

# BREAKING SYMMETRY ON COMPLETE BIPARTITE GRAPHS OF ODD SIZE

Oleg Pikhurko<sup>1</sup>

*Department of Mathematical Sciences*

*Carnegie Mellon University*

*Pittsburgh, PA 15213-3890*

<http://www.math.cmu.edu/~pikhurko/>

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## Abstract

Players  $\mathcal{A}$  and  $\mathcal{B}$  alternatively colour edges of a graph  $G$ , red and blue respectively. Let  $L_{\text{sym}}(G)$  be the largest number of moves during which  $\mathcal{B}$  can keep the red and blue subgraphs isomorphic, no matter how  $\mathcal{A}$  plays.

This function was introduced by Harary, Slany and Verbitsky who in particular showed that for complete bipartite graphs we have  $L_{\text{sym}}(K_{m,n}) = \frac{mn}{2}$  if  $mn$  is even and that  $L_{\text{sym}}(K_{2m+1,2n+1}) \geq \max(m, n)$ . Here we prove that

$$L_{\text{sym}}(K_{2m+1,2n+1}) = O(n), \quad \text{if } m \leq n \leq m^{O(1)},$$

answering a question posed by Harary, Slany and Verbitsky.

## 1. Introduction

Let  $G$  be a graph. The following *symmetry breaking-preserving game* on  $G$  was introduced by Harary, Slany and Verbitsky [1, 2]. We have two players,  $\mathcal{A}$  and  $\mathcal{B}$ , who alternatively select a previously uncoloured edge of  $G$  and colour it red and blue respectively. Player  $\mathcal{A}$  starts the game. A move of  $\mathcal{A}$  followed by a move of  $\mathcal{B}$  is called a *round*. Clearly, we have the same number of red and blue edges after every round. The aim of Player  $\mathcal{B}$  is to keep the red and blue subgraphs isomorphic after every round of the game; as soon as  $\mathcal{B}$  fails to do so, he loses.

Let  $L_{\text{sym}}(G)$  be the maximum number of moves during which  $\mathcal{B}$  can keep the red and blue subgraphs isomorphic, no matter how  $\mathcal{A}$  plays. Equivalently (see [2, Proposition 2.1])

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$L_{\text{sym}}(G)$  is the smallest  $k$  such that  $\mathcal{A}$  can guarantee his win in at most  $k + 1$  moves. It is not quite clear how to define  $L_{\text{sym}}(G)$  in the cases when  $\mathcal{B}$  can preserve the symmetry until the players run out of edges; following [2] we define  $L_{\text{sym}}(G) = \lfloor e(G)/2 \rfloor$  then.

One of the motivations of Harary, Slany and Verbitsky for introducing these notions was that  $L_{\text{sym}}(G)$  is clearly a lower bound on how long the second player can survive in any *graph avoidance game* on  $G$ . (The rules of the avoidance game are the same except that the player who first constructs a monochromatic copy of a certain forbidden subgraph loses.)

As it is observed in [1], if  $G$  has an involutory automorphism  $\psi$  without fixed edges, then  $L_{\text{sym}}(G) = e(G)/2$ . Indeed, every orbit of the induced action  $\psi^*$  on  $E(G)$  consists of two edges, so  $\mathcal{B}$  can use the *copycat strategy* of choosing  $\psi^*(e) \in E(G)$  where  $e$  is the edge previously coloured by  $\mathcal{A}$ .

The determination of  $L_{\text{sym}}(G)$  is suddenly getting rather complicated and deep when we consider graphs which do not admit a copycat strategy but for which this cannot be derived by looking at a part of the graph. For example, when one considers the path  $P_n$  with  $n$  vertices, one has to know (the parity of) its order  $n$  in order to ascertain the existence of an appropriate automorphism  $\psi$ . So, if  $\mathcal{A}$  follows some ‘local’ strategy,  $\mathcal{B}$  might put up a strong resistance by playing copycat on a few separate parts of the graph. The surprising (at least to me) result of Harary, Slany and Verbitsky [2, Corollary 3.7 & Proposition 3.8] states that

$$(0.5 + o(1)) \log_2 n \leq L_{\text{sym}}(P_{2n}) \leq (3.5 + o(1)) (\log_2 n)^2.$$

Its proof exploits some beautiful connections to the so-called *Ehrenfeucht-Fraïssé game*.

