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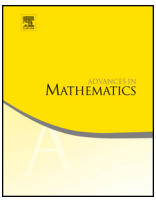

Circle squaring with pieces of small boundary and low Borel complexity

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ABSTRACT

Tarski's Circle Squaring Problem from 1925 asks whether it is possible to partition a disk in the plane into finitely many pieces and reassemble them via isometries to yield a partition of a square of the same area. It was finally resolved by Laczkovich in 1990 in the affirmative. Recently, several new proofs have emerged which achieve circle squaring with better structured pieces: namely, pieces which are Lebesgue measurable and have the property of Baire (Grabowski–Máthé–Pikhurko) or even are Borel (Marks–Unger).

In this paper, we show that circle squaring is possible with Borel pieces of positive Lebesgue measure whose boundaries have upper Minkowski dimension less than 2 (in particular, each piece is Jordan measurable). We also improve the Borel complexity of the pieces: namely, we show that each piece can be taken to be a Boolean combination of F_σ sets. This is a consequence of our more general result that applies to any two bounded subsets of $\mathbb{R}^k$, $k \geqslant 1$, of equal positive measure

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whose boundaries have upper Minkowski dimension smaller than k.

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1. Introduction

Tarski's Circle Squaring Problem [34] from 1925 asks if a circle (i.e., a circular disk) and a square of the same area in $\mathbb{R}^2$ are *equidecomposable*, that is, whether we can partition the circle into finitely many pieces and apply some isometry to each piece to get a partition of the square. This question was answered affirmatively some 65 years later by Laczkovich who showed in a deep and groundbreaking paper [22] that, in fact, it is possible to square a circle using translations only.

The Axiom of Choice plays a crucial role in his proof and, consequently, the pieces of his circle squaring could not be guaranteed to have any discernible regularity properties. A notable problem (mentioned by e.g., Wagon [37, Appendix C] or Laczkovich [22, Section 10]) has been to determine whether there exist circle squarings using "better" structured pieces. Recently, Grabowski, Máthé and Pikhurko [15] proved that the pieces of a circle squaring can simultaneously be Lebesgue measurable and have the property of Baire. Then, Marks and Unger [28] proved that the pieces can be made Borel. (Let us assume in this paper that the disk and the square are closed and thus Borel sets.) In fact, Marks and Unger [28, Section 7] showed that the pieces of a circle squaring can be chosen to be in $\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}^0_4)$, where $\boldsymbol{\Sigma}^0_i$ is the i-th additive class of the standard Borel hierarchy (see e.g., [18, Section 11.B]) and $\boldsymbol{\mathcal{B}}(\mathcal{F})$ denotes the algebra generated by a set family $\mathcal{F}$ (that is, the family of all Boolean combinations of elements from $\mathcal{F}$). For some generalisations and simplifications of the above results, we refer the reader to Cieśla and Sabok [6] and Bowen, Kun and Sabok [3,4].

Shortly after his circle-squaring paper, Laczkovich [23,24] proved a far-reaching generalisation. Before stating it, let us set up some notation. Fix $k \geqslant 1$. Let λ denote the Lebesgue measure on $\mathbb{R}^k$ and let ∂X denote the (topological) boundary of $X \subseteq \mathbb{R}^k$. Recall that the *upper Minkowski dimension*, sometimes called *box* or *grid dimension*, of $X \subseteq \mathbb{R}^k$ is

$$\dim_{\square}(X) := \limsup_{\delta \to 0^+} \frac{\log(N_\delta(X))}{\log(\delta^{-1})}, \tag{1.1}$$

where $N_\delta(X)$ is the number of boxes from the regular grid in $\mathbb{R}^k$ of side-length δ that intersect X.

Theorem 1.1 *(Laczkovich [23,24]). If $k \geqslant 1$ and $A, B \subseteq \mathbb{R}^k$ are bounded sets such that $\lambda(A) = \lambda(B) > 0$, $\dim_{\square}(\partial A) < k$ and $\dim_{\square}(\partial B) < k$, then A and B are equidecomposable by translations.*

The subsequent papers [15,28] on circle squaring in fact prove appropriate "constructive" versions of Theorem 1.1 and derive the corresponding circle squaring results as special cases. The aim of this paper is to extend this line of research. In the context of circle squaring, we prove the following result which decreases the Borel complexity of the pieces by two hierarchy levels and ensures that the boundary of each piece is "small" in a strong sense.

Theorem 1.2. *In $\mathbb{R}^2$, a closed disk and a closed square of the same area can be equidecomposed using translations so that every piece has boundary of upper Minkowski dimension at most* 1.987*, belongs to $\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}^0_2)$ (i.e., is a Boolean combination of F_σ sets), and has positive Lebesgue measure.*

Recall that a subset $X \subseteq \mathbb{R}^k$ is *Jordan measurable* if its indicator function is Riemann integrable. An equivalent definition is that X is bounded and $\lambda(\partial X) = 0$. It easily follows that any bounded set $X \subseteq \mathbb{R}^k$ with $\dim_\square(\partial X) < k$ is Jordan measurable. Therefore, Theorem 1.2 implies that circle squaring is possible with Jordan measurable pieces, which addresses questions by Laczkovich [27] and Máthé [29, Question 6.2]. An advantage of a Jordan measurable circle squaring is that an arbitrarily large portion of it can be described in an error-free way with finitely many bits of information. Namely, for every $\varepsilon > 0$, if n is large enough, then at most εn^2 boxes of the regular $n \times n$ grid on the equidecomposed unit square can intersect more than one piece and thus $O(n^2)$ bits are enough to describe our equidecomposition up to a set of measure at most ε. (Furthermore, the dimension estimate of Theorem 1.2 shows that ε, as a function of $n \to \infty$, can be taken to be $n^{-0.013+o(1)}$.)

We will obtain Theorem 1.2 as a special case of the following general result. For a set $A \subseteq \mathbb{R}^k$, let $\mathcal{T}_A := \{A + \boldsymbol{t} : \boldsymbol{t} \in \mathbb{R}^k\}$ consist of all its translations. For a family $\mathcal{F}$ of sets, let $\boldsymbol{\Sigma}(\mathcal{F})$ be the collection of all countable unions of sets in $\mathcal{F}$. Also, recall that $\boldsymbol{\Sigma}^0_1$ stands for the collection of open sets in $\mathbb{R}^k$.

Theorem 1.3. *If $k \geqslant 1$ and $A, B \subseteq \mathbb{R}^k$ are bounded sets such that $\lambda(A) = \lambda(B) > 0$, $\dim_\square(\partial A) < k$ and $\dim_\square(\partial B) < k$, then A and B are equidecomposable by translations so that all the following statements hold simultaneously:*

(a) *for some $\zeta > 0$ that depends on k, $\dim_\square(\partial A)$ and $\dim_\square(\partial B)$ only, the topological boundary of each piece has upper Minkowski dimension at most $k - \zeta$,*
(b) *each piece belongs to $\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}(\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}^0_1 \cup \mathcal{T}_A \cup \mathcal{T}_B)))$,*
(c) *if*

$$\lambda\left(\left\{\boldsymbol{t} \in \mathbb{R}^k : (A + \boldsymbol{t}) \cap B \neq \emptyset \text{ and } \lambda\left((A + \boldsymbol{t}) \cap B\right) = 0\right\}\right) = 0 \tag{1.2}$$

(that is, the set of vectors $\boldsymbol{t} \in \mathbb{R}^k$ such that $(A+\boldsymbol{t})\cap B$ is non-empty and Lebesgue-null has measure 0), then each piece has positive Lebesgue measure.

In fact, an explicit expression for a possible parameter $\zeta > 0$ in Part (a) as a function of k, $\dim_\square(\partial A)$ and $\dim_\square(\partial B)$ is given in (7.11). For circle squaring, the bound in (7.11) states that any $\zeta < 1/73$ suffices. Interestingly, by Part (a), each non-null piece of the equidecomposition individually satisfies the hypothesis of the theorem and is therefore equidecomposable to a ball or a cube of the same measure. Moreover, each non-null piece automatically has non-empty interior. (Note that a Jordan measurable subset of $\mathbb{R}^k$ is null if and only if it has empty interior.)

If the sets A and B in Theorem 1.3 are Borel, then, by Part (b), all pieces of the equidecomposition can be taken to be Borel with a good control over their Borel complexity (that improves upon the analogous result in [28] by two hierarchy levels). For example, if each of A and B is simultaneously a G_δ set and an F_σ set, that is, belongs to $\mathbf{\Delta}^0_2 := \{X \in \mathbf{\Sigma}^0_2 : \mathbb{R}^k \setminus X \in \mathbf{\Sigma}^0_2\}$ (in particular, if each of A and B is open or closed), then every piece in the equidecomposition is a Boolean combination of F_σ sets, which follows from

$$\boldsymbol{\mathcal{B}}(\mathbf{\Sigma}(\boldsymbol{\mathcal{B}}(\mathbf{\Delta}^0_2))) = \boldsymbol{\mathcal{B}}(\mathbf{\Sigma}(\mathbf{\Delta}^0_2)) \subseteq \boldsymbol{\mathcal{B}}(\mathbf{\Sigma}(\mathbf{\Sigma}^0_2)) = \boldsymbol{\mathcal{B}}(\mathbf{\Sigma}^0_2),$$

where we use the easy facts that $\mathbf{\Delta}^0_2$ is closed under Boolean combinations and $\mathbf{\Sigma}^0_2$ is closed under countable unions.

Regarding Part (c) of Theorem 1.3, note that it is impossible to guarantee that all pieces in Theorem 1.3 have positive measure in general. For example, if A contains a point $\boldsymbol{x}$ such that the distance from $\boldsymbol{x}$ to $A \setminus \{\boldsymbol{x}\}$ is greater than the diameter of B, then every equidecomposition of A to B must include the set $\{\boldsymbol{x}\}$ as a single piece. However, the extra assumption of Part (c) applies in many natural cases: for example, it holds for a ball and a cube in any dimension (with arbitrary subsets of their boundaries removed), or for any open sets A and B.

It seems difficult to weaken the assumptions of Theorem 1.3 in some substantial way. Laczkovich [25, Corollary 3.5] showed that, for every $k \geqslant 1$, there is a bounded subset $A \subseteq \mathbb{R}^k$ which is a countable union of pairwise disjoint convergent cubes but which is not equidecomposable to a single cube by translations. In particular, even in Theorem 1.1, one cannot replace the assumption that $\dim_\square(\partial A), \dim_\square(\partial B) < k$ by requiring that, for example, the Hausdorff dimension of the boundaries is at most $k - 1$. Another family that refutes various extensions of Theorem 1.1 (and thus of Theorem 1.3) comes from the work of Laczkovich [26, Theorem 3] who showed that for every $k \geqslant 2$ there are continuum many *Jordan domains* (i.e. homeomorphic images of the closed ball), each of volume 1 with everywhere differentiable boundary, so that no two are equidecomposable using any amenable subgroup of isometries. (In particular, this applies to the group $\mathbb{R}^k$ of translations for any k and the full group of isometries of $\mathbb{R}^2$, which are amenable.)

On the other hand, if $k \geqslant 3$ and one allows all orientation-preserving isometries of $\mathbb{R}^k$, then the obvious necessary conditions for a set to be equidecomposable to a cube using Lebesgue (resp. Baire) measurable pieces turn out to be sufficient, see Grabowski,

Máthé and Pikhurko [16, Corollary 1.10]. However, nothing like this is known for Borel and Jordan measurable equidecompositions.

1.1. Historical background

Tarski's Circle Squaring Problem has its roots in the theory of paradoxical decompositions. The most famous result in this area is undoubtedly the theorem of Banach and Tarski [2] that, if $k \geqslant 3$ and $A, B \subseteq \mathbb{R}^k$ are bounded and have non-empty interior, then A and B are equidecomposable. As a special case, one obtains the striking Banach–Tarski Paradox that a solid ball in $\mathbb{R}^3$ admits an equidecomposition to two disjoint copies of itself.

The assumption $k \geqslant 3$ is necessary in the Banach–Tarski Theorem, as Banach [1] proved that the Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^2$ can be extended to a finitely additive measure on all subsets which is invariant under isometries. Thus, in $\mathbb{R}^2$, there cannot exist equidecomposable sets of different Lebesgue measure. The theory of amenable groups, pioneered by von Neumann [30], originated as an attempt to obtain a deeper group-theoretic understanding of the change in behaviour between dimensions two and three; the key difference turns out to be that the group of isometries of $\mathbb{R}^k$ is amenable for $k \in \{1, 2\}$ but not for $k \geqslant 3$. Generally, if A and B are Lebesgue measurable subsets of $\mathbb{R}^k$ which are equidecomposable using an amenable subgroup of isometries, then they must have the same measure. Note that the group of translations of $\mathbb{R}^k$ is amenable for all $k \geqslant 1$, and so the condition that $\lambda(A) = \lambda(B)$ is necessary in Theorem 1.1 in general.

Decades before Laczkovich squared the circle, Dubins, Hirsch and Karush [11] proved that a disk is not *scissor-congruent* to a square, meaning that there does not exist a squaring of the circle with pieces that are interior-disjoint Jordan domains, even if boundaries are ignored. This is in strong contrast to the case of polygons: the classical Wallace–Bolyai–Gerwein Theorem states that it is possible to dissect a polygon into finitely many pieces using straight lines and, ignoring boundaries, reassemble them to form any polygon of the same area; see e.g., [36, pp. 34–35]. Another related result, by Gardner [14], asserts that there is no solution to the Circle Squaring Problem using isometries from a locally discrete subgroup of isometries.

For more background, we refer the reader to the monograph of Tomkowicz and Wagon [36], whose Chapter 9 is dedicated to circle squaring.

1.2. Some ideas behind the proof of Theorem 1.3

Let us give a very high-level outline of the proof of Theorem 1.3; all formal definitions will appear later in the paper. A more detailed sketch of the partial result that all pieces can be made Jordan measurable is given in Section 3.

Like in the previous work, we assume that A and B are subsets of the torus $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$ and do all translations modulo 1. We pick a suitable (somewhat large) integer d and vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d \in \mathbb{T}^k$ satisfying certain conditions (that are satisfied with

positive probability by random vectors). Let G_d be the graph on $\mathbb{T}^k$ where we connect $\boldsymbol{u}$ to $\boldsymbol{u}+\sum_{i=1}^d n_i\boldsymbol{x}_i$ for each non-zero $(n_1,\dots,n_d)\in\{-1,0,1\}^d$. Assuming that $\boldsymbol{x}_1,\dots,\boldsymbol{x}_d$ do not satisfy any linear dependencies with rational coefficients, each component of G_d is a (3^d-1)-regular graph on a copy of $\mathbb{Z}^d$.

Our aim is to "construct" a bijection from A to B such that, for some constant r, each element of A is moved by the bijection by distance at most r within the graph G_d. Such a bijection naturally gives an equidecomposition between A and B that uses at most $(2r+1)^d$ pieces. As was observed by Marks and Unger [28], the problem of finding such a bijection can be reduced to finding a uniformly bounded integer-valued flow within the graph G_d, where the demand is 1 on A, -1 on B and 0 elsewhere (see Lemma 2.16 here).

As one of the first steps of their proof, Marks and Unger [28] constructed a real-valued (i.e. not necessarily integer-valued) flow f_∞ which satisfies this demand; see Lemma 3.1. The flow f_∞ is defined to be a pointwise limit of a sequence of flows f_m that are locally constructed from A and B. Since the collection of subsets of $\mathbb{T}^k$ with boundary of upper Minkowski dimension at most $k-\zeta$ is not a σ-algebra, we should not use the values of f_∞ if we want to produce pieces with this structure. Instead, we work with the locally defined approximations f_m.

For flow rounding (that is, making all flow values integral), we construct Jordan measurable subsets $J_1,J_2,\dots$ of $\mathbb{T}^k$ such that their union $\bigcup_{i=1}^\infty J_i$ is co-null in $\mathbb{T}^k$ and $(J_i)_{i=1}^\infty$ is a *toast sequence* (see Definition 2.14), roughly meaning that each J_i induces only finite (in fact, uniformly bounded) components in G_d and the graph boundaries of all components arising this way are well separated from each other. In fact, each set J_i is a finite union of *strips*, i.e., sets of the form $[a,b)\times[0,1)^{k-1}$; in particular, it is Borel and its boundary is $(k-1)$-dimensional. The idea of using toast sequences to construct satisfying assignments was previously applied to many problems in descriptive combinatorics (with the exact definition of "toast sequence" often being problem-specific). For a systematic treatment of this idea for general actions of $\mathbb{Z}^d$, we refer the reader to Grebík and Rozhoň [17].

We can view the toast sequence $(J_1,J_2,\dots)$ as a process where, at time i, vertices of the set J_i arrive and our algorithm has to decide the value of the final integer flow f on every edge with at least one vertex in this set. We are not allowed to look into the future nor modify any already defined values of the flow f. We prove that, if all things are set up carefully, then this is indeed possible to do and, in fact, there are some constants m_i and R_i such that the value of f on any edge $xy\in E(G_d)$ intersecting J_i can be computed only from the current picture in the R_i-neighbourhood of $\{x,y\}$ in G_d and the values of the approximation f_{m_i} of f_∞ there. Here, a key challenge is that, when we round the flow on J_i, we have only incomplete information (namely, the flow f_{m_i} which meets the demands only within some small error). The idea that allows us to overcome this difficulty is that, if the cumulative error of f_{m_i} on each component of J_i is small, then whenever our algorithm encounters some inconsistency, it can round it to the nearest integer and produce values that are in fact perfectly compatible with all past and future choices of the algorithm.

The proof coming from the above arguments, with a careful choice of how the size of the components of J_i can grow with i, produces a partial equidecomposition between A and B so that the topological boundaries of the pieces as well as the unmatched part of A and of B have upper Minkowski dimension less than k. Thus if we can extend this equidecomposition to all of A and B (even by using the Axiom of Choice), then we can achieve Part (a) of Theorem 1.3.

However, these ideas do not seem to be enough to yield pieces which satisfy Parts (a) and (b) of the theorem simultaneously. If A and B are Borel, then all pieces in the partial equidecomposition are Borel, but we did not see a way to match the remaining parts of A and B in a Borel way by using the results of Marks and Unger [28] as a "black box". So, in order to achieve Part (b) of Theorem 1.3, we essentially run the proof from [28] on the complement of $\bigcup_{i=1}^{\infty} J_i$, making sure that, when we define the sets J_i in the first place, we leave enough remaining "wiggle" space. Running the Borel proof on the remaining set is probably the most technical part of this paper. This extra work is worth the effort though, since our modification of both the construction and analysis of Marks and Unger [28] also allows to reduce the Borel complexity of the obtained pieces.

For Part (c), we choose $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d \in \mathbb{T}^k$ so that, additionally, for every vector $\boldsymbol{t}$ which is an integer linear combination of them, the intersection $(A + \boldsymbol{t}) \cap B$ is null only if it is empty. Recall that our equidecomposition translates each $\boldsymbol{u} \in A$ by a vector of the form $\sum_{i=1}^{d} n_i \boldsymbol{x}_i$ where $n_1, n_2, \dots, n_d$ are integers between $-r$ and r, for some r. Denote by $\boldsymbol{t}_1, \dots, \boldsymbol{t}_N$ the vectors $\boldsymbol{t}$ of this form for which $(A + \boldsymbol{t}) \cap B$ is non-empty (and thus non-null). To achieve Part (c), for each $1 \leqslant i \leqslant N$, we pre-select a small non-null subset $A_{\boldsymbol{t}_i}$ of $(B - \boldsymbol{t}_i) \cap A$ to be a part of the piece of the equidecomposition that is translated according to $\boldsymbol{t}_i$. This is done in such a way that the sets $A_{\boldsymbol{t}_1} + \boldsymbol{t}_1, \dots, A_{\boldsymbol{t}_N} + \boldsymbol{t}_N$ are pairwise disjoint and our proof of Parts (a) and (b) still applies to the remaining sets $A \setminus \bigcup_{i=1}^{N} A_{\boldsymbol{t}_i}$ and $B \setminus \bigcup_{i=1}^{N} (A_{\boldsymbol{t}_i} + \boldsymbol{t}_i)$, using the same translation vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_N$. Thus, all pieces are non-null by construction.

1.3. Organisation

As Theorem 1.2 is a direct consequence of Theorem 1.3 and the bound on ζ in (7.11), all our focus will be on proving Theorem 1.3.

In Section 2, we present some notation, introduce a few of the tools used in the paper and prove some auxiliary results. In particular, we present the reduction of Marks and Unger [28] that it is enough to construct a bounded integer-valued flow f from A to B in G_d with appropriate properties (Lemma 2.16). This lemma will also take care of Part (c) of the theorem.

Given these preliminaries, we can give a more detailed outline of the proof of Theorem 1.3 in Section 3. There, we will mostly concentrate on the (less technical) special case of making the pieces just Jordan measurable. We will conclude that section by discussing some of the additional ideas which allow us to control the dimension of the boundaries and the Borel complexity of the pieces.

Then we turn our attention to proving Theorem 1.3 in earnest. In Section 4, we describe the procedure from [28] for constructing a real-valued flow from A to B in the graph G_d. In Section 5, we construct the toast sequences that are needed in the proof. In Section 6 we show how to transform a converging sequence of real-valued flows into a bounded integer-valued flow using a toast sequence. In Section 7, we define our final flow f (that produces the required equidecomposition via Lemma 2.16) and analyse the upper Minkowski dimension of the boundaries of the pieces (in Section 7.1) as well as the Borel complexity of the pieces (in Section 7.2).

Even though many of our auxiliary results can be strengthened or generalised in various ways, we try to present just the simplest versions that suffice for proving Theorem 1.3, since its proof is already very intricate and delicate. In fact, we sometimes present a separate (sketch) proof of a special case before giving the general proof, when we believe that this improves readability.

2. Preliminaries

We denote by $\mathbb{R}$ and $\mathbb{Z}$ the sets of reals and integers, respectively. For $x \in \mathbb{R}$, we define its *nearest integer* $[x]$ to be equal to $\lfloor x \rfloor$ if $x - \lfloor x \rfloor < 1/2$ and $\lceil x \rceil$ otherwise. The set of all values assumed by a function f is denoted by $\mathrm{range}(f)$. When we write e.g., $(a_i, b_i)_{i=1}^{\infty}$, we mean the sequence $(a_1, b_1, a_2, b_2, \ldots)$. The indicator function of a set X is denoted by $\mathbb{1}_X$. By log we mean the natural logarithm.

2.1. Discrepancy bounds for the torus

The *k-torus* $\mathbb{T}^k$ is the quotient group of $(\mathbb{R}^k, +)$ modulo $\mathbb{Z}^k$. When we write the sum of $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{T}^k$, we take it modulo 1 and thus $\boldsymbol{u} + \boldsymbol{v}$ is an element of $\mathbb{T}^k$. In particular, the translation of $Y \subseteq \mathbb{T}^k$ by a vector $\boldsymbol{t} \in \mathbb{T}^k$,

$$Y + \boldsymbol{t} := \{\boldsymbol{y} + \boldsymbol{t} : \boldsymbol{y} \in Y\} \subseteq \mathbb{T}^k,$$

is done modulo 1. Similarly, scalar multiples of vectors in $\mathbb{T}^k$ are also taken modulo 1.

For notational convenience, we may occasionally identify $\mathbb{T}^k$ with $[0,1)^k$. While this identification is not a topological homeomorphism, it does preserve the set families that are of interest to us, namely, sets that are Borel, are Jordan measurable, have boundary of upper Minkowski dimension at most $k-\zeta$, etc. The restriction of the Lebesgue measure on $\mathbb{R}^k$ to the torus will be denoted by the same symbol λ. We have $\lambda(\mathbb{T}^k) = 1$.

Given a finite set $F \subseteq \mathbb{T}^k$ and a Lebesgue measurable set $X \subseteq \mathbb{T}^k$, the *discrepancy of F relative to X* is defined to be

$$D(F, X) := \big|\, |F \cap X| - |F| \cdot \lambda(X) \,\big|. \tag{2.1}$$

In other words, $D(F, X)$ is the deviation between $|F \cap X|$ and the expected size of this intersection if F were a uniformly random subset of $\mathbb{T}^k$ of cardinality $|F|$.

A key tool used in this paper, as well as in [15,28], is the following discrepancy lemma of Laczkovich [23], whose proof sketch can be found in Appendix A.

Lemma 2.1 *(Laczkovich [23, Proof of Theorem 3]; see also [15, Lemma 2.4]). Let X be a measurable subset of $\mathbb{T}^k$ such that $k-1 \leqslant \dim_\square(\partial X) < k$, let d be a positive integer such that*

$$d > \frac{k}{k - \dim_\square(\partial X)}$$

and let $\varepsilon \in \mathbb{R}$ be such that

$$0 < \varepsilon < \frac{d(k - \dim_\square(\partial X)) - k}{k}.$$

If $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ are chosen uniformly at random from $\mathbb{T}^k$ and independently of one another then, with probability 1, there exists $c > 0$ such that, for every $\boldsymbol{u} \in \mathbb{T}^k$ and every integer $n \geqslant 0$, we have

$$D\left(N_n^+[\boldsymbol{u}], X\right) \leqslant c \cdot (n+1)^{d-1-\varepsilon},$$

where we define

$$N_n^+[\boldsymbol{u}] := \left\{ \boldsymbol{u} + \sum_{i=1}^{d} n_i \boldsymbol{x}_i : (n_1, \dots, n_d) \in \{0, \dots, n\}^d \right\}, \tag{2.2}$$

calling it a discrete $(n+1)$-cube *(or just a* discrete cube*).*

Essentially, this lemma says that, if a set $X \subseteq \mathbb{T}^k$ has "small" topological boundary, then, for d sufficiently large, the discrepancy of any large discrete cube given by typical vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d \in \mathbb{T}^k$ with respect to X is significantly smaller than the number of points on the combinatorial boundary of the cube.

2.2. Graph-theoretic definitions

A (simple undirected) *graph* is a pair $G = (V, E)$ where the elements of V are called *vertices* and E is a collection of unordered pairs $\{u, v\}$ of vertices called *edges*. For brevity, an edge $\{u, v\} \in E$ is written uv or, equivalently, vu. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

A vertex u is said to be *adjacent to* (or *a neighbour of*) a vertex v if $uv \in E(G)$. Given a set $S \subseteq V(G)$, the *subgraph of G induced by S*, denoted as $G \restriction S$, is the graph with vertex set S and edge set $\{uv \in E(G) : u, v \in S\}$.

Given a graph G and $u, v \in V(G)$, we let $\mathrm{dist}_G(u, v)$ denote the *graph distance from u to v in G*, i.e., the fewest number of edges in a path from u to v in G. If no such path

exists, then $\mathrm{dist}_G(u,v) := \infty$. Also, note that $\mathrm{dist}_G(u,u) = 0$ for every vertex u. For sets $S, T \subseteq V(G)$, we let

$$\mathrm{dist}_G(S,T) := \min\{\mathrm{dist}_G(u,v) : u \in S \text{ and } v \in T\}$$

and, for $w \in V(G)$, we write $\mathrm{dist}_G(w,T)$ to mean $\mathrm{dist}_G(\{w\},T)$. Given $u \in V(G)$, the *connected component of G containing u* is the set

$$[u]_G := \{v \in V(G) : \mathrm{dist}_G(u,v) < \infty\}.$$

We say that G is *connected* if $[u]_G = V(G)$ for every (equivalently, some) vertex $u \in V(G)$.

Given a graph G and $u \in V(G)$, let

$$N_G[u] := \{v \in V(G) : \mathrm{dist}_G(u,v) \leqslant 1\}$$

denote the *(closed) neighbourhood of u in G*. Also, for a set $S \subseteq V(G)$, let $N_G[S] := \bigcup_{u\in S} N_G[u]$ be the *(closed) neighbourhood of S in G*. The *degree* of a vertex $u \in V(G)$ is defined to be

$$\deg_G(u) := |N_G[u] \setminus \{u\}| = |N_G[u]| - 1.$$

We say that G is *locally finite* if $\deg_G(u)$ is finite for every $u \in V(G)$. For $d \in \mathbb{N}$, we say that G is *d-regular* if $\deg_G(u) = d$ for all $u \in V(G)$.

The *edge boundary* of $S \subseteq V(G)$ in G is the set

$$\partial_E S := \{uv \in E(G) : u \in S \text{ and } v \in V(G) \setminus S\}. \tag{2.3}$$

2.3. Network flows

A *flow* in a graph G is a function $f : V(G) \times V(G) \to \mathbb{R}$ such that

$$f(u,v) = -f(v,u) \text{ for all } u,v \in V(G), \text{ and}$$
$$f(u,v) = 0 \text{ if } uv \notin E(G).$$

The quantity $f(u,v)$ is called the *flow from u to v under f*. Given a finite set $S \subseteq V(G)$, the *flow out of S under f* is defined to be

$$f^{\mathrm{out}}(S) := \sum_{\substack{u\in S \\ v\in V(G)\setminus S}} f(u,v).$$

Note that, in this paper, we will deal with locally finite graphs only, and so the flow out of a finite set will always be well-defined. For a vertex $u \in V(G)$, the *flow out of u under*

f is $f^{\text{out}}(u) := f^{\text{out}}(\{u\})$. The following is an easy consequence of the definition of a flow.

