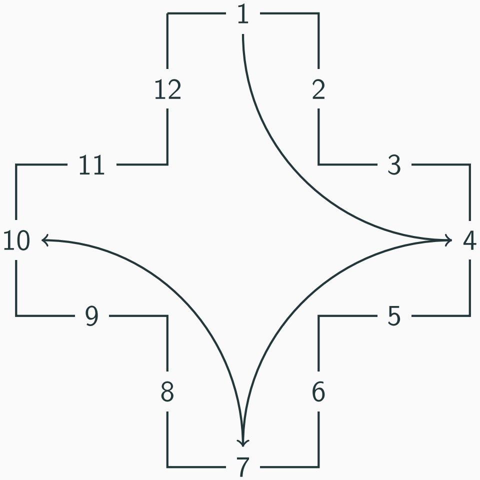

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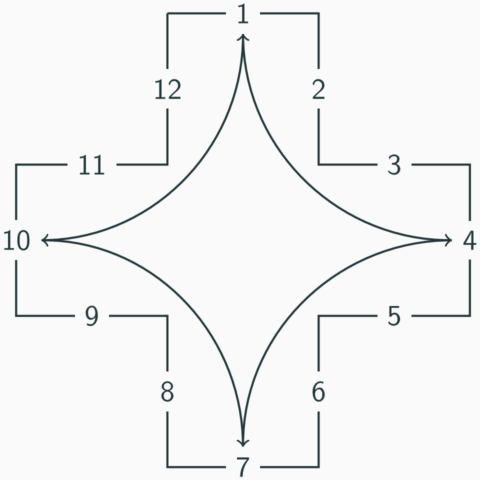

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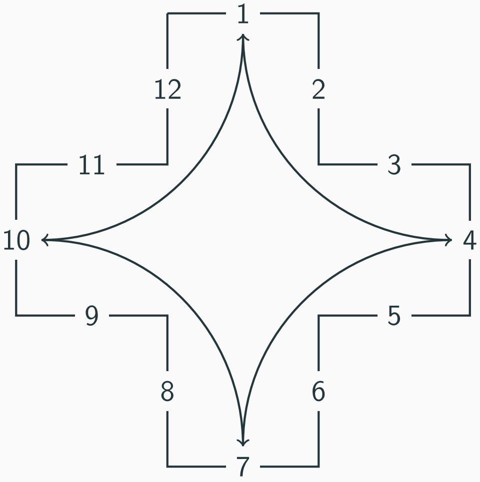
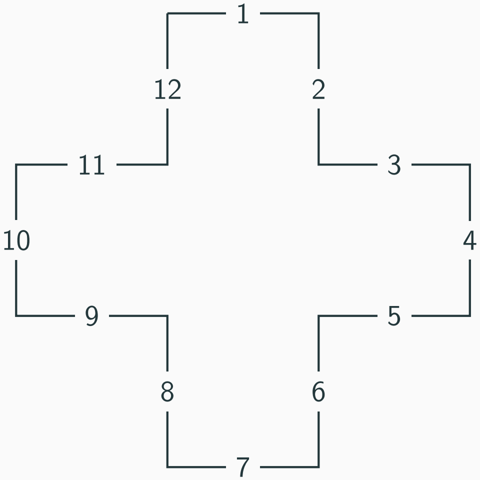
An Application in Industry



