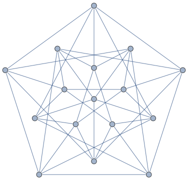
Minimising the Number of Cliques

Oleg Pikhurko and Emil R. Vaughan



C&C 2013

Method: Flag Algebras

Method: Flag Algebras

- Large graph G

Method: Flag Algebras

- Large graph G
- Inequalities between subgraph densities

Method: Flag Algebras

- Large graph G
- Inequalities between subgraph densities
- Razborov'07: Flag algebras

Method: Flag Algebras

- Large graph G
- Inequalities between subgraph densities
- Razborov'07: Flag algebras
  - Asymptotically true inequalities (as $v(G) \to \infty$)

Method: Flag Algebras

- Large graph G
- Inequalities between subgraph densities
- Razborov'07: Flag algebras
  - Asymptotically true inequalities (as $v(G) \to \infty$)
  - Formal rules for deriving them

Method: Flag Algebras

- Large graph G
- Inequalities between subgraph densities
- Razborov'07: Flag algebras
  - Asymptotically true inequalities (as $v(G) \to \infty$)
  - Formal rules for deriving them
  - One aspect: semi-definite programming

Recent Results Obtained with Flag Algebras

Recent Results Obtained with Flag Algebras

- Structures: graphs, digraphs, k-uniform hypergraphs, permutations, subgraphs of the hypercube Q_n, ...

Recent Results Obtained with Flag Algebras

- Structures: graphs, digraphs, k-uniform hypergraphs, permutations, subgraphs of the hypercube Q_n, ...
- Contributors: R.Baber, J.Balogh, J.Cummings, S.Das, V.Falgas-Ravry, R.Glebov, A.Grzesik, H.Hatami, J.Hirst, J.Hladký, P.Hu, H.Huang, T.Klimosova, D.Král', L.Kramer, B.Lidicky, N.Linial, C.-H.Liu, J.Ma, L.Mach, E.Marchant, R.Martin, H.Naves, S.Niess, S.Norine, Y.Peled, F.Pfender, O.Pikhurko, A.Razborov, C.Reiher, J.-S.Sereni, K.Spengler, B.Sudakov, J.Talbot, A.Treglown, E.Vaughan, J.Volec, P.Whalen, Z.Yilma, M.Young, ...

Turán Problem via Flag Algebras

Turán Problem via Flag Algebras

- Turán function:

$$\mathrm{ex}(n, K_3) = \max\{e(G) : v(G) = n,\ G \not\supseteq K_3\}$$

Turán Problem via Flag Algebras

- Turán function:

$$\mathrm{ex}(n, K_3) = \max\{e(G) : v(G) = n,\ G \not\supseteq K_3\}$$

- Construction: $\mathrm{ex}(n, K_3) \geq e(T_n^2) = \lfloor n^2/4 \rfloor$

Turán Problem via Flag Algebras

- Turán function:

$$\mathrm{ex}(n, K_3) = \max\{e(G) : v(G) = n,\ G \not\supseteq K_3\}$$

- Construction: $\mathrm{ex}(n, K_3) \geq e(T_n^2) = \lfloor n^2/4 \rfloor$
- $T_n^2 = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

Subgraph Densities

Subgraph Densities

- K_3-free G of order $n \to \infty$

Subgraph Densities

- K_3-free G of order $n \to \infty$
- $\#F = \#(F, G)$

Subgraph Densities

- K_3-free G of order $n \to \infty$
- $\#F = \#(F, G) =$ # induced copies of F in G

Subgraph Densities

- K_3-free G of order $n \to \infty$
- $\#F = \#(F, G) =$ # induced copies of F in G
  - $e(G) = \#(K_2, G)$

Subgraph Densities

- K_3-free G of order $n \to \infty$
- $\#F = \#(F, G) = $ # induced copies of F in G
  - $e(G) = \#(K_2, G)$
- Density of F:

$$\phi(F) = \#F / \binom{n}{v(F)}$$

Subgraph Densities

- K_3-free G of order $n \to \infty$
- $\#F = \#(F, G) =$ # induced copies of F in G
  - $e(G) = \#(K_2, G)$
- Density of F:

$$\phi(F) = \#F / \binom{n}{v(F)} = \mathbf{Prob}\{G[\text{random } v(F)\text{-set}] \cong F\}$$

Obtaining Inequalities

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- $0 \leq \mathbf{v}_x A \mathbf{v}_x^T$

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- $0 \leq \mathbf{v}_x A \mathbf{v}_x^T$
- Average over x (and ignore $o(1)$):

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- $0 \leq \mathbf{v}_x A \mathbf{v}_x^T$
- Average over x (and ignore $o(1)$):

$$\frac{1}{n}\sum_x \alpha \frac{d^2(x)}{(n-1)^2}$$

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- $0 \leq \mathbf{v}_x A \mathbf{v}_x^T$
- Average over x (and ignore $o(1)$):

$$\frac{1}{n}\sum_x \alpha \frac{d^2(x)}{(n-1)^2} \quad = \quad \frac{\alpha}{n^3}\sum_x \left(\sum_{y\sim x} 1\right)\left(\sum_{z\sim x} 1\right)$$

