Simple strategies versus optimal schedules in multi-agent patrolling

Akitoshi Kawamura and Makoto Soejima

University of Tokyo

Abstract. Suppose that we want to patrol a fence (line segment) using k mobile agents with given speeds v_1, \ldots, v_k so that every point on the fence is visited by an agent at least once in every unit time period. A simple strategy where the ith agent moves back and forth in a segment of length $v_i/2$ patrols the length $(v_1 + \cdots + v_k)/2$, but it has been shown recently that this is not always optimal. Thus a natural question is to determine the smallest c such that a fence of length $c(v_1 + \cdots + v_k)/2$ cannot be patrolled. We give an example showing $c \geq 4/3$ (and conjecture that this is the best possible).

We also consider a variant of this problem where we want to patrol a circle and the agents can move only clockwise. We can patrol a circle of perimeter rv_r by a simple strategy where the r fastest agents move at the same speed. We give an example where we can achieve the perimeter of $1.05 \max_r rv_r$ (and conjecture that this constant can be arbitrary big).

We propose another variant where we want to patrol a single point under the constraint that each agent $i=1,\ldots,k$ can visit the point only at a predefined interval of a_i or longer. This problem can be reduced to the discretized version where the a_i are integers and the goal is to visit the point at every integer time. It is easy to see that this discretized patrolling is impossible if $1/a_1+\cdots+1/a_k<1$, and that there is a simple strategy if $1/a_1+\cdots+1/a_k\geq 2$. Thus we are interested in the smallest c such that patrolling is always possible if $1/a_1+\cdots+1/a_k\geq c$. We prove that $\alpha\leq c<1.546$, where $\alpha=1.264\ldots$ (and conjecture that $c=\alpha$). We also discuss the computational complexity of related problems.

1 Introduction

In patrolling problems, a set of mobile agents are deployed in order to protect or supervise a given area, and the goal is to leave no point unattended for a long period of time. Besides being a well-studied task in robotics and distributed algorithms, patrolling raises interesting theoretical questions [8]. Recent studies [2, 5, 3] have shown that finding an optimal strategy is not at all straightforward, even when the terrain to be patrolled is as simple as it could be. We continue this line of research in three basic settings: patrolling a line segment, a circle, and a point. We will be particularly interested in the ratio by which the best schedule could outperform the simple strategy for each problem.

1.1 Fence patrolling

In 2011, Czyzowicz et al. [2] proposed the following problem:

Fence Patrolling Problem. We want to patrol a fence (line segment) using k mobile agents. We are given the speed limits of the agents v_1, \ldots, v_k and the *idle time* T > 0. For each point x on the fence and time $t \in \mathbb{R}$, there must be an agent who visits the point x during the interval [t, t + T). How long can the fence be?

Formally, a fence is an interval [0, L], and a *schedule* is a k-tuple (a_1, \ldots, a_k) of functions, where each $a_i : \mathbb{R} \to \mathbb{R}$ satisfies $|a_i(s) - a_i(t)| \le v_i \cdot |s - t|$ for all s, $t \in \mathbb{R}$. It *patrols* the fence with idle time T if for any time $t \in \mathbb{R}$ and any location $x \in [0, L]$, there are an agent i and a time $t' \in [t, t + T)$ such that $a_i(t') = x$.

Note that if we can patrol a fence of length L with idle time T, we can patrol a fence of length αL with idle time αT by scaling, for any $\alpha > 0$. Thus, we are only interested in the ratio of L and T. Unless stated otherwise, we fix the idle time to T=1.

In Section 2, we will prove that any schedule can be approximated arbitrarily closely by a periodic schedule. Thus, for any $\varepsilon > 0$, we can find in finite time (though not efficiently) a schedule that is $1 - \varepsilon$ times as good as any schedule.

Czyzowicz et al. [2] discussed the following simple strategy that patrols a fence of length $(v_1 + \cdots + v_k)/2$ (with idle time 1), and proved that no schedule can patrol more than twice as long a fence as this strategy:

Partition-based strategy. Divide the fence into k segments, the ith of which has length $v_i/2$. The agent i moves back and forth in the ith segment.

They conjectured that this gives the optimal schedule. However, Kawamura and Kobayashi [5] exhibited a setting of speed limits v_1, \ldots, v_k and a schedule that patrols a fence slightly longer than the partition-based strategy. Thus, the following natural question arises: what is the biggest ratio between the optimal schedule and partition-based strategy? Formally, we want to determine the smallest constant c such that no schedule can patrol a fence that is c times as long as the partition-based strategy does.

Czyzowicz et al.'s result [2] says that $1 \le c \le 2$, and their conjecture was that c = 1. Kawamura and Kobayashi's example shows that $c \ge 42/41$. Later this lower bound was improved to 25/24 [1, 3]. In Section 3, we will further improve the lower bound to 4/3. We conjecture that c = 4/3.

1.2 Unidirectional circle patrolling

In Section 4, we will discuss another problem proposed by Czyzowicz et al. [2]:

Unidirectional Circle Patrolling Problem. We want to patrol a circle using k mobile agents. We are given the speed limits v_1, \ldots, v_k of the agents. For each point x on the circle and time $t \in \mathbb{R}$, there must be an agent who visits the point x during the interval [t, t+1). Each agent i is allowed to move along

the circle in clockwise direction with arbitrary speed between 0 and its speed limit v_i , but it is not allowed to move in the opposite direction. How long can the perimeter of the circle be?

They conjectured that the following strategy is optimal:

Runners strategy. Without loss of generality, we can assume that $v_1 \ge \cdots \ge v_k$. If all the fastest r agents move at constant speed v_r and placed equidistantly, we can patrol a perimeter of length rv_r . By choosing the optimal r, we can achieve the perimeter $\max_r rv_r$.

However, Dumitrescu et al. [3] constructed an example where this strategy is not optimal. We conjecture that this is not even a constant-ratio approximation strategy. Formally, we conjecture that for any constant c, there exist v_1, \ldots, v_k such that we can patrol a perimeter of $c \max_r rv_r$. We will define a problem that is equivalent to this conjecture. Also, we will prove that this is true for c = 1.05.