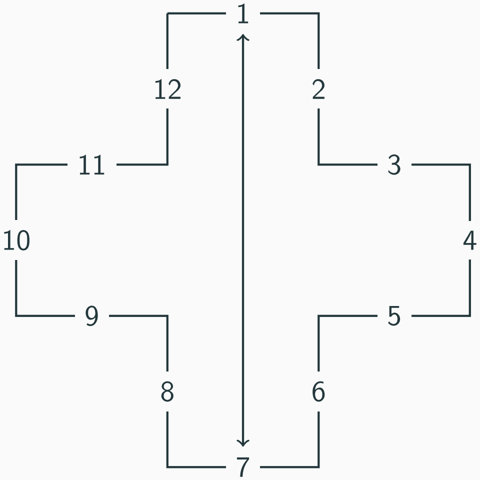
- Identity: x_1^{12}
- 90°: x_4^3

An Application in Industry



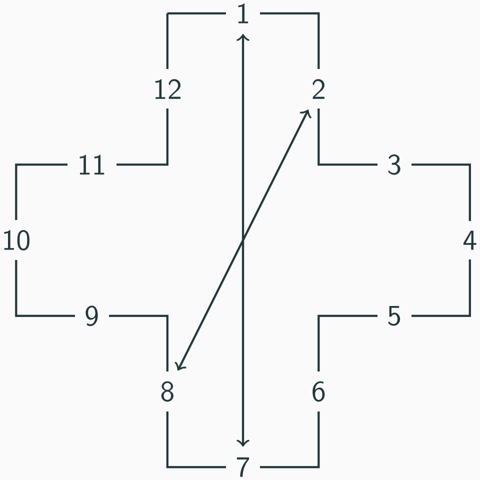
- Identity: x_1^{12}
- 90°: x_4^3
- 180°:

An Application in Industry



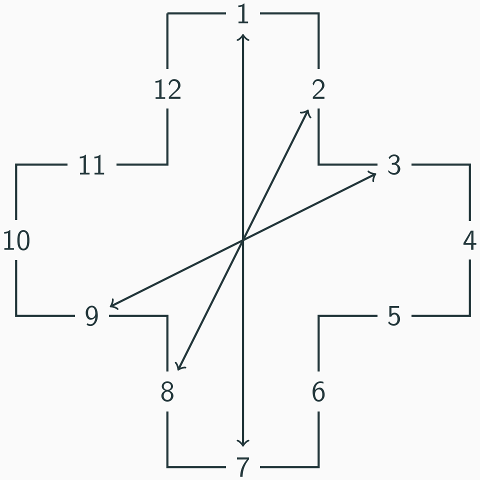
- Identity: x_1^{12}
- 90°: x_4^{3}
- 180°:

An Application in Industry



- Identity: x_1^{12}
- 90°: x_4^3
- 180°:

An Application in Industry



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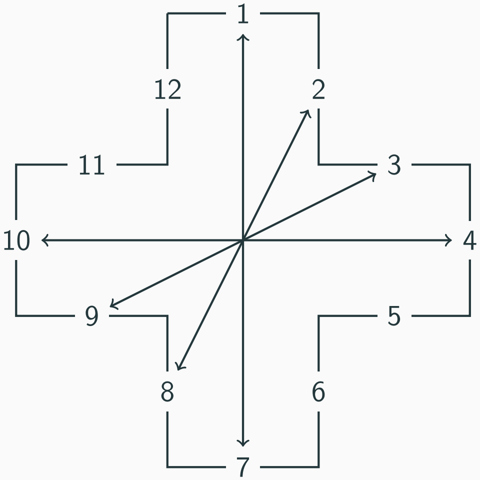

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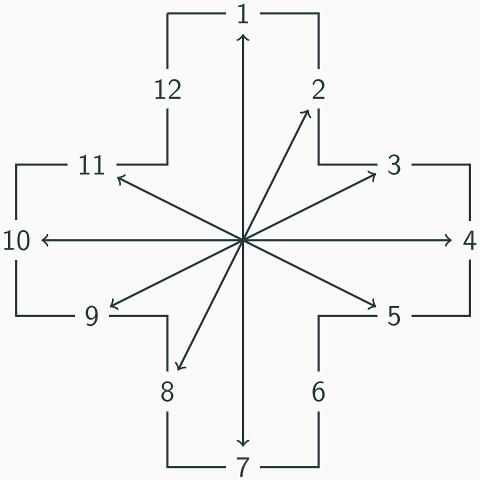
An Application in Industry



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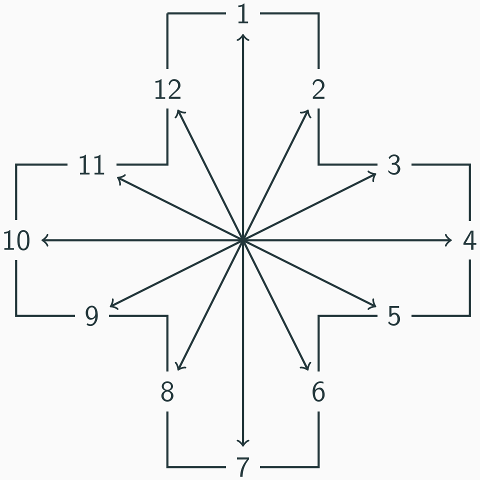
An Application in Industry