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- $0 \leq \mathbf{v}_x A \mathbf{v}_x^T$
- Average over x (and ignore $o(1)$):

$$\begin{aligned} \frac{1}{n}\sum_x \alpha \frac{d^2(x)}{(n-1)^2} &= \frac{\alpha}{n^3}\sum_x \left(\sum_{y\sim x} 1\right)\left(\sum_{z\sim x} 1\right) \\ &= \frac{\alpha}{6\binom{n}{3}} \times 2\,\#P_3 \end{aligned}$$

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- $0 \leq \mathbf{v}_x A \mathbf{v}_x^T$
- Average over x (and ignore $o(1)$):

$$\begin{aligned} \frac{1}{n}\sum_x \alpha \frac{d^2(x)}{(n-1)^2} &= \frac{\alpha}{n^3}\sum_x \left(\sum_{y\sim x} 1\right)\left(\sum_{z\sim x} 1\right) \\ &= \frac{\alpha}{6\binom{n}{3}} \times 2\,\#P_3 \;=\; \frac{\alpha}{3}\phi(P_3) \end{aligned}$$

Obtaining Inequalities

- $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- $0 \leq \mathbf{v}_x A \mathbf{v}_x^T$
- Average over x (and ignore $o(1)$):

$$\frac{1}{n}\sum_x \alpha \frac{d^2(x)}{(n-1)^2} = \frac{\alpha}{n^3}\sum_x \left(\sum_{y\sim x} 1\right)\left(\sum_{z\sim x} 1\right)$$
$$= \frac{\alpha}{6\binom{n}{3}} \times 2\,\#P_3 = \frac{\alpha}{3}\phi(P_3)$$

- $0 \leq (\frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$

Bounding Edge Density from Above

Bounding Edge Density from Above

- $0 \leq (\frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$

Bounding Edge Density from Above

- $0 \leq (\frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$

$\phi(K_2)$

Bounding Edge Density from Above

- $0 \leq (\frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$

$$\phi(K_2) \quad = \quad \frac{2}{3}\phi(P_3) + \frac{1}{3}\phi(\bar{P}_3)$$

Bounding Edge Density from Above

- $0 \leq (\frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$

$$
\begin{aligned}
\phi(K_2) &= \tfrac{2}{3}\phi(P_3) + \tfrac{1}{3}\phi(\bar{P}_3) \\
&\leq (\tfrac{2}{3} + \tfrac{2\beta}{3} + \tfrac{\alpha}{3})\phi(P_3) + (\tfrac{1}{3} + \tfrac{2\beta}{3} + \tfrac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)
\end{aligned}
$$

Bounding Edge Density from Above

- $0 \leq (\frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$

$$\begin{aligned}\phi(K_2) &= \tfrac{2}{3}\phi(P_3) + \tfrac{1}{3}\phi(\bar{P}_3) \\ &\leq (\tfrac{2}{3} + \tfrac{2\beta}{3} + \tfrac{\alpha}{3})\phi(P_3) + (\tfrac{1}{3} + \tfrac{2\beta}{3} + \tfrac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)\end{aligned}$$

- Note: $\phi(P_3) + \phi(\bar{P}_3) + \phi(\bar{K}_3) = 1$

SDP Formulation

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
  - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \leq \delta$

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
  - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \leq \delta$
  - $\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3} \leq \delta$

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
  - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \leq \delta$
  - $\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3} \leq \delta$
  - $\gamma \leq \delta$

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
  - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \leq \delta$
  - $\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3} \leq \delta$
  - $\gamma \leq \delta$
  - $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
  - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \leq \delta$
  - $\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3} \leq \delta$
  - $\gamma \leq \delta$
  - $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- Solution: $\delta = 1/2$ and $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
  - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \leq \delta$
  - $\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3} \leq \delta$
  - $\gamma \leq \delta$
  - $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- Solution: $\delta = 1/2$ and $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\phi(K_2) \leq \frac{1}{2}$

SDP Formulation

- $\phi(K_2) \leq (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
    - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \leq \delta$
    - $\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3} \leq \delta$
    - $\gamma \leq \delta$
    - $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- Solution: $\delta = 1/2$ and $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$

SDP Formulation

- $\phi(K_2) \le (\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3})\phi(P_3) + (\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3})\phi(\bar{P}_3) + \gamma\phi(\bar{K}_3)$
- Minimise δ:
  - $\frac{2}{3} + \frac{2\beta}{3} + \frac{\alpha}{3} \le \delta$
  - $\frac{1}{3} + \frac{2\beta}{3} + \frac{\gamma}{3} \le \delta$
  - $\gamma \le \delta$
  - $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \succeq 0$
- Solution: $\delta = 1/2$ and $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\phi(K_2) \le \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- Asymptotic result: $\mathrm{ex}(n, K_3) \le (\frac{1}{2} + o(1))\binom{n}{2}$