1.3 Point patrolling

In Section 5, we propose a new problem that we call Point Patrolling Problem. In a sense, this is a simplification of the Fence Patrolling Problem. In this problem, agents patrol a single point instead of a fence. In this case, it is natural to set a lower bound on the intervals between two consecutive visits by an agent instead of restricting its speed. Formally, we study the following problem:

Point Patrolling Problem. We want to patrol a point using k mobile agents. We are given the lower bounds a_1, \ldots, a_k on the intervals between two consecutive visits of the agents. A *schedule* is a k-tuple of sets $S_1, \ldots, S_k \subseteq \mathbb{R}$, where S_i means the set of times at which the ith agent visits the point. Thus, if t_1 and t_2 are two distinct elements of S_i , they must satisfy $|t_1 - t_2| \ge a_i$. This schedule patrols the point with idle time t if for any time $t \in \mathbb{R}$, there are an agent t and a time $t' \in [t, t+T)$ such that $t' \in S_i$. How small can the idle time be?

It turns out that this problem can be reduced to a decision problem that asks whether it is possible to visit the point at each integer time under the constraint that each agent $i=1,\ldots,k$ can visit the point only at a predefined interval of at least $a_i \in \mathbb{N}$. We will see the relation between the amount $1/a_1 + \cdots + 1/a_k$ and this problem.

In Section 6, we will analyze the complexity of this discretized problem.

2 Zigzag schedules for fence patrolling

In the following two sections, we will discuss the Fence Patrolling Problem.

In this section, we prove that, for the purpose of discussing upper limit of the length of the fence, we may restrict attention to periodic schedules.

The movement of an agent during time interval $[t_{\text{start}}, t_{\text{end}}] \subseteq \mathbb{R}$ is represented by a function $a : [t_{\text{start}}, t_{\text{end}}] \to \mathbb{R}$. This function is called a (v, ξ) -zigzag movement, for $v, \xi > 0$ (Figure 1), if there are integers $p_0, p_1, p_2, p_3 \in \mathbb{Z}$ such that the agent

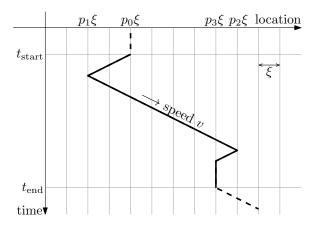


Fig. 1. A (v, ξ) -zigzag movement on a time interval $[t_{\text{start}}, t_{\text{end}}]$.

- starts at time t_{start} at location $p_0\xi$,
- moves at speed v until it reaches $p_1\xi$,
- moves at speed v until it reaches $p_2\xi$,
- moves at speed v until it reaches $p_3\xi$,
- and then stays there until time $t_{\rm end}$.

For this movement to be possible, the entire route must be short enough to be travelled with speed v; that is,

$$|p_0 - p_1|\xi + |p_1 - p_2|\xi + |p_2 - p_3|\xi \le \tau v, \tag{1}$$

where $\tau := t_{\text{end}} - t_{\text{start}}$ is the length of the time interval.

We prove in the next lemma that any schedule can be converted into one that consists of zigzag movements without deteriorating the idle time too much.

Lemma 1. For any positive constants δ , τ , v > 0, we have the following for all sufficiently small $\xi > 0$. For any function $a: [t_{\text{start}}, t_{\text{end}}] \to \mathbb{R}$ on an interval of length τ such that $|a(s) - a(t)| \le v \cdot |s - t|$ for all $s, t \in \mathbb{R}$, there is a (v, ξ) -zigzag movement $a': [(1 + \delta)t_{\text{start}}, (1 + \delta)t_{\text{end}}] \to \mathbb{R}$ such that

$$-a'((1+\delta)t_{\text{start}}) = \left\lfloor \frac{a(t_{\text{start}})}{\xi} \right\rfloor \xi \text{ and } a'((1+\delta)t_{\text{end}}) = \left\lfloor \frac{a(t_{\text{end}})}{\xi} \right\rfloor \xi;$$

$$-any \text{ location visited by } a \text{ is visited by } a' \text{ (that is, for each } t \in [t_{\text{start}}, t_{\text{end}}]$$

$$there \text{ is } t' \in [(1+\delta)t_{\text{start}}, (1+\delta)t_{\text{end}}] \text{ such that } a'(t') = a(t)).$$

Proof. Let $\xi \leq (t_{\rm end} - t_{\rm start})v\delta/5$. Suppose that a takes its minimum and maximum at $t_{\rm min}$, $t_{\rm max} \in [t_{\rm start}, t_{\rm end}]$, respectively. We may assume that $t_{\rm min} \leq t_{\rm max}$ (the other case can be treated similarly). Define a' to be the (v, ξ) -zigzag movement specified by

$$p_0 = \left\lfloor \frac{a(t_{\text{start}})}{\xi} \right\rfloor, \quad p_1 = \left\lfloor \frac{a(t_{\text{min}})}{\xi} \right\rfloor, \quad p_2 = \left\lceil \frac{a(t_{\text{max}})}{\xi} \right\rceil, \quad p_3 = \left\lfloor \frac{a(t_{\text{end}})}{\xi} \right\rfloor \quad (2)$$

(see the beginning of Section 2 for the meaning of these numbers). This is indeed possible: we have (1) for $\tau = (1 + \delta)(t_{\text{end}} - t_{\text{start}})$ because

$$(p_{0} - p_{1})\xi + (p_{2} - p_{1})\xi + (p_{2} - p_{3})\xi$$

$$\leq (a(t_{\text{start}}) - a(t_{\text{min}}) + \xi) + (a(t_{\text{max}}) - a(t_{\text{min}}) + 2\xi)$$

$$+ (a(t_{\text{max}}) - a(t_{\text{end}}) + 2\xi)$$

$$= (a(t_{\text{start}}) - a(t_{\text{min}})) + (a(t_{\text{max}}) - a(t_{\text{min}})) + (a(t_{\text{max}}) - a(t_{\text{end}})) + 5\xi$$

$$\leq (t_{\text{min}} - t_{\text{start}})v + (t_{\text{max}} - t_{\text{min}})v + (t_{\text{end}} - t_{\text{max}})v + 5\xi$$

$$= (t_{\text{end}} - t_{\text{start}})v + 5\xi \leq (1 + \delta)(t_{\text{end}} - t_{\text{start}})v.$$
(3)

It is straightforward to see that this zigzag movement has the claimed properties.

