# Combinatorial Games

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Each of the following games are played by two players, **I** and **II**. Player **I** always starts and players alternate turns until one of them wins or loses. **If a player cannot make a move then they lose**.

## 1 Warmup

**Question 1.1.** Start with a pile of $k$ coins and a set $S \subseteq \{1, \ldots, k\}$. On a player's turn they *must* remove $x$ coins from the pile where $x \in S$.

(a) Show that **I** has a winning strategy when $k = 20$ and $S = \{1, 2, 3, 4, 5\}$.

(b) Show that **I** has a winning strategy when $k = 11$ and $S = \{1, 4, 5\}$.

(c) Who has a winning strategy when $k = 2014$ and $S = \{1, 4, 5\}$?

## 2 To think about

**Question 2.1.** Start with several piles of coins. On a player's turn they *must* remove some non-zero number of coins from *one* pile.

(a) Show that **I** has a winning strategy when the piles start with 2, 3 and 3 coins respectively.

(b) Show that **I** has a winning strategy when the piles start with 2, 3 and 4 coins respectively.

(c) Who has a winning strategy when when the piles start with 2012, 2013 and 2014 coins respectively?

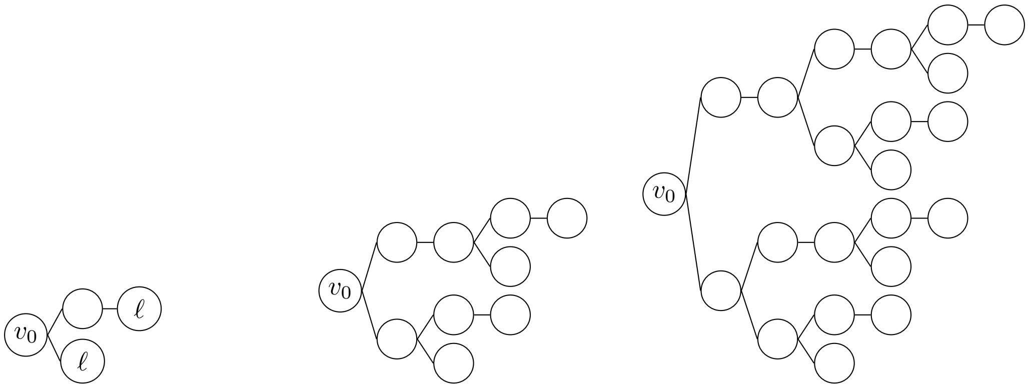
**Question 2.2.** Start with a rooted tree $G$; that is, a connect graph which has no cycles and a special vertex $v_0$ called the *root*. On a player's turn they *must* remove an edge of $G$ along with the connected component *not* containing $v_0$.

Let $G_1$ be the rooted tree shown in Figure 1a. Let $G_k$ be the rooted tree obtained by identifying each leaf vertex of $G_1$ with the root vertex of a copy of $G_{k-1}$. In Figure 1a, the leaf vertices of $G_1$ have been labelled by an $\ell$ and $G_2$ and $G_3$ are shown in Figure 1b and Figure 1c respectively for reference.

(a) Show that **I** has a winning strategy when $G = G_2$.

(b) Show that **I** has a winning strategy when $G = G_3$.

(c) Who has a winning strategy when $G = G_{2014}$?




(a) $G_1$ (b) $G_2$ (c) $G_3$

Figure 1: $G_k$ is obtained by identifying each leaf vertex of $G_1$ with the root vertex of a copy of $G_{k-1}$.

# 3 Homework

**Question 3.1.** Start with a row of coins, each showing either Heads or Tails. On a player's turn they *must* turn over one coin showing Heads and *may* turn over a second coin to the *left* of it.

(a) Show that **II** has a winning strategy when the initial row of coins is HHH.

(b) Show that **I** has a winning strategy when the initial row of coins is HTHTTH.

(c) Suppose that $0 \leq i < j < k$ and that the initial row of coins is

$$\underbrace{T \cdots T}_{i} H \underbrace{T \cdots T}_{j} H \underbrace{T \cdots T}_{k} H.$$

For what values of $i$, $j$ and $k$ does **I** have a winning strategy?

**Question 3.2.** Start with a row of $n$ boxes. On a player's turn they *must* write either an **S** or **O** in *one* box. A player wins if at the end of their turn the sequence **SOS** appears in consecutive boxes.

(a) Show that if $n = 4$ and **I** starts by writing an **S** in the first box then **II** has a winning strategy.

(b) Show that **I** has a winning strategy when $n = 7$.

(c) Who has a winning strategy when $n = 2014$?