Complete bipartite graphs of even size clearly admit a copycat strategy. The following argument from [2] shows that

$$L_{\text{sym}}(K_{m,n}) \geq \max\left(\frac{m-1}{2}, \frac{n-1}{2}\right), \quad \text{for odd } mn. \quad (1)$$

Indeed, let  $X \subset V(K_{m,n})$  be the bigger part of  $K_{m,n}$ . Starting with  $\psi$  being the identity automorphism,  $\mathcal{B}$  uses the  $\psi$ -copycat strategy, that is, responds with  $\psi^*(e)$  to the previous move  $e$ . This works unless  $\psi^*(e) = e$  in which case  $\mathcal{B}$  locally modifies  $\psi$  so that it exchanges  $x$  and  $y$  now, where  $\{x\} = X \cap e$  and  $y \in X$  is a vertex not incident to any coloured edge. Such  $y$  always exists during the first  $(|X| - 1)/2$  rounds during which  $\psi$  remains an involutory automorphism of  $K_{m,n}$  swapping the blue and red subgraphs.

Harary, Slany and Verbitsky [2, Question 4.3] asked for the rate of growth of the function  $L_{\text{sym}}(K_{n,n})$  for odd  $n$ . The following more general result implies that  $L_{\text{sym}}(K_{m,n})$  grows linearly in  $n$  if  $m \leq n \leq m^{O(1)}$  and  $mn$  is odd.

**Theorem 1** *Let odd integers  $m$  and  $n$  and an integer  $k \geq 8$  satisfy*

$$51 \leq m \leq n \leq \frac{(m - 2k - 3)^k}{(k + 1)!m}. \quad (2)$$

Then we have

$$L_{\text{sym}}(K_{m,n}) \leq k(n - m + 3) + 2m + 14. \quad (3)$$

In particular, if  $m = n$ , then letting  $k = 8$  we obtain

$$L_{\text{sym}}(K_{n,n}) \leq 2n + 38, \quad \text{for odd } n \geq 51. \quad (4)$$

In Section 3 we present the bound (3) as a more digestible, explicit function of  $m$  and  $n$ .

Let us describe the main idea behind the proof of Theorem 1. In outline,  $\mathcal{A}$  builds a red graph which has no non-trivial automorphism. If  $\mathcal{B}$  has not lost yet, the isomorphism  $\psi$  between the red and blue graphs is unique and  $\mathcal{B}$  is *forced* to play the  $\psi^*$ -copycat strategy now. As the total number of edges is odd,  $\psi^*$  has at least one odd orbit  $D \subset E(G)$ . No matter how  $D$  has been coloured, Player  $\mathcal{A}$  can beat the  $\psi^*$ -copycat strategy on  $D$  with at most two moves. So, if  $\psi$  remains the unique isomorphism during these two moves of  $\mathcal{A}$ , then  $\mathcal{B}$  loses.

This method might be applicable to many graphs with odd size. Here is its concrete realisation for complete bipartite graphs.

## 2. Proof of Theorem 1

We will describe the appropriate strategy of  $\mathcal{A}$  which consists of a few phases. Assume that  $\mathcal{B}$  keeps the red and blue subgraphs isomorphic throughout our strategy.

Let  $V \cup V' = V(K_{m,n})$  be the vertex classes, where  $|V| = m$ .

PHASE 1.  $\mathcal{A}$  builds a red cycle of length  $2l \in [2m - 6, 2m - 2]$ , say visiting vertices  $x_1, x'_1, \dots, x_l, x'_l$  in this order, plus one edge from some vertex  $y_0 \in Y$  to  $X'$ , where we denote  $X = \{x_1, \dots, x_l\} \subset V$ ,  $X' = \{x'_1, \dots, x'_l\} \subset V'$ ,  $Y = V \setminus X$ , and  $Y' = V' \setminus X'$ .