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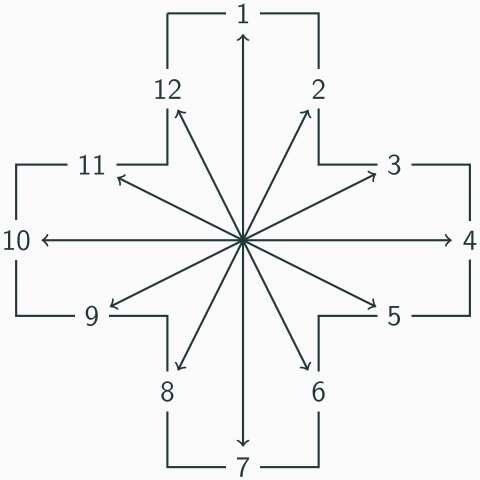
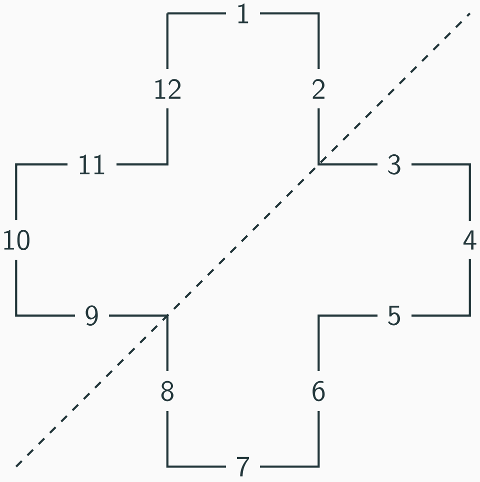
- 180°:

An Application in Industry



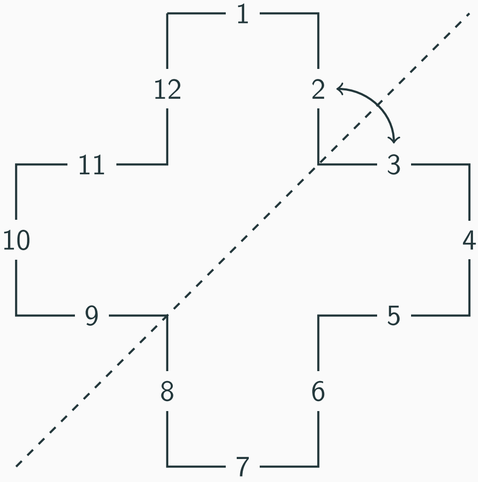
- Identity: x_1^{12}
- 90°: x_4^3
- 180°: x_2^6

An Application in Industry



- Identity: x_1^{12}
- 90°: x_4^3
- 180°: x_2^6
- Diagonal:

An Application in Industry



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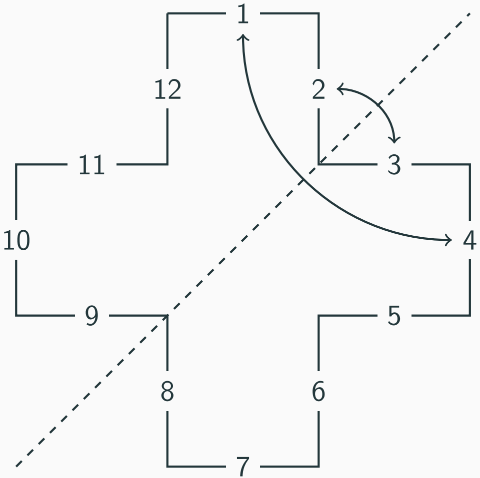
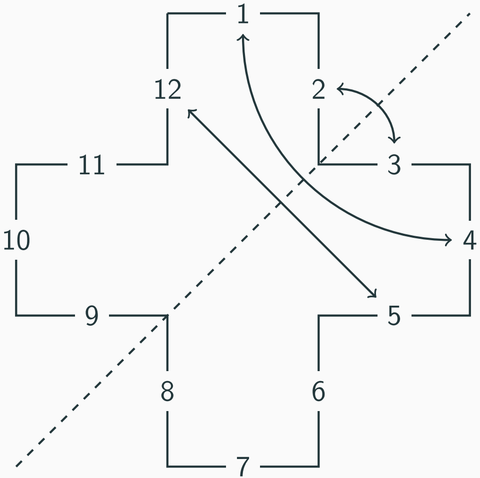
- 90°: x_4^3
- 180°: x_2^6
- Diagonal:

