Fence patrolling by mobile agents with distinct speeds*

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Abstract

Suppose we want to patrol a fence (line segment) using k mobile agents with given speeds $v_1, \ldots, v_k$ so that every point on the fence is visited by an agent at least once in every unit time period. Czyzowicz et al. conjectured that the maximum length of the fence that can be patrolled is $(v_1 + \cdots + v_k)/2$, which is achieved by the simple strategy where each agent i moves back and forth in a segment of length $v_i/2$. We disprove this conjecture by a counterexample involving $k = 6$ agents. We also show that the conjecture is true for $k \leq 3$.

1 Introduction

Patrolling is a well-studied task in robotics. A set of mobile agents move around a given area to protect or supervise it, with the goal of ensuring that each point in the area is visited frequently enough [2, 3, 5, 9, 13]. While many authors study heuristic patrolling strategies for various settings and analyze their performance through experiment, recent studies on theoretical optimality of strategies have revealed that there are interesting questions and intricacies even in the simplest settings [5, 11].

One of the fundamental problems considered by Czyzowicz et al. [5] is to patrol a line segment (called the *fence*) using k mobile agents with given speeds. They showed that the simple partition-based strategy, which is used as parts of many strategies in more general problems [2, 3, 9, 11], is optimal in this setting for $k = 2$. They conjectured that it is also optimal for every k. In this paper, we prove that the conjecture holds for $k = 3$ (Section 3), but fails in general (Section 2).

Formal description of fence patrolling. We are given a line segment of length l, which is identified with the interval $[0, l]$. A set of points (mobile agents) $a_1, a_2, \ldots, a_k$ move along the segment. They can move in both directions, and can pass one another. The speed of each agent a_i may vary during its motion, but its absolute value is bounded

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by the predefined maximum speed v_i. The position of agent a_i at time t is denoted $a_i(t)$. Thus, the motion of the agent a_i is described by a function $a_i\colon [0,\infty) \to [0,l]$ satisfying $|a_i(t) - a_i(t+\epsilon)| \le v_i \cdot \epsilon$ for any $t \ge 0$ and $\epsilon > 0$. A *strategy* (or *schedule*) is given by a k-tuple of such functions a_i.

For a position $x \in [0,l]$ and time $t^* \in [0,\infty)$, the agent a_i is said to *cover* $(x;t^*)$ if $a_i(t) = x$ for some $t \in [t^*-1, t^*)$. A strategy is said to *patrol* the segment $[0,l]$ if for any $x \in [0,l]$ and $t^* \in [1,\infty)$, some agent a_i covers $(x;t^*)$.

Given the speeds $v_1, \ldots, v_k$, we want a strategy that patrols the longest possible fence. This is equivalent, through scaling, to fixing the length of the fence and minimizing the time, often called the *idle time*, during which some point is left unattended by any agent.

The partition-based strategy. An obvious strategy for fence patrolling is as follows: partition the fence $[0,l]$ into k segments, proportionally to the maximum speeds $v_1, \ldots, v_k$, and let each agent a_i patrol the ith segment by alternately visiting both endpoints with its maximum speed. We call this the *partition-based strategy*.

Since each agent a_i can patrol a segment of length $v_i/2$, the partition-based strategy can patrol a segment of length $l = (v_1 + \cdots + v_k)/2$. Czyzowicz et al. [5] observed that this is optimal when $k = 2$. They conjectured that it is also the case for every k, that is, a segment of length $l > (v_1 + \cdots + v_k)/2$ cannot be patrolled.

In this paper, we disprove this conjecture by demonstrating $k = 6$ agents that patrol a fence of length greater than $(v_1 + \cdots + v_k)/2$ (Theorem 1). On the other hand, we show that the partition-based strategy is optimal when $k = 3$ (Theorem 4).

2 The partition-based strategy is not always optimal

Fig. 1 shows six agents with speeds 1, 1, 1, 1, 7/3, 1/2 who patrol a fence of length 7/2. The fence is placed horizontally and time flows upwards. The region covered by each agent is shown shaded (i.e., the agent itself moves along the lower edge of each shaded band of height 1). This strategy is periodic in the sense that each agent repeats its motion every 7 unit times. The four agents with speed 1, shown in the diagram on the left, visit the two endpoints alternately. The region covered by them is shown again by the dotted lines in the middle diagram, where another agent with speed 7/3 covers most of the remaining region, but misses some small triangles. They are covered by the last agent with speed 1/2 in the diagram on the right. Note that the partition-based strategy with these agents would only patrol the length $(1+1+1+1+7/3+1/2)/2 = 41/12 < 7/2$. Thus,

Theorem 1. *There are settings of agents' speeds for which the partition-based strategy is not optimal.*

Note that the above example for $k = 6$ agents easily implies the non-optimality of the partition-based strategy for each $k \ge 6$: we can, for example, modify the above strategy by extending the fence to the right and adding a seventh agent who is just fast enough to cover the extended part by moving back and forth.

