

## Game chromatic index of graphs with given restrictions on degrees

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## abstract

Given a graph  $G$  and an integer  $k$ , two players alternatively color the edges of  $G$  using  $k$  colors so that adjacent edges get different colors. The *game chromatic index*  $\chi_g(G)$  is the minimum  $k$  for which the first player has a strategy that ensures that all edges of  $G$  get colored.

The trivial bounds are  $\chi_g(G) \leq \Delta(G) + 1$ , where  $\Delta(G)$  denotes the maximal degree of  $G$ . Lam, Shiu, and Xu and, independently, Bartnicki and Grytczuk asked whether there is a constant  $C$  such that  $\chi_g(G) \leq C \Delta(G)$  for every graph  $G$ . We show that the answer is in the negative by constructing graphs  $G$  such that  $\chi_g(G) \geq 1.008 \Delta(G)$  and  $\chi_g(G) \geq 1$ . On the other hand, we show that for every  $\epsilon > 0$  there is  $\delta > 0$  such that for any graph  $G$  with  $\Delta(G) \geq 1/\epsilon$ , we have  $\chi_g(G) \leq (1 + \epsilon) \Delta(G)$ , where  $v(G)$  denotes the number of vertices of  $G$ .

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## 1. Introduction

Let a graph  $G$  and a positive integer  $k$  be given. Two players, called Alice and Bob, alternatively color a previously uncolored edge of  $G$  in one of the colors from  $\{1, \dots, k\}$  so that no two adjacent edges have the same color. Thus, at any moment of the game, the current partial coloring of  $E(G)$  is a proper edge coloring. The game can end in two different ways. Either all edges of  $G$  are colored (and then Alice is the winner) or the uncolored edge picked by a player cannot be properly colored (and then Bob wins).

Let us agree that Alice starts the game. (In fact, all theorems stated in this paper will remain valid for the version where we let Bob start the game.) The *game chromatic index*  $\chi_g(G)$  is the smallest  $k$  such that Alice has a winning strategy. This parameter has been previously studied by Lam, Shiu and Xu [9], Cai and Zhu [6], Erdős, Faigle, Hochstättler, and Kern [8], Andres [1], Bartnicki and Grytczuk [2], and others.

This is a variation of the *game chromatic number* which is analogously defined for the game where nodes (not edges) are colored. The latter parameter is much better studied; we refer the reader to Bohman, Frieze, and Sudakov [5] for some history and references on the game chromatic number.

The trivial bounds on the game chromatic index are

$$\chi_g(G) \leq \Delta(G) + 1; \quad (1)$$

where  $\Delta(G)$  denotes the maximal degree of  $G$ .

Unfortunately, the game chromatic index seems hard to analyze. For example, a player's move can easily harm that player later in the game. Also, it is not clear if there is any useful 'potential' function that measures a player's progress. Therefore, we settle for the modest task of getting a constant factor improvement over the trivial bounds (1) when  $\Delta(G)$  is large.

Lam, Shiu and Xu [9, Question 1] and, independently, Bartnicki and Grytczuk (Problem 1 in the preprint version of [2]) asked whether there is a constant  $C$  such that  $\chi_g(G) \leq C \Delta(G)$  for an arbitrary graph  $G$ . In Section 2 we show that the answer to this question is in the negative. Namely, we construct, for every sufficiently large  $d$ , a graph  $G$  with  $\Delta(G) = d$  and  $\chi_g(G) \geq 1.008 d$ .

On the other hand, the lower bound in (1) is attainable for some graphs. A trivial example is  $G \supseteq K_{1,d}$ . However, we believe that large minimal degree  $\delta(G)$  will force  $\phi_g(G)$  to be well above  $\delta(G)$ . Namely, we make the following conjecture.

**Conjecture 1.** *There are  $\epsilon > 0$  and  $d_0$  such that any graph  $G$  with  $\delta(G) \geq d_0$  satisfies  $\phi_g(G) \geq (1 + \epsilon) \delta(G)$ .*

Of course, the conclusion of Conjecture 1 is interesting only when  $\delta(G) < (1 + \epsilon) \phi_g(G)$ , that is, when all degrees are fairly close to each other.

From the other direction, we show in Section 3 that for any  $\epsilon > 0$  there is  $\delta > 0$  such that any graph  $G$  with  $\delta(G) \geq (1 + \epsilon) \phi_g(G)$  satisfies  $\phi_g(G) < (1 + 2\epsilon) \delta(G)$ . (Here,  $v(G)$  denotes the number of vertices of  $G$ .) Surprisingly, this is done by letting Alice play randomly, see Section 3 for details. While probabilistic intuition and reasoning often help in the analysis of combinatorial games, see e.g. Beck [3], there are not many examples where non-trivial results are obtained by actually introducing randomness into a player's strategy. Such examples were discovered by Spencer [11], Bednarska and Śuczak [4], Pluhár [10], and others. Our proof of the upper bound fits into this category.

The restriction  $\delta(G) \geq (1 + \epsilon) \phi_g(G)$  in the above result is needed in order to make our proof work. We do not believe that there is anything special about the constant  $1 + \epsilon$  here. We conjecture that a much stronger claim is true.