Interpreting the Solution

Interpreting the Solution

- Recall $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

Interpreting the Solution

- Recall $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\mathbf{v}_x A \mathbf{v}_x^T = \frac{1}{2(n-1)^2}(d(x) - \bar{d}(x))^2$

Interpreting the Solution

- Recall $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\mathbf{v}_x A \mathbf{v}_x^T = \frac{1}{2(n-1)^2}(d(x) - \bar{d}(x))^2$
- Human proof:

Interpreting the Solution

- Recall $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\mathbf{v}_x A \mathbf{v}_x^T = \frac{1}{2(n-1)^2}(d(x) - \bar{d}(x))^2$
- Human proof:
  - $0 \leq \sum_x \left(d(x) - \bar{d}(x)\right)^2$

Interpreting the Solution

- Recall $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\mathbf{v}_x A \mathbf{v}_x^T = \frac{1}{2(n-1)^2}(d(x) - \bar{d}(x))^2$
- Human proof:
  - $0 \leq \sum_x \left(d(x) - \bar{d}(x)\right)^2$
  - Expand and re-group terms

Interpreting the Solution

- Recall $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
- $\mathbf{v}_x A \mathbf{v}_x^T = \frac{1}{2(n-1)^2}(d(x) - \bar{d}(x))^2$
- Human proof:
  - $0 \leq \sum_x \left(d(x) - \bar{d}(x)\right)^2$
  - Expand and re-group terms
  - $\#K_2 \leq \frac{n^2}{4} - \frac{1}{n}\#\bar{P}_3 + O(n)$

Structure of Almost Extremal Graphs

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\; \phi(\bar{P}_3) = o(1)$

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\; \phi(\bar{P}_3) = o(1)$
- Induced Removal Lemma:
  G is $o(n^2)$-close in edit distance to $\bar{P}_3$-free

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\; \phi(\bar{P}_3) = o(1)$
- Induced Removal Lemma:
  G is $o(n^2)$-close in edit distance to $\bar{P}_3$-free
- $\bar{P}_3$-free $\;\Rightarrow\;$ complete multipartite

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\; \phi(\bar{P}_3) = o(1)$
- Induced Removal Lemma:
  G is $o(n^2)$-close in edit distance to $\bar{P}_3$-free
- $\bar{P}_3$-free $\;\Rightarrow\;$ complete multipartite
- K_3-free $\;\Rightarrow\;$ at most 2 parts

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\; \phi(\bar{P}_3) = o(1)$
- Induced Removal Lemma:
  G is $o(n^2)$-close in edit distance to $\bar{P}_3$-free
- $\bar{P}_3$-free $\;\Rightarrow\;$ complete multipartite
- K_3-free $\;\Rightarrow\;$ at most 2 parts
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\;$ parts almost equal

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\; \phi(\bar{P}_3) = o(1)$
- Induced Removal Lemma:
  G is $o(n^2)$-close in edit distance to $\bar{P}_3$-free
- $\bar{P}_3$-free $\;\Rightarrow\;$ complete multipartite
- K_3-free $\;\Rightarrow\;$ at most 2 parts
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\;$ parts almost equal
- Stability (Erdős'67, Simonovits'68):
  $\forall$ almost extremal G_n is $o(n^2)$-close to T_n^2

Structure of Almost Extremal Graphs

- $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\,\phi(\bar{P}_3)$
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\; \phi(\bar{P}_3) = o(1)$
- Induced Removal Lemma:
  G is $o(n^2)$-close in edit distance to $\bar{P}_3$-free
- $\bar{P}_3$-free $\;\Rightarrow\;$ complete multipartite
- K_3-free $\;\Rightarrow\;$ at most 2 parts
- $\phi(K_2) \approx \frac{1}{2} \;\Rightarrow\;$ parts almost equal
- Stability (Erdős'67, Simonovits'68):
  $\forall$ almost extremal G_n is $o(n^2)$-close to T_n^2
- More work: exact result for $n \geq n_0$

Further Ways of Obtaining Inequalities

Further Ways of Obtaining Inequalities

- Recall:

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- Extensions:

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- Extensions:
  - $\mathbf{v}_x$: densities of k-vertex graphs rooted at x

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \le \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- Extensions:
  - $\mathbf{v}_x$: densities of k-vertex graphs rooted at x
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at edge xy

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- Extensions:
  - $\mathbf{v}_x$: densities of k-vertex graphs rooted at x
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at edge xy
    - Average over $E(G)$ (assuming $e(G) = \Omega(n^2)$)

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- Extensions:
  - $\mathbf{v}_x$: densities of k-vertex graphs rooted at x
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at edge xy
    - Average over $E(G)$ (assuming $e(G) = \Omega(n^2)$)
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at non-edge xy

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- Extensions:
  - $\mathbf{v}_x$: densities of k-vertex graphs rooted at x
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at edge xy
    - Average over $E(G)$ (assuming $e(G) = \Omega(n^2)$)
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at non-edge xy
    - Average over $E(\overline{G})$ (assuming $e(\overline{G}) = \Omega(n^2)$)

Further Ways of Obtaining Inequalities

- Recall:
  - $0 \leq \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T)$
  - $\mathbf{v}_x = (\frac{d(x)}{n-1}, \frac{\bar{d}(x)}{n-1}), \quad x \in V(G)$
- Extensions:
  - $\mathbf{v}_x$: densities of k-vertex graphs rooted at x
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at edge xy
    - Average over $E(G)$ (assuming $e(G) = \Omega(n^2)$)
  - $\mathbf{v}_{xy}$: k-vertex graphs rooted at non-edge xy
    - Average over $E(\overline{G})$ (assuming $e(\overline{G}) = \Omega(n^2)$)
  - ...

