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Mini-Workshop: Descriptive Combinatorics, LOCAL Algorithms and Random Processes (hybrid meeting)

Organized by
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13 February – 19 February 2022

Abstract. The aim of this mini-workshop was to discover and deepen connections between the fields of *descriptive combinatorics*, *distributed computing* and *random processes*. The common link is played by the so-called *local coloring problems* on graphs, where the validity of solution can be checked locally, and the common interest can be phrased as the following central question: *Is it possible to produce a solution to a given local problem efficiently?* While all three areas possess a solid background that was achieved by decades of intense research, a systematic study of formal connections between them is a recent emerging phenomenon. This approach has already proved to be very fruitful: several open questions in each of the fields were solved by means and techniques of the other two. The purpose of this meeting is to bring together researchers in all three fields in order to explore these exciting connections.

Mathematics Subject Classification (2010): 03E15, 68Qxx, 60Gxx, 37Axx, 05C15.

Introduction by the Organizers

The mini-workshop *Descriptive Combinatorics, LOCAL Algorithms and Random Processes*, organised by Jan Grebík (Coventry), Oleg Pikhurko (Coventry) and Anush Tserunyan (Montreal), was held 13–19 February 2022. The meeting ran in a hybrid format.

The main purpose of the mini-workshop was to bring together researchers from descriptive combinatorics, distributed computing and random processes, and communicate the recent progress on the formal connections between the fields as well as possible future directions of research. The schedule was designed accordingly.

The first day of the meeting was dedicated to introductory talks that gave an overall summary of each field (Brandt, Holroyd, Marks). The second day was devoted to techniques connected to the Lovász Local Lemma (Bernshteyn, Grabowski, Rozhoň). These talks were complemented by evening open problems sessions. In the rest of the week, the program was a mixture of talks, where recent results in each field were presented (Hutchcroft, Spinka, Thornton, Vidnyanszky), and working in group sessions, where various topics suggested by participants were discussed.

An important class of problems on graphs is the class of so-called *locally checkable labelling problems (LCL)*, that is, for which the validity of a given candidate solution can be checked *locally* (i.e. in a constant neighbourhood of each vertex). This framework captures the classical colouring problems, matching problems, etc. The existence of a solution of a given LCL is studied in various settings and with various constraints. In particular, very often the combinatorics behind more abstract problems translate to an LCL on some auxiliary graph. Next, we describe three seemingly different fields for which, as it turned out recently, LCLs play the role of a common link.

Distributed computing The LOCAL model of computing, introduced by Linial [8], is motivated by understanding distributed algorithms on graphs. As an input we have a large graph, with each node knowing its size, n, and perhaps some other parameter like the maximum degree Δ. In case of randomized algorithms, each node has an access to a random string, while in case of deterministic algorithms, each node starts with a unique identifier from a range of size polynomial in n. In one round, each node can exchange any message with its neighbours and can perform an arbitrary computation. We want to find a solution to a problem in as few communication rounds as possible. There is a rich theory of distributed algorithms and the local complexity of many problems is fairly well understood. For example, if the input graph is a d-dimensional grid, or a finite tree, then we have a complete picture of possible complexity classes as $n \to \infty$.

Random processes In recent years, *factors of iid* processes on various graphs attracted a lot of attention in combinatorics, probability, ergodic theory and statistical physics. Factors of iid processes are randomized algorithms, where each vertex v outputs a solution which is a measurable function of the random strings on vertices of the whole underlying graph rooted at v. An important example is the result of Lyons and Nazarov [9], who showed that perfect matching can be described as a factor of iid process on an infinite Δ-regular tree. If instead of exploring the whole underlying graph, we require that the algorithm with probability 1 finishes after exploring finite neighbourhood of a given vertex, we end up with the definition of *finitary* factors of iid processes. Interestingly, this notion is nothing else than so-called *uniform local randomized algorithms* on finite graphs, where the vertices do not know the size of the graph. The exact relationship between the classes of LCLs that admit fiid or ffiid solutions is currently known only in the easiest settings of oriented paths.

Descriptive combinatorics Recent results on Tarski's Circle Squaring Problem, whether one can partition disk into finitely many pieces and rearrange them using isometries to form a square, (Grabowski, Máthé and Pikhurko [4], Marks and Unger [10], Máthé, Noel and Pikhurko [11]) are highlights of a field called *descriptive combinatorics*. This field focuses on finding constructive solutions to combinatorial-type questions on definable graphs. We are given a graph (typically, with uncountably many connected components, each being a countable graph of bounded degree) on a topological or measure space and have to find a Borel or measurable vertex labelling that solves a given LCL. For example, in the case of the Circle Squaring Problem, the vertices of the underlying graph are the points of a disk and a square in $\mathbb{R}^2$ of the same area, the edges correspond to the selected isometries by which the pieces will be moved, and one has to construct a perfect matching.

It has been known for quite some time that the theory of random processes, especially the factors of iid random processes, is connected with local algorithms that solve *approximately* a given LCL. The translation of *exact* solutions turns out to be more challenging. For example, Holroyd, Schramm and Wilson [7] were already aware that the classical algorithm of Linial for finding a maximal independent set in bounded degree graphs has its "uniform" version, that corresponds to finitary factors of iid processes with tower tail decay. Recently, Bernshteyn [1] found a formal translation of results from distributed computing to descriptive combinatorics that connects classes of LCLs that can be solved by the so-called Lovász Local Lemma (LLL), an extremely useful tool for showing the existence of various objects in finite combinatorics. The work of Gao, Jackson, Krohne and Seward [3], on continuous combinatorics on Abelian groups, showed stark similarity with local algorithms in the $\log^* n$ regime on grids. The papers of Bernshteyn [2], Grebík and Rozhoň [6, 5], and others give rise to a dictionary that formally connects these fields through a common theory of locality.

Overall, the meeting was a great success as all three communities had enough space to discuss their motivations, perspectives and important open problems. We hope that this will lead to a fruitful future collaboration between and within the fields, and that the vast amount of striking results from last few years will keep growing.

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Mini-Workshop (hybrid meeting): Descriptive Combinatorics, LOCAL Algorithms and Random Processes

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Abstracts

An introduction to the LOCAL model

Sebastian Brandt

The LOCAL model provides a formal setting for studying how fast a given graph problem can be solved by a certain type of distributed algorithm, called a *LOCAL algorithm*. There are two equivalent ways to formalize the LOCAL model.