Observation 2.2. *Given a locally finite graph G, a flow f in G and a finite set $S \subseteq V(G)$, it holds that $f^{\text{out}}(S) = \sum_{u\in S} f^{\text{out}}(u)$.* □

When it is clear from the context, we may view f^{out} just as a function $V(G) \to \mathbb{R}$; for example, when we write $\|f^{\text{out}} - g\|_\infty$ for some $g : V(G) \to \mathbb{R}$, we mean the supremum of $|f^{\text{out}}(u) - g(u)|$ over $u \in V(G)$.

Given a function $\chi : V(G) \to \mathbb{R}$, which we call a *demand function*, a *χ-flow in G* is a flow f such that $f^{\text{out}}(u) = \chi(u)$ for every $u \in V(G)$. For $S, T \subseteq V(G)$, a *flow from S to T in G* is a $(\mathbb{1}_S - \mathbb{1}_T)$-flow in G.

It is well-known that, if χ is an integer-valued demand function for a finite graph G, then a real-valued χ-flow in G can be converted into an integer-valued χ-flow in G by changing the flow along each edge by less than 1. This is known as the Integral Flow Theorem. It seems to have been first observed by Dantzig and Fulkerson [9] using ideas of Dantzig [8]; see [33, Corollary 10.3a] and the discussion in [33, p. 64]. The following version can be derived from the finite case via a standard compactness (i.e., Axiom of Choice) argument; for a proof see e.g., [28, Corollary 5.2].

Theorem 2.3 *(Integral Flow Theorem). Let G be a locally finite graph and let $\chi : V(G) \to \mathbb{Z}$ be an integer-valued demand function. Then, for every χ-flow g in G, there exists an integer-valued χ-flow f in G with $|f(u, v) - g(u, v)| < 1$ for all $u, v \in V(G)$.* □

2.4. The setting

At this point, we are ready to make some key definitions and assumptions that will apply throughout the rest of the paper.

Let $k \geqslant 1$ be integer, and let A and B be sets that satisfy the assumptions of Theorem 1.3. Thus $A, B \subseteq \mathbb{R}^k$ are bounded sets that have the same positive Lebesgue measure (i.e., $\lambda(A) = \lambda(B) > 0$) and boundary of the upper Minkowski dimension less than k (i.e., $\dim_\square(\partial A) < k$ and $\dim_\square(\partial B) < k$).

By scaling A and B by the same factor, we can assume that each of them has diameter less than $1/2$ in the ℓ_∞-norm. Furthermore, by translating them, we can assume that A and B are disjoint subsets of $[0, 1)^k$. If A and B are equidecomposable using translations inside $\mathbb{T}^k$, then they are also equidecomposable using translations in $\mathbb{R}^k$ with exactly the same pieces (by the diameter assumption). From now on, we always assume that we are working in the setting of the k-torus.

In order to satisfy Lemma 2.1 and to optimise the bound on the upper Minkowski dimension of the boundaries of the final pieces, we make the following assignments. First, fix ϵ so that

$$0 < \epsilon < k - \max\{\dim_\square(\partial A), \dim_\square(\partial B)\}, \tag{2.4}$$

where we should choose ϵ close to the upper bound for optimality. Then we set

$$d := \lfloor k/\epsilon \rfloor + 1, \tag{2.5}$$

that is, d is the smallest integer greater than k/ϵ. We also define

$$\varepsilon := (d\epsilon - k)/k. \tag{2.6}$$

Note that, if the boundaries of A and B have upper Minkowski dimension $k-1$, as is the case for most simple "geometric" sets like balls and cubes, then we can take $d = k+1$ and let ϵ and ε be close to 1 and $1/k$, respectively.

For each integer $i \geqslant 0$, define

$$r'_i := 100^{2^{i+1}-1} \tag{2.7}$$

and, for each integer $i \geqslant 1$, define

$$r_i := 100^{2^{i+1}-2}. \tag{2.8}$$

Next, we recursively define $q'_0 < q_1 < q'_1 < q_2 < \dots$ and $t_1 < t'_1 < t_2 < t'_2 < \dots$ by setting $q'_0 := 0$ and, for each $i \geqslant 1$,

$$\begin{aligned} t_i &:= 2r_i + 4q'_{i-1} + 4, \\ q_i &:= t_i + 2q'_{i-1} + 4, \\ t'_i &:= 4r'_i/5 + 2q_i, \\ q'_i &:= t'_i + 2q_i + 4. \end{aligned} \tag{2.9}$$

We fix, for the rest of the paper, some vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d \in \mathbb{T}^k$ that satisfy all of the following properties:

(2.10) the conclusion of Lemma 2.1, with respect to the above d and ε, holds for some $c > 0$ (which is also fixed throughout the paper) for both $X := A$ and $X := B$;

(2.11) the projections of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ on the first coordinate are linearly independent over the rationals;

(2.12) if the extra assumption of Part (c) of Theorem 1.3 (that is, (1.2)) holds, then no integer combination of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ belongs to the "bad" set

$$\{\boldsymbol{t} \in \mathbb{T}^k : (A + \boldsymbol{t}) \cap B \text{ is non-empty and } \lambda\text{-null}\};$$

(2.13) the conclusion of Lemma 7.5 (to be stated in Section 7.1) holds.

This is possible since, if $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ are chosen uniformly at random from $\mathbb{T}^k$ independently of one another, then each of Properties (2.10)–(2.12) holds with probability 1, while Lemma 7.5 is satisfied with positive probability. For example, Property (2.12) holds almost surely since, for each non-zero $(n_1, \dots, n_d) \in \mathbb{Z}^d$, the vector $\sum_{i=1}^d n_i \boldsymbol{x}_i$ is a uniform element of $\mathbb{T}^k$ while the "bad" set has measure 0 by (1.2); also, the zero vector is "good" by our assumption that $A \cap B = \emptyset$.

Remark 2.4. If the reader is interested only in a version of Theorem 1.3 where the conclusion of Part (a) is weakened to requiring that the pieces are just Jordan measurable, then we could have chosen an arbitrary sufficiently fast growing sequence $r'_0 \ll r_1 \ll r'_1 \ll r_2 \ll \dots$ and ignored Property (2.13), which would not be needed. This is why the (somewhat technical) statement of Lemma 7.5 is postponed to Section 7.1, where the dimension of the boundaries of the obtained pieces is analysed.

We denote elements of $\mathbb{Z}^d$ as vectors accented by an arrow, e.g., $\vec{n}$, to distinguish them from vectors in $\mathbb{T}^k$ which are typeset in boldface. For $\vec{n} \in \mathbb{Z}^d$ and $1 \leqslant i \leqslant d$, let n_i denote the i-th coordinate of $\vec{n}$. Consider the action $a : \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ defined by

$$\vec{n} \cdot_a \boldsymbol{u} := \boldsymbol{u} + \sum_{i=1}^d n_i \boldsymbol{x}_i, \quad \text{for } \vec{n} \in \mathbb{Z}^d \text{ and } \boldsymbol{u} \in \mathbb{T}^k.$$

This action is continuous and preserves the measure λ.

Next, we recall the definition of the graph G_d from Subsection 1.2. Namely, the vertex set of G_d is $\mathbb{T}^k$ and the neighbours of $\boldsymbol{u} \in \mathbb{T}^k$ are exactly the vectors $\boldsymbol{u} + \sum_{i=1}^d n_i \boldsymbol{x}_i$ for non-zero $\vec{n} \in \{-1, 0, 1\}^d$. Equivalently, G_d is the $(3^d - 1)$-regular Schreier graph associated to the action a with respect to the symmetric set $\{\vec{n} \in \mathbb{Z}^d : \|\vec{n}\|_\infty = 1\}$. The edge set of G_d, when viewed as a symmetric binary relation on the (standard Borel) space $\mathbb{T}^k$, is a closed and thus Borel set. This means that G_d is a *Borel graph*, a fundamental object of study in *descriptive graph combinatorics* (see e.g., the survey of this field by Kechris and Marks [19]). Note that the action a is free by our choice of the vectors $\boldsymbol{x}_i$, namely by (2.11). Therefore, the subgraph of G_d induced by any component of G_d is nothing else than a copy of $\mathbb{Z}^d$ in which two elements are adjacent if they are at ℓ_∞-distance 1.

We pause to remind the reader of the role of the graph G_d in the equidecomposition.

Remark 2.5. The translation vectors used in the constructed equidecomposition will be of the form $\sum_{i=1}^d n_i \boldsymbol{x}_i$ where $\vec{n} \in \mathbb{Z}^d$ and $\|\vec{n}\|_\infty$ is bounded by a large constant r depending only on A and B (which we do not attempt to estimate). Then an equidecomposition is equivalent to a bijection from A to B where each element of A is matched to an element of B at distance at most r from it in G_d. Given such a matching, each piece of the equidecomposition is indexed by an integer vector in $\{-r, \dots, r\}^d$, where the piece corresponding to $\vec{n}$ is the set of all $\boldsymbol{u} \in A$ such that $\boldsymbol{u}$ is matched to $\boldsymbol{u} + \sum_{i=1}^d n_i \boldsymbol{x}_i \in B$.

2.5. Some auxiliary structures in G_d

Here we present some useful building blocks and definitions. Note that, once d and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ are fixed, the definitions and constructions in this section do not depend on the sets A and B.

At certain points during the construction, we have to make arbitrary "tie-breaking" choices in a consistent manner. In doing so, it is useful to endow $\mathbb{T}^k$ with a partial order in which each component of G_d is totally ordered. We will use the *lexicographic ordering* where, for distinct $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{T}^k$, we write $\boldsymbol{u} \prec_{\text{lex}} \boldsymbol{v}$ to mean $[\boldsymbol{u}]_{G_d} = [\boldsymbol{v}]_{G_d}$ and the first non-zero entry of $\vec{n}$ is positive, where $\vec{n}$ is the (unique) element of $\mathbb{Z}^d$ with

$$\boldsymbol{v} - \boldsymbol{u} = \sum_{i=1}^{d} n_i \boldsymbol{x}_i.$$

This naturally extends to a partial order on $\left(\mathbb{T}^k\right)^t$ for any $t \geqslant 1$ where $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_t) \prec_{\text{lex}} (\boldsymbol{v}_1, \dots, \boldsymbol{v}_t)$ if there exists $1 \leqslant i \leqslant t$ such that $\boldsymbol{u}_i \prec_{\text{lex}} \boldsymbol{v}_i$ and $\boldsymbol{u}_j = \boldsymbol{v}_j$ for all $1 \leqslant j \leqslant i-1$.

Given $\boldsymbol{u} \in \mathbb{T}^k$, we write $N_{G_d}[\boldsymbol{u}]$ simply as $N[\boldsymbol{u}]$. For $n \geqslant 1$, define the *n-neighbourhood* of $\boldsymbol{u}$ to be

$$N_n[\boldsymbol{u}] := \left\{\boldsymbol{v} \in \mathbb{T}^k : \text{dist}_{G_d}(\boldsymbol{u}, \boldsymbol{v}) \leqslant n\right\}.$$

Note that $N_n[\boldsymbol{u}] = N^+_{2n}[\boldsymbol{v}]$, where $\boldsymbol{v} := \boldsymbol{u} - n(\boldsymbol{x}_1 + \ldots + \boldsymbol{x}_d)$ and $N^+_r[\boldsymbol{v}]$ is defined as in (2.2). Thus, $N_n[\boldsymbol{u}]$ (resp. $N^+_n[\boldsymbol{u}]$) is the set of vertices which can be reached from $\boldsymbol{u}$ by taking at most n steps in any (resp. "completely non-negative") directions in G_d. Given a set $S \subseteq \mathbb{T}^k$ and an integer $n \geqslant 1$, let

$$N_n[S] := \bigcup_{\boldsymbol{u} \in S} N_n[\boldsymbol{u}] \quad \text{and} \quad N^+_n[S] := \bigcup_{\boldsymbol{u} \in S} N^+_n[\boldsymbol{u}].$$

We say that a set $Y \subseteq \mathbb{T}^k$ *r-locally depends* on (or is an *r-local function* of) $X_1, \dots, X_m \subseteq \mathbb{T}^k$ if the inclusion of $\boldsymbol{u} \in \mathbb{T}^k$ in Y depends only on the intersection of $X_1, \dots, X_m$ with $N_r[\boldsymbol{u}]$, by which we mean that it depends only on the sequence

$$\left(\left\{\vec{n} \in \mathbb{Z}^d : \|\vec{n}\|_\infty \leqslant r,\, \boldsymbol{u} + n_1\boldsymbol{x_1} + \ldots + n_d\boldsymbol{x_d} \in X_j\right\}\right)_{j=1}^{m}.$$

This is equivalent to Y being some Boolean combination of the sets of form $X_j + n_1\boldsymbol{x}_1 + \ldots + n_d\boldsymbol{x}_d$ for $1 \leqslant j \leqslant m$ and $\vec{n} \in \mathbb{Z}^d$ with $\|\vec{n}\|_\infty \leqslant r$. When the specific value of r is unimportant or clear from context, we simply say that Y *locally depends* on (or is a *local function* of) $X_1, \dots, X_m$. Here is a trivial but very useful observation.

Observation 2.6. *If Y locally depends on some finite sequence of sets from an algebra $\mathcal{A}$ on $\mathbb{T}^k$ which is* invariant *under the action a (that is, $A \pm \boldsymbol{x}_i \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $i \in \{1, \dots, d\}$), then Y also belongs to $\mathcal{A}$.* □

Note that the families of subsets of $\mathbb{T}^k$ that are Jordan measurable or have boundary of upper Minkowski dimension at most some given constant are both examples of a-invariant algebras.

If f is a real-valued flow in G_d, then for $\vec{\gamma} \in \mathbb{Z}^d$ with $\|\vec{\gamma}\|_\infty = 1$ and $\ell \in \mathbb{R}$, we define

$$Z^f_{\vec{\gamma},\ell} := \left\{ \boldsymbol{v} \in \mathbb{T}^k : f\left(\boldsymbol{v}, \vec{\gamma} \cdot_a \boldsymbol{v}\right) = \ell \right\}. \tag{2.14}$$

Observation 2.7. *Given a flow f in G_d and $\vec{\gamma} \in \mathbb{Z}^d$ with $\|\vec{\gamma}\|_\infty = 1$, the sets $Z^f_{\vec{\gamma},\ell}$ for $\ell \in \mathbb{R}$ with $|\ell| \leqslant \|f\|_\infty$ partition $\mathbb{T}^k$. If the flow f assumes finitely many possible values, then only finitely many of these sets are non-empty.* □

A flow f with finite range will be identified with the finite sequence of sets $Z^f_{\vec{\gamma},\ell}$ for all possible choices of $\vec{\gamma}$ and ℓ. Thus we will say that a flow f is an *r-local function* of $X_1, \ldots, X_m$ to mean that each set $Z^f_{\vec{\gamma},\ell}$ is an r-local function of $X_1, \ldots, X_m$. In the other direction, r-local dependence of a set Y on a flow f means r-local dependence on the sequence of the sets $Z^f_{\vec{\gamma},\ell}$ (that is, the membership condition for $\boldsymbol{u} \in \mathbb{T}^k$ to be an element of Y can be determined from the flow values on the edges intersecting $N_r[\boldsymbol{u}]$).

Definition 2.8. Define a *strip* to be a subset of $\mathbb{T}^k$ of the form $[a,b) \times [0,1)^{k-1}$ for some $0 \leqslant a < b \leqslant 1$. The *width* of the strip $[a,b) \times [0,1)^{k-1}$ is defined to be $b - a$.

Note that any translation of a strip by a vector can be written as a union of at most two strips and that any Boolean combination of strips can be written as a union of finitely many disjoint strips. For these reasons, strips are particularly convenient to work with.

Definition 2.9. A set $X \subseteq \mathbb{T}^k$ is said to be *r-discrete (in G_d)* if $\mathrm{dist}_{G_d}(\boldsymbol{x}, \boldsymbol{y}) > r$ for any distinct $\boldsymbol{x}, \boldsymbol{y} \in X$ and *maximally r-discrete (in G_d)* if it is maximal under set inclusion with respect to this property.

Of course, the above properties are not affected when we replace X by any translate. The following fact is a consequence of (2.11), one of our assumptions on $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$.

Observation 2.10. *For any $r \geqslant 1$ there exists $\delta > 0$ such that every set $X \subseteq \mathbb{T}^k$ whose projection onto the first coordinate has diameter at most δ is r-discrete.* □

We will also need the following lemma (whose main proof idea goes back to Kechris, Solecki and Todorcevic [21, Proposition 4.2]).

Lemma 2.11. *For $r \geqslant 0$, let $C_1, \ldots, C_m$ be subsets of $\mathbb{T}^k$ such that every C_i is r-discrete in G_d and $\bigcup_{i=1}^m C_i = \mathbb{T}^k$. Then there exists a maximally r-discrete set X which is an $r(m-1)$-local function of $C_1, \ldots, C_m$.*

Proof. Let $C'_1 := C_1$ and, for $2 \leqslant i \leqslant m$, define

$$C'_i := C_i \setminus N_r \left[\bigcup_{j=1}^{i-1} C'_j \right].$$

Let $X := C'_1 \cup \ldots \cup C'_m$. By construction, and because the sets $C_1, \ldots, C_m$ are r-discrete in G_d, the set X is also r-discrete in G_d. Since the sets $C_1, \ldots, C_m$ cover $\mathbb{T}^k$, it follows that X is maximally r-discrete in G_d.

Since $X = C'_1 \cup \ldots \cup C'_m$, it is enough to show that each C'_i is an $r(i-1)$-local function of $C_1, \ldots, C_i$. We use induction on $i = 1, \ldots, m$. This is true for $i = 1$ since the set $C'_1 = C_1$ is a 0-local function of C_1. For each $2 \leqslant i \leqslant m$, the set C'_i is a Boolean combination of C_i and the sets $C'_1, \ldots, C'_{i-1}$ translated by vectors of the form $\sum_{i=1}^{d} n_i \boldsymbol{x}_i$ where $\|\vec{n}\|_\infty \leqslant r$. So, by induction, C'_i is an $r(i-1)$-local function of $C_1, \ldots, C_i$, as desired. □

The following corollary is easily derived from Observation 2.10 and Lemma 2.11.

Corollary 2.12. *For any $r \geqslant 0$ there exists a set $X \subseteq \mathbb{T}^k$ which is maximally r-discrete in G_d and can be expressed as a union of finitely many disjoint strips in $\mathbb{T}^k$.* □

Definition 2.13. For $S \subseteq \mathbb{T}^k$, let $\mathrm{comp}(S)$ be the collection of all components of $G_d \restriction S$.

The following definition describes the types of structures that we will use to round a sequence of real-valued flows to an integer-valued one. We use the word "toast" which seems to be the standard term for structures of this type now.

Definition 2.14. We say that a sequence $(D_1, D_2, \ldots)$ of subsets of $\mathbb{T}^k$ is a *toast sequence (in G_d)* if the following three conditions are satisfied for all $i \geqslant 1$:

(2.15) the elements of $\mathrm{comp}(D_i)$ are finite and have uniformly bounded cardinality,
(2.16) any two distinct elements of $\mathrm{comp}(D_i)$ are at distance at least 3 in G_d, and
(2.17) for $1 \leqslant j < i$, every $S \in \mathrm{comp}(D_j)$ satisfies that either $N_2[S] \subseteq D_i$ or $\mathrm{dist}_{G_d}(S, D_i) \geqslant 3$.

Let us make a brief contextual remark. A concept which is ubiquitous in descriptive graph combinatorics is the notion of hyperfiniteness. Namely, a Borel graph G is said to be *hyperfinite* if its edge set can be written as an increasing union of edge sets of Borel graphs with finite components; see, e.g., [12, Section 7.2] or [20, Section II.6]. In this context, a Borel toast sequence $(D_1, D_2, \ldots)$ gives a specific type of a hyperfiniteness certificate $(D_1, D_1 \cup D_2, \ldots)$ (or, more precisely, the corresponding sequence of the induced edge sets) for the graph $G_d \restriction \bigcup_{i=1}^{\infty} D_i$; see, e.g., [7,13].

Remark 2.15. Note that our definition of a toast sequence allows *holes* in D_i, that is, finite components of $G_d \restriction (\mathbb{T}^k \setminus D_i)$. (In fact, it even allows D_i to have two holes at distance 2 in G_d, even though this can be shown not occur in the toasts constructed in Section 5.) While it should be possible to get rid of all holes by modifying our constructions in Section 5.2 (as in the approach taken by Marks and Unger [28]), we found it easier instead to write our proofs so that they apply to toast sequences with holes.

2.6. The setting (continuation)

Here we define two sequences of sets, with these definitions applying throughout the paper. Namely, for each $i \geqslant 1$, we fix two sets X_i and Y_i, each of which is a union of finitely many disjoint strips, so that X_i is maximally r_i-discrete and Y_i is maximally r_i'-discrete. This is possible by Corollary 2.12.

For Part (b) of Theorem 1.3, we will need the additional assumption that

$$\operatorname{dist}_{G_d}(\partial Y_i, \partial Y_j) = \infty, \quad \text{for all } i \neq j, \tag{2.18}$$

that is, each component of G_d can intersect the topological boundary of Y_i for at most one value of i. Since each Y_i is a finite union of strips, the projection on the first coordinate of the union of all components of G_d that intersect the topological boundary ∂Y_i is a countable set. Thus, if $Y_1, Y_2, \dots$ are translated by uniformly random vectors in $\mathbb{T}^k$ which are independent of one another and of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$, then, almost surely, the translated countable sets inside $\mathbb{T}^1$ are pairwise disjoint and (2.18) holds.

To achieve Part (a) of Theorem 1.3 with good quantitative bounds on the dimension of the boundaries, we additionally require by our assumption (2.13) that the sets X_i satisfy Lemma 7.6, which basically requires that X_i is an $r_i^{d+1+o(1)}$-local function of a single strip. This extra assumption directly gives an upper bound on the number of strips that make up X_i; also, it will allow us to analyse the boundary of sets which are local functions depending on some sets X_i.

2.7. Equidecompositions from integer-valued flows

We will need the following lemma, whose main idea is inspired by the proof sketch in [28, Remark 6.2]. In the context of Theorem 1.3, this lemma shows that it is enough to find a bounded integer-valued flow f from A to B with the desired regularity properties.

Lemma 2.16. *Let f be a bounded integer-valued flow from A to B in G_d. Then there is an integer R and an equidecomposition between A and B such that each piece is an R-local function of f, A, B and finitely many strips. Moreover, if (1.2) (that is, the extra assumption of Part (c) of Theorem 1.3) holds, then we can additionally require that each piece contains a Lebesgue measurable subset of positive measure.*

Proof. Using Corollary 2.12, let W_r, for each integer $r \geqslant 1$, be a maximally r-discrete subset which is a finite union of strips. For each $\boldsymbol{v} \in \mathbb{T}^k$, let $\eta_r(\boldsymbol{v})$ be the vertex $\boldsymbol{u} \in W_r$ such that $\mathrm{dist}_{G_d}(\boldsymbol{v}, \boldsymbol{u})$ is minimised and, among all vertices of W_r at the minimum distance from $\boldsymbol{v}$, the vertex $\boldsymbol{u}$ comes earliest under $\prec_{\mathrm{lex}}$. For each $r \geqslant 1$ and $\boldsymbol{u} \in W_r$, let

$$V_r(\boldsymbol{u}) := \{\boldsymbol{v} \in \mathbb{T}^k : \eta_r(\boldsymbol{v}) = \boldsymbol{u}\}.$$

The sets $V_r(\boldsymbol{u})$ for $\boldsymbol{u} \in W_r$ clearly partition $\mathbb{T}^k$. They can be thought of as "Voronoi cells" generated by W_r with respect to the graph distance in G_d, where ties are broken using $\prec_{\mathrm{lex}}$. Since W_r is a maximal r-discrete set, every element of $V_r(\boldsymbol{u})$ is at distance at most r from $\boldsymbol{u} \in W_r$ and thus the diameter of every Voronoi cell is at most $2r$.

Since W_r is r-discrete, we have

$$N_{\lfloor r/2 \rfloor}[\boldsymbol{u}] \subseteq V_r(\boldsymbol{u})$$

for every $\boldsymbol{u} \in W_r$. Combining this with Lemma 2.1, we see that

$$\min_{\boldsymbol{u} \in W_r} \min\{\, |V_r(\boldsymbol{u}) \cap A|, |V_r(\boldsymbol{u}) \cap B| \,\} = \Omega(r^d)$$

as $r \to \infty$. It is not hard to argue (see, e.g., the proof of Lemma 5.13) that $|\partial_E V_r(\boldsymbol{u})| = O(r^{d-1})$ as $r \to \infty$ where the implicit constant depends on d only. (Recall that ∂_E denotes the edge boundary of a set, as defined in (2.3).) Thus, if r is sufficiently large with respect to $\|f\|_\infty$, then

$$\min\{\, |V_r(\boldsymbol{u}) \cap A|, |V_r(\boldsymbol{u}) \cap B| \,\} \geqslant \sum_{\boldsymbol{v}\boldsymbol{w} \in \partial_E V_r(\boldsymbol{u})} |f(\boldsymbol{v}, \boldsymbol{w})| \tag{2.19}$$

for every $\boldsymbol{u} \in W_r$. We fix r large enough so that (2.19) holds for every $\boldsymbol{u} \in W_r$; to achieve the "moreover" part of the lemma, we will need the slightly stronger inequality (2.20) stated later.

For each pair $\boldsymbol{u}, \boldsymbol{u}' \in W_r$, define

$$F(\boldsymbol{u}, \boldsymbol{u}') := \sum_{\substack{\boldsymbol{v} \in V_r(\boldsymbol{u}) \\ \boldsymbol{w} \in V_r(\boldsymbol{u}')}} f(\boldsymbol{v}, \boldsymbol{w}),$$

that is, $F(\boldsymbol{u}, \boldsymbol{u}')$ is the total flow sent by f from the Voronoi cell of $\boldsymbol{u}$ to that of $\boldsymbol{u}'$. Define $A(\boldsymbol{u}, \boldsymbol{u}') := \emptyset$ for every pair of distinct $\boldsymbol{u}, \boldsymbol{u}' \in W_r$ with $F(\boldsymbol{u}, \boldsymbol{u}') = 0$. Given $\boldsymbol{u} \in W_r$, there are finitely many $\boldsymbol{u}' \in W_r \setminus \{\boldsymbol{u}\}$ for which $F(\boldsymbol{u}, \boldsymbol{u}') \neq 0$. For every such $\boldsymbol{u}'$, one by one in order prescribed by $\prec_{\mathrm{lex}}$, we define $A(\boldsymbol{u}, \boldsymbol{u}')$ to be the set of those $\max\{0, F(\boldsymbol{u}, \boldsymbol{u}')\}$ elements of $V_r(\boldsymbol{u}) \cap A$ that have not already been assigned to $A(\boldsymbol{u}, \boldsymbol{u}'')$ for some $\boldsymbol{u}'' \prec_{\mathrm{lex}} \boldsymbol{u}'$ and, subject to that, are minimal under $\prec_{\mathrm{lex}}$. Note that, by (2.19), this is always possible. Similarly, we define $B(\boldsymbol{u}, \boldsymbol{u}')$ to be the set of those

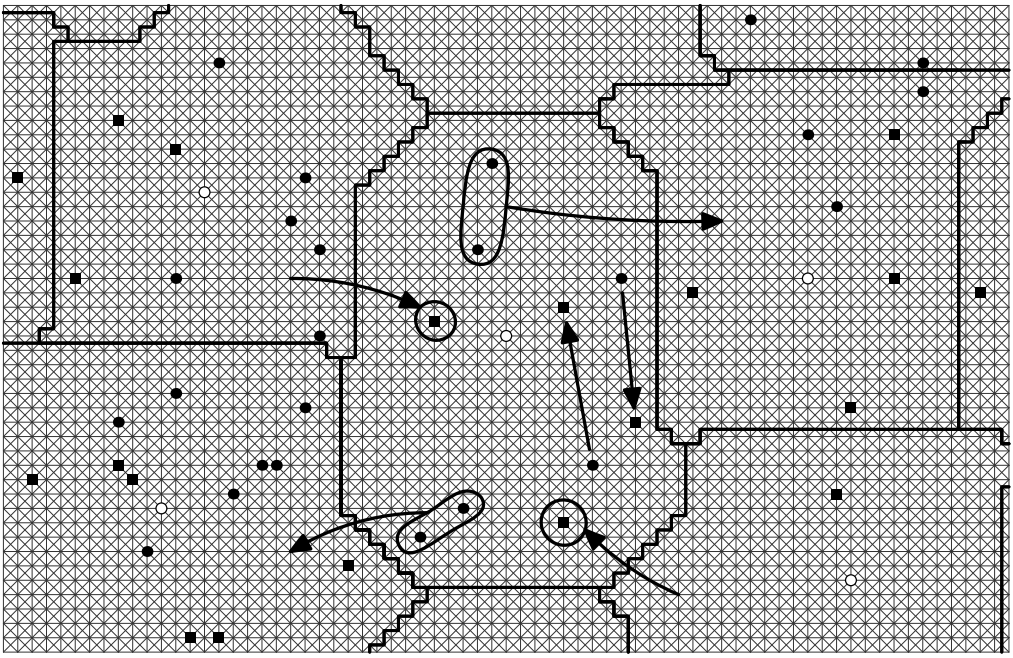

Fig. 1. An illustration of the construction of the equidecomposition from an integer-valued flow f in Lemma 2.16 in the simplified setting $d = 2$. White nodes are elements of a maximally 19-discrete set and the boundaries of the induced Voronoi cells are in bold. Elements of A and B correspond to black round and square nodes, respectively. The total flow from the central Voronoi cell to the bottom-left cell is 2; hence the first two elements of A in the lexicographic order in the central Voronoi cell are associated to that cell. After distributing elements of A and B to the neighbouring cells, the two remaining (lexicographically latest) elements of A and of B in the central cell are mapped to one another.

