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Patrolling a Border

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Patrolling games were recently introduced to model the problem of protecting the nodes of a network from an attack. Time is discrete and in each time unit the Patroller can stay at the same node or move to an adjacent node. The Attacker chooses when to attack and which node to attack and needs m consecutive time units to carry it out. The Attacker wins if the Patroller does not visit the chosen node while it is being attacked; otherwise, the Patroller wins. This paper studies the patrolling game where the network is a line graph of n nodes, which models the problem of guarding a channel or protecting a border from infiltration. We solve the patrolling game for any values of m and n, providing an optimal Patroller strategy, an optimal Attacker strategy, and the value of the game (optimal probability that the attack is intercepted).

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1. Introduction

This paper applies the theory of patrolling games on networks, as introduced by Alpern et al. (2011), to the classical problems of guarding a channel or patrolling a border. We adopt a discrete model in which an antagonistic Attacker has the choice of *where* to attempt an infiltration: in particular, which node of the line graph L_n (nodes $1, 2, \dots, n$) to attack. He can also choose *when* to attack, i.e., any discrete time interval of m consecutive periods, where m represents the difficulty of infiltration. To thwart the attack or attempted infiltration, the Patroller walks along the line, hoping to be at the attacked node at some time within the attack period. If she thus intercepts the attack, she wins the game; otherwise, the attack is successful and she loses (the Attacker wins). This is a win-lose game, which is a finite zero-sum game with payoff 1 to the maximizing Patroller if she win and otherwise has payoff 0. Clearly, both players must adopt mixed strategies. The solution depends in a delicate way on the two parameters n and m and hence covers an infinite number of cases.

The situation is familiar from the context of national border security as many borders have well defined end points (the United States–Mexican or United States–Canadian borders, the former border between East and West Berlin, the “Greenline” in Cyprus patrolled by United Nations peacekeeping troops). However, it is also familiar in more abstract settings, as in the military operations research problem of “patrolling a channel” introduced in the classic early text of Morse and Kimball (1951) in which an aircraft patrols a channel with a view to intercepting a submarine. Related problems are described in our literature review.

Our discrete model considers a setting for which the border can be penetrated at designated points along its length. A classic example would be checkpoints on the Berlin Wall, but for a more contemporary example consider the problem faced by an airport dog patrol that has to cover a bank of security checkpoints at the entrance to the departure lounge. Consider also the example that when animals cross a river, there are limited locations where the slope down from the higher land is such that river level can be reached. These locations are well known to the Patrollers (predators in that context). A final interpretation is that improvised explosive devices(IEDs) can be placed at certain points along a road where they are hidden from sight—they must be found before they explode.

We view this paper as having a similar role to that of Gal (1979) in the related field of search games for an immobile hider on a network, with our Patroller analogous to his Searcher and our Attacker analogous to his Hider. In that field, early work began in a rough way for general networks, and then went forward to special classes of networks: trees and Eulerian. These were then put together to cover *weakly cyclic* networks and then *weakly Eulerian* networks by Gal (2000) and others. See Alpern (2013) for a discussion of this history. The introduction of patrolling games in Alpern et al. (2011) gave a rough outline of the patrolling game theory for general graphs and started the particular theory by covering cycle graphs. This paper gives a complete solution for the line graph. We plan to extend this work to trees in the first instance and then to more general classes of graphs. But we also believe that the line graph is of interest in itself because it models the classic problem of guarding a border.

The reason that the results in this paper are complex and varied is that the "obvious" solution for patrolling a line—simply going back and forth—is not optimal. It leaves nodes near the ends especially vulnerable to attack. See Figure 3 for an illustration. An anecdote from the second author illustrates this problem: "I was watching my daughter taking a group swimming lesson. The teacher lined up seven children along the short side of the pool and went back and forth, taking a child for a short swim before replacing her back on the poolside. After a while I realized that my daughter (at position 7 on the end) was getting only half the attention of her neighbour, who always got to swim immediately before and after my daughter. However I was unable to convince the swimming teacher that there was a problem with her method." The significance of this anecdote for patrolling the line is not the frequency with which a patrol visits an end but rather the gap between visits, particularly the size of this gap relative to m, the length of an attack.

Game theory plays an important role in the study of security problems; for instance, the work of Pita et al. (2008) created randomized security policies at Los Angeles Airport. Other important applications of game theory to security problems can also be found in Baykal-Gürsoy et al. (2014) and Fokkink and Lindelauf (2013).

2. Literature Review

We divide our literature review into two parts. First we review the literature on preventing an infiltrator from crossing a boundary or perimeter, or guarding a channel, that relates to the line graph on which we concentrate. Then we review the literature on patrolling a general graph or network that has developed since our introduction of patrolling games in Alpern et al. (2011).

2.1. Guarding a Channel or Infiltrating a Border

Since the original work in the classic text of Morse and Kimball (1951), several authors have modeled the problem of guarding a channel or border. Washburn (1982) estimates the detection probability obtainable by a channel Patroller. Baston and Bostock (1987) consider a problem where the Attacker has to cross from the left side to the right side of a rectangle while avoiding static blocks of the Patroller (in our terminology). This work is extended in Baston and Kikuta (2009) to the case where the Attacker has a nonzero width (maybe in our format this might correspond to requiring attacks on adjacent nodes rather than at a single node). Baston and Kikuta (2004) consider the possibility of several attackers. Washburn (2010) considers conditions under which the Attacker can get through observable moving barriers in the case of a line or circle.

Collins et al. (2013) suppose that only some portions of a boundary of a region are important to protect and show how multiple patrollers should optimally patrol individual sections of the search space separately. Zoroa et al. (2012) also consider a guarding a boundary (cycle graph), but against multiple attacks. Chung et al. (2011) consider multiple patrollers of a channel with periodic trajectories. Szechtman et al. (2008) model the problem faced by a moving Patroller (a sensor) on a border trying to detect infiltrators who arrive according to a Poisson process.

None of these papers use our graph patrol model, but they give fairly similar models to ours of the infiltration game on a line or circle.

2.2. Patrolling Games on a Graph

Since our introduction of patrolling games in Alpern et al. (2011), a number of papers have a Patroller on a general graph or network. Lin et al. (2013) have a graph model similar to ours and apply approximate methods for a wider class of problems than ours. They consider targets (nodes) that can have different values. Their algorithms seem to work very well for complex problems, for both random and strategic attacks. This work is further extended in Lin et al. (2014). Basilico et al. (2012, 2015) apply simulation techniques to large scale problems, also obtaining robust algorithms. Hochbaum et al. (2014) use a Stackelberg approach to solve the games where the attacks are nuclear threats on edges of the network, with theoretical results on the k-vehicle rural Chinese Postman Path.

3. Model and Preliminary Results

This section recalls the definition of patrolling games on general graphs and specializes to the case of a line graph. Some general properties of the game for general graphs are stated, and some new ones established.

3.1. Defining the Discrete Patrolling Game

As introduced in Alpern et al. (2011), a patrolling game is based on the following given data: a graph Q with n nodes N and edges E, an attack duration m and a time horizon $\mathcal{T} = \{1, 2, \ldots, T\}$ of length T. The Attacker chooses an attack (i, I) where i is the attacked node and $I \subset \mathcal{T}$ is a subinterval of size m, which implies that $T \geqslant m$. The time duration m represents the time required to carry out an attack, or perhaps if G is a border, the time required cross it. The Patroller chooses a patrol, a walk $w\colon \mathcal{T} \to N$. This means that $w(t)$ and $w(t+1)$ are the same or adjacent nodes. The patrolling game is a win-lose game; the Patroller wins if he successfully intercepts the attack, that is, if $w(t) = i$ for some $t \in I$. The Attacker wins if he is undisturbed while carrying out his attack. In zero-sum notation, the Patroller is the maximizer, with payoff 1 if she wins and payoff 0 otherwise. Thus the value $V = V(Q)$ of the game is the optimal probability that the Patroller intercepts the attack. In this paper we shall solve the patrolling game on the line graph L_n, with nodes $N = \{1, 2, \ldots, n\}$ and consecutive numbers considered adjacent. The two significant parameters will be the size n of the line and the attack duration m—the time horizon T will not be important as the solution will be constant for T sufficiently large, namely, for $T \geqslant 2m$. This is because some optimal attacker strategies take $2m$ periods to complete and none of them need more than $2m$ periods to complete. If $T < m$ the attacker will never succeed because there is not enough time to carry out the attack. For $m \leqslant T < 2m$, then in some cases the attackers optimal strategy becomes unavailable. In these cases the probability of a successful attack is bounded above by the values that we give in this paper since the value of the game is nonincreasing with T (see Alpern et al. 2011, Proposition 3.1).

It is clear that in patrolling games it is not sufficient for the players to adopt pure strategies. However, finite mixtures of pure strategies are sufficient. In some cases, for example in describing mixed attack strategies, it is useful to mention multiple attacks with the understanding that each of these is adopted with a given probability.