In the standard message-passing formulation, each vertex of the input graph instance is considered as a computational entity, and these entities work together to compute a solution to the given problem. More specifically, the computation proceeds in synchronous rounds; in each round, each vertex sends an arbitrarily large message to each of its neighbors, and, after receiving the messages sent by its neighbors, performs some arbitrarily complex internal computation. Each vertex executes the same algorithm governing the choice of messages to be sent and internal computation to be performed. The actions of a vertex specified by the algorithm—which we will call a LOCAL algorithm—may depend on the initial knowledge of the vertex or the information received in previous rounds. The initial knowledge of a vertex consists of its own degree, the number n of vertices in the input graph (and, optionally, the maximum vertex degree Δ), some symmetry-breaking information specified below, and potentially some problem-specific input, such as a color list in a list coloring problem. Each vertex has to decide at some point to terminate, upon which it is required to provide a local output such that the collection of the local outputs of all vertices constitutes a correct solution to the given problem. For instance, if the considered problem is a coloring problem, then each vertex has to output a color such that the induced global coloring satisfies the color constraints specified by the problem.

In order to avoid trivial impossibility results based on the fact that each vertex executes the same algorithm, each vertex is initially equipped with the aforementioned symmetry-breaking input. In the case of a *deterministic* LOCAL algorithm, this input is given by assigning a unique identifier from $\{1, \dots, n^c\}$ to each vertex, for some sufficiently large constant c. In the case of a (Monte-Carlo) *randomized* LOCAL algorithm, this input is given by providing each vertex with a private random bit string. A randomized LOCAL algorithm is considered correct if, for each input (n-vertex) graph, the probability that the output produced by the algorithm is incorrect is at most $1/n$.

The complexity of a LOCAL algorithm on a given instance is the number of synchronous rounds until the last vertex terminates. More generally, the complexity of a LOCAL algorithm parameterized by one or more parameters is a function in the respective parameters that maps each parameter combination to the worst-case complexity of the algorithm over all instances exhibiting this parameter combination. The complexity of a problem is the complexity of an optimal algorithm for the problem. Usually, the studied complexities are functions in n, or n and Δ.

As the size of the messages sent by the vertices is unlimited, we can assume w.l.o.g. that, in each round, each vertex simply sends all information accumulated so far, which results in each vertex knowing precisely its T-hop neighborhood in the input graph after T rounds. Hence, a T-round algorithm can be equivalently defined as a function from the space of all possible T-hop neighborhoods of a vertex to the space of all possible local vertex outputs, which yields the aforementioned equivalent definition of the LOCAL model. A simple corollary of this observation is that each problem that admits a correct solution can be solved with complexity $O(n)$ in the LOCAL model; hence, ideally we would like to design much faster algorithms, e.g., with polylogarithmic or even smaller complexities. In the following, unless specified otherwise, we will restrict attention to deterministic LOCAL algorithms.

A simple first question to ask is the following: how fast we can find a proper vertex coloring, provided that the input graph is guaranteed to be a path? If we restrict the color palette to 2 colors, a simple propagation argument shows that the problem has complexity $\Omega(n)$—roughly speaking, a faster algorithm cannot exist as otherwise two distant vertices will decide on their local output based on entirely disjoint information while the correctness of the local outputs depends on the parity of their distance. However, if we allow 3 colors, the problem can already be solved very fast due to an algorithm by Cole and Vishkin [1]: the vertices interpret their unique identifiers as a coloring (with a large color palette) and then stepwise reduce the size of the palette by choosing new, smaller colors that preserve the correctness of the coloring in each round. As the size of the palette can be reduced roughly logarithmically in each round, a coloring with a constant number of colors can be obtained in $O(\log^* n)$ rounds; a subsequent standard greedy color reduction of one color per round then yields a $(\Delta + 1)$-coloring in an additional constant number of rounds. Since $\Delta + 1 = 3$ in paths, the overall complexity of 3-coloring in paths is $O(\log^* n)$, which is tight as shown by Linial [2].

Interestingly, due to the fact that the correctness of the aforementioned algorithm is guaranteed also on rooted trees, this approach can be extended to general graphs by executing Δ 3-coloring algorithms in parallel: essentially, each of the Δ algorithms is executed on a rooted tree obtained by letting each vertex choose one incident edge and orient it outwards. If each vertex chooses each incident edge in at least one of the algorithms, then any two adjacent vertices are guaranteed to have a different color in the output of at least one algorithm; interpreting the Δ-tuple of colors resulting from executing the Δ algorithms as a 3^Δ-coloring, the same greedy color reduction as before yields a correct $(\Delta + 1)$-coloring, with an overall complexity of $O(\log^* n + 3^\Delta)$. Using the concept of cover-free families, Linial improved on this approach by showing that an $O(\Delta^2)$-coloring can be obtained in $O(\log^* n)$ rounds [2]; combining this with the fastest currently known algorithm for reducing the number of colors from $\Theta(\Delta^2)$ to $\Delta + 1$, which has a complexity of $O(\sqrt{\Delta \log \Delta})$ [3], shows that the complexity of $(\Delta + 1)$-coloring is in $O(\log^* n) + \tilde{O}(\sqrt{\Delta})$. As a function in n, the state-of-the-art upper bound of $O(\log^3 n)$ is achieved by a very recent result by Ghaffari and Kuhn [4]. In contrast,

on the lower bound side nothing better is known than the 30-year-old $\Omega(\log^* n)$ lower bound, leading us to one of the big open problems in the LOCAL model.

Open Problem 1. *What is the* LOCAL *complexity of* $(\Delta+1)$*-coloring?*

The widespread interest in this question does not only stem from the considerable gap between lower and upper bounds, but also from the fact that coloring is a very useful subroutine used in many LOCAL algorithms. In particular, a vertex coloring provides a simple recipe for parallelizing sequential procedures where the "correctness" of each separate step only depends on a constant-hop neighborhood of the vertex at which it is executed: first compute a coloring where any two vertices of the same color are sufficiently far apart, and then iterate through the color classes and compute the local outputs of all vertices of the same color at the same time. As the execution of a LOCAL algorithm on some constant power of the input graph can be simulated by a LOCAL algorithm executed on the input graph with only a constant overhead, this approach yields algorithms with complexities of the form $O(\log^* n)+\Delta^c$, for some constant c, for all problems that admit a sequential greedy algorithm with the aforementioned correctness property, such as *maximal matching* or *maximal independent set*. For the well-studied bounded-degree setting that specifies that Δ is a constant, this implies that all such problems can be solved in $O(\log^* n)$ rounds.