An Application in Industry



- Identity: x_1^{12}
- 90°: x_4^3
- 180°: x_2^6
- Diagonal:

An Application in Industry



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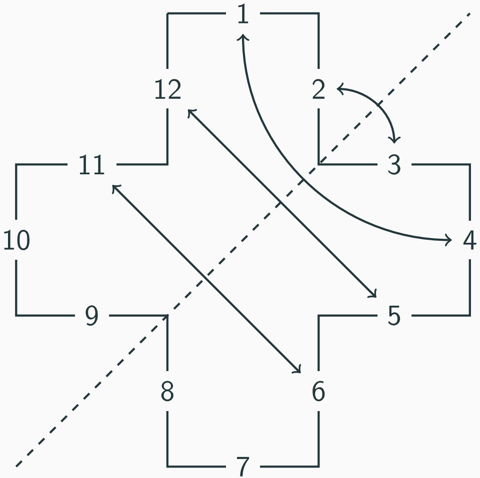

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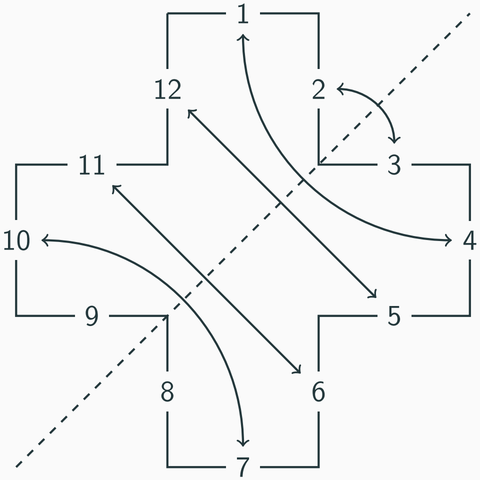

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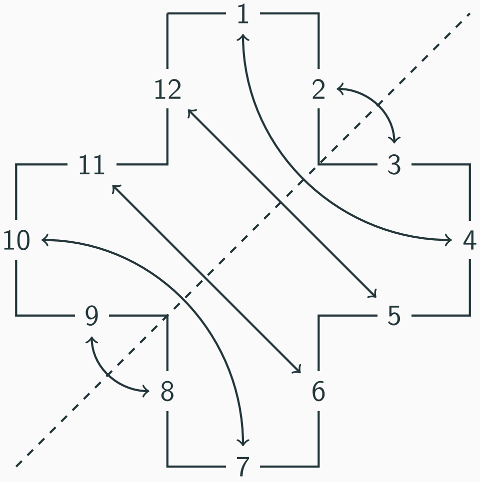

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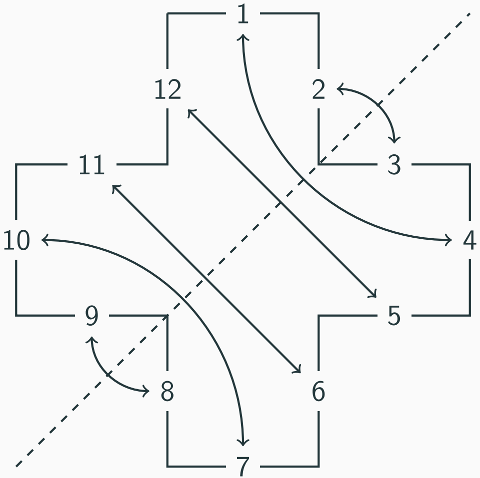

