Spherical set avoiding a prescribed set of angles

Evan DeCorte (McGill)
Oleg Pikhurko (Warwick)

$|\mathrm{Aut}(\mathrm{K}_5)|$*-th Anniversary of Kazimierz Kuratowski*

From Gil Kalai's blog

How Large can a Spherical Set Without Two Orthogonal Vectors Be?

Posted on May 22, 2009

The problem

Problem: Let A be a measurable subset of the d-dimensional sphere $S^d = \{x \in \mathbf{R}^{d+1} : \|x\| = 1\}$. Suppose that A **does not contain two orthogonal vectors.** How large can the d-dimensional volume of A be?

A Conjecture

Conjecture: The maximum volume is attained by two open caps of diameter $\pi/4$ around the south pole and the north pole.

For simplicity, let us normalize the volume of S^d to be 1.

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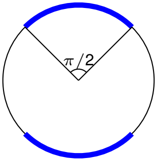
- $\alpha_1 = 1/2$: 

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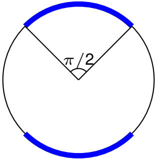

- Conjecture (Kalai'09): two opposite caps are optimal

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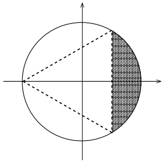

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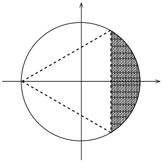

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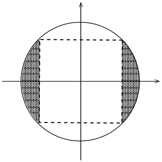

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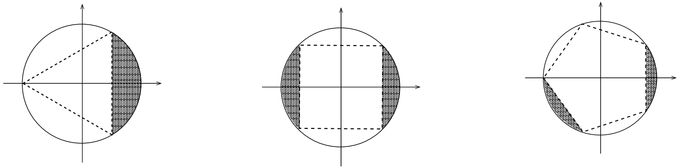

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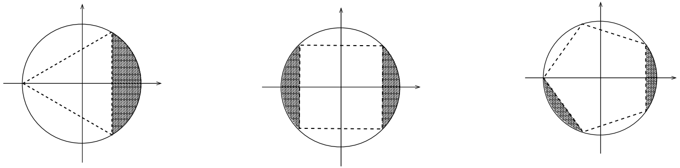

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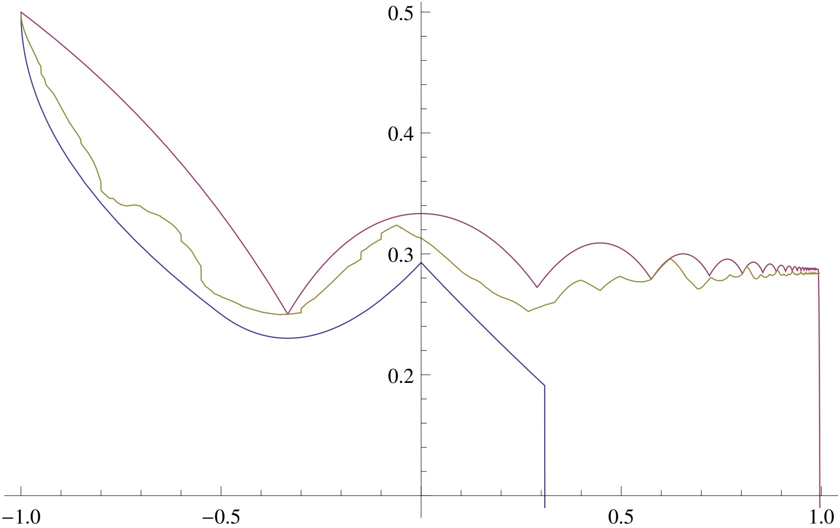
- Bachoc-Nebe-Oliveira Filho-Vallentin'09: $\theta(\mathcal{S}_{2,t})$

Bounds on $\alpha_2(t)$



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  - Finite-dim B s.t. $\|B - A\| < \varepsilon / C$
  - $\|Bf_i - Bf\| < \varepsilon$
  - $\|Af_i - Af\| \leq \|Af_i - Bf_i\| + \|Bf_i - Bf\| + \|Bf - Af\| < 3\varepsilon$

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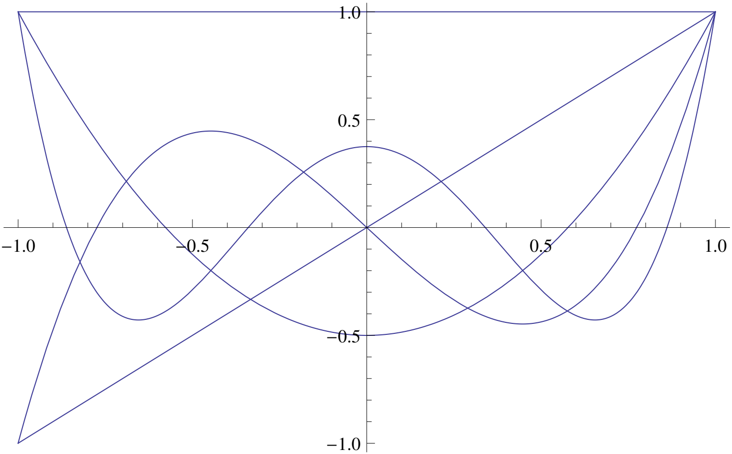
Gegenbauer Polynomials ($n = 2$)



First five

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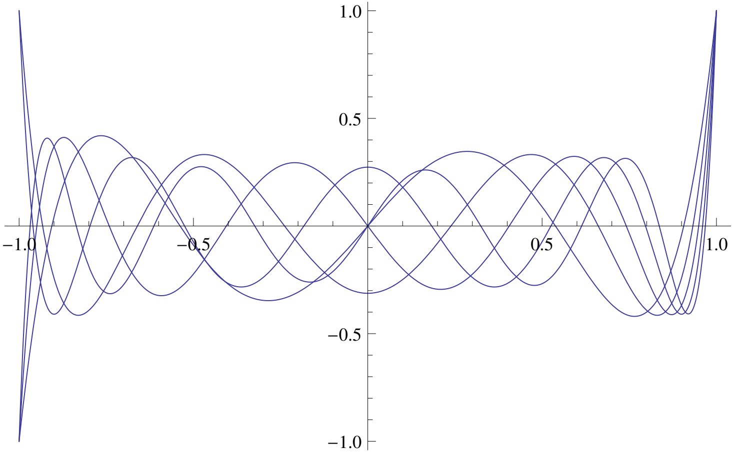

Next five

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Thank you!