$\max\{0, -F(\boldsymbol{u}, \boldsymbol{u}')\}$ $\prec_{\mathrm{lex}}$-minimal elements of $V_r(\boldsymbol{u}) \cap B$ that have not already been assigned to $B(\boldsymbol{u}, \boldsymbol{u}'')$ for some $\boldsymbol{u}'' \prec_{\mathrm{lex}} \boldsymbol{u}'$. Finally, for each $\boldsymbol{u} \in W_r$, define $A(\boldsymbol{u}, \boldsymbol{u})$ to be the set of those vertices in $V_r(\boldsymbol{u}) \cap A$ that have not been assigned to $A(\boldsymbol{u}, \boldsymbol{u}')$ for any $\boldsymbol{u}' \in W_r \setminus \{\boldsymbol{u}\}$ and define $B(\boldsymbol{u}, \boldsymbol{u})$ similarly. We have by construction that

$$|A(\boldsymbol{u}, \boldsymbol{u}')| = |B(\boldsymbol{u}', \boldsymbol{u})|, \quad \text{for all distinct } \boldsymbol{u}, \boldsymbol{u}' \in W_r,$$

and this holds also for $\boldsymbol{u} = \boldsymbol{u}'$ since f is a flow from A to B. The final equidecomposition assigns, for all $\boldsymbol{u}, \boldsymbol{u}' \in W_r$, the vertices of $A(\boldsymbol{u}, \boldsymbol{u}')$ to those of $B(\boldsymbol{u}', \boldsymbol{u})$ in the order prescribed by $\prec_{\mathrm{lex}}$; see Fig. 1 for an illustration.

Since each cell has diameter at most $2r$, this yields an equidecomposition by translations in which the translation vectors are of the form $\sum_{i=1}^{d} n_i \boldsymbol{x}_i$ with $\|\vec{n}\|_\infty \leqslant 4r + 1$. Thus the total number of pieces is finite. Furthermore, the vector by which we translate any $\boldsymbol{v} \in V_r(\boldsymbol{u}) \cap A$ depends only on the situation inside $V_r(\boldsymbol{u})$, its adjacent Voronoi cells and their adjacent Voronoi cells, and the values of f at their boundary edges. All these cells are contained entirely inside $N_{6r+2}[\boldsymbol{v}]$, and so the statement holds with $R := 6r + 2$.

The "moreover" part of the lemma is only needed for Part (c) of Theorem 1.3. Its proof may be skipped by the reader interested only in the other parts of Theorem 1.3. In

brief, the new idea is to arrange that the piece corresponding to each translation vector $\boldsymbol{t} \in \mathbb{T}^k$ that is used in the equidecomposition is pre-assigned a small non-null subset $A_{\boldsymbol{t}} \subseteq A$. Of course, this is possible only if $(A+\boldsymbol{t}) \cap B$ has positive measure, so we have to exclude all vectors violating this.

Let us provide the details. For each integer $r \geqslant 1$, let W_r, η_r and V_r be as in the proof of the first part. Now, fix an integer r such that, for every $\boldsymbol{u} \in W_r$, it holds that

$$\min\left\{ |V_r(\boldsymbol{u}) \cap A|, |V_r(\boldsymbol{u}) \cap B| \right\} \geqslant \sum_{\boldsymbol{vw} \in \partial_E V_r(\boldsymbol{u})} (|f(\boldsymbol{v},\boldsymbol{w})|+1), \tag{2.20}$$

that is, we require a slightly stronger bound than the one in (2.19).

Let T be the set of all vectors of the form $\boldsymbol{t} = \sum_{i=1}^{d} n_i \boldsymbol{x}_i$ such that $\vec{n} \in \mathbb{Z}^d$ with $\|\vec{n}\|_\infty \leqslant 10r$ and $(A+\boldsymbol{t}) \cap B \neq \emptyset$. Since (1.2) holds, we have by (2.12) that, for every $\boldsymbol{t} \in T$, the set $(A+\boldsymbol{t}) \cap B$ has positive measure. (Recall that the sets A and B satisfying the assumptions of Theorem 1.3 are necessarily Lebesgue measurable.)

Claim 2.16.1. *There exists a sequence $(A_{\boldsymbol{t}})_{\boldsymbol{t}\in T}$ of pairwise disjoint measurable non-null subsets of A such that the union $\bigcup_{\boldsymbol{t}\in T} A_{\boldsymbol{t}}$ is $(100r)$-discrete in G_d and, for all $\boldsymbol{t} \in T$, $A_{\boldsymbol{t}} + \boldsymbol{t} \subseteq B$.*

Proof. We use a simple greedy argument. By Observation 2.10, we can choose γ sufficiently small so that every strip of width at most γ is $(100r)$-discrete. Assume further that γ is chosen small enough so that

$$\gamma(200r+1)^d |T| < \lambda\left((A+\boldsymbol{t}) \cap B\right) = \lambda\left(A \cap (B-\boldsymbol{t})\right)$$

for every $\boldsymbol{t} \in T$. Label the elements of T by $\boldsymbol{t}_1, \dots, \boldsymbol{t}_{|T|}$ in an arbitrary fashion.

Start by taking $S_{\boldsymbol{t}_1}$ to be a strip of width γ such that the intersection

$$A_{\boldsymbol{t}_1} := A \cap (B-\boldsymbol{t}_1) \cap S_{\boldsymbol{t}_1}$$

is non-null. Clearly, the measure of $A_{\boldsymbol{t}_1}$ is at most the measure of $S_{\boldsymbol{t}_1}$, which is γ.

Now, let $2 \leqslant i \leqslant |T|$ and assume that each of the sets $A_{\boldsymbol{t}_1}, \dots, A_{\boldsymbol{t}_{i-1}}$ has measure at most γ. Then, by our choice of γ, the set

$$(A \cap (B-\boldsymbol{t}_i)) \setminus \left(\bigcup_{j=1}^{i-1} N_{100r}\left[A_{\boldsymbol{t}_j}\right] \right)$$

has positive measure. Take any strip $S_{\boldsymbol{t}_i}$ of width γ that has non-null intersection with the above set and let $A_{\boldsymbol{t}_i}$ be this intersection.

The collection $\{A_{\boldsymbol{t}_i} : 1 \leqslant i \leqslant |T|\}$ has all of the desired properties, simply by construction, finishing the proof of the claim. □

Let $(A_{\boldsymbol{t}})_{\boldsymbol{t}\in T}$ be the sequence returned by the claim. Observe that, for any distinct $\boldsymbol{t}, \boldsymbol{t}' \in T$, the sets $A_{\boldsymbol{t}} + \boldsymbol{t}$ and $A_{\boldsymbol{t}'} + \boldsymbol{t}'$ are disjoint; otherwise, by the definition of T, the sets $A_{\boldsymbol{t}}$ and $A_{\boldsymbol{t}'}$ would be at distance at most $20r$ in G_d, contradicting the fact that $\bigcup_{\boldsymbol{t}\in T} A_{\boldsymbol{t}}$ is $(100r)$-discrete in G_d. Thus, the sets $A_{\boldsymbol{t}}$ for $\boldsymbol{t} \in T$ form an equidecomposition of the subset

$$A^* := \bigcup_{\boldsymbol{t}\in T} A_{\boldsymbol{t}}$$

of A to the subset

$$B^* := \bigcup_{\boldsymbol{t}\in T} (A_{\boldsymbol{t}} + \boldsymbol{t})$$

of B using translations in T.

Now, for each $\boldsymbol{t} \in T$ and each vertex $\boldsymbol{v} \in A_{\boldsymbol{t}}$, take a shortest path from $\boldsymbol{v}$ to $\boldsymbol{v} + \boldsymbol{t}$ in G_d, chosen to be $\prec_{\text{lex}}$-minimal among all such paths, and let $f_{\boldsymbol{v}}$ be the (unique) $\{0, \pm 1\}$-valued flow in G_d from $\{\boldsymbol{v}\}$ to $\{\boldsymbol{v} + \boldsymbol{t}\}$ supported on the pairs which form edges of this path. Since the set $\bigcup_{\boldsymbol{t}\in T} A_{\boldsymbol{t}}$ is $(100r)$-discrete, any two such paths are edge disjoint; in fact they are at least $80r$ apart in G_d. Thus the flow

$$f^* := \sum_{\boldsymbol{v}\in\bigcup_{\boldsymbol{t}\in T} A_{\boldsymbol{t}}} f_{\boldsymbol{v}}$$

satisfies $\|f^*\|_\infty \leqslant 1$. Also, letting $f' := f - f^*$, we see that (2.20) implies that, for every $\boldsymbol{u} \in W_r$,

$$\min\{\, |V_r(\boldsymbol{u}) \cap A|, |V_r(\boldsymbol{u}) \cap B| \,\} \geqslant \sum_{\boldsymbol{v}\boldsymbol{w}\in\partial_E V_r(\boldsymbol{u})} |f'(\boldsymbol{v}, \boldsymbol{w})|,$$

since at most one path in the support of f^* can intersect $\partial_E V_r(\boldsymbol{u})$. Clearly, f' is a flow from $A \setminus A^*$ to $B \setminus B^*$ in G_d. By the proof of the first part of the lemma with f' in the place of f, we see that there is an equidecomposition from $A \setminus A^*$ to $B \setminus B^*$ in which each $\boldsymbol{v} \in A \setminus A^*$ is associated to a point in $B \setminus B^*$ which is either in the same Voronoi cell as $\boldsymbol{v}$ or one which neighbours it. In particular, all of the translation vectors used in the equidecomposition are contained in T. Moreover, by making the $\prec_{\text{lex}}$-smallest choices inside each Voronoi cell, we can assume that each piece is an $O(r)$-local function of $f, W_r, A, B, A_{\boldsymbol{t}_1}, \ldots, A_{\boldsymbol{t}_{|T|}}$. In turn, each set $A_{\boldsymbol{t}_i}$ is a $O(r)$-local function of A, B and the strips $S_{\boldsymbol{t}_1}, \ldots, S_{\boldsymbol{t}_{|T|}}$.

Taking, for each $\boldsymbol{t} \in T$, the union of $A_{\boldsymbol{t}}$ and the (possibly empty) piece of the equidecomposition from $A \setminus A^*$ to $B \setminus B^*$ corresponding to $\boldsymbol{t}$ yields an equidecomposition of A to B using translation vectors in T such that every piece is non-null, as required. □

3. Proof outline

With some preliminaries covered by Section 2, we can now provide a more detailed outline of the proof of Theorem 1.3 than the one given in Section 1.2. In order to illustrate how the various arguments fit together, we will first sketch the construction of an equidecomposition with Jordan measurable pieces (which are not necessarily Borel) and then outline how to build upon it to get the full result. Our approach mostly follows the general recipe established by Marks and Unger [28], some aspects of which can be traced back to the ideas of Laczkovich [22–24] and Grabowski, Máthé and Pikhurko [15]. A major challenge here is that the family of Jordan measurable subsets of $\mathbb{T}^k$ is not a σ-algebra (although it is an algebra). Therefore, there are a few key differences in our implementation and analysis of this strategy which are necessary to achieve Jordan measurable pieces.

3.1. Real-valued flows

Following Marks and Unger [28, Section 4], we construct a sequence $f_1, f_2, \dots$ of bounded real-valued flows in G_d which converge uniformly to a bounded real-valued flow f_∞ from A to B. The following lemma summarises the key properties of $f_1, f_2, \dots$ that we will need. Recall that c and ε are fixed quantities which were defined in Section 2.4.

Lemma 3.1 *(Marks and Unger [28]). There exist flows $f_1, f_2, \dots$ in G_d such that for all $m \geqslant 1$ the following statements hold with $f_0 := 0$ being the flow which is identically zero:*

(3.1) $2^{2dm} f_m$ *is integer-valued,*
(3.2) $\|f_m^{\text{out}} - \mathbb{1}_A + \mathbb{1}_B\|_\infty \leqslant \frac{2c}{2^{m(1+\varepsilon)}}$,
(3.3) $\|f_m - f_{m-1}\|_\infty \leqslant \frac{2c}{2^{d+\varepsilon(m-1)}}$, *and*
(3.4) f_m *is a* $(2^m - 1)$*-local function of* A *and* B.

The proof of the lemma follows that of a similar result in [28, Section 4]; we include the proof in Section 4 for completeness. The rough idea is as follows. Suppose that we have fixed a partition of $[\boldsymbol{u}]_{G_d} \cong \mathbb{Z}^d$ into discrete 2^m-cubes. Within each cube Q, we define the flow f_m so that it cancels as much as possible between the positive demand $\mathbb{1}_A$ and the negative demand $-\mathbb{1}_B$, and spreads the rest uniformly over Q. A bit more formally, the restriction of f_m to Q is an $(\mathbb{1}_A - \mathbb{1}_B + \xi(Q))$-flow, where $\xi(Q) := (|A \cap Q| - |B \cap Q|)/|Q|$. Also, f_m is zero on all edges between distinct cubes of the partition. Assuming that the fixed dyadic partitions are aligned for different values of m, we can construct such f_m incrementally from f_{m-1}. As long as we do this increment in a "reasonable" way, the discrepancy bound of Lemma 2.1 gives a good upper bound on $\|f_m - f_{m-1}\|_\infty$. Unfortunately, one cannot take a perfect dyadic decomposition of each component of G_d in a constructive way for a generic choice of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$. However, the convenience of working with flows (versus graph matchings) is that we can always take their convex

combinations. So, in order to make the construction of each f_m local, one can simply take the average over all 2^{dm} possible partitions of each component (a copy of $\mathbb{Z}^d$) into a grid of discrete 2^m-cubes.

Each flow f_m assumes finitely many values by (3.1) and is a local function of the Jordan measurable sets A and B. Since the right hand side of (3.3) is summable, the sequence $f_1, f_2, \dots$ converges uniformly to a bounded flow in G_d, which we denote by f_∞. By (3.2), the limit f_∞ is a flow from A to B.

With Lemma 3.1 in hand, the goal of the next two steps is to transform the sequence $f_1, f_2, \dots$ into a bounded integer-valued flow f from A to B in G_d. Notice that, if we were indifferent about the structure of the pieces of the final equidecomposition, then we could just apply the Integral Flow Theorem to f_∞ to obtain such a flow f and then feed it into Lemma 2.16. In particular, this approach is sufficient for proving Theorem 1.1. However, this would use the Axiom of Choice and yield virtually no structural guarantees on the pieces of the equidecomposition. The main purpose of the next two steps, therefore, is to obtain an integer-valued flow from A to B in a more careful manner which allows us to analyse the obtained pieces.

3.2. Toast sequences

Recall that the notion of a toast sequence was defined in Definition 2.14. Informally, if $(J_i)_{i=1}^\infty$ is a toast sequence, then one can view J_i as a collection of bounded and well separated connected subgraphs of G_d that arrive at time i so that every component S of vertices that arrived in an earlier stage is either entirely inside J_i or entirely outside J_i, including some "padding".

The following lemma provides a simple construction of a toast sequence which covers $\mathbb{T}^k$ up to a null set. (This will be sufficient for our sketch of the construction of an equidecomposition with Jordan measurable pieces.)

Lemma 3.2. *There exists a toast sequence $(J_1, J_2, \dots)$ in G_d such that $\lambda\left(\bigcup_{i=1}^\infty J_i\right) = 1$ and, for each $i \geqslant 1$, the set J_i is a union of finitely many disjoint strips.*

Proof Sketch. The construction is slightly more complicated than is necessary, in order for the sets J_i to be defined in the same way as they will be in the proof of Theorem 1.3.

Recall that $r'_0 < r_1 < r'_1 < r_2 < r'_2 < \dots$ is a rapidly increasing sequence of integers and $X_i \subseteq \mathbb{T}^k$ is a maximally r_i-discrete set which is a union of finitely many strips. Define I_i to be the set of all $\boldsymbol{v} \in \mathbb{T}^k$ for which there exists $\boldsymbol{u} \in X_i$ such that

$$\operatorname{dist}_{G_d}(\boldsymbol{v}, \boldsymbol{u}') \geqslant \operatorname{dist}_{G_d}(\boldsymbol{v}, \boldsymbol{u}) + 5r'_{i-1}, \quad \text{for every } \boldsymbol{u}' \in X_i \setminus \{\boldsymbol{u}\}.$$

Informally speaking, the components of I_i are the Voronoi cells of $X_i \subseteq V(G_d)$, as defined in the proof of Lemma 2.16, except we retract somewhat from their graph boundaries. It is not hard to show that the diameter of each component of I_i is at most $2r_i$ and every two components are at distance at least $5r'_{i-1}$ (see Lemma 5.4).

We use induction on i to define J_i. Fix $i \geqslant 1$ and assume that the sets J_j, for all $j < i$, have already been defined. We obtain J_i as the result of the following procedure. Initialise $J_i := I_i$. Then, while there exist $j < i$ and a component S of J_j such that $\mathrm{dist}_{G_d}(S, J_i) \leqslant q'_{j-1} + 4$ and $N_{q'_{j-1}+4}[S] \not\subseteq J_i$, we add to J_i all elements of $N_{q'_{j-1}+4}[S]$. (Recall that q'_i was defined by (2.9), being slightly larger than $4r'_i/5$.)

Since the sequence $r'_0 < r_1 < r'_1 < \dots$ increases sufficiently rapidly, this procedure eventually terminates and in fact stays within the q'_{i-1}-neighbourhood of I_i (Lemma 5.6). Thus, when we construct J_i from I_i, the original components of I_i do not merge and stay well separated, being at distance at least $5r'_{i-1} - 2q'_{i-1}$ from each other. (This stronger separation property is not needed here, but will be useful in the general proof.) The other properties of a toast sequence from Definition 2.14 hold because, whenever some earlier component S' (that is, $S' \in \mathrm{comp}(J_j)$ with $j < i$) comes too close to a currently defined component S of J_i, we add S' with some padding to S (with the new enlarged set S still being connected).

We trivially have $\lambda(J_i) \geqslant \lambda(I_i)$. On the other hand, the measure of I_i can be shown to be least $1 - O(r'_{i-1}/r_i)$ (see Lemma 5.13), which approaches 1 as $i \to \infty$.

Whether or not a vertex $\boldsymbol{u} \in \mathbb{T}^k$ is contained in J_i can be determined by the structure of the sets $X_1, \dots, X_i$ at bounded distance (say, $4r_i$) from $\boldsymbol{u}$ in G_d (see Lemma 5.11); thus, J_i is a local function of $X_1, \dots, X_i$. So, it can be written as a union of finitely many disjoint strips by Observation 2.6 (since such sets form an a-invariant algebra), finishing our proof sketch. □

3.3. Integer-valued flow

The last remaining step is to use the flows $(f_m)_{m=1}^{\infty}$ from Lemma 3.1 and the toast sequence $(J_i)_{i=1}^{\infty}$ from Lemma 3.2 to construct an integer-valued bounded flow f from A to B that will be used as input to Lemma 2.16 to equidecompose A and B. Unfortunately, the construction of f is somewhat involved, even in the context of Jordan measurable pieces.

Although this is not strictly necessary, we keep the perspective that a new set of vertices J_i arrives at time i and we have to fix for good the value of f on every new edge intersecting J_i, being compatible with all previous choices. It is enough to concentrate on constructing the final values of f on the edges in $\partial_E J_i$, that is, on all new boundary edges. Indeed, since every component S of J_i is finite, there are only finitely many possible extensions of f to a uniformly bounded integer flow inside S and, if at least one exists, then the lexicographically smallest extension is a local function of the boundary flow values, $J_1, \dots, J_i, A$ and B whose radius is at most the diameter of S. A bit later, we will address the problem of certifying the existence of such an extension.

So, take any $S \in \mathrm{comp}(J_i)$, i.e., a component of $G_d \restriction J_i$ which must be finite by the definition of a toast sequence. Initially let f be f_{m_i} on $\partial_E S$, where m_i is sufficiently large integer depending on the maximum diameter of the components of J_i. We repeat the following for every "connected" part $P \subseteq \partial_E S$ of the edge boundary of S. Let S' consist

of the vertices "enclosed" by P; thus, S' is either a hole of S or the set S with all its holes filled. By adjusting the value of f on one edge of P, we make the total flow out of S' to be $|A\cap S'|-|B\cap S'|$. Since $f^{\mathrm{out}}_{m_i}$ is a very good approximation to $\mathbb{1}_A-\mathbb{1}_B$, we need to adjust the flow by less than $1/2$. Using Lemma 6.4, we find a sequence $(\boldsymbol{u}_1\boldsymbol{v}_1,\dots,\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1})$ in P so that each edge in P appears at least once but at most constant number of times while $\boldsymbol{u}_j\boldsymbol{v}_j$ and $\boldsymbol{u}_{j+1}\boldsymbol{v}_{j+1}$ are in a triangle in G_d for each $1\leqslant j\leqslant t-2$. Make f to be integer on $\boldsymbol{u}_1\boldsymbol{v}_1$ by adding a 0-flow (i.e., a flow ϕ such that $\phi^{\mathrm{out}}:V(G_d)\to\mathbb{R}$ is the zero function) on the unique triangle containing both $\boldsymbol{u}_1\boldsymbol{v}_1$ and $\boldsymbol{u}_2\boldsymbol{v}_2$. (Note that the third edge of this triangle is not in $\partial_E S$ and thus we do not care about the flow through it being integral at this stage.) By adding a 0-flow along the triangle containing both $\boldsymbol{u}_2\boldsymbol{v}_2$ and $\boldsymbol{u}_3\boldsymbol{v}_3$, we make the flow value on $\boldsymbol{u}_2\boldsymbol{v}_2$ integer, and so on. We repeat this procedure $t-2$ times in total, ensuring that the final flow f is integer-valued on all edges of P except perhaps $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$. Note that this part of the process does not change the flow out of any vertex under f. So $f^{\mathrm{out}}(S')$ keeps its (integer) value and thus the final flow has to be integer also on $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$. Clearly, the definition of f can be made local by fixing a consistent rule for choosing the adjustment, the edge sequence and the values of the added 0-flows.

If we had used f_∞ as the initial values of f on P, then the initial adjustment step would not be necessary, and the obtained rounding algorithm would be essentially the same as used by Marks and Unger [28]. Unfortunately, if we wish to have Jordan measurable pieces, then we should (and, in fact, we do) avoid using the values of f_∞ (which is the pointwise limit of $f_1,f_2,\dots$) when defining f on $\bigcup_{i=1}^\infty \partial_E J_i$. However, we can freely use f_∞ to define, in a way that parallels the construction of f from f_{m_i}, a real-valued flow which certifies via the Integral Flow Theorem (Theorem 2.3) that the constructed integer flow values on $\bigcup_{i=1}^\infty \partial_E J_i$, can be extended to a bounded integer $(\mathbb{1}_A-\mathbb{1}_B)$-flow on all edges of G_d (see Lemma 6.9 for details). As mentioned above, given that such an extension exists, we can choose the lexicographically minimal one on each finite component of the remaining graph; this includes all edges inside $\bigcup_{i=1}^\infty J_i$. Here, crucially, the choice of the extension is independent of f_∞.

Now, during the construction, each of the (finitely many) sets $Z^f_{\vec{\gamma},\ell}\cap\left(\bigcup_{j=1}^i J_j\right)$ that describe the flow f on edges incident to the vertices which have arrived by time i (where $Z^f_{\vec{\gamma},\ell}$ is as in (2.14)), only grows in time, since the values of f fixed at time j are never overwritten later. These sets are Jordan measurable (as local functions of $f_{m_1},\dots,f_{m_i}$ and $J_1,\dots,J_i$) and their union over all values of i and ℓ covers the co-null subset $\bigcup_{i=1}^\infty J_i$ of $\mathbb{T}^k$. Thus, for every ℓ, the set $Z^f_{\vec{\gamma},\ell}\cap\left(\bigcup_{j=1}^\infty J_j\right)$ is Jordan measurable by the following lemma (applied with $Z:=\mathbb{T}^k$).

Lemma 3.3. *Let $Z_1,\dots,Z_N$ be pairwise disjoint subsets of a Jordan measurable set $Z\subseteq\mathbb{T}^k$. If, for every real $\varepsilon>0$, there are Jordan measurable subsets $Z'_j\subseteq Z_j$ such that $\lambda(Z\setminus\bigcup_{j=1}^N Z'_j)<\varepsilon$, then all of the sets $Z_1,\dots,Z_N$ are Jordan measurable.*

Proof. By adding $\mathbb{T}^k \setminus Z$ as an extra set and increasing N by 1, we can assume that $Z = \mathbb{T}^k$. For each integer $i \geqslant 1$, let $Z_{1,i} \subseteq Z_1, \dots, Z_{N,i} \subseteq Z_N$ be Jordan measurable sets such that $\lambda(Z \setminus \bigcup_{j=1}^{N} Z_{j,i}) < 1/i$ and, for $1 \leqslant j \leqslant N$, we let $U_{j,i}$ be the interior of $Z_{j,i}$. Also, define $U_j := \bigcup_{i=1}^{\infty} U_{j,i}$. Since $Z_{j,i}$ is Jordan measurable, we have that $U_{j,i}$ has the same Lebesgue measure as $Z_{j,i}$. Thus, the sets $U_1, \dots, U_N$ are pairwise disjoint, open and cover $\mathbb{T}^k$ up to a null set. For any $1 \leqslant j \leqslant N$, since U_j is open and contained in Z_j, the boundary of Z_j is disjoint from U_j. Also, for any $j' \neq j$, since the set $U_{j'}$ is open and disjoint from Z_j, we see that the boundary of Z_j is also disjoint from $U_{j'}$. Therefore, for each $1 \leqslant j \leqslant N$, the boundary of Z_j is contained in the complement of $\bigcup_{i=1}^{N} U_i$ and therefore has measure zero. Thus the set Z_j is Jordan measurable. □

It also follows from the above lemma that the remaining set $\mathbb{T}^k \setminus \bigcup_{i=1}^{\infty} J_i$ is Jordan measurable and, being also a null set, has null closure. Thus, without destroying Jordan measurability, we can use the Axiom of Choice to define a suitable integer-valued flow on all edges inside this set, using f_∞ in the same way as above to certify the existence via the Integral Flow Theorem.

Finally, we apply Lemma 2.16 to f to get an equidecomposition between A and B as an R-local function of f (that is, the sets $Z^f_{\vec{\gamma},\ell}$) and finitely many strips; thus, the pieces are Jordan measurable.

Remark 3.4. We could have slightly restructured the proof so that whenever the partial integer-valued flow f is defined in the whole R-neighbourhood of some $\boldsymbol{u} \in A$, where R is the constant returned by Lemma 2.16, then we assign $\boldsymbol{u}$ to the appropriate part of the final equidecomposition. Then the obtained Jordan measurable pieces will be incrementally growing as we process $J_1, J_2, \dots$ one by one, exhausting the set A up to measure zero. Thus, we could have applied Lemma 3.3 directly to the pieces (with $Z := A$).

3.4. Making the pieces Borel

Let us now discuss how the approach to obtaining a Jordan measurable equidecomposition described above can be built upon to obtain the stronger conditions of Theorem 1.3.

The main issue to address is the use of the Axiom of Choice in the final step, which must be avoided in order to yield Part (b) of the theorem. If A and B are Borel, then a natural idea for obtaining simultaneously Jordan and Borel pieces is to try to use the existence of a Jordan measurable equidecomposition together with the equidecomposition of Marks and Unger [28]; however, we did not see a way of doing this directly. Instead, we follow essentially the same proof as outlined above (with all locally defined structures being Borel if the sets A and B are) except that we use a version of Marks and Unger's proof in [28] to round the flow values inside $\mathbb{T}^k \setminus \bigcup_{i=1}^{\infty} J_i$ in a way that preserves Borel structure. The Jordan measurability of the final pieces is preserved since, in this extra stage, we do not change any flow value on any edge intersecting $\bigcup_{i=1}^{\infty} J_i$.