Studying the bounded-degree setting is also very interesting from a complexity-theoretic viewpoint. A long line of work, executed primarily over the last 7 years, has finally resulted in a complete classification of all possible complexity classes that so-called *locally checkable labeling (LCL) problems*—problems that, roughly speaking, can be defined by specifying a finite set of "allowed configurations" that the output in the neighborhood of each vertex must form—can exhibit on trees (see, e.g., [5]): the complexity of each LCL problem on trees falls into one of the five classes $O(1)$, $\Theta(\log^* n)$, $\Theta(\log n)$, $\Theta(n^{1/c})$ for some constant c (both deterministic and randomized), and $\Theta(\log n)$ deterministic and $\Theta(\log\log n)$ randomized. Moreover, each of the classes is nonempty, i.e., there is some LCL problem exhibiting the prescribed complexity. However, decidability on which side of the gap between two of the complexity classes a given problem falls is only known for the gaps above $\Theta(\log n)$. In particular, the following simple question is wide open.

Open Problem 2. *Is it decidable whether, on trees, a given LCL problem can be solved in a constant number of rounds?*

A similar classification as for trees is known for general (bounded-degree) graphs; however, in this case the classification is not complete yet, hinging on the complexity of a polynomially relaxed version of the algorithmic Lovász local lemma [6].

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An introduction to Measurable graph combinatorics

Andrew Marks

Many fields of mathematics study combinatorial problems on infinite graphs. Often the vertices of these graphs have some natural topological structure or measure and we would like the solution to satisfy some additional measurability conditions. Measurable graph combinatorics is the study of these types of problems. It uses methods and techniques from probability, combinatorics, ergodic theory, descriptive set theory, and computer science. We introduce this area with a particular focus on the study of *Borel graphs*: graphs $G = (V, E)$ whose vertex set V is the elements of a Polish space, and whose edge relation E is a Borel subset of $V \times V$.

We also discuss some open problems:

- Is there a Borel version of the Lovász Local Lemma for graphs generated by actions of amenable groups?
- The Borel Ruziewicz problem: is Lebesgue measure the only finitely additive measure defined on the Borel subsets of the n-sphere for $n \geq 2$?
- Does every bounded degree Borel graph have an unfriendly Borel coloring?
- Does every Borel graph generated by n functions have Borel chromatic number at most $(2n + 1)$ if it has a finite Borel coloring?
- If G is a Borel graph of degree at most d, does G have a Baire measurable $(d + 1)$-edge coloring?
- Let a be the Bernoulli shift of $\mathbb{F}_2$ with its usual generating set and μ the usual product measure. Let G be the Schreier graph of this action. What is the measurable independence number and measurable chromatic number of this graph?

Problem Session

Jan Grebík

We collect several interesting problems connected with the main theme of this workshop.

As a motivation, we first discuss a folklore result that has been known in the context of descriptive combinatorics, distributed computing and random processes. Namely, any local problem on oriented path is either trivial, solvable by standard methods in each field, or global. Moreover, given a local problem, it is decidable

to determine what situation occurs. We discuss the proof of this result as well as possible generalizations to other graphs.

Next, we collect the open problems, see also [1, 2, 3, 4, 5, 6, 7, 8]:

Finitary fiid The *finitary* factor of iid formalism [8], or equivalently the model of *uniform* local algorithms [7], offers finer scale to measure complexity of local problems than the one provided by the LOCAL model of distributed computing. Namely, there are examples of local problems that admit finitary factor of iid solution with non-trivial tail decay but are global in the standard LOCAL model of distributed computing [7, 3].

- Describe all complexity classes in the finitary factor of iid setting, e.g., speed-up results, new examples of local problems.

Derandomization There are three types of derandomization: (a) the "classical" derandomization, when a randomized algorithm yields a deterministic algorithm of the same complexity (known for grids in the LOCAL model), (b) the "intermediate" derandomization, when the existence of an effective randomized LOCAL algorithm yield Borel solution (known for subexponential growth graphs [1]), and (c) the "higher" one, when factor of iid solution yields Borel measurable solution (known for oriented paths).

- Describe when a graph admit any of the derandomization (a)–(c), e.g., do we have (a) and (c) for subexponential growth graphs?

Lovász Local Lemma What are the limits of the LLL technique [1, 2, 4].

- Can we have a Borel LLL for amenable graphs and a Continuous LLL for subexponential growth graphs?

Particular local problems

- Can we find a perfect matching on trees as a finitary factor of iid?
- A coloring with two colors (on a regular tree) is called *unfriendly* if at least half of the neighbors of every vertex v receive a different color than v does. Is there always a Borel unfriendly coloring?

General questions

- Is there a general complexity theory of local problems on general classes of graphs?
- What techniques in one field admit interpretation in one of the other fields?

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Moser – Tardos algorithm with a small amount of random bits

Łukasz Grabowski
(joint work with Endre Csóka, András Máthé, Oleg Pikhurko, Konstantinos Tyros)

We are interested in vertex colouring problems on graphs. An instance of such a problem consists of a digraph G, a natural number $b \in \mathbb{N}$ which is the number of colours which we use, and a system of constraints $\mathbf{R}$ which we call a *local rule*, i.e. for $x \in V(G)$ we have that $\mathbf{R}(x)$ is a set of b-valued functions defined on the out-neighbourhood of x. We say that $f \in b^{V(G)}$ *satisfies* $\mathbf{R}$ if for every $x \in V(G)$ the restriction of f to the out-neighbourhood of x belongs to $\mathbf{R}(x)$.

One of the most useful tools which allows to deduce that a colouring which satisfies a given local rule exists, is the *Lovász Local Lemma* (LLL for short) first proved in [8]. Let us state the version of it which follows from [15].

Let G be a digraph and let $\mathrm{Rel}(G)$ be the symmetric digraph whose vertex set is $V(G)$ and such that there is an edge between x and y if there is z such that $(x,z),(y,z) \in E(G)$ (we allow x and y to be equal, so there may be self-loops in $\mathrm{Rel}(G)$).