The following lemma says that any agent's movement can be replaced, without changing the visited region, by a zigzag movement that takes only slightly longer.

For positive ξ , $\tau > 0$, a schedule (a_1, \ldots, a_k) (for k agents with speed limits v_1, \ldots, v_k) is called a (ξ, τ) -zigzag schedule if the movement of each agent $i = 1, \ldots, k$ during each time interval $[m\tau, (m+1)\tau], m \in \mathbb{Z}$, is a (v_i, ξ) -zigzag.

Lemma 2. For any $\varepsilon > 0$ and speeds $v_1, \ldots, v_k > 0$, there are $\xi > 0$ and $\tau' > 0$ satisfying the following. Suppose that there is a schedule for a set of agents with speed limits v_1, \ldots, v_k that patrols a fence with some idle time T > 0. Then there is a (ξ, τ') -zigzag schedule for the same set of agents that patrols the same fence with idle time $T(1 + \varepsilon)$.

Proof. We show that it suffices to let ξ be so small that we have the claim of Lemma 1 for

$$\delta = \frac{\varepsilon}{2}, \qquad \tau = \frac{T\varepsilon}{4(1+\delta)} \tag{4}$$

and for all speeds $v = v_i$, and to let $\tau' = (1 + \delta)\tau$.

Using the schedule (a_1, \ldots, a_k) that we start with, we define the claimed (ξ, τ') -zigzag schedule (a'_1, \ldots, a'_k) as follows. For each agent i and each $m \in \mathbb{Z}$, we define a'_i on the time interval $[m\tau', (m+1)\tau']$ to be the zigzag movement obtained by Lemma 1 from the movement a_i during $[m\tau, (m+1)\tau]$. This defines a_i consistently (at multiples of τ) because of the first property in Lemma 1.

To see that this schedule (a'_1, \ldots, a'_k) patrols the fence as claimed, suppose that a location on the fence is left unvisited by the schedule (a'_1, \ldots, a'_k) during a time interval $[\underline{t}, \overline{t}]$ of length $T(1 + \varepsilon)$, and hence during its subinterval $[\lceil \underline{t}/\tau' \rceil \tau', \lfloor \overline{t}/\tau' \rfloor \tau']$. By the second property in Lemma 1, this point is also left unvisited by the schedule (a_1, \ldots, a_k) during the time interval $[\lceil \underline{t}/\tau' \rceil \tau, \lfloor \overline{t}/\tau' \rfloor \tau]$, whose length is

$$\left(\left\lfloor \frac{\overline{t}}{\tau'} \right\rfloor - \left\lceil \frac{\underline{t}}{\tau'} \right\rceil \right) \tau \ge \left(\frac{\overline{t}}{\tau'} - \frac{\underline{t}}{\tau'} - 2 \right) \tau \ge \left(\frac{T(1+\varepsilon)}{\tau'} - 2 \right) \tau = \frac{T(1+\varepsilon)}{1+\delta} - 2\tau = T.$$
(5)

The next lemma says that a zigzag schedule can be made periodic without changing the idle time.

Lemma 3. Suppose that there is a (ξ, τ) -zigzag schedule for a set of agents that patrols a fence with some idle time. Then there is a periodic (ξ, τ) -zigzag schedule for the same agents that patrols the same fence with the same idle time.

Proof. Let [a,b] be the fence, k be the number of agents, and T be the idle time. We may assume that in the given $((\xi,\tau)$ -zigzag) schedule, every agent stays within $[A\xi,B\xi]$, where $A=\lfloor a/\xi\rfloor$ and $B=\lceil b/\xi\rceil$, i.e., it never goes far off the fence. In such a schedule, the movement of each agent during each time interval $[m\tau,(m+1)\tau], m\in\mathbb{Z}$, is specified by a quadruple of intergers $p_0,p_1,p_2,p_3\in\{A,A+1,\ldots,B\}$, and hence there are at most $(B-A+1)^4$ possible such movement.

Let $Q = \lceil T/\tau \rceil$. For each $m \in \mathbb{Z}$, there are at most $(B - A + 1)^{4kQ}$ possible ways that the k agents can move during the time interval $[m\tau, (m+Q)\tau]$. Since this is finite, there are integers m_0 , m_1 with $m_0 < m_1$ such that in the given (ξ, τ) -zigzag schedule, all agents move during the time interval $[m_1\tau, (m_1 + Q)\tau]$ in exactly the same way as they did during $[m_0\tau, (m_0 + Q)\tau]$. Consider the periodic schedule, with period $(m_1 - m_0)\tau$, where each agent perpetually repeats its motion during $[m_0\tau, m_1\tau]$ in the original schedule. This schedule patrols the fence, because the motion of the agents during any time period of length T is identical to their motion in the original schedule in a length-T subinterval of $[m_0\tau, (m_1 + Q)\tau]$.

Using the above lemmas, we obtain an algorithm that solves the Fence Patrolling Problem with arbitrarily high precision in the following sense.

Theorem 4. There exists an algorithm that, given v_1, \ldots, v_k , T and $\varepsilon > 0$, finds a schedule that patrols a fence of length at least $1 - \varepsilon$ times the length of the fence patrolled by the same agents using any schedule.

Proof. Suppose that there is a schedule that patrols a fence of length L with idle time T using these agents. By Lemma 2, there is a (ξ, τ) -zigzag schedule that patrols a fence of length $(1 - \varepsilon)L$, for some ξ , $\tau > 0$ determined by the inputs ε and v_1, \ldots, v_k . By Lemma 3, there is a (ξ, τ) -zigzag schedule with period p that patrols the same length $(1 - \varepsilon)L$, for some p > 0 determined by the inputs. Since there are only finitely many such schedules, we can check all of them in a finite amount of time.