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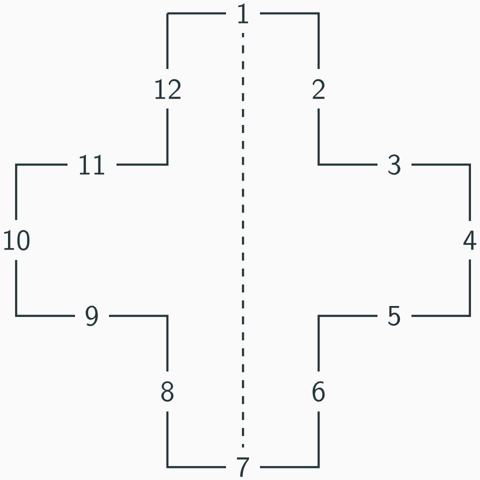
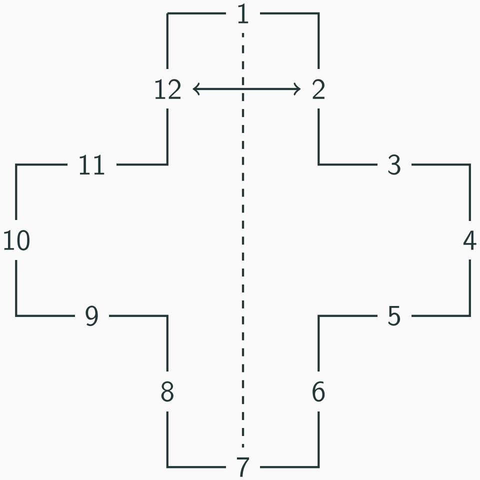
An Application in Industry



- Identity: x_1^{12}
- 90°: x_4^3
- 180°: x_2^6
- Diagonal: x_2^6
- Vertical:

An Application in Industry



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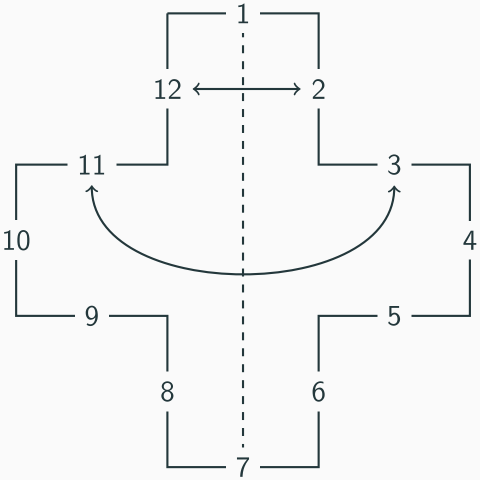

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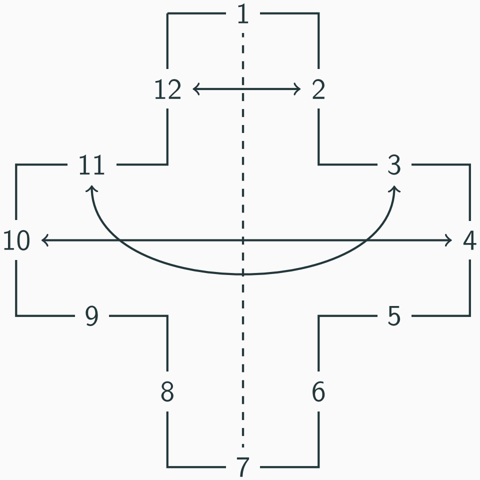

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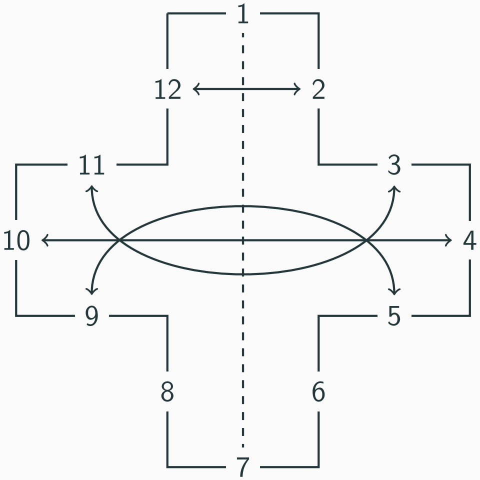