**Conjecture 2.** *There is  $\epsilon > 0$  such that for an arbitrary graph  $G$  we have  $\phi_g(G) \geq (1 + \epsilon) \delta(G)$ .*

## 2. Lower bounds

**Theorem 3.** *For every sufficiently large integer  $d$ , there is a graph  $G$  with maximum degree at most  $d$  such that  $\phi_g(G) > 1.008d$ .*

**Proof.** Let  $d \geq 4$ ,  $\delta \geq 1$ ,  $\delta \geq 3$ , and  $\delta \geq 25$ . Let  $d$  be sufficiently large and let  $n \geq bd = 3d^{2/3}$ .

We define a graph  $G$  of order  $n$  and maximum degree at most  $d$  as follows. Take two disjoint sets  $A$  and  $B$  of sizes  $d$  and  $b \leq n - d$  respectively. The vertex set of  $G$  is  $A \cup B$ . Put a complete bipartite graph between  $A$  and  $B$ . Let  $A$  be an independent set. Let  $G[B]$ , the subgraph of  $G$  induced by  $B$ , be a random graph with each pair of  $B$  being an edge with probability  $p \geq 1 - \delta \geq 1/4$ , independently of the other pairs.

Let the acronym *whp* (with high probability) mean 'with probability  $1 - o(1/n)$ '.

By the Chernoff bound [7], *whp* the  $G$ -degree  $d_x$  of every  $x \in V(G)$  satisfies  $|d_x - n| \leq n^{2/3}$ . In particular, the maximal degree of  $G$  is at most  $d$ . Also, *whp* every subset  $X$  of  $B$  spans at least  $p \binom{|X|}{2} \geq n^{5/3}$  edges. (Indeed, any fixed  $X$  violates this inequality with probability  $o(1/n)$  by the Chernoff bound.) Fix any  $G[B]$  that satisfies these two conditions.

Let  $k \geq 1.008d$  be an arbitrary integer and let  $n \geq k$ . In order to prove Theorem 3 we have to show that Bob has a winning strategy for the pair  $(G, k)$ . By (1), it is enough to consider only those  $k$  that are at least  $\delta(G)$ .

At the start of the game, Bob picks some  $l \leq n$  special colors, say  $1, \dots, l$ . (Note that the assumption  $k \geq \delta(G)$  implies that  $l \leq k$ .) His strategy consists of two stages. At each round of Stage 1, Bob tries to color some (arbitrary) edge inside  $B$  with one of the special colors. If this is impossible (that is, the endpoints of every uncolored edge of  $G[B]$  see all special colors between the two of them), then Stage 1 is over. In Stage 2 Bob plays arbitrarily.

Let us show that Bob necessarily wins. Suppose on the contrary that all edges of  $G$  get colored by the end of the game.

Let Stage 1 have  $n^2$  rounds. Suppose that Alice plays  $n^2$  times inside  $B$  (and  $n^2$  times between  $A$  and  $B$ ).

Let us analyze the moment when Stage 1 ends. Take any special color  $i \in [l]$ . Let  $X_i \subseteq A \cup B$  be the set of vertices that are adjacent to an edge of color  $i$ . Let  $|X_i \cap B| = n_i$ . We have

$$p \binom{n_i}{2} \geq n^{5/3} \implies \frac{p}{2} \binom{n_i}{2} \geq o(1/n^2) \implies n_i \geq \frac{2}{p} \binom{n_i}{2} \geq o(n^2/2). \quad (2)$$

Indeed,  $G[B \cap X_i]$  has at least  $p \binom{n_i}{2} \geq n^{5/3}$  edges and all of them must be colored. On the other hand, Bob can color at most  $l \cdot n = 2$  edges of  $G[B \cap X_i]$  because he uses only the  $l$  special colors in Stage 1 (and each color class is a matching).

Inequality (2), which is quadratic in  $n_i$ , implies that

$$n_i \leq \frac{p \binom{n_i}{2} \leq 8p \cdot n}{2p} \leq o(1/n). \quad (3)$$

Also, we have

$$2 \cdot n^2 \leq \sum_{i \in [l]} |X_i \cap B| \leq 3 \sum_{i \in [l]} |X_i \cap A|/2. \quad (4)$$

Indeed, Bob increases  $\sum_{i \in [l]} |X_i \cap B| - 3 \sum_{i \in [l]} |X_i \cap A|/2$  by 2 in each his move in Stage 1. If Alice uses a special color on a  $G[A, B]$ -edge, then it does not change in this round. If Alice does something else, which happens at least  $n^2$  times, then increases by at least 2 during the round.

For  $i \in \mathbb{I}$ , let  $Y_i$  be the set of vertices covered by the edges of Color  $i$  after the game ends. Let  $n^2 \geq \sum_{i \in \mathbb{I}} |Y_i|$ . At most  $n^2$  edges can be colored with special colors in Stage 2, because an edge colored  $i \in \mathbb{I}$  in Stage 2 must intersect  $B \cap X_i$ . (Recall that  $A$  is an independent set in  $G$ .) Thus we have

$$\sum_{i \in \mathbb{I}} |Y_i \setminus B| \leq \sum_{i \in \mathbb{I}} |Y_i \setminus A_j| \leq \sum_{i \in \mathbb{I}} |X_i \setminus B_j| \leq \sum_{i \in \mathbb{I}} |X_i \setminus A_j| \leq 2n^2. \quad (5)$$