Theorem 1 (Lovász Local Lemma [15])**.** *Let G be a digraph and let Δ be the maximal vertex degree in $\mathrm{Rel}(G)$. If for every $x \in V(G)$ we have*

$$1 - \frac{|\mathbf{R}(x)|}{b^{|\operatorname{Var}(x)|}} < \frac{1}{e\Delta}, \tag{2}$$

where $\mathrm{Var}(x)$ is the out-neighbourhood of x, then there exists $f \in b^{V(G)}$ which satisfies $\mathbf{R}$.

Remark 3. *Frequently, LLL is stated for a "satisfying assignment of variables" instead of a satisfying colouring. Let us explain this alternative point of view and why it is equivalent to the formulation with satisfying colourings.*

Let W be a set variables and let C be a set of logical clauses, each of which may incorporate some of the variables in W. Then we are asking whether there exists an assignment $a \in b^W$ of values to the variables which makes all of the clauses true. The translation from such a "satisfying assignment" problem to a digraph colouring problem is by considering the bipartite digraph G with $V(G) := C \sqcup W$, where $(x,y) \in C \times W$ is an edge if and only if y is a variable which appears in the clause x. For $x \in W$ the set $\mathrm{Var}(x)$ is empty and we set $\mathbf{R}(x) := b^{\emptyset}$, so that $\mathbf{R}(x)$ does not impose any restrictions on satisfying colourings. For $x \in C$ we let $\mathbf{R}(x)$ be the set of those assignments on $\mathrm{Var}(x)$ which make the clause x satisfied.

In the graph $\mathrm{Rel}(G)$ *two clauses are adjacent if and only if they share a common variable, while all elements of* W *are isolated vertices of* $\mathrm{Rel}(G)$.

One of the key developments related to LLL is the Moser-Tardos Algorithm (MTA for short) studied by Moser and Tardos [14] (with a different version analysed earlier by Moser [13]). The MTA is a randomised algorithm for *finding* a satisfying colouring under the assumption that (2) holds. If we restrict attention to a class of colouring problems where the gap in (2) is at least a fixed constant $c > 0$, then the MTA finds a satisfying assignment on a graph G after using $O(|V(G)|)$ random bits in expectation.

Main result. It is convenient to fix a natural number $b > 1$ which is the number of colours in our colouring problems. With this in mind, a *colouring problem* is a pair $(G, \mathbf{R})$, where G is a digraph and $\mathbf{R}$ is a local rule on G.

The main aim of this work is to present a parallel version of the MTA and to show that it has the following property. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a *subexponential function*, i.e. such that $\lim_{n\to\infty} \frac{f(n)}{a^n} = 0$ for all $a > 1$. Consider the class $\mathcal{C}$ of graphs in which the balls of radius r contain at most $f(r)$ vertices. For example, $\mathcal{C}$ could be the class of graphs in which the balls of radius n contain at most $2(n+1)^2$ vertices, in which case $\mathcal{C}$ would contain all subgraphs of the infinite 2-dimensional grid. Let us also fix some $c > 0$ and let $\mathcal{P}$ be the class of colouring problems $(G, \mathbf{R})$ such that $G \in \mathcal{C}$ and

$$\max_{x\in V(G)} \left(1 - \frac{|\mathbf{R}(x)|}{b^{\deg(x)}}\right) \leq \frac{1}{e\Delta} - c.$$

Then there exists a constant $K > 0$ such that for every colouring problem $(G, \mathbf{R}) \in \mathcal{P}$ the expected number of random bits which the algorithm uses is at most K. In particular, the expected number of used random bits is independent from the number of vertices in G. This is the crucial difference when compared to previous algorithmic versions of LLL. As a consequence, we obtain a sequential **deterministic** algorithm which runs in time $O(|V(G)|)$ for colouring problems $(G, \mathbf{R})$ in any class $\mathcal{P}$ of the kind we have just described.

Descriptive Combinatorics. Our results can be also used to derive a Borel version of the Lovász Local Lemma. In brief, we say that a colouring problem $(G, \mathbf{R})$ is *Borel* if the vertex set $V(G)$ is a standard Borel space and both the edge set $E(G) \subset V(G)^2$ and the function $x \mapsto \mathbf{R}(x)$ are Borel. We show that if G has a uniformly subexponential growth and $(G, \mathbf{R})$ is a Borel colouring problem for which the inequality (2) holds, then there is a satisfying colouring $V(G) \to b$ which is a Borel function. This result appeared as the main result in the preprint [7, Theorem 4.7], which is superseded by this work.

Other descriptive versions of LLL were proved by Kun [11] and Bernshteyn [1, 2, 3]. For a quick comparison with our Borel LLL, let us mention that the versions proved in [11, 1, 2] allow a null-set of errors, and the version in [3], while producing an error-free satisfying colouring which is a continuous function, requires much stronger condition than (2).

On the other hand, our version of LLL requires that the underlying graph has subexponential growth. This might seem like a very strong assumption, especially since other descriptive versions of LLL mentioned in the previous paragraph do not need it. However, the assumption of subexponential growth cannot be removed in the Borel context in full generality (see [12, 5, 4]). There is a considerable number of recent articles where graphs of subexponential growth are studied from the point of view of descriptive combinatorics, see e.g. [6, 9, 10, 16].

Our Borel LLL from [7] has already found some interesting applications. For example Bernshteyn [2, Theorem 2.15] used it to show that, for graphs of subexponential growth, if a colouring problem on a graph G can be solved by a randomised LOCAL algorithm in $O(\log |V(G)|)$ rounds, then the corresponding Borel colouring problem admits a Borel solution.

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The Lovász Local Lemma in Descriptive Combinatorics

Anton Bernshteyn

The *Lovász Local Lemma* (the *LLL* for short) is a powerful probabilistic tool developed by Erdős and Lovász in the 1970s [7]. Since then, the LLL has found a multitude of applications in combinatorics [1, 9]. Recently, it has also been applied to problems in other areas, such as topological dynamics, ergodic theory, and descriptive set theory. These recent applications necessitated the development of analogs of the LLL that can work in the Borel, continuous, or measurable context [2, 3, 4, 6]. The aim of this talk is to provide a survey of these analogs.