The red/blue subgraph will have maximum degree at most 2 at any moment before the end of Phase 1, so in particular  $\mathcal{A}$  can easily create a red path of length  $2m - 7$  by choosing vertices  $x_1, x'_1, \dots, x_{m-3}, x'_{m-3}$  one by one in this order. If the edge  $\{x_1, x'_{m-3}\}$  is available, then  $\mathcal{A}$  colours it and we are done because the addition of an edge between  $X'$  and  $Y$  is always possible as  $m - 3 > 2$ . Otherwise,  $\mathcal{A}$  extends the path by two more edges. Now, if  $\{x_1, x'_{m-2}\}$  is available, then  $\mathcal{A}$  selects it and we are done again. Otherwise,  $\mathcal{A}$  can add  $x_{m-1}$  to the path: indeed, the vertex  $x'_{m-2}$  sends a blue edge to  $x_1$ , so it sends at most one blue edge to  $V \setminus X$ . If  $\{x_{m-1}, x'_1\}$  is available, then  $\mathcal{A}$  selects it, obtaining the desired configuration (up to relabelling). Otherwise, the edge  $\{x_{m-1}, x'_1\}$  is blue and  $\mathcal{A}$  can extend the path to  $x'_{m-1}$ . In the next move  $\mathcal{A}$  colours the edge  $\{x_1, x'_{m-1}\}$  which cannot be blue because we have already encountered two blue edges incident to

$x_1$ . Finally,  $\mathcal{A}$  connects some  $y_0 \in Y$  to  $X'$ , obtaining the desired configuration in all cases.

PHASE 2.  $\mathcal{A}$  connects  $Y$  to  $X'$ .

The vertex  $y_0 \in Y$  will play a special role. Assume that  $x'_1$  is the vertex in  $X'$  connected to  $y_0$  by a red edge.

Consider first the case when  $y_0$  is incident to at least one blue edge. By reversing, if necessary, the direction of the red  $2l$ -cycle, we can assume that  $y_0$  has at least one blue neighbour *outside*  $\{x'_1, \dots, x'_{10}\}$ . In the next five moves  $\mathcal{A}$  colours  $\{y_0, x'_i\}$  for the smallest possible index  $i \geq 2$  each time. Suppose that the last such edge was  $\{y_0, x'_s\}$ . When  $\mathcal{A}$  was colouring it, at most 5 edges incident to  $y_0$  were blue of which at most 4 lie in  $X'_0 = \{x'_1, \dots, x'_s\}$ . Hence  $s \leq 10$ .

In the next three moves  $\mathcal{A}$  colours  $\{y_0, x'_i\}$ , where  $i$  is the smallest available index with  $i \geq 2s$  except, when colouring the third edge in the case  $s = 6$ , the additional condition on  $i$  is that the indexes of the last three red neighbours of  $y_0$  do not form an arithmetic progression. (Note that for  $s \geq 7$  the indexes of the *first* six neighbours of  $y_0$  cannot form a six-term arithmetic progression as  $s \leq 10$ .)

Let  $t > 2s$  be the largest index of a red neighbour of  $y_0$  and let  $X'_1 = \{x'_{2s}, \dots, x'_t\}$ . We claim that  $t \leq 26$ . If  $s \geq 7$ , then  $y_0$  sends at most  $8 - (s - 6) = 14 - s$  blue edges to  $X'_1$  because  $y_0$  sends  $s - 6$  blue edges to  $X'_0$ ; thus

$$t - 2s + 1 = |X'_1| \leq 3 + (14 - s), \quad (5)$$

which gives the required in view of  $s \leq 10$ . If  $s = 6$ , then we have to add 1 to the bound (5), still obtaining the claimed inequality  $t \leq 26$ .

