Problem solving seminar
Test 2 - solutions

2. Prove that for integers $n \geq k \geq 1$ the number

$$\frac{\gcd(n,k)}{n}\binom{n}{k}$$

is an integer.

Solution. There exist integers a, b such that $\gcd(n,k) = an + bk$. Thus, it suffices to show that $\frac{k}{n}\binom{n}{k}$ is an integer. To this end, observe that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$. $\square$

3. Consider the following game played by two players, Alice and Bob. Alice starts and players alternate turns until one of them wins or loses. If a player cannot make a move then they *lose*.

Start with two piles containing m and n coins respectively. On a player's turn they *must* remove all the coins from one pile and then they *must* divide the other pile into two new *non-empty* piles. Who has a winning strategy when $m = 2014$ and $n = 2014$?

Solution. Answer: Alice has a winning strategy if and only if at least one of the numbers m, n is even.

If, say m is even, $m = 2k$ for some $k \geq 1$, then Alice discards the n pile and divides the m pile into two piles: $2k-1$ and 1. If $2k-1 = 1$, Bob can't make a move, so he loses. If not, he must divide the $2k-1$ pile, hence producing an even pile for Alice, etc. Eventually, he loses.

If neither m nor n is even and at least one number is greater than 1, then Alice produces an even pile for Bob, so he will be in the winning position described above. $\square$