Our extra steps can be summarised as follows. With sets J_i defined as above and fixed for good (where the role of the parameter r'_{i-1} is to provide enough "wiggle space" between the components of J_i), we interleave the sequence $(J_1, J_2, \dots)$ with sets $K_1, K_2, \dots$ to create an augmented toast sequence $(J_i, K_i)_{i=1}^{\infty}$. (Recall that $(J_i, K_i)_{i=1}^{\infty}$ is a shorthand for $(J_1, K_1, J_2, K_2, \dots)$.) Simultaneously, we will construct sets $L_1, L_2, \dots$ so that $(K_i, L_i)_{i=1}^{\infty}$ is also a toast sequence. The purpose of the sets $L_1, L_2, \dots$ will be to cover all of the points of $\mathbb{T}^k$ that are "left behind" by the construction of $J_1, J_2, \dots$ in a Borel way, while the purpose of K_i is to provide some extra "cushioning" around J_i to protect the flow values on edges intersecting J_i from being influenced by L_i. The sequence $(K_1, K_2, \dots)$ can be seen as a mediator between the competing goals of $J_1, J_2, \dots$ (to keep the boundaries of the final pieces small) and $L_1, L_2, \dots$ (to cover all of $\mathbb{T}^k$ in a Borel way).

We build K_i by initialising $K_i := N_2[J_i]$ and then iteratively adding (with an appropriate padding) all components of J_j, K_j or L_j with $j < i$ that come too close. As before, this process when started at any $S \in \mathrm{comp}(J_i)$ goes only a small distance away from S and the obtained component S' of K_i satisfies $S' \subseteq N_{q'_{i-1}+2}[S]$ (Lemma 5.7) while, trivially, $S' \supseteq N_2[S]$. Thus every component S of J_i is "mimicked" by a component S' of K_i. When we construct L_i, we start with $L_i := N_{2r'_i/5}[Y_i]$ (where Y_i is maximally r'_i-discrete) and keep adding (with some padding) components of K_j for $j \leqslant i$ and L_j for $j < i$ that come too close. Note that each component S of J_j with $j \leqslant i$ is "protected" in this process by the component S' of K_j that mimics it: if L_i comes too close to S then $S' \supseteq S$ with some extra padding is added to L_i. What we have achieved is that the toast sequence $(K_i, L_i)_{i=1}^{\infty}$ covers all vertices in $\bigcup_{i=1}^{\infty} L_i \supseteq \bigcup_{i=1}^{\infty} N_{2r'_i/5}[Y_i]$. Using a compactness argument, we will show that, by applying this construction with a modified version of Y_i, we can ensure that all vertices of $\mathbb{T}^k$ are covered; see the discussion following Lemma 5.14.

Unfortunately, $(J_i, K_i, L_i)_{i=1}^{\infty}$ is not, in general, a toast sequence. Also, we do not want any $S \in \mathrm{comp}(K_j) \cup \mathrm{comp}(L_j)$ to "interfere" with what happens inside J_i for some $i > j$. Our solution is, essentially, to remove all such conflicting components S, obtaining sets $\widetilde{K}_j$ and $\widetilde{L}_j$. The new sequence $(J_i, \widetilde{K}_i, \widetilde{L}_i)_{i=1}^{\infty}$ is then a toast sequence that does not break up any finite components coming from the toast sequence $(J_i)_{i=1}^{\infty}$; also, it still covers every vertex of $\bigcup_{i=1}^{\infty} L_i$ (see Lemma 5.10). The proofs of these claims rely on the fact that the components of K_j "mimic" the components of J_j.

We define the desired integer-valued Borel flow f by using the same construction as in Section 3.3, except that we replace the toast sequence $(J_i)_{i=1}^{\infty}$ by $(J_i, \widetilde{K}_i, \widetilde{L}_i)_{i=1}^{\infty}$.

3.5. Reducing/analysing the complexity of the pieces

The idea to use f_{m_i} instead of f_∞ in the construction of the integer-valued flow f is, essentially by itself, enough to save one level of the Borel hierarchy when compared to the proof of Marks and Unger [28]. There are a couple of places where we change their construction or its analysis in order to drop down by another level of Borel complexity.

(In particular, we have to use a different toast sequence $(J_i, \widehat{K}_i, \widehat{L}_i)_{i=1}^{\infty}$, some modification of $(J_i, \widetilde{K}_i, \widetilde{L}_i)_{i=1}^{\infty}$, for the final rounding.) These changes are incorporated in the main construction, while the Borel complexity analysis is postponed until Section 7.2 for the clarity of presentation.

Our approach to bounding the upper Minkowski dimension (defined in (1.1)) of the boundaries of the pieces is, for given $\delta > 0$, to choose some index $i = i(\delta)$ and run our algorithm to process $J_1, \dots, J_i$. Although we do not know the final equidecomposition at this moment, we may nonetheless determine for some elements of A to which final piece they will belong. This defines the current partial pieces $A_1, \dots, A_N$ (that give an equidecomposition of some subset of A to a subset of B). Now, we estimate the number of boxes of a regular grid with side length δ that intersect the currently unassigned part of A or the boundary of any current piece A_j. (Thus, our dimension estimates depend only on $J_1, J_2, \dots$ and not on how $K_1, L_1, K_2, L_2, \dots$ are built around them.) Recall by Lemma 3.2 that each set J_j is a finite union of strips, whose number we can estimate if the sets X_i are carefully chosen (namely, as in Lemma 7.6). Also, we can estimate a radius R so that the current pieces $A_1, \dots, A_N$ can be R-locally determined from A, B and $J_1, \dots, J_i$. Thus each A_j is a Boolean combination of $(2R+1)^d$ translates of each of these sets, and the number of boxes that intersect its boundary ∂A_j can be bounded above by the number of boxes intersecting the boundary of at least one of these translates. Furthermore, every other box that may potentially intersect the boundary of a final piece has to be a subset of $A' := A \setminus (\bigcup_{j=1}^{N} A_j)$ and their number can be upper bounded by $\lambda(A')/\delta^k$ by the trivial volume argument. Thus we have to control our parameters carefully to get a good balance between the minimal distance between the components of J_i (in order to control the measure of the leftover part A' of A) and their maximum diameter (as our local rule processing J_i has to use radius at least as large as this diameter).

4. Proof of Lemma 3.1

Our goal in this section is to present, for the sake of completeness, the (somewhat rephrased) construction of Marks and Unger [28] of a sequence $f_1, f_2, \dots$ of real-valued flows in G_d which converge uniformly to a bounded flow f_∞ from A to B in G_d. For the reader's convenience, let us repeat the statement of the result that we will use.

Lemma 3.1 *(Marks and Unger [28]). There exist flows $f_1, f_2, \dots$ in G_d such that for all $m \geqslant 1$ the following statements hold with $f_0 := 0$ being the flow which is identically zero:*

(3.1) $2^{2dm} f_m$ *is integer-valued,*
(3.2) $\|f_m^{\mathrm{out}} - \mathbb{1}_A + \mathbb{1}_B\|_\infty \leqslant \frac{2c}{2^{m(1+\varepsilon)}}$,
(3.3) $\|f_m - f_{m-1}\|_\infty \leqslant \frac{2c}{2^{d+\varepsilon(m-1)}}$, *and*
(3.4) f_m *is a* $(2^m - 1)$*-local function of* A *and* B.

Proof of Lemma 3.1. First, we need to introduce a few definitions (that apply only in this section). Given a discrete cube Q in G_d (as defined in (2.2)), define

$$\xi(Q) := \frac{|Q \cap A| - |Q \cap B|}{|Q|}.$$

In other words, $\xi(Q)$ measures the difference between the number of vertices of A and B in Q normalised by the number of vertices in Q. In particular, if Q is a discrete 1-cube, i.e., $Q = \{\boldsymbol{u}\}$ for some $\boldsymbol{u} \in \mathbb{T}^k$, then $\xi(Q) = \mathbb{1}_A(\boldsymbol{u}) - \mathbb{1}_B(\boldsymbol{u})$.

By our assumption (2.10), it holds for every discrete 2^m-cube Q that

$$|\xi(Q)| \leqslant \frac{2c(2^m)^{d-1-\varepsilon}}{2^{dm}} = \frac{2c}{2^{m(1+\varepsilon)}}. \tag{4.1}$$

Given $\boldsymbol{u} \in \mathbb{T}^k$, $\vec{\gamma} \in \{-1,0,1\}^d$ and $n \in \mathbb{Z}$, let $\phi_{\boldsymbol{u},n\vec{\gamma}}$ be the unique $\{-1,0,1\}$-valued $\big(\mathbb{1}_{\{\boldsymbol{u}\}} - \mathbb{1}_{\{(n\vec{\gamma})\cdot\boldsymbol{u}\}}\big)$-flow in G_d supported on the edges of the path

$$\boldsymbol{u},\ \vec{\gamma}\cdot\boldsymbol{u},\ (2\vec{\gamma})\cdot\boldsymbol{u},\ \ldots,\ (n\vec{\gamma})\cdot\boldsymbol{u}.$$

In other words, $\phi_{\boldsymbol{u},n\vec{\gamma}}$ sends a unit of flow from $\boldsymbol{u}$ to $(n\vec{\gamma})\cdot\boldsymbol{u}$ along the straight-line path (whose direction is $\vec{\gamma}$ and graph length is $|n|$). In particular, if $n\vec{\gamma}$ is the zero vector, then the flow $\phi_{\boldsymbol{u},n\vec{\gamma}}$ is identically zero. Also, for an integer $m \geqslant 0$ and a discrete 2^{m+1}-cube Q containing $\boldsymbol{u} \in \mathbb{T}^k$, let

$$\phi_{\boldsymbol{u},Q} := \frac{1}{2^d} \sum_{\substack{\vec{\gamma}\in\{-1,0,1\}^d \\ (2^m\vec{\gamma})\cdot\boldsymbol{u}\in Q}} \phi_{\boldsymbol{u},2^m\vec{\gamma}}.$$

Note that the above sum contains exactly 2^d terms. Informally speaking, $\phi_{\boldsymbol{u},Q}$ spreads one unit of demand from $\boldsymbol{u}$ uniformly among $Q \cap ((2^m\mathbb{Z}^d) \cdot_a \boldsymbol{u})$, the set of all 2^d points of Q that, when we view Q as a subset of $\mathbb{Z}^d$, are congruent to $\boldsymbol{u}$ modulo 2^m. For every discrete 2^{m+1}-cube Q, let $\mathcal{P}(Q)$ be the unique partition of Q into 2^d discrete 2^m-cubes. Given $\boldsymbol{u} \in \mathbb{T}^k$, let $\mathcal{Q}_n(\boldsymbol{u})$ denote the set of all discrete n-cubes that contain $\boldsymbol{u}$. See Fig. 2 for an illustration of some of these definitions.

We are now ready to define the flows f_m. We use induction on m and start by defining $f_0 := 0$ to be the identically zero flow. For $m \geqslant 1$, we define $f_m := f_{m-1} + \theta_m$, where

$$\theta_m := \frac{1}{2^{dm}} \sum_{\boldsymbol{u}\in\mathbb{T}^k} \sum_{C\in\mathcal{Q}_{2m-1}(\boldsymbol{u})} \xi(C) \sum_{\substack{\text{discrete } 2^m\text{-cube } Q \\ \mathcal{P}(Q)\ni C}} \phi_{\boldsymbol{u},Q}. \tag{4.2}$$

Let us provide a probabilistic interpretation that motivates the definition in (4.2) and relates it to the proof outline of Section 3.1. Suppose that we wish to compute θ_m and f_m inside the component $[\boldsymbol{v}]_{G_d}$ of some vertex $\boldsymbol{v} \in \mathbb{T}^k$. Take a random partition $\mathcal{G}_m$ of

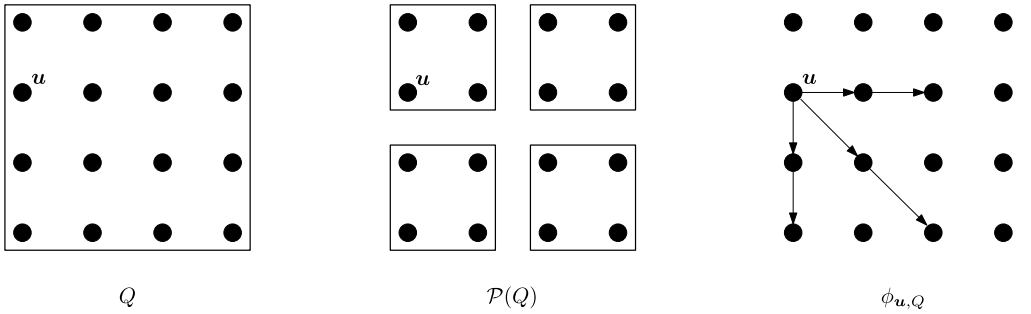

Fig. 2. An example, for $d = 2$, of a 4-cube Q, its dyadic partition $\mathcal{P}(Q)$ into 2-subcubes and the flow $\phi_{\boldsymbol{u},Q}$ for a vertex $\boldsymbol{u} \in Q$ (where each arrow represents a flow of value 1/4 in that direction).

the orbit $[\boldsymbol{v}]_{G_d}$, which is a copy of $\mathbb{Z}^d$, into a 2^m-*grid* (that is, a regular grid of discrete 2^m-cubes), where all 2^{dm} choices of $\mathcal{G}_m$ are equally likely. By defining

$$\mathcal{G}_i := \bigcup \{\mathcal{P}(Q) : Q \in \mathcal{G}_{i+1}\}$$

inductively for $i = m-1, \dots, 0$, we get the dyadic refinements of $\mathcal{G}_m$ down to the (deterministic) partition $\mathcal{G}_0$ into discrete 1-cubes (i.e., singletons). Note that we work inside one component only since, for generic $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$, there cannot exist a measurable choice of a 2^m-grid inside each orbit by ergodicity considerations.

Starting with $f_{\mathcal{G}_0} := 0$ being the zero flow, obtain inductively for $i = 1, 2, \dots, m$, the flow $f_{\mathcal{G}_i}$ from $f_{\mathcal{G}_{i-1}}$ by adding

$$\theta_{\mathcal{G}_i} := \sum_{Q \in \mathcal{G}_i} \sum_{\boldsymbol{u} \in Q} \left(\mathbb{1}_A(\boldsymbol{u}) - \mathbb{1}_B(\boldsymbol{u}) - f^{\mathrm{out}}_{\mathcal{G}_{i-1}}(\boldsymbol{u}) \right) \phi_{\boldsymbol{u},Q}. \tag{4.3}$$

Note that $f_{\mathcal{G}_i}$ and $\theta_{\mathcal{G}_i}$ do not depend on m (for $m \geqslant i$). Inside each 2^i-cube $Q \in \mathcal{G}_i$, the increment flow $\theta_{\mathcal{G}_i}$ spreads the current demand error $\mathbb{1}_A - \mathbb{1}_B - f^{\mathrm{out}}_{\mathcal{G}_{i-1}}$ uniformly inside each congruence class modulo 2^{i-1}. Of course, if the error $\mathbb{1}_A - \mathbb{1}_B - f^{\mathrm{out}}_{\mathcal{G}_{i-1}}$ is constant on each cube in $\mathcal{P}(Q)$ for some $Q \in \mathcal{G}_i$ then $\theta_{\mathcal{G}_i}$ spreads this error evenly inside Q. Thus an easy induction on $i = 1, 2, \dots, m$ shows that $\mathbb{1}_A - \mathbb{1}_B - f^{\mathrm{out}}_{\mathcal{G}_i}$ is constant on every cube $Q \in \mathcal{G}_i$. Moreover, since $f_{\mathcal{G}_i}$ is zero on the edge boundary of $Q \in \mathcal{G}_i$ (and thus sends no flow out of Q), this constant is $\xi(Q)$, that is,

$$\mathbb{1}_A(\boldsymbol{u}) - \mathbb{1}_B(\boldsymbol{u}) - f^{\mathrm{out}}_{\mathcal{G}_i}(\boldsymbol{u}) = \xi(Q), \quad \text{for all } Q \in \mathcal{G}_i \text{ and } \boldsymbol{u} \in Q. \tag{4.4}$$

Thus we have

$$\theta_{\mathcal{G}_i} := \sum_{Q \in \mathcal{G}_i} \sum_{C \in \mathcal{P}(Q)} \xi(C) \sum_{\boldsymbol{u} \in C} \phi_{\boldsymbol{u},Q}, \quad \text{for every } 1 \leqslant i \leqslant m \tag{4.5}$$

(and for $i = m$ we arrive at the definition in (4.2)).

It follows from (4.2) and (4.5) that, inside the component $[\boldsymbol{v}]_{G_d}$, the flow θ_i for each $1 \leqslant i \leqslant m$ is the expectation of $\theta_{\mathcal{G}_i}$ over a uniformly random 2^m-grid $\mathcal{G}_m$, since $\mathcal{G}_i = \mathcal{P}^{m-i}(\mathcal{G}_m)$ is a uniformly random 2^i-grid. Note that, by the linearity of expectation, f_m inside $[\boldsymbol{v}]_{G_d}$ is the expectation of $f_{\mathcal{G}_m}$.

Now we are ready to verify all four conclusions, (3.1)–(3.4), of Lemma 3.1.

Conclusion (3.1) states that $2^{2dm} f_m$ is integer-valued. Indeed, the only non-integer factors in (4.2) in addition to 2^{-dm} are $2^{-d(m-1)}$ (from the definition of $\xi(C)$) and 2^{-d} (from the definition of $\phi_{\boldsymbol{u},Q}$), proving (3.1).

Next, let estimate how close f_m^{out} is from the desired demand $\mathbb{1}_A - \mathbb{1}_B$. By taking the expectation over $\mathcal{G}_m$ of (4.4) for $i := m$, we get that

$$\mathbb{1}_A(\boldsymbol{u}) - \mathbb{1}_B(\boldsymbol{u}) - f_m^{\text{out}}(\boldsymbol{u}) = \frac{1}{2^{dm}} \sum_{Q \in \mathcal{Q}_{2^m}(\boldsymbol{u})} \xi(Q), \tag{4.6}$$

for all $m \geqslant 0$ and $\boldsymbol{u} \in [\boldsymbol{v}]_{G_d}$. As $\boldsymbol{v} \in \mathbb{T}^k$ was arbitrary, (4.6) holds for every $\boldsymbol{u} \in \mathbb{T}^k$. Now, (4.6) and (4.1) imply that, for every $\boldsymbol{u} \in \mathbb{T}^k$,

$$\left| f_m^{\text{out}}(\boldsymbol{u}) - \mathbb{1}_A(\boldsymbol{u}) + \mathbb{1}_B(\boldsymbol{u}) \right| = \left| \sum_{Q \in \mathcal{Q}_{2^m}(\boldsymbol{u})} \frac{\xi(Q)}{2^{dm}} \right| \leqslant 2^{dm} \cdot \frac{2c}{2^{m(1+\varepsilon)}} \cdot \frac{1}{2^{dm}} = \frac{2c}{2^{m(1+\varepsilon)}},$$

which is exactly the bound of Conclusion (3.2).

For (3.3), we need to compute the maximum flow along any given edge under θ_m. Let us re-write the right-hand side of (4.2) using only straight-line paths:

$$\theta_m = \frac{1}{2^{dm}} \sum_{\boldsymbol{u} \in \mathbb{T}^k} \sum_{C \in \mathcal{Q}_{2^{m-1}}(\boldsymbol{u})} \xi(C) \sum_{\substack{\vec{\gamma} \in \{-1,0,1\}^d \\ \vec{\gamma} \neq (0,\dots,0)}} 2^{|\{i:\gamma_i = 0\}|} \frac{\phi_{\boldsymbol{u}, 2^{m-1}\vec{\gamma}}}{2^d}. \tag{4.7}$$

(Note that, for a discrete 2^{m-1}-cube C, the quantity $2^{|\{i:\gamma_i=0\}|}$ is exactly the number of discrete 2^m-cubes Q such that $\mathcal{P}(Q)$ contains both Q and $(2^{m-1}\vec{\gamma}) \cdot_a Q$.) For any given edge $\boldsymbol{vw}$, there is a unique $\vec{\gamma} \in \{-1,0,1\}^d$ such that $\boldsymbol{w} = \vec{\gamma} \cdot \boldsymbol{v}$. The number of vertices $\boldsymbol{u}$ such that either $\phi_{\boldsymbol{u}, 2^{m-1}\vec{\gamma}}$ or $\phi_{\boldsymbol{u}, -2^{m-1}\vec{\gamma}}$ are non-zero on the pair $(\boldsymbol{v}, \boldsymbol{w})$ is precisely $2 \cdot 2^{m-1} = 2^m$. Since $\vec{\gamma}$ is non-zero, we have $2^{|\{i:\gamma_i=0\}|} \leqslant 2^{d-1}$. So, by (4.1) and (4.7), we have

$$|\theta_m(\boldsymbol{v}, \boldsymbol{w})| \leqslant 2^{-dm} \cdot 2^m \cdot 2^{(m-1)d} \cdot \frac{2c}{2^{(m-1)(1+\varepsilon)}} \cdot \frac{2^{d-1}}{2^d} = \frac{2c}{2^{d+\varepsilon(m-1)}}.$$

Thus indeed, as stated by Conclusion (3.3), we have $\|f_m - f_{m-1}\|_\infty \leqslant \frac{2c}{2^{d+\varepsilon(m-1)}}$.

Finally, let us prove (3.4) which states that f_m is a $(2^m - 1)$-local function of A and B. Since this is trivially true for the zero flow f_0, it is enough to prove that, for every $m \geqslant 1$, the flow θ_m is a $(2^m - 1)$-local function of A and B. Since every discrete 2^m-cube containing a vertex $\boldsymbol{v}$ lies inside the $(2^m - 1)$-neighbourhood of $\boldsymbol{v}$, it follows

from (4.2) that the values of the flow θ_m on the edges incident to $\boldsymbol{v}$ depend only on intersections of A and B with $N_{2^m-1}[\boldsymbol{v}]$. Thus $f_m = \theta_1 + \ldots + \theta_m$ is indeed a (2^m-1)-local function of A and B, as required.

This completes the proof of Lemma 3.1. □

Note that the sequence $f_1, f_2, \ldots$ is clearly convergent as the expression on the right side of the inequality (3.3) is summable as a function of m. We can therefore define f_∞ to be the pointwise limit of $f_1, f_2, \ldots$. The following corollary is immediate by applying the triangle inequality and summing the bound in (3.3).

Observation 4.1. *For any $m \geqslant 0$,*

$$\|f_\infty - f_m\|_\infty \leqslant \sum_{i=m}^{\infty} \|f_{i+1} - f_i\|_\infty \leqslant \sum_{i=m}^{\infty} \frac{2c}{2^{d+\varepsilon i}} = \frac{c\,2^{1+\varepsilon}}{2^{d+\varepsilon m}(2^\varepsilon - 1)}. \quad \square$$

5. Toast sequences

Our next goal is to formally describe the construction of the toast sequence $(J_1, J_2, \ldots)$ from the proof of Lemma 3.2 as well as some other toast sequences built around it.

5.1. Some preliminaries

Let us say that a set A *cuts* a set B if both $B \setminus A$ and $B \cap A$ are non-empty. The following definition is key to describing our constructions.

Definition 5.1. For $i \geqslant 1$, let $D = (D_1, D_2, \ldots, D_i)$ be a sequence of subsets of $\mathbb{T}^k$ and let $b \geqslant 0$. Define $\mathcal{C}_b(D) \subseteq \mathbb{T}^k$ to be the set constructed as follows:

(5.1) initialise $\mathcal{C}_b(D) := D_i$,
(5.2) while there exist $1 \leqslant j \leqslant i-1$ and $S \in \mathrm{comp}(D_j)$ such that $\mathcal{C}_b(D)$ cuts $N_b[S]$, add all vertices of $N_b[S]$ to $\mathcal{C}_b(D)$.

It may help the reader to have the following informal description of the step in (5.2) in mind: the current set $\mathcal{C}_b(D)$ iteratively swallows each set $N_b[S]$ that it intersects (unless the whole set $N_b[S]$ is already inside it). Note that the order in which we perform the operations in (5.2) does not affect the final set $\mathcal{C}_b(D)$. Intuitively, Definition 5.1 is designed so that the boundaries of the components of the final set $\mathcal{C}_b(D)$ are well separated from $D_1 \cup \cdots \cup D_{i-1}$. Unfortunately, neither some separation between the components of the new set $\mathcal{C}_b(D)$ nor their boundedness holds automatically (as two distinct components of D_i may grow too close to each other, or even merge); this will require proofs based on various extra properties of the input sets D_j.

The next lemma will be used to demonstrate that if the diameter of each component of $G_d \restriction D_j$ is not too large and the distance between them is not too small, then the

obtained set $\mathcal{C}_b(D)$ can be added without violating Properties (2.15) and (2.16) from the definition of a toast sequence. Note that the lemma holds without assuming any bound on the distance between distinct components of $G_d \restriction D_i$.

Lemma 5.2. *Let $D = (D_1, D_2, \dots, D_i)$ be a sequence of subsets of $\mathbb{T}^k$, let $b \geqslant 0$, and let $(a_1, a_2, \dots, a_{i-1})$ and $(b_0, b_1, \dots, b_{i-1})$ be sequences of non-negative integers such that*

$$b_j \geqslant a_j + 2b_{j-1} + 2b, \quad \text{for all } 1 \leqslant j \leqslant i-1. \tag{5.3}$$

If, for every $1 \leqslant j \leqslant i-1$, every component of $G_d \restriction D_j$ has diameter at most a_j in G_d and the distance in G_d between any two components of $G_d \restriction D_j$ is greater than $b_{j-1}+2b$, then $\mathcal{C}_b(D) \subseteq N_{b_{i-1}}[D_i]$.

Proof. Take any vertex $\boldsymbol{w} \in \mathcal{C}_b(D) \setminus D_i$. By definition of $\mathcal{C}_b(D)$, there exists a sequence $S_1, S_2, \dots, S_n$ of distinct subsets of $\mathbb{T}^k$ such that

(5.4) for each $1 \leqslant \ell \leqslant n$, there exists $1 \leqslant j \leqslant i-1$ such that S_ℓ is a component of $G_d \restriction D_j$,
(5.5) $\operatorname{dist}_{G_d}(S_1, D_i) \leqslant b$,
(5.6) $\operatorname{dist}_{G_d}(S_\ell, S_{\ell+1}) \leqslant 2b$ for $1 \leqslant \ell < n$, and
(5.7) $\operatorname{dist}_{G_d}(S_n, \boldsymbol{w}) \leqslant b$.

Given such a vertex $\boldsymbol{w}$ and sets $S_1, \dots, S_n$, let $\boldsymbol{u} \in D_i$ be such that $\operatorname{dist}_{G_d}(\boldsymbol{u}, S_1) \leqslant b$. Our aim is to prove, by induction on i, that, for every such $\boldsymbol{w}$ and $\boldsymbol{u}$, we have $\operatorname{dist}_{G_d}(\boldsymbol{u}, \boldsymbol{w}) \leqslant b_{i-1}$. In the base case $i = 1$, the statement is true vacuously as $\mathcal{C}_b(D) = D_1$ (and so no such $\boldsymbol{w}$ can exist).

So, suppose that $i \geqslant 2$. If none of the sets $S_1, \dots, S_n$ is a component of $G_d \restriction D_{i-1}$, then the sequence $S_1, \dots, S_n$ actually certifies that $\boldsymbol{w} \in \mathcal{C}_b(D_1, D_2, \dots, D_{i-2}, \{\boldsymbol{u}\})$ and we have that $\operatorname{dist}_{G_d}(\boldsymbol{u}, \boldsymbol{w}) \leqslant b_{i-2} \leqslant b_{i-1}$ by induction on i.

Next, suppose that there is a unique index ℓ such that S_ℓ is a component of $G_d \restriction D_{i-1}$. Let $\boldsymbol{y}$ be a vertex of S_ℓ at distance at most $2b$ from $S_{\ell-1}$ (or, at distance at most b from $\boldsymbol{u}$ in the case $\ell = 1$) and let $\boldsymbol{z}$ be a vertex of S_ℓ at distance at most $2b$ from $S_{\ell+1}$ (or at distance at most b from $\boldsymbol{w}$ in the case $\ell = n$). The sequences $S_{\ell-1}, S_{\ell-2}, \dots S_1$ and $S_{\ell+1}, S_{\ell+2}, \dots, S_n$ certify that $\boldsymbol{u} \in \mathcal{C}_b(D_1, D_2, \dots, D_{i-2}, N_b[\boldsymbol{y}])$ and $\boldsymbol{w} \in \mathcal{C}_b(D_1, D_2, \dots, D_{i-2}, N_b[\boldsymbol{z}])$ respectively. So, by the inductive hypothesis and (5.3),

$$\begin{aligned}
\operatorname{dist}_{G_d}(\boldsymbol{u}, \boldsymbol{w}) &\leqslant \operatorname{dist}_{G_d}(\boldsymbol{u}, \boldsymbol{y}) + \operatorname{dist}_{G_d}(\boldsymbol{y}, \boldsymbol{z}) + \operatorname{dist}_{G_d}(\boldsymbol{z}, \boldsymbol{w}) \\
&\leqslant (\operatorname{dist}_{G_d}(\boldsymbol{u}, N_b[\boldsymbol{y}]) + b) + a_{i-1} + (\operatorname{dist}_{G_d}(N_b[\boldsymbol{z}], \boldsymbol{w}) + b) \\
&\leqslant (b_{i-2} + b) + a_{i-1} + (b_{i-2} + b) \;\leqslant\; b_{i-1},
\end{aligned}$$

as desired.