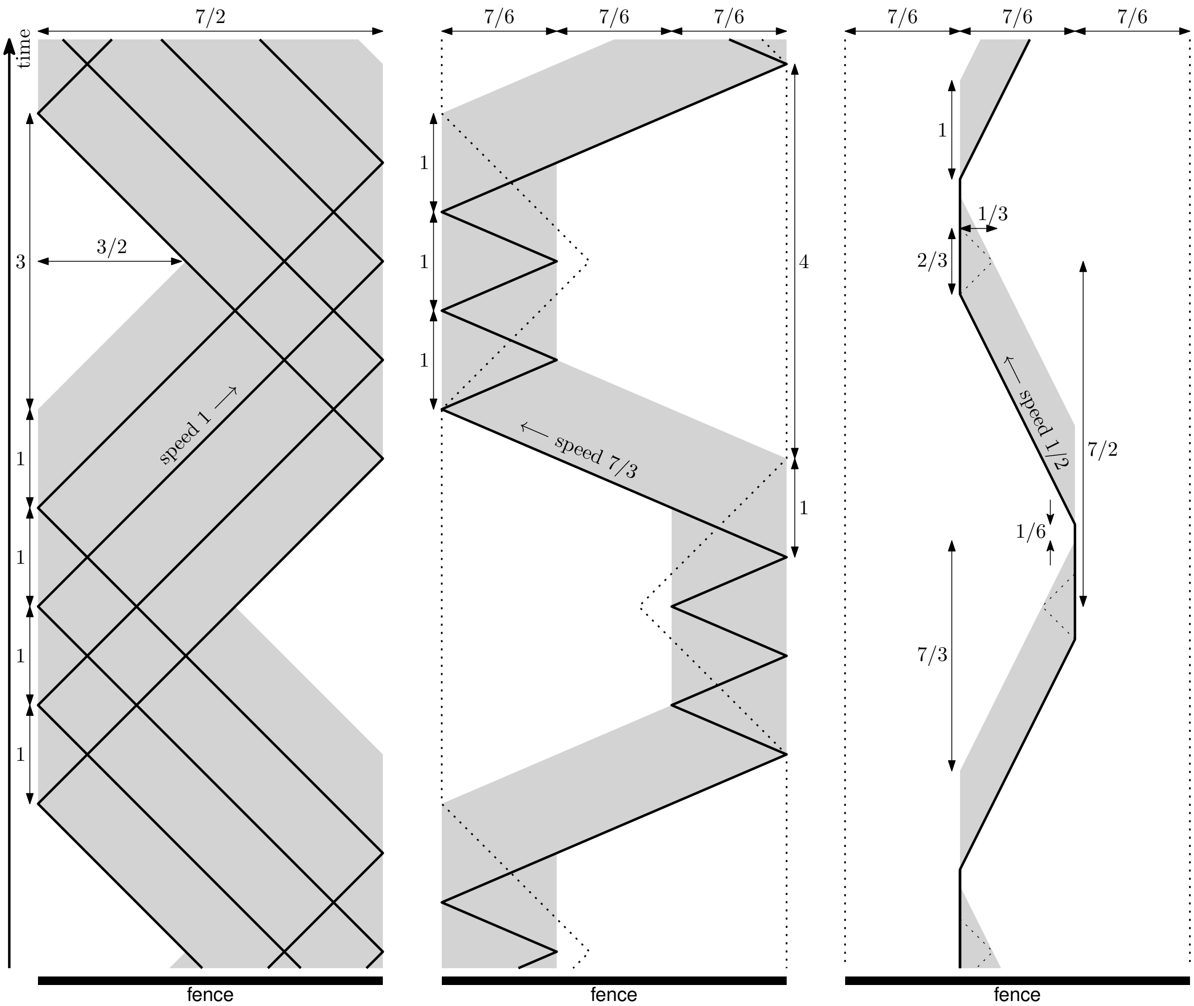

Figure 1: Six agents patrolling a longer fence than they would with the partition-based strategy.