The total number of edges of special colors at the end of the game is, by (4) and (5),

$$\begin{aligned} \frac{1}{2} \sum_{i \in \mathbb{I}} |Y_{ij}| &\leq \frac{1}{2} \sum_{i \in \mathbb{I}} |Y_i \setminus A_j \cap Y_i \setminus B_j| \\ &\leq \frac{2}{3} \sum_{i \in \mathbb{I}} |Y_i \setminus B_j| \leq \frac{B}{3} n^2 \leq \frac{2}{3} n^2 \leq \frac{B}{3} n^2 \leq o(n^2). \end{aligned}$$

Each of the remaining  $k - 1 \leq \frac{1}{n}$  colors is used on at most  $n^2$  edges. Since the total number of edges of  $G$  is  $E \leq 2 \leq o(1/n^2)$ , we have

$$\frac{2}{3} \leq \frac{B}{3} \leq \frac{1}{2} \leq o(1):$$

By dividing by  $\frac{1}{2}$ , re-arranging, and using the definition of  $\rho$  and Inequality (3), we obtain

$$- \leq o(1) - 1 \leq \frac{2B}{3} - \frac{4}{3} - \frac{2 \leq \rho^2 \leq 8p}{3p} \leq - \quad (6)$$

By taking the derivative with respect to  $B$  of the right-hand side of (6), one can conclude that the minimum over all real  $B$  is attained when  $B \geq 3^2 = 8p$ . Substituting this into (6) and using the known values of the constants, we obtain that

$$- \leq o(1) - \frac{3781}{3750} > 1.0082: \quad (7)$$

This contradiction shows that Bob wins, finishing the proof of [Theorem 3](#).

**Remark.** In order to have a rigorous proof of [Theorem 3](#) checkable by hand, we used rational numbers for all fixed constants. These choices are not optimal (given the stated inequalities) but are good rational approximations of such. In particular, the bound (7) can be slightly improved. Further improvements can be obtained by using more sophisticated strategies for Bob in Stage 1. Unfortunately, the analysis becomes too messy while the new bounds seem still to be very close to 1. Therefore we settled for the current version.

### 3. Upper bounds

Here we are going to prove the upper bound on  $\chi_u^0$  promised in the introduction. Our result will be stronger if we give Bob the freedom to skip moves. Namely, we consider the following new game, studied by Andres [1].

Let  $G$  and  $k$  be given. Bob and Alice alternatively make moves. Bob starts. In his move, Bob can either properly color an uncolored edge or skip (that is, not color any edge at all). Alice, however, always has to properly color an uncolored edge. As in the old version, any moment of the game gives a partial proper coloring of  $E(G)$  and Alice wins if the whole graph is colored at the end. Let the *upper game chromatic index*  $\chi_u^0(G)$  be the smallest  $k$  such that Alice has a winning strategy.

Since Bob is allowed to miss his first turn, we have  $\chi_u^0(G) \leq \chi_u^0(G)$  for any graph  $G$ . Here we prove the following upper bound on  $\chi_u^0(G)$ .

**Theorem 4.** For every  $\epsilon > 0$  there is  $\delta > 0$  such that any graph  $G$  with  $\chi_u^0(G) \geq 1 = 2 \leq \chi_u^0(G)$  satisfies

$$\chi_u^0(G) \leq 2 + \delta \chi_u^0(G). \quad (8)$$

The rest of Section 3 is dedicated to proving [Theorem 4](#).

Let us first specify some notation we are going to use. We abbreviate an unordered pair  $\{x, y\}$  as  $xy$ .

Suppose that we fix players' strategies and observe the game. Let us agree that we immediately stop the game if there is an uncolored edge incident to all possible colors. (Then no player will be able to color it and Bob automatically wins.) A *round* consists of a move of Bob (possibly skipped) followed by a move of Alice. Let  $A_r$  and  $B_r$  denote the sets of edges colored by Alice and Bob respectively after the initial  $r$  rounds. If the game ended earlier, before  $r$  full rounds were completed, let  $A_r$  and  $B_r$  denote the final edge-sets colored by Alice and Bob respectively.

Let  $C_r \subseteq A_r \cup B_r$  consist of the colored edges after Round  $r$ . Let  $c \in E(G) \setminus C_r$  be the (possibly partial) coloring constructed at the end of the game. Let  $C_r^0$  be the set of colored edges before Alice's move in Round  $r$ . For a vertex  $x$  of  $G$ , let  $C_r \cdot x$  be the set of the colors of the  $C_r$ -edges incident to  $x$ :

$$C_r \cdot x = \{c \in c \mid xy \in C_r \text{ for some } y\}.$$

The sets  $A_r \cdot x$ ,  $B_r \cdot x$ , and  $C_r^0 \cdot x$  are defined analogously.

**Proof of Theorem 4.** We will use various positive constants, whose dependences are as follows

$$c_1, c_2, c_3, c_4, c_5, \epsilon > 0;$$

where  $a \gg b$  means that  $b$  is sufficiently small depending on  $a$ . It is enough to prove Theorem 4 for all sufficiently large  $n$ . Indeed, for any order- $n$  graph  $G$  we have  $|G| \leq n$  and  $|G| \geq \frac{1}{2} \sum_v d_v \geq \frac{1}{2} \sum_v 1 = \frac{n}{2}$ ; thus the theorem becomes valid for every  $n \geq n_0$  if  $\epsilon$  is reduced below  $1/n_0$ .

