Minimising the Number of Cliques

Oleg Pikhurko and Emil R. Vaughan

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- Conlon'12: $\mu_k \geq (2.18 + o(1))^{-k^2}$

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- Nikiforov'05:
  $\alpha(G_n) < 3$ & regular $\Rightarrow \#(K_4, G) \geq (\frac{3}{25} + o(1))\binom{n}{4}$

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- P.-Vaughan $\geq$'13: $(3,4)$, $(3,5)$, $(3,6)$, $(3,7)$, $(4,3)$, $(5,3)$, $(6,3)$ & $(7,3)$

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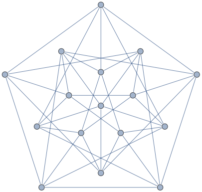

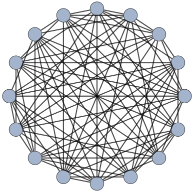

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  - One aspect: semi-definite programming

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- Asymptotic result: $\mathrm{ex}(n, K_3) \leq (\frac{1}{2} + o(1))\binom{n}{2}$

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  - $\#K_2 \leq \frac{n^2}{4} - \frac{1}{n}\#\overline{P}_3 + O(n)$

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  $\forall$ almost extremal G_n is $o(n^2)$-close to T_n^2

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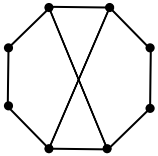

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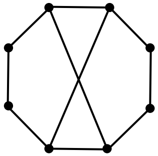

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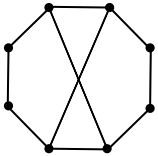

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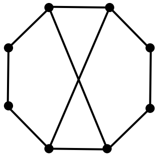

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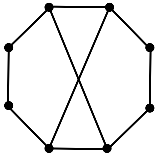

- $\ell_0 = \max\{\ell : c_{3,\ell} = (\ell - 1)^{-2}\}$
  - Nikiforov'01: $\ell_0 < \infty$
  - Das-Huang-Ma-Naves-Sudakov'13: $\ell_0 < 2074$
  - P.-Vaughan $\geq$'13: $\ell_0 \geq 7$

Thank you!