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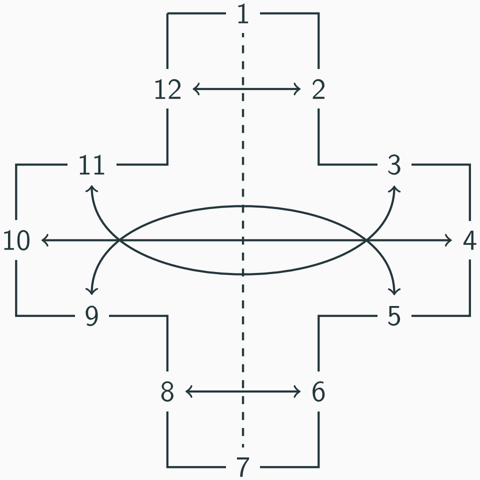

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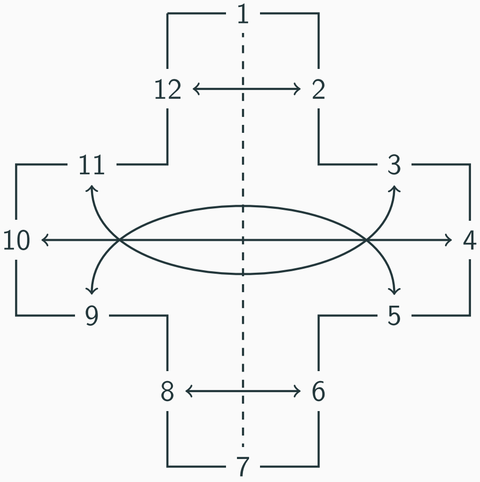

- Identity: x_1^{12}
- 90°: x_4^3
- 180°: x_2^6
- Diagonal: x_2^6
- Vertical: $x_1^2 x_2^5$

An Application in Industry

From this, we get the cycle index:

$$P = \frac{1}{8}(x_1^{12} + 2x_4^3 + 3x_2^6 + 2x_1^2 x_2^5)$$

An Application in Industry

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Using Pólya's Enumeration Theorem, we'll plug in the colors r, g, b, y:

$$x_k = r^k + g^k + b^k + y^k$$

So looking at each term individually:

$$\begin{aligned}
x_1^{12} &= (r + g + b + y)^{12} \\
x_4^3 &= (r^4 + g^4 + b^4 + y^4)^3 \\
x_2^6 &= (r^2 + g^2 + b^2 + y^2)^6 \\
x_1^2x_2^5 &= (r + g + b + y)^2(r^2 + g^2 + b^2 + y^2)^5
\end{aligned}$$

An Application in Industry

- $P = \frac{1}{8}(x_1^{12} + 2x_4^3 + 3x_2^6 + 2x_1^2 x_2^5)$

So the coefficient for the term $r^3 g^3 b^3 y^3$ from P will be...

$$\frac{1}{8}\binom{12}{3,3,3,3} = 46200$$

Therefore, we have found that there are 46,200 different ways to design the symbol!

An Interesting Find

- While working on this example, we discovered that the group of symmetries for the edges of a regular cross is actually the dihedral group D_4, like that of a square.
- This is interesting! So we started looking into the effects of dihedral groups acting on polygons.

Results

So we just saw an example which used Polya's Theorem to color the twelve sides of a cross.

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Our Question

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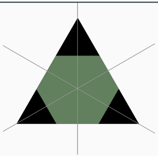
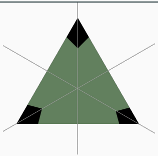
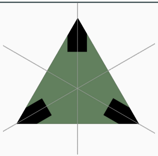
  - In the Cross example we had k=4 and n=3.
- The creation of a formula is dependent upon the action.

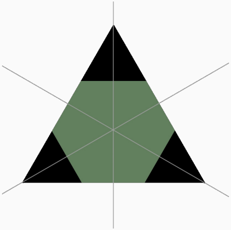
Base Case

- To investigate how dihedral groups act on polygons we decided to start out with $D_3 \circlearrowright 3n$-gon.

n	nk	cycle index	image
1	3	$\frac{1}{6}\left(x_1^3 + 2x_3^1 + 3x_1^1x_2^1\right)$	triangle
2	6	$\frac{1}{6}\left(x_1^6 + 2x_3^2 + 3x_1^2x_2^2\right)$	
3	9	$\frac{1}{6}\left(x_1^9 + 2x_3^3 + 3x_1^1x_2^4\right)$	
4	12	$\frac{1}{6}\left(x_1^{12} + 2x_3^4 + 3x_1^2x_2^5\right)$	