Another example involving more agents but perhaps simpler is shown in Fig. 2, where six agents with speed 5 and three with speed 1 patrol a fence of length 50/3 using a periodic strategy, with period 10/3. Here, the six fast agents in the first diagram work in two groups of three in a synchronized way. The region covered by them is shown again in the second diagram in dotted lines, where the missed small triangular regions are covered by the three slow agents. The partition-based strategy would only achieve 33/2.

3 Cases where the partition-based strategy is optimal

Before proving the optimality of the partition-based strategy for three agents (Section 3.3), we briefly discuss the much simpler cases of equal-speed agents (Section 3.1) and two agents (Section 3.2).

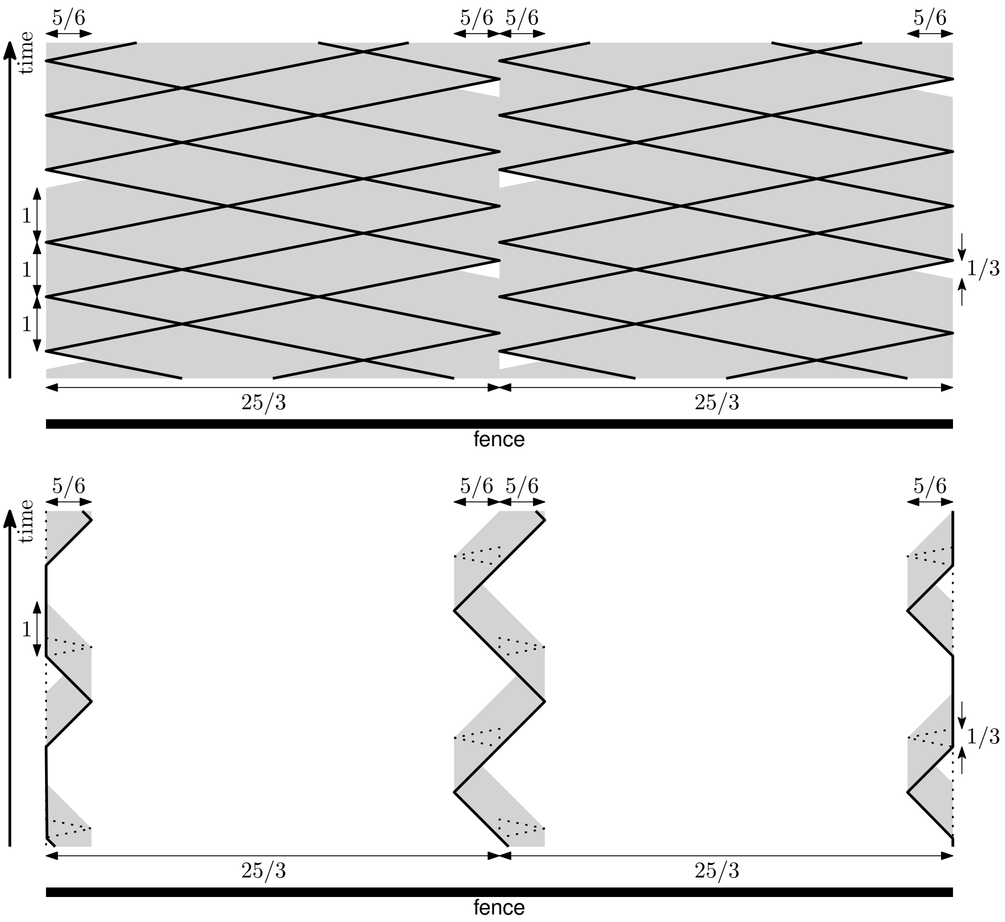

Figure 2: Six agents with speed 5 (top) and three agents with speed 1 (bottom) together patrolling a longer fence than they would with the partition-based strategy.

3.1 Agents with equal speeds

In the homogeneous setting where all agents have the same speed v, it is relatively easy to prove that the partition-based strategy is optimal. This is true more generally when there are regions that do not have to be visited frequently [3], as well as in related settings where the time and locations are discretized in a certain way [11, Section III]. For the sake of completeness, we provide a short proof for our setting:

Theorem 2. *If all agents have the same speed, the partition-based strategy is optimal.*