Let  $n$  be sufficiently large. Let the asymptotic notation, like  $O(1/n)$ , refer to the case that  $n \rightarrow \infty$  while  $c_1$ , etc., are fixed. Let  $G$  be an arbitrary graph of order  $n$  and maximum degree  $d \leq \frac{1}{2} C \sqrt{n}$ . Let  $k \geq 2/\epsilon$ .

Here is the strategy of Alice.

She makes two types of moves: *R-moves* (or *random moves*) and *S-moves* (or *set moves*). If Bob skipped his move, then Alice makes an R-move. An R-move consists of selecting an uncolored edge, uniformly at random from all uncolored edges of  $G$ . (The coloring rule, which is the same for both R-moves and S-moves, will be described shortly.) If Bob selected an edge  $xy$  in the previous move, then Alice throws a biased coin. With probability  $1 - c_1$ , she makes an R-move. With probability  $c_1/2$ , she makes an S-move at  $x$ , that is, picks a random uncolored edge at  $x$ . (If all edges at  $x$  have already been colored, then Alice makes an R-move instead.) With probability  $c_1/2$ , she makes an S-move at  $y$ , that is, picks a random uncolored edge at  $y$  (or makes an R-move if all edges at  $y$  have already been colored).

The rules for selecting Alice's edge  $uv$  are different for these two types of moves but the coloring rule is the same: the color  $c(uv)$  is chosen uniformly at random from all admissible colors (that is, from the set  $\{k \in \mathbb{N} : C_r(u) \cap C_r(v) = \emptyset\}$ , where  $r$  is the number of the current round). There is always at least one available color for the edge  $uv$ , for otherwise we would have already stopped the game and declared Bob to be the winner. Let  $R_r$  and  $S_r$  denote the sets of Alice's R-moves and S-moves respectively after  $r$  rounds. Thus, for every  $r$ ,

$$A_r \subseteq R_r \cup S_r.$$

Note that if Bob has a winning strategy, then (since this is a complete information game) Bob has a *deterministic* winning strategy. Hence, in order to prove the theorem, it is enough to show that, for any fixed (deterministic) strategy of Bob, this random strategy of Alice has non-negative probability of winning.

So let us fix some strategy of Bob and let Alice play as above. Let

$$D \subseteq V(G) \text{ with } |D| \geq \frac{1}{2} \epsilon n;$$

where  $d_x$  denotes the degree of  $x$ . Clearly, if we take an edge not entirely inside  $D$ , then at most  $d \leq \frac{1}{2} \epsilon \sqrt{n} < k$  colors are forbidden, so this edge can always be colored. For  $x \in D$  let

$$r_x = \min_{j \in A_r} |j \cap C_r(x)| \quad (9)$$

Here is an informal description why Alice wins whp. We will show that whp two vertices of  $D$  will share at least  $\epsilon n$  common colors before the game ends. Indeed, if this occurs, then Alice wins because every edge gets colored: the number of forbidden colors is at most  $d \leq \frac{1}{2} \epsilon \sqrt{n} < k$ . In fact, we show that this event occurs early in the game, after at most  $r$  rounds, where  $r$  can be set to be, for example,  $4c_2 n^2$ . Since the set  $C_r$ , containing at most  $2r$  edges, is small, when Alice colors a random edge incident to a vertex  $x \in D$  in some Round  $i$ , the color of this edge is spread on almost all the set  $\{k \in \mathbb{N} : C_i(x) \cap C_i(y) = \emptyset\}$ . Hence, it is enough to show that whp each  $|j \cap C_r(x)|$ ,  $x \in D$  is fairly large. To this end observe that if  $|j \cap C_r(x)|$  is small, then  $|j \cap C_r(x)|$  is large because then any R-move had a chance at least  $\frac{1}{2} \epsilon \sqrt{n}$  to pick an edge at  $x$ ; otherwise  $|j \cap C_r(x)|$  is large, being whp at least  $c_1/4 \sqrt{n}$ . This is why we need an occasional S-move: to prevent Bob from claiming almost all edges at some vertex  $x \in D$ .

Let us present a rigorous proof. We define a family of 'bad' events and establish the following two properties. *Property I*: the expected value of the sum of the indicator functions of bad events is  $o(1/n)$ . *Property II*: if none of the bad events occurs then Alice necessarily wins. Then the theorem clearly follows.

All our bad events will be split into a few families. For each family we immediately analyze Property I, leaving the proof of Property II until the very end. For the notational convenience, we identify each event with its indicator function.

### 3.1. The first family

Each event  $B_{z,m,r}^1$  of this family is indexed by a triple  $(z, m, r)$ , where  $z \in D$  and  $0 \leq r < m \leq \epsilon n$ . Informally speaking, if none of the events  $B_{z,m,r}^1$  occurs, then the number of A-edges that hit a vertex  $z \in D$  at any interval of the game is not much smaller than the expected value, provided that the final degree of  $z$  is not too big.

