GEOMETRY

Warm-up

1. Let P_0, P_1, P_2 be the vertices of a triangle T in the plane and P a point in its interior. Prove that there exists $\lambda_i \in (0,1)$ such that $\lambda_0 + \lambda_1 + \lambda_2 = 1$ and

$$P = \lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2$$

Moreover, if T_i is the triangle obtained from T by replacing P_i by P. Show that

$$\lambda_i = \frac{Area(T_i)}{Area(T)}$$

State and prove an analogous result in any dimension.

Solution.

Claim. *Let S be the n-simplex defined by the points $P_0, P_1, ..., P_n$ in $\mathbb{R}^n$ not all belonging to the same hyperplane and P a point in its interior. Then, there exist $\lambda_i \in (0,1)$, $i = 0, 1, ..., n$, such that $\lambda_0 + \lambda_1 + ... + \lambda_n = 1$ and*

$$P = \lambda_0 P_0 + \lambda_1 P_1 + ... + \lambda_n P_n$$

Moreover, if S_i is the n-simplex obtained from S by replacing P_i by P and $v(S)$ denote the volume of S, then

$$\lambda_i = \frac{v(S_i)}{v(S)}$$

We proceed by induction. The case case $n = 1$ is well known: $P = \lambda_0 P_0 + \lambda_1 P_1$ with $\lambda_1 = \frac{|P-P_0|}{|P_1-P_0|}$. Now suppose it is true for n.

For the inductive step assume $P_0, ..., P_n$ are in the hyperplane $H = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and let P_{n+1} be elsewhere. Denote by Q the intersection of the line $\overline{P_{n+1}P}$ with H, then we know $P = \lambda Q + \lambda_{n+1} P_{n+1}$

$$Q = \mu_0 P_0 + \mu_1 P_1 + ... + \mu_n P_n$$

with $\lambda + \lambda_{n+1} = 1 = \mu_0 + ... + \mu_n$, hence

$$P = \lambda_0 P_0 + \lambda_1 P_1 + ... + \lambda_{n+1} P_{n+1}$$

where $\lambda_i = \lambda \mu_i$ for $i = 0, ..., n$. From this it can be checked that $\lambda_0 + ... + \lambda_{n+1} = 1$.

Moreover, if we call h and h_{n+1} the distance from P and P_{n+1} to H (“the heights of the $(n+1)$-simplexes S and S_{n+1}”) then

$$\frac{v(S_{n+1})}{v(S)} = \frac{v(T)h}{v(T)h_{n+1}} = \frac{h}{h_{n+1}} = \frac{|P_{n+1} - P|}{|P_{n+1} - Q|} = \lambda_{n+1}$$

where T is the n-simplex defined by $P_0, ..., P_n$ and we have use similarity to obtain $\frac{h}{h_{n+1}} = \frac{|P_{n+1}-P|}{|P_{n+1}-Q|}$.

Since the point P_{n+1} can be choosen arbitrarily, this finish the proof. ■

Homework

1. Let l be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to l is greater than or equal to two times the distance between X and P. If the distance from P to l is $d > 0$, find the volume of S.

   Solution.
   Choose coordinates such that l is the x-axis and $P = (0, 0, d)$, for a point $X = (x, y, z)$ we have that the condition of being in S is

   $$\sqrt{y^2 + z^2} \geq 2\sqrt{x^2 + y^2 + (z - d)^2}$$

   which is equivalent to

   $$\begin{aligned} 0 &\geq 4x^2 + 3y^2 + 3z^2 - 8zd + 4d^2 \\ 0 &\geq 4x^2 + 3y^2 + 3(z - \frac{4}{3}d)^2 - \frac{16}{3}d^2 + 4d^2 \\ \frac{4}{3}d^2 &\geq 4x^2 + 3y^2 + 3(z - \frac{4}{3}d)^2 \\ 1 &\geq \left(\frac{x}{\frac{d}{\sqrt{3}}}\right)^2 + \left(\frac{y}{\frac{2d}{3}}\right)^2 + \left(\frac{z - \frac{4d}{3}}{\frac{2d}{3}}\right)^2 \end{aligned}$$

   The latter represents a solid ellipsoid and hence S has volume equal to $\frac{4}{3}\pi\frac{d}{\sqrt{3}}\frac{2d}{3}\frac{2d}{3} = \frac{16\pi}{27\sqrt{3}}d^3$. $\blacksquare$

2. Let us choose arbitrarily n vertices of a regular $2n$-gon and colour them red. The remaining vertices are coloured blue. We arrange all red-red distances into a non-decreasing sequence and do the same with the blue-blue distances. Prove that the sequences are equal.

