Continuous Patrolling and Hiding Games

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Abstract

We present two zero-sum games modeling situations where one player attacks (or hides in) a finite dimensional nonempty compact set, and the other tries to prevent the attack (or find him). The first game, called patrolling game, corresponds to a dynamic formulation of this situation in the sense that the attacker chooses a time and a point to attack and the patroller chooses a trajectory to maximize the probability of finding the attack point in a given time. Whereas the second game, called hiding game, corresponds to a static formulation since both the searcher and the hider choose simultaneously a point and the searcher maximizes the probability of being at distance less than a given threshold of the hider.

1 Introduction

Patrolling games were introduced by Alpern et al. [4] to deal with the security of vulnerable facilities. Examples of situations given in [4] include security guards patrolling a museum or an art gallery and antiterrorist officers patrolling an airport or shopping mall. The authors consider the case of patrolling a graph in discrete time. That is, an attacker chooses a node and a (discrete) time to attack, and a patroller walks from one node to an other (also at discrete times). After the beginning of the attack, the patroller has a fixed time m to visit the attacked node, otherwise he loses. A companion article [3], is dedicated to the complete resolution of patrolling a discrete line.

As suggested in [4], the present article considers a continuous time formulation of patrolling games: the attacker chooses a point in a search space, which is a nonempty compact subset of $\mathbb{R}^n$, and a time. The patroller walks continuously with speed less than 1 and wins if he detects the attack before it succeeds, otherwise he loses.

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In the present article, existence of the value of such games is established. A formulation of the value for a special class of search spaces, namely, Eulerian networks, is given, as well as an asymptotic result, when m goes to 0, for the value of a large class of search spaces in $\mathbb{R}^2$.

In a second part, hiding games are studied. In a hiding game, the searcher and the hider choose a point in a search space. The searcher wins if the distance between the two players is less than some detection radius, and loses otherwise.

Two links between patrolling games and hiding games are presented. We show the existence of the value in hiding games, and give an asymptotic formulation of the value, when the detection radius goes to 0, for a large class of search spaces as well as an example proving that this result cannot be extended.

Such games were introduced by Ruckle [13], through examples (hiding in a sphere, hiding in a disc, among others). Danskin [6] improved substantially the resolution of the game on a disc, he called the cookie-cutter game, however the solution is not complete and no progress have been made since then, see [1, 16]. Hiding games have also been considered on graphs [5], producing upper and lower bounds on the value, as well as optimal strategies for particular classes of graphs. [5] also provides a review about games dealing with search and security aspects.

Finally, let us mention that games on a convex closed bounded set where the payoff is the distance between the two players, introduced by Karlin [10] have been extensively studied. The value of such games is known in a rather general setting, consult [9]. Though these games resemble to hiding games to a certain extant, the lack of continuity of the payoff in hiding games makes their analysis much more involved.

In all the article, $\mathbb{R}^n$ is endowed with a norm denoted $\|\cdot\|$, which induces a metric d. For all $x \in \mathbb{R}^n$ and $r > 0$, the closed ball of center x with radius r is denoted $B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$. For all Lebesgue measurable set $B \subset \mathbb{R}^n$, $\lambda(B)$ denotes the Lebesgue measure on $\mathbb{R}^n$ of the set B. Finally, $\lambda(B_r)$ denotes the Lebesgue measure of any ball of radius r.

Let X be a topological space, the set of Borel probability measures on X is denoted $\Delta(X)$, and the set of probability measures on Y with finite support is denoted $\Delta_f(Y)$.

2 Patrolling games

2.1 The model

The patrolling game is a triplet (Q, m, r) where the search space Q is a nonempty compact set included in $\mathbb{R}^n$. $m \geq 0$ is the attack duration, and $r \geq 0$ is the radius of detection. The attacker's set of pure strategies is

$\mathcal{A} = Q \times \mathbb{R}_+$. An element of $\mathcal{A}$ is called an attack. The patroller's set of pure strategies is $\mathcal{W} = \{w : \mathbb{R}_+ \to Q \mid w \text{ is } 1 - \text{Lipschitz continuous}\}$. An element of $\mathcal{W}$ is called a walk. The payoff to the patroller is given by

$$g_{m,r}(w,(y,t)) = \begin{cases} 1 & \text{if } d(y, w([t,t+m])) \leq r \\ 0 & \text{else.} \end{cases}$$

2.2 The value of patrolling games

In this section we make sure that the value of a patrolling game exists.

First, let us define a metric on the set $\mathcal{W}$. For $n \in \mathbb{N}$ define $K_n = [0, n]$.

$$\begin{array}{rcccc} D & : & \mathcal{W} \times \mathcal{W} & \to & \mathbb{R}_+ \\ & & (f,g) & \mapsto & \sum_{n=1}^{\infty} \frac{1}{2^n} \sup\limits_{x \in K_n} \|f(x) - g(x)\|. \end{array}$$

is a metric on $\mathcal{W}$.

The topology induced by the metric D on $\mathcal{W}$ is called the topology of compact convergence. Let us recall the following fact about this topology, see [12].

Proposition 1 (Application of theorem 46.2 in [12])**.** *Let Q be a search space. A sequence $f_n : \mathbb{R}_+ \to Q$ of functions converges to the function f in the topology of compact convergence if and only if for each compact subspace K of $\mathbb{R}_+$, the sequence $f_n|_K$ converges uniformly to $f|_K$.*

The following corollary follows from Sion's theorem [14] and is given in [15].

Corollary 1 (Proposition A.10 in [15])**.** *Let (X, Y, g) be a zero-sum game such that:*

- *X is a compact metric space,*
- *for all $y \in Y$, the function $g(\cdot, y)$ is upper semi-continuous.*

The game $(\Delta(X), \Delta_f(Y), g)$ has a value v and player 1 has an optimal strategy, i.e.

$$v = \max_{\mu \in \Delta(X)} \inf_{y \in Y} \int_X g(x,y) d\mu(x) = \inf_{\nu \in \Delta_f(Y)} \max_{x \in X} \int_Y g(x,y) d\nu(y).$$

It is now possible to give the following result on the existence of the value in patrolling games.

Proposition 2. *The patrolling game (Q, m, r) played with mixed strategies has a value denoted $V_Q(m, r)$. Moreover the patroller has an optimal strategy and the attacker has an ε-optimal strategy with finite support, i.e.*

$$\begin{aligned} V_Q(m,r) &= \max_{\mu \in \Delta(\mathcal{W})} \inf_{(y,t) \in \mathcal{A}} \int_{\mathcal{W}} g_{m,r}(w,(y,t)) d\mu(w) \\ &= \inf_{\nu \in \Delta_f(\mathcal{A})} \max_{w \in \mathcal{W}} \int_{\mathcal{A}} g_{m,r}(w,(y,t)) d\nu(y,t). \end{aligned}$$

Proof of proposition 2. By Ascoli's theorem (application of theorem 47.1 in [12]), $\mathcal{W}$ is compact for the topology of compact convergence.

Moreover, for all $(y,t) \in \mathcal{A}$ the function $g_{m,r}(\cdot,(y,t))$ is upper semi-continuous.

The conclusion follows from corollary 1. $\square$

We now give some properties of the function V_Q.

Proposition 3. *Let Q be a search space. The function*

$$\begin{array}{cccc} V_Q & : & \mathbb{R}_+ \times \mathbb{R}_+ & \to & [0,1] \\ & & (m,r) & \mapsto & V_Q(m,r) \end{array}$$

is

i) non decreasing in m and r,

ii) upper semi-continuous in r for all m.

