Problem solving seminar
Test 1 - solutions

1. Let $P : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial with integer coefficients. Suppose that $|P(z)| \leq 2$ for every $z \in \mathbb{C}$ with $|z| = 1$. Prove that P has at most 4 nonzero coefficients.

Solution. Let $P(z) = a_n z^n + \ldots + a_1 z + a_0$. Note that

$$4 \geq \frac{1}{2\pi} \int_{|z|=1} |P(z)|^2 \mathrm{d}z = \frac{1}{2\pi} \int_{|z|=1} P(z)\bar{P(z)}\mathrm{d}z = \sum_{k,l=0}^{n} a_k a_l \int_{|z|=1} z^k \bar{z}^l \mathrm{d}z = \sum_{k=0}^{n} a_k^2.$$

Therefore only at most 4 a_k's can be nonzero. □

2. A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coefficient 1. By its vertex we mean the point $(-a/2, -a^2/4 + b)$. Three standard parabolas with vertices V_1, V_2, V_3 intersect pairwise at points A_1, A_2, A_3. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x-axis.

Prove that standard parabolas with vertices $s(A_1)$, $s(A_2)$, $s(A_3)$ intersect pairwise at the points $s(V_1)$, $s(V_2)$, $s(V_3)$.

Solution. First we show that the standard parabola with vertex V contains point A if and only if the standard parabola with vertex $s(A)$ contains point $s(V)$.

Let $A = (a, b)$ and $V = (v, w)$. The equation of the standard parabola with vertex $V = (v, w)$ is $y = (x - v)^2 + w$, so it contains point A if and only if $b = (a - v)^2 + w$. Similarly, the equation of the parabola with vertex $s(A) = (a, -b)$ is $y = (x - a)^2 - b$; it contains point $s(V) = (v, -w)$ if and only if $-w = (v - a)^2 - b$. The two conditions are equivalent.

Now assume that the standard parabolas with vertices V_1 and V_2 , V_1 and V_3, V_2 and V_3 intersect each other at points A_3, A_2, A_1, respectively. Then, by the statement above, the standard parabolas with vertices $s(A_1)$ and $s(A_2)$, $s(A_1)$ and $s(A_3)$, $s(A_2)$ and $s(A_3)$ intersect each other at points V_3, V_2, V_1, respectively, because they contain these points. □