Measurable Combinatorics

Oleg Pikhurko
University of Warwick

"Interactions with Combinatorics", Birmingham, 30 June 2017

Motivation

Motivation

- Scientific curiosity

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs
  - $\chi(G) \leq k$

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs
  - $\chi(G) \leq k$ iff $\forall$ finite $U \subseteq V$ $\chi(G[U]) \leq k$

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs
  - $\chi(G) \leq k$ iff $\forall$ finite $U \subseteq V$ $\chi(G[U]) \leq k$
- Extra structure

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs
  - $\chi(G) \leq k$ iff $\forall$ finite $U \subseteq V$ $\chi(G[U]) \leq k$
- Extra structure
  - End compactification (Freudenthal'31, ...)

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs
  - $\chi(G) \leq k$ iff $\forall$ finite $U \subseteq V$ $\chi(G[U]) \leq k$
- Extra structure
  - End compactification (Freudenthal'31, ...)
  - Martin boundary (Martin'41, Doob'59, Hung'60, ...)

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs
  - $\chi(G) \le k$ iff $\forall$ finite $U \subseteq V$ $\chi(G[U]) \le k$
- Extra structure
  - End compactification (Freudenthal'31, ...)
  - Martin boundary (Martin'41, Doob'59, Hung'60, ...)
- V: measure space

Motivation

- Scientific curiosity
- Extend known results to infinite graphs $G = (V, E)$
- All degrees uniformly bounded
- Axiom of Choice $\rightsquigarrow$ finite graphs
  - $\chi(G) \le k$ iff $\forall$ finite $U \subseteq V$ $\chi(G[U]) \le k$
- Extra structure
  - End compactification (Freudenthal'31, ...)
  - Martin boundary (Martin'41, Doob'59, Hung'60, ...)
- V: measure space
  - Allow only measurable $c : V \to [k]$

Finite graphs

Finite graphs

- Finite graph G

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \ldots, \phi_k : V \to V$ st
    $$E = \{\{x, y\} : x \neq y \ \& \ \exists\, i \ \phi_i(x) = y\}$$

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \ldots, \phi_k : V \to V$ st
    $$E = \{\{x, y\} : x \neq y \ \&\ \exists\, i\ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \ldots, \phi_k : V \to V$ st
    $$E = \{\{x, y\} : x \neq y \ \& \ \exists\, i \ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition
    - Matching $\leftrightarrow$ involution

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \ldots, \phi_k : V \to V$ st
$$E = \{\{x, y\} : x \neq y \ \& \ \exists\, i \ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition
    - Matching $\leftrightarrow$ involution
    - Cycles (with orientations) $\leftrightarrow$ bijection

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \dots, \phi_k : V \to V$ st
    $$E = \{\{x, y\} : x \neq y \ \& \ \exists\, i \ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition
    - Matching $\leftrightarrow$ involution
    - Cycles (with orientations) $\leftrightarrow$ bijection
    - $E =$ union of edges

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \dots, \phi_k : V \to V$ st
$$E = \{\{x, y\} : x \neq y \ \& \ \exists\, i \ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition
    - Matching $\leftrightarrow$ involution
    - Cycles (with orientations) $\leftrightarrow$ bijection
    - $E =$ union of edges / matchings

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \ldots, \phi_k : V \to V$ st
    $$E = \{\{x, y\} : x \neq y \ \& \ \exists\, i \ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition
    - Matching $\leftrightarrow$ involution
    - Cycles (with orientations) $\leftrightarrow$ bijection
    - $E =$ union of edges / matchings / cycles

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \ldots, \phi_k : V \to V$ st
  $$E = \{\{x, y\} : x \neq y \ \& \ \exists\, i \ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition
    - Matching $\leftrightarrow$ involution
    - Cycles (with orientations) $\leftrightarrow$ bijection
    - $E =$ union of edges / matchings / cycles
  - μ: uniform measure on V

Finite graphs

- Finite graph G
  - Bijections $\phi_1, \ldots, \phi_k : V \to V$ st
    $$E = \{\{x, y\} : x \neq y \ \&\ \exists\, i\ \phi_i(x) = y\}$$
    - Edge $\leftrightarrow$ transposition
    - Matching $\leftrightarrow$ involution
    - Cycles (with orientations) $\leftrightarrow$ bijection
    - $E =$ union of edges / matchings / cycles
  - μ: uniform measure on V
  - Measure-preserving invertible maps $\phi_i : V \to V$