3.2. Decomposition Results

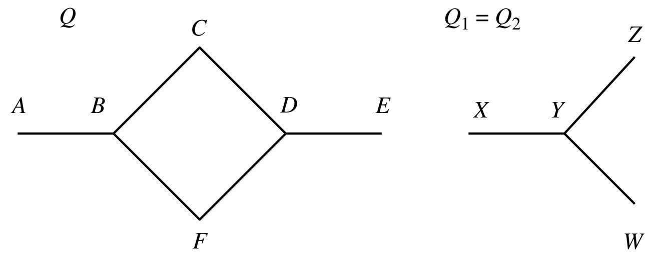
A notion introduced in Alpern et al. (2011) is the decomposition of a graph $Q = (N, E)$ into simpler graphs $Q_j = (N_j, E_j)$ where $\bigcup N_j = N$ and nodes in Q_j are adjacent if they are adjacent in Q. The Patroller has the option of choosing each graph Q_j with some probability and then patrolling optimally on Q_j with some probability p_j. Thus an attack on a node in Q_j will be intercepted with probability at least $p_j V_j$, where V_j is the value of the patrolling game on Q_j. By choosing the p_j to equalize these probabilities, the following result was obtained as Alpern et al. (2011, Lemma 6).

Lemma 1 (Patroller Decomposition Lemma). *Suppose a graph Q is decomposed into graphs Q_j, $j = 1, \ldots, J$. Then the value V of the patrolling game on Q satisfies*

$$V \geqslant \frac{1}{\sum_{j=1}^{J} 1/V_j}.$$

We now introduce a new decomposition result from the point of view of the attacker, which will be needed in later sections. First observe that for any graph Q the value $V(Q)$ can be written as a rational number a/b, where the Attacker has an optimal strategy of equiprobably choosing among b attacks, no more than a of which can be intercepted by a single patrol. To see this, observe that because the patrolling game is a finite game with rational payoffs, it has a rational value and the Attacker can choose among the finite number of attacks such that the probability of choosing attack j is a rational number p_j. Write the rationals $p_1, p_2, \ldots$ as $p_j = a_j/b$ with a common denominator b. In this way the Attacker chooses equiprobably between the b attacks of which a_j attacks are of type j.

Figure 1. Q can be decomposed as $Q_1 \cup Q_2$.



Now suppose that the nodes N of a graph Q are the union of two node sets N_1 and N_2. It turns out that attack strategies on the associated graphs Q_1 and Q_2 can be usefully combined to form an attack strategy on N if they agree on the intersection nodes $N_1 \cap N_2$. This is analogous to the result that two continuous functions form a continuous function on the union of their domains as long as they agree on the intersection. We also need a condition that attacked nodes that are not in the intersection are not too close to the intersection. The precise formulation is as follows.

Lemma 2 (Attacker Decomposition Lemma). *Suppose the graph $Q = (N, E)$ is the union of two connected graphs Q_i, $i = 1, 2$, whose intersection is the set of nodes S. That is, $N = N_1 \cup N_2$, $N_1 \cap N_2 = S$. For $i = 1, 2$, write $V_i = V(Q_i) = a/b_i$ as rationals with the same numerator so that there are optimal attack strategies on Q_i which equiprobably use b_i attacks (some with duplication) of which at most a can be intercepted by a single patrol. Suppose the following two conditions hold:*

1. *The two optimal attack strategies on Q_1 and Q_2 have the same c pure attacks (at the same times, same nodes) on S.*

2. *If a patrol intercepts one of b_1 attacks in Q_1 and one of the b_2 attacks in Q_2, then one of these must be in S.*

Then the value V of the patrolling game on Q satisfies the inequality

$$V \leqslant \frac{a}{b_1 + b_2 - c} = \frac{1}{1/V_1 + 1/V_2 - c/a}.$$

Proof. It is easy to combine the two optimal attack strategies for Q_1 and Q_2 as a feasible attack strategy for Q. Simply make all the b_1 attacks on Q_1 and all the b_2 attacks on Q_2, equiprobably, without duplicating the c attacks on S. This makes $b_1 + b_2 - c$ attacks, by item 1. By item 2, the number of these that can be intercepted by a single patrol is still at most a. □

To illustrate Lemma 2 with an example, consider $Q = Q_1 \cup Q_2$ with $m = 4$ as in Figure 1, with $S = \{C, F\} =$

$\{Z, W\}$. Clearly $V(Q_1) = V(Q_2) = 2/3 = a/b$ because the Attacker can attack equiprobably at the three nodes X, Z, W at the same time, say 1, and no patrol can intercept more than two of them. This gives $c = 2$. The Patroller can equiprobably adopt the periodic patrols *XYZYX*, *ZYWYZ*, *WYXYW*, two of which intercept any attack. It follows from the above result that

$$V(Q) \leqslant \frac{a}{b+b-c} = \frac{2}{3+3-2} = \frac{1}{2}.$$

Note that the Patroller cannot use Lemma 1 to obtain an optimal strategy because that estimate gives only

$$V(Q) \geqslant \frac{1}{1/(2/3)+1/(2/3)} = \frac{1}{3}.$$

In fact, $V(Q) = 1/2$, as can be seen by considering that the Patroller equiprobably adopts the periodic walk *ABCBA* and the three others symmetric to it. An attack at any node is intercepted by two of these four patrols.

3.3. Oscillations and Random Oscillations

Oscillations will play an important role in patrolling the line graph L_n.

Definition 3. An *oscillation* on a subinterval $L_j = \{k+1, \dots, k+j\}$ for any $0 \leqslant k \leqslant n-j$ of the line graph L_n is a walk starting at any node in L_j, going in either direction (unless starting at an endpoint), and turning around whenever reaching an endpoint of L_j (reflecting). Note that there are two oscillations starting at each interior node of L_j and one starting at each end node. In total this makes $2j-2$ oscillations. A *random oscillation* is an equiprobable choice between these $2j-2$ oscillations.

The use of oscillations in patrolling strategies will become clearer later in this section. Note that a random oscillation is a mixed strategy.

There are two general observations about oscillations. The first is that if n is sufficiently small with respect to m, then an oscillation on L_n intercepts all possible attacks on L_n. The second is that when n is large with respect to m, a random oscillation has constant probability of intercepting attacks near the middle of L_n, tailing off at the ends. (This can be seen later in Equation (1) on the calculation of $\omega(i)$, as plotted in Figure 3.)

To analyze the first observation, note that when leaving a node of L_j an oscillation is away for at most $2(j-2)+1 = 2j-3$ periods before returning, with this maximum achieved when the node is an endpoint. The first term $2(j-2)$ counts the periods away at one of the $j-2$ interior points of L_j, and the second term counts the single period the oscillation is away at the opposite endpoint. Thus if $m > 2j-3$, the attack interval is too long to be contained in one of these periods away from a node, and will be intercepted. Thus we have shown the following.

Lemma 4 (Oscillation Lemma). *Suppose that* $j \leqslant (m+2)/2$. *Then*

1. *An oscillation on* L_j *intercepts any attack on* L_j, *and therefore*
2. $V(L_j) = 1$ (*any oscillation is a winning pure strategy*).

Observe that the second part of the lemma shows that we need only consider cases $n > (m+2)/2$ or equivalently $n > \hat{m} = \lfloor (m+2)/2 \rfloor$ (where $\lfloor k \rfloor$ and $\lceil k \rceil$ are respectively the floor and ceiling of an integer k) as the Patroller can surely win otherwise.

To analyze the second observation (on random oscillations), we calculate the probability that a random oscillation on L_n intercepts an attack at node i.

Suppose the attack is at a node i near the middle of L_n as pictured in Figure 2(a) for $i = 5$. The oscillations can be viewed as clockwise movements of period $2\ (n-1)$, starting at a random node. We label nodes on the circle where the oscillation is moving left with a star. Suppose the attack is at node i on the line starting at some time t. Then it will be intercepted by oscillations located at the $2m$ (=6) nodes labeled with x or y at time t. For example the oscillation at location $i-1$ (=4) at time t will intercept the attack at time $t+1$. Thus here the intercepting oscillations are described by two disjoint arcs of size m (=3) and since the sample space has size $2\ (n-1)$ the probability that an attack at $i = 5$ is intercepted is given by $2m/(2(n-1)) = m/(n-1)$. When i is small with respect to m, the two arcs (determined by the nodes labeled x and those labeled y) overlap, as shown in Figure 2(b), where there are 5 starting points which intercept an attack on node 2, which gives value $5/18$ where $18 = 2(10) - 2$ is the total number of oscillations.

Figure 2. (a) An attack at $i = 5$ is intercepted with probability $6/18$, $m = 3$. (b) An attack at $i = 2$ is intercepted with probability $5/18$, $m = 3$.

