Problem Solving: Analysis

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March 2014

Section A (Warm Up)

Question 1

Let $f : [0, \infty) \to [0, \infty)$ be a differentiable function satisfying the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} f^2(t) \leq 2f(t) \tag{1}$$

for all $t \in \mathbb{R}$. Show that $f(t) \leq t + f(0)$ for all $t \in [0, \infty)$. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies (1), what can you say about $\{t : f(t) < 0\}$?

Solution

Let $0 \leq a < b$ and assume that $f(t) > 0$ if $t \in (a, b)$. Clearly $\frac{\mathrm{d}f}{\mathrm{d}t} \leq 1$ on (a, b) and so $f(t) \leq t - a + f(a)$. Since f is continuous the set $\{t : f(t) = 0\}$ is closed and so the set $\{t : f(t) > 0\}$ is a (countable) union of disjoint open intervals. Let (a, b) be one such interval, then by the above $f(t) \leq t - a + f(a) \leq t + f(0)$.

For $f : \mathbb{R} \to \mathbb{R}$ differentiable and satisfying (1), $\{t : f(t) < 0\}$ must be an interval of the form $(-\infty, x)$. If not then there exists an interval (a, b) on which f is negative and $f(a) = 0$. Now by (1), $f' \geq 1$ on this interval. Since $f(t) = t - (a + b)/2$ for some $t \in (a, b)$ it follows that f has a zero in $(a, (a + b)/2)$ contradicting negativity.

Question 2

Does there exist a differentiable function $f : \mathbb{R} \to (0, \infty)$ such that $f'(t) = f \circ f(t)$ for all $t \in \mathbb{R}$?

Solution (2002 Day 1, Q2)

No such function exists, for it must be increasing and so the formula for the derivative gives $f'(t) \geq f(0)$. Therefore for $t \leq 0$ we have $f(t) \leq f(0) + tf(0)$ contradicting positivity.

Question 3

Suppose $f : [0, 1] \to \mathbb{R}$ is nowhere monotone (i.e. not monotone on any open interval). Show that the set $\{x : f \text{ has a local minimum at } x\}$ is dense in $[0, 1]$.

Solution (2000 Day 2, Q2)

Consider an interval $(a, b) \subset [0, 1]$. Since f is nowhere monotone there must exist points $a \leq \alpha < \beta < \gamma \leq b$ such that $f(\alpha) > f(\beta) < f(\gamma)$. Then $f\big|_{[\alpha,\gamma]}$ attains its global minimum value in (α, γ) and thus f has local minima in (a, b). This is enough since (a, b) was arbitrary.

Section B

Question 4

Evaluate

$$\lim_{t\uparrow 1}(t-1)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n}$$

Solution (2001 Day 1, Q3)

$$\begin{aligned}\lim_{t\uparrow 1}(t-1)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n} &= \lim_{t\uparrow 1}\frac{t-1}{-\log t}\lim_{t\uparrow 1}\sum_{n=1}^{\infty}\frac{-\log t}{1+t^{-n}} = \lim_{t\uparrow 1}\sum_{n=1}^{\infty}\frac{-\log t}{1+e^{-n\log t}}\\ &= \lim_{h\downarrow 0}\sum_{n=1}^{\infty}\frac{h}{1+e^{nh}} = \int_0^{\infty}\frac{1}{1+e^x}\mathrm{d}x = \int_0^{\infty}\frac{e^{-x}}{1+e^{-x}}\mathrm{d}x = \log 2\end{aligned}$$

Question 5

Let $f:[a,b]\to[a,b]$ be continuous and fix $p\in[a,b]$. Define a sequence $(p_n)_{n=0}^{\infty}$ by $p_0=p$, $p_{n+1}=f(p_n)$ and let $S=\{p_n\}$. Show that if S is closed then it must be finite.

Solution (2002 Day 1, Q4)

Suppose that S is closed but infinite, then there exists $N\geq 0$ such that p_N is an accumulation point of S. Now if $p_{n_k}\to p_N$ then $p_{n_k+1}\to f(p_N)$ hence all points p_n with $n\geq N$ are accumulation points (in fact, let's assume that all points are accumulation points without loss of generality). Now if you know the Baire Category theorem this is enough because each set $\{p_N\}$ is nowhere dense since it is an accumulation point, but S is closed (hence complete) and is a countable union of the sets $\{p_N\}$.

To prove the contradiction from first principles, for $k\geq 1$ let n_k be the least index such that $p_{n_k}\in S\setminus\bigcup_{\ell=1}^{k-1}I_\ell$ and choose[1] an open interval I_k centred on p_{n_k} omitting infinitely many points of S. Now WLOG p_{n_k} converges to some $p_N\in S$ by closedness, but for some k we must have $p_N\in I_k$ which is a contradiction.

Homework

Question 6

Fix $T:[-2,2]\to[-1,1]$ and define $T_n:[-2,2]\to[-1,1]$ by $T_1=T$, $T_{n+1}=T\circ T_n$. Find the pointwise limit of T_n if it exists in the case $T(x)=\sin(\frac{\pi}{2}x)$. i.e. find $\lim_{n\to\infty}T_n(x)$ for all x. What happens for $T(x)=\cos(\frac{\pi}{2}x)$?

Solution

For $T(x)=\sin(\frac{\pi}{2}x)$ we clearly have $T_n(-2)=T_n(0)=T_n(2)$ and $T_n(1)=1=-T_n(-1)$ for all n. Fix $x\in(0,2)\setminus\{1\}$ $T(x)\in(0,1)$ so WLOG $x\in(0,1)$. Now $1>T(x)>x$ for all $x\in(0,1)$ so $T_n(x)$ converges to some point $y\in(0,1]$. T is continuous so $y=\lim_{n\to\infty}T_n=\lim_{n\to\infty}T_{n+1}(y)=T(y)>y$ if $y<1$, hence $y=1$. A similar argument shows that $T_n(x)\to -1$ as $n\to\infty$ for $x\in(-2,0)\setminus\{1\}$.

[1] If we are more careful, we don't need the axiom of choice for this argument!

If $T(x) = \cos(\frac{\pi}{2}x)$ the sequence doesn't converge pointwise e.g. consider $x = 0$. It appears (I haven't found a proof) that the sequence (T_n) decomposes into two convergent subsequences. More generally what happens if $T(x) = \sin(\frac{\pi}{2}(x + \alpha))$ with $\alpha \neq 0$?

Question 7

Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(x)f(yf(x)) = f(x + y)$$

for all $x, y > 0$.

Solution (2000 Day 2, Q5)

Suppose that $f(x) > 1$ then setting $y = \frac{x}{f(x)-1}$ gives a contradiction, so $f(x) \in (0, 1]$ for all $x > 0$. In particular $f(x) \leq f(x + y)$ for all $x, y > 0$ so f is decreasing. If $f(x) = 1$ then $f(x) \leq f(x + y) = f(y) \leq 1$ for all $y > 0$ so $f \equiv 1$.

If $f(x) < 1$ for some x then f must be strictly decreasing (otherwise we can construct x so that $f(x) = 1$) so f is injective. Now we see that

$$f(x)f(yf(x)) = f(x + y) = f(x + yf(x) + y(1 - f(x))) = f(yf(x))f((x + y(1 - f(x)))f(yf(x)))$$

which gives, by injectivity:

$$x = (x + y(1 - f(x)))f(yf(x)).$$

Hence, setting $x = 1$, $z = yf(1)$ and $a = \frac{1-f(1)}{f(1)}$ we have

$$f(z) = \frac{1}{1 + az}$$

for all $z > 0$.