Graphings

Graphings

- Graphing $(V, E, \mathcal{A}, \mu)$:

Graphings

- Graphing $(V, E, \mathcal{A}, \mu)$:
  - (V, E): graph

Graphings

- Graphing $(V, E, \mathcal{A}, \mu)$:
  - (V, E): graph
  - $(V, \mathcal{A}, \mu)$: stardard probability space

Graphings

- Graphing $(V, E, \mathcal{A}, \mu)$:
  - (V, E): graph
  - $(V, \mathcal{A}, \mu)$: stardard probability space
    - eg unit interval $[0, 1]$

Graphings

- Graphing $(V, E, \mathcal{A}, \mu)$:
  - (V, E): graph
  - $(V, \mathcal{A}, \mu)$: stardard probability space
    - eg unit interval $[0, 1]$
  - $\exists$ measure-preserving maps $\phi_1, \ldots, \phi_k : V \to V$

Graphings

- Graphing $(V, E, \mathcal{A}, \mu)$:
  - (V, E): graph
  - $(V, \mathcal{A}, \mu)$: stardard probability space
    - eg unit interval $[0, 1]$
  - $\exists$ measure-preserving maps $\phi_1, \ldots, \phi_k : V \to V$
    $E = \{\, \{x, y\} : x \neq y \;\; \& \;\; \exists\, i \;\; \phi_i x = y \,\}$

Graphings

- Graphing $(V, E, \mathcal{A}, \mu)$:
    - (V, E): graph
    - $(V, \mathcal{A}, \mu)$: stardard probability space
        - eg unit interval $[0, 1]$
    - $\exists$ measure-preserving maps $\phi_1, \ldots, \phi_k : V \to V$
      $E = \{ \{x, y\} : x \neq y \ \& \ \exists i \ \ \phi_i x = y \}$
- $\Delta(G) \leq 2k$

More motivation

More motivation

- Limit objects for Benjamini-Schramm convergence

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x
    - Rooted graph, up to isomorphism

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x
    - Rooted graph, up to isomorphism
- Graphing $(V, E, \mathcal{A}, \mu)$:

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x
    - Rooted graph, up to isomorphism
- Graphing $(V, E, \mathcal{A}, \mu)$:
  - Sample $x \in V$ using μ

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x
    - Rooted graph, up to isomorphism
- Graphing $(V, E, \mathcal{A}, \mu)$:
  - Sample $x \in V$ using μ
  - Output radius-r neighbourhood of x in (V, E)

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x
    - Rooted graph, up to isomorphism
- Graphing $(V, E, \mathcal{A}, \mu)$:
  - Sample $x \in V$ using μ
  - Output radius-r neighbourhood of x in (V, E)
- Property testing

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x
    - Rooted graph, up to isomorphism
- Graphing $(V, E, \mathcal{A}, \mu)$:
  - Sample $x \in V$ using μ
  - Output radius-r neighbourhood of x in (V, E)
- Property testing
- Not unique

More motivation

- Limit objects for Benjamini-Schramm convergence
- Finite graph:
  - Uniform vertex x
  - Output radius-r neighbourhood of x
    - Rooted graph, up to isomorphism
- Graphing $(V, E, \mathcal{A}, \mu)$:
  - Sample $x \in V$ using μ
  - Output radius-r neighbourhood of x in (V, E)
- Property testing
- Not unique
- Aldous-Lyons'01: Is every graphing approximable by finite graphs ?

Example: irrational rotation $\mathcal{R}_\alpha$

Example: irrational rotation $\mathcal{R}_\alpha$

- Fix irrational $\alpha \in \mathbb{R}$

Example: irrational rotation $\mathcal{R}_\alpha$

- Fix irrational $\alpha \in \mathbb{R}$
- $V := [0, 1)$

Example: irrational rotation $\mathcal{R}_\alpha$

- Fix irrational $\alpha \in \mathbb{R}$
- $V := [0, 1)$
- $T : V \to V, \quad x \mapsto x + \alpha \pmod 1$

Example: irrational rotation $\mathcal{R}_\alpha$

- Fix irrational $\alpha \in \mathbb{R}$
- $V := [0, 1)$
- $T : V \to V, \quad x \mapsto x + \alpha \pmod 1$
- $E := \{\{x, Tx\} : x \in V\}$

Example: irrational rotation $\mathcal{R}_\alpha$

- Fix irrational $\alpha \in \mathbb{R}$
- $V := [0, 1)$
- $T : V \to V, \quad x \mapsto x + \alpha \pmod 1$
- $E := \{\{x, Tx\} : x \in V\}$
- Graphing $\mathcal{R}_\alpha$