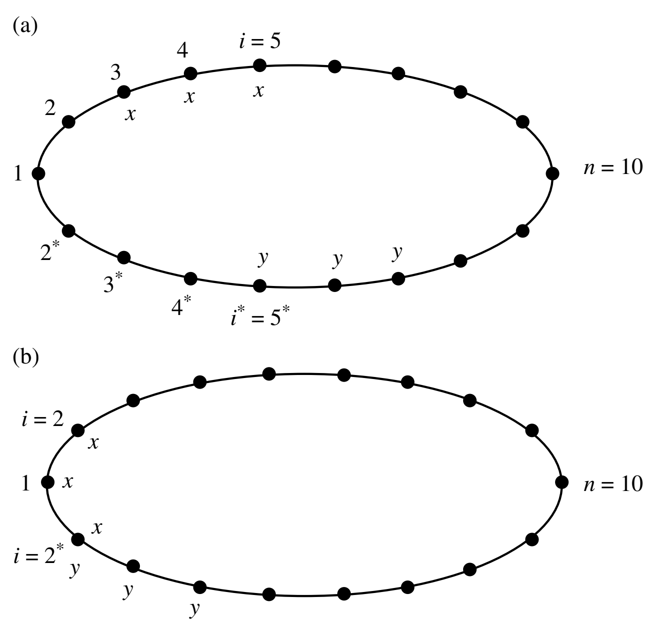
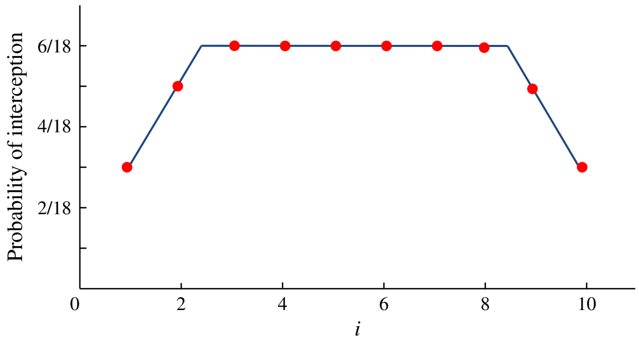


Figure 3. (Color online) Probabilities $\omega(i)$ of intercepting attack at i for $n=10$, $m=3$.



Thus the probability that an attack at node i of L_n is intercepted by a random oscillation on L_n is given by $\omega(i)$, where

$$\omega(i)=\frac{\min(m+2(i-1),2m)}{2(n-1)},\quad \text{for } i\leqslant\frac{n+1}{2},$$
$$\text{with}\quad \omega(i)=\omega(1+(n-i)),\quad \text{for } i>\frac{n+1}{2}. \tag{1}$$

For the case $n=10$ and $m=3$ illustrated in the two previous figures, the interception probabilities $\omega(i)$ are shown in Figure 3.

3.4. Independence and Covering Strategies

We now describe a pair of strategies that give further bounds on the value of the game. A patrol w is called *intercepting* if it intercepts every attack on a node that it visits. We emphasize that whether or not w is intercepting depends on the value of m. For example, Lemma 4 says that if $j\leqslant(m+2)/2$, then any oscillation on L_j is intercepting. A set of intercepting patrols is called a *covering set* if every node of Q is visited by at least one of the patrols. The *covering number* $\mathscr{C}_m$ is the minimum cardinality of any covering set. We use the subscript m to indicate the dependence of the covering number on the length of the attack period. We define the *covering strategy* as an equiprobable choice between the intercepting patrols of a minimum covering set. Note that intercepting patrols may visit overlapping sets. For example when $n=5$ and $m=4$ the oscillations on $\{1,2,3\}$ and on $\{3,4,5\}$ form a covering set. But so do the oscillations on $\{1,2,3\}$ and $\{4,5\}$.

A set of nodes is called an *independent* set if no two attacks at two nodes of the set taking place in the same time interval (simultaneously) can be intercepted by the same patrol. This means that the nodes are at least m edges apart. The *independence number* $\mathscr{I}_m=\mathscr{I}_m(Q)$ is the cardinality of a maximum independent set. We define the Attacker's *independent strategy* as an equiprobable choice of attacks on a maximum independent set during a fixed time interval.

These notions of the independence and covering number are taken from Alpern et al. (2011) and they can be seen to be equivalent to the well-known graph-theoretic definitions of independence and covering numbers for a suitably defined hypergraph.

These strategies give the following bounds, obtained in Alpern et al. (2011, Lemma 12).

Lemma 5 (Covering-Independence Lemma). *The value satisfies*

$$\frac{1}{\mathscr{C}_m}\leqslant V\leqslant\frac{1}{\mathscr{I}_m}, \tag{2}$$

where the upper bound is guaranteed by the Attacker using the independent strategy and the lower bound is guaranteed by the Patroller using the covering strategy.

We also have $\mathscr{I}_m\leqslant\mathscr{C}_m$. This follows from Lemma 5, but it can also be argued using the definition of independent and covering sets: each node in an independent set is covered by at least one patrol from a covering set and at the same time each intercepting patrol in a covering set cannot cover more than one node from an independent set; thus there are no fewer intercepting patrols than there are independent nodes.

3.5. Division of (n,m) Space and Main Results

The aim of this paper is to solve the patrolling game on the line L_n with attack duration m for arbitrary values of n and m. The solution comes in several types, according to a partition of (n,m) space. For $n\leqslant(m+2)/2$ the game is trivial as the Patroller can intercept every possible attack by simply going back and forth between the end nodes (see Lemma 4). Thus we assume throughout that $n\geqslant(m+2)/2$. For $n<m+1$ the solution is also fairly simple, as shown in Theorem 16 Alpern et al. (2011): the optimal strategy for the Attacker is to attack simultaneously at the ends (*diametrical strategy*); the optimal strategy for the Patroller is the random oscillation and the value of the game is given by

$$V(L_n)=\frac{m}{2(n-1)},\quad \text{when } \frac{m+2}{2}\leqslant n<m+1. \tag{3}$$

For the purposes of this paper, we partition the (n,m) space into five regions: regions S_1 and S_2 are described above and we partition the remaining region $n>m+1$ into three further regions that will have separate solution types. (Actually, the previous paper covered the case $n=m+1$ as well, but we shall cover that case differently in this paper.)

$S_1=\{(n,m)\colon n<(m+2)/2\}$,

$S_2=\{(n,m)\colon (m+2)/2\leqslant n<m+1\}$,

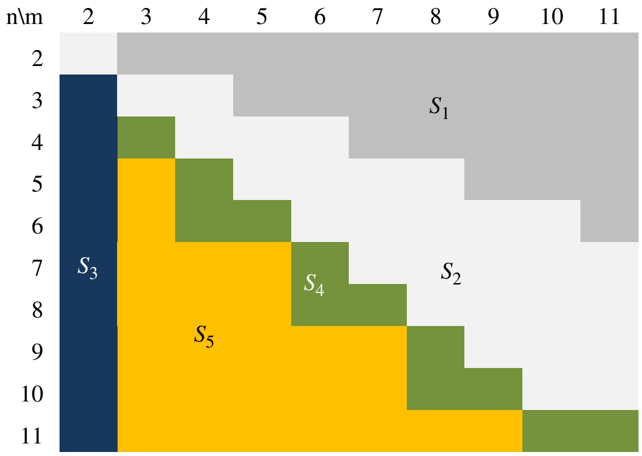
$S_3=\{(n,m)\colon m=2,\ n\geqslant 3\}$,

$S_4=\{(n,m)\colon n=m+1$, or $n=m+2$ and $m\geqslant 3$ is even$\}$,

$S_5=\{(n,m)\colon n\geqslant m+3$, or $n=m+2$ and $m\geqslant 3$ is odd$\}$.

The partition of (n,m) space into these regions is shown in Figure 4.

Figure 4. (Color online) (n, m) space partitioned in sets S_1, S_2, S_3, S_4, and S_5.



The two cases S_3 and S_4 will be solved by similar techniques (a kind of covering-independence argument) in Section 4, summarized as Theorem 10. The remaining case S_5 will be solved in Sections 5 and 6, summarized as Theorem 13, with detailed proofs of the subcases outlined in Section 6 and proved in the online appendix (available as supplemental material at https://doi.org/10.1287/opre.2016.1511).

In the following result two derived parameters that are important are $\rho = (n-1) \bmod m$ and $\bar{V} = m/(n+m-1)$. Further, S_1 is trivial; S_2 has been proved in Alpern et al. (2011); and S_3, S_4, S_5 cover all the results of this paper.

Theorem 6. *The solution to the patrolling game on the line graph L_n for attack duration m, is given in the following five cases.*

$(n, m) \in S_1$: *The value of the game is given by $V = 1$; the oscillation strategy is optimal for the Patroller and all strategies are optimal for the Attacker.*

$(n, m) \in S_2$: *The value of the game is given by $V = m/2(n-1)$; the random oscillation strategy is optimal for the Patroller and equiprobably attacking at the endpoints (diametrical) is optimal for the Attacker.*

$(n, m) \in S_3$: *The value of the game is given by $V = 1/\lceil n/2 \rceil$; the covering strategy is optimal for the Patroller and the independent strategy is optimal for the Attacker.*

$(n, m) \in S_4$: *The value of the game is $V = \frac{1}{2} = 1/\mathcal{I}_m = 1/\mathcal{C}_m$, and the covering and independent strategies are optimal for the players.*

$(n, m) \in S_5$: *The value of the game is $\bar{V} = m/(n+m-1)$, the end-augmented oscillation strategy (as defined later in Definition 11) is optimal for the Patroller, and the optimal Attacker strategy depends on m and the parameter $\rho = (n-1) \bmod m$.*

4. Optimal Patroller and Attacker Strategies on S_3 and S_4

In this section, we solve the patrolling game on L_n for sets S_3 and S_4. The main tool is Lemma 5.

4.1. Independence and Covering Numbers for the Line

For the game played on L_n we can explicitly calculate the independence number and covering number, as defined in Subsection 3.2. This will be useful in the next section when we give optimal strategies for the patroller. We denote the independence number and covering number of L_n by $\mathcal{I}_{m,n}$ and $\mathcal{C}_{m,n}$, respectively.

