GEOMETRY
29 Jan 2014

Warm-up

1. Let P_0, P_1, P_2 be the vertices of a triangle T in the plane and P a point in its interior. Prove that there exists $\lambda_i \in (0,1)$ such that $\lambda_0 + \lambda_1 + \lambda_2 = 1$ and

$$P = \lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2$$

Moreover, if T_i is the triangle obtained from T by replacing P_i by P. Show that

$$\lambda_i = \frac{Area(T_i)}{Area(T)}$$

State and prove an analogous result in any dimension.

Miscellaneous

1. The vertices of a triangle ABC lie on the hyperbola $xy = 1$. Show that the orthocentre of ABC lies on that hyperbola as well.

2. The n-dimensional unit cube C is divided into 2^n rectangular boxes by n hyperplanes $P_1, P_2, ..., P_n$ such that each face of C is parallel to exactly one of these hyperplanes. The boxes are painted either white or black in such a way that neighbouring boxes have different colours. Assume that the sum of the volumes of the white boxes is equal to the sum of the volumes of the black boxes. Prove that at least one of the hyperplanes $P_1, P_2, ..., P_n$ bisects C.

3. Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

Homework

1. Let l be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to l is greater than or equal to two times the distance between X and P. If the distance from P to l is $d > 0$, find the volume of S.

2. Let us choose arbitrarily n vertices of a regular $2n$-gon and colour them red. The remaining vertices are coloured blue. We arrange all red-red distances into a non-decreasing sequence and do the same with the blue-blue distances. Prove that the sequences are equal.

3. Let $n \in \mathbb{N}$, an *n-simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, ..., P_n$ called its vertices which do not all belong to the same hyperplane. For every n-simplex S we denote $v(S)$ the volume of S, and we write $C(S)$ for the centre of the unique sphere containing all the vertices of S. Suppose that P is a point inside an n-simplex S. Let S_i be the n-simplex obtained from S by replacing its i-th vertex by P. Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + ... + v(S_n)C(S_n) = v(S)C(S)$$