Problem solving seminar
Final Test - solutions

1. Let $s_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$. Prove that $s_n(x) \ln\big(s_n(x)\big) > x s_{n-1}(x)$ for $n \geq 1$, $x > 0$.

Solution. We proceed by induction. For $n = 1$ the assertion follows by the standard estimate $(1+x)\ln(1+x) > x$. Suppose the assertion holds for some $n \geq 1$. Consider the function

$$h(x) = s_{n+1}(x) \ln\big(s_{n+1}(x)\big) - x s_n(x).$$

Since $s'_{n+1}(x) = s_n(x)$ we have

$$h'(x) = s_n(x) \ln\big(s_{n+1}(x)\big) - x s_{n-1}(x) > s_n(x) \ln\big(s_n(x)\big) - x s_{n-1}(x) > 0$$

by the inductive assumption. It means that h is increasing, thus, $h(x) > h(0) = 0$ for $x > 0$. $\square$

2. Suppose that $A = [a_{ij}]_{i,j=1}^n$ is a symmetric $n \times n$ matrix, $n > 1$, with nonnegative entries for which

$$\sum_{i,j=1}^{n} x_i x_j a_{ij} \leq -\theta \sum_{i=1}^{n} x_i^2 \quad \text{whenever } \sum_{i=1}^{n} x_i = 0$$

for some $\theta > 0$. Prove that all eigenvalues of A have absolute value at least θ.

Solution. The matrix A is symmetric, so its eigenvalues are real and it is diagonalizable in an orthonormal basis. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A with corresponding eigenvectors $u_1, \ldots, u_n$ which form an orthonormal basis of $\mathbb{R}^n$.

First we show that $\lambda_2 \leq -\theta$. Let $H = \{x \in \mathbb{R}^n,\ \sum x_i = 0\}$ be the $n-1$ dimensional subspace on which A has eigenvalues at most $-\theta$ by the assumption. Since $H \cap \operatorname{span}\{u_1, u_2\} \neq \{0\}$, take $v \neq 0$ from this intersection, $v = \alpha_1 u_1 + \alpha_2 u_2$. We get

$$\frac{Av \cdot v}{v \cdot v} \leq -\theta$$

as $v \in H$ (by $\cdot$ we mean the standard scalar product on $\mathbb{R}^n$). On the other hand,

$$\frac{Av \cdot v}{v \cdot v} \geq \frac{\lambda_1 |\alpha_1|^2 + \lambda_2 |\alpha_2|^2}{|\alpha_1|^2 + |\alpha_2|^2} \geq \lambda_2.$$

Now we show that $\lambda_1 \geq \theta$. Observe that $\operatorname{tr} A = \lambda_1 + \lambda_2 + \ldots + \lambda_n \leq \lambda_1 + (n-1)\lambda_2$, so $\lambda_2 \leq -\theta$ yields $\operatorname{tr} A \leq \lambda_1 - (n-1)\theta$. Since A has nonnegative entries, $\operatorname{tr} A \geq 0$, hence $\lambda_1 \geq (n-1)\theta \geq \theta$. $\square$

3. Each square of the usual 8×8 chessboard is initially either white or black. At each step we choose a subboard of size 3×3 or 4×4 and reverse the colour of each square of the subboard. Is it true that no matter what the initial colouring of the chessboard is, there exists a sequence of steps such that the chessboard becomes all white.

Solution. No!

Since there are $6 \cdot 6$ subboards of size 3×3 and $5 \cdot 5$ subboards of size 4×4, there are 61 possible choices for a subboard. Note that choosing subboard A and then B yields the same colouring of the chessboard as choosing B and A in two consecutive steps. Moreover, if we choose a certain subboard an even number of times, it does not change the colouring. Therefore, any sequence of steps is determined by saying which subboards are chosen the odd number of times, so there are 2^{61} sequences of steps which are not equivalent. Thus, given the initial colouring, we can produce only $2^{61} - 1$ other colourings. Since there are 2^{64} distinct possible initial colourings, there is one which cannot be obtained starting from the white chessboard. Equivalently, there is an initial colouring such that no matter what we do, we will not get the white chessboard. $\square$

4. Suppose that $\mathcal{S}$ is a family of circles in $\mathbb{R}^2$, such that the intersection of any two contains at most one point. Prove that the set M of those points that belong to at least two different circles from $\mathcal{S}$ is countable.

Solution. For every $x \in M$ choose circles $S, T \in \mathcal{S}$ such that $S \neq T$ and $x \in S \cap T$; denote by U, V, W the three components of $\mathbb{R}^2 \setminus (S \cup T)$, where the notation is such that $\partial U = S$, $\partial V = T$ and x is the only point of $\overline{U} \cap \overline{V}$, finally we choose points with rational coordinates $u \in U$, $v \in V$, $w \in W$ obtaining $f(x) = (\{u, v\}, w)$. We claim that the operation f is 1-1, this will finish the proof since the set of triplets with rational coordinates is countable.

To prove the claim, suppose that from some $x' \in M$ we arrived at the same $(\{u, v\}, w)$ using circles S', T' and components U', V', W' of $\mathbb{R}^2 \setminus (S' \cup T')$. Since $S \cap S'$ contains at most one point and since $u \in U \cap U' \neq \varnothing$ we have that $U \subset U'$ or $U' \subset U$, similarly for V's and W's. Exchanging the role of x and x' and/or of U's and V's if necessary, there are only two cases to consider:

1. $U \subset U'$ and $V \subset V'$. Recall that $x \in \overline{U} \cap \overline{V}$ is unique and similarly $x' \in \overline{U'} \cap \overline{V'}$, hence $x = x'$.
2. $U \subset U'$ and $V' \subset V$ and $W \subset W'$. We get from $W \subset W'$ that $U' \subset \overline{U \cap V}$; so since U' is open and connected and $\overline{U} \cap \overline{V}$ is just one point, we infer $U = U'$ and we are back in the previous case.

$\square$

5. Prove that for every positive integer n the number

$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1$$

is composite.

Solution. Let $m = \max\{j \geq 0,\ 2^j | n\}$, $k = 2^m$. Then n/k is odd, $10^n/k = 2^{n-m}5^n$ is even as $n - m \geq 2^m - m \geq 1$, and $10^{10^{10^n}}$ is even. Thus,

$$\begin{aligned} 10^n &= (10^k)^{n/k} \equiv (-1)^{\text{odd}} \equiv -1 \pmod{10^k + 1}, \\ 10^{10^n} &= (10^k)^{10^n/k} \equiv (-1)^{\text{even}} \equiv 1 \pmod{10^k + 1}, \\ 10^{10^{10^n}} &= (10^k)^{10^{10^n}/k} \equiv (-1)^{\text{even}} \equiv 1 \pmod{10^k + 1}, \end{aligned}$$

hence $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is divisible by $10^k + 1$. $\square$