Proof. We proceed by induction on the number k of agents. We may assume that the agents never switch positions, so that $a_1(t) \leq \cdots \leq a_k(t)$ for all t. This is because two agents passing each other could as well just turn back. Under this assumption, the agent a_1 must visit the point 0 once in every unit time, and hence is confined to the interval $[0, v/2]$. The rest of the fence must be patrolled by the other $k-1$ agents, who, by the induction hypothesis, cannot do better than the partition-based strategy which patrols the length $(k-1)v/2$. Thus the total length is bounded by $v/2 + (k-1)v/2 = kv/2$. $\square$

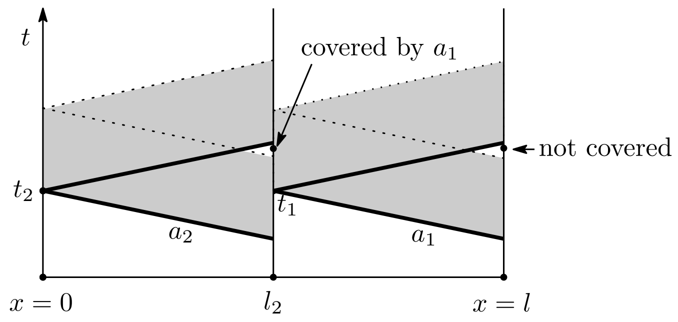

Figure 3: Proof of Theorem 3

3.2 Two agents

Although the optimality of the partition-based strategy for two agents was already pointed out in [5], we present an alternative proof here. Some ideas in the proof will be used for three agents (Section 3.3) and also for the weighted setting (Section 4).

Theorem 3. *For two agents, the partition-based strategy is optimal.*

Proof. Suppose that this was false. That is, suppose that there is a strategy where agents a_1 and a_2 patrol $[0, l]$ for some $l > (v_1 + v_2)/2$. We may assume that $v_1 \geq v_2$. Let $l_i = v_i l/(v_1 + v_2)$ for $i = 1, 2$. Note that $l = l_1 + l_2$, and that it takes time longer than $1/2$ for agent a_i to travel the distance l_i.

For any time $t \geq 0$, each agent must visit an endpoint (0 or l) some time after t. To see this, let $t_0 > t$ be a time at which the endpoint 0 is visited. Then $(l; t_0 + 1/2)$ cannot be covered by this same agent, and thus is covered by the other agent.

Hence, the slower agent a_2 visits an endpoint, say 0, at some time $t_2 > 1$. This implies that $(l_2; t_2 + 1/2)$ cannot be covered by a_2. It must therefore be covered by a_1, that is, a_1 must visit l_2 at some time $t_1 \in [t_2 - 1/2, t_2 + 1/2)$. This implies that $(l; t_1 + 1/2)$ is not covered by a_1. But it is not covered by a_2 either, because $t_1 + 1/2 \in [t_2, t_2 + 1)$ and the agent a_2 cannot travel the distance $l_1 + l_2$ in unit time (see Fig. 3). This is a contradiction. □

3.3 Three agents

In this section, we show that Czyzowicz et al.'s conjecture is true for three agents:

Theorem 4. *For three agents, the partition-based strategy is optimal.*

For a contradiction, suppose that agents a_1, a_2, a_3 with speeds $v_1 \geq v_2 \geq v_3$ patrol $[0, l]$, where $l > (v_1 + v_2 + v_3)/2$. For $i = 1, 2, 3$ let $l_i = v_i l/(v_1 + v_2 + v_3)$, so that $l = l_1 + l_2 + l_3$ and $l_i > v_i/2$. We start with some lemmas about the coverage of endpoints.

Lemma 5. *For any $t^* \geq 0$, at least two different agents visit 0 after the time t^*, and at least two different agents visit l after the time t^*.*

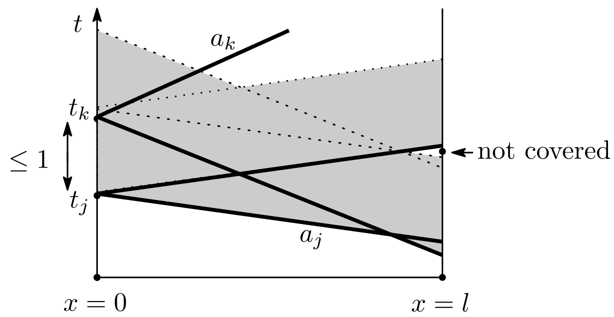

Figure 4: Proof of Lemma 6

Proof. Let $\{i, j, k\} = \{1, 2, 3\}$, and assume that a_i is the only agent that visits 0 after time t^*. This forces it to stay (after time $t^* + 1/2$) in the part $[0, l_i]$, so the remaining part $[l_i, l]$ of length $l_j + l_k$ has to be patrolled by a_j and a_k, contradicting Theorem 3. The same argument applies to the other endpoint l. □

