Mearurable circle squaring

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DIAMANT Symposium, 25 November 2016

Equidecomposability

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  - Exercise: $[0, 1] \sim [0, 1)$

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- Tomkowicz-Wagon'16: *"The Banach-Tarski Paradox"*

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- Tarski'38: no invariant mean $\Leftrightarrow$ paradoxical

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- Laczkovich'90: YES!

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- Laczkovich'90,92: $A, B \subseteq \mathbb{R}^k$, $\lambda(A) = \lambda(B) > 0$, bounded, $\dim_{\square}(\partial A), \dim_{\square}(\partial B) < k \ \Rightarrow\ A \overset{\mathrm{Tr}}{\sim} B$

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- Banach-Tarki'24: Banach-Tarski with Lebesgue pieces (but countably many)

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- Elek-Lippner'10: $\exists$ measurable $\mathcal{M}_i$ without augmenting paths of length $\leq 2i-1$

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- If
  1. $\lambda(\{a \in A : \text{missed by } \mathcal{M}_i\}) \to 0$
  2. $\sum_{i=1}^{\infty} \lambda(\{a \in A : \mathcal{M}_i, \mathcal{M}_{i+1} \text{differ on } a\}) < \infty$

  then $\mathcal{M}$ is a perfect matching a.e.

Borel-Cantelli-type argument

- $\mathcal{M}_1, \mathcal{M}_2, \ldots$
- $\mathcal{M} := \cup_{i=1}^{\infty} \cap_{j=1}^{n} M_j \subseteq E$
- If
  1. $\lambda(\{a \in A : \text{missed by } \mathcal{M}_i\}) \to 0$
  2. $\sum_{i=1}^{\infty} \lambda(\{a \in A : \mathcal{M}_i, \mathcal{M}_{i+1} \text{differ on } a\}) < \infty$

  then $\mathcal{M}$ is a perfect matching a.e.
- Lyons-Nazarov'11: Automatically holds if $\mathcal{G}$ is a bipartite expander.

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- Minimum number of pieces for circle squaring?
    - Laczkovich'92: *"a rough estimate"* is 10^{40}
    - Lower bound:
        - 3 (if rotations allowed)
        - 4 (for translations only)

Thank you!