Lemma 7. *When Q is the line graph L_n, the covering and independence numbers are given by*

$$\mathcal{C}_{m,n} = \left\lceil \frac{n}{\lfloor m/2 \rfloor + 1} \right\rceil \quad \textit{and} \quad \mathcal{I}_{m,n} = \left\lfloor \frac{n+m-1}{m} \right\rfloor. \tag{4}$$

Proof. From the Oscillation Lemma (Lemma 4, part 1), we see that if $n = c\hat{m}$, where $\hat{m} = \lfloor (m+2)/2 \rfloor$, we can cover L_n with c disjoint intercepting patrols, and hence $\mathcal{C}_{m,n} \leqslant c = n/\hat{m}$. This clearly still holds for smaller n, $n \leqslant c\hat{m}$. Since an intercepting patrol cannot cover an interval of size larger than $\hat{m}$, it follows that

$$\mathcal{C}_{m,n} = \left\lceil \frac{n}{\hat{m}} \right\rceil = \left\lceil \frac{n}{\lfloor m/2 \rfloor + 1} \right\rceil.$$

To calculate the independence number, we obtain a maximally independent set by placing i attacks at $1, 1+m, 1+2m, \ldots, 1+(i-1)m$, where

$$1 + (i-1)m \leqslant n \leqslant 1 + im, \quad \text{and hence}$$

$$i = \mathcal{I}_{m,n} = \left\lfloor \frac{n+m-1}{m} \right\rfloor. \quad \square$$

Since we know $\mathcal{I}_{m,n}$ and $\mathcal{C}_{m,n}$, we have bounds on the value V from Lemma 5. A particularly useful and easy application of Lemmas 5 and 7 occurs when the fraction involved in the formula (4) for I_m is an integer, that is, when $n = qm + 1$ for some integer q and so $(n+m-1)/m = (q+1)m/m = q+1$. In this case we have $\bar{V} \equiv m/(n+m-1) = 1/I_m$. For purposes related to the statement of Theorem 13, we write the condition $n = qm+1$ as $\rho \equiv (n-1) \bmod m = 0$.

Corollary 8. *If $\rho \equiv (n-1) \bmod m = 0$, then the independent strategy ensures an expected payoff not exceeding $\bar{V} \equiv m/(n+m-1)$, and hence $V \leqslant \bar{V}$.*

Proof. By Lemma 7 it is sufficient to prove that $1/\mathcal{I}_{m,n} = \bar{V}$. However the hypothesis implies that $(n+m-1)/m$ is an integer, and so $\mathcal{I}_{m,n}$ can be calculated from Lemma 4 as $\mathcal{I}_{m,n} = 1/\lfloor (n+m-1)/m \rfloor = 1/\bar{V}$. $\square$

It turns out that $\bar{V} \equiv m/(n+m-1)$ is the value of the game for $n \geqslant m+3$, and the above result provides the (tight) upper bound.

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Table 1. $\mathscr{C}_{m,n}-\mathscr{I}_{m,n}$

n\m	2	3	4	5	6	7	8	9	10
3	0								
4	0	0							
5	0	1	0						
6	0	1	0	0					
7	0	1	1	1	0				
8	0	1	1	1	0	0			
9	0	2	$\boxed{0}$	1	1	1	0		
10	0	1	1	2	1	1	0	0	
11	0	2	1	1	1	1	1	1	0

4.2. When Are $\mathscr{C}_{m,n}$ and $\mathscr{I}_{m,n}$ Equal?

According to the Lemma 5, the value of the patrolling game on the line is simply $V=1/\mathscr{I}_{m,n}=1/\mathscr{C}_{m,n}$ when $\mathscr{C}_{m,n}=\mathscr{I}_{m,n}$, with the optimal mixed strategies for the Patroller and Attacker being the covering strategy and the independent strategy, respectively. As formula (4) in Lemma 7 gives $\mathscr{C}_{m,n}$ and $\mathscr{I}_{m,n}$ in terms of m and n, it is not difficult to identify all the cases in which the patrolling game on the line can be simply solved in this manner. In particular we find $\mathscr{C}_{m,n}=\mathscr{I}_{m,n}$ on S_3 and S_4 (and in one other case). We begin by calculating Table 1 with entries $\mathscr{C}_{m,n}-\mathscr{I}_{m,n}$ for some small values of n and m, $n\geqslant m+1$.

The 0s in Table 1 obviously correspond to the $\mathscr{C}_{m,n}=\mathscr{I}_{m,n}$ cases. They come in the four types: a diagonal of 0s with $n=m+1$; a diagonal $n=m+2$ with alternating 0s and 1s, the column for $m=2$, and the apparently anomalous 0 at $(9,4)$. We explain these 0s and furthermore show that there are no others, in the following result.

Proposition 9. *Suppose $m\geqslant 2$ and $n\geqslant m+1$. Then the covering number $\mathscr{C}_{m,n}$ and independence number $\mathscr{I}_{m,n}$ are equal in the following four cases:*

(a) $m=2$ $(\mathscr{C}_{m,n}=\mathscr{I}_{m,n}=\lceil n/2\rceil)$,
(b) $n=m+1$ $(\mathscr{C}_{m,n}=\mathscr{I}_{m,n}=2)$,
(c) $n=m+2$ *and m is even* $(\mathscr{C}_{m,n}=\mathscr{I}_{m,n}=2)$,
(d) $n=9$ *and* $m=4$ $(\mathscr{C}_{m,n}=\mathscr{I}_{m,n}=3)$.

Furthermore,

(e) $\mathscr{C}_{m,n}>\mathscr{I}_{m,n}$ *in all other cases.*

Proof. Case (a) follows from the observation that for $m=2$, by Lemma 7, $\mathscr{C}_{m,n}=\lceil n/2\rceil=\lfloor(n+1)/2\rfloor=\mathscr{I}_{m,n}$.

For cases (b) and (c) we have following: If $n=m+1$, then $\mathscr{I}_{m,n}=\lfloor((m+1)+m-1)/m\rfloor=\lfloor 2\rfloor=2$ and if $n=m+2$, we have $\mathscr{I}_{m,n}=\lfloor((m+2)+m-1)/m\rfloor=\lfloor(2m+1)/m\rfloor=\lfloor 2+1/m\rfloor=2$. If m is even, then $\mathscr{C}_{m,n}=\lceil 2n/(m+2)\rceil$. If also $n=m+1$, then $\mathscr{C}_{m,n}=\lceil(2m+2)/(m+2)\rceil=\lceil 2-2/(m+2)\rceil=2$, and if instead $n=m+2$, then $\mathscr{C}_{m,n}=\lceil 2(m+2)/(m+2)\rceil=2$. If m is odd and $n=m+1$, then $\mathscr{C}_{m,n}=\lceil 2(m+1)/(m+1)\rceil=2$.

Case (d) follows from Lemma 7, which implies that $\mathscr{C}_{4,9}=\mathscr{I}_{4,9}=3$.

The proof of result (e) is in the appendix since we don't use that in the rest of the paper. $\square$

Figure 5. Solution for $m=4$, $n=9$.

$$\overleftrightarrow{\underset{x}{1}\ 2\ 3}\quad \overleftrightarrow{4\ \underset{x}{5}\ 6}\quad \overleftrightarrow{7\ 8\ \underset{x}{9}}$$

Note the one additional 0 for $m=4$ and $n=9$, where $\mathscr{C}_{m,n}=\mathscr{I}_{m,n}=3$ and $V=1/3$. Here the independent set is $\{1,5,9\}$, and the covering intervals are $[1,3]$, $[4,6]$, and $[7,9]$, as shown in Figure 5. Thus the attacks are equiprobable on the independent set (denoted by x below) and the Patroller oscillates equiprobably on the three stated intervals.

Combining Proposition 9 with Lemma 5 in cases S_3 and S_4 we get that the value of the game is $1/\mathscr{C}_{m,n}$.

Theorem 10. *For S_3 and S_4 the value of the patrolling game on L_n is $1/\mathscr{C}_{m,n}$, which for S_3 is $\lceil n/2\rceil$ and for S_4 is $1/2$. Furthermore, the optimal strategies are the covering and the independent strategies.*

This establishes the first two cases of our main theorem (Theorem 6).

It is interesting to see how the part of the above result, for the case $n=m+2$ where m is even, can be obtained from the Patroller Decomposition Lemma (Lemma 1). We decompose L_n into two copies of $L_{n/2}$, that is, into L_a and L_b where $a=b=n/2$. Since $n/2=(m+2)/2$, it follows from Lemma 4 that $V(L_a)=1$. Hence the Patroller Decomposition Lemma says that $V(L_n)\geqslant 1/(1/V(L_a)+1/V(L_b))=1/2$.

5. Optimal Patroller Strategy on S_5

This section considers the Patroller's strategy when the parameters (n,m) belong to the set S_5, that is, $n\geqslant m+3$, or $n=m+2$ and $m>2$ is odd. In the first instance, the Patroller might consider simply adopting a random oscillation on the full line. However, as seen in Figure 3, this strategy gives a poor interception probability near the ends. Thus one solution, which turns out to be optimal, is to add to the mixed Patroller strategy random oscillations on intervals of size $\hat{m}=\lfloor(m+2)/2\rfloor$. Since in particular $\hat{m}\leqslant(m+2)/2$, the first part of the Oscillation Lemma (Lemma 4) shows that the oscillation on such intervals intercept any attack in that interval (they are intercepting patrols).