Proof. Since i) is direct we only prove ii). For all $(m,r) \in \mathbb{R}_+^2$,

$$\begin{aligned} V_Q(m,r) &= \max_{\mu \in \Delta(\mathcal{W})} \inf_{(y,t)\in\mathcal{A}} \int_{\mathcal{W}} g_{m,r}(w,(y,t))d\mu(w) \\ &= \inf_{(y,t)\in\mathcal{A}} \int_{\mathcal{W}} g_{m,r}(w,(y,t))d\mu^*(w), \end{aligned}$$

where $\mu^* \in \Delta(\mathcal{W})$ is an optimal strategy of the patroller. Let $(y,t) \in \mathcal{A}$ and $m \geq 0$. For all $w \in \mathcal{W}$, the function $r \mapsto g_{m,r}(w,(y,t))$ is clearly upper semi-continuous. Let $r_n \to r$, then by Fatou's lemma,

$$\begin{aligned} \limsup_n \int_W g_{m,r_n}(w,(y,t))d\mu^*(w) &\leq \int_W \limsup_n g_{m,r_n}(w,(y,t))d\mu^*(w) \\ &\leq \int_{\mathcal{W}} g_{m,r}(w,(y,t))d\mu^*(w). \end{aligned}$$

Thus the function $r \mapsto \int_{\mathcal{W}} g_{m,r}(w,(y,t))d\mu^*(w)$ is upper semi-continuous. Hence

$$V_Q(m,\cdot) : r \mapsto \inf_{(y,t)\in\mathcal{A}} \int_{\mathcal{W}} g_{m,r}(w,(y,t))d\mu^*(w)$$

is upper semi-continuous. $\square$

Let us now make two remarks about the function which maps any search space Q to $V_Q(m,r)$, the value of the patrolling game (Q,m,r).

Remark 1. *i) Let $m,r \geq 0$, and Q_1, Q_2 be two search spaces, it is clear that if $Q_1 \subset Q_2$ then the attacker is better off in Q_2 hence $V_{Q_1}(m,r) \geq V_{Q_2}(m,r)$.*

ii) In general $Q \mapsto V_Q(m,r)$ is not continuous with respect to the Hausdorff metric between nonempty compact sets. An example such that $Q \mapsto V_Q(0,1)$ is not continuous is postponed to the next section, see proposition 9.

2.3 A general upper bound

As in [2] let us define the maximum rate at which the patroller discovers new points of Q. Together with a strategy of the attacker (called uniform), this yields an upper bound on the values patrolling games.

Definition 1. *The maximum discovery rate is given by*

$$\rho = \sup_{w \in \mathcal{W}, t>0} \frac{\lambda(w([0,t]) + B_r(0)) - \lambda(B_r)}{t}.$$

Remark 2. *Let (Q, m, r) be a patrolling game. If Q has nonempty interior in $\mathbb{R}^2$ endowed with the Euclidean norm, then $\rho = 2r$.*

Indeed, since Q has nonempty interior let $x \in Q$ and $s > 0$ such that $B_s(x) \subset Q$. Define $w(t) = x + t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $t \in [0, s]$ (and arbitrarily such that $w \in \mathcal{W}$ for $t > s$). Then

$$\frac{\lambda(w([0,s]) + B_r(0)) - \lambda(B_r)}{s} = \frac{2rs}{s} = 2r,$$

and it is clear that this is the maximum.

Definition 2. *Let Q be a search space such that $\lambda(Q) > 0$. The attacker's uniform strategy on Q, denoted a_λ, is a random choice of the attack point a at time 0 such that for all measurable sets $B \subset Q$,*

$$a_\lambda(B) = \frac{\lambda(B)}{\lambda(Q)}.$$

The following proposition gives a general upper bound for patrolling games which search space have nonzero Lebesgue measure.

Proposition 4. *Let Q be a search space such that $\lambda(Q) > 0$. Then*

$$V_Q(m, r) \leq \frac{m\rho + \lambda(B_r)}{\lambda(Q)}.$$

Proof. For all $w \in \mathcal{W}$,

$$\frac{\lambda\Big(((w([0,m]) + B_r(0)) \setminus B_r(w(0))\Big)}{m} = \frac{\lambda(w([0,m]) + B_r(0)) - \lambda(B_r)}{m} \leq \rho.$$

Thus,

$$\begin{aligned}\int_{\mathcal{A}} g_{m,r}(w, (a,t)) da_\lambda(a,t) &= a_\lambda(w([0,m]) + B_r(0)) \\ &= \frac{\lambda(w([0,m]) + B_r(0))}{\lambda(Q)} \\ &\leq \frac{m\rho + \lambda(B_r)}{\lambda(Q)}.\end{aligned}$$

$\square$

2.4 Patrolling a network

Networks are a particular class of search spaces. We follow the construction of [7].

Definition 3. *Let (V, E, l) be a weighted undirected graph, V is the finite set of nodes and E the finite set of edges which elements $e \in E$ are associated to a length $l(e) \in \mathbb{R}_+$. An edge $e \in E$ linking the two nodes s and t is also denoted (s, t).*

We identify the elements of V with the vectors of the canonical basis of $\mathbb{R}^{|V|}$. The network generated by (V, E) is the set of points

$$\mathcal{N} = \{(s, t, \alpha) \mid \alpha \in [0, 1] \text{ and } (s, t) \in E\},$$

where $(s, t, \alpha) = \alpha s + (1 - \alpha)t$.

We now define a natural metric d on $\mathcal{N}$.

Definition 4. *A network $\mathcal{N}$ is endowed with a metric d as follows. Let u_1 and u_2 be two points of the same edge (s, t). There exist $\alpha_1, \alpha_2 \in [0, 1]$ such that $u_1 = (s, t, \alpha_1)$ and $u_2 = (s, t, \alpha_2)$. $d(u_1, u_2)$ is given by $d(u_1, u_2) = l(s, t) \times |\alpha_1 - \alpha_2|$.*

If u and v are not in the same edge, consider the set of paths $P(u, v)$ between u and v as the set of all sequences $(u_1, \ldots, u_n)$, $n \in \mathbb{N}^*$ such that $u_1 = u$, $u_n = v$ and such that for all $i \in \{1, \ldots, n - 1\}$, u_i and u_{i+1} belong to the same edge. The distance $d(u, v)$ is then defined as:

$$d(u, v) = \inf_{(u_1,\ldots,u_n) \in P(u,v)} \sum_{i=1}^{n-1} d(u_i, u_{i+1}).$$

Finally, we define the Lebesgue measure on $\mathcal{N}$.

Definition 5. *Let $u_1 = (s, t, \alpha_1)$ and $u_2 = (s, t, \alpha_2)$, suppose $\alpha_1 < \alpha_2$. The set*

$$[u_1, u_2] = \{(s, t, \alpha) \mid \alpha \in [\alpha_1, \alpha_2]\}$$

is called an interval. An interval $[u_1, u_2]$ can be isometrically identified with the real interval $[\alpha_1 l(s, t), \alpha_2 l(s, t)]$.

As a subset of $\mathcal{N}$ can be identified with a finite union of subsets of intervals, the Lebesgue measure on $\mathcal{N}$ is defined as a natural extension of the Lebesgue measure on a real interval.

For a particular class of networks called Eulerian, it is possible to compute the value and optimal strategies of the game when the detection radius is zero. Let us first introduce some definitions.

Definition 6. *Let $u \in \mathcal{N}$ and $\pi = (u_1, u_2, \ldots, u_{n-1}, u_n) \in P(u, u)$. If*

$$\bigcup_{k=1}^{n-1} [u_k, u_{k+1}] = \mathcal{N}$$

then π is called a tour. Moreover, if

$$\sum_{k=1}^{n-1} \lambda([u_k, u_{k+1}]) = \lambda(\mathcal{N}),$$

then π is called an Eulerian tour.