Lemma 6. *For any $t^* \geq 0$, each agent visits at least one of 0 and l after the time t^*.*

Proof. Let $\{i, j, k\} = \{1, 2, 3\}$, and assume that a_i does not visit 0 after t^*. By Lemma 5, both a_j and a_k visit 0 infinitely often after t^*. Thus, $a_j(t_j) = a_k(t_k) = 0$ for some $t_j, t_k > t^* + 1/2$ with $t_j \leq t_k \leq t_j + 1$ (see Fig. 4). The pair $(l; t_j + l/v_j)$ is not covered by a_j, because $(t_j + l/v_j) - (t_j - l/v_j) > 1$. It is not covered by a_k either, because

$$\left(t_j + \frac{l}{v_j}\right) - \left(t_k - \frac{l}{v_k}\right) > t_j + \frac{v_j + v_k}{2v_j} - t_k + \frac{v_j + v_k}{2v_k} = (t_j - t_k) + 1 + \frac{1}{2}\left(\frac{v_k}{v_j} + \frac{v_j}{v_k}\right) \geq 1.$$

Hence, it must be covered by a_i, which means that a_i visits l after the time t^*. □

Lemma 7. *Suppose that $a_2(t_2) = a_3(t_3) = 0$ (resp. $= l$) for some $t_2, t_3 > 1$. Then,*

- *$a_1(t_1) = 0$ (resp. $= l$) for some $t_1 \in (t_2, t_3)$ if $t_2 \leq t_3$, and*
- *$a_1(t_1) = 0$ (resp. $= l$) for some $t_1 \in (t_3, t_2)$ if $t_2 \geq t_3$.*

Proof. Assume that there are $t_3 \geq t_2 > 1$ such that $a_2(t_2) = a_3(t_3) = 0$ and $a_1(t_1) \neq 0$ for any $t_1 \in (t_2, t_3)$. We may then retake t_2 and t_3, if necessary, and have $t_3 - t_2 \leq 1$ (see Fig. 5). By the same argument as the proof of Lemma 6, the pair $(l_2 + l_3; t_2 + (l_2 + l_3)/v_2)$ is covered by neither a_2 nor a_3. More precisely, it is not covered by a_2, because $\big(t_2 + (l_2 + l_3)/v_2\big) - \big(t_2 - (l_2 + l_3)/v_2\big) > 1$, and it is not covered by a_3 either, because

$$\left(t_2 + \frac{l_2 + l_3}{v_2}\right) - \left(t_3 - \frac{l_2 + l_3}{v_3}\right) > t_2 + \frac{v_2 + v_3}{2v_2} - t_3 + \frac{v_2 + v_3}{2v_3} = (t_2 - t_3) + 1 + \frac{1}{2}\left(\frac{v_3}{v_2} + \frac{v_2}{v_3}\right) \geq 1.$$

Hence, it must be covered by a_1, which means that $a_1(t_1) = l_2 + l_3$ for some $t_1 \in [t_2 + (l_2 + l_3)/v_2 - 1, t_2 + (l_2 + l_3)/v_2)$. Since $v_1 \geq v_2 \geq v_3$, $(l; t_1 + l_1/v_1)$ is covered by none of a_1, a_2, and a_3, which is a contradiction.

The argument is similar when $t_2 \geq t_3$ and when $a_2(t_2) = a_3(t_3) = l$. □