In previous work [2, 5], a schedule was defined as functions on the halfline $[0, +\infty)$ (instead of $\mathbb R$) and the requirement for patrolling was that each location be visited in every length-T time interval contained in this halfline. Note that the argument for Lemmas 2 and 3 in this section stays valid when we start with a patrolling schedule on $[0, +\infty)$ in this sense. In particular, a patrolling schedule on $[0, +\infty)$ can be converted to a (periodic) schedule on $\mathbb R$ without essentially worsening the idle time. Therefore, the ratio bound that we are interested in (the constant c in Section 1.1) is not affected by our slight deviation in the definition.

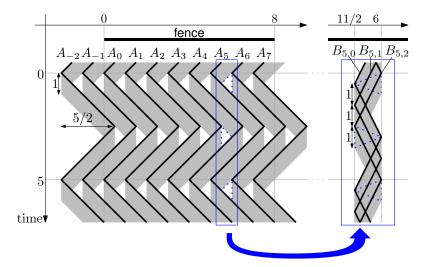


Fig. 2. The strategy in the proof of Theorem 5 when n=3 and L=8. The trajectories of the agents are the thick solid lines, and the regions they cover (the points that have been visited during the past unit time, see appendix) are shown shaded. The n+L-1 faster agents $A_{-n+1}, \ldots, A_{L-1}$ (left) move back and forth with period 2n-1, but leave some triangular regions (dotted) uncovered. These regions are covered by the nL slow agents $B_{0,0}, \ldots, B_{n-1,L-1}$ (right; scaled up horizontally for clarity).

3 A schedule patrolling a long fence

In this section, we will prove that for any c < 4/3, there exists a schedule that patrols a fence c times as long as the partition-based strategy. This improves the same claim for c < 25/24 established previously [1, 3].

Theorem 5. For any c < 4/3, there are settings of speed limits v_1, \ldots, v_k and a schedule that patrols a fence of length $c(v_1 + \cdots + v_k)/2$ (with idle time 1).

Proof. We construct, for any positive integers n and L, a schedule that patrols a fence of length L with idle time 1 using n+L-1 agents with speed 1 and nL agents with speed 1/(2n-1). Note that with the partition-based strategy, the same set of agents would patrol (with idle time 1) a fence of length $\frac{1}{2}(n+L-1+nL/(2n-1))$. The ratio between L and this approaches 4/3 when $1 \ll n \ll L$, and hence we have the theorem.

The schedule that proves our claim is as follows (Figure 2):

- Each of the n+L-1 agents A_i (-n < i < L) with speed 1 visits the locations i and i+n-1/2 alternately (at its maximal speed); it is at location i at time 0. (This means that some agents occasionally step out of the fence [0, L]; to avoid this, we could simply modify the schedule so that they stay at the end of the fence for a while.)

- Each of the nL agents $B_{i,j}$ $(0 \le i < L, 0 \le j < n)$ with speed 1/(2n-1) visits the locations i+1/2 and i+1 alternately (at its maximal speed); it is at location i+1/2 at time j+1/2.

We will show that this schedule indeed patrols a fence. We say that an agent covers a point $(x,t) \in [0,L] \times \mathbb{R}$ if it visits the location x during the time interval [t-1,t]. We want to prove that every (x,t) is covered by some agent. In this schedule, every agent has a period of 2n-1. Thus, in this proof, we consider the time modulo 2n-1. Consider two cases:

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Case I: k \le x \le k + 1/2 for some k = 0, \ldots, L - 1.
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For each integer $0 \le i < n$, agent A_{k-i} leaves x = k at t = i, moves to the right with speed 1, and reaches x = k + 1/2 at t = i + 1/2. These agents cover a parallelogram-shaped region whose vertices are (x, t) = (k, 0), (k, n), (k + 1/2, 1/2), (k + 1/2, n + 1/2).

For each integer $0 \le i < n$, agent A_{k-i} leaves x = k + 1/2 at t = 2n - 1 - (i + 1/2), moves to the left with speed 1, and reaches x = k at t = 2n - 1 - i. These agents cover a parallelogram-shaped region whose vertices are (x, t) = (k, n), (k, 2n), (k + 1/2, n - 1/2), (k + 1/2, 2n - 1/2).

These two parallelograms cover the entire region.

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Case II: k + 1/2 \le x \le k + 1 for some k = 0, ..., L - 1.
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For each integer $0 \le i < n$, agent A_{k-i} leaves x = k+1/2 at t = i+1/2, moves to the right with speed 1, and reaches x = k+1 at t = i+1. These agents cover a parallelogram-shaped region whose vertices are (x,t) = (k+1/2,1/2), (k+1/2,n-1/2), (k+1,1), (k+1,n).

For each integer $0 \le i < n$, agent A_{k-i} leaves x = k+1 at t = 2n-1-(i+1), moves to the left with speed 1, and reaches x = k+1/2 at t = 2n-1-(i+1/2). These agents cover a parallelogram-shaped region whose vertices are (x,t) = (k+1/2, n+1/2), (k+1/2, 2n-1/2), (k+1,n), (k+1,2n-1).

The only regions that are *not* covered by these parallelograms are:

- Region P: Triangle-shaped region whose vertices are (x,t) = (k+1,0), (k+1,1), (k+1/2,1/2).
- Region Q: Triangle-shaped region whose vertices are (x,t) = (k,n-1/2), (k,n+1/2), (k+1/2,n).

For each integer $0 \le i < n$, agent $B_{k,i}$ leaves x = k+1/2 at t = i+1/2, moves to the right with speed 1, and reaches x = k+1 at t = i+n. These agents cover a parallelogram-shaped region whose vertices are (x,t) = (k+1/2,1/2), (k+1/2,n+1/2), (k+1,n), (k+1,2n). In particular, these agents entirely cover Region Q.