Remark 3. *Let $\pi = (u_1, u_2, \ldots, u_{n-1}, u_n)$ be an Eulerian tour in a network $\mathcal{N}$. Then there exist $k \in \{1, \ldots, n\}$ and $\pi' = (u_{i_1}, \ldots, u_{i_k}) \subset \pi$ such that any two edges $(u_{i_j}, u_{i_{j+1}})$, $j \in \{1, \ldots, k-1\}$ are distinct.*

Definition 7. *A network $\mathcal{N}$ is called Eulerian if there exists an Eulerian tour in $\mathcal{N}$.*

Example 1. *Figure 1 and figure 2 give two examples of networks. $\mathcal{N}_1$ is an Eulerian network with Eulerian tour $\pi_1 = (u_1, u_2, u_3, u_4, u_5, u_6, u_3, u_7, u_1)$. Whereas $\mathcal{N}_2$ is not an Eulerian network.*

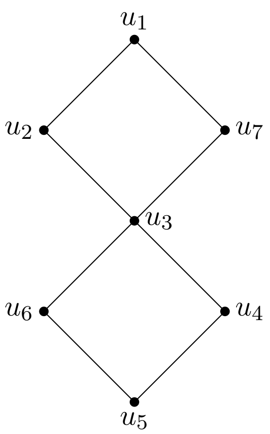

Figure 1: The network $\mathcal{N}_1$ is Eulerian

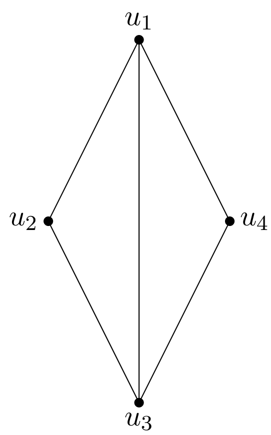

Figure 2: The network $\mathcal{N}_2$ is not Eulerian

When $\mathcal{N}$ is an Eulerian network, it is possible to define a parametrization of $\mathcal{N}$.

Definition 8. *Let $\mathcal{N}$ be an Eulerian network. A continuous function w from $[0, \lambda(\mathcal{N})]$ to $\mathcal{N}$ such that*

i) $w(0) = w(\lambda(\mathcal{N}))$,

ii) w is surjective.

iii) $\forall t_1, t_2 \in [0, \lambda(\mathcal{N})]$ $\lambda(w([t_1, t_2])) = |t_1 - t_2|$ (the speed of w is 1).

is called a parametrization of $\mathcal{N}$.

Moreover such a w can be extended to a $\lambda(\mathcal{N})$-periodic function on $\mathbb{R}$ which is still denoted w.

Lemma 1. *Let $\mathcal{N}$ be an Eulerian network, then there exists a parametrization w of $\mathcal{N}$.*

Proof. w is constructed in the following way. Let $\pi = (u_1, u_2, \ldots, u_{n-1}, u_n)$, $u_1 = u_n = u \in V$ be an Eulerian tour as in remark 3.

If $t \in [0, l(u_1, u_2)]$ then

$$w(t) = \left(u_1, u_2, \frac{t}{l(u_1, u_2)}\right).$$

Else, suppose $n \geq 3$. For all $k \in \{2, \ldots, n-1\}$ if

$$t \in \left(\sum_{i=1}^{k-1} l(u_i, u_{i+1}), \sum_{i=1}^{k} l(u_i, u_{i+1})\right]$$

then

$$w(t) = \left(u_k, u_{k+1}, \frac{t - \sum_{i=1}^{k-1} l(u_i, u_{i+1})}{l(u_k, u_{k+1})}\right).$$

It is not difficult to verify that such w is appropriate. $\square$

Remark 4. *Let $\pi = (u_1, u_2, \ldots, u_{n-1}, u_n)$ be any path such that $u_k \neq u_{k+1}$ for all $k \in \{1, \ldots n-1\}$. The construction made in the proof of lemma 1 yields a continuous function on $\left[0, \sum_{k=1}^{n-1} \lambda([u_k, u_{k+1}])\right]$ which has speed 1.*

It is said to be the walk induced by π on $\left[0, \sum_{k=1}^{n-1} \lambda([u_k, u_{k+1}])\right]$.

Definition 9. *Suppose $\mathcal{N}$ is an Eulerian network. Let w be a parametrization of $\mathcal{N}$. Denote $(w_{t_0})_{t_0 \in [0, \lambda(\mathcal{N})]}$ the family of $\lambda(\mathcal{N})$-periodic walks such that*

$$w_{t_0}(\cdot) = w(t_0 + \cdot).$$

The patroller's uniform strategy p_λ is the uniform choice of $t_0 \in [0, \lambda(\mathcal{N})]$.

Proposition 5. *If $\mathcal{N}$ is an Eulerian network, then*

$$V_{\mathcal{N}}(m, 0) = \min\left(\frac{m}{\lambda(\mathcal{N})}, 1\right).$$

Moreover the attacker's and the patroller's uniform strategies are optimal.

Proof. It is clear that if $m \geq \lambda(\mathcal{N})$, the patroller guarantees 1 by playing a parametrization w of $\mathcal{N}$, see definition 8 and lemma 1. Suppose thus that $m < \lambda(\mathcal{N})$.

Let $(y,t) \in \mathcal{N} \times \mathbb{R}_+$ be a pure strategy of the attacker. By lemma 8 ii) there exists $t_y \in [0, \lambda(\mathcal{N})]$ such that $w(t_y) = y$.

Now let $t_0 \in [t_y - t - m, t_y - t]$. Then $w_{t_0}(t_y - t_0) = w(t_y) = y$. And $t_y - t_0 \in [t, t+m]$. Hence $y \in w_{t_0}([t, t+m])$.

$$\begin{aligned}\mathbb{P}_{p_\lambda}(y \in w_{t_0}([t,t+m])) &\geq \mathbb{P}_{p_\lambda}(t_0 \in [t_y - t - m, t_y - t]) \\ &= \frac{m}{\lambda(\mathcal{N})}.\end{aligned}$$

The other inequality follows from proposition 4 since in this case, ρ equals 1. $\square$

Let us now present some considerations on the value of a game played over a non Eurelian network.

Example 2. *We consider again the network $\mathcal{N}_2$ represented in figure 2. In this example, we take $l(u_1,u_2) = l(u_2,u_3) = l(u_1,u_4) = l(u_4,u_3) = 1/2$ and $l(u_1,u_3) = 1$. Notice that $\lambda(\mathcal{N}_2) = 3$. We give the following bounds on the value of $(\mathcal{N}_2, m, 0)$:*

$$V_{\mathcal{N}_2}(m,0) \begin{cases} = \frac{m}{3} \text{ if } m \leq 2 \\ \in \left[\frac{5m-2}{3(m+2)}, 1 - \frac{1}{3}\left(\frac{4-m}{2}\right)^2\right] \text{ if } m \in \left[2, \frac{10}{3}\right] \\ \in \left[\frac{14-2m}{3(6-m)}, 1 - \frac{1}{3}\left(\frac{4-m}{2}\right)^2\right] \text{ if } m \in \left[\frac{10}{3}, 4\right] \\ = 1 \text{ if } m \geq 4. \end{cases}$$

First case: $m \leq 2$. *Recall that for all $m \geq 0$, by proposition 4,*

$$V_{\mathcal{N}_2}(m,0) \leq \frac{m}{3}.$$

Let us now introduce a particular class of walks as follows. Let $\pi^1 = (u_1,u_2,u_3,u_1,u_4,u_3)$ and $\pi^2 = (u_3,u_2,u_1,u_3,u_4,u_1)$ be two paths. π^1 and π^2 induce to walks on $[0,3]$, respectively denoted w^1 and w^2. For all $u \in \mathcal{N}_2 \setminus \{u_1,u_3\}$ and all $i \in \{1,2\}$ there exists a unique $t_u^i \in [0,3]$ such that $w^i(t_u^i) = u$. Now for all $u \in \mathcal{N}_2 \setminus \{u_1,u_3\}$ and all $t \in \mathbb{R}_+$, define

$$w_u^1(t) = \begin{cases} w^1(t + t_u^1) \text{ if } t \in [0, 3 - t_u^1] \\ w^2\left(t - (3(2k+1) - t_u^1)\right) \text{ if } t \in (3(2k+1) - t_u^1, 3(2k+2) - t_u^1] \\ w^1\left(t - (3(2k+2) - t_u^1)\right) \text{ if } t \in (3(2k+2) - t_u^1, 3(2k+3) - t_u^1] \end{cases}$$

for all $k \in \mathbb{N}$. And analogously for w_u^2. Denote μ^0 the uniform choice of a walk in $(w_u^i)_{u \in \mathcal{N}_2 \setminus \{u_1,u_3\}}^{i \in \{1,2\}}$.

