Exercise 3

1 Pre-set

1. Consider the system

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k.$$

If $x \in \mathcal{X}$, $\mathcal{X} = \{x| \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}$, sketch the set $\mathcal{X}$ and pre-set of $\mathcal{X}$. Compute the successor state of a few points in $\text{pre}(\mathcal{X})$ to convince yourself it is correct.

2. Consider the system $x_{k+1} = 0.5x_k + u_k$. Calculate and plot the pre-set if $x \in [-2, 2]$ and $u \in [-1, 1]$.

2 Invariant set

1. Consider the system $x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k$ and the constraint$\{x| \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}$. Calculate and plot the maximum invariant set.

2. Consider the nonlinear system $x_{k+1} = x_k^2 + u_k$, where $x \in [-2, 2]$, $u \in [-1, 1]$. Calculate the maximum control invariant set. (Hint: Consider a sequence, $a_{k+1} = \sqrt{a_k + 1}$, if $a_0 \geq \frac{1+\sqrt{5}}{2} \approx 1.618$, then $a_{k+1} \leq a_k$, and $\lim_{k\to\infty} a_k = \frac{1+\sqrt{5}}{2}$)

3. Given a system $x_{k+1} = f(x_k)$, which has a Lyapunov function V. (Recall a Lyapunov function has the property $V(f(x_k)) \leq V(x_k)$). Prove that the set $\{x|V(x) \leq \alpha\}$ is invariant for all $\alpha \geq 0$.

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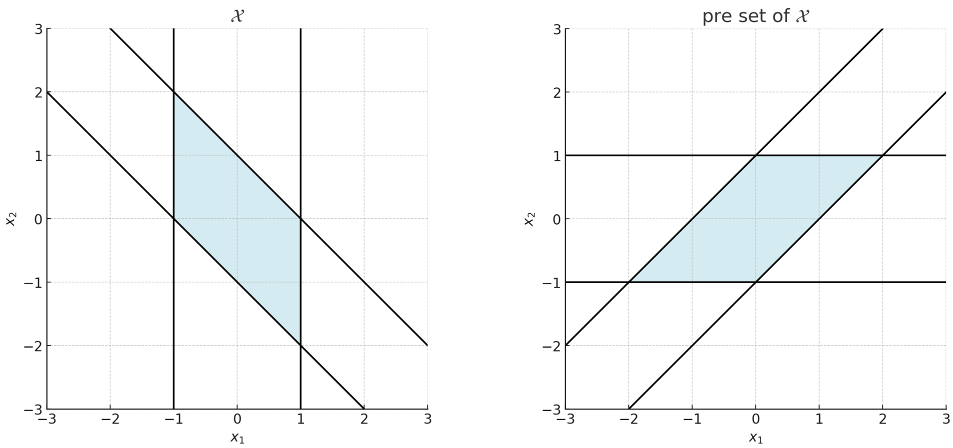
1. Consider the system

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k.$$

If $x \in \mathcal{X}$, $\mathcal{X} = \{x | \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}$, sketch the set $\mathcal{X}$ and pre-set of $\mathcal{X}$. Compute the successor state of a few points in $\text{pre}(\mathcal{X})$ to convince yourself it is correct.

See the Fig. 1



Original set. Pre set.

Figure 1: Solution of problem 1,1.

2. Consider the system $x_{k+1} = 0.5x_k + u_k$. Calculate and plot the pre-set if $x \in [-2, 2]$ and $u \in [-1, 1]$.

For the pre set, we have

$$\begin{bmatrix} 0.5 & 1 \\ -0.5 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

Project the feasible set onto the x-axis, we conclude that the pre set for x is $[-6, 6]$.

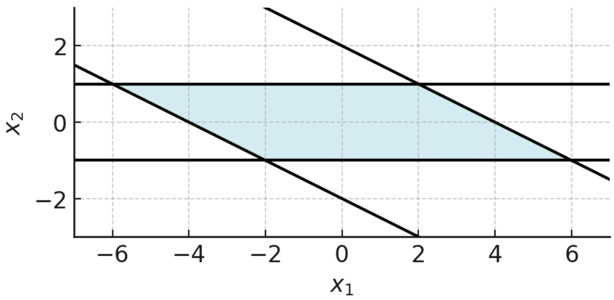


Figure 2: Solution of problem 1,2.

