Harmonic Analysis

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1 Fourier Series

The theory of Fourier series was developed initially to solve boundary value problems using trigonometric series. The early pioneers of Fourier series were d'Alembert, Bernoulli, Euler and Clairaut, who worked on the problem of the vibrating string and held the belief that arbitrary periodic functions could be represented as sums of sines and cosines. Fourier developed his heuristic further while trying to solve the heat equation. It was further developed by Laplace and Dirichlet.

1.1 The n-Torus $\mathbb{T}^n$

Let $\mathbb{R}$ be the additive group of real numbers and let $\mathbb{Z}$ be the additive group of integers. Let us define an equivalence relation (Verify!) $\sim$ as follows

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}^n, \text{ for } x, y \in \mathbb{R}^n.$$

As the $\sim$ is an equivalence relation it partitions $\mathbb{R}^n$ into equivalence classes, the n-torus $\mathbb{T}^n$ can be defined as the set $\mathbb{R}^n/\mathbb{Z}^n$ of all such equivalence classes. The n-Torus is an additive group. The identity element coincides with every vector in $\mathbb{Z}^n$. To avoid multiple appearances of the identity element we can think of the n-torus as $[-1/2, 1/2]^n$.

The n-torus can also be thought of as the follwing subset of $\mathbb{C}^n$

$$\{(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}) \in \mathbb{C}^n : (x_1, \dots, x_n) \in [0, 1]^n\}.$$

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Functions on $\mathbb{T}^n$ are functions f on $\mathbb{R}^n$ that satisfy $f(x+m) = f(x)$ for all $m \in ZZ^n$ and $x \in RR^n$. These functions are called 1-*periodic* in each coordinate (2π-*periodic* if we take $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$). The Haar measure on the n-torus is the restriction of