A convenient framework for the LLL is given by the formalism of constraint satisfaction problems. Let X be a set and let $k \in \mathbb{N}^+$ be a positive integer. As usual, we use the identification $k = \{0, 1, \dots, k-1\}$. By a *k-coloring* of X we mean a function $f\colon X \to k$. A *constraint* is a set B of functions $\mathrm{dom}(B) \to k$, where $\mathrm{dom}(B)$ is some finite subset of X, called the *domain* of B. A k-coloring $f\colon X \to k$ *violates* a constraint B if $f|_{\mathrm{dom}(B)} \in B$; otherwise, f *satisfies* B. A *constraint satisfaction problem* (a *CSP* for short) is a set $\mathcal{B}$ of constraints. We write $\mathcal{B}\colon X \to^? k$ to indicate that $\mathcal{B}$ is a CSP on X. A *solution* to a CSP $\mathcal{B}\colon X \to^? k$ is a k-coloring $f\colon X \to k$ that satisfies every constraint $B \in \mathcal{B}$.

Many problems in combinatorics can naturally be interpreted as seeking solutions to specific CSPs. Therefore, it is desirable to have easily verifiable sufficient conditions that guarantee that a given CSP has a solution. The LLL provides one such condition. For a CSP $\mathcal{B}\colon X \to^? k$, we define two numerical parameters, $\mathsf{p}(\mathcal{B})$ and $\mathsf{d}(\mathcal{B})$, as follows:

$$\mathsf{p}(\mathcal{B}) \coloneqq \sup_{B\in\mathcal{B}} \frac{|B|}{k^{|\mathrm{dom}(B)|}},$$
$$\mathsf{d}(\mathcal{B}) \coloneqq \sup_{B\in\mathcal{B}} |\{B' \in \mathcal{B} : B' \neq B \text{ and } \mathrm{dom}(B') \cap \mathrm{dom}(B) \neq \emptyset\}|.$$

The LLL can now be stated as follows:

Theorem 1 (Lovaśz Local Lemma)**.** *Let $\mathcal{B}\colon X \to^? k$ be a CSP. Suppose that*

$$e\,\mathsf{p}(\mathcal{B})\,(\mathsf{d}(\mathcal{B})+1) < 1, \tag{2}$$

where $e = 2.71\dots$ is the base of the natural logarithm. Then $\mathcal{B}$ has a solution.

Note that Theorem 1 holds even when X is an infinite set and $\mathcal{B}$ is an infinite collection of constraints. However, the proof of Theorem 1 in the infinite case uses a compactness argument. In practice, it is often desirable to have a solution f with some additional regularity properties, which a compactness argument cannot provide. For example, if X is equipped with a measure μ, we may want to find a solution $f\colon X \to k$ to $\mathcal{B}$ that furthermore is μ-measurable. For such problems, it may be necessary to replace the LLL condition 2 with a stronger one or to impose additional requirements on the structure of the CSP $\mathcal{B}$.

In this talk, we focus on three main regularity notions: continuity, Borelness, and measurability. Table 1 provides a summary of some of the known results (each of which holds under appropriate topological/measure-theoretic assumptions on X

TABLE 1. Does a CSP have a well-behaved solution?

Assumptions	Continuous?	Borel?	Measurable?
$e\mathsf{p}(\mathsf{d}+1) < 1$	No	No	Open; yes with an ε error
$2^{15}\mathsf{p}(\mathsf{d}+1)^8 < 1$	No	No	Yes
$\mathsf{p}2^{\mathsf{d}} \leq 1$	No	No	Yes (for large d)
$\mathsf{p}2^{\mathsf{d}} < 1$	Yes	Yes	Yes
$e\mathsf{p}(\mathsf{d}+1) < 1$ and the dependency graph has growth rate $\exp(o(r))$	Open	Yes	Yes
$e\mathsf{p}(\mathsf{d}+1) < 1$ and the dependency graph has growth rate $\exp(r^{o(1)})$	Yes	Yes	Yes
$\mathsf{p}\,\mathsf{vdeg}^{\mathsf{ord}} < 1$	Yes	Yes	Yes

and $\mathcal{B}$). Some of these results require additional assumptions on the growth rate of the dependency graph $G_{\mathcal{B}}$ of $\mathcal{B}$, which is defined as follows: the vertex set of $G_{\mathcal{B}}$ is $\mathcal{B}$, and there is an edge between two distinct constraints B, $B' \in \mathcal{B}$ if and only if $\mathrm{dom}(B) \cap \mathrm{dom}(B') \neq \emptyset$. In particular, $\mathsf{d}(\mathcal{B})$ is exactly the maximum degree of the graph $G_{\mathcal{B}}$. The last line of Table 1 uses two alternative parameters $\mathsf{vdeg}(\mathcal{B})$ and $\mathsf{ord}(\mathcal{B})$ in place of $\mathsf{d}(\mathcal{B})$. These parameters are given by the following definitions:

$$\mathsf{vdeg}(\mathcal{B}) := \sup_{x \in X} |\{B \in \mathcal{B} : x \in \mathrm{dom}(B)\}|,$$
$$\mathsf{ord}(\mathcal{B}) := \sup_{B \in \mathcal{B}} |\mathrm{dom}(B)|.$$

Many of the results mentioned in Table 1 are intimately related to analogous facts in distributed computing [5, 8, 10]. Indeed, in many cases the proof of an LLL-like theorem in the Borel or measurable context relies in a crucial way on a distributed algorithm that applies in similar circumstances.

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Lovász Local Lemma in Distributed Algorithms

Václav Rozhoň

Lovász local lemma is one of the most useful tools to solve so-called locally checkable problems (LCLs, coloring problems). The problem is as follows. Imagine a graph G of maximum degree Δ whose nodes correspond to *bad events*. Each event $\mathcal{E}(u)$ at node u is independent on all other events, except of the events $\mathcal{E}(v)$ for a neighbor v of u.

Here is an example of an instance of Lovász local lemma. Consider a Δ-regular tree and the local problem of orienting each edge in one direction so that no non-leaf node is a sink. That is, no non-leaf node should have outdegree at least one. This problem is known as the sinkless orientation problem. Orienting each edge at random, with both orientations having probability $1/2$, we have a bad event for each node that it is a sink: This bad event has probability $1/2^{\Delta}$. Since the Lovász Local Lemma admits an efficient local distributed algorithm in this setup (see the next paragraphs), we are automatically getting a solution also for the sinkless orientation problem.

