Exercise 6

1 Robust Invariant Set

1. Consider the following scalar autonomous discrete-time system

$$x^+ = 0.5x + w$$

under the constraint $x \in \mathbb{X} = [-10, 10]$ and subject to a bounded disturbance $w \in \mathbb{W} = [-1, 1]$. Compute the maximum robust invariant set.

2. Suppose that $\mathbb{Y}$ is a robust control invariant set for the system $x^+ = Ax + Bu + w$ subject to the state and input constraints $\mathbb{X}$ and $\mathbb{U}$ and the noise $w \in \mathbb{W}$, where 0 is in $\mathbb{W}$, $\mathbb{X}$ and $\mathbb{U}$. For which of the following scenarios is $\mathbb{Y}$ still a robust control invariant set for the resulting system?

   (a) Disturbance set changes to $\alpha\mathbb{W}$ for some $\alpha \in \mathbb{R}_+$.

   (b) Input constraint set changes to $\alpha\mathbb{U}$ for some $\alpha \in \mathbb{R}_+$.

2 Robust linear MPC for nonlinear system

Consider the following nonlinear discrete-time dynamic system

$$x^+ = rx(1 - x^2) + u = f(x, u)$$

subject to the state and input constraints $x \in \mathbb{X} = [-0.5, 0.5]$ and $u \in \mathbb{U} = [-1, 1]$. Treat r symbolically as a constant $r \in (0, 0.5)$. Design a robust linear MPC controller for this nonlinear system to regulate it to the origin with the following steps:

1. Find values a, b by linearizing the system around the origin and the smallest $\bar{w}$ such that the evolution of the following uncertain linear dynamic system contains that of the nonlinear system above for all $x \in \mathbb{X}, u \in \mathbb{U}$.

$$x^+ = ax + bu + w \qquad w \in \mathbb{W} = \bar{w} \cdot [-1, 1]$$

2. Compute the maximum robust invariant set $\mathbb{X}_f$ for the linearized system within the constraint set $\mathbb{X}$ for the control law $u = 0$. Use the value $\bar{w} = 0.1$ for this question.

3. Design a set $C(x)$ such that any controller $u(x) \in C(x)$ will ensure robust constraint satisfaction in two stages of open-loop control of the linearized system. Use the value $\bar{w} = 0.1$ for this question.
Hint: the set $C(x)$ has the shape of

$$C(x) := \left\{ u_0 \middle| \exists u_1 \text{ s.t. } -h \leq G \begin{bmatrix} u_0 \\ u_1 \\ x \end{bmatrix} \leq h \right\}$$

where x is the current state, u_0 is the next input to apply, and u_1 is the planned input in the next stage. The problem is converted to finding a proper matrix-vector pair of (G, h).

Exercise 6

1 Robust Invariant Set

1. Consider the following scalar autonomous discrete-time system

$$x^+ = 0.5x + w$$

under the constraint $x \in \mathbb{X} = [-10, 10]$ and subject to a bounded disturbance $w \in \mathbb{W} = [-1, 1]$. Compute the maximum robust invariant set.

$$\begin{aligned}
\Omega_0 &= [-10, 10] \\
\text{pre}^{\mathcal{W}}(\Omega_0) &= \{x \mid -10 \le 0.5x + w \le 10 \text{ for all } -1 \le w \le 1\} \\
&= \{x \mid -20 - 2w \le x \le 20 + 2w \text{ for all } -1 \le w \le 1\} \\
&= \{x \mid -18 \le x \le 18\} \\
\Omega_1 &= [-10, 10] \cap [-18, 18] = [-10, 10]
\end{aligned}$$

Since $\Omega_1 = \Omega_0$, the computation has already converged. Therefore, the maximum robust invariant set is $\mathcal{O}_\infty^{\mathcal{W}} = [-10, 10]$.

2. Suppose that $\mathbb{Y}$ is a robust control invariant set for the system $x^+ = Ax + Bu + w$ subject to the state and input constraints $\mathbb{X}$ and $\mathbb{U}$ and the noise $w \in \mathbb{W}$, where 0 is in $\mathbb{W}$, $\mathbb{X}$ and $\mathbb{U}$. For which of the following scenarios is $\mathbb{Y}$ still a robust control invariant set for the resulting system?

(a) Disturbance set changes to $\alpha\mathbb{W}$ for some $\alpha \in \mathbb{R}_+$.

By definition of $\mathbb{Y}$, $\forall x \in \mathbb{Y} \subseteq \mathbb{X}$, there exists an input $u \in \mathbb{U}$ ensuring that $\forall w \in \mathbb{W}$, the next state $x^+ = Ax + Bu + w$ will be also in the set $\mathbb{Y}$.
If $\alpha \le 1$, then $\alpha\mathbb{W} \subseteq \mathbb{W}$, so $\forall w \in \alpha\mathbb{W} \subseteq \mathbb{W}$, the same input u can still ensure $x^+ \in \mathbb{Y}$.
If $\alpha > 1$, then there exists $w \in \alpha\mathbb{W}$ but $w \notin \mathbb{W}$, so it is NOT guaranteed that an input $u \in \mathbb{U}$ can always be found to ensure $x^+ \in \mathbb{Y}$.
In short, if $\mathbb{W}$ becomes larger, then we might not have enough capability from $u \in \mathbb{U}$ to counter-act.

(b) Input constraint set changes to $\alpha\mathbb{U}$ for some $\alpha \in \mathbb{R}_+$.