Properties of $\mathcal{R}_\alpha$

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha)$

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha) \geq 3$:

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha) \geq 3$:
  - $V = A \cup B$

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha) \geq 3$:
  - $V = A \cup B$
  - $T(A) = B$

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha) \geq 3$:
  - $V = A \cup B$
  - $T(A) = B$
  - $\mu(A) = \mu(B) = 1/2$

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha) \geq 3$:
  - $V = A \cup B$
  - $T(A) = B$
  - $\mu(A) = \mu(B) = 1/2$
  - Contradicts ergodicity of T

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha) \geq 3$:
  - $V = A \cup B$
  - $T(A) = B$
  - $\mu(A) = \mu(B) = 1/2$
  - Contradicts ergodicity of T
- $\chi_\mu(\mathcal{R}_\alpha) = 3$

Properties of $\mathcal{R}_\alpha$

- Each component: infinite 2-regular path
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- Measurable chromatic number $\chi_\mu(\mathcal{R}_\alpha) \geq 3$:
  - $V = A \cup B$
  - $T(A) = B$
  - $\mu(A) = \mu(B) = 1/2$
  - Contradicts ergodicity of T
- $\chi_\mu(\mathcal{R}_\alpha) = 3$
- $\chi_{\mathrm{Borel}}(\mathcal{R}_\alpha) = 3$

Borel and measurable set

Borel and measurable set

- $\{\text{Borel}\} \stackrel{\text{def}}{=}$ smallest σ-agebra $\supseteq$ open sets

Borel and measurable set

- $\{\text{Borel}\} \stackrel{\text{def}}{=}$ smallest σ-agebra $\supseteq$ open sets
- Some permitted operations:

Borel and measurable set

- $\{\text{Borel}\} \stackrel{\text{def}}{=}$ smallest σ-agebra $\supseteq$ open sets
- Some permitted operations:
  - Boolean operations

Borel and measurable set

- $\{\text{Borel}\} \stackrel{\text{def}}{=}$ smallest σ-agebra $\supseteq$ open sets
- Some permitted operations:
  - Boolean operations
  - Countably many iterations

Borel and measurable set

- $\{\text{Borel}\} \stackrel{\text{def}}{=}$ smallest σ-agebra $\supseteq$ open sets
- Some permitted operations:
  - Boolean operations
  - Countably many iterations
  - Colour of every vertex eventually stabilises $\Rightarrow$ Borel

Borel and measurable set

- $\{\text{Borel}\} \stackrel{\text{def}}{=}$ smallest σ-agebra $\supseteq$ open sets
- Some permitted operations:
  - Boolean operations
  - Countably many iterations
  - Colour of every vertex eventually stabilises $\Rightarrow$ Borel
- (Lebesgue) measurable $\stackrel{\text{def}}{=}$ Borel $\triangle$ measure-0

Borel and measurable set

- {Borel} $\stackrel{\text{def}}{=}$ smallest σ-agebra $\supseteq$ open sets
- Some permitted operations:
  - Boolean operations
  - Countably many iterations
  - Colour of every vertex eventually stabilises $\Rightarrow$ Borel
- (Lebesgue) measurable $\stackrel{\text{def}}{=}$ Borel $\triangle$ measure-0
  - Stabilises a.e. $\Rightarrow$ measurable

Greedy colourings Borel way

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Proof:

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\mathrm{Borel}} \leq \Delta + 1$
- Proof:
  - WLOG $V = [0, 1]$

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\mathrm{Borel}} \le \Delta + 1$
- Proof:
  - WLOG $V = [0, 1]$
  - Borel vertex colouring $f : V \to \{0, 1\}^{<\omega}$

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Proof:
  - WLOG $V = [0, 1]$
  - Borel vertex colouring $f : V \to \{0, 1\}^{<\omega}$
  - $f(x) :=$ min prefix $b_1 \dots b_k$ separating $x = 0.b_1 b_2 b_3 ...$ from $N(x)$

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \le \Delta + 1$
- Proof:
  - WLOG $V = [0, 1]$
  - Borel vertex colouring $f : V \to \{0, 1\}^{<\omega}$
  - $f(x) :=$ min prefix $b_1 \dots b_k$ separating $x = 0.b_1 b_2 b_3 ...$ from $N(x)$
  - Define $c : V \to [\Delta + 1]$:

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Proof:
  - WLOG $V = [0, 1]$
  - Borel vertex colouring $f : V \to \{0, 1\}^{<\omega}$
  - $f(x) :=$ min prefix $b_1 \dots b_k$ separating $x = 0.b_1 b_2 b_3 \dots$ from $N(x)$
  - Define $c : V \to [\Delta + 1]$:
    - For $i \in \{0, 1\}^{<\omega}$

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\mathrm{Borel}} \leq \Delta + 1$
- Proof:
  - WLOG $V = [0, 1]$
  - Borel vertex colouring $f : V \to \{0, 1\}^{<\omega}$
  - $f(x) :=$ min prefix $b_1 \dots b_k$ separating $x = 0.b_1 b_2 b_3 \dots$ from $N(x)$
  - Define $c : V \to [\Delta + 1]$:
    - For $i \in \{0, 1\}^{<\omega}$
      Colour each $x \in f^{-1}(i)$ with smallest allowed colour

Greedy colourings Borel way

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Proof:
  - WLOG $V = [0, 1]$
  - Borel vertex colouring $f : V \to \{0, 1\}^{<\omega}$
  - $f(x) :=$ min prefix $b_1 \ldots b_k$ separating $x = 0.b_1 b_2 b_3 \ldots$ from $N(x)$
  - Define $c : V \to [\Delta + 1]$:
    - For $i \in \{0, 1\}^{<\omega}$
      Colour each $x \in f^{-1}(i)$ with smallest allowed colour
- Descriptive set theory

Some known results

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\mathrm{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\mathrm{Borel}} \leq 2\Delta - 1$

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic
- Vizing'64, Gupta'66: $\chi' \leq \Delta + 1$

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic
- Vizing'64, Gupta'66: $\chi' \leq \Delta + 1$
- Abert'10, Marks'13: Measurable Vizing's theorem?

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic
- Vizing'64, Gupta'66: $\chi' \leq \Delta + 1$
- Abert'10, Marks'13: Measurable Vizing's theorem?
- Csóka, Lippner & P.'16:

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic
- Vizing'64, Gupta'66: $\chi' \leq \Delta + 1$
- Abert'10, Marks'13: Measurable Vizing's theorem?
- Csóka, Lippner & P.'16:
  - $\chi'_\mu \leq \Delta + O(\sqrt{\Delta})$

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic
- Vizing'64, Gupta'66: $\chi' \leq \Delta + 1$
- Abert'10, Marks'13: Measurable Vizing's theorem?
- Csóka, Lippner & P.'16:
  - $\chi'_\mu \leq \Delta + O(\sqrt{\Delta})$
  - Bipartite $\Rightarrow$ $\chi'_\mu \leq \Delta + 1$

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \le \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \le 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic
- Vizing'64, Gupta'66: $\chi' \le \Delta + 1$
- Abert'10, Marks'13: Measurable Vizing's theorem?
- Csóka, Lippner & P.'16:
  - $\chi'_\mu \le \Delta + O(\sqrt{\Delta})$
  - Bipartite $\quad\Rightarrow\quad \chi'_\mu \le \Delta + 1$
- Marks'16: Acyclic $\mathcal{G}$ with $\chi_{\text{Borel}} = \Delta + 1$

Some known results

- Kechris, Solecky & Todorcevic'99: $\chi_{\text{Borel}} \leq \Delta + 1$
- Corollary: Borel chromatic index $\chi'_{\text{Borel}} \leq 2\Delta - 1$
- Marks'16: $2\Delta - 1$ is attainable
  - bipartite & acyclic
- Vizing'64, Gupta'66: $\chi' \leq \Delta + 1$
- Abert'10, Marks'13: Measurable Vizing's theorem?
- Csóka, Lippner & P.'16:
  - $\chi'_\mu \leq \Delta + O(\sqrt{\Delta})$
  - Bipartite $\Rightarrow$ $\chi'_\mu \leq \Delta + 1$
- Marks'16: Acyclic $\mathcal{G}$ with $\chi_{\text{Borel}} = \Delta + 1$
- Conley, Marks & Tucker-Drob'16: Measurable Brooks' theorem for $\Delta \geq 3$

Example: group actions

Example: group actions

- Group Γ

Example: group actions

- Group Γ
- Measure-preserving action $\Gamma \curvearrowright (V, \mathcal{A}, \mu)$

Example: group actions

- Group Γ
- Measure-preserving action $\Gamma \curvearrowright (V, \mathcal{A}, \mu)$
- Finite generating set $S \subseteq \Gamma$, $S = S^{-1}$