For each integer $0 \le i < n$, agent $B_{k,i}$ leaves x = k+1 at t = i+n, moves to the right with speed 1, and reaches x = k+1/2 at t = i+2n-1/2. These agents cover a parallelogram-shaped region whose vertices are (x,t) = (k+1/2,1/2), (k+1/2,n+1/2), (k+1,-n+1), (k+1,1). In particular, these agents entirely cover Region P.

Therefore, the entire region is covered by at least one agent.

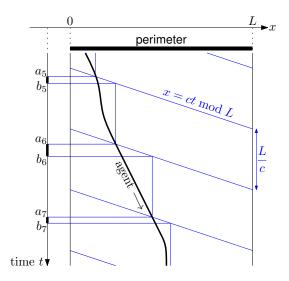


Fig. 3. Time intervals $[a_i, b_i]$ that are covered by an agent.

We conjecture that this constant 4/3 is the best possible. That is,

Conjecture 6. No schedule can patrol a fence that is more than 4/3 times as long as the partition-based strategy.

4 Circle patrolling

We start by defining (c, k)-sequences, whose existence is closely related to the Circle Patrolling Problem as we will show in Lemma 7.

For a real number c > 1 and a positive integer k, a (c, k)-sequence is a k-tuple of sets $S_1, \ldots, S_k \subseteq \mathbb{R}$ with $S_1 \cup \cdots \cup S_k = \mathbb{R}$ such that for each i,

- 1. the set S_i is a union of non-overlapping intervals $S_i = \bigcup_{j \in \mathbb{Z}} [a_{i,j}, b_{i,j}];$
- 2. the length of each interval in S_i is at most 1/(ci-1), i.e., $b_{i,j}-a_{i,j} \leq 1/(ci-1)$;
- 3. the distance between two consecutive intervals in S_i is exactly 1, i.e., $a_{i,j+1} b_{i,j} = 1$.

Lemma 7. *Let* c > 1.

- 1. If k agents with speed limits $1, \ldots, 1/k$ can patrol a circle of perimeter c, then there is a (c,k)-sequence.
- 2. If there is a (c,k)-sequence, then k agents with speed limits $1, \ldots, 1/k$ can patrol a circle of perimeter c/2.

Proof. Consider a circle with perimeter L. We say an agent covers $t \in \mathbb{R}$ if it visits the point $ct \mod L$ at least once during the time interval [t, t + L/c] (Figure 3). It is straightforward to show that for any possible movement of an agent with speed limit v < c, the set of $t \in \mathbb{R}$ covered by this agent is a union of disjoint intervals $\bigcup_{i \in \mathbb{Z}} [a_i, b_i]$ such that

$$b_i - a_i \le \frac{vL}{c(c-v)}, \qquad a_{i+1} - b_i = \frac{L}{c}. \tag{6}$$

These are also sufficient conditions in the sense that for any a_i and b_i satisfying (6), there is a movement that covers $\bigcup_{i\in\mathbb{Z}}[a_i,b_i]$; that is, the agent can travel fast enough to be at the point $ca_i \mod L$ at time a_i for every $i\in\mathbb{Z}$. We now prove the claims.

- 1. Define S_i as the set of numbers t that are covered by the agent with speed limit 1/i. Then, S_1, \ldots, S_k is a (c, k)-sequence.
- 2. Let S_1, \ldots, S_k be a (c, k)-sequence. Then, we can define a schedule on a circle of perimeter c/2 such that the agent of speed limit 1/i covers all $t \in S_i$. By the definition of "cover", for each t, at least one agent visits ct during the time interval [t, t + 1/2]. Note that ct and ct + nL refer to the same point for each integer n. Thus ct is visited at least once during the time interval [(ct + nL)/c, (ct + nL)/c + 1/2] = [t + n/2, t + n/2 + 1/2] for each integer n. This implies that in each unit time interval each point is visited by an agent.

In particular, the runners strategy for circle patrolling is not a constant-ratio approximation strategy if and only if for any constant c, there exists k such that a (c, k)-sequence exists.

Theorem 8. There exist v_1, \ldots, v_k and a schedule that patrols a circle with perimeter $1.05 \max_r rv_r$.

Proof. By Lemma 7, it suffices to prove the existence of a (2.1, k)-sequence for some k. We have found a (2.1, 122)-sequence (S_1, \ldots, S_{122}) using a computer program which, for each $i = 1, 2, \ldots$, chooses an S_i such that

- it satisfies conditions 1-3 in the definition of (2.1, k)-sequences;
- both ends of each interval in S_i are multiples of 1/400;
- it has a period of 500 (that is, $t \in S_i$ if and only if $t + 500 \in S_i$);
- the interior of S_i does not intersect $S_0 \cup \cdots \cup S_{i-1}$;
- it is (one of) the biggest among those satisfying the above conditions. Formally, choose a S_i that maximizes the number of t such that $0 \le t < 500$, t is a multiple of 1/400, and $[t, t + 1/400] \subseteq S_i$.

The sequence (S_1, \ldots, S_{122}) that we obtained in this way was then verified to cover \mathbb{R} ; it can be found at data.txt in the ancillary files section.

We conjecture that for any constant c, there exist an integer k and a (c,k)-sequence. Equivalently,

Conjecture 9. The runners strategy does not have a constant approximation ratio. Formally, for any constant c, there exist v_1, \ldots, v_k and a schedule that patrols a circle with perimeter $c \max_r r v_r$.

5 Point patrolling

In this section, we will discuss Point Patrolling Problem. First, let's see that this problem can be reduced to a problem in which time is also discrete. Consider a decision version of this problem. That is, you are given T, and you need to decide whether the idle time can be at most T. We can reduce the original problem to this decision problem by binary search. This decision problem can be discretized in the following way:

Discretized Point Patrolling Problem. There are k agents and they want to patrol a point. We are given positive integers a_1, \ldots, a_k . The interval between two consecutive visits by the ith agent must be at least a_i . A schedule is called good if at each integer time the point is visited by at least one agent. Determine whether there exists a good schedule.

For simplicity, we call (a_1, \ldots, a_k) good if there exists a good strategy in Discretized Point Patrolling Problem, and otherwise call it bad.