Rounding Floating-Point Solution

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} \; ?$$

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} ?$$

- Suppose $A \succeq 0$ proves $\phi(K_2) \leq \frac{1}{2}$

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} \; ?$$

- Suppose $A \succeq 0$ proves $\phi(K_2) \leq \frac{1}{2}$
- If $\varepsilon = \frac{1}{2} - (\frac{2}{3} + 2\beta + \alpha) > 0$:

Rounding Floating-Point Solution

- $\phi(K_2) \le (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} \, ?$$

- Suppose $A \succeq 0$ proves $\phi(K_2) \le \frac{1}{2}$
- If $\varepsilon = \frac{1}{2} - (\frac{2}{3} + 2\beta + \alpha) > 0$:
  - Then $\phi(K_2) \le \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3) - \varepsilon\,\phi(P_3)$

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} ?$$

- Suppose $A \succeq 0$ proves $\phi(K_2) \leq \frac{1}{2}$
- If $\varepsilon = \frac{1}{2} - (\frac{2}{3} + 2\beta + \alpha) > 0$:
  - Then $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3) - \varepsilon\,\phi(P_3)$
  - False for $K_{n/2,n/2}$!

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} ?$$

- Suppose $A \succeq 0$ proves $\phi(K_2) \leq \frac{1}{2}$
- If $\varepsilon = \frac{1}{2} - (\frac{2}{3} + 2\beta + \alpha) > 0$:
  - Then $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3) - \varepsilon\,\phi(P_3)$
  - False for $K_{n/2,n/2}$!
- Likewise: $3\gamma = \frac{1}{2}$

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} ?$$

- Suppose $A \succeq 0$ proves $\phi(K_2) \leq \frac{1}{2}$
- If $\varepsilon = \frac{1}{2} - (\frac{2}{3} + 2\beta + \alpha) > 0$:
  - Then $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3) - \varepsilon\,\phi(P_3)$
  - False for $K_{n/2,n/2}$!
- Likewise: $3\gamma = \frac{1}{2}$

- $\phi(K_2) + \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T) = \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3)$

Rounding Floating-Point Solution

- $\phi(K_2) \leq (\frac{2}{3}+2\beta+\alpha)\phi(P_3)+(\frac{1}{3}+2\beta+\gamma)\phi(\bar{P}_3)+3\gamma\phi(\bar{K}_3)$
- How to round

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0.500007 & -0.499997 \\ -0.499997 & 0.500012 \end{pmatrix} \text{ ?}$$

- Suppose $A \succeq 0$ proves $\phi(K_2) \leq \frac{1}{2}$
- If $\varepsilon = \frac{1}{2} - (\frac{2}{3} + 2\beta + \alpha) > 0$:
  - Then $\phi(K_2) \leq \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3) - \varepsilon\,\phi(P_3)$
  - False for $K_{n/2,n/2}$!
- Likewise: $3\gamma = \frac{1}{2}$
- $\phi(K_2) + \mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T) = \frac{1}{2} - \frac{1}{3}\phi(\bar{P}_3) \;\Rightarrow\; A \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = 0$

Clique Minimisation Problem

Clique Minimisation Problem

- Independence number:

$$\alpha(G) = \min\{m : \#(\overline{K}_m, G) > 0\}$$

Clique Minimisation Problem

- Independence number:

$$\alpha(G) = \min\{m : \#(\overline{K}_m, G) > 0\}$$

- Erdős'62:

$$f(n, k, \ell) = \min\{\#(K_k, G) : \alpha(G) < \ell,\ v(G) = n\}$$

Clique Minimisation Problem

- Independence number:

$$\alpha(G) = \min\{m : \#(\overline{K}_m, G) > 0\}$$

- Erdős'62:

$$f(n, k, \ell) = \min\{\#(K_k, G) : \alpha(G) < \ell,\ v(G) = n\}$$

- Motivation: Ramsey numbers

Clique Minimisation Problem

- Independence number:

$$\alpha(G) = \min\{m : \#(\overline{K}_m, G) > 0\}$$

- Erdős'62:

$$f(n, k, \ell) = \min\{\#(K_k, G) : \alpha(G) < \ell,\ v(G) = n\}$$

- Motivation: Ramsey numbers
  - $n \geq R(k, \ell) \iff f(n, k, \ell) > 0$

Ramsey Multiplicity Problem

Ramsey Multiplicity Problem

- Erdős'62: $m(k, n) = \text{min}\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$
- Erdős'62: Is $\mu_k = 2^{1-\binom{k}{2}}$?

