Moser-Tardos Algorithm with small number of random bits

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- Sketch of proof

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  - Compactness $\;\Rightarrow\;$ $\mathbb{Z}^d$ can be coloured $\square$

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- Flood of follow-up results

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  - Expected running time $O(|\mathrm{Var}|)$

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  - Bernshteyn'20: CSPs of LOCAL complexity $O(\log n)$ admit Borel solutions on subexp growth graphs

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  - Deterministic $O(n)$ algorithm on subexp growth inputs

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  - $|\{C : \mathrm{dist}(C, C') \leqslant r\}| > (1+\delta)^r \quad \Rightarrow\Leftarrow$

Thank you!