Theorem 10. Agents with intervals (a_1, \ldots, a_k) can achieve the idle time of T for the (non-discretized) Point Patrolling Problem if and only if $(\lceil a_1/T \rceil, \ldots, \lceil a_k/T \rceil)$ is good.

Proof. Suppose that agents with intervals (a_1, \ldots, a_k) can achieve the idle time of T. Let $(k_i, t_i)_{i \in \mathbb{Z}}$ be one such schedule: that is, for each integer i, the agent k_i visits the point at time t_i , and $(t_i)_{i \in \mathbb{Z}}$ is an non-decreasing sequence. It is easy to see that $(k_i, \lfloor t_i/T \rfloor)_{i \in \mathbb{Z}}$ is a good schedule for Discretized Point Patrolling Problem.

Conversely, suppose that $(\lceil a_1/T \rceil, \ldots, \lceil a_n/T \rceil)$ is good. That is, there is a good schedule where at each time $i \in \mathbb{Z}$, the point is visited by an agent k_i . Then it is easy to see that $(k_i, iT)_{i \in \mathbb{Z}}$ is a valid schedule with idle time T for the non-discretized problem.

This discretized problem can be solved in $O(k \prod_{i=1}^k a_i)$ time. Construct a graph with $\prod_{i=1}^k a_i$ vertices. Each vertex of the graph is labeled with a sequence of integers (b_1,\ldots,b_k) such that $0 \le b_i < a_i$ for all i. This vertex means that $\min\{(\text{current time}) - (\text{the last visit time by agent } i), a_i - 1\} = b_i$. If $b_r = a_r - 1$, add an edge from (b_1,\ldots,b_k) to $(\min\{b_1+1,a_1-1\},\ldots,\min\{b_{r-1}+1,a_{r-1}-1\},0,\min\{b_{r+1}+1,a_{r+1}-1\},\ldots,\min\{b_k+1,a_k-1\})$. A valid schedule corresponds to an infinite path in this graph. Thus, (a_1,\ldots,a_k) is good if and only if this graph contains an infinite path. Since this graph is finite, this can be checked by finding a cycle in the graph.

However, this algorithm is slow. In order to design a fast approximation algorithm, first we give a sufficient condition for (a_1, \ldots, a_k) to be bad:

Theorem 11. If $\sum_{i=1}^{k} 1/a_i < 1$, (a_1, \ldots, a_k) is bad.

Proof. Let M be a sufficiently big integer. Out of any consecutive M integer times, the ith agent can visit the point at most $\lceil M/a_i \rceil$ times. If (a_1, \ldots, a_k) is good, the sum of $\lceil M/a_i \rceil$ must be at least M, but this contradicts $\sum_{i=1}^k 1/a_i < 1$ when M is sufficiently big.

On the other hand, the following gives a sufficient for (a_1, \ldots, a_k) to be good when a_1, \ldots, a_k are powers of 2:

Lemma 12. If
$$\sum_{i=1}^{k} 1/2^{b_i} \ge 1$$
, $(2^{b_1}, \dots, 2^{b_k})$ is good.

Proof. We prove the lemma by induction of k. Since $\sum_{i=1}^{k} 1/2^{b_i} \ge 1$, at least one of the following conditions hold:

- For some $i, b_i = 0$. In this case, $(2^{b_1}, \dots, 2^{b_k})$ is obviously good.
- There exist distinct i, j such that $b_i = b_j = t$. Let S be a set of integers. If an agent with interval 2d can visit the point at all elements in S, there exists a schedule of two agents with intervals d such that for each element in S, at least one agent visits the point. Thus, we can replace two agents with intervals 2^t with an agent with interval 2^{t-1} . This replacement doesn't change the inverse sum of intervals, and by the assumption of the induction $(2^{b_1}, \ldots, 2^{b_k})$ is good.

We can design a polynomial time 2-approximation algorithm for the (non-discretized) Point Patrolling Problem using the previous two lemmas.

Let a_1, \ldots, a_k be the input and x be the optimal idle time. As we noted above, $(\lceil a_1/x \rceil, \ldots, \lceil a_k/x \rceil)$ is good. Thus, by theorem $11, x/a_1 + \cdots + x/a_k \ge 1$.

Let y be a number that satisfies $y/a_1 + \cdots + y/a_k = 1$. Let b_i be an integer that satisfies $a_i/2y \leq 2^{b_i} \leq a_i/y$. Since $1/2^{b_1} + \cdots + 1/2^{b_k} \geq y/a_1 + \cdots + y/a_k = 1$, $(2^{b_1}, \ldots, 2^{b_k})$ is good. Since $a_i/2y \leq 2^{b_i}$ for each i, $(\lceil a_1/2y \rceil, \ldots, \lceil a_k/2y \rceil)$ is also good and we can achieve the idle time of 2y. This is at most twice bigger than the optimal idle time x.

In the remaining part of this section, we focus on the relation between Discretized Point Patrolling Problem and the amount $\sum_{i=1}^{k} 1/a_i$.

Theorem 13. If
$$\sum_{i=1}^{k} 1/a_i \ge 2$$
, $(a_1, ..., a_k)$ is good.

Proof. Let b_i be an integer that satisfies $a_i \leq 2^{b_i} < 2a_i$. Since $\sum_{i=1}^k \frac{1}{2^{b_i}} \geq \sum_{i=1}^k \frac{1}{2a_i} \geq 1$, by lemma 12, $(2^{b_1}, \dots, 2^{b_k})$ is good. Therefore, (a_1, \dots, a_k) is also good.

This constant 2 can be improved, as shown in Theorem 15 below.

Lemma 14. If (a_1,\ldots,a_k) is bad and $\sum_{i=1}^k \frac{1}{a_i} = t$ and $a_i \leq 2M$ for all i, there exists a bad (b_1,\ldots,b_m) such that $\sum_{i=1}^m \frac{1}{b_i} \geq \frac{M+1}{M+2}t - \frac{1}{M+2}$ and $b_i \leq M$ for all i.