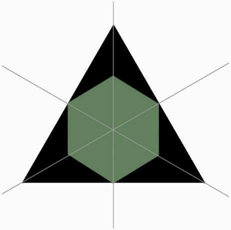
Understanding the Different Cases Concerning Flips

This is a hexagon inscribed inside a triangle:



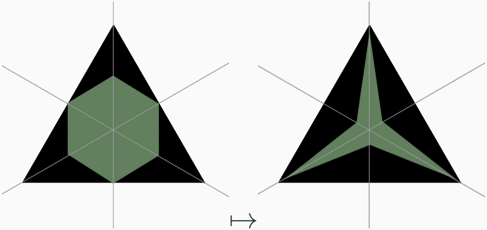
$\frac{1}{6}\left(x_1^6 + 2x_3^2 + 3x_1^2x_2^2\right)$

But so is this:

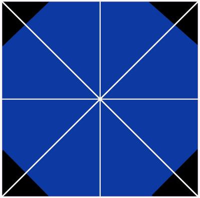
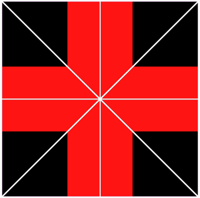
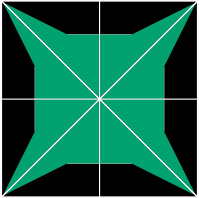


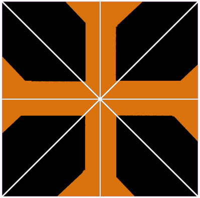
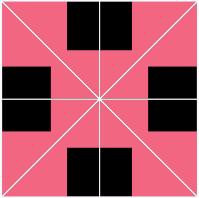
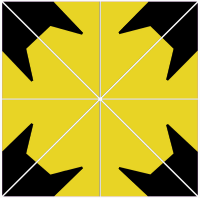
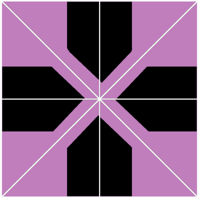
$\frac{1}{6}\left(x_1^6 + 2x_3^2 + 3x_2^3\right)$

Thankfully we are able to draw the second picture another way.



Both of these shapes have the same cycle index.

n	nk	Form 1	Form 2
2	8		
3	12		

n	nk	Form 1	Form 2
5	20		
6	24		

Cycle index of a triangle: $\frac{1}{6}(x_1^3 + 2x_3^1 + 3x_1^1 x_2^1)$

n	Form 1	Form 2
2	$\frac{1}{6}(x_1^6 + 2x_3^2 + 3x_1^2 x_2^2)$	$\frac{1}{6}(x_1^6 + 2x_3^2 + 3x_2^3)$
3	$\frac{1}{6}(x_1^9 + 2x_3^3 + 3x_1^1 x_2^4)$	$\frac{1}{6}(x_1^9 + 2x_3^3 + 3x_1^1 x_2^4)$
4	$\frac{1}{6}(x_1^{12} + 2x_3^4 + 3x_1^2 x_2^5)$	$\frac{1}{6}(x_1^{12} + 2x_3^4 + 3x_2^6)$
5	$\frac{1}{6}(x_1^{15} + 2x_3^5 + 3x_1^1 x_2^7)$	$\frac{1}{6}(x_1^{15} + 2x_3^5 + 3x_1^1 x_2^7)$

Table 1: $D_3 \circlearrowright 3n$-gon

The Pattern: $D_3 \circlearrowright n3$-gon

$$\frac{1}{6}\left(x_1^{3n} + (3-1)x_3^{n} + \begin{cases} 3x_1^1 x_2^{(3n-1)/2} & \text{n odd, V-E flip} \\ 3x_1^2 x_2^{(3n-2)/2} & \text{n even, E-E flip} \\ 3x_1^2 x_2^{(3n)/2} & \text{n even, V-V flip} \end{cases}\right)$$

The Pattern:r $D_k \circlearrowright$ nk-gon with k-prime

Formula

$$\frac{1}{2k}\left(x_1^{kn} + (k-1)x_k^{n} + \begin{cases} kx_1^1 x_2^{(kn-1)/2} & \text{n odd, V-E flip} \\ kx_1^2 x_2^{(kn-2)/2} & \text{n even, E-E flip} \\ kx_1^2 x_2^{(kn)/2} & \text{n even, V-V flip} \end{cases}\right)$$

