Problem solving seminar
Final Test

Instructions

1. Work independently
2. Time: **240 minutes**
3. Books, notes, and calculators **are not allowed**
4. Please write down your solutions for Problems 1, 2, 3, 4, 5 on **individual** sheets
5. Please write down your name in **capital** letters, please write down your **e-mail** as well
6. Each question is worth 10 points.

Good luck!

Questions

1. Let $s_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$. Prove that $s_n(x) \ln\big(s_n(x)\big) > x s_{n-1}(x)$ for $n \geq 1$, $x > 0$.

2. Suppose that $\theta > 0$, $A = [a_{ij}]_{i,j=1}^{n}$ is a symmetric $n \times n$ matrix, $n > 1$, with nonnegative entries for which

$$\sum_{i,j=1}^{n} x_i x_j a_{ij} \leq -\theta \sum_{i=1}^{n} x_i^2 \quad \text{whenever} \quad \sum_{i=1}^{n} x_i = 0.$$

Prove that all eigenvalues of A have absolute value at least θ.

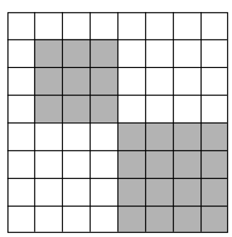

Figure 1: Examples of subboards.

3. Each square of the usual 8×8 chessboard is initially either white or black. At each step we choose a subboard of size 3×3 or 4×4 and reverse the colour of each square of the subboard. Is it true that no matter what the initial colouring of the chessboard is, there exists a sequence of steps such that the chessboard becomes all white?

4. Suppose that $\mathcal{S}$ is a family of circles in $\mathbb{R}^2$ such that the intersection of any two contains at most one point. Prove that the set M of those points that belong to at least two different circles from $\mathcal{S}$ is countable.

5. Prove that for every positive integer n the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is composite.