We state the Patroller strategy more precisely below.

Definition 11. We define the *end-augmented oscillation strategy* on L_n as follows. With probability $p=(n-1)/(m+n-1)$ the Patroller adopts the random oscillation on L_n and with probability $q=m/(2(m+n-1))$ adopts any oscillation on the left interval of size $\hat{m} = \lfloor (m+2)/2 \rfloor$ and also with probability q adopts any oscillation on the right interval of size $\hat{m}$. (Note $p + 2q = 1$.) The left interval of size $\hat{m}$ is the interval $\{1, \ldots, \hat{m}\}$ and the right interval is $\{n - \hat{m} + 1, \ldots, n\}$.

Lemma 12. *For $n \geqslant m + 2$, the end-augmented oscillation strategy ensures that the value V satisfies $V \geqslant m/(n + m - 1) \equiv \bar{V}$.*

Proof. We give the proof for even m, where $\hat{m} = (m+2)/2$. Note that the end oscillations intercept every attack in their interval and that the full random oscillation intercepts an attack at node i with probability $\omega(i)$ as defined in (1). Thus an attack at any node $i \leqslant \hat{m}$ or $i > n - \hat{m}$ (in the two end intervals) is intercepted with probability

$$\begin{aligned} q \cdot 1 + p \cdot \omega(i) &\geqslant q \cdot 1 + p \cdot \omega(1) \\ &= \frac{m}{2(m+n-1)} + \frac{n-1}{(m+n-1)} \frac{m}{2(n-1)} \\ &= \bar{V}. \end{aligned}$$

An attack at any middle node $\hat{m} < i \leqslant n - \hat{m}$ is intercepted with probability $\omega(i) = m/(n-1)$ if the Patroller is adopting the full random oscillation on L_n, and thus with probability

$$p \cdot \frac{m}{n-1} = \frac{n-1}{(m+n-1)} \frac{m}{n-1} = \bar{V}.$$

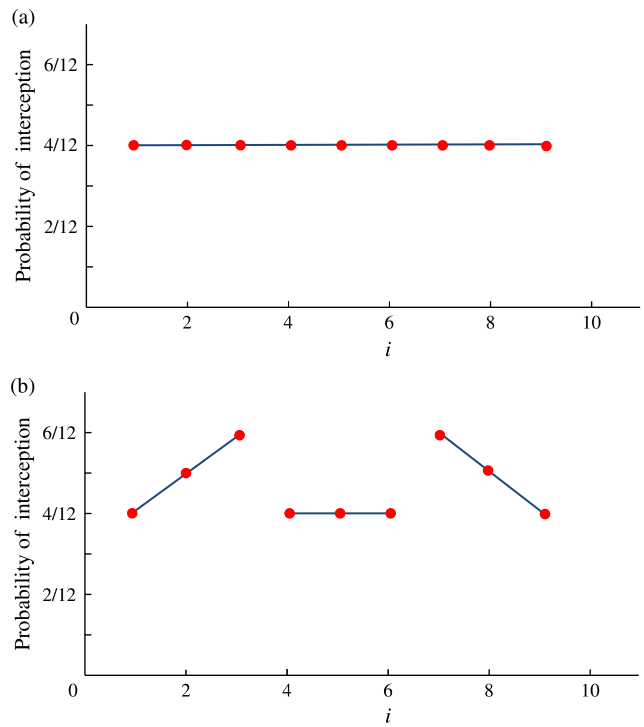
The proof for odd m, where $\hat{m} = (m+1)/2$, is similar. □

It is interesting to compare the end-augmented oscillation strategy in the case $n = 9$ and $m = 4$ to the covering strategy, which we saw was optimal for these parameters in Section 4.2. The end-augmented oscillation strategy chooses random oscillations on L_9 with probability $p = 2/3$ and with probability $q = 1/6$ adopts an oscillation on each of the intervals $[1, 3]$ and $[7, 9]$. In Figure 6 we show the probability the attacks at each of the nodes are intercepted when the Attacker uses each strategy. Even though both strategies are optimal, the end-augmented oscillation strategy weakly dominates as it detects attacks at some nodes with probability strictly higher than $1/3$, whereas the covering strategy detects attacks at all nodes with probability exactly $1/3$.

6. Optimal Attacker Strategies on S_5

This section states the main theorem for the patrolling game on the line for S_5. We consider five types of attack strategies (A, B, C, D, E) and show that one of these always guarantees that $V \leqslant \bar{V} \equiv m/(m+n-1)$. Since we already showed in the previous section that the Patroller can ensure that $V \geqslant \bar{V}$ by adopting the end-augmented oscillation, this will establish that $V = \bar{V}$. We have already shown (Corollary 8) that if $\rho \equiv n - 1 \bmod m = 0$ then $V \leqslant 1/\mathcal{I}_{m,n} = \bar{V}$ and thus the independent attack strategy (A) is optimal. We

Figure 6. (Color online) Probability of interception of an attack at node i for (a) the covering strategy and (b) the end-augmented oscillation strategy ($n = 9$, $m = 4$).



will describe the other four attack strategies (B–E) in the subsections below. Each of these attack strategies is optimal for certain pairs of m and n, as described in the following main result.

Theorem 13. *Consider the patrolling game on the line L_n with attack duration m. If $n \geqslant m + 3 \geqslant 6$, or $n = m + 2$ and m is odd, the value is given by $V = \bar{V} \equiv m/(m + n - 1)$ and the end-augmented oscillation is an optimal patrol. The type of optimal attack strategy is either the independent attack* (A), *the horizontal attack* (B), *the vertical attack* (C), *the zig–zag attack* (*D*), *or the extended zig–zag attack* (E), *depending on the values of m and $\rho \equiv (n-1) \bmod m$ as follows.*

The partition of (m, ρ) space into regions where the various attack types are optimal can be seen in Tables 2 and 3. The pattern of letters is easy to describe. In Figure 3, aside from the top row of A's and the second row of alternating B and E, the remaining grid can be viewed as a chessboard, with the corner $(2, 3)$ a white square. Then all the black

Table 2. The partition of (m, ρ) space depending on optimality of attack strategies.

$m\backslash\rho$	0	1	even, $\geqslant 2$	odd, $\geqslant 3$
Even	A	E	B	D
Odd	A	B	C	B

Table 3. Partition of (m,ρ) space into optimal attack types.

ρ/m	3	4	5	6	7	8	9	10	11
0	A	A	A	A	A	A	A	A	A
1	B	E	B	E	B	E	B	E	B
2	C	B	C	B	C	B	C	B	C
3		D	B	D	B	D	B	D	B
4			C	B	C	B	C	B	C

squares have B and the white squares alternate C and D along diagonals. Clearly the squares with $\rho \geqslant m$ are empty because ρ is a number modulo m.

6.1. Outline of Proof of Theorem 13

We already know from Lemma 12 that $V \geqslant \bar{V} = m/(n+m-1)$, which means that the Patroller can guarantee winning with probability $\bar{V}$. Next we need to show that in all cases there is an Attacker mixed strategy that intercepts any attack with probability at least $\bar{V}$.

The proof of Theorem 13 will be carried out in five parts, corresponding to each of the cases in Table 3. As we will see below, the case A with $\rho=0$ is easily proved. For each pair n, m with $n > m+2$ and $\rho \geqslant 1$ we decompose the line L_n into two lines $Q_1 = L_r$ (with nodes $1,\ldots,r$) and $Q_2 = L_{qm+1}$ (with nodes $r,\ldots,n$) that overlap at the single node r. Thus $n = qm+r$ and the choice of r determines q. For cases B, C, and D, we take $r = m+\rho+1$, and for case E, we take $r = 2m+\rho+1$. In each case we explicitly define a family of $b_1 = r+m-1$ attacks on $Q_1 = L_r$, of which at most m can be intercepted by a single patrol (walk) and such that there are exactly m attacks at node r, either (i) one each at times $1,2,\ldots,m$ or (ii) two each at times $2,\ldots,m$ (this requires m is even). We then use Lemma 15 to cover the graph $Q_2 = L_{qm+1}$ (where q is defined by the equation $n = qm+r$) with $b_2 = (q+1)m$ attacks, of which at most m can be intercepted by any patrol and such that the attacks at the end node r are of the same type A(i) or A(ii) as the $c = m$ attacks on $Q_1 = L_r$. As in the proof of Lemma 2, with $a = m$, this gives a total of

$$b_1 + b_2 - m = (r+m-1) + (q+1)m - m = n+m-1$$

attacks on L_n, of which at most m can be intercepted by a single patrol. Or to use the inequality in Lemma 2, we have

$$\begin{aligned} V(L_n) &\leqslant \frac{1}{1/V(L_r) + 1/V(L_{qm+1}) - m/m} \\ &= \frac{1}{(r+m-1)/m + (q+1)m/m - m/m} \\ &= \frac{m}{(r+m-1)+(q+1)m-m} \\ &= \frac{m}{n+m-1}. \end{aligned}$$

The significance of this Attacker decomposition is that we only have to describe the optimal attack strategy on the short line L_r rather than on the full line L_n.

Thus the proof of Theorem 13 reduces to Lemmas 15, 19, 22, 25, and 28, which are dealt with in separate subsections below.