Finally, we consider the remaining case that there are two indices $1 \leqslant \ell < \ell' \leqslant n$ such that S_ℓ and $S_{\ell'}$ are distinct components of $G_d \restriction D_{i-1}$. Choose these indices so that $\ell' - \ell$ is minimised. Let $\boldsymbol{y}$ be a vertex of $N_b[S_\ell]$ which is at distance at most b from $S_{\ell+1}$ and let $\boldsymbol{z}$ be a vertex of $N_b[S_{\ell'}]$ at distance at most b from $S_{\ell'-1}$. The sequence $S_{\ell+1}, S_{\ell+2}, \dots, S_{\ell'-1}$ certifies that $\boldsymbol{z}$ is in $\mathcal{C}_b(D_1, \dots, D_{i-2}, \{\boldsymbol{y}\})$. By the inductive hypothesis,

$$\operatorname{dist}_{G_d}(S_\ell, S_{\ell'}) \leqslant \operatorname{dist}_{G_d}(\boldsymbol{y}, \boldsymbol{z}) + 2b \leqslant b_{i-2} + 2b,$$

which contradicts the hypotheses of the lemma. This completes the proof. □

5.2. Constructions of toast sequences

We now construct some toast sequences that we use in the proof of Theorem 1.3. We remark that, as constructed, they will not immediately cover all of $\mathbb{T}^k$. However, at the end of the section, we will give a compactness argument of Boykin and Jackson [5] (see also Marks and Unger [28, Lemma A.2]) which allows us to produce one toast sequence that covers all of the vertices (see Lemma 5.15).

We will use the global parameters $r_i, r'_i, q_i, q'_i, t_i, t'_i$ that were defined in Section 2.4. For the reader's convenience, we repeat these definitions here. Namely, we set

$$r'_i := 100^{2^{i+1}-1}, \quad i \geqslant 0, \tag{2.7}$$

$$r_i := 100^{2^{i+1}-2}, \quad i \geqslant 1. \tag{2.8}$$

Thus $r'_0 = 100$ while $r_i = (r'_{i-1})^2$ and $r'_i = 100 r_i$ for each $i \geqslant 1$. Then we defined $q'_0 := 0$ and, for $i \geqslant 1$,

$$\begin{aligned} t_i &:= 2r_i + 4q'_{i-1} + 4, \\ q_i &:= t_i + 2q'_{i-1} + 4, \\ t'_i &:= 4r'_i/5 + 2q_i, \\ q'_i &:= t'_i + 2q_i + 4. \end{aligned} \tag{2.9}$$

Note that, within additive $O(\sqrt{r_i})$ as $i \to \infty$, the last four values are $2r_i$, $2r_i$, $84r_i$ and $88r_i$ respectively. It is tedious, but not hard, to verify that, with the specific definitions above (or for the sufficiently fast-growing sequences as in Remark 2.4, with r'_i divisible by 5), the following inequalities hold for all $i \geqslant 1$:

$$r'_i - 2 > q'_i > t'_i > q_i > t_i > r_i > 5r'_{i-1}, \tag{5.8}$$

$$r'_i \geqslant 15 q_i + 25, \tag{5.9}$$

$$q'_{i+1} \geqslant 2r_{i+1} + 6q_i + 4. \tag{5.10}$$

These are some key inequalities that will be used when establishing various properties of the constructed sets in Section 5.3.

Recall that, for each $i \geqslant 1$, X_i is maximally r_i-discrete and Y_i is maximally r'_i-discrete such that both these sets are the unions of finitely many disjoint strips. For each $i \geqslant 1$, define

$$I_i := \bigcup_{\boldsymbol{u} \in X_i} I_i^{\boldsymbol{u}},$$

where $I_i^{\boldsymbol{u}}$ denotes the set of all $\boldsymbol{v} \in \mathbb{T}^k$ such that

$$\operatorname{dist}_{G_d}(\boldsymbol{v}, \boldsymbol{u}') \geqslant \operatorname{dist}_{G_d}(\boldsymbol{v}, \boldsymbol{u}) + 5r'_{i-1}, \quad \text{for every } \boldsymbol{u}' \in X_i \setminus \{\boldsymbol{u}\}.$$

The set $I_i^{\boldsymbol{u}}$ can be viewed as the "partial Voronoi cell" of $\boldsymbol{u} \in X_i$.

Now, inductively for $i = 1, 2, \ldots$, define

$$J_i := \mathcal{C}_2(N_{q'_0+2}[J_1], \ldots, N_{q'_{i-2}+2}[J_{i-1}], I_i), \tag{5.11}$$

$$K_i := \mathcal{C}_2(K_1, L_1, \ldots, K_{i-1}, L_{i-1}, N_2[J_i]), \tag{5.12}$$

$$L_i := \mathcal{C}_2(K_1, L_1, \ldots, K_{i-1}, L_{i-1}, K_i, N_{2r'_i/5}[Y_i]). \tag{5.13}$$

We will show that $(J_1, K_1, J_2, K_2, \ldots)$ and $(K_1, L_1, K_2, L_2, \ldots)$ are toast sequences. In particular, this implies that $(J_1, J_2, \ldots)$ is a toast sequence. Along the way, we will also prove some key properties of these sequences of sets which will be applied later in the paper to prove Theorem 1.3.

5.3. Properties of the constructed sequences

We begin with the following statement, which simply follows by construction.

Lemma 5.3. *For every $i \geqslant 1$, we have $I_i \subseteq J_i$, $N_2[J_i] \subseteq K_i$ and $N_{2r'_i/5}[Y_i] \subseteq L_i$.* □

Next, we prove a lemma regarding the structure of I_i.

Lemma 5.4. *The sets $I_i^{\boldsymbol{u}}$ for $\boldsymbol{u} \in X_i$ are exactly the components of $G_d \restriction I_i$ and the distance in G_d between any two of these sets is at least $5r'_{i-1}$. Also, for each $\boldsymbol{u} \in X_i$, the diameter of $G_d \restriction I_i^{\boldsymbol{u}}$ is at most $2r_i$ and $I_i^{\boldsymbol{u}} \cap X_i = \{\boldsymbol{u}\}$. Furthermore, I_i is a $2r_i$-local function of X_i.*

Proof. Take any $\boldsymbol{u} \in X_i$ and $\boldsymbol{v} \in I_i^{\boldsymbol{u}}$. It holds by $r_i \geqslant 5r'_{i-1}$ that $\boldsymbol{u} \in I_i^{\boldsymbol{u}}$. Every vertex on a shortest path from $\boldsymbol{v}$ to $\boldsymbol{u}$ in G_d is contained in $I_i^{\boldsymbol{u}}$ so this set is connected. Since X_i is maximally r_i-discrete and $\boldsymbol{u}$ is the element of X_i at minimum distance from $\boldsymbol{v}$, we have that $\operatorname{dist}_{G_d}(\boldsymbol{u}, \boldsymbol{v}) \leqslant r_i$. Thus, the diameter of $G_d \restriction I_i^{\boldsymbol{u}}$ is at most $2r_i$.

Now, for distinct $\boldsymbol{u}, \boldsymbol{w} \in X_i$ in the same component of G_d, let $\boldsymbol{v}_0\boldsymbol{v}_1 \dots \boldsymbol{v}_n$ be the shortest path in G_d from $I_i^{\boldsymbol{u}}$ to $I_i^{\boldsymbol{w}}$. Then we have by definition that

$$\begin{aligned}\operatorname{dist}_{G_d}(\boldsymbol{v}_0, \boldsymbol{u}) + 2n &\geqslant \operatorname{dist}_{G_d}(\boldsymbol{v}_n, \boldsymbol{u}) + n \geqslant \operatorname{dist}_{G_d}(\boldsymbol{v}_n, \boldsymbol{w}) + 5r'_{i-1} + n \\ &\geqslant \operatorname{dist}_{G_d}(\boldsymbol{v}_0, \boldsymbol{w}) + 5r'_{i-1} \geqslant \operatorname{dist}_{G_d}(\boldsymbol{v}_0, \boldsymbol{u}) + 10r'_{i-1}\end{aligned}$$

and so $n \geqslant 5r'_{i-1}$. Therefore, the sets $I_i^{\boldsymbol{u}}$, for $\boldsymbol{u} \in X_i$, are the components of $G_d \restriction I_i$ and the distance in G_d between any two such components is at least $5r'_{i-1}$.

Finally, in order to decide if some $\boldsymbol{v} \in \mathbb{T}^k$ belongs to I_i, it is enough to compute the distance d from $\boldsymbol{v}$ to X_i in G_d and then check if the $(d + 5r'_{i-1} - 1)$-neighbourhood of $\boldsymbol{v}$ contain a unique element of X_i. Since $d \leqslant r_i$, the set I_i is a local function of X_i of radius $r_i + 5r'_{i-1} \leqslant 2r_i$, finishing the proof of the lemma. □

The following lemma holds because Y_i is maximally r'_i-discrete; we omit its easy proof.

Lemma 5.5. *Every component of $G_d \restriction N_{2r'_i/5}[Y_i]$ has diameter at most $4r'_i/5$ and contains exactly one vertex of Y_i. Also, the distance in G_d between any two such components is at least $r'_i/5$.* □

The proofs of the next two lemmas are the main applications of Lemma 5.2.

Lemma 5.6. *For $i \geqslant 1$, we have that $J_i \subseteq N_{q'_{i-1}}[I_i]$, the distance in G_d between any two distinct components of $G_d \restriction J_i$ is least $5r'_{i-1} - 2q'_{i-1}$, and every component of $G_d \restriction J_i$ has diameter at most $2r_i + 2q'_{i-1}$ and contains exactly one vertex of X_i.*

Proof. We proceed by induction on i. The base case $i = 1$ follows from Lemma 5.4 as $J_1 = I_1$ and $q'_0 = 0$. Let $i \geqslant 2$. We would like to apply Lemma 5.2 with $b := 2$ to the sequences

$$(N_{q'_0+2}[J_1], \dots, N_{q'_{i-2}+2}[J_{i-1}], I_i),\ (t'_1, \dots, t'_{i-1}) \text{ and } (q'_0, \dots, q'_{i-1}). \tag{5.14}$$

Let us check the assumptions of Lemma 5.2. By induction, for every $1 \leqslant j \leqslant i-1$, every component of $G_d \restriction N_{q'_{j-1}+2}[J_j]$ has diameter at most $(2r_j + 2q'_{j-1}) + 2(q'_{j-1} + 2) = t_j < t'_j$ and the distance in G_d between any two such components is at least $(5r'_{j-1} - 2q'_{j-1}) - 2(q'_{j-1} + 2)$, which is strictly larger than $q'_{j-1} + 4$ by (5.8). Also, the distance between any two components of $G_d \restriction I_i$ is at least $5r'_{i-1} > q'_{i-1} + 4$ while the inequality in (5.3) for $1 \leqslant j \leqslant i - 1$ (which states that $q'_j \geqslant t'_j + 2q'_{j-1} + 4$) holds by $q_j > q'_{j-1}$. Thus Lemma 5.2 applies to the sequences in (5.14) and gives that $J_i \subseteq N_{q'_{i-1}}[I_i]$, proving the first stated property.

This routinely implies all other claims about J_i. Indeed, it follows from Lemma 5.4 that the distance in G_d between different components of $G_d \restriction J_i$ is at least $5r'_{i-1} - 2q'_{i-1}$ (note that $5r'_{i-1} - 2q'_{i-1} \geqslant 2$) and each component has diameter at most $2r_i + 2q'_{i-1}$.

Finally, each component $S \in \mathrm{comp}(J_i)$ is built from $I_i^{\boldsymbol{u}}$ for some $\boldsymbol{u} \in X_i$ and, by the separation bounds proved above, $\boldsymbol{u}$ is the unique vertex in $S \cap X_i$. □

Lemma 5.7. *For each $i \geqslant 1$, it holds that*

(5.15) *$K_i \subseteq N_{q'_{i-1}+2}[J_i]$, the distance in G_d between any two distinct components of $G_d \restriction K_i$ is least $5r'_{i-1} - 4q'_{i-1} - 4$, and every component of $G_d \restriction K_i$ has diameter at most t_i and contains exactly one vertex of X_i.*

(5.16) *$L_i \subseteq N_{2r'_i/5+q_i}[Y_i]$, the distance in G_d between any two distinct components of $G_d \restriction L_i$ is least $r'_i/5 - 2q_i$, and every component of $G_d \restriction L_i$ has diameter at most t'_i and contains exactly one vertex of Y_i.*

Proof. We use induction on i. The base case $i = 1$ holds for K_i by Lemma 5.6 since $K_1 = N_2[J_1]$ and $q'_0 = 0$. Let $i \geqslant 2$. In the i-th step of induction, we prove the claims about L_{i-1} and K_i. Analogously to the proof of Lemma 5.6, we will apply the inductive hypotheses and Lemma 5.2 with $b := 2$ to respectively

- $(K_1, L_1, \dots, K_{i-1}, N_{2r'_{i-1}/5}[Y_{i-1}])$, $(t_1, t'_1, \dots, t'_{i-2}, t_{i-1})$ and $(q'_0, q_1, \dots, q'_{i-2}, q_{i-1})$,
- $(K_1, L_1, \dots, L_{i-1}, N_2[J_i])$, $(t_1, t'_1, \dots, t_{i-1}, t'_{i-1})$ and $(q'_0, q_1, \dots, q_{i-1}, q'_{i-1})$.

Let us check the assumptions of Lemma 5.2 for the first triple of sequences. By induction, for every $1 \leqslant j \leqslant i-1$, every component of $G_d \restriction K_j$ has diameter at most t_j and the distance in G_d between any two such components is at least $5r'_{j-1} - 4q'_{j-1} - 4$, which is strictly more than $q'_{j-1} + 4$ by (5.8). Similarly, for each $1 \leqslant j \leqslant i-2$, every component of $G_d \restriction L_j$ has diameter at most t'_j and the distance in G_d between any two such components is at least $r'_j/5 - 2q_j$, which is strictly larger than $q_j + 4$ by (5.9). Also, the inequalities in (5.3) are routine to verify.

So, Lemma 5.2 applies and gives that $L_{i-1} \subseteq N_{q_{i-1}}[N_{2r'_{i-1}/5}[Y_{i-1}]]$. This implies all other stated properties of L_{i-1}. Indeed, by Lemma 5.5, the distance between any two components of $G_d \restriction L_{i-1}$, is at least $r'_{i-1}/5 - 2q_{i-1}$, and each component has diameter at most $4r'_{i-1}/5 + 2q_{i-1} = t'_{i-1}$ and contains exactly one vertex from Y_i, proving all claims about L_{i-1}.

By $r'_{i-1}/5 - 2q_{i-1} > q_{i-1} + 4$, we can apply Lemma 5.2 also to the second triple of sequences to derive that $K_i \subseteq N_{q'_{i-1}+2}[J_i]$. It follows from Lemma 5.6 that the distance between any two components of $G_d \restriction K_i$ is at least $(5r'_{i-1} - 2q'_{i-1}) - 2(q'_{i-1} + 2) = 5r'_{i-1} - 4q'_{i-1} - 4$, while each component has diameter at most $2r_i + 2q'_{i-1} + 2(q'_{i-1} + 2) = t_i$ and contains exactly one element of X_i. □

We will need to refer a few times to the following, not entirely trivial result so we state it as a separate lemma.

Lemma 5.8. *Let $i > j \geqslant 1$. If $S' \in \mathrm{comp}(K_j)$ and $T \in \mathrm{comp}(J_i)$ are at distance at most 2 in G_d, then $N_2[S'] \subseteq T$.*

Proof. Observe that, by Lemma 5.7, we have $S' \subseteq K_j \subseteq N_{q'_{j-1}+2}[J_j]$. Take a component S of $G_d \restriction J_j$ such that $S'' := N_{q'_{j-1}+2}[S]$ intersects S'. Let us show that, in fact, $S' \subseteq S''$. Suppose on the contrary that we have some $\boldsymbol{u} \in S' \setminus S''$. Again by Lemma 5.7, $\boldsymbol{u}$ is within distance $q'_{j-1}+2$ to some component $X \in \mathrm{comp}(J_j)$ that has to be different from S. But then the distance between X and S is, by the triangle inequality when we go via S', at most $2(q'_{j-1}+2)+t_i < 5r'_{j-1} - 2q'_{j-1}$, contradicting Lemma 5.6.

By the definition of J_i, every component of $N_{q'_{j-1}+2}[J_j]$ that is within distance 2 of $T \in \mathrm{comp}(J_i)$ is included with its 2-neighbourhood into T. Since $S'' \subseteq N_{q'_{j-1}+2}[J_j]$, the set $N_2[S'']$ is included into T. Thus T contains $N_2[S'] \subseteq N_2[S'']$, as desired. □

We are now in position to prove the following lemma.

Lemma 5.9. *Both of $(J_1, K_1, J_2, K_2, \ldots)$ and $(K_1, L_1, K_2, L_2, \ldots)$ are toast sequences.*

Proof. The fact that the components of the subgraphs of G_d induced by each of J_i, K_i and L_i are of uniformly bounded cardinality and are pairwise separated by distance at least three follows from Lemmas 5.6 and 5.7. So, Properties (2.15) and (2.16) of Definition 2.14 hold for each of the two sequences.

Let us check Property (2.17) for the sequence $(J_i, K_i)_{i=1}^{\infty}$. Note some asymmetry in (2.17), which states that if two components from two different times are at distance at most 2 then the 2-neighbourhood of the *earlier* one is contained in the *later* one. There are some cases to consider. First, take any $S \in \mathrm{comp}(J_i)$. Suppose on the contrary to (2.17) that S is at distance at most 2 from $X = J_j$ for $j > i$ or $X = K_j$ for $j \geqslant i$ but $N_2[S]$ is a not a subset of X. Let $T \in \mathrm{comp}(X)$ satisfy $\mathrm{dist}_{G_d}(S, T) \leqslant 2$. By construction, T does not cut $N_2[S]$ as otherwise this set would be added into T (since T was built after S). Hence we have that $T \subseteq N_2[S]$. This is impossible for $j > i$ because the diameter of S is at most $2r_i + 2q'_{i-1}$ by Lemma 5.6 while T contains a ball of radius $\lfloor (r_j - 5r'_{i-1})/2 \rfloor$ centred at the unique point of $X_j \cap T$ by the definition of $I_j \subseteq J_j \subseteq K_j$. Also, the case $X = K_i$ leads to a contradiction. Indeed, when constructing $T \in \mathrm{comp}(K_i)$ we started with $N_2[T']$ for some $T' \in \mathrm{comp}(J_i)$, By Lemmas 5.4, 5.6 and 5.7, we must have $T' = S$. Thus $K_i \supseteq T \supseteq N_2[T']$ contains $N_2[S]$, a contradiction. Second, take any $S \in \mathrm{comp}(K_i)$. No set J_j with $j > i$ can violate (2.17) by Lemma 5.8. Also, any $T \in \mathrm{comp}(K_j)$ with $j > i$ and $\mathrm{dist}_{G_d}(S, T) \leqslant 2$ contains $N_2[S]$ (again since $T \subseteq N_2[S]$ is impossible by the diameter argument). Thus (2.17) holds for $(J_i, K_i)_{i=1}^{\infty}$.

Likewise, Property (2.17) can be verified for the sequence $(K_i, L_i)_{i=1}^{\infty}$, using additionally the fact that each component of $G_d \restriction L_i$ contains a ball of radius $2r'_i/5$.

Thus each of $(J_i, K_i)_{i=1}^{\infty}$ and $(K_i, L_i)_{i=1}^{\infty}$ is a toast sequence. □

For each $i \geqslant 1$, define $\widetilde{K}_i$ from K_i (resp. $\widetilde{L}_i$ from L_i) by removing the vertex sets of all components S such that there is $j > i$ with J_j (resp. K_j) being at distance at most 2 from S in the graph G_d. By Lemma 5.8, it holds that, in fact,

$$\widetilde{K}_i = K_i \setminus \left(\bigcup_{j=i+1}^{\infty} J_j \right), \tag{5.17}$$

that is, we could have equivalently defined $\widetilde{K}_i$ from K_i by removing all individual vertices that belong to some J_j with $j > i$. Likewise (with the claim following trivially from our definitions), we have

$$\widetilde{L}_i = L_i \setminus \left(\bigcup_{j=i+1}^{\infty} K_j \right). \tag{5.18}$$

The following lemma records some properties of these sets, in particular showing that they together with J_i's form a toast sequence without losing any single vertex of $\bigcup_{i=1}^{\infty} L_i$. Moreover, the structure of the toast sequence $(J_1, J_2, \ldots)$ is preserved in a very strong way.

Lemma 5.10. *The following properties hold.*

(5.19) *For every $i \geqslant 1$, we have $\mathrm{comp}(\widetilde{K}_i) \subseteq \mathrm{comp}(K_i)$ and $\mathrm{comp}(\widetilde{L}_i) \subseteq \mathrm{comp}(L_i)$.*

(5.20) $\bigcup_{i=1}^{\infty}(J_i \cup \widetilde{K}_i \cup \widetilde{L}_i) \supseteq \bigcup_{i=1}^{\infty} L_i$.

(5.21) *For every integers $i, j \geqslant 1$, if $T \in \mathrm{comp}(J_j)$ and $S \in \mathrm{comp}(\widetilde{L}_i)$ are at distance at most 2, then $i \geqslant j$ and $N_2[T] \subseteq S$.*

(5.22) *The sequence $(J_1, \widetilde{K}_1, \widetilde{L}_1, J_2, \widetilde{K}_2, \widetilde{L}_2, \ldots)$ is a toast sequence.*

Proof. Conclusion (5.19) is a trivial consequence of the definition of $\widetilde{K}_i$ and $\widetilde{L}_i$.

For (5.20), take any $\boldsymbol{u} \in L_i \setminus \widetilde{L}_i$. We have to show that $\boldsymbol{u} \in \bigcup_{j=1}^{\infty}(J_j \cup \widetilde{K}_j)$. By (5.18), there is $j > i$ with $\boldsymbol{u} \in K_j$. If $\boldsymbol{u} \in \widetilde{K}_j$ then we are done; otherwise the vertex $\boldsymbol{u}$ lies inside some J_h with $h > j$ by (5.17), proving (5.20) in either case.

For (5.21), note that $i \geqslant j$ for otherwise the component S of L_i (recall that $\mathrm{comp}(\widetilde{L}_i) \subseteq \mathrm{comp}(L_i)$ by (5.19)) would be removed when constructing $\widetilde{L}_i$ from L_i. By the definition of K_j, we have that $K_j \supseteq N_2[J_j]$. Thus K_j contains every vertex of the connected set $N_2[T]$ which in turn intersects S. Thus by the construction of the set L_i, its component S must contain $N_2[T]$ as a subset, proving (5.21).

For (5.22), note by (5.19) that no new components (and thus no new boundaries) are created when we construct $\widetilde{K}_j$ from K_j and $\widetilde{L}_j$ from L_j. Since $(J_1, K_1, J_2, K_2, \ldots)$ and $(K_1, L_1, K_2, L_2, \ldots)$ are toast sequences by Lemma 5.9, the only remaining case that requires some checking is that no conflict to the definition of a toast sequence (namely,

Property (2.17) of Definition 2.14) can come from $S \in \mathrm{comp}(\widetilde{L}_i)$ and $C \in \mathrm{comp}(J_j)$. This has already been taken care of by (5.21). □

5.4. Some structure of the constructed sets

The next two lemmas will be useful in analysing the structure of the pieces in the final equidecomposition.

Lemma 5.11. *For each integer $i \geqslant 1$, the set J_i is a $4r_i$-local function of $X_1, \dots, X_i$.*

Proof. We use induction on i with the base case $i = 1$ following from $J_1 = I_1$ being in fact an $(r_1 + 5r_0)$-local function of X_1. Let $i \geqslant 2$. In order to decide if some $\boldsymbol{v} \in \mathbb{T}^k$ is included into J_i, it is enough to know all elements $\boldsymbol{u} \in X_i$ at distance at most $r_i + q'_{i-1}$ from $\boldsymbol{v}$ and the component of $G_d \restriction J_i$ containing each such $\boldsymbol{u}$. By Lemma 5.6, the component $S \in \mathrm{comp}(J_i)$ containing $\boldsymbol{u} \in X_i$ has diameter at most $2r_i + 2q'_{i-1}$ and contains no other elements of X_i. So to build S for given $\boldsymbol{u} \in X_i$, we need to know only the sets $N_{q'_0+2}[J_1], \dots, N_{q'_{i-2}+2}[J_{i-1}]$ inside $N_{2r_i+2q'_{i-1}}[\boldsymbol{u}]$. By induction, each of these sets is a $4r_{i-1}$-local function of $X_1, \dots, X_{i-1}$. The lemma follows since $(r_i + q'_{i-1}) + (2r_i + 2q'_{i-1}) + 4r_{i-1} \leqslant 4r_i$. □

Lemma 5.12. *Each of the sets J_i, K_i and L_i for $i \geqslant 1$ can be expressed as a union of finitely many disjoint strips.*

Proof. Observe that each of the sets J_i, K_i and L_i is a local function of $X_1, Y_1, \dots, X_i, Y_i$. Lemma 5.11 proves this for J_i and its argument can be easily adapted to K_i and L_j using the uniform diameter bounds of Lemma 5.7. Since every set X_j and Y_j is in turn a finite union of disjoint strips (by our choices in Section 2.6) and such unions form an invariant algebra, the lemma holds by Observation 2.6. □

Let us remark that the sets $\widetilde{K}_i$ and $\widetilde{L}_i$ need not satisfy Lemma 5.12, that is, they are not in general finite unions of strips.

Lemma 5.13. *We have*

$$\lambda(I_i) = 1 - O\left(r'_{i-1}/r_i\right)$$

as $i \to \infty$, where the constant factor is bounded by a function of d only.

Proof. Note that I_i is a union of finitely many disjoint strips and, therefore, it is measurable. For any pair of distinct vertices $\boldsymbol{u}, \boldsymbol{u}' \in X_i$, let $W_i(\boldsymbol{u}, \boldsymbol{u}')$ be the set of all $\boldsymbol{v} \in \mathbb{T}^k$ such that

$$\max\left\{\mathrm{dist}_{G_d}(\boldsymbol{u}, \boldsymbol{v}), \mathrm{dist}_{G_d}(\boldsymbol{u}', \boldsymbol{v})\right\} \leqslant r_i + 5r'_{i-1} \text{ and}$$

$$|\mathrm{dist}_{G_d}(\boldsymbol{u},\boldsymbol{v}) - \mathrm{dist}_{G_d}(\boldsymbol{u}',\boldsymbol{v})| \leqslant 5r'_{i-1}.$$

Define W_i to be the union of $W_i(\boldsymbol{u},\boldsymbol{u}')$ over all pairs of distinct vertices $\boldsymbol{u},\boldsymbol{u}' \in X_i$. The set W_i can be written as a Boolean combination of translates of X_i, and so it is measurable. By the definition of I_i and the fact that X_i is maximally r_i-discrete, we have that $\mathbb{T}^k \setminus I_i \subseteq W_i$. So, it suffices to prove that $\lambda(W_i) = O(r'_{i-1}/r_i)$.

First, for each $\boldsymbol{u} \in X_i$, let us bound the number of $\boldsymbol{u}' \in X_i$ such that $W_i(\boldsymbol{u},\boldsymbol{u}')$ is non-empty. Given any such $\boldsymbol{u}'$, we have

$$\mathrm{dist}_{G_d}(\boldsymbol{u},\boldsymbol{u}') \leqslant 2r_i + 10r'_{i-1}.$$

Therefore,

$$N_{\lfloor r_i/2\rfloor}[\boldsymbol{u}'] \subseteq N_{2r_i+\lfloor r_i/2\rfloor+10r'_{i-1}}[\boldsymbol{u}].$$

Since the set X_i is r_i-discrete, the sets $N_{\lfloor r_i/2\rfloor}[\boldsymbol{u}']$ are disjoint for different $\boldsymbol{u}' \in X_i$. So the number of such $\boldsymbol{u}'$ is at most

$$\frac{\left(4r_i + 2\lfloor r_i/2\rfloor + 20r'_{i-1} + 1\right)^d}{\left(2\lfloor r_i/2\rfloor + 1\right)^d},$$

which can be upper bounded above by a constant that depends on d only.