It is not difficult to check that μ^0 guarantees $m/3$ to the patroller (moreover μ^0 gives a payoff of $m/3$ for every $(y,t) \in \mathcal{A}$). Hence

$$V_{\mathcal{N}_2}(m,0) = \frac{m}{3}.$$

Second case: $2 < m < 4$. *We give the computation for $m = 3$. The walks w^3, w^4 and w^5 hereafter can be easily adapted and similar strategies can be used to derive the bounds for all $m \in (2,4)$.*

Let us define three paths π^3, π^4 and π^5 as in figure 3, 4 and 5 respectively. That is, $\pi^3 = (u_1,u_2,u_3,u_5,u_3,u_1,u_6,u_1)$, $\pi^4 = (u_1,u_7,u_1,u_3,u_8,u_3,u_4,u_1)$ and $\pi^5 = (u_1,u_2,u_3,u_{10},u_3,u_4,u_1,u_9,u_1)$. Where $u_5 = (u_3,u_4,1/2)$, $u_6 = (u_1,u_4,1/2)$, $u_7 = (u_1,u_2,1/2)$, $u_8 = (u_2,u_3,1/2)$, $u_9 = (u_1,u_3,1/4)$ and $u_{10} = (u_1,u_3,3/4)$.

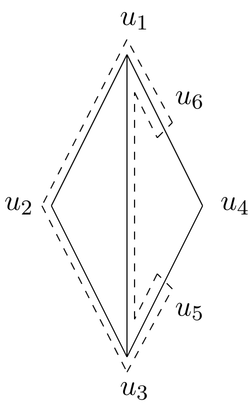

Figure 3: The path π^3

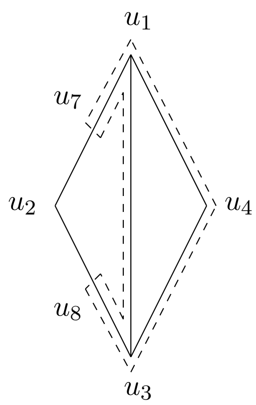

Figure 4: The path π^4

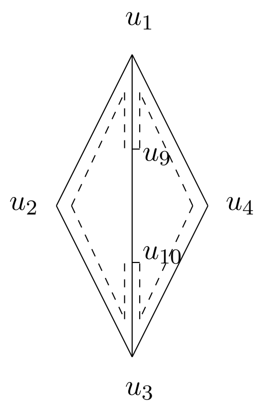

Figure 5: The path π^5

π^3, π^4 and π^5 induce three 3-periodic walks, denoted respectively w^3, w^4 and w^5. These are such that for $i \in \{3,4,5\}$, w^i intercepts any attack on $w^i([0,3])$ with probability 1.

With a slight abuse of notation, for $y \in [0,1/2]$, denote y the point (u_1,u_3,y) and $1-y$ the point $(u_1,u_3,1-y)$. By symmetry it is enough to consider attacks occurring at y. Moreover, μ^0, w^3, w^4 and w^5 make the patroller time indifferent, hence we only consider attacks at time 0.

μ^0 intercepts the attack $(y,0)$ with probability $1 - \frac{1-2y}{6} = \frac{5}{6} + \frac{y}{3}$. Indeed, only the walks w^1_u, such that u belongs to the open interval $(y, 1-y) = \{(u_1,u_3,\alpha) \mid \alpha \in (y,1-y)\}$ do not intercept the attack. Finally, define $\widetilde{\mu} = \frac{1}{15}(\delta_{w^3} + \delta_{w^4} + \delta_{w^5}) + \frac{4}{5}\mu^0$, where δ_w is the Dirac measure at $w \in \mathcal{W}$.

At any time, an attack at $y \leq \frac{1}{4}$ is intercepted by $\widetilde{\mu}$ with probability

$$\frac{1}{15} \cdot 3 + \frac{4}{5}\left(\frac{5}{6} + \frac{y}{3}\right) \geq \frac{3}{15} + \frac{4}{5} \cdot \frac{5}{6} = \frac{13}{15}.$$

An attack at $y > 1/4$ is intercepted by $\widetilde{\mu}$ with probability

$$\frac{1}{15} \cdot 2 + \frac{4}{5}\left(\frac{5}{6} + \frac{y}{3}\right) \geq \frac{2}{15} + \frac{4}{5}\left(\frac{5}{6} + \frac{1}{12}\right) = \frac{13}{15}.$$

Hence

$$V_{\mathcal{N}_2}(3,0) \geq \frac{13}{15}.$$

Define the following attack $\widetilde{a}$: choose uniformly a point in $\mathcal{N}_2 \times [0,3]$. $(u_1, u_2, u_3, u_1, u_4, u_3, u_2, u_1, u_3, u_4, u_1)$ induces a 6-periodic walk w^6 which is a best reply for the patroller. Moreover $g_{3,0}(w^6, \widetilde{a}) = 11/12$.

Hence

$$V_{\mathcal{N}_2}(3,0) \leq \frac{11}{12}.$$

Third case: $m \geq 4$. The tour $(u_1, u_2, u_3, u_1, u_4, u_3, u_1)$ induces a 4-periodic walk which guarantees 1 to the patroller. Hence

$$V_{\mathcal{N}_2}(m,0) = 1.$$

2.5 Patrolling a simple search space in $\mathbb{R}^2$

Let us recall the notion of bounded variation of a function.

Definition 10. *Let $a > 0$. Let $f : [0,a] \to \mathbb{R}^n$ be a continuous function. Then the total variation of f is the quantity:*

$$TV(f) = \sup\left\{\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_2 \;\middle|\; n \in \mathbb{N}^*, 0 = t_0 < t_1 < \cdots < t_n = a\right\}.$$

If $TV(f) < +\infty$, then f is said to have bounded variation.

The next definition introduces a classical assumption on the boundary of a search space. This is a weak assumption already made in [2, 8].

Definition 11. *Let $a > 0$, let f_1 and f_2 be two continuous functions from $[0,a]$ to $\mathbb{R}$ such that $f_1 \geq f_2$, $f_1 \neq f_2$, and f_1 and f_2 have bounded variation. Then the nonempty compact set*

$$\{(x,t) \in [0,a] \times \mathbb{R} \mid f_2(x) \leq t \leq f_1(x)\}$$

is called an elementary search space.

A finite union of elementary search spaces such that any two elementary search spaces have disjoint interiors is called a simple search space.

Definition 12. *Let Q be a search space. A continuous function $L : [0,1] \to Q$ such that $L(0) = L(1)$ is called an r-tour if for any $x \in Q$ there exists $l \in L([0,1])$ such that $d(x,l) \leq r$.*

Lemma 2 (Lemma 3.39 in [2])**.** *Let $Q \subset \mathbb{R}^2$ be a simple search space. Endow Q with the Euclidean norm. Then for any $\varepsilon > 0$ there is an $r_\varepsilon > 0$ such that for any $r < r_\varepsilon$ there exists an r-tour $L : [0,1] \to Q$ such that*

$$TV(L) \leq (1+\varepsilon)\frac{\lambda(Q)}{2r}.$$

Remark 5. *$TV(L)$ can be interpreted as a quantification of the length of $L([0,1])$.*

The next lemma gives a parametrization of $L([0,1])$ in terms of walks.

Lemma 3. *Let L be an r-tour as in lemma 2. Then for all $\varepsilon' > 0$ there exists $w : [0, TV(L)+\varepsilon'] \to L([0,1])$ continuous such that:*

i) $w(0) = w(TV(L)+\varepsilon')$,

ii) *w is surjective,*

iii) *w is 1-Lipschitz continuous,*

iv) $TV(w) = TV(L)$.

w is extended to a $(TV(L)+\varepsilon')$-periodic function on $\mathbb{R}$, which is still denoted w.