In general, whether the Lovász local lemma can be solved with an efficient distributed local algorithm depends on the maximum bad event probability p; the problem can be solved in $O(\log^* n)$ distributed rounds if $p < 2^{-\Delta}$ and in $\operatorname{poly}\log\log(n)$ randomized rounds (or $\operatorname{poly}\log(n)$ deterministic rounds) if $p < 1/\Delta^C$ for some large enough constant C. These and related results are proven in a series of exciting new developments [CHL$^+$18, FG17, GHK18, RG20, GGR21, BBH$^+$19, BMU19, BGR20].

In this talk, we will see how above results are proven and how they fit the general theory of distributed local algorithms [CKP19, CP19].

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Continuity of the phase transition for the Ising model on nonamenable groups: an application of FIIDs

Tom Hutchcroft

We prove that the Ising model undergoes a continuous phase transition on any nonamenable Cayley graph. The proof builds on earlier work [2, 3] analyzing critical Bernoulli bond percolation in the same context, in which one deduces upper bounds on the tail of the volume of a critical cluster using the *two-ghost inequality*, a universal upper bound on the probability of having two large clusters next to each other that is related to the seminal work of Aizenman, Kesten, and Newman [4]. Our theorems are proven by applying this methodology to the *double random current* representation of the Ising model, and a key difficulty is that this model does *not* satisfy the FKG inequality. We circumvent this issue by using the spectral theory of automorphism-invariant processes on nonamenable groups, and in particular the monotonicity of the spectral radius under factors.

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Finitary factors of iid processes

Yinon Spinka

Consider two processes $X = (X_v)_{v\in\mathbb{Z}^d}$ and $Y = (Y_v)_{v\in\mathbb{Z}^d}$ whose distributions are invariant under translations. We are concerned with codings (factor maps) from Y to X. Suppose that $X \in A^{\mathbb{Z}^d}$ and $Y \in B^{\mathbb{Z}^d}$ where A and B are countable sets. A *coding* from Y to X is a measurable function $\varphi\colon B^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ which is translation-equivariant (i.e., commutes with translations) and satisfies that $\varphi(Y) = X$ almost surely. The coding φ is *finitary* if there almost surely exists a finite number R such that X_0 is determined by $(Y_v)_{|v|\le R}$. The coding radius R is a random variable whose value depends on the realization of Y. Finitary codings can be defined similarly when the underlying state spaces A and B are more general (e.g., $[0,1]$). We say that X is a *(finitary) factor* of Y is there exists a (finitary) coding from Y to X. Given a process X, we say that X is *ffiid* if it is a finitary factor of some i.i.d. (independent and identically distributed) process Y.

The study of finitary codings attracts researchers from various fields. The subject started in ergodic theory and is a classical topic in this respect. Later, connections were discovered between finitary codings and phase transitions in statistical mechanics. In addition, finitary codings have an algorithmic aspect to them and are closely linked to problems of exact sampling (perfect simulation) and distributed randomized algorithms.

We are concerned with the question of whether particular processes are ffiid. This problem is sometimes referred to as *sampling* or *simulating*, in contrast to the problem of *constraint solving* or *labeling*, where one is concerned with the possibility (or lack thereof) of constructing certain combinatorial structures (e.g., coloring or perfect matching) as ffiid's, with no regard to the measure induced on them. The question of whether a given process is ffiid or not has received much attention. While the theory of (non-finitary) factors is rather well established, with several useful necessary and sufficient conditions for being a factor of an i.i.d. process, the finitary counterpart of the theory is lacking. In particular, there are no known useful necessary and sufficient conditions for a general process to be ffiid (even in one dimension). We begin by discussing some results in this direction and then move on to discuss certain aspects of efficiency.

Finitely dependent processes. We start with the class of finitely dependent processes. Roughly speaking, a process is *finitely dependent* if its restriction to regions which are sufficiently separated are independent. Precisely, a process $X = (X_v)_{v\in\mathbb{Z}^d}$ is finitely dependent if there exists a positive integer k such that $(X_u)_{u\in U}$ and $(X_v)_{v\in V}$ are independent whenever $U, V \subset \mathbb{Z}^d$ have $\operatorname{dist}(U,V) > k$, where $\operatorname{dist}(U,V) := \min_{u\in U, v\in V} |u - v|$. Trivial examples of finitely dependent invariant processes are i.i.d. processes. Other examples are so-called block factors (finitary

factors with bounded coding radius) of i.i.d. processes. More interesting examples, which are not block factors of i.i.d. processes, are the 1-dependent 4-coloring and 2-dependent 3-coloring of $\mathbb{Z}$ constructed by Holroyd and Liggett [5]. A result of Smorodinsky [8] shows that finitely dependent invariant processes on $\mathbb{Z}$ are ffiid. This was extended by the author to all dimensions and in fact holds for any transitive amenable graph [9].

The proof in [9] shows that for one dimensional processes the coding radius R satisfies $\mathbb{P}(R > r) \leq \frac{C}{r}$, which falls just short of showing that the coding radius has finite expectation. An interesting open problem is whether there exists a finitely dependent invariant process which cannot be expressed as an ffiid with finite expected coding radius. The 1-dependent 4-coloring and 2-dependent 3-coloring from [5] are potential such candidates (see [4]).