Proof. Without loss of generality, we can assume that $a_1 \leq \cdots \leq a_k$. Let r be an integer that satisfies $a_r \leq M < a_{r+1}$. First, define $c := (c_1, \ldots, c_k)$ as follows:

- If $i \leq r$ or i r is even, $c_i = a_i$.
- Otherwise, $c_i = a_{i+1}$.

For all $i \ c_i \ge a_i$, so c is bad. Also, we can bound the inverse sum of c: $\sum_{i=1}^k 1/c_i = \sum_{i=1}^k 1/a_i - (1/a_{r+1} - 1/a_{r+2}) - (1/a_{r+1} - 1/a_{r+2}) \cdots \ge \sum_{i=1}^k 1/a_i - \frac{1}{M+1} = t - \frac{1}{M+1}$.

Next, we construct $b:=(b_1,\ldots,b_m)$ from c. First, we add all elements in c that is at most M to b. Other elements in c can be divided into pairs of same integers. If c contains the pair (x,x), we add an integer $\lceil x/2 \rceil$ to b. Two agents with intervals (x,x) can work as a single agent with interval $\lceil x/2 \rceil$, so b is also bad. This process reduces the inverse sum by the factor of at most $\frac{M+2}{M+1}$. Thus, we can bound the inverse sum of b as follows: $\sum_{i=1}^k 1/b_i \leq \frac{M+1}{M+2} \sum_{i=1}^k 1/c_i \leq \frac{M+1}{M+2} t - \frac{1}{M+2}$.

Theorem 15. If $\sum_{i=1}^{k} 1/a_i > 1.546$, (a_1, \ldots, a_k) is good.

Proof. Suppose that there exists bad (a_1,\ldots,a_k) such that $\sum_{i=1}^k 1/a_i > 1.546$ and $a_i \leq 12 \cdot 2^r$ for all i. By using the previous lemma r times, we can prove that there exists (b_1,\ldots,b_m) such that $\sum_{i=1}^m 1/b_i > f(r)$ and $b_i \leq 12$ for all i, where $f(r) = (\cdots((1.546 \cdot \frac{12 \cdot 2^{r-1}+1}{12 \cdot 2^{r-1}+2} - \frac{1}{12 \cdot 2^{r-2}+2}) \cdot \frac{12 \cdot 2^{r-2}+1}{12 \cdot 2^{r-2}+2} - \frac{1}{12 \cdot 2^{r-2}+2}) \cdots) \cdot \frac{13}{14} - \frac{1}{14}$. We verified that f(r) > 1.1822 for all r, hence $\sum_{i=1}^m 1/b_i > 1.1822$. There are finite number of (b_1,\ldots,b_m) that satisfy $b_1 \leq \ldots \leq b_m$ and $\sum_{i=1}^m 1/b_i \geq 1.1822 > \sum_{i=1}^{m-1} 1/b_i$ and $b_i \leq 12$ for all i. We verified that all these cases are good. This is a contradiction

On the other hand, we can prove that the constant cannot be smaller than $\sum_{i=0}^{\infty} 1/(2^i+1) = 1.264...$

Theorem 16. $(2, 3, 5, \dots, 2^k + 1)$ is bad.

Proof. We will prove the following stronger proposition: at least one of integers $1, \ldots, 2^{k+1}$ is not visited by agents with intervals $(2, 3, 5, \ldots, 2^k + 1)$. We prove this by induction of k.

When k=0, this is trivial. Suppose that this is correct for k=t-1. Then, at least one of $1,\ldots,2^t$ is not visited by agents $(2,3,5,\ldots,2^{t-1}+1)$, and at least one of $2^t+1,\ldots,2^{t+1}$ is not visited by agents $(2,3,5,\ldots,2^{t-1}+1)$. If all of $1,\ldots,2^{t+1}$ are visited by agents $(2,3,5,\ldots,2^t+1)$, the agent 2^t+1 must visit at least twice. However, there are 2^t integers between consecutive two visits of this agent, and by the assumption of the induction, at least one of these 2^t integers is not visited.

We suspect that this cannot be improved:

Conjecture 17. Let $\alpha := \sum_{i=1}^{\infty} 1/(2^i + 1) \approx 1.264$. If $\sum_{i=1}^{k} 1/a_i > \alpha$, (a_1, \dots, a_k) is good.

6 Complexity of problems related to point patrolling

In previous sections, we discussed approximation algorithms of patrolling problems. This is because patrolling problems look unsolvable in polynomial time. In this section, we will try to justify this intuition. Ideally, we should prove NP hardness of patrolling problems, but we failed to prove that. Instead, we will prove NP completeness of problems related to Discretized Point Patrolling Problem.

We conjecture that even in the special case where $\sum_{i=1}^{k} 1/a_i = 1$, there is no pseudo-polynomial time algorithm for the Discretized Point Patrolling Problem. It turns out that this special case is closely related to a well-studied object called *Disjoint Covering Systems*.

A set of pairs of integers (m_i, r_i) is called a *Disjoint Covering System* [6] if for all integer x, there exists unique i such that $x \equiv r_i \pmod{m_i}$.

We define a new decision problem:

Disjoint Covering System Problem. You are given a list of integers (m_1, \ldots, m_k) . Determine whether there exists a list of integers (r_1, \ldots, r_k) such that (m_i, r_i) is a disjoint covering system.

This problem is equivalent to the special case of the Discretized Point Patrolling Problem:

Theorem 18. Suppose that $\sum_{i=1}^{k} 1/m_i = 1$. Then there exists a list of integers (r_1, \ldots, r_k) such that (m_i, r_i) is a disjoint covering system if and only if (m_1, \ldots, m_k) is good.

Proof. We will prove that if $\sum_{i=1}^{k} 1/m_i = 1$, a schedule of Discretized Point Patrolling Problems corresponds to a Disjoint Covering System. As discussed in Section 5, we can assume that the schedule is periodic. Let C be the period. During one period, agent i can visit the point at most C/m_i times. Since $\sum_{i=1}^{k} C/m_i = C$, for each i, agent i must visit the point exactly C/m_i times. Thus, there exists r_i such that agent i visits the point at time t if and only if $t \equiv r_i \pmod{m_i}$. Therefore a schedule corresponds to a Disjoint Covering System, and we have proved the theorem.

