Problem solving seminar
Homework II - Solutions

1. Given $\alpha > 0$ find inf and sup of $\int_0^1 xf(x)\mathrm{d}x$ subject to integrable functions $f\colon [0,1] \longrightarrow [0,\infty)$ with $\int_0^1 f(x)\mathrm{d}x = \alpha$.

Solution. Obviously $\int_0^1 xf(x)\mathrm{d}x \geq 0$ and $\int_0^1 xf(x)\mathrm{d}x \leq \int_0^1 f(x)\mathrm{d}x = \alpha$ for any integrable $f\colon [0,1] \longrightarrow [0,\infty)$ with $\int_0^1 f(x)\mathrm{d}x = \alpha$.

Functions $f_\epsilon(x) = \frac{\alpha}{\epsilon}\mathbf{1}_{[0,\epsilon]}(x)$, $g_\epsilon(x) = \frac{\alpha}{\epsilon}\mathbf{1}_{[1-\epsilon,1]}(x)$ with $\epsilon \to 0$ show that the inf and the sup are respectively equal to 0 and α. $\square$

2. Let $\phi\colon [0,\infty) \longrightarrow \mathbb{R}$ be a convex function and $\phi(0) = 0$, $\phi(x) \xrightarrow[x\to+\infty]{} +\infty$. Prove that for every integer $n \geq 0$,

$$\int_0^\infty t^n e^{-\phi(t)}\mathrm{d}t \leq n!\left(\int_0^\infty e^{-\phi(t)}\mathrm{d}t\right)^{n+1}.$$

Solution. Define $\alpha > 0$ by $1/\alpha = \int_0^\infty e^{-\alpha t}\mathrm{d}t = \int_0^\infty e^{-\phi(t)}\mathrm{d}t$. Then, as in the class, we show that the function

$$h(t) = \int_t^\infty \left(e^{-\alpha s} - e^{-\phi(s)}\right)\mathrm{d}s, \qquad t \geq 0,$$

is nonnegative (briefly, $h(0) = 0, h(\infty) = 0$, h' changes sign (positive then negative), so h increases and then decreases, consequently $h \geq 0$). To finish the proof it is enough to integrate by parts

$$\begin{aligned}
\int_0^\infty t^n e^{-\phi(t)} &= \int_0^\infty \left(n\int_0^t s^{n-1}\right)e^{-\phi(t)} = \int_0^\infty ns^{n-1}\left(\int_s^\infty e^{-\phi(t)}\right) \\
&\leq \int_0^\infty ns^{n-1}\left(\int_s^\infty e^{-\alpha t)}\right) = \int_0^\infty s^n e^{-\alpha s} = \frac{1}{\alpha^{n+1}}\int_0^\infty s^n e^{-s} \\
&= \frac{n!}{\alpha^{n+1}}.
\end{aligned}$$

$\square$

3. Let $f\colon [0,1] \longrightarrow [0,\infty)$ be a nonincreasing concave function such that $f(0) = 1$. Prove that for every integer $n \geq 3$,

$$\frac{n-1}{n}\left(\int_0^1 f(x)^{n-2}\mathrm{d}x\right)^2 \geq \int_0^1 xf(x)^{n-2}\mathrm{d}x.$$

Solution. Since f is concave and nonincreasing, we have $1 - x \leq f(x) \leq 1$ for $x \in [0,1]$. Therefore, there exists a real number $\alpha \in [0,1]$ such that for $g(x) = 1 - \alpha x$ we have

$$\int_0^1 f(x)^{n-2}\mathrm{d}x = \int_0^1 g(x)^{n-2}\mathrm{d}x.$$

Clearly, we can find a number $c \in [0,1]$ such that $f(c) = g(c)$. Since f is concave and g is affine, we have $f(x) \geq g(x)$ for $x \in [0,c]$ and $f(x) \leq g(x)$ for $x \in [c,1]$. Hence,

$$\begin{aligned}
\int_0^1 x(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \leq &\int_0^c c(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \\
&+ \int_c^1 c(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \ = \ 0.
\end{aligned}$$

We conclude that it suffices to prove the desired inequality for the function g, which is by simple computation equivalent to

$$\frac{1}{\alpha^2 n(n-1)}\left(1-(1-\alpha)^{n-1}\right)^2 \geq \frac{1}{\alpha^2}\Bigg(\frac{1}{n-1}\left(1-(1-\alpha)^{n-1}\right) \\ -\frac{1}{n}\left(1-(1-\alpha)^{n}\right)\Bigg).$$

To finish the proof one has to perform a short calculation and use Bernoulli's inequality. $\square$