   Solution.
   Let $d_1 < d_2 < ... < d_n$ be the possible distances between any two points of the $2n$-gon, and denote by a_i the number of red-red d_i-distances, b_i the number of blue-blue d_i-distances and c_i the number of red-blue d_i-distances. Consider two cases:

   - $i = n$. In this case the d_n's are diameters so there are only n of them and more importantly, they do not have vertices in common. Now, we count the number of red vertices in terms of a_n and c_n, there are 2 red vertices for each red-red diameter and only one for each red-blue, hence in total there are $2a_n + c_n = n$ red vertices. Similarly, there are $2b_n + c_n = n$ blue vertices. Therefore $a_n = b_n$.
   - $1 \leq i < n$. We proceed as in the previous case, but now taking into account that each vertex is counted doubly as it belongs to two (and only two) d_i-distances. Thus, the number of red vertices is $\frac{2a_i + c_i}{2} = n$. Analogously, the number of blue vertices is $\frac{2b_i + c_i}{2} = n$. So $a_i = b_i$.

   Therefore $a_i = b_i$ for $i = 1, ..., n$ and the sequences are equal. $\blacksquare$

3. Let $n \in \mathbb{N}$, an *n-simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, ..., P_n$ called its vertices which do not all belong to the same hyperplane. For every n-simplex S we denote $v(S)$ the volume of S, and we write $C(S)$ for the centre of the unique sphere containing all the vertices of S. Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + ... + v(S_n)C(S_n) = v(S)C(S)$$

Solution.
By the warm-up exercise we know $P = \lambda_0 P_0 + ... + \lambda_n P_n$ and $Q = \mu_0 Q_0 + ... + \mu_n Q_n$ (where $Q_i = C(S_i)$), with $\lambda_i = \frac{v(S_i)}{v(S)}$. So it is enough to prove $\lambda_i = \mu_i$.

The key observation is that Q belongs to the hyperplanes orthogonal to $P_i - P_j$, similarly Q_i belongs to the hyperplanes orthogonal to $P - P_j$, hence

$$\begin{aligned} (P_i - P) \cdot (Q_j - Q_k) &= 0 \\ (Q_i - Q) \cdot (P_j - P_k) &= 0 \end{aligned}$$

for $i \neq j \neq k$. Now, write

$$\begin{aligned} \lambda_0(P_0 - P) + ... + \lambda_n(P_n - P) &= 0 \\ \mu_0(Q_0 - Q) + ... + \mu_n(Q_n - Q) &= 0 \end{aligned}$$

and take the dot product with $Q_i - Q_j$ and $P_i - P_j$, respectively. The terms distinct from i and j get cancelled, hence we obtain

$$\begin{aligned} \lambda_i(P_i - P) \cdot (Q_i - Q_j) + \lambda_j(P_j - P) \cdot (Q_i - Q_j) &= 0 \\ \mu_i(Q_i - Q) \cdot (P_i - P_j) + \mu_j(Q_j - Q) \cdot (P_j - P_i) &= 0 \end{aligned}$$

Thus it is sufficient to prove the equality of the coefficients accompanying the λ's and μ's. Using bilinearity it is easy to check that $(P_i - P) \cdot (Q_i - Q_j) = (Q_i - Q) \cdot (P_i - P_j)$ if and only if $(P_i - P) \cdot (Q_j - Q) = (P_j - P) \cdot (Q_i - Q)$.

Now,

$$0 = (P_i - P) \cdot (Q_j - Q_k) = (P_i - P) \cdot ((Q_j - Q) - (Q_k - Q))$$

so $(P_i - P) \cdot (Q_j - Q) = (P_i - P) \cdot (Q_k - Q)$. Similarly $(Q_i - Q) \cdot (P_j - P) = (Q_i - Q) \cdot (P_k - P)$.

Using these relations (after suitably relabelling the indexes) we have

$$\begin{aligned} (P_i - P) \cdot (Q_j - Q) &= (P_i - P) \cdot (Q_k - Q) \\ &= (P_j - P) \cdot (Q_k - Q) \\ &= (P_j - P) \cdot (Q_i - Q) \end{aligned}$$

Hence (λ_i, λ_j) and (μ_i, μ_j) obey the same linear equation[1] and so they are proportional. But they add up to 1, therefore $\lambda_i = \mu_i$. ■

[1]For each i we can always find a non-trivial linear equation since $\{P_0 - P, ..., P_n - P\}$ contains a basis.