Here a formal definition of  $B_{z,m,r}^1$ . If the game ends before Round  $m$ , we set  $B_{z,m,r}^1 \equiv \emptyset$ . So suppose that the game continues for at least  $m$  rounds. We set  $B_{z,m,r}^1 \equiv \emptyset$  if  $C_m(z) \leq 4c_2 d$  or if

$$|A_m(z) \cap A_r(z)| \leq \frac{m - r/2 - 4c_1}{n} \quad (10)$$

Otherwise, we set  $B_{z,m,r}^1 \equiv 1$ .

Let us show that

$$\sum_{z \in D} \sum_{0 \leq r < m} \mathbb{E}[B_{z,m,r}^1] \leq o(1). \quad (11)$$

We fix  $z, m, r$  and estimate the probability of  $B_{z,m,r}^1$ . Consider a moment when Alice is about to make a move in Round  $i$ , with  $r < i \leq m$ . Assume that  $z$  is currently incident to fewer than  $4c_2d$  colored edges for otherwise we necessarily have  $C_m(z) \leq 4c_2d$  and  $B_{z,m,r}^1 = 0$ . The probability of Alice's making an R-move is at least  $1 - c_1$ , whether or not Bob skipped his previous move. We have at least  $d - z/4c_2d \geq 1 - 2'' - 4c_2/d$  uncolored edges at  $z$  and at most  $dn/2$  edges in total. Hence, the probability of increasing the number of A-edges at  $z$  is at least

$$(1 - c_1) \cdot \frac{1 - 2'' - 4c_2/d}{dn/2} > \frac{2 - 4c_1}{n} \geq p$$

at each round. Hence, if we assume that  $C_i(z) \leq 4c_2d$  for each  $i \in [r, m]$ , then the left-hand side of (10) can be bounded from below by coupling with the  $(m - r + 1)p$ -Binomial variable. The Chernoff bound implies that the probability that (10) fails is exponentially small in  $n$ . Since the number of choices of the triple  $(z, m, r)$  is  $O(n^5)$ , the inequality (11) follows.

### 3.2. The second family

Here we define the event  $B_{fx,yg}^2$ , where  $x, y \in D, x \neq y$ . Using our convention, we will abbreviate it as  $B_{xy}^2$ . Roughly speaking, we observe the game for the initial  $r$  rounds for some  $r$ . Suppose that  $x$  and  $y$  have not acquired at least  $d$  common colors yet. Furthermore, suppose that one of them (say  $x$ ) is incident to at least  $c_3d$  uncolored edges whose other endpoint does not see at least  $c_3d$  of the colors appearing at  $y$ . Then it is very unlikely that, in next  $c_4dn$  rounds,  $x$  gets almost none of the colors that were present at  $y$  at Round  $r$ .

Here is a formal definition of  $B_{xy}^2$ . If  $B_{u,m,r}^1 = 1$  for some  $0 \leq r < m \leq e \cdot G$  and  $u \in fx, yg$ , then we immediately set  $B_{xy}^2 = 0$ . So let us suppose otherwise. Let  $r = \max\{r : x/r \leq y/r\}$ , where  $r \cdot z$  is the function defined by (9). If  $r$  is undefined (i.e. the game stops before each of  $x$  and  $y$  gets  $C_r$ -degree at least  $c_2d$ ), then we set  $B_{xy}^2 = 0$ . (It will be the case that some other bad events will be 'responsible' for this.) If

$$|C_r(x) \setminus C_r(y)| \leq d; \quad (12)$$

then we set  $B_{xy}^2 = 0$ , so let us suppose that (12) does not hold. We define

$$Z_{x,y} = \{z : xz \in E(G) \cap C_r, |C_r(y) \cap C_r(z)| \leq c_3d\}; \quad (13)$$

$$Z_{y,x} = \{z : yz \in E(G) \cap C_r, |C_r(x) \cap C_r(z)| \leq c_3d\}; \quad (14)$$

If  $\max\{|Z_{x,y}|, |Z_{y,x}|\} < c_3d$ , then we set  $B_{xy}^2 = 0$ . So suppose otherwise and let  $fu, vg \in fx, yg$  satisfy  $|Z_{u,v}| \geq c_3d$ . (If both assignments  $u \in x$  and  $u \in y$  work, we can agree that e.g.  $u$  is the smaller of  $x$  and  $y$  with respect to some fixed linear order on  $V(G)$ .)

Let us observe the game until Round  $m$ , where

$$m \leq r + c_4dn; \quad (15)$$

If the game ends before Round  $m$ , then we do the following: Set  $B_{xy}^2 = 1$  if the edge  $xy$  is uncolorable at the end (possibly one of a few uncolorable edges) and set  $B_{xy}^2 = 0$  otherwise (that is, if  $xy$  is colored or can be properly colored in the final position).

So, suppose that the game lasts at least until Round  $m$ . If

$$|C_m(x) \setminus C_m(y)| \leq d; \quad (16)$$

we set  $B_{xy}^2 = 0$ ; otherwise, we set  $B_{xy}^2 = 1$ . This finishes the description of the event  $B_{xy}^2$ .

Let us prove that

$$\sum_{x,y \in D} \mathbb{E}[B_{xy}^2] \leq o(1). \quad (17)$$

Let us fix  $xy \in D^2$  and estimate the probability of  $B_{xy}^2$ . We analyze the game, starting from Round  $r$  and assuming that the previous development of the game does not rule out  $B_{xy}^2$  yet. In particular, we have defined  $u, v$  with  $fu, vg \in fx, yg$  and  $|Z_{u,v}| \geq c_3d$ .