If  $y_0$  is not incident to a blue edge at the end of Phase 1, then this stays so, in whichever manner we add red edges incident to  $y_0$ . In particular,  $\mathcal{A}$  can connect  $y_0$  to

$$\{x'_1, x'_2, \dots, x'_6, x'_{12}, x'_{13}, x'_{15}\}$$

and we have  $s = 6$  and  $t = 15$ .

Next,  $\mathcal{A}$  connects the vertices of  $Y \setminus \{y_0\}$  to  $X'$ , by five edges each, so that distinct vertices of  $Y$  have disjoint neighbourhoods in  $X'$ , which is possible as  $l > 2 \cdot 9 + 5 \cdot (|Y| - 1)$ .

The first two phases last for  $r_2 = 2l + 9 + 5(m - l - 1)$  rounds.

PHASE 3. If  $m = n = l + 1$ , then  $\mathcal{A}$  connects the (unique) vertex of  $Y'$  to arbitrary 12 vertices of  $X$ . Otherwise,  $\mathcal{A}$  picks vertices of  $Y'$  one by one and connects each selected vertex by  $k$  edges to  $X$ . Suppose that  $\mathcal{A}$  has already dealt with  $y'_1, \dots, y'_i \in Y'$  and connected the next vertex  $y' \in Y'$  to a set  $H \subset X$  of size  $h \in [0, k - 1]$ . Let  $\Gamma_{\text{red}}(y)$  (resp.  $\Gamma_{\text{blue}}(y)$ ) denotes the red (resp. blue) neighbourhood of a vertex  $y$ .

We claim that  $f$ , the number of vertices in  $F = \Gamma_{\text{blue}}(y') \cap X$ , will be at most  $k + 1$  when we deal with the vertex  $y'$ . This is true for  $i \leq 3$  when the maximum degree of the blue graph is at most  $\max(k, 9)$ . If  $i \geq 4$ , then the isomorphism between the red and blue graphs must respect the classes  $V$  and  $V'$  because the non-trivial red component has at least  $l + i > |V|$  vertices in  $V'$ . So the blue degree of  $y'$  is at most the maximum red degree of a vertex in  $V'$  which is  $k$ , giving the desired bound on  $f$ .

Here is the strategy:  $\mathcal{A}$  connects  $y'$  to a vertex  $x \in X \setminus (F \cup H)$  such that  $\mu(H \cup \{x\})$  is minimum, where

$$\mu(Z) = \sum_{j=1}^i c_{|Z \setminus \Gamma_{\text{red}}(y'_j)|}, \quad Z \subset X,$$

where  $c_0 = l$ ,  $c_1 = 1$ , and all other  $c$ 's are zero. In other words,

$$\mu(Z) = l \left| \left\{ j \in [i] \mid Z \subset \Gamma_{\text{red}}(y'_j) \right\} \right| + \left| \left\{ j \in [i] \mid |Z \setminus \Gamma_{\text{red}}(y'_j)| = 1 \right\} \right|.$$

It is easy to see that

$$\begin{aligned} \sum_{x \in X \setminus (F \cup H)} \mu(H \cup \{x\}) &\leq \sum_{\substack{j \in [i] \\ H \subset \Gamma_{\text{red}}(y'_j)}} (c_0(k - h) + c_1(l - k)) \\ &+ \sum_{\substack{j \in [i] \\ |H \setminus \Gamma_{\text{red}}(y'_j)| = 1}} c_1(k - h + 1) \leq (k - h + 1)\mu(H), \end{aligned}$$

by straightforwardly comparing the corresponding terms. Hence,  $\mathcal{A}$  can choose an  $x \in X \setminus (F \cup H)$  such that

$$\mu(H \cup \{x\}) \leq \frac{k - h + 1}{l - f - h} \mu(H) \leq \frac{k - h + 1}{m - 2k - 3} \mu(H).$$

As  $\mu(\emptyset) = li < mn$ , the inequality

$$\mu(H) < \frac{(k + 1)!}{(m - 2k - 3)^k} mn \leq 1$$

holds when we reach the case  $|H| = k$ . As  $\mu$  is integer-valued, it must be the case that  $\mu(H) = 0$ , which means by the definition that  $|H \setminus \Gamma_{\text{red}}(y'_j)| \geq 2$  for any  $j \in [i]$ . Thus  $\mathcal{A}$  can ensure that the hamming distance between any two sets in  $\{\Gamma_{\text{red}}(y') \mid y' \in Y'\}$  is at least 4 at the end of Phase 3.

