Flag Algebra Method in Extremal Combinatorics

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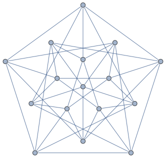

KTH, 29 January 2014

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  - Formal rules for deriving them
  - One aspect: semi-definite programming

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- Contributors: R.Baber, J.Balogh, J.Cummings, S.Das, V.Falgas-Ravry, R.Glebov, A.Grzesik, H.Hatami, J.Hirst, J.Hladký, P.Hu, H.Huang, T.Klimosova, D.Král', L.Kramer, B.Lidicky, N.Linial, C.-H.Liu, J.Ma, L.Mach, E.Marchant, R.Martin, H.Naves, S.Niess, S.Norine, Y.Peled, F.Pfender, O.Pikhurko, A.Razborov, C.Reiher, J.-S.Sereni, K.Spengler, B.Sudakov, J.Talbot, A.Treglown, E.Vaughan, J.Volec, P.Whalen, Z.Yilma, M.Young, ...

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- $T_n^2 = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

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- Note: $\phi(P_3) + \phi(\bar{P}_3) + \phi(\bar{K}_3) = 1$

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- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3)$
- Asymptotic result: $\mathrm{ex}(n, K_3) \leq (\frac{1}{2} + o(1))\binom{n}{2}$

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- More work: exact result for $n \geq n_0$

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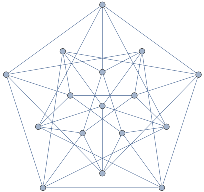

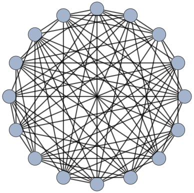

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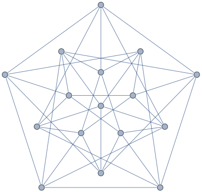

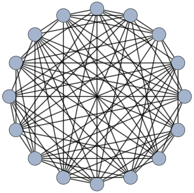

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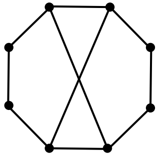

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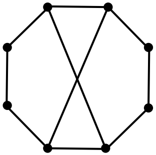

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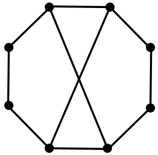

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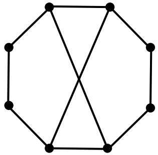

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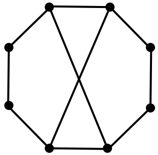

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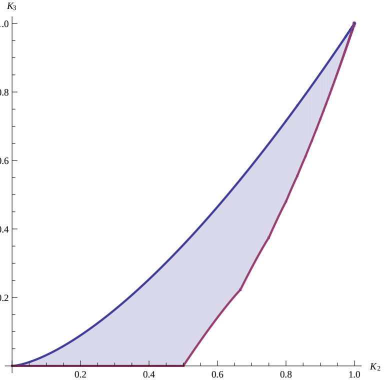

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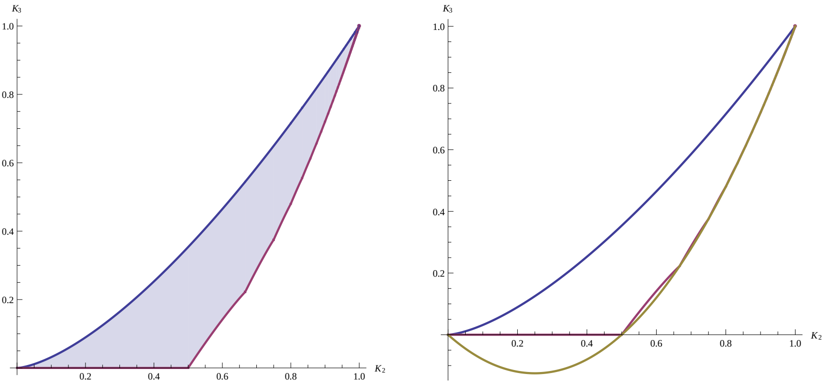

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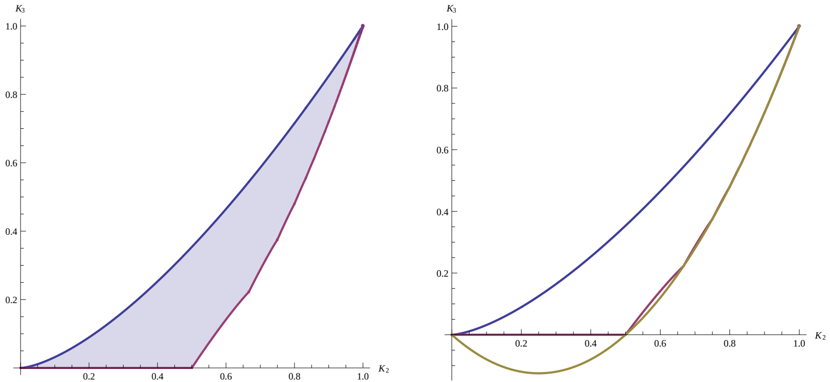

- Upper bound: Kruskal'63, Katona'66

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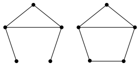
- Special graphs F_1 and F_2: 

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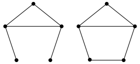
- Special graphs F_1 and F_2: 
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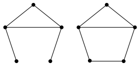
- Special graphs F_1 and F_2: 
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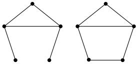
- Special graphs F_1 and F_2: 
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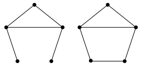
- Special graphs F_1 and F_2: 
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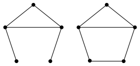
- Special graphs F_1 and F_2: 
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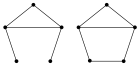
- Special graphs F_1 and F_2: 
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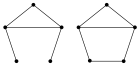
- Special graphs F_1 and F_2: 
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Thank you!