6.2. Case A: Refining the Independent Attack Strategy

For the case $\rho = 0$, that is, $n = qm+1$, we have already shown in Corollary 8 that the independent strategy is optimal, and so in this case $V \leqslant 1/\mathcal{I}_{m,n} = \bar{V}$. However, for later purposes it is useful and maybe necessary to have two additional optimal strategies, which we will call A(i) and A(ii).

Definition 14 (Attack Strategies A(i) and A(ii)). The attack strategies A(i) and A(ii) on L_n, $n = qm+1$ (where $\rho = 0$) each place m attacks at each node $km+1$, $k = 0,1,2,\ldots,q$, so $(q+1)m$ attacks in all, equiprobably. Strategy A(i) starts the m attacks at each node at times $1,2,\ldots,m$. Strategy A(ii) is only defined when m is even and starts the m attacks with two each at times $2,4,\ldots,m$.

Lemma 15. *Assume $n = qm+1$, or equivalently $\rho = 0$. Of the $(q+1)m$ attacks in the attack strategy A(i), at most m can be intercepted by a single patrol. If m is even, the same statement holds for the attack strategy A(ii).*

Proof. The claim of the lemma is easy to verify directly because if a patrol is last at an attacked node $km+1$ at time t (intercepting at most t attacks there), then the earliest it can arrive at an "adjacent" attacked node $(k\pm 1)m+1$ is at time $t+m$. In this case it can intercept attacks there starting after time t, so at most $m-t$ such attacks. Thus in total it can intercept at most $t+(m-t) = m$ attacks. ☐

The lemma can also be established by induction on q, using the Attacker Decomposition Lemma (Lemma 2).

The importance of these two attack strategies on L_{qm+1} is that all of the later attack strategies B, C, D, E attack one of the end nodes of L_r in the same manner as either A(i) or A(ii) and consequently the overlapping node of the decomposition of L_n into L_{qm+1} and L_r, $qm+r = n$, satisfies the condition on S in the Attacker Decomposition Lemma.

Example 16 ($n = 13$, $m = 4$). In Figure 7 we illustrate the attack strategies A(i) and A(ii) on L_{13}, where $13 = mq+1$ with $m = 4$ and $q = 3$. In each case, we show with a dotted line a patrol that clearly intercepts only four of the attacks.

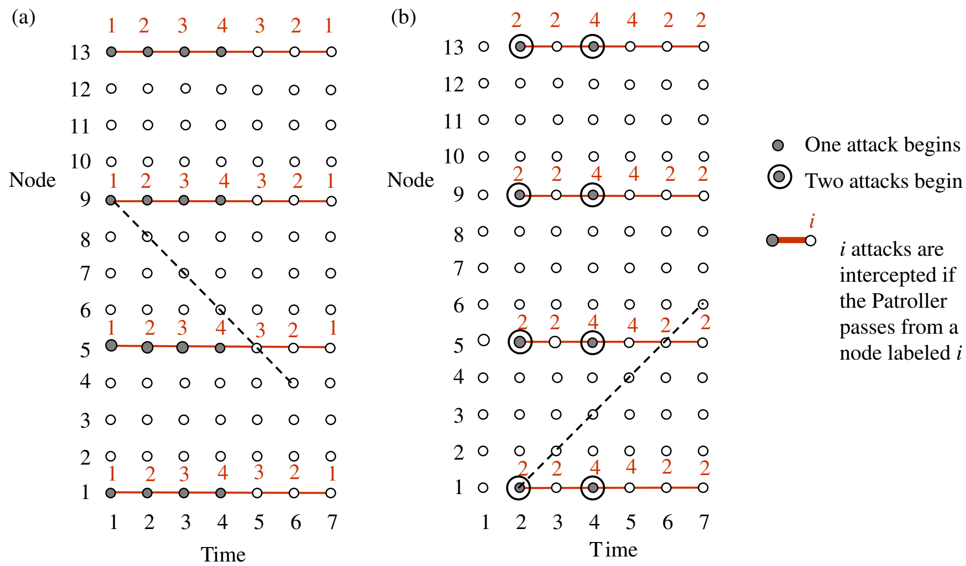
The attack strategy A(i) will be used for cases B and C and the attack strategy A(ii) for cases D and E.

6.3. Case B of $m+\rho$ Even, $\rho > 0$: Horizontal Attack Strategies

We first consider the horizontal strategy (B), which is optimal when both m and ρ are positive and even or when they are both positive and odd. In this case we define r by

$$r = m+\rho+1, \quad \text{which is odd and denote by,}$$
$$k = (1+r)/2, \quad \text{the middle node of } L_r.$$

Figure 7. (Color online) Attack strategies (a) A(i) and (b) A(ii) on L_n for $n = 13$, $m = 4$.



Note that L_r has even length $2d = r - 1$, where

$$d = (m + \rho)/2.$$

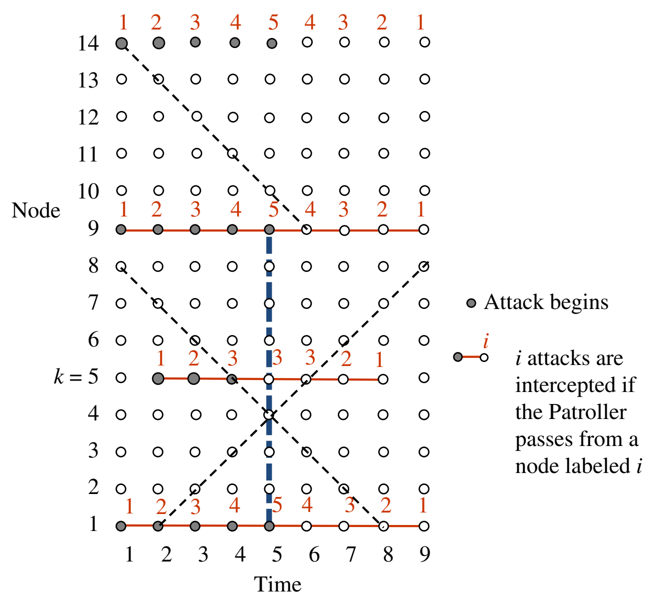
In this case the optimal attack strategy on L_r is as follows.

Definition 17 (Horizontal Strategy). The horizontal attack strategy on an odd size interval L_r consists of $2m + \rho = r + m - 1$ attacks. Of these, m start at each end (nodes 1 and r) at times $1, 2, \ldots, m$. Also there are ρ attacks at the middle node, node k, at the middle ρ of the times $1, \ldots, m$. These times go from time $M - \delta$ to time $M + \delta$, where $M = (1 + m)/2$ and $\delta = (\rho - 1)/2$.

Note that the horizontal strategy has a time symmetry property, in that for any given node the number of attacks that are taking place at a time t is equal to the number of attacks taking place at time $2m - t$ (note that we do not mean the number of attacks that *start* at time t). In other words there is symmetry around time $t = m$. In fact, all the attack strategies in cases B, C, D, and E have a similar time symmetry property, and we will exploit this to simplify the analysis.

Example 18 ($n = 14$, $m = 5$, $r = 9$). We note that $\rho = 13 \bmod 5 = 3$ and hence $r = 5 + 3 + 1 = 9$, as shown in Figure 8. In Figure 8 the top six nodes $9 - 14$ represent $L_6 = L_{m+1}$ with attack strategy A(ii). Using the decomposition result, we take $q = 1$. The middle node of L_r is $k = (1 + 9)/2 = 5$. Thus the attacks on L_r for the horizontal strategy are at the ends, nodes 1 and 9 and at the middle node $k = 5$. At the ends, the start times of the attacks (indicated by solid circles in the space-time diagram) are $1, 2, 3, 4, 5 = m$. The middle of these times is $M = 3$ and the "radius" $\delta = (\rho - 1)/2 = 1$. Hence the middle attacks go from the integers $M - \delta = 2$ to $M + \delta = 4$. In all there are $r + m - 1 = 13$ attacks. From Figure 8, it is easy to see that no more than

Figure 8. (Color online) Horizontal strategy on L_r for $n = 14$, $m = 5$, $r = 9$.



$m = 5$ of these can be intercepted by a single patrol. Note that an attack whose start is indicated by a solid circle will be intercepted by a walk that reaches that height (node) at a time no more than $m - 1 = 4$ periods after the start of the attack. The number of attacks intercepted by a walk arriving at an attacked node $(1, 5, 9)$ is indicated by an integer on top of the node. For example the patrol (walk) staying at node 1 till time 2 and then moving to the right in each period (shown by a dashed line in the figure) intercepts two attacks at node 1, three attacks at middle node 5 and no attacks at node 9. By our decomposition result, we only need to show that no patrol can intercept more than

$m = 5$ of the attacks on L_r. Note also that the time symmetry property holds: the number of attacks taking place at a node x at time t is equal to the number of attacks at x taking place at time $2m - t = 10 - t$. Because of this vertical line of symmetry shown in Figure 8 at $t = m = 5$, the second dashed line in the figure starting at node 8 and going to node 5 and then node 1 intercepts the same number of attacks as the first dashed line.

The importance of the horizontal strategy lies in the following.

Lemma 19. *Consider the horizontal strategy on L_r, $r = m+\rho+1$ for Case B: $m+\rho$ even. Of the $2m+\rho = r+m-1$ attacks, at most m can be intercepted by any walk w of the Patroller.*

The proof is in the appendix.