Proof. Let $\begin{array}{cccc} f & : & [0,1] & \to & [0, TV(L)+\varepsilon'] \\ & & s & \mapsto & TV(L|_{[0,s]}) + \varepsilon' s. \end{array}$ The function f is increasing and continuous on $[0,1]$, hence f is an homeomorphism. Define $w = L \circ f^{-1}$ on $[0, TV(L)+\varepsilon']$.

It is not difficult to prove that such w verify i), ii) and iv). To show that iii) holds, first remark that if $s_1, s_2 \in [0,1]$ and $s_1 \leq s_2$ then $TV\left(L|_{[0,s_2]}\right) \geq TV\left(L|_{[0,s_1]}\right) + TV\left(L|_{[s_1,s_2]}\right)$. Hence for all $t_1, t_2 \in [0, TV(L)+\varepsilon']$

$$\begin{aligned} \|w(t_1) - w(t_2)\|_2 &\leq TV\left(L|_{[f^{-1}(t_1), f^{-1}(t_2)]}\right) \\ &\leq \left| TV\left(L|_{[0,f^{-1}(t_2)]}\right) - TV\left(L|_{[0,f^{-1}(t_1)]}\right) \right| \\ &\leq |t_1 - t_2|. \end{aligned}$$

□

Proposition 6. *If Q is a simple search space endowed with the Euclidean norm, then*

$$V_Q(m,r) \sim \frac{2rm}{\lambda(Q)},$$

as r goes to 0.

Proof. Let L and w be as in lemma 2 and lemma 3 respectively. For all $t_0 \in [0, TV(L) + \varepsilon']$ define $w_{t_0}(\cdot) = w(t_0 + \cdot)$.

Let $(l, t) \in L([0,1]) \times \mathbb{R}_+$. By lemma 3 ii), there exists $t_l \in [0, TV(L) + \varepsilon']$ such that $w(t_l) = l$.

Now let $t_0 \in [t_l - t - m, t_l - t]$. Then $w_{t_0}(t_l - t_0) = w(t_l) = l$. And $t_l - t_0 \in [t, t+m]$. Hence $l \in w_{t_0}([t, t+m])$.

Suppose t_0 is chosen uniformly on $[0, TV(L) + \varepsilon']$. By lemma 3 iii) this is an admissible strategy for the patroller denoted p_λ. Let $(y,t) \in \mathcal{A}$ be a pure strategy of the attacker. Then

$$\mathbb{P}_{p_\lambda}(d(y, w_{t_0}([t, t+m])) \leq r) \geq \mathbb{P}_{p_\lambda}(l \in w_{t_0}([t, t+m])),$$

where $l \in L([0,1])$ is such that $d(y, l) \leq r$

$$\begin{aligned} &\geq \mathbb{P}_{p_\lambda}(t_0 \in [t_l - t - m, t_l - t]) \\ &= \frac{m}{TV(L) + \varepsilon'} \\ &\geq \frac{m}{\frac{2r}{(1+\varepsilon)\lambda(Q)} + \varepsilon'}. \end{aligned}$$

Thus for all $\varepsilon' > 0$ the patroller guarantees $\frac{m}{\frac{2r}{(1+\varepsilon)\lambda(Q)} + \varepsilon'}$, hence

$$V_Q(m, r) \geq \frac{2rm}{(1+\varepsilon)\lambda(Q)} \sim \frac{2rm}{\lambda(Q)}$$

as r goes to 0.

Together with proposition 4, which in this context yields $V_Q(m,r) \leq \frac{2rm + \lambda(B_r)}{\lambda(Q)} \sim \frac{2rm}{\lambda(Q)}$ as r goes to 0 (see remark 2), the result is proven. □

3 Hiding games

3.1 The model

A hiding game is given by a couple $H = (Q, r)$, where Q is a search space and $r \geq 0$ is the detection radius. The set of pure strategies of both players, called searcher and hider, is Q. The payoff to the searcher is given by

$$h_r(x,y) = \begin{cases} 1 & \text{if } \|x - y\| \leq r \\ 0 & \text{else.} \end{cases}$$

We will see later that these games have a value.

Hiding games correspond to two possible variants of patrolling games.

In the first one, consider a patrolling game $P = (Q, 0, r)$. Then for all $w \in \mathcal{W}$ and $(y,t) \in \mathcal{A}$

$$g_{0,r}(w, (y,t)) = \begin{cases} 1 & \text{if } d(y, w(t)) \leq r \\ 0 & \text{else.} \end{cases}$$

Thus to a strategy $x \in Q$ of the searcher in H is associated in P the constant strategy $w \in \mathcal{W}$ equal to x. And to a strategy $y \in Q$ of the hider in H is associated the strategy $(y, 0) \in \mathcal{A}$ in P. Any quantity guaranteed by the searcher in H is guaranteed by the patroller in P. Conversely, any quantity guaranteed by the hider in H is guaranteed by the attacker in P. Thus, since H has a value, the values of these two games are the same.

In [4] the authors suggest the study of patrolling games in which the patroller may be informed of the presence of the attacker. The second interpretation corresponds to a variant of patrolling games in which the patroller is informed of the point of attack when the attack occurs. Let Q be a search space, and m be the attack duration. The detection radius r is taken equal to 0. The payoff of this game is

$$g_{m,0}(w,(y,t)) = \begin{cases} 1 & \text{if } d(y, w([t, t+m])) \leq 0 \\ 0 & \text{else.} \end{cases}$$

This game is denoted P'. In P', if the patroller's strategy is to choose a point and not move until the attack, then go to the attack point, the attacker is time-indifferent. In particular, the attacker has a best response in the set of attacks occurring at time 0. Finally, if the attack occurs at time 0, the patroller has a best response consisting in choosing (possibly at random) a point in Q and going directly to the attack point when he is informed of the attack.

Thus, with the same mappings of strategies in the hiding game $H' = (Q, m)$ to strategies in P' as before, any quantity guaranteed by the searcher in H' is guaranteed by the patroller in P'. Conversely, any quantity guaranteed by the hider in H' is guaranteed by the attacker in P'. Thus, since H' has a value, the values of these two games are the same.

3.2 The value of hiding games

The following results show the existence of the value in hiding games and give some of its basic properties.

Proposition 7. *The hiding game (Q, r) played in mixed strategies has a value denoted $V_Q(r)$. Moreover the searcher has an optimal strategy and the hider has an ε-optimal strategy with finite support.*

Proof. Follows directly from corollary 1. □

Proposition 8. *Let Q be a search space. The function*

$$\begin{array}{rcccc} V_Q & : & \mathbb{R}_+ & \to & [0,1] \\ & & r & \mapsto & V_Q(r) \end{array}$$

is

i) non decreasing,

ii) upper semi-continuous.

Proof. Let Q be a search space. We have seen that for all $r \geq 0$, $V_Q(r) = V_Q(0, r)$. Hence the results follow directly from proposition 3. $\square$

The following example was first solved by Ruckle [13].

Example 3. *Let Q be an the $[0,1]$ interval endowed with the ∞-norm, then* $V_Q(r) = \begin{cases} \min\left(\lceil \frac{1}{2r} \rceil^{-1}, 1\right) & \textit{if } r > 0 \\ 0 & \textit{else.} \end{cases}$

Indeed, it is clear when r equals 0 and $r \geq 1/2$. Let $n \in \mathbb{N}^*$ and suppose $r \in \left[\frac{1}{2(n+1)}, \frac{1}{2n}\right)$. Then the patroller guarantees $\frac{1}{n+1}$ by choosing equiprobably a point in $\left\{\frac{1+2k}{2(n+1)}\right\}_{0 \leq k \leq n}$. And the attacker, choosing equiprobably a point in $\left\{\frac{(2+\varepsilon)k}{2(n+1)}\right\}_{0 \leq k \leq n}$, with $0 < \varepsilon \leq 2/n$, also guarantees $\frac{1}{n+1}$.