Example: group actions

- Group Γ
- Measure-preserving action $\Gamma \curvearrowright (V, \mathcal{A}, \mu)$
- Finite generating set $S \subseteq \Gamma$, $S = S^{-1}$
- $E := \{\, \{x, \gamma.x\} : x \in V,\ \gamma \in S,\ x \neq \gamma.x\}$

Example: group actions

- Group Γ
- Measure-preserving action $\Gamma \curvearrowright (V, \mathcal{A}, \mu)$
- Finite generating set $S \subseteq \Gamma$, $S = S^{-1}$
- $E := \{ \{x, \gamma.x\} : x \in V,\ \gamma \in S,\ x \neq \gamma.x\}$
- Graphing $(V, E, \mathcal{A}, \mu)$

Example: group actions

- Group Γ
- Measure-preserving action $\Gamma \curvearrowright (V, \mathcal{A}, \mu)$
- Finite generating set $S \subseteq \Gamma$, $S = S^{-1}$
- $E := \{ \{x, \gamma.x\} : x \in V,\ \gamma \in S,\ x \neq \gamma.x\}$
- Graphing $(V, E, \mathcal{A}, \mu)$
- $\mathcal{R}_\alpha$: $\mathbb{Z} \cong \langle T \rangle$ acting on $[0, 1)$

Example: group actions

- Group Γ
- Measure-preserving action $\Gamma \curvearrowright (V, \mathcal{A}, \mu)$
- Finite generating set $S \subseteq \Gamma$, $S = S^{-1}$
- $E := \{ \{x, \gamma.x\} : x \in V,\ \gamma \in S,\ x \neq \gamma.x\}$
- Graphing $(V, E, \mathcal{A}, \mu)$
- $\mathcal{R}_\alpha$: $\mathbb{Z} \cong \langle T \rangle$ acting on $[0, 1)$
- $\forall$ graphing is realisable this way

Example: group actions

- Group Γ
- Measure-preserving action $\Gamma \curvearrowright (V, \mathcal{A}, \mu)$
- Finite generating set $S \subseteq \Gamma$, $S = S^{-1}$
- $E := \{\, \{x, \gamma.x\} : x \in V,\ \gamma \in S,\ x \neq \gamma.x\}$
- Graphing $(V, E, \mathcal{A}, \mu)$
- $\mathcal{R}_\alpha$: $\mathbb{Z} \cong \langle T \rangle$ acting on $[0, 1)$
- $\forall$ graphing is realisable this way
- Measurable group theory

Equidecomposability

Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable $(A \sim B) \stackrel{\text{def}}{=}$

Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$) $\stackrel{\text{def}}{=}$
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$

Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$) $\stackrel{\text{def}}{=}$
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

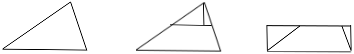
Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$) $\stackrel{\text{def}}{=}$
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

  st $\forall\, i\ \ \exists$ isometry γ_i with $A_i = \gamma_i(B_i)$

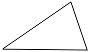
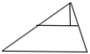
Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$) $\stackrel{\text{def}}{=}$
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

  st $\forall\, i\ \ \exists$ isometry γ_i with $A_i = \gamma_i(B_i)$
- Example: 

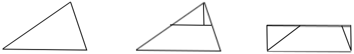
Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable $(A \sim B) \stackrel{\text{def}}{=}$
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

  st $\forall\, i\ \ \exists$ isometry γ_i with $A_i = \gamma_i(B_i)$
- Example:   
- Wallace 1807, Bolyai 1832, Gerwien 1832:
  equi-area polygons are dissection congruent

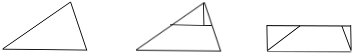
Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$) $\stackrel{\text{def}}{=}$
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

  st $\forall\, i\ \ \exists$ isometry γ_i with $A_i = \gamma_i(B_i)$
- Example: 
- Wallace 1807, Bolyai 1832, Gerwien 1832: equi-area polygons are dissection congruent
  - Tarski'24: equidecomposable

Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$) $\stackrel{\text{def}}{=}$
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

  st $\forall\, i\ \ \exists$ isometry γ_i with $A_i = \gamma_i(B_i)$
- Example: 
- Wallace 1807, Bolyai 1832, Gerwien 1832:
  equi-area polygons are dissection congruent
  - Tarski'24: equidecomposable
  - Exercise: $[0,1] \sim [0,1)$

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow A \sim B$

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball
    - Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball
    - Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$
- $k \leq 2$:

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball
    - Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$
- $k \leq 2$:
  - Banach'23: Measurable $A, B \Rightarrow \mu(A) = \mu(B)$

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball
    - Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$
- $k \leq 2$:
  - Banach'23: Measurable $A, B \Rightarrow \mu(A) = \mu(B)$
  - Tarski'25: Is $\mathbb{B}^2$ equidecomposable to a square ?