The first three phases took  $r_3 = r_2 + 12$  rounds if  $n = m = l + 1$  and  $r_3 = r_2 + k(n - l)$  rounds otherwise.

**PHASE 4.**  $\mathcal{A}$  adds at most two more edges and wins.

This phase needs some analysis before we can specify the moves of  $\mathcal{A}$ . Let  $A_i$  (resp.  $B_i$ ) consist of red (resp. blue) edges after  $i$  rounds, viewed as a subset of  $E(K_{m,n})$ . Thus

$A = A_{r_3}$  is the red graph at the end of Phase 3. Note that  $X \cup X'$  spans in  $A$  an induced  $2l$ -cycle which we denote by  $C \subset A$ .

**Claim 1.** Let  $A'$  be obtained by adding to  $A$  at most two edges of the encompassing graph  $K_{m,n}$ . Let  $\phi : V(K_{m,n}) \rightarrow V(K_{m,n})$  be a bijection such that  $\phi^*(A) \subset A'$ , where  $\phi^*$  denotes the induced action on 2-point sets. Then  $\phi(y_0) = y_0$ ,  $\phi(Y) = Y$ ,  $\phi(X) = X$ ,  $\phi(Y') = Y'$  and  $\phi(X') = X'$ .

If, furthermore,  $\phi^*(C) = C$ , then  $\phi$  is the identity map.

*Proof.* As  $A, A'$  are connected bipartite graphs, we have  $\phi(V) = V$  or  $\phi(V) = V'$ .

Let us show first that  $\phi(V) = V$ , which needs justification when  $|V| = |V'|$ . If  $|Y'| = 1$ , then the (unique) vertex of  $Y'$  of degree at least 12 must be preserved by  $\phi$  as any other vertex has  $A'$ -degree at most 11; thus  $\phi(V) = V$ , as required. Suppose that  $|Y'| = |Y| \geq 2$ . The vertices in  $X$  are incident to at most  $4 + |Y'| \leq 7 < k$   $A'$ -edges each. Thus  $V$  contains at most one vertex of  $A'$ -degree at least  $k$  and  $\phi$  must map  $Y'$  into  $V'$ . Now it follows that  $\phi(V) = V$  and  $\phi(V') = V'$ .

As each vertex of  $Y'$  has degree at least 8 while the  $A'$ -degrees in  $X'$  are all at most 5, we conclude that  $\phi(Y') = Y'$ . As each vertex of  $\phi(Y)$  sends at least five edges to  $\phi(X') = X'$ , we have  $\phi(Y) = Y$ . Similarly,  $\phi(y_0) = y_0$ , which proves the first part of the claim.

Suppose furthermore that  $\phi^*(C) = C$ . This means that the restriction of  $\phi$  to  $X \cup X'$  is a cyclic rotation, possibly composed with the reflection  $x_i \mapsto x'_{l-i+1}$ ,  $x'_i \mapsto x_{l-i+1}$ . We are going to show that  $\phi|_{X \cup X'}$  is the identity by considering the neighbourhood of  $y_0$ .