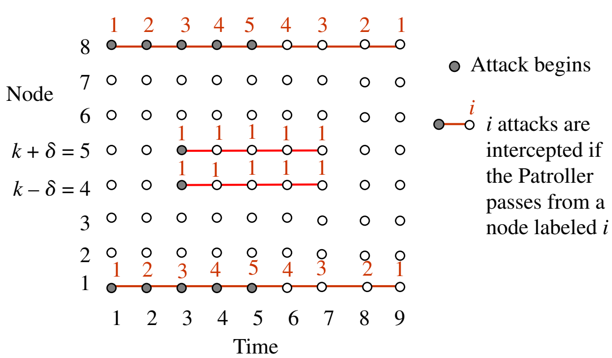
6.4. Case C of m Odd, ρ Even: Vertical Attack

We now consider the case of m odd and ρ even, where we define $r = m+\rho+1$ and $k = (1+r)/2$ as in the previous subsection. Observe that now r is even and k lies between two integers. In this case the following attack strategy is optimal.

Definition 20 (The Vertical Attack Strategy). For m odd and ρ even, the vertical attack strategy consists of $r + m - 1 = 2m + \rho$ attacks. Of these, m start at each end (nodes 1 and r) at times $1, 2, \ldots, m$. Also there are ρ attacks at the middle time $M = (1+m)/2$, one each at the middle ρ nodes of L_r. These middle nodes go from node $k - \delta$ to node $k + \delta$, where $\delta = (\rho - 1)/2$.

Example 21 ($n = 13$, $m = 5$). We illustrate the vertical strategy in Figure 9 for the case $n = 13$ and $m = 5$. We have $\rho = 12 \bmod 5 = 2$ and we take $r = m+\rho+1 = 8$ (giving $q = 1$ again). Here the middle of the nodes of $L_r = L_8$ are at $k = 4.5$ and the middle of the time $1, \ldots, m = 5$ is the integer time $M = 3$. The attacks at the ends 1 and 8 of L_8 start at times 1 to $m = 5$ and the attacks at the middle nodes start at time $M = 3$. We have $\delta = (\rho - 1)/2 = 1/2$, so the attacked middle nodes go from $k - \delta = 4$ to $k + \delta = 5$.

Figure 9. (Color online) Vertical strategy on L_r for $n = 13$, $m = 5$, $r = 8$.



As in the previous subsection, the importance of this strategy is shown in the following lemma.

Lemma 22. *Consider the vertical strategy on L_r, $r = m + \rho + 1$ for Case C: ρ even and m odd. Of the $2m + \rho = r + m - 1$ attacks, at most m can be intercepted by any walk w of the Patroller.*

The proof is in the appendix.

6.5. Case D of m Even, $\rho > 1$ Odd: Zig–Zag Attack

In the case D of m even and ρ odd, we write $\rho = 2s + 1$ and take

$$r = m + \rho + 1 = 2k, \quad \text{as } r \text{ is even.} \tag{5}$$

Thus the two middle nodes of L_r are those labeled k and $k + 1$. The middle ρ periods of the time interval $M = \{1, \ldots, m + 1\}$ can be labelled as $t_1, t_2, \ldots, t_\rho$, centered so that $(t_1 + t_\rho)/2 = (1 + (m + 1))/2$, given by

$$t_i = \frac{m - \rho + 1}{2} + i, \quad i = 1, \ldots, \rho. \tag{6}$$

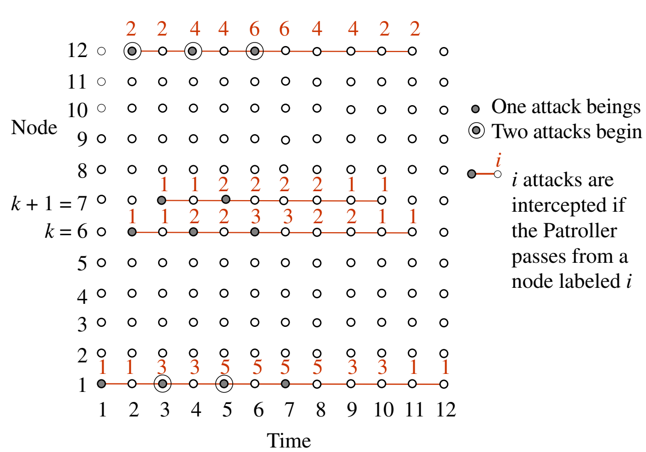
Definition 23 (The Zig–Zag Attack Strategy). For m even, ρ odd and $r = m + \rho + 1 = 2k$, the zig–zag strategy places $r + m - 1 = 2m + \rho$ equiprobable attacks on L_r as follows.

Bottom: We place m attacks on the bottom node 1 one each starting at times 1 and $m + 1$, and two each starting at odd times $3, 5, \ldots, m - 1$.

Top: We place m attacks at the top node r, two each starting at even times $2, 4, \ldots, m$.

Middle: We place ρ attacks at the middle nodes k and $k+1$, starting at the middle times t_i, $i = 1, \ldots, \rho$, given by (6). The attacks at k start at the $s + 1$ odd indexed times t_i, $i = 1, 3, \ldots, \rho$; those at $k + 1$ start at the s even indexed times t_i, $i = 2, 4, \ldots, \rho - 1$.

Figure 10. (Color online) Zig–zag strategy on L_r for $n = 18$, $m = 6$, $r = 12$.



Example 24 ($n = 18$, $m = 6$). We illustrate the zig–zag strategy in Figure 10 for the case $n = 18$ and $m = 6$. We have $\rho = 17 \bmod 6 = 5$ and we take $r = m + \rho + 1 = 12$ (once again giving $q = 1$). The two middle nodes of $L_r = L_{12}$ are at $k = 6$ and $k + 1 = 7$ and the middle time periods are given by $t_1 = 2, t_2 = 3, \ldots, t_5 = 6$. The bottom attacks comprise one starting at times 1 and $m + 1 = 7$ and two starting at the odd times, 3 and 5 between 1 and 7. The top attacks comprise two starting at each of the even times between 2 and $m = 6$. The middle attacks comprise one attack starting at each of the $s + 1 = 3$ odd indexed times t_1, t_3, t_5 at node $k = 6$ and one attack starting at each of the $s = 2$ even indexed times t_2 and t_4 at node $k + 1 = 7$.

The importance of the zig–zag attack lies in the following.

Lemma 25. *Consider the zig–zag strategy on L_r, $r = m + \rho + 1$ for Case B: m even and ρ odd. Of the $r + m - 1$ attacks, at most m can be intercepted by any walk w of the Patroller.*

The proof is in the appendix.

6.6. Case E of m Even, $\rho = 1$: The Extended Zig–Zag Strategy

Finally, we illustrate the case E where $m = 2a$ is even and $\rho = n - 1 \bmod m = 1$. We take $r = 2m + 2$, which differs from the common r value used in the previous three cases. In this case the extended zig–zag attack strategy is optimal. This strategy uses the same $2m$ end attacks as the zig–zag strategy (case D). The middle attacks are on the middle m nodes of L_r defined by

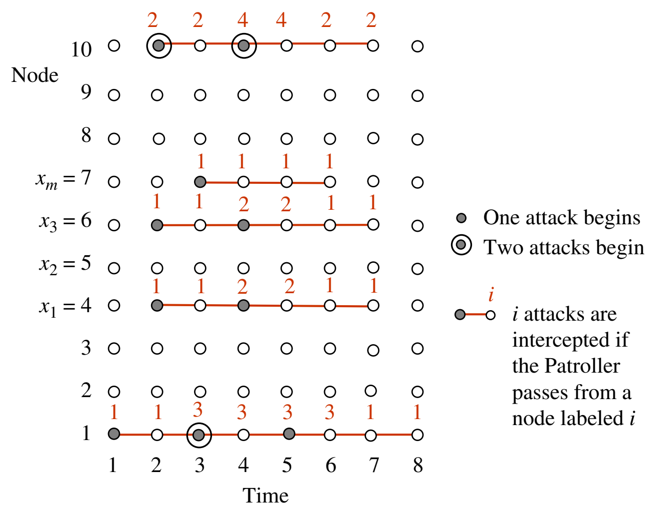
$$x_i = a + 1 + i, \quad i = 1, \ldots, m.$$

For example, the middle 4 nodes of L_{10} with $m = 4$ are $4, 5, 6, 7$.

Definition 26 (Extended Zig-Zag). When $m = 2a$ is even and $\rho = 1$ (Case E), we define the extended zig–zag strategy on L_r, $r = 2m+2$ as follows. We place $r+m-1 = 2m + (m + 1)$ equiprobable attacks, m at each of the ends and $m+1$ in the middle m nodes of L_r. The $2m$ end attacks are placed in the same way as for the zig–zag attack (two each at even times $2, 4, \ldots, m$ at node r; two each at odd times $3, \ldots, m - 1$ and one each at times 1 and $m + 1$ at node 1). The $m + 1 = 2a + 1$ middle attacks are as follows. One attack is at time $a + 1$ at node x_m (the top middle node) and the others are at the nodes $x_1, x_3, \ldots, x_{m-1}$, one each at times a and $a + 2$.