Remark 6. *This simple example shows that in general, V_Q is not lower semi-continuous.*

Proposition 9. *Let $r \geq 0$. The function which maps any search space Q to $V_Q(r)$ is in general not continuous with respect to the Hausdorff metric between nonempty compact sets.*

Proof. Let $D_s = \{x \in \mathbb{R}^2 \mid \|x\|_2 \leq s\}$ be the Euclidean disc of radius $s > 0$ centered at 0. From [6], it is known that

$$V_{D_s}(1) = \begin{cases} 1 \text{ if } s \in [0,1] \\ \frac{1}{\pi} \arcsin\left(\frac{1}{s}\right) \text{ if } s \in \left(1, \sqrt{2}\right] \end{cases}.$$

Hence

$$\lim_{s \to 1, s > 1} V_{D_s}(1) = \frac{1}{2} < 1.$$

The intuition is the following: it is clear that when s equals 1 the searcher guarantees 1 by playing $x = (0,0)$. Suppose now that s equals $1 + \varepsilon$. Then the searcher covers almost all the area of the disc but less than half of its circumference. Hence the hider guarantees $1/2$ by choosing uniformly a point on the boundary of D_s. $\square$

3.3 Equalizing strategies

We now study particular strategies called equalizing, these have been introduced in [5] when the search space is a graph (see definition 7.3 and proposition 7.3). We adapt those considerations to our compact setting. The interest of equalizing strategies lies in the fact that if such a strategy exists, then it is optimal for both player.

Definition 13. *Let Q be a search space. A strategy $\mu \in \Delta(Q)$ is said to be equalizing if the function $\int_Q h_r(x, \cdot) d\mu(x)$ is constant on Q, that is there exists $c \in \mathbb{R}_+$ for all $y \in Q$ $\mu(B_r(y) \cap Q) = c$.*

Proposition 10. *Let $\mu \in \Delta(Q)$. Then μ is an equalizing strategy (with constant payoff c) if and only if μ is optimal for both players (and $V_Q(r) = c$).*

Proof. Suppose $\mu \in \Delta(Q)$ is an equalizing strategy. If the searcher plays μ, then for all $y \in Q$ $\mu(B_r(y) \cap Q) = c$, hence $V_Q(r) \geq c$. Symmetrically, if the hider plays μ, then for all $x \in Q$ $\mu(B_r(x) \cap Q) = c$, hence $V_Q(r) \leq c$, and $V_Q(r) = c$.

Conversely, suppose $\mu \in \Delta(Q)$ is optimal for both players. Then the searcher guaranties $V_Q(r)$ that is for all $y \in Q$ $\mu(B_r(y) \cap Q) \geq V_Q(r)$, and the hider guaranties $V_Q(r)$ that is for all $x \in Q$ $\mu(B_r(x) \cap Q) \leq V_Q(r)$. Hence for all $y \in Q$

$$\mu(B_r(y) \cap Q) = V_Q(r).$$

□

The following game is an example of a hiding game with finite search space without equalizing strategies.

Example 4. *Let $r = 1$ and $Q = \{x_1, x_2, x_3, x_4, x_5\}$ be the finite subset of $\mathbb{R}^2$ such that $x_1 = (0, 0)$, $x_2 = (0, 1)$, $x_3 = (1, 1)$, $x_4 = (1, 0)$ and $x_5 = (1/2, 0)$.*

Denote for $i \in \{1, \dots, 5\}$ $Q_i = \{j \mid \|x_i - x_j\|_2 \leq r\}$. That is $Q_1 = \{1, 2, 4, 5\}$, $Q_2 = \{1, 2, 3\}$, $Q_3 = \{2, 3, 4\}$, $Q_4 = \{1, 3, 4, 5\}$ and $Q_5 = \{1, 4, 5\}$.

The game (Q, r) admits an equalizing strategy if and only if the following system of equations admits a solution $p = (p_i)_{1 \leq i \leq 5}$:

$$\begin{cases} p_i \geq 0 \text{ for all } i \in \{1, \dots, 5\} \\ \sum_{i=1}^{5} p_i = 1 \\ \sum_{i \in Q_1} p_i = \sum_{i \in Q_j} p_i \text{ for all } j \in \{2, \dots, 5\}. \end{cases} \tag{1}$$

That is the equivalent system admit a solution:

$$\begin{cases} Cp = b \\ p \geq 0, \end{cases}$$

where

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

As the only solution to $Cp = b$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, *the system (1) does not admit a solution, hence (Q, r) does not have an equalizing strategy.*

3.4 An asymptotic result for hiding games

The following proposition gives a general upper bound on the value of hiding games.

Proposition 11. *Let Q be a compact set in $\mathbb{R}^n$ such that $\lambda(Q) > 0$. Then*

$$V_Q(r) \leq \frac{\lambda(B_r)}{\lambda(Q)}.$$

Proof. Let the uniform strategy q_λ be such that for all measurable sets $B \subset Q$, $q_\lambda(B) = \lambda(B)/\lambda(Q)$. Using the uniform strategy, the hider guarantees

$$\sup_{x \in Q} \frac{\lambda(B_r(x) \cap Q)}{\lambda(Q)} \leq \frac{\lambda(B_r)}{\lambda(Q)}.$$

□

Theorem 1. *Let Q be a compact subset of $\mathbb{R}^n$. Suppose $\lambda(Q) > 0$. Then*

$$V_Q(r) \sim \frac{\lambda(B_r)}{\lambda(Q)}$$

as r goes to 0.

To prove theorem 1 we first give two lemmas.

Denote $B_r^2(x) = \{y \in \mathbb{R}^n \mid \|x - y\|_2 \leq r\}$ the closed ball of center x with radius r for the Euclidean norm, and $\partial B_r^2(x) = \{y \in \mathbb{R}^n \mid \|x - y\|_2 = r\}$ the sphere of center x with radius r for the Eucliean norm.

Lemma 4. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $c_1, c_2 > 0$ be such that $c_1 \|\cdot\| \leq \|\cdot\|_2 \leq c_2 \|\cdot\|$. Then for all $x \in \partial B_\varepsilon^2(0)$*

$$\limsup_{r \to 0} \frac{\lambda(B_r)}{\lambda(B_\varepsilon^2(0) \cap B_r(x))} \leq \frac{c_2^n}{2c_1^n}.$$

Proof of lemma 4. Let $x \in \partial B_\varepsilon^2(0)$, let $\varepsilon > 0$. From [11] one deduces

$$\lambda\left(B_\varepsilon^2(0) \cap B_r^2(x)\right) = \frac{\pi^{n/2}}{2\Gamma(\frac{n}{2}+1)} \Bigg(r^n I_{1-(\frac{r}{2\varepsilon})^2}\left(\frac{n+1}{2}, \frac{1}{2}\right) + \varepsilon^n I_{(\frac{r}{\varepsilon})^2\left(1-(\frac{r}{2\varepsilon})^2\right)}\left(\frac{n+1}{2}, \frac{1}{2}\right)\Bigg).$$

Where I is the regularized incomplete Beta function: for $a, b > 0$ and $0 < z < 1$, $I_z(a,b) = \frac{B(z;a,b)}{B(a,b)}$. Where $B(z;a,b) = \int_0^z t^{a-1}(1-t)^{b-1}dt$ and $B(a,b) = B(1;a,b)$ is the Beta function.