Recall that the set  $X'_0 = \{x'_1, \dots, x'_s\}$  contains six  $A$ -neighbours of  $y_0$  and  $X'_1 = \{x'_{2s}, \dots, x'_t\}$  the remaining three. The sets  $X'_0$  and  $X'_1$  cannot both intersect  $\phi(X'_0)$  as they are separated by  $s - 1$  other vertices of  $X'$ . Hence,  $\phi(X'_0) \cap X'_1 = \emptyset$  and  $\phi(X'_0) \cap X'_0$  contains at least four  $A$ -neighbours of  $y_0$ . If  $\phi(X'_1)$  is situated at the ‘wrong’ side of  $\phi(X'_0)$ , then (as  $l \geq 2t - 4$ ) we have  $\phi(X'_1) \cap X'_1 = \emptyset$  and at least three  $A'$ -neighbours of  $y_0$  fall outside  $\phi(X'_0 \cup X'_1)$ , a contradiction. Thus  $\phi|_{X \cup X'}$  is a cycle rotation (without any reflection). Moreover, it is the identity rotation for otherwise we obtain the contradiction  $|\phi(\Gamma_{\text{red}}(y_0)) \setminus \Gamma_{\text{red}}(y_0)| \geq 3$ . (The latter inequality holds because in Phase 2 we excluded the possibility that the indexes of the red neighbours of  $y_0$  form two arithmetic progressions.)

Now, for any two vertices of  $Y \cup Y'$  the hamming distance between their  $A$ -neighbourhoods is at least 4. This clearly implies that  $\phi$  is the identity bijection. ■

Claim 1 implies in particular that  $A$  has no non-trivial automorphism. Thus  $\psi = \psi_{r_3}$  is uniquely determined, where  $\psi_r$  denotes the red-blue isomorphism after  $r$  rounds. The bipartition  $V \cup V'$  is clearly preserved (or reversed) by  $\psi$  so we have the induced action  $\psi^*$  on  $E(K_{m,n})$ .

**Claim 2.** For any  $(r_3 + 1)$ st move  $e$  of  $\mathcal{A}$ , the player  $\mathcal{B}$  is forced to reply with  $\psi^*(e)$ .

*Proof.* By Claim 1 any bijection  $\phi$  with  $\phi^*(A) \subset A_{r_3+1}$  preserves  $X \cup X'$ . As  $X \cup X'$  induces in  $A_{r_3+1}$  a cycle with at most one chord added, we have  $\phi^*(C) = C$  and, again by Claim 1,  $\phi$  is the identity map. This means that  $A$  is the only subgraph of  $A_{r_3+1}$  isomorphic to  $A$  and the analogous claim holds for the blue edges. Thus  $\psi_{r_3+1}^*(A) = B_{r_3}$ , which implies that  $\psi_{r_3+1} = \psi$ . Now,  $\psi^*$  must map  $\{e\} = A_{r_3+1} \setminus A$  into  $B_{r_3+1} \setminus B_{r_3}$ , as required. ■

As the total number of edges  $mn$  is odd, some  $\psi^*$ -orbit  $D \subset E(K_{m,n})$  has the odd number of elements.

If at least one element of  $D$  is already coloured, then we can find via a parity argument an uncoloured  $e \in D$  such that  $\psi^*(e)$  is red. Now,  $\mathcal{A}$  colours  $e$  red and  $\mathcal{B}$  loses by Claim 2.

Suppose that no edge of  $D$  has been coloured. If  $D = \{e\}$ , then  $\psi^*(e) = e$  and  $\mathcal{A}$  wins by choosing  $e$  by Claim 2. Suppose that  $|D| > 1$  and let  $e \in D$ . Now  $e' = \psi^*(e)$  and  $e'' = (\psi^*)^{-1}(e)$  are two distinct edges belonging to  $D$ .

Suppose first that  $e = \{x_i, x'_j\}$  and that both  $C \cup \{e, e'\}$  and  $C \cup \{e, e''\}$  have a  $2l$ -cycle passing through the edge  $e$ . Trivial considerations show that

$$\{e', e''\} = \left\{ \{x_j, x'_{i-1}\}, \{x_{j+1}, x'_i\} \right\}. \quad (6)$$

(Of course, we do all index calculations modulo  $l$ .) If  $\psi(V) = V$ , then (6) means that for some  $\delta = \pm 1$  we have  $\psi^\delta(x_i) = x_{j+1}$ ,  $\psi^\delta(x'_j) = x'_i$ ,  $\psi^{-\delta}(x_i) = x_j$  and  $\psi^{-\delta}(x'_j) = x'_{i-1}$ . But then  $\psi^\delta$  maps the red edge  $\{x_j, x'_j\}$  into the red edge  $\{x_i, x'_i\}$ , a contradiction. In the same way we obtain a contradiction in the case  $\psi(V) = V'$ .

