Assignment 7

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> Your solutions must be typed (preferably typeset in LaTeX) and submitted as a hard-copy at the beginning of class on the day its due.

Problem 1: Partition [10 points] Show that the following problem is NP-Complete (Hint: reduce from SubsetSum).

PARTITION: Given a finite set A and a positive integer $s(a)$ associated with each $a \in A$, is there a subset $A' \subseteq A$ such that $\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$?

Solution

We can see that partition is polynomial time since summing A and A' can be both done in $O(n)$ and then comparing the sums together is constant time.

Recalling from SubsetSum, given a set S of positive integers and a target integer $s(a)$, find a subset A of S that sums to t. If s is the sum of member of A, then $X' = X \cup \{s - 2(s(a))\}$ where $s = \sum_{a \in A} x$.

Assume there exists some $S \subset A$ such that $t = \sum_{a \in A} a$.

$$
\begin{aligned}
s - t &= \sum_{a \in A \cup s - 2s(a)} a \\
&= \sum_{a \in A \cup X' \setminus s - 2s(a)} a
\end{aligned}
$$

Therefore $S \cup \{s - 2s(a)\}$ and $X' \setminus S \cup \{s - 2s(a)\}$ form a partition A'.

Suppose A' is partitioned into P_1' and P_2', then there is a sublist A that $\{s - 2s(a)\} \in \sum_{a \in P_1'} a = \sum_{a \in P_2'} a$. Therefore, $\{s - 2s(a)\} \notin \sum_{a \in P_1'} a = s(a)$. Since P_1 is in A', all the numbers summed to $s(a)$ are in A. The SubsetSum is a known NP-Complete problem. Therefore, reducing this problem into partitioning shows that partitioning is NP-Complete.

Problem 2: Fire Houses [10 points] Show that the following problem is NP-Complete (Hint: reduce from 3-SAT or Vertex Cover).

FIREHOUSES: Given an undirected graph G with positive integer distances on the edges, and two integers f and d, is there a way to select f vertices on G on which to locate firehouses, so that no vertex of G is at distance more than d from a firehouse?

Solution

To make the problem more tractable, let's be clear on definitions:

G = (V, E)

V makes up locations in general, including potential locations of firehouses. Assume unselected vertices are houses being covered. Selected vertices are replaced with firehouses.

E makes up the distances between vertices, i.e. locations.

f = $|F|$ where $F \subseteq V$, i.e. where firehouses will be placed.

NP-Completeness

This problem is NP-Complete because we can verify answers easily, in polynomial time. Look at each vertex in F, and mark each vertex you can get to in d distance or less. Afterwards, there should be no unmarked vertices in G. If this is the case, then you have a solution that meets the criteria.

Reduction

You can reduce Set-Cover to the Firehouses problem as follows:

$\forall v \in V$, do a DFS and collect all the vertices on the way where the total edge distance $<= d$. Those vertices make up S_i, a subset of the total set space. We know that $\cup S_i = V$, as each subset contains it's starting vertex, from V. Each subset is "centered" around it's origin vertex and the vertices it can reach. Note that subsets definitely will overlap. We also keep a dictionary mapping each subset to its origin vertex.

With these $|V|$ subsets, we can now put them through set-cover, with minimum size subset $|C| = f$. If we find a solution that fits these requirements, we can get the original vertices back by using the constructed dictionary, thus leaving us with the locations of the firehouses.