By definition of $\mathbb{Y}$, $\forall x \in \mathbb{Y} \subseteq \mathbb{X}$, there exists an input $u \in \mathbb{U}$ ensuring that $\forall w \in \mathbb{W}$, the next state $x^+ = Ax + Bu + w$ will be also in the set $\mathbb{Y}$.
If $\alpha \ge 1$, then $\mathbb{U} \subseteq \alpha\mathbb{U}$, so $\forall w \in \mathbb{W}$, the same input $u \in \mathbb{U} \subseteq \alpha\mathbb{U}$ can still ensure $x^+ \in \mathbb{Y}$.
If $\alpha < 1$, then there exists $u \in \mathbb{U}$ but $u \notin \alpha\mathbb{U}$. So there might exist a disturbance $w \in \mathbb{W}$ that needs $u \in \mathbb{U}$ to ensure robust invariance but $u \notin \alpha\mathbb{U}$, so robust invariance of $\mathbb{Y}$ cannot be guaranteed.
In short, if $\mathbb{U}$ becomes smaller, then our counter-act capability might be shrunk.

2 Robust linear MPC for nonlinear system

Consider the following nonlinear discrete-time dynamic system

$$x^+ = rx(1 - x^2) + u = f(x, u)$$

subject to the state and input constraints $x \in \mathbb{X} = [-0.5, 0.5]$ and $u \in \mathbb{U} = [-1, 1]$. Treat r symbolically as a constant $r \in (0, 0.5)$. Design a robust linear MPC controller for this nonlinear system to regulate it to the origin with the following steps:

1. Find values a, b by linearizing the system around the origin and the smallest $\bar{w}$ such that the evolution of the following uncertain linear dynamic system contains that of the nonlinear system above for all $x \in \mathbb{X}, u \in \mathbb{U}$.

$$x^+ = ax + bu + w \qquad w \in \mathbb{W} = \bar{w} \cdot [-1, 1]$$

The idea is to find $a, b, \bar{w}$ to ensure that $\forall (x,u) \in \mathbb{X}\times\mathbb{U}$, there exists $w \in \mathbb{W}$ such that $ax+bu+w = f(x,u)$.
Rewrite the system as a polynomial: $rx - rx^3 + u$.
Linearize the system around $x = 0, u = 0$:

$$
\begin{aligned}
a &= r - 2rx^2 = r \\
b &= 1
\end{aligned}
$$

The maximum error will occur at the boundary, so we get that $\bar{w} = 0.5r - 0.5r(1-0.5^2) = \frac{r}{8}$.

2. Compute the maximum robust invariant set $\mathbb{X}_f$ for the linearized system within the constraint set $\mathbb{X}$ for the control law $u = 0$. Use the value $\bar{w} = 0.1$ for this question.

Our linearized system is $x^+ = rx + w$, for $|w| \leq 0.1$.
We compute the robust pre-set of an interval $[-\alpha, \alpha]$

$$
\begin{aligned}
\text{pre}([-\alpha,\alpha]) &= \{x | -\alpha \leq rx + w \leq \alpha \forall |w| \leq 0.1\} \\
&= \{x \mid -\alpha + 0.1 \leq rx \leq \alpha - 0.1\} \\
&= \left\{ x \,\middle|\, -\frac{\alpha - 0.1}{r} \leq x \leq \frac{\alpha - 0.1}{r} \right\} \\
&= \left[-\frac{\alpha - 0.1}{r}, \frac{\alpha - 0.1}{r} \right]
\end{aligned}
$$

The pre-set of $\mathbb{X} = [-0.5, 0.5]$ is

$$
\text{pre}\left(-\frac{1}{2}, \frac{1}{2}\right) = \left[-\frac{0.5-0.1}{r}, \frac{0.5-0.1}{r} \right] = \left[-\frac{0.4}{r}, \frac{0.4}{r} \right] \supset X
$$

because $r \in (0, 0.5)$. So, $\mathbb{X}_f = \mathbb{X}$.

3. Design a set $C(x)$ such that any controller $u(x) \in C(x)$ will ensure robust constraint satisfaction in two stages of open-loop control of the linearized system. Use the value $\bar{w} = 0.1$ for this question.
Hint: the set $C(x)$ has the shape of

$$
C(x) := \left\{ u_0 \middle| \exists u_1 \text{ s.t. } \quad -h \leq G \begin{bmatrix} u_0 \\ u_1 \\ x \end{bmatrix} \leq h \right\}
$$

where x is the current state, u_0 is the next input to apply, and u_1 is the planned input in the next stage. The problem is converted to finding a proper matrix-vector pair of (G, h).

u_0 needs to ensure that $x^+ = rx + u_0 + \bar{w} \in [-0.5, 0.5]$ holds. So $-0.4 \leq u_0 + rx \leq 0.4$.
u_0, u_1 need to ensure that $x^{++} = r^2x + ru_0 + u_1 + r\bar{w} + \bar{w} \in [-0.5, 0.5]$ holds. So $-0.4 + 0.1r \leq ru_0 + u_1 + r^2x \leq 0.4 - 0.1r$.
u_0, u_1 need to satisfy the input constraint $u \in [-1, 1]$.
As a result,

$$
G = \left[\begin{array}{c:c:c}
1 & & r \\ \hdashline
r & 1 & r^2 \\ \hdashline
1 & & \\ \hdashline
 & 1 &
\end{array}\right]
\qquad
h = \left[\begin{array}{c}
0.4 \\ \hdashline
0.4 - 0.1r \\ \hdashline
1 \\ \hdashline
1
\end{array}\right]
$$