Now, given a pair of distinct $\boldsymbol{u},\boldsymbol{u}' \in X_i$ in the same component of G_d, let $\vec{n} \in \mathbb{Z}^d$ be such that $\boldsymbol{u}' = \vec{n} \cdot_a \boldsymbol{u}$. Note that $\|\vec{n}\|_\infty \geqslant r_i + 1$ since X_i is r_i-discrete. Then $W_i(\boldsymbol{u},\boldsymbol{u}')$ is the set of all $\boldsymbol{v} \in \mathbb{T}^k$ of the form $\boldsymbol{v} = \vec{m} \cdot_a \boldsymbol{u}$ where

- $\|\vec{m}\|_\infty \leqslant r_i + 5r'_{i-1}$,
- $\|\vec{n} - \vec{m}\|_\infty \leqslant r_i + 5r'_{i-1}$, and
- $\left|\|\vec{m}\|_\infty - \|\vec{n} - \vec{m}\|_\infty\right| \leqslant 5r'_{i-1}$.

For fixed d, the number of vectors $\vec{m}$ of this type is $O\left(r_i^{d-1}r'_{i-1}\right)$. Indeed, there are d^2 ways to choose the indices $j_1, j_2 \in \{1,\dots,d\}$ such that $|m_{j_1}|$ and $|n_{j_2} - m_{j_2}|$ are maximum. In the case that $j_1 = j_2$, it holds that $|m_{j_1} - (n_{j_1} - m_{j_1})| \leqslant 5r'_{i-1}$ so there are $O(r'_{i-1})$ choices for m_{j_1} and $O(r_i)$ choices for m_j for each $j \in \{1,\dots,d\} \setminus \{j_1\}$. On the other hand, if $j_1 \neq j_2$, then the number of choices of m_{j_1} is $O(r_i)$ and, given this choice, the number of choices for $n_{j_2} - m_{j_2}$ (and thus for m_{j_2}) is $O(r'_{i-1})$ and, again, there are $O(r_i)$ choices for m_j for each $j \in \{1,\dots,d\} \setminus \{j_1, j_2\}$.

Suppose now that we split $\mathbb{T}^k$ into Voronoi cells with the points in X_i as the centres, similarly to the proof of Lemma 2.16. Each Voronoi cell contains $\Omega(r_i^d)$ elements and, by the arguments above, $O(r_i^{d-1}r'_{i-1})$ of these points are in W_i. It follows that the measure of W_i is $O(r'_{i-1}/r_i)$. □

5.5. Covering the whole torus

Finally, we apply a compactness argument of Boykin and Jackson [5] to get a toast sequence that covers the whole torus $\mathbb{T}^k$. For each $\vec{p} \in \{0, \dots, 5\}^d$ and $i \geqslant 1$, define

$$Y_i^{\vec{p}} := Y_i - \lfloor r_i'/3 \rfloor \sum_{j=1}^{d} (p_j - 3)\boldsymbol{x}_j.$$

Note that $Y_i^{\vec{p}}$ is simply a shifted version of Y_i and so it inherits the property of being a maximally r_i'-discrete union of finitely many strips. Also, $Y_i^{(3,3,\dots,3)}$ is simply Y_i. We define, for each $i \geqslant 1$,

$$K_i^{\vec{p}} := \mathcal{C}_2(K_1^{\vec{p}}, L_1^{\vec{p}}, \dots, K_{i-1}^{\vec{p}}, L_{i-1}^{\vec{p}}, N_2[J_i]), \text{ and} \tag{5.24}$$

$$L_i^{\vec{p}} := \mathcal{C}_2(K_1^{\vec{p}}, L_1^{\vec{p}}, \dots, K_{i-1}^{\vec{p}}, L_{i-1}^{\vec{p}}, K_i^{\vec{p}}, N_{2r_i'/5}[Y_i^{\vec{p}}]), \tag{5.25}$$

as well as $\widetilde{K}_i^{\vec{p}} \subseteq K_i^{\vec{p}}$ and $\widetilde{L}_i^{\vec{p}} \subseteq L_i^{\vec{p}}$ by removing whole components that are at distance at most 2 from respectively J_j and $K_j^{\vec{p}}$ for some $j > i$. Note that these definitions are exactly the shifted versions of the previous definitions, namely of (5.12), (5.13) and so on, except the same (unshifted) sets J_i are used. (Thus, for example, it is not true in general that e.g., $K_i^{\vec{p}}$ is just a shifted copy of K_i.) Therefore, all of the lemmas proved in Section 5.3 apply equally well to the sets

$$J_1, K_1^{\vec{p}}, \widetilde{K}_1^{\vec{p}}, L_1^{\vec{p}}, \widetilde{L}_1^{\vec{p}}, J_2, K_2^{\vec{p}}, \widetilde{K}_2^{\vec{p}}, L_2^{\vec{p}}, \widetilde{L}_2^{\vec{p}}, \dots,$$

as they apply to the sets $J_1, K_1, \widetilde{K}_1, L_1, \widetilde{L}_1, J_2, K_2, \widetilde{K}_2, L_2, \widetilde{L}_2, \dots$.

The above definitions are motivated by the following result which implies that every vertex of $\mathbb{T}^k$ belongs to $L_i^{\vec{p}}$ for at least one (in fact infinitely many) pairs $(i, \vec{p})$.

Lemma 5.14. *Every element of $\mathbb{T}^k$ is covered by infinitely many of the sets $N_{2r_i'/5}[Y_i^{\vec{p}}]$ for $\vec{p} \in \{0, \dots, 5\}^d$ and integer $i \geqslant 1$.*

Proof. For each vertex $\boldsymbol{v} \in \mathbb{T}^k$, let

$$R_i(\boldsymbol{v}) := \{\vec{n}/r_i' : \vec{n} \cdot_a \boldsymbol{v} \in Y_i\} \subseteq \mathbb{R}^d.$$

Note that, since Y_i is maximally r_i'-discrete, we have that $R_i(\boldsymbol{v})$ contains a point in $[-1, 1]^d$ for all $i \geqslant 1$. Thus, by compactness of $[-1, 1]^d$, the set $R^*(\boldsymbol{v})$ of accumulation points of the sequence $R_1(\boldsymbol{v}), R_2(\boldsymbol{v}), \dots$ in $[-1, 1]^d$ is non-empty.

If $\boldsymbol{v}$ and $\boldsymbol{v}'$ are in the same component of G_d, then $R_i(\boldsymbol{v})$ is the same as $R_i(\boldsymbol{v}')$ shifted by ℓ_∞-distance $\mathrm{dist}_{G_d}(\boldsymbol{v}, \boldsymbol{v}')/r_i'$ in $\mathbb{R}^d$. Since the sequence $r_1', r_2', \dots$ is increasing, this distance tends to zero. Thus the sequences $(R_i(\boldsymbol{v}))_{i=1}^{\infty}$ and $(R_i(\boldsymbol{v}'))_{i=1}^{\infty}$ have the same set

of accumulation points in $\mathbb{R}^d$; in particular, we have $R^*(\boldsymbol{v}) = R^*(\boldsymbol{v}')$. Thus the function R^* is invariant under the action $a : \mathbb{Z}^d \curvearrowright \mathbb{T}^k$.

Define

$$T'_{\vec{p}} := \left\{ \boldsymbol{v} \in \mathbb{T}^k : R^*(\boldsymbol{v}) \cap \prod_{s=1}^{d} \left[\frac{p_s - 3}{3}, \frac{p_s - 2}{3} \right] \neq \emptyset \right\}, \quad \text{for } \vec{p} \in \{0, \dots, 5\}^d.$$

These sets cover the whole torus $\mathbb{T}^k$, since $R^*(\boldsymbol{v})$ is non-empty for every $\boldsymbol{v}$. For each $\vec{p} \in \{0, \dots, 5\}^d$, we let

$$T_{\vec{p}} := T'_{\vec{p}} \setminus \bigcup_{\substack{\vec{q} \in \{0,\dots,5\}^d \\ \vec{q} \prec_{\text{lex}} \vec{p}}} T_{\vec{q}} \tag{5.26}$$

consist of those $\boldsymbol{v}$ for which $\vec{p}$ is the lexicographically smallest vector with $R^*(\boldsymbol{v})$ having non-empty intersection with $\prod_{s=1}^{d} \left[\frac{p_s-3}{3}, \frac{p_s-2}{3} \right]$.

Since the (set-valued) function R^* is constant on each component of G_d, we have that each set $T'_{\vec{p}}$ (and thus each set $T_{\vec{p}}$) is a union of components of G_d, that is, is invariant under the action a. Also, the sets $T_{\vec{p}}$, for $\vec{p} \in \{0, \dots, 5\}^d$, partition $\mathbb{T}^k$.

Now, for any $\boldsymbol{v} \in T_{\vec{p}}$, we have by the definition of $R^*(\boldsymbol{v})$ that there are infinitely many indices i for which there is a vertex $\boldsymbol{u} \in Y_i$ of the form $\boldsymbol{u} = \vec{n} \cdot_a \boldsymbol{v}$ for some $\vec{n}$ such that

$$r'_i \cdot \left(\frac{p_j - 3}{3} - \frac{1}{100} \right) \leqslant n_j \leqslant r'_i \cdot \left(\frac{p_j - 2}{3} + \frac{1}{100} \right), \quad \text{for all } 1 \leqslant j \leqslant d.$$

Thus, for all such i with r'_i sufficiently large, it holds that

$$\operatorname{dist}_{G_d}(\boldsymbol{v}, Y_i^{\vec{p}}) \leqslant \operatorname{dist}_{G_d} \left(\boldsymbol{v}, \boldsymbol{u} - \lfloor r'_i/3 \rfloor \sum_{j=1}^{d} (p_j - 3) \boldsymbol{x}_j \right) \leqslant \frac{r'_i}{3} + \frac{3r'_i}{100} < \frac{2r'_i}{5}.$$

We see that every element of $T_{\vec{p}}$ is covered by $N_{2r'_i/5}[Y_i^{\vec{p}}]$ for infinitely many integers i, proving the lemma. □

Finally, we are ready to construct the toast sequence that will be enough for proving all statements of Theorem 1.3. Namely, for $i \geqslant 1$, define

$$\widehat{K}_i := \bigcup_{\vec{p} \in \{0,\dots,5\}^d} \left(\widetilde{K}_i^{\vec{p}} \cap T_{\vec{p}} \right), \text{ and} \tag{5.27}$$

$$\widehat{L}_i := \bigcup_{\vec{p} \in \{0,\dots,5\}^d} \left(\widetilde{L}_i^{\vec{p}} \cap T_{\vec{p}} \right). \tag{5.28}$$

Since each set $T_{\vec{p}}$ consists of whole components of G_d and the sets $T_{\vec{p}}$ form a partition of the torus $\mathbb{T}^k$, Lemma 5.10 implies the following.

Lemma 5.15. *$(J_1, \widehat{K}_1, \widehat{L}_1, J_1, \widehat{K}_2, \widehat{L}_2, \dots)$ is a toast sequence that completely covers $\mathbb{T}^k$.* □

6. Using toast sequences to round flows

Here we show how one can use a toast sequence to obtain an integer-valued $(\mathbb{1}_A - \mathbb{1}_B)$-flow from a sequence of real-valued flows. The key new challenge is that the used real-valued flows f_i need not meet the demand $\mathbb{1}_A - \mathbb{1}_B$ exactly (only when we pass to the limit as $i \to \infty$) while we are not allowed to access the whole sequence when rounding.

6.1. Dealing with finite connected sets

Here, we present one of the main subroutines, which perturbs a flow along the boundary of a finite set of vertices, attaining the desired properties on all edges except possibly one. Many of the ideas appearing here are borrowed from [28, Section 5].

Definition 6.1. A *triangle* in a graph G is a set $\{u, v, w\}$ of three distinct vertices of G, every pair of which are adjacent in G.

Definition 6.2. Given a graph G and a set $F \subseteq E(G)$, let $\triangle_F$ be the graph with vertex set F where two elements uv and yz of F are adjacent in $\triangle_F$ if they are contained in a common triangle of G.

Definition 6.3. An *Eulerian circuit* in a finite graph G is a sequence $(v_1, v_2, \dots, v_t)$ of vertices of G such that $v_1 = v_t$ and the sequence $(v_1v_2, v_2v_3, \dots, v_{t-1}v_t)$ is an enumeration of the edge set of G.

In the language of graph theory, an Eulerian circuit is a closed walk in a graph which traverses every edge exactly once. Perhaps the most classical result in graph theory is Euler's Theorem from 1736 which says that a finite graph G has an Eulerian circuit if and only if it is connected and all of its vertices have even degree; see e.g. [10, Theorem 1.8.1]. The following lemma highlights a small technical advantage of defining the graph G_d in terms of the set $\{\vec{\gamma} \in \mathbb{Z}^d : \|\vec{\gamma}\|_\infty = 1\}$ of generators of $\mathbb{Z}^d$ as opposed to the standard basis (although one can also work in the latter graph, see e.g. [6, Section 6]). Recall that the edge-boundary $\partial_E S$ of $S \subseteq \mathbb{T}^k$ consists of the edges of G_d with exactly one vertex in S.

Lemma 6.4 *(Marks and Unger [28, Proof of Lemma 5.6]). If $S \subseteq \mathbb{T}^k$, then every finite component of $\triangle_{\partial_E S}$ has an Eulerian circuit.*

Proof. By Euler's Theorem, it suffices to show that every vertex of $\triangle_{\partial_E S}$ has even degree. Let $\boldsymbol{uv} \in \partial_E S$ with $\boldsymbol{u} \in S$ and $\boldsymbol{v} \notin S$. Note that, for any triangle $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ in G_d containing $\boldsymbol{u}$ and $\boldsymbol{v}$, exactly one of the edges $\boldsymbol{uw}$ or $\boldsymbol{vw}$ is in $\partial_E S$. Also, for any pair of

edges of G_d which are adjacent in $\triangle_{\partial_E S}$, there is a unique triangle which contains them. So, it suffices to prove that $\boldsymbol{uv}$ is contained in an even number of triangles of G_d.

Let $\vec{n}$ be the unique element of $\mathbb{Z}^d$ such that $\|\vec{n}\|_\infty = 1$ and $\boldsymbol{v} = \boldsymbol{u} + \sum_{i=1}^d n_i \boldsymbol{x}_i$. Let $T_0 := \{i : 1 \leqslant i \leqslant d \text{ and } n_i = 0\}$ and $T_1 := \{1, 2, \dots, d\} \setminus T_0$. Note that the number of triangles containing $\boldsymbol{uv}$ is exactly

$$3^{|T_0|} \cdot 2^{|T_1|} - 2.$$

Indeed, the number of choices of $\vec{n}' = (n'_1, \dots, n'_d) \in \mathbb{Z}^d$ such that $\boldsymbol{w} := \boldsymbol{u} + \sum_{i=1}^d n'_i \boldsymbol{x}_i$ forms a triangle with $\boldsymbol{u}$ and $\boldsymbol{v}$ in G_d or satisfies $\boldsymbol{w} \in \{\boldsymbol{u}, \boldsymbol{v}\}$ is exactly $3^{|T_0|}$ (each n'_i for $i \in T_0$ can assume any value in $\{-1, 0, 1\}$) times $2^{|T_1|}$ (there are exactly two possible values for n'_i for each $i \in T_1$).

Since $|T_1| \geqslant 1$, the number of triangles is even, finishing the proof. □

Recall that by a *hole* of $S \subseteq \mathbb{T}^k$ we mean a finite component of $G_d \restriction \left(\mathbb{T}^k \setminus S\right)$. Marks and Unger [28, Proof of Lemma 5.6] use a result of Timár [35] to show that, if S is a finite subset of $\mathbb{T}^k$ with no holes such that $G_d \restriction S$ is connected, then $\triangle_{\partial_E S}$ is connected. Combining this with Lemma 6.4 and Euler's Theorem, we get the following.

Lemma 6.5 *(Marks and Unger [28, Lemma 5.6]). If S is a finite subset of $\mathbb{T}^k$ with no holes and $G_d \restriction S$ is connected, then $\triangle_{\partial_E S}$ has an Eulerian circuit.* □

The following definition is helpful for explaining the way in which we perturb flow values on triangles in G_d.

Definition 6.6. Given a graph G and an ordered triple $(u, v, w) \in V(G)^3$ such that $\{u, v, w\}$ is a triangle in G, define $\circlearrowleft_{u,v,w}$ to be the flow in G such that, for $x, y \in V(G)$,

$$\circlearrowleft_{u,v,w}(x, y) := \begin{cases} 1, & \text{if } (x, y) \in \{(u, v), (v, w), (w, u)\}, \\ -1, & \text{if } (x, y) \in \{(v, u), (w, v), (u, w)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Given a flow ϕ in G_d and a finite subset S of $\mathbb{T}^k$ with no holes such that $G_d \restriction S$ is connected, the operation of *rounding ϕ along the boundary of S* is defined as follows. Using Lemma 6.5, we let $\boldsymbol{u}_1\boldsymbol{v}_1, \dots, \boldsymbol{u}_t\boldsymbol{v}_t$ be an Eulerian circuit in $\triangle_{\partial_E S}$ where, for each $1 \leqslant s \leqslant t$, we have $\boldsymbol{u}_s \in S$ and $\boldsymbol{v}_s \notin S$. Moreover, among all Eulerian circuits, we choose the $\prec_{\text{lex}}$-*minimal* one (that is, one such that $(\boldsymbol{u}_1, \boldsymbol{v}_1, \dots, \boldsymbol{u}_t, \boldsymbol{v}_t)$ is minimal under $\prec_{\text{lex}}$). Note that $\boldsymbol{u}_t = \boldsymbol{u}_1$ and $\boldsymbol{v}_t = \boldsymbol{v}_1$. First, we do the *adjustment* step, where we redefine $\phi(\boldsymbol{u}_{t-1}, \boldsymbol{v}_{t-1})$ by adding $[\phi^{\text{out}}(S)] - \phi^{\text{out}}(S)$ to it and change $\phi(\boldsymbol{v}_{t-1}, \boldsymbol{u}_{t-1})$ accordingly. The flow out of S is now an integer. Then, for each $s = 1, \dots, t-2$, one by one, we let $\boldsymbol{w}_s$ be the (unique) vertex of $\{\boldsymbol{u}_{s+1}, \boldsymbol{v}_{s+1}\}$ such that $\{\boldsymbol{u}_s, \boldsymbol{v}_s, \boldsymbol{w}_s\}$ is a triangle in G_d and redefine ϕ to be

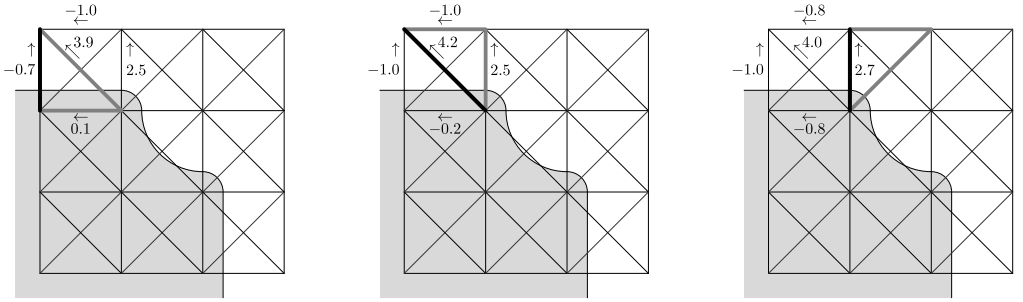

Fig. 3. Two steps of the operation of rounding a flow ϕ along the boundary of a set S. The vertices within the grey region are in S. The edge $\boldsymbol{u}_s\boldsymbol{v}_s$ currently being rounded is depicted by a bold black line and the other two edges of the triangle containing $\boldsymbol{u}_s\boldsymbol{v}_s$ and $\boldsymbol{u}_{s+1}\boldsymbol{v}_{s+1}$ are depicted by a bold grey line. Some edges are labelled with numbers which represent their current flow value in the direction indicated by the arrow.

$$\phi := \phi + ([\phi(\boldsymbol{u}_s, \boldsymbol{v}_s)] - \phi(\boldsymbol{u}_s, \boldsymbol{v}_s))\circlearrowleft_{\boldsymbol{u}_s,\boldsymbol{v}_s,\boldsymbol{w}_s}. \tag{6.1}$$

(See Fig. 3 for an illustration.) Each of the steps in (6.1) preserves the flow out of every vertex. After all $t-2$ steps have been completed, every edge $\boldsymbol{u}_i\boldsymbol{v}_i$ of $\partial_E S$, except possibly $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$, is assigned to an integer flow value by ϕ (because the last triangle update affecting $\phi(\boldsymbol{u}_i\boldsymbol{v}_i)$ makes it integer). However, the total flow out of S is an integer, and so $\phi(\boldsymbol{u}_{t-1}, \boldsymbol{v}_{t-1})$ must be an integer as well.

For future reference, we also define the flow $\mathcal{O}_S^\phi$ (where ϕ in this notation will refer to the initial flow ϕ before any modifications took place), which is the sum of all $t-2$ increments in (6.1). Thus, $\mathcal{O}_S^\phi$ is a 0-flow (i.e., the flow out of every vertex is 0) and, with ϕ referring to its initial value, the flow $\phi + \mathcal{O}_S^\phi$ assumes integer values on all edges of $\partial_E S$ except possibly $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$.

More generally, given a flow ϕ in G_d and a set $D \subseteq \mathbb{T}^k$ such that every set in $\mathrm{comp}(D)$ is finite, the operation of *rounding ϕ along the boundary of D* is defined as follows. First, for every $S \in \mathrm{comp}(D)$ and every hole S' of S, we round ϕ along the boundary of S'. Then, for each $S \in \mathrm{comp}(D)$, we round ϕ along the boundary of the union of S and all of its holes. All of these operations are well-defined and the order in which we perform them does not affect the result. Note that, for every finite connected S, the edge sets $\partial_E S'$, for S' being a hole in S or being S with all its holes filled, partition the edge boundary of S. Indeed, every pair in $\partial_E S$ has exactly one point outside of S, lying in a (unique) component S' of $G_d \restriction (\mathbb{T}^k \setminus S)$. If S' is not a hole of S, then S' is infinite while S'', the complement of S' in the component of G_d containing S, is finite. Trivially, S'' cannot have any holes, so S'' is exactly S with all its holes filled, as desired.

We define

$$\mathcal{O}_D^\phi := \sum_{S\in\mathrm{comp}(D)} \sum_{S'} \mathcal{O}_{S'}^\phi,$$

where S' in the inner sum is a hole of S, or S with all its holes filled.

Next, we show that if, additionally, any two distinct components of $G_d \restriction D$ are at distance at least three[4] in G_d, then rounding along the boundary of such a set D cannot displace the flow values by an arbitrary amount.

Lemma 6.7. *Let ϕ be a flow in G_d and D be a subset of $\mathbb{T}^k$ such that each component of $G_d \restriction D$ is finite and any two distinct components of $G_d \restriction D$ are at distance at least three in G_d. Then, the sum of the absolute values of flow changes on each $\boldsymbol{uv} \in E(G_d)$ when we round ϕ on the boundary of D is at most $(3^d - 2)/2$. (In particular, this also upper bounds the total change on each edge, that is, $\|\mathcal{O}_D^\phi\|_\infty \leqslant (3^d - 2)/2$.)*

Proof. By construction, the flow from $\boldsymbol{u}$ to $\boldsymbol{v}$ changes only if there exists $S \in \mathrm{comp}(D)$ such that either

- $\boldsymbol{uv} \in \partial_E S$,
- $\boldsymbol{u}, \boldsymbol{v} \in N[S] \setminus S$, or
- $\boldsymbol{u}, \boldsymbol{v} \in N[\mathbb{T}^k \setminus S] \cap S$.

Since any two components of D are separated by a distance of at least three in G_d, we see that the choice of the component S is unique.

In the first two cases, we let T be the unique component of $G_d \restriction \left(\mathbb{T}^k \setminus S\right)$ which contains $\{\boldsymbol{u}, \boldsymbol{v}\} \setminus S$. While rounding ϕ on the boundary of T (in the case that T is finite), or on the boundary of the union of S and all of its holes (in the case that T is infinite), the value of $\phi(\boldsymbol{u}, \boldsymbol{v})$ is changed at most once for every triangle containing $\boldsymbol{u}$ and $\boldsymbol{v}$, where we view the last triangle of the Eulerian tour (the one that we have not used for any flow updates) as "responsible" for the initial adjustment of the flow on the second to last edge. The number of such triangles is at most the number of neighbours of $\boldsymbol{u}$ different from $\boldsymbol{v}$, which is at most $3^d - 2$. Also, each time that the flow value of an edge changes, it is displaced by at most $1/2$ (since we round to a nearest integer). The flow from $\boldsymbol{u}$ to $\boldsymbol{v}$ is not changed at any other stage, giving the required.

In the third case, for each $\boldsymbol{w} \notin S$ such that $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a triangle in G_d, there is a unique component T of $G_d \restriction \left(\mathbb{T}^k \setminus S\right)$ containing $\boldsymbol{w}$. This triangle is used at most once when rounding ϕ along the boundary of T (in the case that T is finite) or the union of S and all of its holes (in the case that T is infinite). Thus, each triangle containing $\boldsymbol{u}$ and $\boldsymbol{v}$ contributes at most $1/2$ to the amount by which $\phi(\boldsymbol{u}, \boldsymbol{v})$ changes; there are at most $3^d - 2$ such triangles, and so the proof is complete. □

Next, given a flow ϕ in G_d and a subset S of $\mathbb{T}^k$ such that $G_d \restriction S$ is connected, the operation of *completing ϕ within S* is defined as follows. Consider all integer-valued flows ϕ' in $G_d \restriction S$ such that

[4] This assumption, while convenient, is actually not necessary.

$$\sum_{\boldsymbol{v}\in S}\phi'(\boldsymbol{u},\boldsymbol{v})+\sum_{\boldsymbol{v}\in\mathbb{T}^k\setminus S}\phi(\boldsymbol{u},\boldsymbol{v})=\mathbb{1}_A(\boldsymbol{u})-\mathbb{1}_B(\boldsymbol{u}),\quad\text{for every }\boldsymbol{u}\in S, \tag{6.2}$$

and, given this, $\|\phi'\|_\infty$ is as small as possible. (It will be the case that, whenever we apply this operation, at least one such ϕ' exists.) If S is finite then we choose ϕ' so that the sequence $(\phi'(\boldsymbol{u},\boldsymbol{v}):(\boldsymbol{u},\boldsymbol{v})\in S^2)$ is lexicographically minimised, where the pairs in S are viewed as being ordered according to $\prec_{\text{lex}}$. If S is infinite then we choose ϕ' arbitrarily, using Theorem 2.3 (i.e., using the Axiom of Choice). Change $\phi(\boldsymbol{u},\boldsymbol{v})$ to be equal to $\phi'(\boldsymbol{u},\boldsymbol{v})$ for all $\boldsymbol{u},\boldsymbol{v}\in S$.

6.2. Rounding a sequence of flows on a toast sequence

The following key definition will allow us to round flows.

Definition 6.8. The *rounding* of a sequence $(g_1,g_2,\dots)$ of real-valued flows in G_d on a toast sequence $(D_1,D_2,\dots)$ is the flow f obtained via the following steps.

(i) Initially, let f be the identically zero flow except for each $i\geqslant 1$, $S\in\text{comp}(D_i)$, and $(\boldsymbol{u},\boldsymbol{v})\in\partial_E S$ we define $f(\boldsymbol{u},\boldsymbol{v}):=g_i(\boldsymbol{u},\boldsymbol{v})$ and $f(\boldsymbol{v},\boldsymbol{u}):=g_i(\boldsymbol{v},\boldsymbol{u})$ (that is, we copy the values of g_i on $\partial_E D_i$).
(ii) For each $i\geqslant 1$, round f along the boundary of D_i.
(iii) For each component S of the graph $G':=(\mathbb{T}^k,E(G_d)\setminus\bigcup_{i=1}^\infty\partial_E D_i)$, which is obtained from G_d by removing the edge boundaries of all sets D_i, complete f within S, as specified after (6.2). (If at least one completion step fails, then the whole procedure fails and f is undefined.)

Observe that, since $(D_1,D_2,\dots)$ is a toast sequence, the first two steps in the above definition are well-defined; also, the second step does not depend on the order in which we round the boundaries.

Lemma 6.9. *In the notation of Definition 6.8, suppose that there are a constant C and a sequence $d_1\leqslant d_2\leqslant\dots$ of integers such that*

$$\lim_{i\to\infty}g_i(\boldsymbol{u})=\mathbb{1}_A(\boldsymbol{u})-\mathbb{1}_B(\boldsymbol{u}),\quad\textit{for each }\boldsymbol{u}\in\mathbb{T}^k, \tag{6.3}$$

$\|g_1\|_\infty\leqslant C$ *and, for each $i\geqslant 1$, we have that*

$$(3^d-1)\cdot(d_i+1)^d\cdot\sum_{j=i}^\infty\|g_{j+1}-g_j\|_\infty<1/2, \tag{6.4}$$

and every component S of $G_d\restriction D_i$ has diameter at most d_i.