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$
- Erdős'62: Is $\mu_k = 2^{1-\binom{k}{2}}$?
- Thomason'89: False: $\mu_k \le 0.936 \cdot 2^{1-\binom{k}{2}}$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$
- Erdős'62: Is $\mu_k = 2^{1-\binom{k}{2}}$?
- Thomason'89: False: $\mu_k \leq 0.936 \cdot 2^{1-\binom{k}{2}}$
- Thomason'89: $\mu_4 \leq \frac{1}{33} < \frac{1}{32}$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$
- Erdős'62: Is $\mu_k = 2^{1-\binom{k}{2}}$?
- Thomason'89: False: $\mu_k \leq 0.936 \cdot 2^{1-\binom{k}{2}}$
- Thomason'89: $\mu_4 \leq \frac{1}{33} < \frac{1}{32}$
- Giraud'79: $\mu_4 \geq \frac{1}{46} = 0.0217...$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$
- Erdős'62: Is $\mu_k = 2^{1-\binom{k}{2}}$?
- Thomason'89: False: $\mu_k \leq 0.936 \cdot 2^{1-\binom{k}{2}}$
- Thomason'89: $\mu_4 \leq \frac{1}{33} < \frac{1}{32}$
- Giraud'79: $\mu_4 \geq \frac{1}{46} = 0.0217...$
- Nieß $\geq$'13, Sperfeld $\geq$'13: $\mu_4 \geq 0.0287...$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$
- Erdős'62: Is $\mu_k = 2^{1-\binom{k}{2}}$?
- Thomason'89: False: $\mu_k \leq 0.936 \cdot 2^{1-\binom{k}{2}}$
- Thomason'89: $\mu_4 \leq \frac{1}{33} < \frac{1}{32}$
- Giraud'79: $\mu_4 \geq \frac{1}{46} = 0.0217\ldots$
- Nieß $\geq$'13, Sperfeld $\geq$'13: $\mu_4 \geq 0.0287\ldots$
- Erdős'62: $\mu_k \geq \binom{R(k,k)}{k}^{-1} \geq (4+o(1))^{-k^2}$

Ramsey Multiplicity Problem

- Erdős'62: $m(k,n) = \min\{\#(K_k, G_n) + \#(\overline{K}_k, G_n)\}$
- $\mu_k = \lim_{n\to\infty} m(k,n)/\binom{n}{k}$
- Goodman'56: $\mu_3 = 1/4$
- Erdős'62: Is $\mu_k = 2^{1-\binom{k}{2}}$?
- Thomason'89: False: $\mu_k \le 0.936 \cdot 2^{1-\binom{k}{2}}$
- Thomason'89: $\mu_4 \le \frac{1}{33} < \frac{1}{32}$
- Giraud'79: $\mu_4 \ge \frac{1}{46} = 0.0217\ldots$
- Nieß $\ge$'13, Sperfeld $\ge$'13: $\mu_4 \ge 0.0287\ldots$
- Erdős'62: $\mu_k \ge \binom{R(k,k)}{k}^{-1} \ge (4+o(1))^{-k^2}$
- Conlon'12: $\mu_k \ge (2.18+o(1))^{-k^2}$

Early Results

Early Results

- $f(n, k, \ell) = \min\{\#(K_k, G_n) : \alpha(G_n) < \ell\}$

Early Results

- $f(n,k,\ell) = \min\{\#(K_k, G_n) : \alpha(G_n) < \ell\}$
- Goodman'56: $f(2m,3,3) = \#(K_3, K_m \sqcup K_m)$

Early Results

- $f(n, k, \ell) = \min\{\#(K_k, G_n) : \alpha(G_n) < \ell\}$
- Goodman'56: $f(2m, 3, 3) = \#(K_3, K_m \sqcup K_m)$
- Lorden'62: $f(2m+1, 3, 3) = \#(K_3, K_m \sqcup K_{m+1})$, $m \geq 6$

Early Results

- $f(n,k,\ell) = \min\{\#(K_k, G_n) : \alpha(G_n) < \ell\}$
- Goodman'56: $f(2m,3,3) = \#(K_3, K_m \sqcup K_m)$
- Lorden'62: $f(2m+1,3,3) = \#(K_3, K_m \sqcup K_{m+1})$, $m \geq 6$
- Complement of T_n^2

Early Results

- $f(n, k, \ell) = \min\{\#(K_k, G_n) : \alpha(G_n) < \ell\}$
- Goodman'56: $f(2m, 3, 3) = \#(K_3, K_m \sqcup K_m)$
- Lorden'62: $f(2m + 1, 3, 3) = \#(K_3, K_m \sqcup K_{m+1})$, $m \geq 6$
- Complement of T_n^2
- Turán graph T_n^r: max r-partite order-n graph

Early Results

- $f(n,k,\ell) = \min\{\#(K_k, G_n) : \alpha(G_n) < \ell\}$
- Goodman'56: $f(2m,3,3) = \#(K_3, K_m \sqcup K_m)$
- Lorden'62: $f(2m+1,3,3) = \#(K_3, K_m \sqcup K_{m+1})$, $m \geq 6$
- Complement of T_n^2
- Turán graph T_n^r: max r-partite order-n graph
- Erdős'62: Is $f(n,k,\ell) = \#(K_3, \overline{T}_n^{\ell-1})$?