Markov random fields. Another class of processes of interest are Markov random fields (MRFs), which are the higher dimensional analogue of Markov chains. These are processes $X = (X_v)_{v \in \mathbb{Z}^d}$ with the property that the conditional distribution of the process on any given finite set V given the process outside of V depends only on the values on the boundary of V. In one dimension, a finite-state Markov chain is ffiid if and only if it is ergodic (aperiodic and irreducible) [6], and a countable-state Markov chain is ffiid if and only if it is exponentially ergodic (in the sense that the return time to a fixed state has exponential tails) [1]. The situation in higher dimensions is not fully understood. For the well-known Ising model on $\mathbb{Z}^d$, van den Berg and Steif [2] proved that the plus state Gibbs measure is ffiid if and only if there is no phase coexistence (i.e., the plus and minus states coincide, which is known to occur precisely when the temperature is at or above the critical temperature). The picture is not as complete for general MRFs on $\mathbb{Z}^d$. On the one hand, it is known that if two ergodic MRFs have the same conditional distributions (this corresponds to phase coexistence in models of statistical mechanics) and have finite energy, then neither can be ffiid [2]. In the other direction, it seems plausible that a sufficient condition for an invariant MRF to be ffiid is that it is uniquely specified, meaning that there is no other MRF with the same conditional distributions, perhaps also assuming finite energy. Let us describe a result in this direction. A MRF X is said to satisfy *weak spatial mixing (WSM)* if there exist constants $C, c > 0$ such that the total variation distance between $P^{\tau}_{V,U}$ and $P^{\tau'}_{V,U}$ is at most $Ce^{-c\mathrm{dist}(U,\partial V)}$, for any finite $U \subset V \subset \mathbb{Z}^d$ and any $\tau, \tau' \in A^{\partial V}$, where ∂V is the vertex boundary of V and $P^{\tau}_{V,U}$ is the conditional law of $(X_u)_{u \in U}$ given that $(X_v)_{v \in \partial V} = \tau$. Then any invariant MRF satisfying WSM is ffiid [10]. This is a strengthening of a result of Häggström a and Steif [3] which states that an invariant MRF satisfying a certain "high noise" assumption is ffiid.

An open problem is to determine which MRFs are ffiid. Closing the gap between the known sufficient conditions and the known necessary conditions would be of great interest.

Efficient finitary codings. Once a given process is shown to be ffiid, one may further ask about two aspects of the efficiency of the finitary coding: spatial efficiency as described by the coding radius, and informational efficiency as described by properties of the source i.i.d. process such as its support size or its entropy.

For example, a significant part of the effort in the result of van den Berg and Steif [2] was toward establishing that the high-temperature Ising model is a finitary factor of a *finite-valued* i.i.d. process, as opposed to merely a finitary factor of *some* i.i.d. process (in which case they obtained a coding radius with exponential tails). For the critical Ising model, only the basic finitary factor of i.i.d. was shown and it was asked [2, Question 1] whether there is a finitary coding from a finite-valued i.i.d. process. Meyerovitch and the author [7] obtained the following general result which says that informational efficiency can always be attained and thereby gives a positive answer to the previous question: if a finite-valued process X on $\mathbb{Z}^d$ is ffiid, then it is also a finitary factor of a finite-valued i.i.d. process, which can furthermore be chosen to have entropy arbitrarily close to that of X. On the other hand, it is known that, in the critical case, no finitary coding can be very efficient in terms of the coding radius in the sense that the d-th moment must always be infinite [2].

In some cases, a process is known to be ffiid with good spatial efficiency (e.g., coding radius with exponential tails). Let X be such a process. The results above shows that X can also have good informational efficiency (e.g., finite-valued i.i.d. process with low entropy). Is it the case that both properties can be realized simultaneously? That is, is X a finitary factor of a finite-valued i.i.d. process whose entropy is close to that of X and with a coding radius having exponential tails?

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An algebraic approach to Borel CSPs

Riley Thornton

For a given finite relational structure $\mathcal{D}$, the associated constraint satisfaction problem (or CSP) is defined as follows

Definition 1. $\mathrm{CSP}(\mathcal{D})$ is the set of (codes for) structures which admit a homomorphism into $\mathcal{D}$

These CSPs were introduced by Feder and Vardi in the 1980s as class of problems where complexity questions might be tractable [2]. Many natural problems can be interpreted as CSPs: graph coloring, Boolean satisfaction, solving systems of linear equations, etc. And yet, Feder and Vardi conjectured the complexity behaviour of CSPs should be quite rigid. For instance their CSP Dichotomy Conjecture claimed that $\mathrm{CSP}(\mathcal{D})$ is either polynomial time solvable or NP-complete. Remarkably Feder and Vardi's conjectures have been verified using methods from universal algebra.

Theorem 1 (Bulatov, Zhuk [1][3])**.** For any finite relational structure $\mathcal{D}$, if there is a homomorphism $f : \mathcal{D}^4 \to \mathcal{D}$ so that

$$(*) \qquad (\forall a, e, r)\ f(a, r, e, a) = f(r, a, r, e)$$

then $\mathrm{CSP}(\mathcal{D})$ is polynomial time solvable. Otherwise $\mathrm{CSP}(\mathcal{D})$ is NP-complete.

The present work adapts this algebraic machinery to the Borel setting. We define $\mathrm{CSP}_B(\mathcal{D})$ analogously as the set of (codes for) Borel structures which admit a Borel homomorphism to $\mathcal{D}$.

Theorem 2. If there is no homomorphism $f : \mathcal{D}^4 \to \mathcal{D}$ satisfying $(*)$, then $\mathrm{CSP}_B(\mathcal{D})$ is $\mathbf{\Sigma}^1_2$-complete.

Corollary 1 (P$\neq$NP)**.** If $\mathrm{CSP}(\mathcal{D})$ is NP-complete, then $\mathrm{CSP}_B(\mathcal{D})$ is $\mathbf{\Sigma}^1_2$.

There are many open questions in this directions. Is the converse to the above theorem true? Can we algebraically characterize Borel notions of complexity such effectiviability or the presence of a dichotomy theorem? If $\mathcal{D}$ is a directed graph, can we algebraically characterize when $\mathrm{CSP}(\mathcal{D})$ admits an efficient local algorithm over some class of directed graphs? Does the above theorem hold for all LCLs?

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Complexity in Borel Combinatorics

Zoltán Vidnyánszky

One of the most useful features of descriptive set theory is that it has tools to calculate the descriptive complexity of a subset of a Polish space, this way disproving conjectures that assert a different, simpler description of the set in question. An early example of this phenomenon is due to Mazurkiewicz, who has essentially shown that the set of everywhere differentiable functions in $C[0,1]$ cannot be described in a simpler way than its usual definition, that is, by using a universal quantifier over the reals (see [6, Section 33] for a rich variety of such examples.)

While Borel combinatorics turned out to be an exceptionally elegant way to treat the infinite objects and their relations to their finite counterparts, there are relatively few results with an emphasis on descriptive complexity.

The focus of our talk was to present a couple of results of this sort and to discuss the connections to the LOCAL model of distributed computing.