$$I_{1-\left(\frac{r}{2\varepsilon}\right)^2}\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\int_0^{1-\left(\frac{r}{2\varepsilon}\right)^2} t^{\frac{n-1}{2}}(1-t)^{-1/2}dt}{B(\frac{n-1}{2}, \frac{1}{2})} \to 1,$$

as r goes to 0, since $t \mapsto t^{\frac{n-1}{2}}(1-t)^{-1/2}$ is integrable over $[0,1)$. And

$$\begin{aligned} I_{\left(\frac{r}{\varepsilon}\right)^2\left(1-\left(\frac{r}{2\varepsilon}\right)^2\right)}\left(\frac{n+1}{2}, \frac{1}{2}\right) &= \frac{\int_0^{\left(\frac{r}{\varepsilon}\right)^2\left(1-\left(\frac{r}{2\varepsilon}\right)^2\right)} t^{\frac{n-1}{2}}(1-t)^{-1/2}dt}{B(\frac{n-1}{2}, \frac{1}{2})} \\ &\geq \frac{\frac{2}{n+1}\left(\frac{r}{\varepsilon}\right)^{n+1}\left(1-\left(\frac{r}{2\varepsilon}\right)^2\right)^{\frac{n+1}{2}}}{B(\frac{n-1}{2}, \frac{1}{2})} \\ &\text{since } 1 \leq (1-t)^{-1/2} \text{ when } t \in [0,1) \\ &= \frac{2r^{n+1}}{(n+1)\varepsilon^{n+1}B(\frac{n-1}{2}, \frac{1}{2})} + o(r^{2n+2}) \end{aligned}$$

as r goes to 0. Hence

$$\lambda\left(B_\varepsilon^2(0) \cap B_r^2(x)\right) \geq \frac{\pi^{n/2}}{2\Gamma(\frac{n}{2}+1)}(r^n + o(r^n))$$

as r goes to 0. Moreover

$$\begin{aligned} B_\varepsilon^2(0) \cap B_r(x) &= \{y \in \mathbb{R}^n \mid \|y\|_2 \leq \varepsilon \text{ and } \|x-y\| \leq r\} \\ &\supset \{y \in \mathbb{R}^n \mid \|y\|_2 \leq \varepsilon \text{ and } \|x-y\|_2 \leq c_1 r\}, \end{aligned}$$

and

$$B_r(0) \subset B_{c_2 r}^2(0).$$

Hence

$$\lambda\left(B_\varepsilon^2(0) \cap B_r(x)\right) \geq \lambda\left(B_\varepsilon^2(0) \cap B_{c_1 r}^2(x)\right),$$

and

$$c_2^n \lambda(B_r^2) \geq \lambda(B_r).$$

Finally, since $\lambda(B_r^2) = \frac{\pi^{n/2}r^n}{\Gamma(\frac{n}{2}+1)}$,

$$\frac{\lambda(B_r)}{\lambda\left(B_\varepsilon^2(0) \cap B_r(x)\right)} \leq \frac{c_2^n \lambda(B_r^2)}{\lambda\left(B_\varepsilon^2(0) \cap B_{c_1 r}^2(x)\right)} \to_{r\to 0} \frac{c_2^n}{2c_1^n}.$$

$\square$

The goal of the following lemma is to show that the optimization problem $\min_{x\in B^2_\varepsilon(0)} \lambda((B^2_\varepsilon(0)\cap B_r(x))$ has a solution on $\partial B^2_\varepsilon(0)$, giving a hint to why x is taken in $\partial B^2_\varepsilon(0)$ in lemma 4.

Lemma 5. *Let $\varepsilon, r > 0$ and define*

$$\begin{array}{cccc} \Psi & : & B^2_\varepsilon(0) & \to & [0, \lambda(B^2_\varepsilon)] \\ & & x & \mapsto & \lambda\left(B^2_\varepsilon(0)\cap B_r(x)\right). \end{array}$$

For all r small enough, Ψ is non increasing along radii of $B^2_\varepsilon(0)$. Hence, there exists $r_\varepsilon > 0$ such that for all $r < r_\varepsilon$ there exists $x^* \in \partial B^2_\varepsilon(0)$ such that

$$\Psi(x^*) = \min_{x\in B^2_\varepsilon(0)} \Psi(x).$$

Proof of lemma 5. Let us first prove that Ψ reaches a minimum on $B^2_\varepsilon(0)$.

$$\begin{aligned} \Psi : y \mapsto \lambda(B_r(y)\cap B^2_\varepsilon(0)) &= \int_{B^2_\varepsilon(0)} 1_{B_r(y)}(x) d\lambda(x) \\ &= \int_{B^2_\varepsilon(0)} 1_{B'_r(y)}(x) d\lambda(x), \end{aligned}$$

where $B'_r(y) = \{x \in \mathbb{R}^n \mid \|x - y\| < r\}$, is lower semi-continuous since $y \mapsto 1_{B'_r(y)}(x)$ is lower semi-continuous for all $x \in \mathbb{R}^n$. The conclusion follows from the fact that $B^2_\varepsilon(0)$ is compact.

Now denote, for all $t \in \mathbb{R}_+$, $x_t = (t\ 0\ \dots\ 0)^T \in \mathbb{R}^n$. Let $\varepsilon > 0$ and $r_\varepsilon > 0$ be such that for all $r < r_\varepsilon$ there exists $\underline{t} > 0$ such that $B_r(0) \subset B^2_\varepsilon(x_{\underline{t}})$ and $|y_1| \geq \underline{t}$ for all $y = (y_1\ \dots\ y_n)^T \in B_r(0)$.

Denote $V_t = B^2_\varepsilon(x_t) \cap B_r(0)$. Let $\underline{t} \leq t_1 \leq t_2$ and $y \in V_{t_2}$. Hence y is such that $\|y\| \leq r$ and $\|x_{t_2} - y\|_2 \leq \varepsilon$. Thus $0 \leq \underline{t} - |y_1| \leq t_1 - y_1 \leq t_2 - y_1$. Hence $\|x_{t_1} - y\|_2 \leq \|x_{t_2} - y\|_2$, thus $V_{t_2} \subset V_{t_1}$.

Finally

$$\lambda(V_{t_2}) \leq \lambda(V_{t_1}).$$

By symmetry of $B^2_\varepsilon(0)$ and $B_r(0)$, this setting is without loss of generality and lemma 5 is proved. □

Proof of theorem 1. Let $\varepsilon > 0$, let $r \in (0, \varepsilon)$.

Define

$$Q_\varepsilon = Q + B^2_\varepsilon(0),$$

and

$$I^\varepsilon(r) = \{y \in Q_\varepsilon \mid B_r(y) \subset Q_\varepsilon\}.$$

Define as well

$$\lambda^\varepsilon_{\min}(r) = \min_{y\in Q_\varepsilon} \lambda(B_r(y)\cap Q_\varepsilon).$$

Finally define $\mu \in \Delta(Q_\varepsilon)$ such that for all $B \subset Q_\varepsilon$ measurable

$$\mu(B) = \frac{\lambda\left(B \cap I^\varepsilon(r)\right) \lambda^\varepsilon_{\min}(r) + \lambda(B \cap (Q_\varepsilon \setminus I^\varepsilon(r))\lambda(B_r)}{\lambda(I^\varepsilon(r))\lambda^\varepsilon_{\min}(r) + \lambda(B_r)\lambda(Q_\varepsilon \setminus I^\varepsilon(r))}.$$

Since by definition $\lambda(B_r) \geq \lambda^\varepsilon_{\min}(r)$, for all $x \in Q_\varepsilon$

$$\mu(B_r(x) \cap Q_\varepsilon) \geq \frac{\lambda^\varepsilon_{\min}(r)\lambda(B_r)}{\lambda(I^\varepsilon(r))\lambda^\varepsilon_{\min}(r) + \lambda(B_r)\lambda(Q_\varepsilon \setminus I^\varepsilon(r))}.$$

Because the hider can play in (Q_ε, r) as he would play in (Q, r), $V_{Q_\varepsilon}(r) \leq V_Q(r)$.

By proposition 11,

$$\frac{\lambda^\varepsilon_{\min}(r)\lambda(B_r)}{\lambda(I^\varepsilon(r))\lambda^\varepsilon_{\min}(r) + \lambda(B_r)\lambda(Q_\varepsilon \setminus I^\varepsilon(r))} \leq V_{Q_\varepsilon}(r) \leq V_Q(r) \leq \frac{\lambda(B_r)}{\lambda(Q)}.$$

Dividing by $\lambda(B_r)/\lambda(Q)$

$$\frac{\lambda^\varepsilon_{\min}(r)\lambda(Q)}{\lambda(I^\varepsilon(r))\lambda^\varepsilon_{\min}(r) + \lambda(B_r)\lambda(Q_\varepsilon \setminus I^\varepsilon(r))} \leq \frac{V_Q(r)\lambda(Q)}{\lambda(B_r)} \leq 1.$$

Let us show that

$$\forall \varepsilon > 0 \ \bigcup_{r>0} I^\varepsilon(r) = \mathring{Q}_\varepsilon.$$

Indeed, let $y \in \bigcup_{r>0} I^\varepsilon(r)$. There exists $r > 0$ such that $y \in I^\varepsilon(r)$. Thus there exists $r > 0$ such that $B_r(y) \subset Q_\varepsilon$.

