Subsets of the 2-dimensional sphere that avoid two orthogonal vectors

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BMC2014

From Gil Kalai's blog

How Large can a Spherical Set Without Two Orthogonal Vectors Be?

Posted on May 22, 2009

The problem

Problem: Let A be a measurable subset of the d-dimensional sphere $S^d = \{x \in \mathbf{R}^{d+1} : \|x\| = 1\}$. Suppose that A **does not contain two orthogonal vectors.** How large can the d-dimensional volume of A be?

A Conjecture

Conjecture: The maximum volume is attained by two open caps of diameter $\pi/4$ around the south pole and the north pole.
For simplicity, let us normalize the volume of S^d to be 1.

- Gil Kalai on A Few Mathematical Snapshots from India (ICM2010)
- Asilomar Conference | GPU Enthusiast on Emmanuel Abbe: Erdal Arıkan's Polar Codes
- A Few Mathematical Snapshots from India (ICM2010) | Combinatorics and more on Mabruk Elon, India, and More
- valuevar on The Quantum Debate is Over! (and other Updates)
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- Shmuel Weinberger on The Quantum Debate is Over! (and other Updates)

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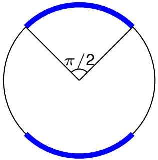
- $\alpha_1 = 1/2$: 

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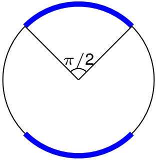

- Conjecture (Kalai'09): two opposite caps are optimal

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- Two caps conjecture $\Rightarrow$ $\chi_m(\mathbb{R}^n) \geq (\sqrt{2} + o(1))^n$

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  - Schmidt'48, Levi'51: cap is optimal

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  - Extra combinatorial contraints

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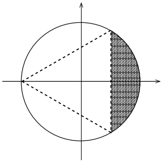

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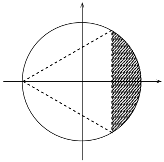

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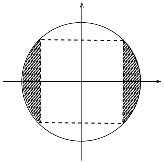

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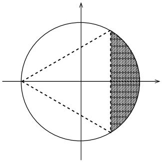

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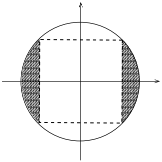

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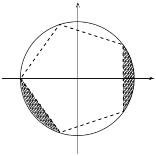

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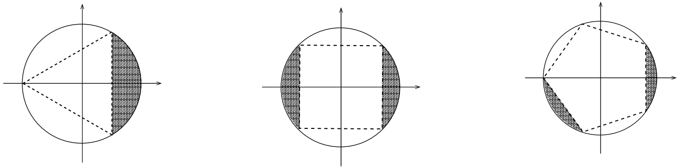

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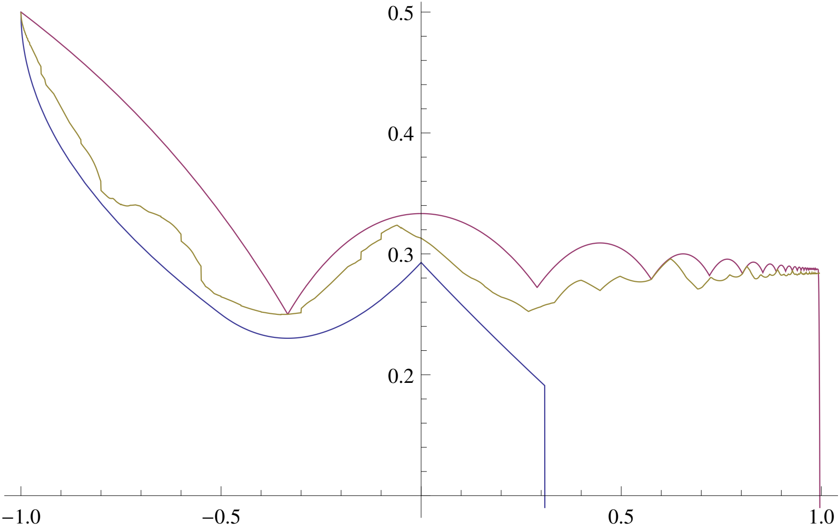
- Bachoc-Nebe-Oliveira Filho-Vallentin'09: $\theta(\mathcal{S}_{2,t})$

Bounds on $\alpha_2(t)$



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- DeCorte-P. $\geq$'14: $\lim_{t \nearrow 1} \alpha_2(t) \geq \alpha(\mathbb{R}^2)$
  - Regularity of Lebesgue measure
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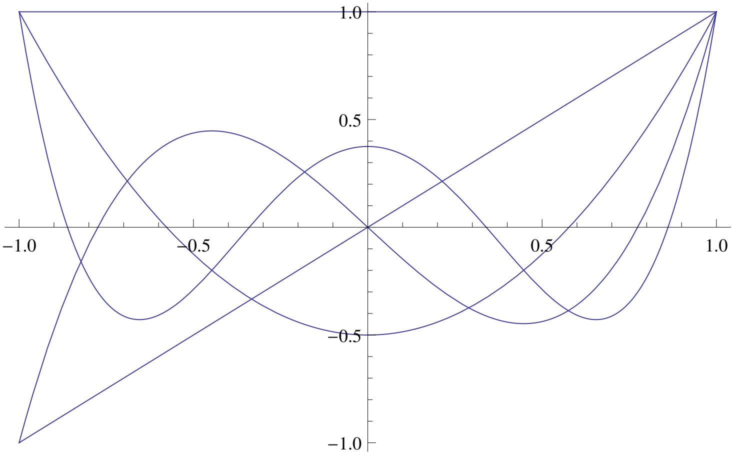
Gegenbauer Polynomials ($n = 2$)



First five

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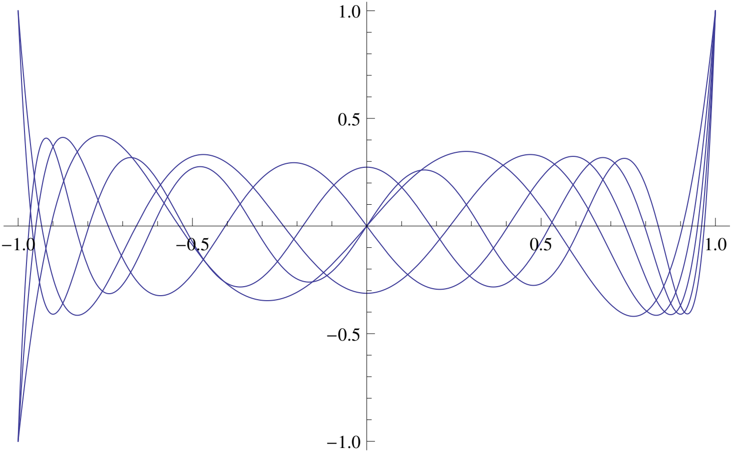

Next five

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