2 Invariant set

1. Consider the system $x_{k+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_k$ and the constraint$\{x| \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}$. Calculate and plot the maximum invariant set.

   See Fig. 3

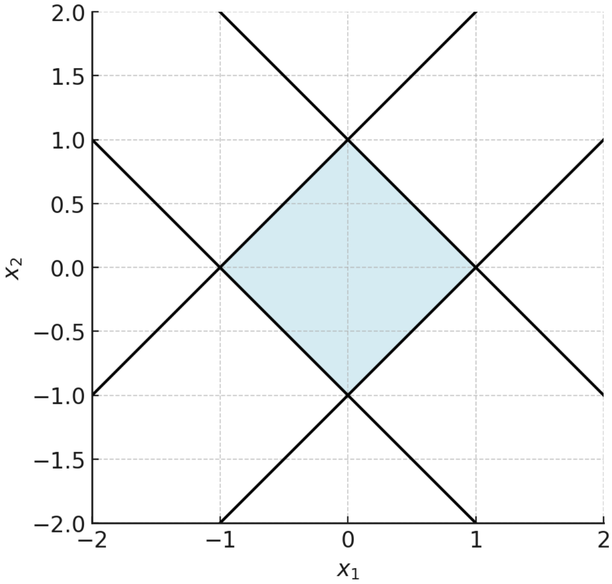


Figure 3: Solution of problem 2,1.

2. Consider the nonlinear system $x_{k+1} = x_k^2 + u_k$, where $x \in [-2, 2]$, $u \in [-1, 1]$. Calculate the maximum control invariant set. (Hint: Consider a sequence, $a_{k+1} = \sqrt{a_k + 1}$, if $a_0 \geq \frac{1+\sqrt{5}}{2} \approx 1.618$, then $a_{k+1} \leq a_k$, and $\lim_{k\to\infty} a_k = \frac{1+\sqrt{5}}{2}$)

   We perform the iteration for calculating the maximum control invariant set. Let Ω_i be the set sequence. $\Omega_0 = [-2, 2]$. $\text{pre}(\Omega_0)$ is the set that $\exists u \in [-1, 1]$ such that $x^2 + u \in [-2, 2]$. Hence $x^2 \in [-3, 3]$, $x \in [-\sqrt{3}, \sqrt{3}]$. $\text{pre}(\Omega_0) = [-\sqrt{3}, \sqrt{3}]$. $\Omega_1 = \Omega_0 \cap \text{pre}(\Omega_0) = [-\sqrt{3}, \sqrt{3}]$. $\text{pre}(\Omega_1) = [-\sqrt{\sqrt{3}+1}, \sqrt{\sqrt{3}+1}]$. $\Omega_2 = \Omega_1 \cap \text{pre}(\Omega_1) = [-\sqrt{\sqrt{3}+1}, \sqrt{\sqrt{3}+1}]$. Hence $\Omega_{k+1} = [-a_k, a_k]$, where $a_{k+1} = \sqrt{a_k + 1}$, $a_0 = \sqrt{3} \geq 1.618$. $\Omega_\infty = [-\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}]$.

3. Given a system $x_{k+1} = f(x_k)$, which has a Lyapunov function V. (Recall a Lyapunov function has the property $V(f(x_k)) \leq V(x_k)$). Prove that the set $\{x|V(x) \leq \alpha\}$ is invariant for all $\alpha \geq 0$.

   We need to prove that $\forall x_k \in \mathcal{X}$, where $\mathcal{X} := \{x|V(x) \leq \alpha\}$, $x_{k+1} \in \mathcal{X}$. If $x_k \in \mathcal{X}$, $V(x_k) \leq \alpha$. Since $V(x)$ is a Lyapunov function, $V(x_{k+1}) \leq V(x_k) \leq \alpha$. Hence $x_{k+1} \in \mathcal{X}$.