Conversely, let $y \in \mathring{Q}_\varepsilon$. There exists $r' > 0$ such that $B'_{r'}(y) \subset \mathring{Q}_\varepsilon$, where $B'_{r'}(y) = \{x \in \mathbb{R}^n \mid \|x - y\| < r'\}$. Take $0 < r < r'$, then $B_r(y) \subset \mathring{Q}_\varepsilon$ hence $y \in I^\varepsilon(r)$.

For all $r_1, r_2 > 0$ such that $r_1 > r_2$ one has $I^\varepsilon(r_1) \subset I^\varepsilon(r_2)$. Hence $\lim_{r \to 0} \lambda(I^\varepsilon(r)) = \lambda(\mathring{Q}_\varepsilon)$.

Letting $r \to 0$, by lemma 4 and lemma 5 one has

$$\frac{\lambda(Q)}{\lambda(\mathring{Q}_\varepsilon) + \frac{c_2^n}{2c_1^n}\lambda(\partial Q_\varepsilon)} \leq \lim_{r \to 0} \frac{V_Q(r)\lambda(Q)}{\lambda(B_r)} \leq 1. \tag{2}$$

Let us show that

$$\bigcap_{\varepsilon>0} \mathring{Q}_\varepsilon = \bigcap_{\varepsilon>0} Q_\varepsilon = Q.$$

Indeed, let $y \in \bigcap_{\varepsilon>0} Q_\varepsilon$. For all $\varepsilon > 0$ $\min_{z \in Q} \|y - z\|_2 \leq \varepsilon$, hence $y \in Q$. Conversely, for all $\varepsilon > 0$ $Q \subset \mathring{Q}_\varepsilon$ hence $Q \subset \bigcap_{\varepsilon>0} \mathring{Q}_\varepsilon$. Moreover for all $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 < \varepsilon_2$ one has $Q_{\varepsilon_1} \subset Q_{\varepsilon_2}$.

Hence $\lim_{\varepsilon\to 0}\lambda(\mathring{Q}_\varepsilon)=\lambda(Q)$, $\lim_{\varepsilon\to 0}\lambda(Q_\varepsilon)=\lambda(Q)$ and $\lambda(\partial Q_\varepsilon)=\lambda(Q_\varepsilon)-\lambda(\mathring{Q}_\varepsilon)$ so

$$\lim_{\varepsilon\to 0}\lambda(\partial Q_\varepsilon)=0.$$

Letting $\varepsilon\to 0$ in equation (2),

$$\frac{\lambda(Q)}{\lambda(Q)}\leq \lim_{r\to 0}\frac{V_Q(r)\lambda(Q)}{\lambda(B_r)}\leq 1.$$

Finally,

$$\lim_{r\to 0}\frac{V_Q(r)\lambda(Q)}{\lambda(B_r)}=1.$$

□

A consequence of theorem 1 is that for a compact Q included in $\mathbb{R}^n$ such that $\lambda(Q)>0$,

$$V_Q(r)\sim r^n\frac{\lambda(B_1)}{\lambda(Q)}$$

as r goes to 0. When $\lambda(Q)=0$, it is not always the case that V_Q admits an equivalent of the form Mr^α, α and $M>0$, as r goes to 0, as the following example shows.

Example 5. *Let $Q\subset[0,1]$ be the following Cantor-type set. Define $C_0=[0,1]$, and for all $n\in\mathbb{N}^*$ $C_n=\frac{1}{4}C_{n-1}\cup\left(\frac{3}{4}+\frac{1}{4}C_{n-1}\right)$. Finally, let $Q=\bigcap_{n\in\mathbb{N}}C_n$.*

Q is compact and $\lambda(Q)=0$. The value of the hiding game played on Q is given by the following formula:

$$V_Q(r)=\begin{cases}\frac{1}{2^n} & \text{if } r\in\left[\frac{1}{2^{2n}},\frac{3}{2^{2n}}\right),n\in\mathbb{N}^*\\ \frac{1}{2^{n-1}} & \text{if } r\in\left[\frac{3}{2^{2n}},\frac{1}{2^{2(n-1)}}\right),n\in\mathbb{N}^*\end{cases}.$$

Indeed, let $\Sigma_1=\{0,1\}$ and for all $n\in\mathbb{N}\setminus\{0,1\}$ let $\Sigma_n=\frac{1}{4}\Sigma_{n-1}\cup\left(\frac{3}{4}+\frac{1}{4}\Sigma_{n-1}\right)$.

For $n\in\mathbb{N}^*$, consider the following strategy σ_n: choose uniformly a point in Σ_n, that is with probability $\frac{1}{|\Sigma_n|}=\frac{1}{2^n}$.

Let $n\in\mathbb{N}^*$ suppose $r\in\left[\frac{1}{2^{2n}},\frac{3}{2^{2n}}\right)$. Then for all point $q\in Q$ there is exactly one point s in Σ_n such that $|q-s|\leq r$. Hence σ_n is an equalizing strategy which guarantees $\frac{1}{2^n}$ to both players.

Let $\Sigma'_1=\{1/4\}$ and for all $n\in\mathbb{N}\setminus\{0,1\}$ let $\Sigma'_n=\frac{1}{4}\Sigma'_{n-1}\cup\left(1-\frac{1}{4}\Sigma'_{n-1}\right)$. For $n\in\mathbb{N}^*$ consider the following strategy σ'_n: choose uniformly a point in Σ'_n, that is with probability $\frac{1}{|\Sigma'_n|}=\frac{1}{2^{n-1}}$.

Suppose now that $r \in \left[\frac{3}{2^{2n}}, \frac{1}{2^{2(n-1)}}\right]$*. Then for all points* $q \in Q$ *there is exactly one point* s *in* Σ'_n *such that* $|q - s| \leq r$*. Hence* σ'_n *is an equalizing strategy which guarantees* $\frac{1}{2^{n-1}}$ *to both players.*

In particular, for all $n \in \mathbb{N}^*$

$$V_Q\left(\frac{1}{2^{2n-1}}\right) = V_Q\left(\frac{1}{2^{2n}}\right) = \frac{1}{2^n}.$$

Let $(r_n)_{n\in\mathbb{N}^*} = \left(\frac{1}{2^n}\right)_{n\in\mathbb{N}^*}$*. Let* $\alpha > 0$*. Then for all* $n \in \mathbb{N}^*$

$$\frac{V_Q(r_{2n-1})}{(r_{2n-1})^\alpha} = 2^{\alpha(2n-1)}V_Q\left(\frac{1}{2^{2n-1}}\right) = \frac{1}{2^\alpha}2^{(2\alpha-1)n},$$

thus

$$\lim_{n\to+\infty}\frac{V_Q(r_{2n-1})}{(r_{2n-1})^\alpha} = \begin{cases} +\infty & \text{if } \alpha > 1/2 \\ \frac{1}{\sqrt{2}} & \text{if } \alpha = 1/2 \\ 0 & \text{if } \alpha < 1/2. \end{cases}$$

And,

$$\frac{V_Q(r_{2n})}{(r_{2n})^\alpha} = 2^{2\alpha n}V_Q\left(\frac{1}{2^{2n}}\right) = 2^{(2\alpha-1)n},$$

thus

$$\lim_{n\to+\infty}\frac{V_Q(r_{2n})}{(r_{2n})^\alpha} = \begin{cases} +\infty & \text{if } \alpha > 1/2 \\ 1 & \text{if } \alpha = 1/2 \\ 0 & \text{if } \alpha < 1/2. \end{cases}$$

Hence, $r \mapsto V_Q(r)$ *does not admit an equivalent of the form* $r \mapsto Mr^\alpha$*,* α *and* M *positive numbers, as* r *goes to* 0*.*

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