Example 27. Let $n = r = 10$ and $m = 4$, so $\rho = 1$. Figure 11 illustrates the $m + n - 1 = 13$ equiprobable attacks, of which no more than $m = 4$ can be intercepted by any walk. The middle attacks are at times $a, a + 1, a + 2$, that is, times $2, 3, 4$. The middle nodes are those from 4 to 7, with two attacks at nodes $x_1 = 4$ and $x_3 = 6$ and one attack at node $x_4 = 7$.

Figure 11. (Color online) Extended zig–zag strategy on L_r for $n = 10$, $m = 4$, $r = 10$.



Lemma 28. *Consider the extended zig–zag strategy on L_r, $r = 2m + 2$ for Case E: m even and $\rho = 1$. Of the $r + m - 1 = 3m + 1$ attacks, at most m can be intercepted by any walk w of the Patroller.*

The proof is in the appendix.

7. Redundant Edges and Perfect Decomposition

In the patrolling literature, for example in Collins et al. (2013), an important qualitative question is whether the network under attack can be defended with nonoverlapping patrols, either by separate patrollers (when there are several) or when the patrols are mixed probabilistically. To deal with this question, we first introduce a new concept in patrolling games on a network $Q = (N, E)$. We say that an edge $e \in E$ of Q is *redundant* if the value of the patrolling game on the graph $Q - e = (N, E - e)$ is the same as that of the original network Q. The Patroller can avoid traversing a redundant edge without reducing her chances of intercepting the attack. In general, removing edges cannot help the Patroller because it reduces her pure strategies, but it can hurt her and reduce the value. Similarly, we say that Q can be *perfectly decomposed* into Q_1 and Q_2 if the inequality in the Patroller Decomposition Lemma (Lemma 1) holds with equality. That is, if

$$V(Q) = \frac{1}{1/V(Q_1) + 1/V(Q_2)}. \tag{7}$$

For the line graphs the decomposition Equation (7) becomes

$$V(L_{a+b}) = \frac{1}{1/V(L_a) + 1/V(L_b)}. \tag{8}$$

Clearly if Q is the line graph L_n and an edge $e = (i, i + 1)$ is redundant, then Equation (8) holds for $a = i$ and $b = n - i$. Note that if L_n is a case of $\mathscr{C}_{m,n} = \mathscr{I}_{m,n}$ for some m, as in Proposition 9 (for example $(n, m) \in S_3 \cup S_4$), then L_n can be perfectly decomposed, possibly in several ways. For example, if $m=4$ and $n=9$ (where $\mathscr{C}_{m,n} = \mathscr{I}_{m,n} = 3$) we can take $a=3$ and $b=6$ and decompose $\{1, 2, \ldots, 9\}$ into $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8, 9\}$. It turns out that there are some classes of parameters $(n, m) \in S_5$ for which $\mathscr{C}_{m,n} \neq \mathscr{I}_{m,n}$ but nevertheless the Patroller can perfectly decomposed the line L_n and patrol each part separately. In this section we completely determine those parameters (n, m).

For example when $m=7$ and $n=m+2=9$, we can perfectly decompose L_9 into L_4 and L_5. We have $V(L_4)=1$, $V(L_5) = 7/(2\cdot 4) = 7/8$ and (8) is satisfied because

$$V(L_9) = \bar{V} = \frac{7}{9+7-1} = \frac{7}{15} = \frac{1}{1+8/7} = \frac{1}{1/V(L_4)+1/V(L_5)}.$$

This is a particular case of the class $n=m+2$, m odd. Thus the two "middle edges," those between nodes 4 and 5 and between nodes 5 and 6, are each redundant. Of course we cannot simultaneously remove both edges without changing the value.

Next consider the same attack duration $m=7$ but now $n=10$. Here we take $a=b=5$, with the same calculation $V(L_a) = 7/8$ and so

$$V(L_{5+5}) = \bar{V} = \frac{7}{10+7-1} = \frac{7}{16} = \frac{1}{8/7+8/7} = \frac{1}{1/V(L_5)+1/V(L_5)}.$$

This is a particular case of the other class, $n=m+3$. In this case the unique middle edge is redundant, the one between 5 and 6.

Figure 12(a) and 12(b) show the value function $V(L_n)$ (top curve) and $\hat{V}(L_n)$ (bottom curve), the best that can be obtained by a nontrivial decomposition of L_n,

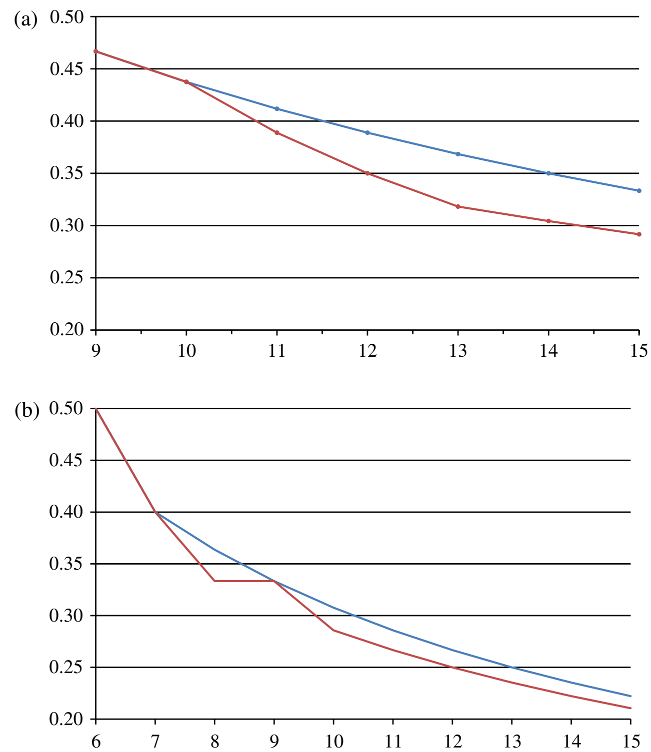
$$\hat{V}(L_n) \equiv \max_{1<a<n} \frac{1}{1/V(L_a)+1/V(L_{n-a})}.$$

The plot on the left considers the case $m=7$ and n from 9 to 15. When n is 9 the lines coincide because $n=m+2$ and m is odd. When n is 10, they again coincide because $n=m+3$. For larger n, decomposing the line is suboptimal. The case of $m=4$ is plotted on the right, with $V(L_n)$ on top and $\hat{V}(L_n)$ below. The curves (defined only for integers) can be seen to intersect at $n=m+3=7$ and for the anomalous case $n=9$ where $\mathscr{C}_{m,n} = \mathscr{I}_{m,n} = 3$.

It turns out that the cases we have just considered, $n = m+2$ when m is odd, and $n=m+3$ without any restriction, together with the anomalous case $n=9$, $m=4$, are exhaustive.

Theorem 29. *For $(n, m) \in S_5$ there are three cases where it can be optimal for the Patroller to decompose the line L_n into two lines L_a and L_b, $a+b=n$, $a \leqslant b$. Otherwise, such decomposition is always suboptimal.*

Figure 12. (Color online) Plots of $V(L_n)$ and $\hat{V}(L_n)$, where $V \geqslant \hat{V}$, (a) $m=1$, $n=9$ to 15; (b) $m=4$, $n=6$ to 15.



1. $n=m+3$: *In this case the perfect decomposition is into two equal sized sets, $a=b=(m+3)/2$ when m is odd or $a=m/2+1$ and $b=m/2+2$ if m is even. If n is even, the middle edge is redundant; if n is odd, the two middle edges are redundant.*

2. *m is odd and $n=m+2$: In this case $a=(m+1)/2$ and $b=(m+3)/2$. The two middle edges are redundant.*

3. *$m=4$ and $n=9$: This is an instance of $\mathscr{C}_{m,n} = \mathscr{I}_{m,n}$. In this case $a=3$ and $b=6$. (Note that here L_6 can be further partioned into two copies of L_3.)*

The proof is in the appendix.

8. Discussion and Conclusions

We see our work as an important building block to building general theory of patrolling games. From an application point of view, one might argue that there are two canonical cases: patrolling a perimeter and patrolling a line, i.e., a border with endpoints. Alpern et al. (2011) provide a complete solution to the first problem, and in this paper we provide a complete solution to the second. As can be seen, the problem poses a surprising level of complexity, and the strategies involved are subtle and sometimes counterintuitive. However, we consider that the work of this paper provides the groundwork necessary to embark on a solution framework for patrolling tree graphs and subsequently for more general graphs consisting of tree-like structures that link up multiple embedded Hamiltonian graphs (this might model the situation of a security guard who has to patrol multiple buildings on a site). We are currently unsure how far analytic results will be possible and how far optimal strategies will have to be computed by black box algorithms. Either way, the results of the current paper can be expected to provide a critical piece of the puzzle. Other interesting avenues for exploration are as follows:

- there could be several patrollers, who may have responsibility for different (perhaps overlapping) sections of the border;
- there may be multiple attackers, who may or may not be able to coordinate their attacks;
- there could be variable (rather than constant) distance between crossing points, or crossing times could be different at different points along the border;
- attackers may be partially detectable (that is, when they start to cross the guard becomes aware and can move towards them);
- patrollers might be restricted to starting at certain locations, known to the attacker; and
- we could consider the problem where the utility to the attacker of a successful attack at node i depends on i.

We leave these as challenges which may be of interest to future researchers in this area.

Supplemental Material

Supplemental material to this paper is available at https://doi.org/10.1287/opre.2016.1511.

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