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow$ $A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball
    - Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$
- $k \leq 2$:
  - Banach'23: Measurable A, B $\Rightarrow$ $\mu(A) = \mu(B)$
  - Tarski'25: Is $\mathbb{B}^2$ equidecomposable to a square ?
  - von Neumann'29: Yes, with affine transformations

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow$ $A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball
    - Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$
- $k \leq 2$:
  - Banach'23: Measurable A, B $\Rightarrow$ $\mu(A) = \mu(B)$
  - Tarski'25: Is $\mathbb{B}^2$ equidecomposable to a square ?
  - von Neumann'29: Yes, with affine transformations
  - Dubins-Hirsh-Karush'63: No, with topological disks

Arbitrary parts, equidecomposing $A, B \subseteq \mathbb{R}^k$

- $k \geq 3$:
  - Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, bounded, non-empty interior $\Rightarrow A \sim B$
    - $\mathbb{B}^k \sim \mathbb{B}^k \sqcup \mathbb{B}^k$, $\mathbb{B}^k$:= k-dim ball
    - Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$
- $k \leq 2$:
  - Banach'23: Measurable $A, B \Rightarrow \mu(A) = \mu(B)$
  - Tarski'25: Is $\mathbb{B}^2$ equidecomposable to a square ?
  - von Neumann'29: Yes, with affine transformations
  - Dubins-Hirsh-Karush'63: No, with topological disks
  - Laczkovich'90: YES!

Measurable pieces?

Measurable pieces?

- Necessary conditions for $A \sim [0,1]^k$:

Measurable pieces?

- Necessary conditions for $A \sim [0,1]^k$:
  - A is measurable

Measurable pieces?

- Necessary conditions for $A \sim [0,1]^k$:
  - A is measurable
  - A is bounded

Measurable pieces?

- Necessary conditions for $A \sim [0, 1]^k$:
  - A is measurable
  - A is bounded
  - $\mu(A) = 1$

Measurable pieces?

- Necessary conditions for $A \sim [0,1]^k$:
  - A is measurable
  - A is bounded
  - $\mu(A) = 1$
  - $\bigcup\{\text{finitely many copies of } A\} \supseteq [0,1]^k$

Measurable pieces?

- Necessary conditions for $A \sim [0,1]^k$:
  - A is measurable
  - A is bounded
  - $\mu(A) = 1$
  - $\bigcup\{\text{finitely many copies of } A\} \supseteq [0,1]^k$
- Grabowski-Máthé-P. $\geq$'17: sufficient for $k \geq 3$

Measurable pieces?

- Necessary conditions for $A \sim [0,1]^k$:
  - A is measurable
  - A is bounded
  - $\mu(A) = 1$
  - $\bigcup\{\text{finitely many copies of } A\} \supseteq [0,1]^k$
- Grabowski-Máthé-P. $\geq$'17: sufficient for $k \geq 3$
- Grabowski-Máthé-P.'16: circle squaring with measurable pieces

Measurable pieces?

- Necessary conditions for $A \sim [0,1]^k$:
  - A is measurable
  - A is bounded
  - $\mu(A) = 1$
  - $\bigcup\{\text{finitely many copies of } A\} \supseteq [0,1]^k$
- Grabowski-Máthé-P. $\geq$'17: sufficient for $k \geq 3$
- Grabowski-Máthé-P.'16: circle squaring with measurable pieces
- Marks-Unger $\geq$'17: circle squaring with Borel pieces

Equidecompositions via matchings

Equidecompositions via matchings

- Isometries $\gamma_1, \ldots, \gamma_n$

Equidecompositions via matchings

- Isometries $\gamma_1, \ldots, \gamma_n$
- Graphing $\mathcal{G}$:

Equidecompositions via matchings

- Isometries $\gamma_1, \ldots, \gamma_n$
- Graphing $\mathcal{G}$:
  - $V := A \sqcup B$

Equidecompositions via matchings

- Isometries $\gamma_1, \ldots, \gamma_n$
- Graphing $\mathcal{G}$:
  - $V := A \sqcup B$
  - $E = \{xy \in A \times B : \exists\, i\ \gamma_i(x) = y\}$

