# Minimusing the number of triangles in graphs of given order and size

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- Maximising $\#K_3$ is easy (Kruskal'63, Katona'68)

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- Lovász-Simonovits'83: Solved $q \leq \varepsilon n^2$

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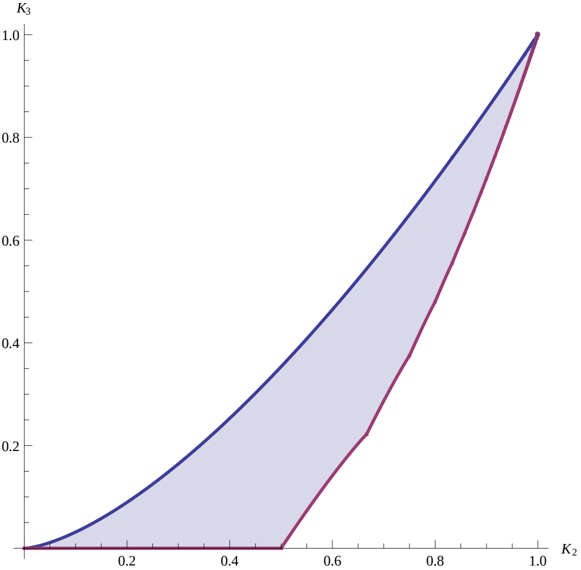
- Nikiforov'11: different proof

# Possible Edge-Triangle Densities in Limit



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- P.-Razborov'17:
  $\forall$ almost extremal $G_n$ is $o(n^2)$-close to some $H_n^a$

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- Open: exact result for all $m$

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- Razborov'07: limit versions $\Rightarrow$ asymptotic result

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# Thank you!

Hope to see you at *“Extremal Combinatorics"*
Warwick, 18-22 September 2017
`http://go.warwick.ac.uk/excomb2017`