

Exercise 4

1 Stable and recursively feasible MPC

Recall the following definitions

$X \oplus Y := \{x + y | x \in X, y \in Y\}$ Minkowski sum of X and Y ,

$\alpha X := \{\alpha x | x \in X\}$ scaling of a set,

if X is convex: $x_1, x_2 \in X \rightarrow \alpha x_1 + (1 - \alpha)x_2 \in X, \forall \alpha \in [0, 1]$ convex set.

1. Let X_1 and X_2 be convex invariant sets for the system $x_{k+1} = Ax_k$. Show that $\alpha X_1 \oplus (1 - \alpha)X_2$ is also an invariant set for any $\alpha \in [0, 1]$.
2. Let $X_1 \subseteq \mathbb{X}$ and $X_2 \subseteq \mathbb{X}$, where X_1, X_2 and \mathbb{X} are convex sets. Show that $\alpha X_1 \oplus (1 - \alpha)X_2 \subseteq \mathbb{X}$ for any $\alpha \in [0, 1]$.
3. Let $V_i(x) := x^\top P_i x$ be a Lyapunov function for the system $x_{k+1} = Ax_k$ for $i = 1, 2$, with a rate of decrease of $x^\top \Gamma x$, i.e.: $V_i(x_{k+1}) - V_i(x_k) \leq -x^\top \Gamma x$. Show that $V(x) = \alpha V_1(x) + (1 - \alpha)V_2(x)$ is also a Lyapunov function with a rate of decrease of $x^\top \Gamma x$ for any $\alpha \in [0, 1]$.
4. Let K be a stabilizing controller for the system $x_{k+1} = Ax_k + Bu_k$, and $X_i \subset \mathbb{X}$ be a convex invariant set for the system $x_{k+1} = (A + BK)x_k$, with $KX_i \subset \mathbb{U}$ for each $i = 1, 2$. $V_i(x) = x^\top P_i x$ are Lyapunov functions for the system $x_{k+1} = (A + BK)x_k$ with a rate of decrease of $Q + K^\top R K$, for some $Q = Q^\top \succ 0$ and $R = R^\top \succ 0$.

$$\begin{aligned}
 J^*(x(t)) &= \min_u \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k + \alpha V_1(x_N) + (1 - \alpha)V_2(x_N), \\
 \text{s.t. } & x_{k+1} = Ax_k + Bu_k, \forall k = 0, \dots, N-1, \\
 & x_k \in \mathbb{X}, \forall k = 1, \dots, N, \\
 & u_k \in \mathbb{U}, \forall k = 0, \dots, N-1, \\
 & x_N \in \alpha X_1 \oplus (1 - \alpha)X_2, \\
 & x_0 = x(t).
 \end{aligned}$$

Prove that this MPC controller is stabilizing and recursively feasible for any $\alpha \in [0, 1]$ by listing sufficient conditions for stability and proving them. You can use the result of the previous three questions.

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Recall the following definitions

$X \oplus Y := \{x + y | x \in X, y \in Y\}$ Minkowski sum of X and Y ,

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if X is convex: $x_1, x_2 \in X \rightarrow \alpha x_1 + (1 - \alpha)x_2 \in X, \forall \alpha \in [0, 1]$ convex set.

1. Let X_1 and X_2 be convex invariant sets for the system $x_{k+1} = Ax_k$. Show that $\alpha X_1 \oplus (1 - \alpha)X_2$ is also an invariant set for any $\alpha \in [0, 1]$.

We want to show that if $x_k \in \alpha X_1 \oplus (1 - \alpha)X_2$, $x_{k+1} \in \alpha X_1 \oplus (1 - \alpha)X_2$. Suppose that $x_k \in \alpha X_1 \oplus (1 - \alpha)X_2$, which means that $\exists x_1 \in X_1, x_2 \in X_2$ such that $x_k = \alpha x_1 + (1 - \alpha)x_2$. $x_{k+1} = Ax_k = \alpha Ax_1 + (1 - \alpha)Ax_2$. Since X_1, X_2 are invariant sets. $Ax_1 \in X_1, Ax_2 \in X_2$. That means $x_{k+1} \in \alpha X_1 \oplus (1 - \alpha)X_2$.

2. Let $X_1 \subseteq \mathbb{X}$ and $X_2 \subseteq \mathbb{X}$, where X_1, X_2 and \mathbb{X} are convex sets. Show that $\alpha X_1 \oplus (1 - \alpha)X_2 \subseteq \mathbb{X}$ for any $\alpha \in [0, 1]$.

If $x \in \alpha X_1 \oplus (1 - \alpha)X_2$, $\exists x_1 \in X_1, x_2 \in X_2$ such that $x = \alpha x_1 + (1 - \alpha)x_2$. Since both $x_1, x_2 \in \mathbb{X}$ and \mathbb{X} is a convex set. By convexity, we conclude $x \in \mathbb{X}$.

3. Let $V_i(x) := x^\top P_i x$ be a Lyapunov function for the system $x_{k+1} = Ax_k$ for $i = 1, 2$, with a rate of decrease of $x^\top \Gamma x$, i.e.: $V_i(x_{k+1}) - V_i(x_k) \leq -x^\top \Gamma x$. Show that $V(x) = \alpha V_1(x) + (1 - \alpha)V_2(x)$ is also a Lyapunov function with a rate of decrease of $x^\top \Gamma x$ for any $\alpha \in [0, 1]$.

To show $V(x)$ is a Lyapunov function with the same decreasing rate, we need to show that $V(x_{k+1}) - V(x_k) \leq -x^\top \Gamma x$. $V(x_{k+1}) = \alpha V_1(x_{k+1}) + (1 - \alpha)V_2(x_{k+1}) = \alpha x_k^\top A^\top P_1 A x_k + (1 - \alpha)x_k^\top A^\top P_2 A x_k$. $V(x_{k+1}) - V(x_k) = \alpha x_k^\top (A^\top P_1 A - I) x_k + (1 - \alpha)x_k^\top (A^\top P_2 A - I) x_k \leq -\alpha x_k^\top \Gamma x_k - (1 - \alpha)x_k^\top \Gamma x_k = -x_k^\top \Gamma x_k$.

4. Let K be a stabilizing controller for the system $x_{k+1} = Ax_k + Bu_k$, and $X_i \subseteq \mathbb{X}$ be a convex invariant set for the system $x_{k+1} = (A + BK)x_k$, with $KX_i \subseteq \mathbb{U}$ for each $i = 1, 2$. $V_i(x) = x^\top P_i x$ are Lyapunov functions for the system $x_{k+1} = (A + BK)x_k$ with a rate of decrease of $Q + K^\top R K$, for some $Q = Q^\top \succ 0$ and $R = R^\top \succ 0$.

$$\begin{aligned}
 J^*(x(t)) = \min_u \quad & \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k + \alpha V_1(x_N) + (1 - \alpha)V_2(x_N), \\
 \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k, \forall k = 0, \dots, N-1, \\
 & x_k \in \mathbb{X}, \forall k = 1, \dots, N, \\
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 & x_N \in \alpha X_1 \oplus (1 - \alpha)X_2, \\
 & x_0 = x(t).
 \end{aligned}$$

Prove that this MPC controller is stabilizing and recursively feasible for any $\alpha \in [0, 1]$ by listing sufficient conditions for stability and proving them. You can use the result of the previous three questions.

Sufficient conditions:

- stage cost is positive definite,
- the terminal set is invariant under the local control law (proved by question 1.1),
- all state and input constraints are satisfied in the terminal set (proved by question 1.2),
- the terminal cost is a Lyapunov function in the terminal set (proved by question 1.3).