We will observe the game in Rounds  $r + 1$  to  $m^0$ , where

$$m^0 \leq \min\{m, r^0\}; \quad (18)$$

where  $r^0$  is the total number of the rounds until the game stops.

Since  $m^0 \leq r \leq m \leq r + c_4 d n$ , we have, for any  $i \in \text{Tr} \subset [1; m^0]$ , that

$$|Z_{i,u,v}^0| \leq \frac{m}{c_3 d} \leq \frac{r}{c_3 d} + \frac{c_4 d n}{c_3 d} = \frac{c_3 d}{4} + \frac{c_4 d n}{c_3 d},$$

where we define  $Z_i^0 \subseteq \{z \in Z_{i,u,v} \mid |J_{C_r, v} \cap C_i^0(z)| \geq c_3 d/2\}$ . Indeed, every vertex in  $Z_{i,u,v} \cap Z_i^0$  must gain at least  $c_3 d/2$  colored edges in Rounds  $r \in [1; i]$ , resulting in the first inequality. The final inequality follows from  $c_4 \leq c_3$ . Thus

$$|Z_i^0| \leq |Z_{i,u,v}| + \frac{c_3 d}{4} \leq \frac{3c_3 d}{4}. \quad (19)$$

Let  $I$  consist of those  $i \in \text{Tr} \subset [1; m^0]$  such that in Round  $i$  Alice colors an edge between  $u$  and  $Z_i^0$ . Let  $E_0$  be the event that  $B_{xy}^2$  occurs and  $|I| \leq c_5 d$ . Let us show that the probability of each of  $E_0$  and  $B_{xy}^2 \cap E_0$  (given the previous history up to Round  $r$ ) is exponentially small in  $n$ —this will prove (17) because there are  $O(n^2)$  choices of  $xy$  in total.

Let us analyze  $E_0$  first. We will make use of the following coupling. Let  $p_0 \in [c_3/2, 1/2]$ . Let  $X \in \{0, 1\}^n$  be an infinite 0=1-sequence where each entry is 1 with probability  $p_0$ , independently of the other entries. Initially we set  $k \in [1]$ .

Let us observe the rounds one by one as the game progresses. Let  $i \in [r, \infty)$  be number of the current round.

Suppose first that, in this Round  $i$ , Alice has selected and is about to color an edge  $uz$  with some  $z \in Z_i^0$ . Let  $W \subseteq \text{Tr} \cap C_i^0(u) \setminus C_i^0(z)$  consist of all available colors for  $uz$ . Let  $W^0 \subseteq W \setminus C_i^0(v)$  consist of those available colors that are also present at  $v$ . Alice increases the number of common colors at  $u$  and  $v$  with probability  $p \in [p_0, 1]$   $|W^0|/|W|$ .

By the definition of  $Z_i^0$  there are at least  $c_3 d/2$  colors of  $C_r \setminus C_i^0(v)$  that are absent in  $C_i^0(z)$ . Assume that at most  $d$  of these colors are present in  $C_i^0(u)$  for otherwise (16) holds,  $B_{xy}^2 \subseteq 0$ , and  $E_0 \subseteq 0$ . Hence,  $|W^0| \geq c_3 d/2 - d$  and  $p \in [p_0, 1]$   $|W^0|/|W| \geq c_3 d/2 - d$ .

Our coupling requires that the edge  $uz$  is colored with a color from  $W^0$  whenever the  $k$ -th element  $X_k$  of  $X$  is 1. This can be achieved, for example, as follows. If  $X_k \in [1]$ , Alice picks a random color from  $W^0$ . If  $X_k \in 0$ , then Alice picks, with probability  $p \in [p_0, 1]$   $|W^0|/|W|$  a random color from  $W^0$  and with probability  $1 - p \in [0, 1 - p_0]$  a random color from  $W \setminus W^0$ . This gives the uniform distribution on the set  $W$  of all available colors. Indeed, any two colors both from  $W^0$  or from  $W \setminus W^0$  are equally likely to be picked while the probability of selecting a color from  $W^0$  is exactly

$$p_0 \in [1 - p_0, p_0] \frac{p}{1 - p_0} \in [p, 1] \frac{|W^0|}{|W|}.$$

Now, we increase  $k$  by 1 so that the new (unexposed) value of  $X_k$  is independent of the previous history. Continue the game.

If, in Round  $i$ , Alice does not color an edge  $uz$  with  $z \in Z_i^0$ , then we do not do anything (and do not increase the counter  $k$ ).

It follows that if  $E_0$  occurs, then the first  $c_5 d$  elements of  $X$  contain at most  $d$  ones. By the Chernoff bound, this has exponentially small in  $n$  probability, giving the desired result. (Note that we do not have to take the union bound over all choices of  $I$  since we were feeding in the bits of  $X$  only when there was demand.)

Next, we analyze  $E_0 \cap B_{xy}^2$ , the event that  $B_{xy}^2$  occurs and  $|I| \leq c_5 d$ . We split it further into two complementary sub-events  $E_0 \cap E_1$   $E_2$  depending respectively on whether or not Bob colors at least  $c_3 d/4$  edges incident to  $u$  in Rounds  $r \in [1; m^0]$ , where  $m^0$  is defined by (18).