Asymptotic densities

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \le (\ell-1)^{1-k}$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \le (\ell - 1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \le (\ell - 1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \leq (\ell-1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:
    - Replace each $ij \in E(F)$ by complete graph $K[V_i, V_j]$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \le (\ell-1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:
  - Replace each $ij \in E(F)$ by complete graph $K[V_i, V_j]$
  - Replace each $i \in V(F)$ by clique on $K[V_i]$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \leq (\ell-1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:
  - Replace each $ij \in E(F)$ by complete graph $K[V_i, V_j]$
  - Replace each $i \in V(F)$ by clique on $K[V_i]$
- Uniform expansion $U_n(F)$: $\forall i,j \ \ |V_i - V_j| \leq 1$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \le (\ell-1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:
  - Replace each $ij \in E(F)$ by complete graph $K[V_i, V_j]$
  - Replace each $i \in V(F)$ by clique on $K[V_i]$
- Uniform expansion $U_n(F)$: $\forall i,j \ \ |V_i - V_j| \le 1$
- Nikiforov'01: $f(n,4,3) \le \#(K_4, U_n(C_5))$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \leq (\ell-1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:
  - Replace each $ij \in E(F)$ by complete graph $K[V_i, V_j]$
  - Replace each $i \in V(F)$ by clique on $K[V_i]$
- Uniform expansion $U_n(F)$: $\forall i,j \ \ |V_i - V_j| \leq 1$
- Nikiforov'01: $f(n,4,3) \leq \#(K_4, U_n(C_5))$
  - $c_{4,3} \leq 3/25$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \le (\ell-1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:
  - Replace each $ij \in E(F)$ by complete graph $K[V_i, V_j]$
  - Replace each $i \in V(F)$ by clique on $K[V_i]$
- Uniform expansion $U_n(F)$: $\forall i,j \;\; |V_i - V_j| \le 1$
- Nikiforov'01: $f(n,4,3) \le \#(K_4, U_n(C_5))$
  - $c_{4,3} \le 3/25 < 1/8$

Asymptotic densities

- Nikiforov'01: $c_{k,\ell} = \lim_{n\to\infty} f(n,k,\ell)/\binom{n}{k}$
- Turán graph: $c_{k,\ell} \leq (\ell-1)^{1-k}$
- Nikiforov'01: Strict for all but finitely many (k,ℓ)
- Expansion of F:
  - Replace each $ij \in E(F)$ by complete graph $K[V_i, V_j]$
  - Replace each $i \in V(F)$ by clique on $K[V_i]$
- Uniform expansion $U_n(F)$: $\forall i,j \;\; |V_i - V_j| \leq 1$
- Nikiforov'01: $f(n,4,3) \leq \#(K_4, U_n(C_5))$
  - $c_{4,3} \leq 3/25 < 1/8$
- Nikiforov'05:
  $\alpha(G_n) < 3$ & regular $\;\Rightarrow\; \#(K_4, G) \geq (\frac{3}{25} + o(1))\binom{n}{4}$

Complete Solutions (for $n \geq n_0$)

Complete Solutions (for $n \geq n_0$)

- Das-Huang-Ma-Naves-Sudakov'13: $(3, 4)$ & $(4, 3)$

Complete Solutions (for $n \geq n_0$)

- Das-Huang-Ma-Naves-Sudakov'13: $(3, 4)$ & $(4, 3)$
- P.-Vaughan $\geq$'13:

$$\begin{aligned} (k, 3) &: \quad 4 \leq k \leq 7 \\ (3, \ell) &: \quad 4 \leq \ell \leq 7 \end{aligned}$$

New Values of $c_{k,\ell}$

New Values of $c_{k,\ell}$

- $c_{3,\ell} = (\ell - 1)^2$ for $4 \leq \ell \leq 7$

New Values of $c_{k,\ell}$

- $c_{3,\ell} = (\ell - 1)^2$ for $4 \leq \ell \leq 7$
  - $\overline{T}_n^{\ell-1}$

New Values of $c_{k,\ell}$

- $c_{3,\ell} = (\ell - 1)^2$ for $4 \leq \ell \leq 7$
  - $\overline{T}_n^{\ell-1}$
- $c_{4,3} = 3/25$

New Values of $c_{k,\ell}$

- $c_{3,\ell} = (\ell - 1)^2$ for $4 \leq \ell \leq 7$
  - $\overline{T}_n^{\ell-1}$
- $c_{4,3} = 3/25$
- $c_{5,3} = 31/625$

New Values of $c_{k,\ell}$

- $c_{3,\ell} = (\ell - 1)^2$ for $4 \leq \ell \leq 7$
  - $\overline{T}_n^{\ell-1}$
- $c_{4,3} = 3/25$
- $c_{5,3} = 31/625$
  - Uniform expansion of C_5