Cycle index of a square: $\frac{1}{8}(x_1^4 + 2x_4^1 + x_2^2 + x_1^2x_2^1 + 2x_2^2)$

n	Form 1	Form 2
2	$\frac{1}{8}(x_1^8 + 2x_4^2 + x_2^4 + 4x_1^2x_2^3)$	$\frac{1}{8}(x_1^8 + 2x_4^2 + x_2^4 + 4x_2^4)$
3	$\frac{1}{8}(x_1^{12} + 2x_4^3 + x_2^6 + 2x_1^2x_2^5 + 2x_2^6)$	$\frac{1}{8}(x_1^{12} + 2x_4^3 + x_2^6 + 2x_1^2x_2^5 + 2x_2^6)$
4	$\frac{1}{8}(x_1^{16} + 2x_4^4 + x_2^8 + 4x_1^2x_2^7)$	$\frac{1}{8}(x_1^{16} + 2x_4^4 + x_2^8 + 4x_2^8)$
5	$\frac{1}{8}(x_1^{20} + 2x_4^5 + x_2^{10} + 2x_1^2x_2^4 + 2x_2^{10})$	$\frac{1}{8}(x_1^{20} + 2x_4^5 + x_2^{10} + 2x_1^2x_2^4 + 2x_2^{10}$
6	$\frac{1}{8}(x_1^{24} + 2x_4^6 + x_2^{12} + 4x_1^2x_2^{11})$	$\frac{1}{8}(x_1^{24} + 2x_4^6 + x_2^{12} + 4x_2^{12})$

Table 2: $D_4 \circlearrowleft 4n - gon$

Algorithm for constructing the polynomials which represent a dihedral group D_k acting on a $nk - gon$.

So for any D_n acting on a $nk - gon$ we can produce the flips using cases:

- $k/2x_1^2x_2^{(kn-2)/2} + k/2x_2^{kn/2}$ n odd, k even (V-V flips + E-E flips)
- $kx_1^1x_2^{nk/2}$ n odd (V-E flips)
- $kx_1^2x_2^{(kn-2)/2}$ n even (E-E flips)
- $kx_2^{kn/2}$ n odd (V-V flips)

Theorem: Rotation Monomials

The monomials for $C_k \circlearrowright k$-gon:

$$\sum_{m=1}^{k} X_{k/gcd(m,k)}^{gcd(m,k)}$$
$$= \sum_{m|k} \varphi(m) X_m^{nk/m}$$

Recall φ, the Euler-Phi Function:

$$\varphi : \mathbb{N} \mapsto \mathbb{N}$$

$\varphi(m)$= the number of positive integers less than m which are co-prime to m.

Result

General Formula $D_k \circlearrowright nk$-gon

$$\frac{1}{2k}\left(\sum_{m|k}\varphi(m)X_m^{nk/m}+\begin{cases} k/2x_1^2x_2^{(kn-2)/2}+k/2x_2^{kn/2} & \text{n odd, k even (V} \\ kx_1^1x_2^{(kn-1)/2} & \text{n odd, (V-E flip)} \\ kx_1^2x_2^{(kn-2)/2} & \text{n even, (E-E flip} \\ kx_1^2x_2^{(kn)/2} & \text{n even, (V-V flip} \end{cases}\right.$$

Further Questions to Explore...

- Reflection groups of a Tetrahedron and a Cube, and making cycle indexes for them.
    - the reflections of symmetries can be viewed when drawn in 2-D
- Applications in Graph Theory, such as counting the number of unlabeled cubic graphs.
- Applications of Pólya Theory on chemical isomers.
    - chemical compounds, instead of colors

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Questions?

	e	180 °	120°, 240°	90 °, 270°	60°, 300°	30°,150°,210°,330°
1	x_1^1					
2	x_1^2	x_2^1				
3	x_1^3		$2x_3^1$			
4	x_1^4	x_2^2		$2x_4^1$		
6	x_1^6	x_2^3	$2x_3^2$		$2x_6^1$	
12	x_1^{12}	x_2^6	$2x_3^4$	$2x_4^3$	$2x_6^2$	$4x_{12}^1$