Problems from class with solutions

Problem 1 We say that a real $c \in (0,1]$ is a *chord* of a function $f : [0,1] \to \mathbb{R}$ if there is $x \in [0, 1-c]$ with $f(x+c) = f(x)$. Call $c \in (0,1]$ a *universal chord* if it is a chord of every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$. Which reals are universal chords?

Solution: Answer: $c \in (0,1]$ is a universal chord if and only $c = 1/n$ for some integer $n \geq 1$.

First, let us show that $1/n$ for any integer $n \geq 1$ is a universal chord. Take any function f as in the definition. Consider the following telescopic sum:

$$\sum_{i=1}^{n} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right) = f(1) - f(0) = 0.$$

Suppose that no summand is 0 for otherwise we are done. Then there are two summands with different signs. We see that the continuous function $g(x) = f(x+c) - f(x)$ defined on $[0, 1-c]$ changes sign, so it must be 0 at some point by the Intermediate Value Theorem.

Conversely, suppose that $c \in (0,1]$ is not a reciprocal of an integer. Let $g(x) = |\sin(\frac{\pi x}{c})|$ for $x \in \mathbb{R}$. This is a continuous c-periodic function $\mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$ and $g(1) \neq 0$. Consider $f(x) := g(x) - xg(1)$. It is a continuous function with $f(0) = f(1) = 0$. Furthermore, for every $x \in [0, 1-c]$ we have

$$f(x+c) - f(x) = g(x+c) - (x+c)g(1) - g(x) + xg(1) = -cg(1) \neq 0,$$

that is, f shows that c is not a universal chord, as claimed. ∎

Problem 2 Construct a function $f : [0,1] \to [0,1]$ such that $f(\,(a,b)\,) = [0,1]$ for every $0 \leq a < b \leq 1$.

Solution: We write all reals in binary, agreeing that in the ambiguous cases like

$$0.b_1 \ldots b_k 1000\ldots = 0.b_1 \ldots b_k 01111\ldots,$$

we always use the latter representation.

On input $x = 0.b_1 b_2 \ldots$ written in binary, if there is no k such that $b_{2k} = b_{2k+2} = b_{2k+4} = \ldots = 0$, let $f(x) = 0$. Otherwise, take smallest such k and output $0.b_{2k+1} b_{2k+3} \ldots$

Take arbitrary $0 \leq a < b \leq 1$ and $y \in [0,1]$. We have to show that there is $x \in (a,b)$ with $f(x) = y$. Write $(a+b)/2$ in binary: $(a+b)/2 = 0.c_1 c_2 c_3 \ldots$. Take an even integer m such that $2^{-m} < (b-a)/4$; then any real that begins as $0.c_1 \ldots c_m \ldots$ is necessarily in (a,b). Write $y = 0.y_1 y_2 \ldots$ in binary. Let $x = 0.c_1 \ldots c_m 1111\ldots$ if $y = 0$ and $x = 0.c_1 \ldots c_m y_1 0 y_2 0 y_3 0 \ldots$. It is easy to check that x has the required properties. ∎

Problem 3 Prove that the polynomial $p(x) = 2\,\frac{x^{2015}}{2015!} + \sum_{i=0}^{2014} \frac{x^i}{i!}$ cannot have all real roots.

Solution: If the claim is false, then the derivative p' has all real roots by Rolle's theorem. This implies that p'' has all real roots, and so on. Repeating this 2013 times, we obtain that $p^{(2013)}(x) = x^2 + x + 1$ has all reals roots, a contradiction. ∎

Homework problems (due at the class on 21 January)

Problem 4 Call a real $c \in (0,1)$ a *semi-universal chord* if, for every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$, c or $1-c$ is a chord of f. Which reals in $(0,1)$ are semi-universal chords?

Solution: Answer: every $c \in (0,1)$ is a semi-universal chord.