In order to analyze  $E_1$ , consider the first  $I \subseteq [c_3 d/4, c_4 d n]$  moves of Bob after Round  $r$  incident to  $u$ . Let us consider Alice's move in any such Round  $i$ , when Bob has just colored an edge at  $u$ . Of all  $G$ -edges between  $u$  and  $Z_i^0$ , at most  $c_5 d$  edges are colored by Alice (we can assume this for otherwise  $|I| \leq c_5 d$ ) and, trivially, at most  $I \leq c_3 d/4$  edges are colored by Bob. Hence, by (19), the probability that Alice picks an edge between  $u$  and  $Z_i^0$  in Round  $i$  is at least

$$\frac{c_1}{2} \geq \frac{3c_3 d/4 - c_5 d}{c_3 d/4} \geq \frac{c_1 c_3}{5}.$$

Similarly to above, we can couple this with an infinite 0=1-sequence  $X \in \{0, 1\}^n$ , whose each entry is 1 with probability  $c_1 c_3/5$ , where we read the next bit of  $X$  after those moves of Bob that touch  $u$ . It follows that if  $E_1 \subseteq 1$ , then there are less than  $c_5 d$  ones among the first  $I$  members of  $X$ . The Chernoff bound shows that the probability of this (and thus of  $E_1$ ) is exponentially small in  $n$ .

Let us analyze  $E_2$ .

Suppose first that  $m^0 < m$  (and that  $B_{xy}^2 \subseteq 1$ ). Recall that  $m$  and  $m^0$  are defined by (15) and (18). Then  $xy$  is uncolorable in the final position, so at least  $1/d$  edges incident to  $u$  get colored. In particular, at least  $|Z_{m^0}^0|/d$  edges between  $u$  and  $Z_{m^0}^0$  are colored. Alice colors at most  $c_5 d$  of these edges since  $E_0$  does not occur, while Bob colors at most  $c_3 d/4$  of these edges since  $E_1$  does not occur. By (19), we conclude that  $c_5 d \leq c_3 d/4 - 3c_3 d/4 = -d$ , a contradiction.

Hence, assume that  $m^0 \geq m$ , that is, the game lasts for at least  $m$  rounds. We observe  $c_4 d n$  rounds after Round  $r$  and, in each Round  $i$  with  $r < i \leq m$ , the probability of Alice's hitting an edge between  $u$  and  $Z_i^0$  is at least

$$1 - c_1/5 \geq \frac{3c_3 d/4 - c_5 d}{c_3 d/4} \geq \frac{c_1 c_3}{5}$$

because neither  $E_0$  nor  $E_1$  occurs. Again, the probability of fewer than  $c_5 d$  successes (which is needed to avoid  $|I| \leq c_5 d$ ) is exponentially small. This completely proves (17).

### 3.3. The third family

Its events  $B_a^3$  are indexed by a color  $a \in [k]$ . The event  $B_a^3$  occurs if and only if the graph  $R_m$  contains at least  $37c_2n$  edges of color  $a$ , where  $m \leq 4c_2n^2c$ . (Recall that  $R_m$  consists of all R-moves of Alice made during the first  $m$  rounds.)

Initially, let us set  $H \subseteq E(G)$  and let  $X$  be an infinite 0=1-sequence where each entry is 1 with probability  $3/n$  independently of the other entries. Let the game last for  $r^0$  rounds.

We observe Alice's moves until Round  $m^0$ , where  $m^0 \leq \min\{m; r^0\}$ . Let us consider a moment when Alice has just selected an R-edge  $xy$  and is about to color it in some Round  $i$ . If there are less than  $n/3$  available colors for the edge  $xy$  at the current moment, then we just add the pair  $xy$  to  $H$  and proceed with the game. Suppose that there are more than  $n/3$  available colors. The probability of selecting the color  $a$  for  $c \cdot xy$  is at most  $3/n$ . We read the next unexposed bit of  $X$ . Our coupling requires that if it is 0, then Alice does not select color  $a$  for  $c \cdot xy$ .

Consider the partial coloring right after Round  $m^0$ . Let  $Y$  consist of vertices of  $G$  of  $C_{m^0}$ -degree at least  $n/3$ . We have  $|Y| \cdot n/3 \leq 2m^0$ ; thus  $|Y| \leq 24c_2n$ . Every edge  $xy$  of  $H$  has to intersect  $Y$  for otherwise the number of available colors at  $xy$ , even at Round  $m^0$ , is at least  $n/3 - |Y| \geq n/3 - 24c_2n > n/3$ , a contradiction. Since each color class is a matching, the number of color- $a$  edges inside  $H$  is at most  $|Y|$ . It follows that if  $B_a^3 \subseteq 1$ , then the first  $m$  entries of  $X$  contain at least  $37c_2n - 24c_2n \geq 13c_2n$  ones. By the Chernoff bound this has probability exponentially small in  $n$ . Hence  $\sum_{a \in [k]} E(B_a^3) \leq o(1)$ .

### 3.4. The fourth family

Let  $0 \leq m \leq e(G)$ .

