Assignment 1

Due: January 16, 2020

Problem 1: Asymptotic Analysis Practice

(a) [5 points] Prove or disprove that $\log_k n \in O(\lg n)$ for any $k > 1$. (Note that lg refers to $\log_2$)

Solution

Another way of stating $\log_k n \in O(\lg n)$ for any $k > 1$ is:

$$\log_k n \leq \lg n \text{for any} k > 1$$
$$\frac{\lg n}{\lg k} \leq \frac{\lg n}{\lg 2}$$

Since $\lg 2 = 1$,

$$\frac{\lg n}{\lg k} \leq \lg n$$
$$\left(\frac{1}{\lg k}\right) \lg n \leq \lg n$$
$$\left(\frac{1}{\lg k}\right) \leq 1 \text{for any} k > 1$$

(b) [5 points] The following recurrence relation solves to $O(n \lg^2 n)$. Prove this by substition. Do not use the Master method.

$$T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$$
$$T(1) = 0$$

Solution

$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n\lg n \\
T(\frac{n}{2}) &= 2[2T\left(\frac{n}{4}\right) + \frac{n}{2}\lg\frac{n}{2}] + n\lg n \\
T(n) &= 2^k T\left(\frac{n}{2^k}\right) + k\left(n\lg n\right)
\end{aligned}
$$

Assume that:

$$T\left(\frac{n}{k}\right) = T\left(1\right)$$

That implies:

$$
\begin{aligned}
\frac{n}{2^k} &= 1 \\
n &= 2^k \\
k &= \lg n
\end{aligned}
$$

Substituting $k = \lg n$

$$T(n) = 2^{\lg n} T\left(\frac{n}{2^{\lg n}}\right) + \left(\lg n\right)\left(n\lg n\right)$$

Simplifying...

$$
\begin{aligned}
T(n) &= nT\left(1\right) + n\lg^2 n \\
&= O(n\lg^2 n)
\end{aligned}
$$

(c) [5 points] Suppose that $f(n)$ and $g(n)$ are non-negative functions. Prove or disprove the following: if $f(n) \in O(g(n))$ then $2^{f(n)} \in O(2^{g(n)})$.

Problem 2: Peak-finding Given a set of real numbers stored in an array A find the index of a *Peak*, where a *Peak* is defined as an element that is larger or equal to the both the elements on its sides. (Note: you only need to return *a* peak, not the highest one.)

Example array:

-2	6	-1	4	9	-5	5

Assuming the array one based it has peaks at indices $\{2, 5, 7\}$.

(a) [5 points] Give a linear time algorithm to solve this problem. (This should be obvious)

(b) [10 points] Give a $O(\log n)$ time algorithm to solve this problem.

(c) [10 points] What if instead of a simple array you are given a square matrix, where a *Peak* is now defined as an element larger or equal to its four neighbours. Give a $O(n\log n)$ solution to this variant of the problem.