Let us prove this. Take an arbitrary function f as in the problem. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := f(\{x\})$, where $\{x\}$ is the fractional part of $x \in \mathbb{R}$. It is a 1-periodic function defined on the whole of $\mathbb{R}$. The new function g is continuous because $f(0) = f(1)$. Now, $h(x) := g(x+c) - g(x)$ is also 1-periodic and continuous. It satisfies

$$\int_0^1 h(x)\, dx = \int_c^{1+c} g(x)\, dx - \int_0^1 g(x)\, dx = 0.$$

Thus, by the continuity of h, there must be x such that $h(x) = 0$, that is, $g(x+c) = g(x)$. By replacing x by its fractional part, assume $x \in [0,1)$. Suppose that $x + c > 1$ for otherwise c is a chord of f, as required. But then, with $y := x + c - 1 = \{x + c\}$, we have $y + (1-c) = x < 1$ and

$$f(y) = f(\{x+c\}) = g(x+c) = g(x) = f(x),$$

so $1-c$ is a chord, finishing the proof. ∎

Problem 5 Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Prove that there exists $c \in (a,b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Solution: This is Cauchy's theorem.

If $g(a) = g(b)$, take any $c \in (a,b)$ st $g'(c) = 0$, which exists by Rolle's theorem. Otherwise, apply Rolle's theorem to

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)),$$

getting the required.

As an alternative solution, apply Rolle's Theorem to

$$m(x) := \begin{vmatrix} f(x) & g(x) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix}$$

∎

Problem 6 Let $b > a > 0$ and let $f : [a, b] \to \mathbb{R}$ be a continous function, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{1}{a-b}(af(b) - b(f(a)) = f(c) - cf'(c).$$

Solution: Apply Cauchy's theorem to the functions $f(x)/x$ and $1/x$. We conclude that there is $c \in (a, b)$ such that

$$\left(\frac{f(b)}{b} - \frac{f(a)}{a}\right)\left(-\frac{1}{c^2}\right) = \left(\frac{1}{b} - \frac{1}{a}\right)\frac{cf'(c) - f(c)}{c^2},$$

implying the required. ∎

Problem 7 Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeroes (including multiplicities) of $f_k := \frac{d^k}{dt^k} f$ in the half-open real interval $[0, 1)$. Prove that $N_0 \le N_1 \le N_2 \le \ldots$ and that $N_k = 2N$ for all large k.

Solution: This is Putnam Problem 2000:B3.

Recall a special case of Rolle's theorem: if $f(t)$ is differentiable, then between any two zeroes of $f(t)$ there exists a zero of $f'(t)$. This also applies when the zeroes are not all distinct: if f has a zero of multiplicity m at $t = x$, then f' has a zero of multiplicity at least $m - 1$ there.

Thus, if $0 \le t_0 \le t_1 \le \cdots \le t_r < 1$ are the roots of f_k in $[0, 1)$, then f_{k+1} has a root in each of the intervals $(t_0, t_1), (t_1, t_2), \ldots, (t_{r-1}, t_r)$, as long as we adopt the convention that the empty interval (t, t) actually contains the point t itself. There is also a root in the "wrap-around" interval (t_r, t_0). Thus $N_{k+1} \ge N_k$.

Next, note that if we set $z = e^{2\pi it}$ and use the identity $\sin x = (\mathrm{e}^x - \mathrm{e}^{-x})/(2i)$, then we get that

$$f_k(t) = \frac{1}{2i}\sum_{j=1}^{N}(2\pi ij)^{4k} a_j (z^j - z^{-j})$$

is equal to z^{-N} times a polynomial of degree $2N$. Hence as a function of z, it has at most $2N$ roots; therefore $f_k(t)$ has at most $2N$ roots in $[0, 1)$. That is, $N_k \le 2N$ for all N.

To establish that $N_k = 2N$ for all large k, we make precise the observation that

$$f_k(t) = \sum_{j=1}^{N}(2\pi ij)^k a_j \sin(2\pi jt)$$

is dominated by the term with $j = N$. At the points $t = (2i+1)/(2N)$ for $i = 0, 1, \ldots, N-1$, we have $N^k a_N \sin(2\pi Nt) = \pm N^k a_N$. If k is chosen large enough so that

$$|a_N|N^k > |a_1|1^k + \cdots + |a_{N-1}|(N-1)^k,$$

then $f_k((2i+1)/2N)$ has the same sign as $a_N \sin(2\pi N a)$, which is to say, the sequence $f_k(1/2N), f_k(3/2N), \ldots$ alternates in sign. Thus between these points (again including the "wrap-around" interval) we find $2N$ sign changes of f_k. Therefore $N_k = 2N$. ∎