New Values of $c_{k,\ell}$

- $c_{3,\ell} = (\ell - 1)^2$ for $4 \leq \ell \leq 7$
  - $\overline{T}_n^{\ell-1}$
- $c_{4,3} = 3/25$
- $c_{5,3} = 31/625$
  - Uniform expansion of C_5
- $c_{6,3} = 19211/2^{20} = 19211/1048576$

New Values of $c_{k,\ell}$

- $c_{3,\ell} = (\ell - 1)^2$ for $4 \leq \ell \leq 7$
    - $\overline{T}_n^{\ell-1}$
- $c_{4,3} = 3/25$
- $c_{5,3} = 31/625$
    - Uniform expansion of C_5
- $c_{6,3} = 19211/2^{20} = 19211/1048576$
- $c_{7,3} = 98491/2^{24} = 98491/16777216$

Construction for $c_{6,3}$ and $c_{7,3}$

Construction for $c_{6,3}$ and $c_{7,3}$

- Clebsch graph L:

Construction for $c_{6,3}$ and $c_{7,3}$

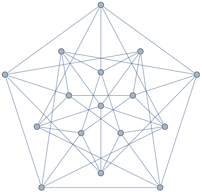
- Clebsch graph L:
  - $V = \{\mathbf{X} \in \{0,1\}^5 : \mathrm{dist}(\mathbf{X}, \mathbf{0}) \text{ is even}\}$

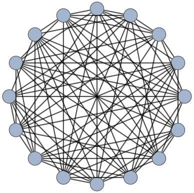
Construction for $c_{6,3}$ and $c_{7,3}$

- Clebsch graph L:
  - $V = \{\mathbf{X} \in \{0,1\}^5 : \mathrm{dist}(\mathbf{X}, \mathbf{0}) \text{ is even}\}$
  - $\mathbf{X} \sim \mathbf{Y}$ iff $\mathrm{dist}(\mathbf{X}, \mathbf{Y}) = 4$

Construction for $c_{6,3}$ and $c_{7,3}$

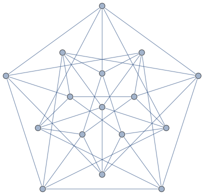
- Clebsch graph L:
  - $V = \{\mathbf{X} \in \{0,1\}^5 : \mathrm{dist}(\mathbf{X}, \mathbf{0}) \text{ is even}\}$
  - $\mathbf{X} \sim \mathbf{Y}$ iff $\mathrm{dist}(\mathbf{X}, \mathbf{Y}) = 4$
- L and $\overline{L}$:

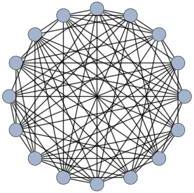




Construction for $c_{6,3}$ and $c_{7,3}$

- Clebsch graph L:
  - $V = \{\mathbf{X} \in \{0,1\}^5 : \mathrm{dist}(\mathbf{X}, \mathbf{0}) \text{ is even}\}$
  - $\mathbf{X} \sim \mathbf{Y}$ iff $\mathrm{dist}(\mathbf{X}, \mathbf{Y}) = 4$
- L and $\overline{L}$:





- Upper bound: $U_n(\overline{L})$

Computer’s Part

Computer's Part

- $c_{3,7}$: 12,338 constraints

Computer's Part

- $c_{3,7}$: 12,338 constraints
- Emil Vaughan's package `Flagmatic`

Computer's Part

- $c_{3,7}$: 12,338 constraints
- Emil Vaughan's package `Flagmatic`

Human Part: Stability

Human Part: Stability

- Stability: above ideas + eigenvalues

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$
  - $\phi(K_7) = p(y_1, \ldots, y_{16})$, polynomial of degree 7

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$
  - $\phi(K_7) = p(y_1, \ldots, y_{16})$, polynomial of degree 7
- Which expansion of $\overline{K}_2$ minimise K_3-density?

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$
  - $\phi(K_7) = p(y_1, \ldots, y_{16})$, polynomial of degree 7
- Which expansion of $\overline{K}_2$ minimise K_3-density?
  - $p(y_1, y_2) = y_1^3 + y_2^3$

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$
  - $\phi(K_7) = p(y_1, \ldots, y_{16})$, polynomial of degree 7
- Which expansion of $\overline{K}_2$ minimise K_3-density?
  - $p(y_1, y_2) = y_1^3 + y_2^3$
  - $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$
  - $\phi(K_7) = p(y_1, \ldots, y_{16})$, polynomial of degree 7
- Which expansion of $\overline{K}_2$ minimise K_3-density?
  - $p(y_1, y_2) = y_1^3 + y_2^3$
  - $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
  - $\mathbf{v}_x = (y_1, y_2)$ or (y_2, y_1)

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$
  - $\phi(K_7) = p(y_1, \ldots, y_{16})$, polynomial of degree 7
- Which expansion of $\overline{K}_2$ minimise K_3-density?
  - $p(y_1, y_2) = y_1^3 + y_2^3$
  - $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
  - $\mathbf{v}_x = (y_1, y_2)$ or (y_2, y_1)
  - $\mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T) = 0$