Then the following statements hold:

(6.5) *all completion steps succeed,*
(6.6) *the obtained function f is an integer-valued flow from A to B,*
(6.7) $\|f\|_\infty \leqslant C + (3^d + 2)/2$,
(6.8) *the restriction of f to the edges intersecting D_i is a $(d_i + 1)$-local function of $g_1, \dots, g_i$ and $D_1, \dots, D_i$.*

Proof. By combining the fact that $\|g_1\|_\infty \leqslant C$ with the bound in (6.4) in the case $i = 1$, we see that the sequence $g_1, g_2, \dots$ converges uniformly to a bounded function. Let f_∞ denote the pointwise limit of this sequence. By (6.3), the function f_∞ is an $(\mathbb{1}_A - \mathbb{1}_B)$-flow.

Let us show (6.5), that is, that all completion steps succeed. For this, we introduce one more operation. Given two flows ϕ and ψ in G_d and a finite subset S of $\mathbb{T}^k$ with no holes such that $G_d \restriction S$ is connected, the operation of *equalising ϕ to ψ along the boundary of S* is defined as follows. Let $(\boldsymbol{u}_1\boldsymbol{v}_1, \dots, \boldsymbol{u}_t\boldsymbol{v}_t)$ with $\boldsymbol{u}_1, \dots, \boldsymbol{u}_t \in S$ be the lexicographically minimal Eulerian circuit in $\triangle_{\partial S}$ (which exists by Lemma 6.5). For each $s = 1, \dots, t-2$, one by one, we let $\boldsymbol{w}_s \in \{\boldsymbol{u}_{s+1}, \boldsymbol{v}_{s+1}\}$ be such that $\{\boldsymbol{u}_s, \boldsymbol{v}_s, \boldsymbol{w}_s\}$ is a triangle in G_d and redefine ϕ to be

$$\phi := \phi + (\psi(\boldsymbol{u}_s, \boldsymbol{v}_s) - \phi(\boldsymbol{u}_s, \boldsymbol{v}_s))\circlearrowleft_{\boldsymbol{u}_s, \boldsymbol{v}_s, \boldsymbol{w}_s}. \tag{6.9}$$

Note that this operation makes the new value of ϕ on $(\boldsymbol{u}_s, \boldsymbol{v}_s)$ to be equal to $\psi(\boldsymbol{u}_s, \boldsymbol{v}_s)$. Also, we define $\mathcal{E}_S^{\phi,\psi}$ (where ϕ stands for its initial value) as the difference between the final and initial flows ϕ during this process, that is, $\mathcal{E}_S^{\phi,\psi}$ is the sum of all $t-2$ increments in (6.9). Thus $\mathcal{E}_S^{\phi,\psi}$ is a 0-flow such that $\phi + \mathcal{E}_S^{\phi,\psi}$ is equal to ψ on every pair in ∂S except possibly the pair $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$. More generally, for a set $D \subseteq \mathbb{T}^k$ such that every component of $G_d \restriction D$ is finite, *equalising ϕ to ψ along the boundary of D* is defined as follows. First, for each component S of $G_d \restriction D$ and each hole S' of S, we equalise ϕ to ψ along the boundary of S'. Then, for every component S of $G_d \restriction D$, we equalise ϕ to ψ along the boundary of the union of S and all of its holes. Also, define

$$\mathcal{E}_D^{\phi,\psi} := \sum_{S \in \mathrm{comp}(D)} \sum_{S'} \mathcal{E}_{S'}^{\phi,\psi},$$

where S' in the inner sum is a hole of S, or S with all holes filled.

The proof of the following claim follows similar lines as that of Lemma 6.7, so we give only a very brief proof sketch.

Claim 6.9.1. *Let ϕ and ψ be flows in G_d, M be a non-negative integer and D be a subset of $\mathbb{T}^k$ such that $|\partial_E S| \leqslant M$ for every component S of $G_d \restriction D$ and any two components of $G_d \restriction D$ are at distance at least three in G_d. Then $\|\mathcal{E}_D^{\phi,\psi}\|_\infty \leqslant M\|\phi - \psi\|_\infty$.*

Sketch of Proof. Note that, before the s-th step as in (6.9), the current value of ϕ on $\boldsymbol{u}_{s+1}\boldsymbol{v}_{s+1}$ is still the original value. Thus, during the step, its absolute value increases by

at most $|\psi(\boldsymbol{u}_s, \boldsymbol{v}_s) - \phi(\boldsymbol{u}_s, \boldsymbol{v}_s)|$, which in turn is at most $s\|\phi - \psi\|_\infty$ by an easy induction on s. The claim now follows since we do at most $M - 1$ steps. □

This claim gives that, for every $S \in \mathrm{comp}(D_i)$, if S' is a hole of S or S will all holes filled then

$$\|\mathcal{E}_S^{f_\infty, g_i}\|_\infty \leqslant 1/2. \tag{6.10}$$

Indeed, $|S'| \leqslant (d_i+1)^d$ since the projection of S', when viewed as a subset of $\mathbb{Z}^d$, on each coordinate is at most $d_i + 1$ by the diameter assumption. Thus $|\partial_E S'| \leqslant (3^d - 1)(d_i+1)^d$. Also,

$$\|f_\infty - g_i\|_\infty \leqslant \sum_{j=i}^{\infty} \|g_{j+1} - g_j\|_\infty < \frac{1/2}{(3^d - 1)(d_i + 1)^d} \tag{6.11}$$

by (6.4). Therefore, (6.10) follows from Claim 6.9.1.

Now, consider the real-valued flow

$$h := f_\infty + \sum_{i=1}^{\infty} \mathcal{E}_{D_i}^{f_\infty, g_i} + \sum_{i=1}^{\infty} \mathcal{O}_{D_i}^{g_i}, \tag{6.12}$$

which will be used to certify that all completion steps succeed. This function h is a flow from A to B since it is obtained from the $(\mathbb{1}_A - \mathbb{1}_B)$-flow f_∞ by adding a 0-flow. Take any S which is, for some $T \in \mathrm{comp}(D_i)$, either T with all holes filled or a hole of T. By Lemma 6.5, the graph $\triangle_{\partial_E S}$ has an Eulerian circuit; let $(\boldsymbol{u}_1\boldsymbol{v}_1, \ldots, \boldsymbol{u}_t\boldsymbol{v}_t)$ with $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t \in S$ be the $\prec_{\mathrm{lex}}$-minimal one. By the definition of $\mathcal{E}_{D_i}^{f_\infty, g_i}$, the flows

$$f' := f_\infty + \mathcal{E}_{D_i}^{f_\infty, g_i}$$

and g_i coincide on all pairs in $\partial_E S$ except possibly $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$. Also, we have by (6.11) that

$$\left| g_i^{\mathrm{out}}(S) - |A \cap S| + |B \cap S| \right| = \left| g_i^{\mathrm{out}}(S) - f_\infty^{\mathrm{out}}(S) \right| \leqslant |\partial_E S| \cdot \|f_\infty - g_i\|_\infty < 1/2.$$

Recall that, initially, f was set to be g_i on $\partial_E S$. Thus the adjustment step of making $f^{\mathrm{out}}(S)$ integer by adjusting its value on $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$ by at most $1/2$ makes $f^{\mathrm{out}}(S)$ to be equal to $|A \cap S| - |B \cap S|$, which in turn is equal to the flow out of S by f' since it is obtained from the $(\mathbb{1}_A - \mathbb{1}_B)$-flow f_∞ by adding a 0-flow. Thus f after the adjustment is equal to f' on the edge $\boldsymbol{u}_{t-1}\boldsymbol{v}_{t-1}$ (as they are already equal on every other edge in $\partial_E S$). Thus when we round f along the boundary of S, we add the same $t - 2$ increments as we would do for f', that is, $\mathcal{O}_S^{f'} = \mathcal{O}_S^{g_i}$. Hence, the flows $\mathcal{O}_{D_i}^{f'}$ and $\mathcal{O}_{D_i}^{g_i}$ coincide on $\partial_E S$. We do not modify the values of f on $\partial_E S$ during any other steps, so they are the same as the values of $f' + \mathcal{O}_{D_i}^{f'}$ (which are in turn the same as the values of h

on $\partial_E S$). Thus the final flow f coincides with h on $\partial_E D_i$ (for every i) and the Integral Flow Theorem (Theorem 2.3) shows that each completion step works, that is, (6.2) can always be satisfied. Thus Conclusion (6.5) holds.

Conclusion (6.6) is a direct consequence of the previous conclusion since the completion step is applied to every component obtained by removing $\bigcup_{i=1}^{\infty} \partial_E D_i$ from $E(G_d)$.

For (6.7), recall that the Integral Flow Theorem (Theorem 2.3) guarantees that the produced integer-valued flow differs from the given real-valued flow by at most 1 on every edge. Thus, by Lemma 6.7 and by (6.10), we have

$$\|f\|_\infty \leqslant \|h\|_\infty + 1 \leqslant \|f_\infty\|_\infty + \frac{3^d - 2}{2} + \frac{1}{2} + 1 \leqslant C + \frac{1}{2} + \frac{3^d + 1}{2} = C + \frac{3^d + 2}{2},$$

as desired.

For (6.8), take any $\boldsymbol{u} \in D_i$. Its component $S \in \mathrm{comp}(D_i)$ has diameter at most d_i and thus entirely lies inside the d_i-neighbourhood of $\boldsymbol{u}$. Since $\boldsymbol{u}$ has access to every vertex of S, it also has access to the flow values on all edges in $\partial_E S$ (because, by our conventions, the flow value on an edge is encoded in each of its endpoints). Since $(D_i)_{i=1}^{\infty}$ is a toast sequence, the choices of how the flow f is modified on S during the rounding along $\partial_E S$ or any completion inside S are well-defined functions of the restrictions of $g_1, \dots, g_i$ and $D_1, \dots, D_{i-1}$ to S. Thus the restriction of the final flow f to the edges intersecting S is indeed a (d_i+1)-local function of $g_1, \dots, g_i$ and $D_1, \dots, D_i$. (Note that we add 1 to d_i so that each endpoint of an affected edge, which includes points in $N_1[S] \setminus S$, can compute the new flow value on the edge.) □

7. Proof of Theorem 1.3

We are now ready to prove our main result, Theorem 1.3.

For each $i \geqslant 1$, let m_i and m_i' be the minimal non-negative integers such that

$$(3^d - 1) \cdot (t_i + 1)^d \cdot \left(\frac{c\, 2^{1+\varepsilon}}{2^{d+\varepsilon m_i}(2^\varepsilon - 1)} \right) < 1/2, \text{ and} \tag{7.1}$$

$$(3^d - 1) \cdot (t_i' + 1)^d \cdot \left(\frac{c\, 2^{1+\varepsilon}}{2^{d+\varepsilon m_i'}(2^\varepsilon - 1)} \right) < 1/2. \tag{7.2}$$

The main purpose of these bounds is to certify inequalities of the type given by (6.4).

Let $f = f_{J\widehat{K}\widehat{L}}$ be the rounding of the sequence $(f_{m_i}, f_{m_i}, f_{m_i'})_{i=1}^{\infty}$ of flows on the toast sequence $(J_i, \widehat{K}_i, \widehat{L}_i)_{i=1}^{\infty}$. (Recall that the sets $\widehat{K}_i$ and $\widehat{L}_i$ were defined before Lemma 5.15 and $(J_i, \widehat{K}_i, \widehat{L}_i)_{i=1}^{\infty}$ is a toast sequence by the lemma; also, f_m for $m \geqslant 0$ is the flow returned by Lemma 3.1.) We will use this flow f to obtain the desired equidecomposition between A and B.

Remark 7.1. If the reader is interested only in Parts (a) and (c) of Theorem 1.3 then it suffices to take for f the flow f_J, which we define to be the rounding of the se-

quence $(f_{m_1}, f_{m_2}, \dots)$ on the toast sequence $(J_1, J_2, \dots)$. By construction (or, formally, by Lemma 7.3), f and f_J are the same except on the set $\mathbb{T}^k \setminus \bigcup_{i=1}^{\infty} J_i$ which is small in many respects (e.g., its closure is null).

For reader's convenience, let us summarise some key properties of the flow $f = f_{J\widehat{K}\widehat{L}}$ in Lemma 7.2 below. This lemma is stated in a more general form so that it also applies to some other auxiliary flows (in particular to f_J when we take $(J_i, \emptyset, \emptyset)_{i=1}^{\infty}$ for the toast sequence $(J_i', K_i', L_i')_{i=1}^{\infty}$). Recall that, for each $\vec{p} \in \{0, \dots, 5\}^d$, the set $T_{\vec{p}}$ (defined in (5.26)) is a union of whole components of G_d; moreover, these sets partition $\mathbb{T}^k$.

Lemma 7.2. *Fix any $\vec{p} \in \{0, \dots, 5\}^d$. Suppose that for each $i \geqslant 1$, J_i', K_i' and L_i' are obtained from respectively J_i, $K_i^{\vec{p}}$ and $L_i^{\vec{p}}$ by removing the vertex sets of some components so that $(J_i', K_i', L_i')_{i=1}^{\infty}$ is a toast sequence. Let f' be the rounding of the flows $(f_{m_i}, f_{m_i}, f_{m_i'})_{i=1}^{\infty}$ on $(J_i', K_i', L_i')_{i=1}^{\infty}$. Then the following statements hold:*

(7.3) *the function f' is an integer-valued flow in G_d from A to B,*
(7.4) $\|f'\|_\infty \leqslant \frac{c\, 2^{1+\varepsilon}}{2^d(2^\varepsilon - 1)} + \frac{3^d+2}{2}$,
(7.5) *there is a $(2^{m_i} + t_i)$-local function of $J_1', \dots, J_i', K_1', \dots, K_i'$, A and B that coincides with f' on all edges of G_d intersecting $T_{\vec{p}} \cap \left(\bigcup_{j=1}^{i} K_j'\right)$,*
(7.6) *there is a $(2^{m_i'} + t_i')$-local function of $J_1', \dots, J_i', K_1', \dots, K_i', L_1', \dots, L_i'$, A and B that coincides with f' on all edges of G_d intersecting $T_{\vec{p}} \cap \left(\bigcup_{j=1}^{i} L_j'\right)$.*

Moreover, if for every $C \in \mathrm{comp}(J_i')$ and $S \in \mathrm{comp}(K_j') \cup \mathrm{comp}(L_j')$ at distance at most 2 we have that $j \geqslant i$ and $N_2[C] \subseteq S$, then

(7.7) *the flow f' coincides with g on every edge intersecting $\bigcup_{i=1}^{\infty} J_i'$, where g is the flow obtained by rounding $(f_{m_i})_{i=1}^{\infty}$ on $(J_i')_{i=1}^{\infty}$,*
(7.8) *there is a $(2^{m_i} + 2r_i + 2q_{i-1}')$-local function of $J_1', \dots, J_i'$, A and B that coincides with f' on all edges of G_d intersecting $T_{\vec{p}} \cap \left(\bigcup_{j=1}^{i} J_j'\right)$.*

Proof. We just apply Lemma 6.9 to the flows $(f_{m_i}, f_{m_i}, f_{m_i'})_{i=1}^{\infty}$ and the toast sequence $(J_i', K_i', L_i')_{i=1}^{\infty}$ that were used to define f'. Lemmas 5.6 and 5.7 show that we can take

$$(2r_i + 2q_{i-1}', t_i, t_i')_{i=1}^{\infty} \tag{7.9}$$

for the sequence $(d_i)_{i=1}^{\infty}$ that upper bounds the diameters of finite components in our toast sequence.

By (3.3), we have for all $m \geqslant 0$ that

$$\|f_m\|_\infty \leqslant \|f_0\|_\infty + \sum_{i=1}^{m} \|f_i - f_{i-1}\|_\infty \leqslant \sum_{i=1}^{m} \frac{2c}{2^{d+\varepsilon(i-1)}} \leqslant \frac{c\, 2^{1+\varepsilon}}{2^d(2^\varepsilon - 1)} \tag{7.10}$$

that is, the flows f_m have their ℓ_∞-norm uniformly bounded. Furthermore, Assumption (6.3) of Lemma 6.9 holds by (3.2) while Assumption (6.4) holds (with respect to the sequence in (7.9)) by our choice of m_i and m'_i in (7.1) and (7.2).

Thus Lemma 6.9 applies to the flow f' with C being the expression in the right-hand side of (7.10).

The claims in (7.3) and (7.4) follow from respectively Conclusions (6.6) and (6.7) of Lemma 6.9.

Recall that f_{m_i} is a $(2^{m_i}-1)$-local function of A and B by Conclusion (3.4) of Lemma 3.1. Thus we can replace a local function of f_{m_i} by a local function of A and B, with just increasing the locality radius by $2^{m_i}-1$. Thus (7.5) holds by Conclusion (6.8) of Lemma 6.9, according to which it is enough to increase the locality radius by the diameter bound plus 1. The same argument applied to $f_{m'_i}$ shows that (7.6) holds.

It remains to show that the last two claims of the lemma hold, under the additional assumption. For (7.7), take any edge $\boldsymbol{uv}$ intersecting some J'_i. Suppose first that $\boldsymbol{uv} \in \partial_E J'_i$. Let S be the (unique) component of $G \restriction J'_i$ such that $\boldsymbol{uv} \in \partial_E S$. The value of g (as well as the value of f') on $\boldsymbol{uv}$ is the one which is assigned during the rounding of f_{m_i} along the boundary of S. Thus g and f' assume the same value on $\boldsymbol{uv}$, as desired. So suppose that both $\boldsymbol{u}$ and $\boldsymbol{v}$ are in J'_i. Let S be the component of $G \restriction J'_i$ containing $\boldsymbol{u}$ and $\boldsymbol{v}$. Consider any $T \in \mathrm{comp}(K'_j) \cup \mathrm{comp}(L'_j)$ which is at distance at most 2 from S. By the extra assumption, we have that $j \geqslant i$ and $N_2[S] \subseteq T$. Thus $\partial_E S$ separates $\boldsymbol{uv}$ from $\partial_E T$. This means that when we remove the boundary edge set $\bigcup_{j=1}^{\infty}(\partial_E J'_j \cup \partial_E K'_j \cup \partial_E L'_j)$ from $E(G_d)$, the component containing $\boldsymbol{uv}$ will be the same when we remove only $\bigcup_{j=1}^{\infty} \partial_E J'_j$. Thus, indeed f' assumes the same value as g on $\boldsymbol{uv}$. This proves (7.7).

Finally, (7.8) follows from (7.7) by Conclusion (6.8) of Lemma 6.9 applied directly to the flow g. □

Observe that, for every $\vec{p} \in \{0,\dots,5\}^d$, Lemma 7.2 (when we take the toast sequence $(J_i, \widetilde{K}_i^{\vec{p}}, \widetilde{L}_i^{\vec{p}})_{i=1}^{\infty}$ for $(J'_i, K'_i, L'_i)_{i=1}^{\infty}$) applies to the restriction of the flow $f = f_{J\widehat{K}\widehat{L}}$ to the set $T_{\vec{p}}$ (which consists of whole components of G_d). Indeed, $\widehat{K}_i \cap T_{\vec{p}} = \widetilde{K}_i^{\vec{p}} \cap T_{\vec{p}}$ and $\widehat{L}_i \cap T_{\vec{p}} = \widetilde{L}_i^{\vec{p}} \cap T_{\vec{p}}$ while $\widetilde{K}_i^{\vec{p}}$ and $\widetilde{L}_i^{\vec{p}}$ are obtained from respectively $K_i^{\vec{p}}$ and $L_i^{\vec{p}}$ by removing whole components. We conclude by Conclusions (7.3) and (7.4) of Lemma 7.2 that f is a bounded integer flow from A to B. Also, the extra assumption of Lemma 7.2 holds by Lemma 5.8 and (5.21); in particular, the following (obvious from the construction) result is formally proved by Conclusion (7.7).

Lemma 7.3. *The flows f and f_J coincide on every edge intersecting $\bigcup_{i=1}^{\infty} J_i$.* □

Lemma 2.16 can be applied to the integer $(\mathbb{1}_A - \mathbb{1}_B)$-flow f, producing an equidecomposition between A and B. We claim that this equidecomposition satisfies all claims of Theorem 1.3. Since Part (c) of Theorem 1.3 is already taken care of by the second claim of Lemma 2.16, it remains to prove the remaining two parts.

Recall that each piece returned by Lemma 2.16 is a local function of A, B, finitely many strips and the flow f. Since the corresponding target families in Parts (a) and (b) of Theorem 1.3 form a-invariant algebras (that contain all strips as well as our sets A and B), it is enough by Observation 2.6 to prove that each set of the form $Z^f_{\vec{\gamma},\ell}$ (as defined in (2.14)) for $\vec{\gamma} \in \{-1,0,1\}^d$ and integer ℓ with $|\ell| \leqslant \|f\|_\infty$ belongs to the corresponding algebra.

7.1. Dimension of boundaries

Here we prove Part (a) of Theorem 1.3 that the upper Minkowski dimension of the boundary of the obtained pieces is strictly smaller than k. Specifically, we show that, for every $\vec{\gamma} \in \{-1,0,1\}^d$ and $|\ell| \leqslant \|f\|_\infty$, the upper Minkowski dimension of the boundary of $Z^f_{\vec{\gamma},\ell}$ is at most $k-\zeta$, where

$$\zeta := \frac{\epsilon}{4d\max\{d/\varepsilon,\, d+1\}+1} > 0. \tag{7.11}$$

(Recall that ϵ, d and ε were defined in (2.4), (2.5) and (2.6), respectively.) One can show that $d/\varepsilon \geqslant d+1$ if $k \geqslant 2$.

Let us pause for a moment to discuss the quantitative bound on the boundary dimension that we get when $k \geqslant 2$ and $\dim_\square(\partial A) = \dim_\square(\partial B) = k-1$. In this special case, we can choose ϵ to be arbitrarily close to 1, take d to be $k+1$ and ε close to $1/k$. Thus we get that any $\zeta < \frac{1}{4k(k+1)^2+1}$ works here, and so we can obtain pieces whose boundaries have upper Minkowski dimension arbitrarily close to $k - \frac{1}{4k(k+1)^2+1}$. In particular, if $k=2$, as in the case that A is a disk and B is a square, then this quantity evaluates to $145/73 < 1.987$. Thus, this is sufficient for proving Theorem 1.2.

Here we need to use the specific choices $r'_i := 100^{2^{i+1}-1}$ and $r_i := 100^{2^{i+1}-2}$, as defined in (2.7) and (2.8) respectively.

Let us provide some auxiliary results first. A *box* in $\mathbb{T}^k$ is a set of the form $\prod_{i=1}^k [a_i, b_i)$ where $0 \leqslant a_i < b_i \leqslant 1$ for all $1 \leqslant i \leqslant k$.

Lemma 7.4 *(Laczkovich [23, Lemma 2]; see also Schmidt [32, p. 517]). For almost every choice of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ in $\mathbb{T}^k$ and for every $t > 0$ there exists $C > 0$ such that*

$$D\left(N_r^+[\boldsymbol{u}], I\right) \leqslant C \log^{k+d+t}(r) \tag{7.12}$$

for every element $\boldsymbol{u} \in \mathbb{T}^k$, integer $r \geqslant 2$ and box I in $\mathbb{T}^k$.

This result is needed to prove the following lemma, whose conclusion was one of the assumptions (namely, Property (2.13)) made about $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ in Section 2.4.

Lemma 7.5. *There exist positive constants c_1 and C_2 such that the following holds. If, for $i \geqslant 1$, we define $\gamma_i := c_1 r_i^{-d} \log^{-2}(r_i)$, $R_i := C_2 r_i \log^{(k+d+3)/d}(r_i)$ and*

$$Q_i := [0, \gamma_i) \times [0,1)^{k-1}, \tag{7.13}$$

then, with positive probability with respect to uniform independent $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d \in \mathbb{T}^k$*, it holds for every integer* $i \geqslant 1$ *that the strip* Q_i *is* r_i*-discrete in* G_d *while* $N_{R_i}[Q_i] = \mathbb{T}^k$*.*

Proof. Let C_1 be a constant satisfying Lemma 7.4 with $t = 1$ for at least half (in measure) of the choices of $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_d) \in (\mathbb{T}^k)^d$. One way to see its existence (without checking whether the proof of Lemma 7.4 gives some effective bounds on C_1) is as follows. For every real C, the set $\mathcal{X}_C$ of sequences $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_d) \in (\mathbb{T}^k)^d$ for which Lemma 7.4 holds for this C (with $t = 1$) is an analytic subset, since it is the projection of the Borel subset of $(\mathbb{T}^k)^d \times \Omega$ of points satisfying (7.12), where $\Omega \subseteq \mathbb{T}^k \times \{2, 3, \dots\} \times \mathbb{T}^{2k}$ is the subspace encoding all suitable triples $(\boldsymbol{u}, r, I)$ equipped with the standard (Polish) topology. As every analytic subset is universally measurable (see e.g. [18, Theorem 21.10]) and the countable nested union $\bigcup_{i=1}^{\infty} \mathcal{X}_i \subseteq \mathbb{T}^k$ has Lebesgue measure 1, by the countable additivity there is an index i such that the measure of $\mathcal{X}_i$ is at least $1/2$ and we can take C_1 to be equal to this i.

We will use the following estimates that hold by choosing the constants $c_1 > 0$ and $C_2 \gg C_1$ suitably: for every integer $i \geqslant 1$,

$$\sum_{i=1}^{\infty} (2r_i + 1)^d \gamma_i < 1/12 \tag{7.14}$$

and

$$(2R_i + 1)^d \gamma_i > C_1 \log^{k+d+1}(2R_i). \tag{7.15}$$

Let us briefly check that suitable c_1 and C_2 exist. The i-th summand in the left-hand side of (7.14) is at most $(3r_i)^d \cdot c_1 r_i^{-d} \log^{-2}(r_i) \leqslant O(c_1 2^{-2i})$. This is summable, so (7.14) can be satisfied by choosing c_1 sufficiently small. Also, the left-hand size of (7.15) is at least $R_i^d \gamma_i = C_2^d \cdot c_1 \cdot \log^{k+d+1}(r_i)$. As the exponent at the logarithm matches that in the right-hand side of (7.15), this constraint can be satisfied by taking C_2 sufficiently large.

We start by applying a simple union bound argument to estimate the probability that Q_i fails to be r_i-discrete for some $i \geqslant 1$. For $1 \leqslant j \leqslant d$, let $\boldsymbol{x}_j = (x_{j,1}, \dots, x_{j,d})$ with each $x_{j,i} \in [0,1)$. Of course, only the first coordinates $x_{j,1}$ matter for the lemma.

Clearly, Q_i fails to be r_i-discrete if and only if there exists a non-zero vector $\vec{n} \in \{-r_i, \dots, 0, \dots, r_i\}^d$ such that the sum $\sum_{j=1}^d n_j x_{j,1}$ viewed in $\mathbb{R}$ (i.e., not modulo 1) is at distance less than γ_i from an integer. Fix a non-zero vector $\vec{n} = (n_1, \dots, n_d)$. By symmetry, assume that $n_d \neq 0$ and, in fact, $n_d > 0$. Suppose that $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d-1} \in \mathbb{T}^k$ have already been sampled. Then, since $0 \leqslant n_d x_{j,1} < n_d$, there are at most $n_d + 1$ different possible integers that the sum $\sum_{j=1}^d n_j x_{j,1}$ can be within γ_i of and, for each of them, the probability of this event is at most $2\gamma_i/n_d$ and thus the probability of at least one happening is at most $2\gamma_i(n_d + 1)/n_d \leqslant 4\gamma_i$. By the Union Bound over $\vec{n}$, the probability that Q_i is not r_i-discrete is at most $4(2r_i + 1)^d \gamma_i$.

Thus the probability that the set Q_i is not r_i-discrete for some $i \geqslant 1$ is at most $\sum_{i=1}^{\infty} 4(2r_i+1)^d\gamma_i$, which is at most $1/3$ by (7.14).

Now, for the other assertion, take an arbitrary element $\boldsymbol{u} \in \mathbb{T}^k$. By the definition of C_1, with probability at least $1/2$ it holds for every integer i that the discrepancy of $N_{R_i}[\boldsymbol{u}]$ with respect to any strip is at most $C_1 \log^{k+d+1}(2R_i)$. In particular, for the strip Q_i, we have

$$|N_{R_i}[\boldsymbol{u}] \cap Q_i| \geqslant (2R_i+1)^d\gamma_i - C_1 \log^{k+d+1}(2R_i),$$

which is strictly positive by (7.15). From the point of view of Q_i, this states that its translates by integer vectors of ℓ_∞-norm at most R_i under the action $a : \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ cover the whole torus, as required.

