Flag Algebra Method in Combinatorics

Oleg Pikhurko

University of Warwick

RSA, Poznań, 9 August 2013

Many Happy Returns!!!

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Béla

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David

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Mihyun

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RSA

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Binomial$(200, \frac{7}{365})$
participants with unexposed birthdays

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Paul Erdös with some "epsilons" in 1952.
Barbie Benzer, Miriam and Debbie Golomb

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Is there a person with

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Theorem (Ron Graham): $\exists$ directed cycle!

Ron's Proof

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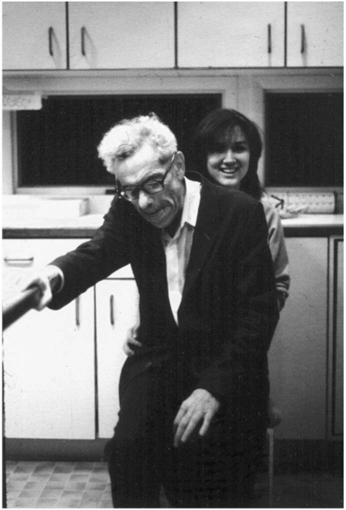
Oxford 1969
Ché Graham (age 7)
Paul Erdos (age ω)



New Jersey 1994
Ché Graham (age 32)
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Flag Algebras

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  - Formal rules for deriving them
  - One aspect: semi-definite programming

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- Structures: graphs, digraphs, k-uniform hypergraphs, permutations, subgraphs of the hypercube Q_n, ...

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- Contributors: R.Baber, J.Balogh, J.Cummings, S.Das, V.Falgas-Ravry, R.Glebov, A.Grzesik, H.Hatami, J.Hirst, J.Hladký, P.Hu, H.Huang, T.Klimosova, D.Král', L.Kramer, B.Lidicky, N.Linial, C.-H.Liu, J.Ma, L.Mach, E.Marchant, R.Martin, H.Naves, S.Niess, S.Norine, Y.Peled, J.Pfender, O.Pikhurko, A.Razborov, C.Reiher, J.-S.Sereni, K.Spengler, B.Sudakov, J.Talbot, A.Treglown, E.Vaughan, J.Volec, P.Whalen, Z.Yilma, M.Young, ...

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- $T_n^2 = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

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- Note: $\phi(P_3) + \phi(\bar{P}_3) + \phi(\bar{K}_3) = 1$

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- $\phi(K_2) \le \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- Asymptotic result: $\mathrm{ex}(n, K_3) \le (\frac{1}{2} + o(1))\binom{n}{2}$

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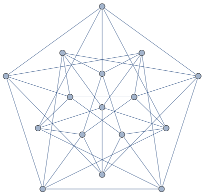

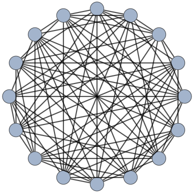

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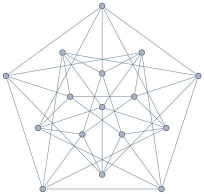

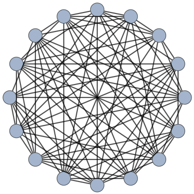

- Upper bound: $U_n(\overline{L})$

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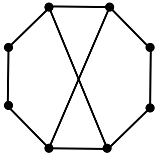

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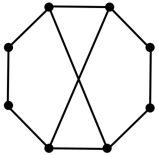

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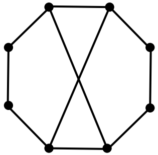

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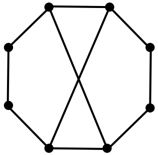

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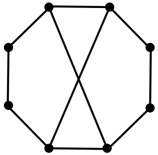

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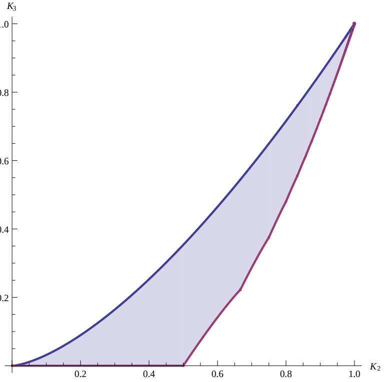

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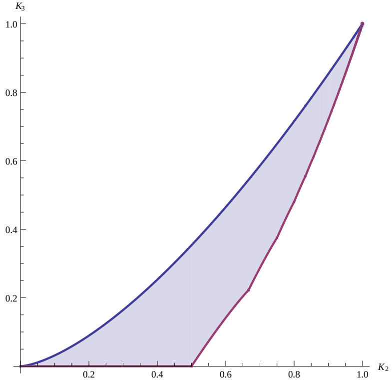