Human Part: Stability

- Stability: above ideas + eigenvalues
- Which expansions of $\overline{L}$ minimise K_7-density?
  - Part ratios $y_1 + \cdots + y_{16} = 1$
  - $\phi(K_7) = p(y_1, \ldots, y_{16})$, polynomial of degree 7
- Which expansion of $\overline{K}_2$ minimise K_3-density?
  - $p(y_1, y_2) = y_1^3 + y_2^3$
  - $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
  - $\mathbf{v}_x = (y_1, y_2)$ or (y_2, y_1)
  - $\mathbb{E}_x(\mathbf{v}_x A \mathbf{v}_x^T) = 0 \quad \Rightarrow \quad y_1 = y_2$

Human Part: Exact Result

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases
- Assumptions:

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases
- Assumptions:
  - Stability ($\forall$ almost extremal G is $o(n^2)$-close to $U_n(F)$)

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases
- Assumptions:
  - Stability ($\forall$ almost extremal G is $o(n^2)$-close to $U_n(F)$)
  - $\forall$ block addition of a vertex x to $U_n(F)$:

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases
- Assumptions:
  - Stability ($\forall$ almost extremal G is $o(n^2)$-close to $U_n(F)$)
  - $\forall$ block addition of a vertex x to $U_n(F)$:
    - $\overline{K_\ell}$ is created

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases
- Assumptions:
  - Stability ($\forall$ almost extremal G is $o(n^2)$-close to $U_n(F)$)
  - $\forall$ block addition of a vertex x to $U_n(F)$:
    - $\overline{K}_\ell$ is created
    - or x is a clone of an existing vertex

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases
- Assumptions:
  - Stability ($\forall$ almost extremal G is $o(n^2)$-close to $U_n(F)$)
  - $\forall$ block addition of a vertex x to $U_n(F)$:
    - $\overline{K}_\ell$ is created
    - or x is a clone of an existing vertex
    - or too many K_k's are created

Human Part: Exact Result

- Exact result: One theorem covering all 8 cases
- Assumptions:
  - Stability ($\forall$ almost extremal G is $o(n^2)$-close to $U_n(F)$)
  - $\forall$ block addition of a vertex x to $U_n(F)$:
    - $\overline{K}_\ell$ is created
    - or x is a clone of an existing vertex
    - or too many K_k's are created
- Conclusion:

$$f(n,k,\ell) = \#(K_k, \text{expansion of } F), \qquad n > n_0$$

Open Problems

Open Problems

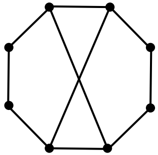
- Conjecture (P.-Vaughan):

$$c_{4,4} = \frac{14 \cdot 2^{1/3} - 11}{192}$$

Open Problems

- Conjecture (P.-Vaughan):

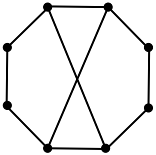
$$c_{4,4} = \frac{14 \cdot 2^{1/3} - 11}{192}$$



Open Problems

- Conjecture (P.-Vaughan):

$$c_{4,4} = \frac{14 \cdot 2^{1/3} - 11}{192}$$

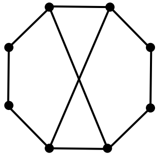


- Das-Huang-Ma-Naves-Sudakov'13:
  $\ell_0 = \max\{\ell : c_{3,\ell} = (\ell - 1)^{-2}\} = ?$

Open Problems

- Conjecture (P.-Vaughan):

$$c_{4,4} = \frac{14 \cdot 2^{1/3} - 11}{192}$$

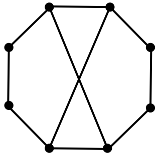


- Das-Huang-Ma-Naves-Sudakov'13:
  $\ell_0 = \max\{\ell : c_{3,\ell} = (\ell - 1)^{-2}\} = ?$
  - Nikiforov'01: $\ell_0 < \infty$

Open Problems

- Conjecture (P.-Vaughan):

$$c_{4,4} = \frac{14 \cdot 2^{1/3} - 11}{192}$$

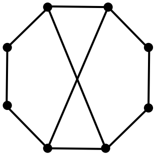


- Das-Huang-Ma-Naves-Sudakov'13:
  $\ell_0 = \max\{\ell : c_{3,\ell} = (\ell - 1)^{-2}\} = ?$
  - Nikiforov'01: $\ell_0 < \infty$
  - Das-Huang-Ma-Naves-Sudakov'13: $\ell_0 < 2074$

Open Problems

- Conjecture (P.-Vaughan):

$$c_{4,4} = \frac{14 \cdot 2^{1/3} - 11}{192}$$



- Das-Huang-Ma-Naves-Sudakov'13:
  $\ell_0 = \max\{\ell : c_{3,\ell} = (\ell - 1)^{-2}\} = ?$
  - Nikiforov'01: $\ell_0 < \infty$
  - Das-Huang-Ma-Naves-Sudakov'13: $\ell_0 < 2074$
  - P.-Vaughan $\geq$'13: $\ell_0 \geq 7$

Thank you!