Equidecompositions via matchings

- Isometries $\gamma_1, \ldots, \gamma_n$
- Graphing $\mathcal{G}$:
  - $V := A \sqcup B$
  - $E = \{xy \in A \times B : \exists\, i\ \gamma_i(x) = y\}$
- Matching $\mathcal{M} \subseteq E \ \longleftrightarrow\$ disjoint $A_1, \ldots, A_n \subseteq A$ st $\gamma_1(A_1), \ldots, \gamma_n(A_n) \subseteq B$ are disjoint

Equidecompositions via matchings

- Isometries $\gamma_1, \ldots, \gamma_n$
- Graphing $\mathcal{G}$:
  - $V := A \sqcup B$
  - $E = \{xy \in A \times B : \exists\, i\ \gamma_i(x) = y\}$
- Matching $\mathcal{M} \subseteq E \ \longleftrightarrow\ $ disjoint $A_1, \ldots, A_n \subseteq A$ st $\gamma_1(A_1), \ldots, \gamma_n(A_n) \subseteq B$ are disjoint
- $\exists$ perfect matching $\ \Rightarrow\ A \sim B$

Measurable maximal matching

Measurable maximal matching

- Greedy Algorithm:

Measurable maximal matching

- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$

Measurable maximal matching

- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$
  - $B_1 := \gamma_1(A_1)$

Measurable maximal matching

- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$
  - $B_1 := \gamma_1(A_1)$
  - $A_2 := (A \setminus A_1) \cap \gamma_2^{-1}(B \setminus B_1)$

Measurable maximal matching

- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$
  - $B_1 := \gamma_1(A_1)$
  - $A_2 := (A \setminus A_1) \cap \gamma_2^{-1}(B \setminus B_1)$
  - $B_2 := \gamma_2(A_2)$

Measurable maximal matching

- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$
  - $B_1 := \gamma_1(A_1)$
  - $A_2 := (A \setminus A_1) \cap \gamma_2^{-1}(B \setminus B_1)$
  - $B_2 := \gamma_2(A_2)$
  - $A_3 := (A \setminus (A_1 \cup A_2)) \cap \gamma_3^{-1}(B \setminus (B_1 \cup B_2))$

Measurable maximal matching

- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$
  - $B_1 := \gamma_1(A_1)$
  - $A_2 := (A \setminus A_1) \cap \gamma_2^{-1}(B \setminus B_1)$
  - $B_2 := \gamma_2(A_2)$
  - $A_3 := (A \setminus (A_1 \cup A_2)) \cap \gamma_3^{-1}(B \setminus (B_1 \cup B_2))$
  - ...

Measurable maximal matching

- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$
  - $B_1 := \gamma_1(A_1)$
  - $A_2 := (A \setminus A_1) \cap \gamma_2^{-1}(B \setminus B_1)$
  - $B_2 := \gamma_2(A_2)$
  - $A_3 := (A \setminus (A_1 \cup A_2)) \cap \gamma_3^{-1}(B \setminus (B_1 \cup B_2))$
  - ...
- Elek-Lippner'10: $\exists$ Borel $\mathcal{M}_i$ without augmenting paths of length $\leq 2i - 1$

Selecting $\gamma_1, \ldots, \gamma_n$

Selecting $\gamma_1, \ldots, \gamma_n$

- For $A, B \subseteq \mathbb{S}^k$, $k \geq 2$:

Selecting $\gamma_1, \ldots, \gamma_n$

- For $A, B \subseteq \mathbb{S}^k$, $k \geq 2$:
  - Margulis'80, Sullivan'81 ($k \geq 4$), Drinfeld'84 ($k = 2, 3$): $\exists\, \rho_1, \ldots, \rho_s$ st $(\mathbb{S}^k, E(\rho_1, \ldots, \rho_s))$ is expander

Selecting $\gamma_1, \ldots, \gamma_n$

- For $A, B \subseteq \mathbb{S}^k$, $k \geq 2$:
  - Margulis'80, Sullivan'81 ($k \geq 4$), Drinfeld'84 ($k = 2, 3$): $\exists\, \rho_1, \ldots, \rho_s$ st $(\mathbb{S}^k, E(\rho_1, \ldots, \rho_s))$ is expander
  - All ℓ-products of ρ_i's, large $\ell = \ell(A, B)$

