Quasirandom Permutations Are Characterised by 4-Point Densities

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- Král'-P. $\geq$'13: $m = 4$ suffices

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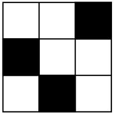

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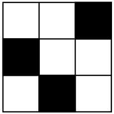

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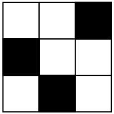
213 $\mapsto$ 

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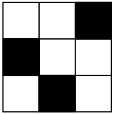
213 $\mapsto$ 

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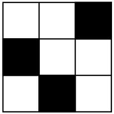
213 $\quad\mapsto\quad$ 

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- Prokhorov'56: subsequence of $\mu_{\Pi_n} \to$ some μ weakly

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- Motivated by graph limits (Lovász-Szegedy-...)

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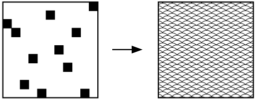

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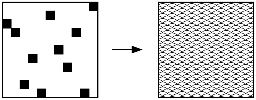

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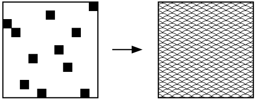

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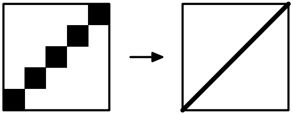

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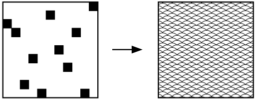

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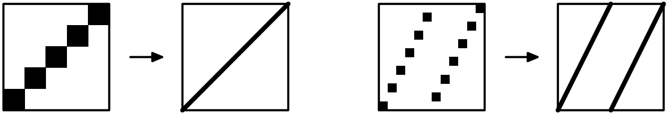

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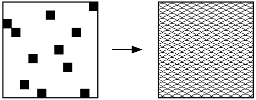

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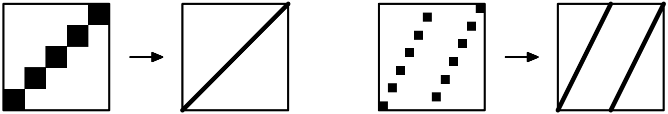

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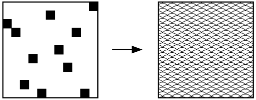

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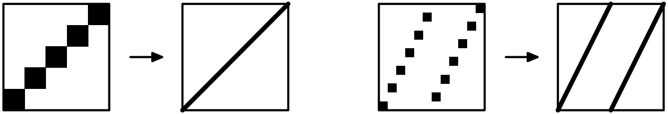

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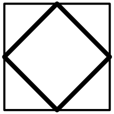

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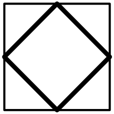

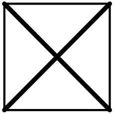

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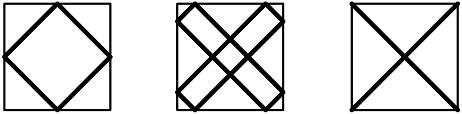

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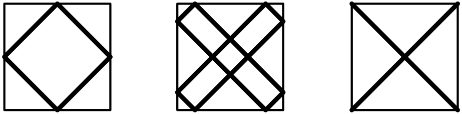

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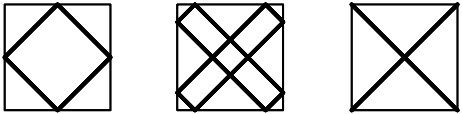

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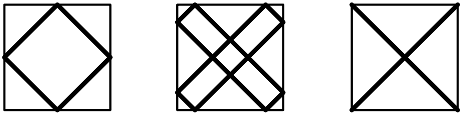

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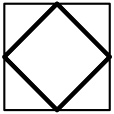

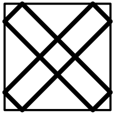

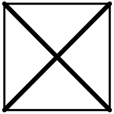

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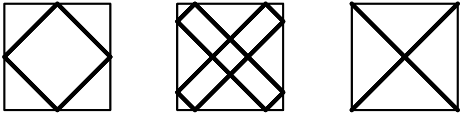

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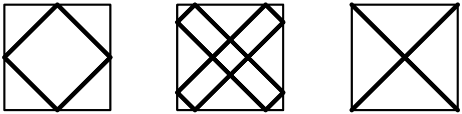

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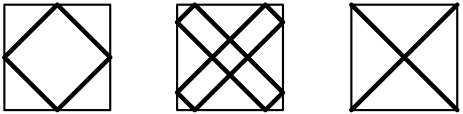

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$$
\begin{aligned}
\frac{1}{81} &= \left(\int F_\mu(X,Y)XY \,\mathrm{d}\mu\right)^2 \\
&\leq \left(\int F_\mu(X,Y)^2 \,\mathrm{d}\mu\right) \cdot \left(\int X^2 Y^2 \,\mathrm{d}\mu\right) \\
&= \frac{1}{9}\left(4 \cdot \int F_\mu(x,y)xy \,\mathrm{d}\lambda - \int (1 - X^2 - X^2) \,\mathrm{d}\mu\right) \\
&\leq \frac{4}{9}\sqrt{\int F_\mu(x,y) \,\mathrm{d}\lambda} \cdot \sqrt{\int x^2 y^2 \,\mathrm{d}\lambda} - \frac{1}{27} = \frac{1}{81}
\end{aligned}
$$

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Thank you!