Let $\mathbf{\Gamma}$ be a family of subsets of Polish spaces. A subset S of a Polish space X is called $\mathbf{\Gamma}$-*complete*, if for any Polish space Y and $T \subseteq Y$ in $\mathbf{\Gamma}$ there is a Borel map $f : Y \to X$ with $f^{-1}(S) = T$. Recall that a subset S of a Polish space X is $\mathbf{\Sigma}^1_2$, if there is a Borel set $B \subset X \times \mathbb{R} \times \mathbb{R}$ with $z \in S \iff \exists x \in \mathbb{R}\, \forall y \in \mathbb{R}\, ((x,y,z) \in B)$. It is well known that $\mathbf{\Sigma}^1_2$-complete sets are not $\mathbf{\Pi}^1_2$, that is, there is no Borel set B with $z \in S \iff \forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, ((x,y,z) \in B)$.

We will denote the *Borel chromatic number* of a Borel graph $\mathcal{G}$ (that is, the minimal n for which $\mathcal{G}$ admits a Borel n-coloring) by $\chi_B(\mathcal{G})$. The following theorem establishes a barrier for the characterization of graphs with certain Borel chromatic numbers.

Theorem 1 (Todorčević-V [9])**.** *Let $n \geq 3$ be a natural number. The set of Borel graphs $\mathcal{G}$ with $\chi_B(\mathcal{G}) \leq n$ is $\mathbf{\Sigma}^1_2$-complete.*

Note that virtually all conjectured characterizations of graphs with Borel chromatic numbers at most n would yield that such graphs form a $\mathbf{\Pi}^1_2$ set, hence this theorem rules out all such conjectures. On the other hand, there is a characterization in the case $n = 2$.

Theorem 2 (Carroy-Miller-Schrittesser-V [5])**.** *There exists a Borel graph $\mathbb{L}_0$ such that for any Borel graph $\mathcal{G}$ we have $\chi_B(\mathcal{G}) \geq 3$ if and only if $\mathbb{L}_0$ admits a Borel homomorphism to $\mathcal{G}$.*

This reflects the intuition coming from the finite case: it is easy to decide whether a graph has an n-coloring for $n = 2$, but it is hard for any $n > 2$.

The proof of Theorem 1 uses the so called *shift-graph* from [7]. Let $S : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ be the *shift-map*, defined by $S(x) = x \setminus \{\min x\}$ (here $[\mathbb{N}]^{\mathbb{N}}$ stands for the collection of infinite subsets of $\mathbb{N}$). Define the shift-graph $\mathcal{G}_S$ by letting $x \mathcal{G}_S y$ iff $y = S(x)$ or $x = S(y)$. It follows from an infinite dimensional generalization of Ramsey's theorem, the Galvin-Prikry theorem, that $\chi_B(\mathcal{G}_S)$ is infinite.

A key weakness of Theorem 1 is that it only establishes the complexity result for subgraphs of $\mathcal{G}_S$, which have finite but unbounded degree. Essentially, the only

known tool to establish complexity results in the bounded degree acyclic context is Marks' determinacy method [8]. Recall that Marks has shown the existence of d-regular acyclic Borel graphs $\mathcal{G}$ with $\chi_B(\mathcal{G}) = d + 1$. However, it was completely unclear, whether there are analogues of the methods developed in [9] to establish a complexity result.

In [2] the connections between the LOCAL model and Borel combinatorics have been investigated. It soon became clear that Marks' method is rather similar to Brandt's automatic speedup theorem, in fact the former could be adapted to the realm of distributed computing. In this adaptation one encounters a subtle problem: Marks' method is based on 2-player games, where the players alternatingly label vertices of a graph with natural numbers; this corresponds to IDs in the LOCAL model. However, in the LOCAL models IDs are unique, while one cannot enforce such property easily in the game method. To resolve this issue, one can use a trick coming from distributed computing [4]: instead of labeling with natural numbers, one builds a homomorphism to graphs (called ID graphs) with large enough girth, hence establishing local injectivity. Transferring back this idea to the Borel context, we arrive to the following concept.

Let $\mathcal{H}$ be a Borel graph and Γ be a group with generating set S. Let $Hom(\Gamma, S, \mathcal{H})$ be the restriction of the Schreier graph corresponding to the left-shift action of Γ on $V(\mathcal{H})^\Gamma$ to the set

$$\{h \in V(\mathcal{H})^\Gamma : h \text{ is a homomorphism from } Cay(\Gamma, S) \text{ to } V(\mathcal{H})\}.$$

It turns out that certain combinatorial properties of $\mathcal{H}$ are reflected in $Hom(\Gamma, S, \mathcal{H})$. Let Γ_3 be the group $2^{\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2}$, that is, the free product of 3 copies of $\mathbb{Z}_2$, and S_3 be its natural generating set.

Theorem 3 (Brandt-Chang-Grebík-Grunau-Rozhoň-V [3])**.** *Let $\mathcal{H}$ be a locally countable Borel graph. Then*

$$\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > 3 \implies \chi_B(Hom(\Gamma_3, S_3, \mathcal{H})) > 3,$$

and

$$\chi_B(\mathcal{H}) \leq 3 \implies \chi_B(Hom(\Gamma_3, S_3, \mathcal{H})) \leq 3,$$

where $\chi_{wpr-\mathbf{\Delta}^1_2}$ stands for a certain version of definable chromatic number.

These ideas allow one to combine the methods form [9] with Marks' method, and hence establish the complexity result for 3-regular acyclic Borel graphs, ruling out for example any generalization of Brook's theorem to the Borel context.

Finally, let us list a couple of natural open problems related to the topic. Since our examples are non-compact, it is conceivable that one can still formulate a characterization result for compact subshifts.

Question 1. *Is the collection of compact free subshifts of $2^{\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2}$ with Borel chromatic number ≤ 3 also Σ^1_2-complete?*

As for further connections between the distributed and descriptive settings one can ask the following.

Question 2. *Is there a way to utilize $\mathcal{G}_S$ and Galvin-Prikry-like theorems in the LOCAL model?*

A natural question is, whether one can relate determinacy arguments to the shift graph.

Question 3. *Is it possible to construct functions with large Borel chromatic number using determinacy arguments? Does it follow from the Axiom of Determinacy over ZF that $\chi(\mathcal{G}_S)$ is infinite?*

Note that it is a long standing open problem, whether AD implies the Ramsey-property for subsets of $[\mathbb{N}]^{\mathbb{N}}$, and this is would be an intermediate result.

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