Selecting $\gamma_1, \ldots, \gamma_n$

- For $A, B \subseteq \mathbb{S}^k$, $k \geq 2$:
  - Margulis'80, Sullivan'81 ($k \geq 4$), Drinfeld'84 ($k = 2, 3$): $\exists\, \rho_1, \ldots, \rho_s$ st $(\mathbb{S}^k, E(\rho_1, \ldots, \rho_s))$ is expander
  - All ℓ-products of ρ_i's, large $\ell = \ell(A, B)$
- For circle squaring

Selecting $\gamma_1, \ldots, \gamma_n$

- For $A, B \subseteq \mathbb{S}^k$, $k \geq 2$:
  - Margulis'80, Sullivan'81 ($k \geq 4$), Drinfeld'84 ($k = 2, 3$): $\exists\, \rho_1, \ldots, \rho_s$ st $(\mathbb{S}^k, E(\rho_1, \ldots, \rho_s))$ is expander
  - All ℓ-products of ρ_i's, large $\ell = \ell(A, B)$
- For circle squaring
  - Work in $[0, 1)^k \pmod 1$

Selecting $\gamma_1, \ldots, \gamma_n$

- For $A, B \subseteq \mathbb{S}^k$, $k \geq 2$:
  - Margulis'80, Sullivan'81 ($k \geq 4$), Drinfeld'84 ($k = 2, 3$): $\exists\, \rho_1, \ldots, \rho_s$ st $(\mathbb{S}^k, E(\rho_1, \ldots, \rho_s))$ is expander
  - All ℓ-products of ρ_i's, large $\ell = \ell(A, B)$
- For circle squaring
  - Work in $[0, 1)^k \pmod 1$
  - Random $\mathbf{x}_1, \ldots, \mathbf{x}_d$, large $d = d(A, B)$

Selecting $\gamma_1, \ldots, \gamma_n$

- For $A, B \subseteq \mathbb{S}^k$, $k \geq 2$:
  - Margulis'80, Sullivan'81 ($k \geq 4$), Drinfeld'84 ($k = 2, 3$): $\exists\, \rho_1, \ldots, \rho_s$ st $(\mathbb{S}^k, E(\rho_1, \ldots, \rho_s))$ is expander
  - All ℓ-products of ρ_i's, large $\ell = \ell(A, B)$
- For circle squaring
  - Work in $[0, 1)^k \pmod 1$
  - Random $\mathbf{x}_1, \ldots, \mathbf{x}_d$, large $d = d(A, B)$
  - Translate by $\{n_1\mathbf{x}_1 + \cdots + n_d\mathbf{x}_d : \|\mathbf{n}\|_\infty < M\}$, $M \gg d$

Open Problems

Open Problems

- Minimum number of pieces for circle squaring?

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger $\geq$'17: $\leq 10^{200}$

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger $\geq$'17: $\leq 10^{200}$
  - Lower bound:

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger $\geq$'17: $\leq 10^{200}$
  - Lower bound:
    - 3 (if rotations allowed)

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger $\geq$'17: $\leq 10^{200}$
  - Lower bound:
    - 3 (if rotations allowed)
    - 4 (for translations only)

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger $\geq$'17: $\leq 10^{200}$
  - Lower bound:
    - 3 (if rotations allowed)
    - 4 (for translations only)
- Banach-Tarski'24: 5 parts for $\mathbb{B}^3 \cong \mathbb{B}^3 \sqcup \mathbb{B}^3$

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger $\geq$'17: $\leq 10^{200}$
  - Lower bound:
    - 3 (if rotations allowed)
    - 4 (for translations only)
- Banach-Tarski'24: 5 parts for $\mathbb{B}^3 \cong \mathbb{B}^3 \sqcup \mathbb{B}^3$
- Grabowski-Máthé-P. $\geq$'17: $\leq 2^{2^{75}}$ for $\mathbb{B}^3 \sim$ cube

Open Problems

- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger $\geq$'17: $\leq 10^{200}$
  - Lower bound:
    - 3 (if rotations allowed)
    - 4 (for translations only)
- Banach-Tarski'24: 5 parts for $\mathbb{B}^3 \cong \mathbb{B}^3 \sqcup \mathbb{B}^3$
- Grabowski-Máthé-P. $\geq$'17: $\leq 2^{2^{75}}$ for $\mathbb{B}^3 \sim$ cube
- Open: $A \sim B,\ \mu(A) = \mu(B) \ \Rightarrow\ A \sim B$ measurably ?

Thank you!