Conjecture 19. Disjoint Covering System Problem is strongly NP-complete. In particular, if this conjecture is true, Point Patrolling Problem is strongly NP-hard. Here a problem is called strongly NP-complete if the problem is NP-complete when the integers in the input are given in unary notation.

We will prove that a similar problem is strongly NP-complete.

A set of pairs of integers (m_i, r_i) is called a *Disjoint Residue Class* [7] if for every integer x, there exists at most one i such that $x \equiv r_i \pmod{m_i}$.

We define a new decision problem in a similar way:

Disjoint Residue Class Problem. We are given a list of integers (m_1, \ldots, m_k) . Determine whether there exists a list of integers (r_1, \ldots, r_k) such that (m_i, r_i) is a disjoint residue class.

Theorem 20. The Disjoint Residue Class Problem is strongly NP-complete.

Proof. The vertex cover problem for triangle-free graphs is known to be NP-complete. We will reduce this problem to the Disjoint Residue Class Problem. Let G = (V, E) be a triangle-free graph, and k be an integer. Let n = |V|, and p_1, \ldots, p_n be the smallest n primes greater than n. Label the vertices of G with p_1, \ldots, p_n . For each $e \in E$, assign a label $m_e := kp_sp_t$, where p_s and p_t are the primes assigned to the endpoints of e. We claim that G has a vertex cover of size $\leq k$ if and only if each edge $e \in E$ can be assigned an integer r_e such that (m_e, r_e) forms a disjoint residue class.

Suppose that $S = \{v_1, \dots, v_k\} \subseteq V$ is a vertex cover. It is possible to assign a pair of integers (a_e, b_e) to each $e \in E$ such that:

- One of the endpoints of e is v_{a_e} .
- $-0 \le b_e < n.$
- No two edges are assigned the same pair of integers.

Then, we choose r_e such that $r_e \equiv a_e \pmod{k}$ and $r_e \equiv b_e \pmod{p_{a_e}}$. Let e_1, e_2 be two distinct edges.

- If $a := a_{e_1} = a_{e_2}$, both m_{e_1} and m_{e_2} are multiples of p_a and $r_{e_1} \neq r_{e_2} \pmod{p_a}$.
- If $a_{e_1} \neq a_{e_2}$, both m_{e_1} and m_{e_2} are multiples of k and $r_{e_1} \neq r_{e_2} \pmod{k}$.

Thus, (m_e, r_e) forms a disjoint residue class.

Conversely, let (m_i, r_i) be a disjoint residue class. Suppose that $e_1, e_2 \in E$ doesn't share a vertex. Assume that $r_1 \equiv r_2 \pmod{k}$. Since $gcd(m_{e_1}, m_{e_2}) = k$, by Chinese remainder theorem, there exists an integer x that satisfies $x \equiv r_1 \pmod{m_{e_1}}$ and $x \equiv r_2 \pmod{m_{e_2}}$. This contradicts the fact that (m_i, r_i) is a disjoint residue class. Therefore $r_1 \neq r_2 \pmod{k}$. This means that if we divide E into k disjoint sets E_1, \ldots, E_k by $r_i \pmod{k}$, any two edges in the same subset shares a vertex. Since G is triangle-free, for each E_i , there must exist a vertex v_i such that all edges in E_i contain v_i, v_1, \ldots, v_k is a vertex cover.

We also obtain an NP-complete problem if we specify the set of times at which the point must be visited:

Generalized Point Patrolling Problem. We are given a finite set of times $S \subseteq \mathbb{Z}$ and integers $a_1, \ldots, a_k > 0$. For each $t \in S$, at least one agent must visit the point at time t. If the ith agent visits the point at two distinct times t_1 and t_2 , we must have $|t_1 - t_2| \ge a_i$. Determine whether this is possible.

Theorem 21. Generalized Point Patrolling Problem is NP-complete.

Proof. Numerical 3-dimensional Matching is known to be a NP-complete problem [4]. We will reduce this problem to Generalized Point Patrolling Problem.

Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n), (z_1, \ldots, z_n), b$ be an instance of numerical 3-dimensional matching. We can assume that $x_1 + \cdots + x_n + y_1 + \cdots + y_n + z_1 + \cdots + z_n = nb$ (otherwise this instance is obvious). Let M be a sufficiently

big integer. There exists a numerical 3-dimensional matching if and only if it is possible to patrol when $S = x_1, \ldots, x_n, M - y_1, \ldots, M - y_n$ and $a_i = M - b + z_i$.

Suppose that $(x_1,\ldots,x_n),(y_1,\ldots,y_n),(z_1,\ldots,z_n),b$ has a numerical 3-dimensional matching. Let p,q be permutations of $(1,\ldots,n)$ that satisfies $x_{p_i}+y_{q_i}+z_i=b$ for each i. Then, the i-th agent can visit the point at time x_{p_i} and $M-y_{q_i}$. This is a valid schedule because $M-y_{q_i}-x_{p_i}=M-(b-z_i)=a_i$.

Conversely, suppose that it is possible to patrol when $S = \{x_1, \ldots, x_n, M - y_1, \ldots, M - y_n\}$ and $a_i = M - b + z_i$. When M is sufficiently big, each agent can visit the point at most twice: once in x_1, \ldots, x_n and once in $M - y_1, \ldots, M - y_n$. There are 2n elements in S and there are n agents, so in a valid schedule each agent must visit the point exactly twice. Let x_{p_i} and $M - y_{q_i}$ be two times that are visited by the i-th agent. Since $M - y_{q_i} - x_{p_i} \ge a_i$, we get $x_{p_i} + y_{q_i} + z_i \le b$. By taking the sum of these inequalities for all i and comparing with $x_1 + \cdots + x_n + y_1 + \cdots + y_n + z_1 + \cdots + z_n = nb$, we get $x_{p_i} + y_{q_i} + z_i = b$. This means that $(x_1, \ldots, x_n), (y_1, \ldots, y_n), (z_1, \ldots, z_n), b$ has a numerical 3-dimensional matching.

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