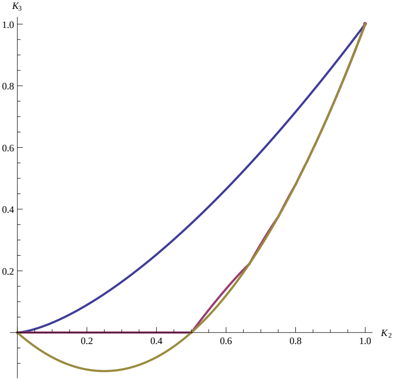

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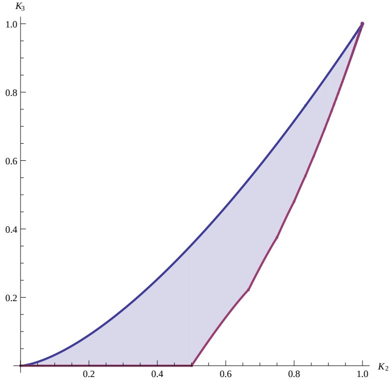

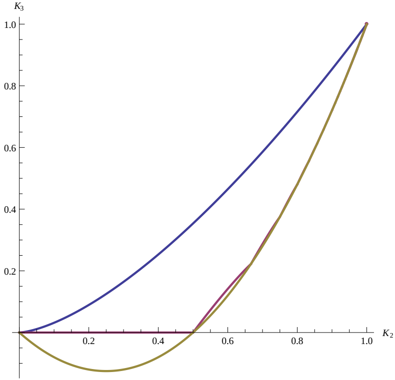

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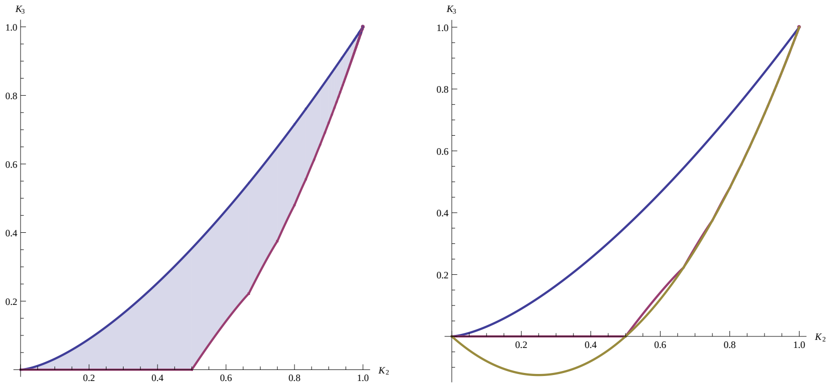

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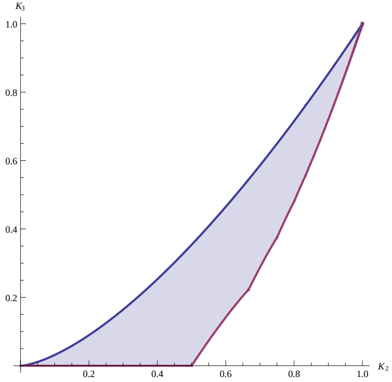

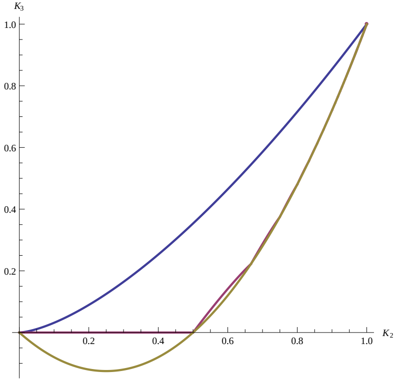

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  - G_n has $\lessapprox cn$ triangles on almost every edge

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- In the graphon language:

$$W(x, y) > 0 \ \Rightarrow \ \int_z W(x, z) W(y, z) \, \mathrm{d}z \le c, \quad \text{a.e. } (x, y)$$

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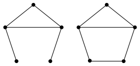
- Special graphs F_1 and F_2: 

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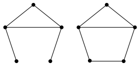
- Special graphs F_1 and F_2: 
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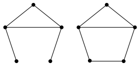
- Special graphs F_1 and F_2: 
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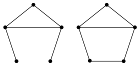
- Special graphs F_1 and F_2: 
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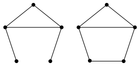
- Special graphs F_1 and F_2: 
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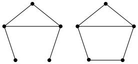
- Special graphs F_1 and F_2: 
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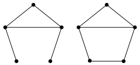
- Special graphs F_1 and F_2: 
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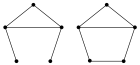
- Special graphs F_1 and F_2: 
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Thank you!

Photos: Gil Kalai's blog, Ron Graham & Math PUrview