Thus the probability that a uniform $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_d) \in (\mathbb{T}^k)^d$ satisfies the lemma is at least $1/2 - 1/3 > 0$, as desired. □

Let us remark that the existence of C_1 in the proof of Lemma 7.5 can be argued without referring to the theory of analytic sets. Informally speaking, to compute the smallest possible $C_1(\boldsymbol{x}_1, \dots, \boldsymbol{x}_d)$ for given $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d \in \mathbb{T}^k$, it is enough to consider only those boxes that have at least one integer combination of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ on each side, plus 2^{2d} choices whether to include that side or not (realisable by making the box infinitesimally larger or smaller in that direction). There are countably many choices here and thus the best possible $C_1 : (\mathbb{T}^k)^d \to [0, \infty]$ is in fact a Borel function, as the supremum of countably many Borel functions.

Lemma 7.6. *If $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ satisfy the conclusion of Lemma 7.5, then, as $i \to \infty$, there is a maximally r_i-discrete set X_i which is an $r_i^{d+1} \log^{O(1)}(r_i)$-local function of the strip Q_i that was defined in (7.13).*

Proof. Let c_1 and C_2 satisfy Lemma 7.5 and let R_i be defined as in the lemma. Thus the set $\mathcal{Q}$ of translates of Q_i by vectors of the form $\sum_{i=1}^d n_i \boldsymbol{x}_i$ with $\vec{n} \in \mathbb{Z}^d$ and $\|\vec{n}\|_\infty \leqslant R_i$ covers $\mathbb{T}^k$. Lemma 2.11, when applied to this covering, yields a maximally r_i-discrete set X_i which is an $r_i(2R_i+1)^d$-local function of the sets in $\mathcal{Q}$. Every such translate is an R_i-local function of Q_i itself. Thus, since $R_i = r_i \log^{O(1)}(r_i)$, we get that X_i is an $r_i^{d+1} \log^{O(1)}(r_i)$-local function of Q_i. □

Recall that our final flow $f = f_{J\widehat{KL}}$ was defined at the beginning of Section 7. Since our forthcoming argument will only use the values of f on edges intersecting $\bigcup_{i=1}^\infty I_i \subseteq \bigcup_{i=1}^\infty J_i$, the reader may equivalently use f_J instead of f here.

Lemma 7.7. *For any $\vec{\gamma} \in \{-1, 0, 1\}^d$, $\ell \in \operatorname{range}(f)$ and $i \geqslant 1$, the set $Z^f_{\vec{\gamma},\ell} \cap I_i$ can be written as a Boolean combination of at most $r_i^{d \max\{d/\varepsilon,\, d+1\}+o(1)}$ translates of A, B and $Q_1, \dots, Q_i$.*

Proof. By (7.1) it holds that $2^{m_i} = \Theta(r_i^{d/\varepsilon})$. By Conclusion (7.8) of Lemma 7.2 (applied to f), each set $Z^f_{\vec{\gamma},\ell} \cap I_i$ is a $O(r_i^{d/\varepsilon})$-local function of A, B and $J_1, \dots, J_i$. By Lemmas 5.4 and 5.11, the sets I_i and J_i are both $O(r_i)$-local functions of $X_1, \dots, X_i$ which in turn, by Lemma 7.6, are $r_i^{d+1+o(1)}$-local functions of $Q_1, \dots, Q_i$. By adding the locality radii, we see that $Z^f_{\vec{\gamma},\ell} \cap I_i$ is a local function of A, B and $Q_1, Q_2, \dots, Q_i$ of radius at most $r_i^{\max\{d/\varepsilon,\, d+1\}+o(1)}$, implying the lemma. □

We are now in position to prove the following result.

Lemma 7.8. *For every $\vec{\gamma} \in \{-1,0,1\}^d$ and $\ell \in \mathrm{range}(f)$, the boundary of $Z^f_{\vec{\gamma},\ell}$ has upper Minkowski dimension at most $k - \zeta$.*

Proof. By the definition of ϵ in (2.4), the upper Minkowski dimension of each of ∂A and ∂B is less than $k - \epsilon$. Thus we can choose $\delta_0 > 0$ to be sufficiently small so that, for all $0 < \delta < \delta_0$,

$$\max\left\{\lambda\left(\{\boldsymbol{x} : \mathrm{dist}_\infty(\boldsymbol{x}, \partial A) \leqslant \delta\}\right),\, \lambda\left(\{\boldsymbol{x} : \mathrm{dist}_\infty(\boldsymbol{x}, \partial B) \leqslant \delta\}\right)\right\} < \delta^\epsilon, \tag{7.16}$$

where dist_∞ denotes the ℓ_∞-distance on the torus. Moreover, we also assume that δ_0 is small enough so that

$$r_1 < \delta_0^{-4\zeta}. \tag{7.17}$$

Let $0 < \delta < \delta_0$ be arbitrary and let i be the unique index so that

$$r_i \leqslant \delta^{-4\zeta} < r_{i+1}. \tag{7.18}$$

Note that such an index i is guaranteed to exist by (7.17). Let $\mathcal{P}$ be a collection of boxes in $\mathbb{T}^k$ of side-length δ such that each point of $\mathbb{T}^k$ is contained in at least one and at most 2^k sets in $\mathcal{P}$. (For example, let $\mathcal{P}$ be the product of such 1-dimensional coverings.) Define

$$\mathcal{Z} := \left\{ Z^f_{\vec{\gamma},\ell} : \vec{\gamma} \in \{-1,0,1\}^d \text{ and } \ell \in \mathrm{range}(f) \right\}.$$

Since f is integer-valued, the set $\mathcal{Z}$ is finite, having at most $3^d(2\|f\|_\infty + 1) < \infty$ elements.

Since we can choose δ_0 arbitrarily small, let us view δ as tending to 0 (and $i \to \infty$) using the asymptotic notation accordingly. Thus, for example, $|\mathcal{Z}| = O(1)$.

Our goal is to show that at most $(1/\delta)^{k-\zeta+o(1)}$ boxes in $\mathcal{P}$ can intersect $\bigcup_{Z\in\mathcal{Z}} \partial Z$. This will be enough for bounding the upper Minkowski dimension of this set, since each box in $\mathcal{P}$ can intersect at most 2^k boxes of a regular δ-grid.

Let $\mathcal{B}$ be the collection of boxes in $\mathcal{P}$ which intersect the topological boundary of $Z \cap I_i$ for some $Z \in \mathcal{Z}$. Recall that i was chosen to satisfy (7.18) while the set I_i was defined in Section 5.2. Let $\mathcal{I}$ be the collection of sets in $\mathcal{P}$ which are not in $\mathcal{B}$ and intersect $\mathbb{T}^k \setminus I_i$.

By construction, $\mathcal{B}\cup\mathcal{I}$ contains every set in $\mathcal{P}$ which intersects the topological boundary of a set in $\mathcal{Z}$. So, it suffices to bound $|\mathcal{B}|$ and $|\mathcal{I}|$.

By Lemma 7.7, any set of the form $Z\cap I_i$ for $Z\in\mathcal{Z}$ can be written as a Boolean combination of at most $r_i^{d\max\{d/\varepsilon,d+1\}+o(1)}$ translates of A, B and $Q_1,\dots,Q_i$. The union of all boxes in $\mathcal{B}$ lies within dist_∞-distance at most δ from the boundary of at least one of these translates. Thus, using the volume bound, the estimate in (7.16) and that, trivially, the measure of $\{\boldsymbol{x}:\mathrm{dist}_\infty(\boldsymbol{x},\partial Q_i)\leqslant\delta\}$ is at most 4δ, we get that

$$\begin{aligned}\delta^k|\mathcal{B}| &\leqslant 2^k\lambda\left(\bigcup_{Z\in\mathcal{Z}}\{\boldsymbol{x}:\mathrm{dist}_\infty(\boldsymbol{x},\partial(Z\cap I_i))\leqslant\delta\}\right)\\ &\leqslant 2^k\,|\mathcal{Z}|\,\left(r_i^{d\max\{d/\varepsilon,\,d+1\}+o(1)}(\delta^\epsilon+i\cdot 4\delta)\right).\end{aligned}$$

By the definitions of ζ and i (that is, by (7.11) and (7.18)) and since $i=O(\log(\log(r_i)))$, we have that $i\delta=o(\delta^\epsilon)$ and

$$\delta^k|\mathcal{B}|\leqslant(\delta^{-4\zeta})^{d\max\{d/\varepsilon,\,d+1\}+o(1)}\cdot\delta^{\epsilon-o(1)}\leqslant\delta^{\zeta-o(1)}.$$

So, indeed, $|\mathcal{B}|\leqslant(1/\delta)^{k-\zeta+o(1)}$.

Since none of the boxes in $\mathcal{I}$ are contained in $\mathcal{B}$, and $\bigcup_{Z\in\mathcal{Z}}Z=\mathbb{T}^k$, all sets in $\mathcal{I}$ must all be contained in the interior of $\mathbb{T}^k\setminus I_i$. Thus, by Lemma 5.13 combined with (2.7) and (2.8), we have that

$$\delta^k|\mathcal{I}|\leqslant 2^k\lambda(\mathbb{T}^k\setminus I_i)=O(r'_{i-1}/r_i)=O\left(r_{i+1}^{-1/4}\right)=O\left((1/\delta)^{-\zeta}\right).$$

This completes the proof of Lemma 7.8. □

Proof of Theorem 1.3(a). Recall that the final equidecomposition is obtained by applying the local function of Lemma 2.16 to f (i.e., to the sets $Z^f_{\vec{\gamma},\ell}$). Clearly, the family of subsets of $\mathbb{T}^k$ whose boundary has upper Minkowski dimension at most $k-\zeta$ is a translation-invariant algebra. Since this algebra contains each set $Z^f_{\vec{\gamma},\ell}$ by Lemma 7.8, it also contains all pieces of the equidecomposition by Observation 2.6. □

7.2. Borel complexity

This part of the proof heavily relies on the properties of various toast sequences that we constructed earlier. First, let us briefly recall some key related definitions and properties. The toast sequence $(J_i,\widehat{K}_i,\widehat{L}_i)_{i=1}^\infty$, as defined before Lemma 5.15, is obtained by taking the toast sequence $(J_i,\widetilde{K}_i^{\vec{p}},\widetilde{L}_i^{\vec{p}})_{i=1}^\infty$ inside each (a-invariant) set $T_{\vec{p}}$, and then taking the union over all $\vec{p}$. In turn, $(J_i,\widetilde{K}_i^{\vec{p}},\widetilde{L}_i^{\vec{p}})_{i=1}^\infty$ (whose properties are summarised in Lemma 5.10) is obtained from $(J_i,K_i^{\vec{p}},L_i^{\vec{p}})_{i=1}^\infty$ by removing those components from sets $K_i^{\vec{p}}$ and $L_i^{\vec{p}}$ that "conflict with later sets". Finally, the sets $K_i^{\vec{p}}$ and $L_i^{\vec{p}}$ are defined

by (5.24) and (5.25) (which are the same as the definitions in (5.12) and (5.13), except we take the "$\vec{p}$-shifted" sets Y_i).

Recall that we have to show that all pieces of the constructed equidecomposition belong to $\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}(\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}^0_1 \cup \mathcal{T}_A \cup \mathcal{T}_B)))$, that is, each piece can be obtained from open sets and translations of the sets A and B by taking Boolean combinations, then countable unions and then Boolean combinations. (In fact, our proof gives a slightly stronger version where open sets can be replaced by strips.)

For $\vec{p} \in \{0,\dots,5\}^d$, let the flow $f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}$ be obtained by rounding the sequence of flows $(f_{m_i}, f_{m_i}, f_{m'_i})_{i=1}^{\infty}$ on the toast sequence $(J_i, \widetilde{K}_i^{\vec{p}}, \widetilde{L}_i^{\vec{p}})_{i=1}^{\infty}$. Since the toast sequences $(J_i, \widehat{K}_i, \widehat{L}_i)_{i=1}^{\infty}$ and $(J_i, \widetilde{K}_i^{\vec{p}}, \widetilde{L}_i^{\vec{p}})_{i=1}^{\infty}$ coincide inside $T_{\vec{p}}$, we have for all $\vec{n}$ and ℓ that

$$Z^f_{\vec{n},\ell} = \bigcup_{\vec{p}\in\{0,\dots,5\}^d} \left(T_{\vec{p}} \cap Z^{f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}}_{\vec{n},\ell}\right). \tag{7.19}$$

So, it suffices to show that $T_{\vec{p}}$ and $Z^{f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}}_{\vec{n},\ell}$ for $\vec{p} \in \{0,\dots,5\}^d$ are all in the set family $\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}(\boldsymbol{\mathcal{B}}(\boldsymbol{\Sigma}^0_1 \cup \mathcal{T}_A \cup \mathcal{T}_B)))$. We start with the former.

Lemma 7.9. *Each of the sets $T_{\vec{p}}$ constructed in the proof of Lemma 5.14 can be written as a Boolean combination of G_δ sets.*

Proof. Here we are going to use the assumption in (2.18) which states that each component of G_d intersects the topological boundary of at most one set Y_i. Thus, if the sets $R_i(\boldsymbol{v})$ for $i \geqslant 1$ in the proof of Lemma 5.14 are re-defined in terms of the interior of Y_i instead of Y_i itself, then, for each $\boldsymbol{v} \in \mathbb{T}^k$, the sets $R_1(\boldsymbol{v}), R_2(\boldsymbol{v}), \dots$ remain unchanged, except for possibly one index $i \geqslant 1$. In particular, the set $R^*(\boldsymbol{v})$ of their accumulation points is unchanged. So, assuming that $R_i(\boldsymbol{v})$ is defined in terms of the interior of Y_i, we have that for any $b, i \geqslant 1$ and $\vec{p} \in \{0,\dots,5\}^d$ the set

$$\left\{\boldsymbol{v} \in \mathbb{T}^k : R_i(\boldsymbol{v}) \cap \prod_{s=1}^{d} \left(\frac{p_s-3}{3} - \frac{1}{b}, \frac{p_s-2}{3} + \frac{1}{b}\right) \neq \emptyset\right\}$$

is open as well. So, for each $\vec{p}$,

$$\begin{aligned} T'_{\vec{p}} &= \left\{\boldsymbol{v} \in \mathbb{T}^k : R^*(\boldsymbol{v}) \cap \prod_{s=1}^{d} \left[\frac{p_s-3}{3}, \frac{p_s-2}{3}\right] \neq \emptyset\right\} \\ &= \bigcap_{b=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} \left\{\boldsymbol{v} \in \mathbb{T}^k : R_i(\boldsymbol{v}) \cap \prod_{s=1}^{d} \left(\frac{p_s-3}{3} - \frac{1}{b}, \frac{p_s-2}{3} + \frac{1}{b}\right) \neq \emptyset\right\} \end{aligned}$$

is a G_δ set. Recall that, by definition (namely by (5.26)), $T_{\vec{p}}$ consists of those $\boldsymbol{v}$ for which $\vec{p}$ is the lexicographically smallest vector with $\boldsymbol{v} \in T'_{\vec{p}}$. Thus $T_{\vec{p}}$ is Boolean combination of these sets. □

Next, let us consider the sets of the form $Z^{f_{J\vec{K}^{\vec{p}}\vec{L}^{\vec{p}}}}_{\vec{n},\ell}$.

Lemma 7.10. *There exists a collection $\mathcal{W}$ of subsets of $\mathbb{T}^k$ such that*

(7.20) *every set in $\mathcal{W}$ is a countable union of Boolean combinations of strips and translates of A and B;*

(7.21) *for every $\vec{n} \in \{-1,0,1\}^d$, $\ell \in \mathrm{range}(f)$ and $\vec{p} \in \{0,\dots,5\}^d$, the set $Z^{f_{J\vec{K}^{\vec{p}}\vec{L}^{\vec{p}}}}_{\vec{n},\ell}$ can be written as a Boolean combination of some sets in $\mathcal{W}$.*

Proof. Recall that the flow f_J is obtained by rounding $(f_{m_i})_{i=1}^\infty$ on $(J_i)_{i=1}^\infty$. For the proof, we need to define two further flows:

- $f_{JK^{\vec{p}}}$ is the rounding of $(f_{m_i}, f_{m_i})_{i=1}^\infty$ on the toast sequence $(J_i, K_i^{\vec{p}})_{i=1}^\infty$, and
- $f_{K^{\vec{p}}L^{\vec{p}}}$ is the rounding of $(f_{m_i}, f_{m'_i})_{i=1}^\infty$ on the toast sequence $(K_i^{\vec{p}}, L_i^{\vec{p}})_{i=1}^\infty$.

Recall that each of $(J_i, K_i^{\vec{p}})_{i=1}^\infty$ and $(K_i^{\vec{p}}, L_i^{\vec{p}})_{i=1}^\infty$ is a toast sequence by Lemma 5.9. Also, Lemma 7.2 (with the used toast sequence $(J'_i, K'_i, L'_i)_{i=1}^\infty$ being respectively $(J_i, K_i^{\vec{p}}, \emptyset)_{i=1}^\infty$ and $(\emptyset, K_i^{\vec{p}}, L_i^{\vec{p}})_{i=1}^\infty$) applies to each of these two new flows.

For every $\vec{n} \in \{-1,0,1\}^d$, $\ell \in \mathrm{range}(f)$, $\vec{p} \in \{0,\dots,5\}^d$ and $i \geqslant 1$, define

$$W^{J}_{\vec{n},\ell,i} := \left\{ \boldsymbol{u} : f_J(\boldsymbol{u}, \vec{n} \cdot_a \boldsymbol{u}) = \ell \text{ and } \{\boldsymbol{u}, \vec{n} \cdot_a \boldsymbol{u}\} \cap \left(\bigcup_{j=1}^{i} J_j \right) \neq \emptyset \right\},$$

$$W^{JK^{\vec{p}}}_{\vec{n},\ell,i} := \left\{ \boldsymbol{u} : f_{JK^{\vec{p}}}(\boldsymbol{u}, \vec{n} \cdot_a \boldsymbol{u}) = \ell \text{ and } \{\boldsymbol{u}, \vec{n} \cdot_a \boldsymbol{u}\} \cap \left(\bigcup_{j=1}^{i} K_j^{\vec{p}} \right) \neq \emptyset \right\},$$

$$W^{K^{\vec{p}}L^{\vec{p}}}_{\vec{n},\ell,i} := \left\{ \boldsymbol{u} : f_{K^{\vec{p}}L^{\vec{p}}}(\boldsymbol{u}, \vec{n} \cdot_a \boldsymbol{u}) = \ell \text{ and } \{\boldsymbol{u}, \vec{n} \cdot_a \boldsymbol{u}\} \cap \left(\bigcup_{j=1}^{i} (K_j^{\vec{p}} \cup L_j^{\vec{p}}) \right) \neq \emptyset \right\}.$$

In other words, these sets are obtained by restricting f_J, $f_{JK^{\vec{p}}}$ and $f_{K^{\vec{p}}L^{\vec{p}}}$ to all edges intersecting respectively $\bigcup_{j=1}^i J_j$, $\bigcup_{j=1}^i K_j^{\vec{p}}$ and $\bigcup_{j=1}^i (K_j^{\vec{p}} \cup L_j^{\vec{p}})$ and encoding the obtained partially defined flows by sequences of sets of vertices similarly as in (2.14). Recall that $J_j \subseteq K_j^{\vec{p}}$ for every j. By Lemma 7.2, each of these newly defined sets is a local function of A, B and finitely many of sets $J_i, K_i^{\vec{p}}, L_i^{\vec{p}}$ while, in turn, each of the sets $J_i, K_i^{\vec{p}}, L_i^{\vec{p}}$ is a finite union of strips by Lemma 5.12. So each set defined above is a Boolean combination of strips and translates of A and B.

Now, for all $\vec{n}$ and ℓ, define

$$W^{J}_{\vec{n},\ell} := \bigcup_{i=1}^{\infty} W^{J}_{\vec{n},\ell,i}, \quad W^{JK^{\vec{p}}}_{\vec{n},\ell} := \bigcup_{i=1}^{\infty} W^{JK^{\vec{p}}}_{\vec{n},\ell,i} \quad \text{and} \quad W^{K^{\vec{p}}L^{\vec{p}}}_{\vec{n},\ell} := \bigcup_{i=1}^{\infty} W^{K^{\vec{p}}L^{\vec{p}}}_{\vec{n},\ell,i}.$$

We define $\mathcal{W}$ to be the collection of all sets of the form $W^{J}_{\vec{n},\ell}$, $W^{JK^{\vec{p}}}_{\vec{n},\ell}$ and $W^{K^{\vec{p}}L^{\vec{p}}}_{\vec{n},\ell}$ for any $\vec{n}, \ell$ and $\vec{p}$. By above, the family $\mathcal{W}$ satisfies (7.20).

By construction (i.e. by Conclusion (7.7) of Lemma 7.2), the flows $f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}$ and f_J coincide on every pair intersecting $\bigcup_{i=1}^{\infty} J_i$. Also, if an edge $\boldsymbol{uv}$ intersects $K_j^{\vec{p}}$ then, by (5.17), the value of $f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}$ on this edge equals the value of $f_{JK^{\vec{p}}}$ unless there exists $i > j$ such that $\boldsymbol{uv}$ intersects J_i when we use the value of f_J (which may happen to coincide with the value of $f_{JK^{\vec{p}}}$ on $\boldsymbol{uv}$). Furthermore, we can drop the restriction that $i > j$ here, since our rounding procedures for any J_i and $J_{i'}$ produce the same value on each edge intersecting $J_i \cap J_{i'}$. Similarly, if an edge intersects $L_j^{\vec{p}}$, then the value of $f_{K^{\vec{p}}L^{\vec{p}}}$ on this edge is retained by $f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}$, unless it is overwritten (to a different value or the same one) due to J_i or $K_i^{\vec{p}}$ with $i > j$ intersecting this edge; again we can drop the restriction that $i > j$ here.

Thus, informally speaking, our flow values are partitioned into the following three types: J-values, K-values not overwritten by J, and L-values not overwritten by J nor K, that is, the order of precedence is J, K, L. Formally, we can express the above partition as

$$Z^{f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}}_{\vec{n},\ell} = W^{J}_{\vec{n},\ell} \cup \left(W^{JK^{\vec{p}}}_{\vec{n},\ell} \setminus \bigcup_{s\in\mathrm{range}(f_J)} W^{J}_{\vec{n},s} \right) \cup \left(W^{K^{\vec{p}}L^{\vec{p}}}_{\vec{n},\ell} \setminus \left(\left(\bigcup_{s\in\mathrm{range}(f_J)} W^{J}_{\vec{n},s} \right) \cup \left(\bigcup_{s\in\mathrm{range}(f_{JK^{\vec{p}}})} W^{JK^{\vec{p}}}_{\vec{n},s} \right) \right) \right),$$

for every non-zero flow value ℓ. Note that each union is over a finite set (since the involved flows are integer-valued and uniformly bounded), thus satisfying (7.21). The case $\ell = 0$ is somewhat special since the union $\bigcup_{i=1}^{\infty}(J_i \cup K_i^{\vec{p}} \cup L_i^{\vec{p}})$ need not cover the whole torus. However, this case also satisfies (7.21) since $Z^{f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}}_{\vec{n},0}$ is the complement of the union of the sets $Z^{f_{J\widetilde{K}^{\vec{p}}\widetilde{L}^{\vec{p}}}}_{\vec{n},\ell}$ over all possible (finitely many) non-zero flow values ℓ. This completes the proof of the lemma. □

Proof of Theorem 1.3(b). Since every strip is the difference of two open sets, Part (b) of Theorem 1.3 now follows by combining (7.19) with Lemmas 7.9 and 7.10. □

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Appendix A. Laczkovich's discrepancy bound

The purpose of this appendix is to sketch a proof of Lemma 2.1. This result is implicit in [23, Proof of Theorem 3] and its proof sketch is given in [15, pp. 677–678]. Since the dependence of d on k and $\dim_\square(\partial X)$ (which is crucial for our estimates in Section 7.1) is not explicitly calculated there, we present a slightly expanded proof sketch.

Let $X \subseteq \mathbb{T}^k$ be a measurable set such that $k-1 \leqslant \dim_\square(\partial X) < k$. The first step is to use the upper Minkowski dimension of the boundary to reduce the proof to bounding discrepancy of $N_r^+[\boldsymbol{u}]$ relative to boxes in $\mathbb{T}^k$. This argument is due to Niederreiter and Wills [31, Kollorar 4]. Recall that d is an integer such that $d > k/(k-\dim_\square(\partial X))$ and ε is a real number satisfying $0 < \varepsilon < (d(k-\dim_\square(\partial X)) - k)/k$. Define

$$\alpha := \frac{(1+\varepsilon)k}{d}.$$

By the definition of upper Minkowski dimension and the fact that $\alpha < k - \dim_\square(\partial X)$, there exists $\delta_0 \in (0,1)$ such that

$$\lambda\left(\{\boldsymbol{x} : \operatorname{dist}_\infty(\boldsymbol{x}, \partial X) \leqslant \delta\}\right) \leqslant \delta^\alpha \tag{A.1}$$

for all $0 < \delta < \delta_0$. Now, for $r \in \mathbb{N}$, choose $\delta \in (0, \delta_0)$ small with respect to r, where the dependence is clarified below. For convenience, let us assume that δ^{-1} is an integer.

Let $\mathcal{P}$ be the partition of $\mathbb{T}^k$ into a grid of δ^{-k} boxes, each with side-length δ. Let $\mathcal{B}$ be the elements of $\mathcal{P}$ which intersect ∂X. We have

$$\delta^k|\mathcal{B}| \leqslant \lambda\left(\{\boldsymbol{x} : \operatorname{dist}_\infty(\boldsymbol{x}, \partial X) \leqslant \delta\}\right) \leqslant \delta^\alpha$$

by (A.1) and so $|\mathcal{B}| \leqslant \delta^{-k+\alpha}$.

Let $\mathcal{I}$ be the elements of $\mathcal{P}$ contained in the interior of X. We then let $\mathcal{I}^*$ be the collection of boxes obtained by starting with $\mathcal{I}$ and iteratively merging two boxes if they have the same projection onto the first $k-1$ coordinates and their closures share a $(k-1)$-dimensional face. For any two distinct boxes in $\mathcal{I}^*$ which have the same projection onto the first $k-1$ coordinates, there must be at least one element in $\mathcal{B}$ "between" them which prevents them from merging. Conversely, each element of $\mathcal{B}$ prevents at most one potential merging. Therefore,

$$|\mathcal{I}^*| \leqslant \delta^{-k+1} + |\mathcal{B}| \leqslant \delta^{-k+1} + \delta^{-k+\alpha} < 2\delta^{-k+\alpha},$$

since $\alpha < 1$ by our assumption that $\dim_\square(\partial X) \geqslant k-1$. Thus, $\mathcal{I}^* \cup \mathcal{B}$ is a covering of X with at most $3\delta^{-k+\alpha}$ boxes such that each of the boxes in $\mathcal{B}$ has measure at most δ^k.

Now, given any finite set $F \subseteq \mathbb{T}^k$, by the triangle inequality applied to the partition $X = (\cup_{I\in\mathcal{I}^*} I) \cup (\cup_{I\in\mathcal{B}}(I \cap X))$, the discrepancy of F relative to X can be bounded as follows:

$$D(F,X) \leqslant \sum_{I\in\mathcal{I}^*} D(F,I) + \sum_{I\in\mathcal{B}} D(F, I\cap X) \tag{A.2}$$

We apply Lemma 7.4 to the first sum on the right side of (A.2), say with $t := 1$, and obtain that

$$\sum_{I\in\mathcal{I}^*} D\left(N_r^+[\boldsymbol{u}], I\right) \leqslant C\log^{k+d+t}(r)\,|\mathcal{I}^*| \leqslant 2C\log^{k+d+t}(r)\delta^{-k+\alpha}.$$

The contribution of $I \in \mathcal{B}$ to the second sum is at most

$$\max\left\{\,|N_r^+[\boldsymbol{u}]\cap(I\cap X)|,\ |N_r^+[\boldsymbol{u}]|\,\mu(X\cap I)\,\right\} \leqslant \max\left\{|N_r^+[\boldsymbol{u}]\cap I|, (r+1)^d\mu(I)\right\},$$

and each term can be bounded from above by $(r+1)^d\delta^k + C\log^{k+d+t}(r)$. Thus, by $|\mathcal{B}| \leqslant \delta^{-k+\alpha}$, we have

$$\sum_{I\in\mathcal{B}} D\left(N_r^+[\boldsymbol{u}], I\right) \leqslant \left((r+1)^d\delta^k + C\log^{k+d+t}(r)\right)\delta^{-k+\alpha}.$$

So, if we set $\delta := (r+1)^{-d/k}$ for $r\to\infty$, then we get

$$D\left(N_r^+[\boldsymbol{u}], X\right) \leqslant (r+1)^{d-\alpha d/k+o(1)} = (r+1)^{d-1-\varepsilon+o(1)}.$$

The extra $o(1)$ term in the exponent can clearly be taken care of by choosing ε sufficiently close to $\frac{d(k-\dim_\square(\partial X))-k}{k}$ in the beginning. Thus, Lemma 2.1 follows from Lemma 7.4.

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