Measure Theory

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1 Measure

A measure can be thought of as a generalisation of area in two dimensions and of volume in three dimensions for higher dimensions. It can be thought of as a function that maps subsets of $\mathbb{R}^n$ to a number (the generalisation of area or volume). In 1984 Banach and Tarski proved the follwing result:

Let U and V be arbitrary bounded open sets in $\mathbb{R}^n$. Then for $n >= 3$ there exists $k \in NN$ and subsets $E_1, \dots, E_k, F_1, \dots, F_k$ such that

- the E_j's are disjoint and their union is U;
- the F_j's are disjoint and their union is V;
- each E_j is congruent to F_j (that is E_j can be transformed into F_j via translations, rotations and reflections) for $j = 1, \dots, k$.

Thus one can cut up a ball of any size into finite number of pieces and rearrange them to form a ball of any arbitrary size. The construction of E_j and F_j depends upon the axiom of choice. This example shows that $\mathbb{R}^n$ contains many strange sets upon which a reasonable notion of measure is impossible to define. We will define a measure on only a class of subsets of $\mathbb{R}^n$.

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1.1 Algebras and Sigma-Algebras

Definition 1 (Algebra of Sets). *Let X be a non-empty set. A collection $\mathcal{A}$ of subsets of X is called an **algebra** of sets if*

- *for every set A in $\mathcal{A}$ the set A^c belongs to $\mathcal{A}$.*
- *for each finite sequence $A_1, \dots, A_n$ of sets that belong to $\mathcal{A}$, the set $\cup_{i=1}^n A_i$ belongs to $\mathcal{A}$.*

It is easy to see that closure under complementation and closure under finite union implies closure under finite intersection. Suppose $A_1, \dots, A_n$ be a sequence of sets then

$$\bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} A_i^c \in \mathcal{A}.$$

Definition 2 (Sigma-Algebra of Sets). *Let X be a non-empty set. A collection $\mathcal{A}$ of subsets of X is called an **sigma-algebra** of sets if*

- *for every set A in $\mathcal{A}$ the set A^c belongs to $\mathcal{A}$.*
- *for each infinite sequence $A_1, A_2, \dots,$ of sets that belong to $\mathcal{A}$, the set $\cup_{n\in\mathbb{N}} A_n$ belongs to $\mathcal{A}$.*

By a similar argument given above we can show that sigma-algebras are also closed under infinite intersection. We observe that if $\mathcal{A}$ is an algebra (or sigma-algebra), then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$, for if $E \in \mathcal{A}$ we have $\emptyset = E \cap E^c$ and $X = E \cup E^c$.

Each sigma-algebra is an algebra on X since the union of finite sequence $A_1, A_2, \dots, A_n$ is same as the sequence $A_1, A_2, \dots, A_n, A_n, \dots$. An algebra $\mathcal{A}$ is a sigma-algebra if $\mathcal{A}$ is closed under countable disjoint unions. Suppose $\{E_n\}_{n\in\mathbb{N}}$. Set

$$F_n = E_n \backslash \left[\bigcup_{1}^{k-1} E_k\right].$$

Then F_n's belong to $\mathcal{A}$ and are disjoint and $\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n$.

1.2 Outer Algebras