Thus, by the first part of Claim 1, we can assume that we can recover  $C$  from knowing, for example,  $A \cup \{e, e'\}$ . Now,  $\mathcal{A}$  selects  $e'$  to which, by Claim 2,  $\mathcal{B}$  is forced to reply by colouring  $\psi^*(e')$ . Then  $\mathcal{A}$  selects  $e \neq \psi^*(e')$ . Claim 1 implies that  $A$  is the unique subgraph of  $A \cup \{e, e'\}$  isomorphic to  $A$  and the argument of Claim 2 shows that  $\mathcal{B}$  loses.

Let us count the total number  $r_4$  of rounds. If  $n = m = l + 1$ , then

$$r_4 \leq 2(m - 1) + 9 + 12 + 2 \leq 3(n - m + k) + 2m + 15,$$

as  $k \geq 8$ . Otherwise,

$$r_4 \leq 2l + 9 + 5(m - l - 1) + k(n - l) + 2 \leq k(n - m + 3) + 2m + 15,$$

where we used the inequality  $l \geq m - 3$ . This finishes the proof of Theorem 1 as  $L_{\text{sym}}(K_{m,n}) \leq r_4 - 1$ .

### 3. Concluding Remarks

It is not hard to convert our strategy into an algorithm which has running time  $O(mn)$  per one move of  $\mathcal{A}$ .

Unfortunately, not for every pair  $(m, n)$  an integer  $k$  satisfying (2) can be found. For example, if  $n > 2^m$ , then any subgraph of  $K_{m,n}$  contains two vertices in  $V'$  with the same neighbourhood in  $V$ , which ruins our strategy. The value of  $n$  can range up to

$$\max_{k \geq 8} \frac{(m - 2k - 3)^k}{(k + 1)!m} = (1.531\dots + o(1))^m.$$

The optimal choice is to take the smallest integer  $k \geq 8$  satisfying (2). If  $m$  is large, then the following formulae can be routinely verified. If  $\log n / \log m < 7 - o(1)$ , then  $k = 8$ . If  $7 - o(1) \leq \log n / \log m = O(1)$ , then

$$\frac{\log n}{\log m} + 1 < k < \frac{\log n}{\log m} + 3.$$

If  $O(\log m) < \log n = o(m)$ , then

$$k = (1 + o(1)) \frac{\log n}{\log m - \log \log n}.$$

If  $(1 + o(1))^m < n < (1.531\dots + o(1))^m$ , then  $k = (x_0 + o(1))m$ , where  $x_0$  is the smallest positive root of equation

$$((1 - 2x)e/x)^x = e^{\log n/m}.$$

The present best known bounds on  $L_{\text{sym}}(K_{n,n})$  for odd  $n$  are given by (1) and (4). Unfortunately, it seems that a minor modification of our method cannot give any considerable improvement on (4) as any graph  $H$  with no non-trivial automorphism has at least  $(1 - o(1))v(H)$  edges. Indeed, if we fix some large constant  $C$ , then  $H$  has  $O(1)$  components with less than  $C$  vertices (as no two can be isomorphic), while the remaining components span at least  $(v(H) - O(1))\frac{c-1}{c}$  edges. Also, one meets difficulties when trying to improve on (1): it is not hard to see that  $\mathcal{A}$  can ensure that within first  $\frac{n+1}{2}$  rounds there was a position when no blue-red isomorphism could be generated by an involutory automorphism of  $K_{n,n}$ . Hence,  $\mathcal{B}$  must go beyond copycat if he wants to survive for more than  $(\frac{1}{2} + o(1))n$  rounds.

In fact, we do not even have any solid conjecture what the value of

$$\lim_{n \rightarrow \infty} \frac{L_{\text{sym}}(K_{2n+1, 2n+1})}{n}$$

is (if the limit exists).



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