The event  $B_m^4$  occurs if and only if  $|S_m| \geq c_1m \geq c_3n^2$ . (Recall that  $S_m$  consists of Alice's S-moves made in the first  $m$  rounds.) Since the probability of increasing the current  $|S_r|$  in any round is at most  $c_1$ , the Chernoff bound easily implies that  $\sum_{m \geq 0} E(B_m^4) \leq o(1)$ .

### 3.5. Putting all together

Let us show that if none of the above bad events occurs, then Alice surely wins. Let us assume on the contrary that the game ends when an edge  $xy \in E(G)$  cannot be properly colored. If there are a few choices for  $xy$ , pick one arbitrarily. (Recall that we stop the game as soon as an uncolorable edge appears.)

This means that  $x, y \in D$  and each of  $x$  and  $y$  is incident to at least  $n/4$  colored edges. Thus  $r \cdot x$  and  $r \cdot y$  are well-defined. Let  $r \leq \max\{r \cdot x; r \cdot y\}$ . Assume  $r \cdot x \leq r \cdot y$ . Thus  $r \leq r \cdot x$  and  $0 \leq |C_r \cdot x| \leq c_2d < 2$ . Since  $xy$  cannot be properly colored at the end of the game, Inequality (12) is false, that is,

$$|C_r \cdot x \setminus C_r \cdot y| < d. \quad (20)$$

Since  $B_{x,r,0}^1 \subseteq 0$  and  $|C_r \cdot x| \leq c_2d \leq 2 < 4c_2d$ , we have by (10) applied to the first  $r$  rounds that

$$c_2d \leq 2 + |A_r \cdot x| \leq \frac{r \cdot 2 + 4c_1}{n} \leq c_5n.$$

It follows from  $c_5 \leq c_2 \leq c_1$  that, for example,

$$r \leq \frac{.1 \leq 3c_1/c_2nd}{2}; \quad (21)$$

**Claim 1.**  $\max\{|Z_{x,y}|; |Z_{y,x}|\} \leq c_3d$ .

**Proof of Claim.** Suppose that the claim is not true. Let  $l \leq |C_r \cdot y|$ . Then, by the definition of  $Z_{x,y}$ , there are at least

$$d_G \cdot x - |C_r \cdot x| - |Z_{x,y}| \geq .1 - 2/d - c_2d \geq 2/c_3d \geq .1 - 2c_2/d;$$

vertices of  $G$  such that, after the  $r$ -th round, each sees at least  $1/c_3d$  colors from  $C_r \cdot y$ . Likewise, there are at least

$$d_G \cdot y - l - |Z_{y,x}| \geq .1 - 2/d - l \leq c_3d \leq .1 - 2c_3/d - l$$

vertices of  $G$ , each seeing at least  $|C_r \cdot x| - c_3d - c_2d \geq c_3d$  colors from  $C_r \cdot x$ . This requires at least

$$\frac{1}{2} \cdot .1 - 2c_2/d - l \leq c_3d \leq .1 - 2c_3/d - l/c_2d \leq c_3d \leq \frac{dl}{2} \leq c_2 \frac{3dl}{2} \leq c_2^2n^2 \quad (22)$$

edges of  $C_r$ , each colored by a color in  $C_r \cdot x \setminus [C_r \cdot y]$ . By (20) and since each color class contains at most  $n/2$  edges, we double-count at most

$$|C_r \cdot x \setminus C_r \cdot y| \leq .n/2 - 2/dn \leq c_2^2n^2 \quad (23)$$

edges. Moreover, by (21) we have  $r \leq 4c_2n^2$  and since no bad event  $B_a^3$  occurs,  $A_r$  can contribute at most

$$|C_r \cdot x \setminus [C_r \cdot y]| \leq .37c_2n - .c_2d \leq 2 \leq l - .37c_2n$$



to (22). Since  $B_r^4$  does not occur,  $S_r$  contributes at most  $c_1 r \subset c_3 n^2$  to (22). Finally,  $|B_r| \leq r$ . From (22) and (23), we obtain that

$$\frac{dl}{2} \subset c_2 \frac{3dl \subset d^2}{2} \subset 2c_2^2 n^2 \subset c_2 d \subset 2 \subset l/ \subset 37c_2 n / \subset c_1 r \subset c_3 n^2 \subset r:$$

Using (21), we obtain (after routine simplifications) that

$$l \subset \frac{d}{2} \subset \frac{3c_2 d}{2} \subset 37c_2 n \subset c_2 \frac{d^2}{2} \subset c_2 \frac{nd}{2} \subset 3c_1 c_2 n^2: \quad (24)$$

This is a contradiction to  $l \subset |C_r| \cdot y \subset c_2 d, d \subset \frac{1}{2} \subset \frac{1}{n}$ , and  $c_1 \subset c_2$ . The claim is proved.

So we can define  $f_u; v \subset f_x; y$  as it is done after (14). Let  $m \subset r \subset bc_4 d n c$ . The game cannot end before Round  $m$  for then  $B_{xy}^2 \subset 1$  (as  $xy$  is responsible for the end of the game). Again, since  $B_{xy}^2 \subset 0$ , we conclude that (16) holds. This means that, after Round  $m$ ,  $x$  and  $y$  share at least  $d$  colors, so there will always be a choice of color for  $xy$ . This contradicts our assumption and proves that Alice wins. This completes the proof of Theorem 4.

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