Combinatorial Games

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Each of the following games are played by two players, **I** and **II**. Player **I** always starts and players alternate turns until one of them wins or loses. **If a player cannot make a move then they lose**.

Question 3.1. Start with a row of coins, each showing either Heads or Tails. On a player's turn they *must* turn over one coin showing Heads and *may* turn over a second coin to the *left* of it.

(a) Show that **II** has a winning strategy when the initial row of coins is HHH.

(b) Show that **I** has a winning strategy when the initial row of coins is $HTHTTH$.

(c) Suppose that $0 \leq i < j < k$ and that the initial row of coins is

$$\underbrace{T \cdots T}_{i} H \underbrace{T \cdots T}_{j} H \underbrace{T \cdots T}_{k} H.$$

For what values of i, j and k does **I** have a winning strategy?

Solution. (a) **I** must start by either turning one or two H to T, leaving at most two H. **II** can respond by turning all the remaining H to T and so wins.

(b) **I** can start by turning coins so that the sequence becomes $HHHTTT$. We may ignore the rightmost sequence of T and consider this as the game from part (a). As **I** is now the second player to move in this game, the same argument as above shows that he has a winning strategy.

(c) We first note that on a player's turn if they reduce the number of H to less than 3 then they lose. Therefore let us write (x, y, z) for the game with H in positions x, y and z.

Claim. II has a winning strategy if and only if

$$x \oplus y \oplus z = 0$$

where $\oplus$ is bitwise XOR. That is, $1 \oplus 2 = 3$, $3 \oplus 3 = 0$ and $3 \oplus 6 = 5$.

Proof. We first prove the reverse direction by assuming that $x \oplus y \oplus z = 0$. Now without loss of generality, we may assume that **I** starts by moving to (x', y, z) and that $y < z$.

In this case, by considering the largest bit of z, we note that

$$0 \leq z \oplus x \oplus x' < z.$$

Hence, **II** can respond by moving to $(x', y, z \oplus x \oplus x')$. Now note that $x' \oplus y \oplus (z \oplus x \oplus x') = x \oplus y \oplus z = 0$.

Thus by repeating this strategy **II** can ensure that at the start of each of **I**'s turns $x \oplus y \oplus z = 0$. Hence, the game must reach a stage at which it is **I**'s turn and the positions are $(1, 2, 3)$, which by (a) **II** wins.

Conversely, if $x \oplus y \oplus z \neq 0$ then without loss of generality $x < y < z$ and **I** can start by moving to $(x, y, x \oplus y)$. **I** can now consider himself as the second player in a game in which $x \oplus y \oplus z = 0$ and so can apply **II**'s winning strategy. Hence if $x \oplus y \oplus z \neq 0$ then **I** wins. $\square$

Now in (c) we start with $x = i + 1$, $y = i + j + 2$ and $z = i + j + k + 3$. In this case,

$$(i + 1) \oplus (i + j + 2) \leq (i + 1) + (i + j + 2) < i + j + k + 3$$

and so we have that $(i+1) \oplus (i+j+2) \oplus (i+j+k+3) \neq 0$. Therefore, by the claim, **I** has a winning strategy for all values of i, j and k satisfying $0 \leq i < j < k$.

Note, in fact in this argument it is sufficient to assume that $i < k$.

Question 3.2. Start with a row of n boxes. On a player's turn they *must* write either an **S** or **O** in *one* box. A player wins if at the end of their turn the sequence **SOS** appears in consecutive boxes.

(a) Show that if $n = 4$ and **I** starts by writing an **S** in the first box then **II** has a winning strategy.

(b) Show that **I** has a winning strategy when $n = 7$.

(c) Who has a winning strategy when $n = 2014$?

Solution. (a) If **I** starts by writing an **S** in the first box then **II** can respond by writing an **S** in the last box. Then no matter what letter **I** writes **II** can complete an **SOS** and so win.

(b) **I** can start by writing an **S** in the center box. Without loss of generality **II** plays to the left. If **I** cannot win immediately on that side then he can write an **S** in the right most box. If **II** now writes anything in either of the boxes between these two **S**s then by Game 3.2a **I** wins. But there are only two other empty boxes so if **II** writes in one of them the **I** can write in the other and so **II** is now forced to write in one of the bad boxes and so **I** wins.

(c) Label a box with an X if regardless what a player writes in that box, the other player can win immediately. We note that a box is labelled with an X if and only if it is part of the sequence **S**XX**S**.

So, as 2014 is even, after some even number of moves all empty boxes will be labelled X. Therefore, it will then be **I**'s turn and either all boxes are filled or he must fill a box labelled X. In either case he loses and so **II** has a winning strategy.