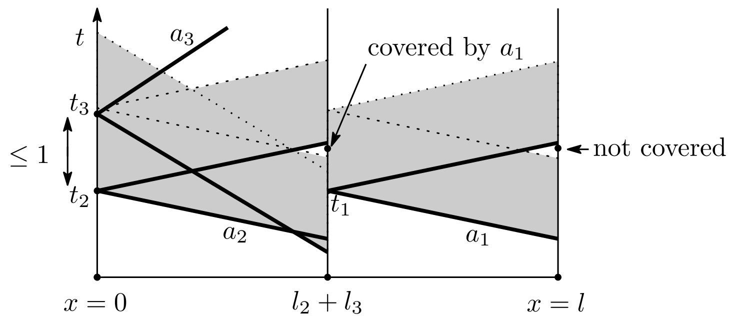

Figure 5: Proof of Lemma 7

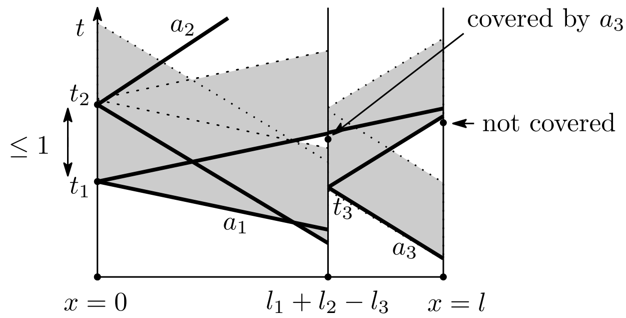

Figure 6: Case of $v_1 \geq 2v_2 + v_3$

By Lemmas 6 and 7, it happens infinitely often that one of the endpoints is visited by a_1 and then immediately by a_2. Let us focus on one occurrence of this event, sufficiently later in time (time $1 + l/v_3$ is enough), which, without loss of generality, happens at the endpoint 0. That is, we fix t_1 and t_2 with $1 + l/v_3 < t_1 \leq t_2 \leq t_1 + 1$ such that $a_1(t_1) = a_2(t_2) = 0$ and no agent visits 0 during the time interval (t_1, t_2). Note that we choose $1 + l/v_3$ so that every value of time appearing in the proof is at least 1. Now we split into two cases.

Case I: $v_1 \geq 2v_2 + v_3$

The pair $(l_1 + l_2 - l_3; t_1 + \frac{l_1+l_2-l_3}{v_1})$ is not covered by a_1, because $\left(t_1 + \frac{l_1+l_2-l_3}{v_1}\right) - \left(t_1 - \frac{l_1+l_2-l_3}{v_1}\right) = 2 \cdot \frac{l_1+l_2-l_3}{v_1} \geq \frac{2l_1}{v_1} > 1$. It is not covered by a_2 either, because

$$\begin{aligned}\left(t_1 + \frac{l_1 + l_2 - l_3}{v_1}\right) - \left(t_2 - \frac{l_1 + l_2 - l_3}{v_2}\right) &> t_1 + \frac{v_1}{2v_1} - t_2 + \frac{v_1 + v_2 - v_3}{2v_2} \\ &= (t_1 - t_2) + 1 + \frac{v_1 - v_3}{2v_2} \geq 1.\end{aligned}$$

Hence, it must be covered by a_3 (Fig. 6), which means that $a_3(t_3) = l_1 + l_2 - l_3$ for some $t_3 \in [t_1 + \frac{l_1+l_2-l_3}{v_1} - 1, t_1 + \frac{l_1+l_2-l_3}{v_1})$.

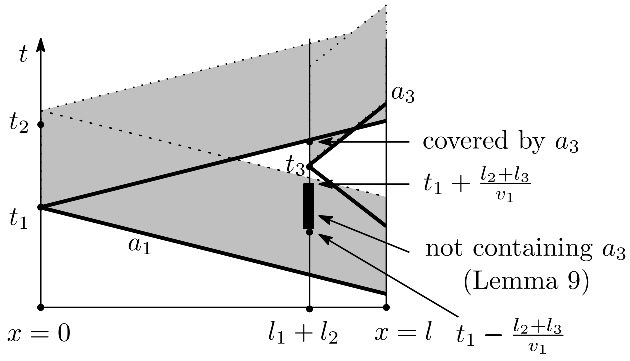

Figure 7: Lemmas 9 and 10

If $t_3 + \frac{2l_3}{v_3} \leq t_1 + \frac{l}{v_1}$, then $(l; t_3 + \frac{2l_3}{v_3})$ is not covered by any of a_1, a_2, a_3 (see Fig. 6). Otherwise, $(l; t_1 + \frac{l}{v_1})$ is not covered by any of a_1, a_2, a_3 (not by a_3 because $(t_1 + \frac{l}{v_1}) - (t_3 - \frac{2l_3}{v_3}) > t_3 - (t_3 - 1) = 1$).

Case II: $v_1 \leq 2v_2 + v_3$

This is the harder case and takes up the rest of this section. Again, let t_1 and t_2 be such that $1 + \frac{l}{v_3} < t_1 \leq t_2 \leq t_1 + 1$ and $a_1(t_1) = a_2(t_2) = 0$.

Lemma 8. *$a_2(t) \neq l$ for any $t \in [t_1 - \frac{l}{v_1}, t_1 + \frac{l}{v_1}]$.*

Proof. The pair $(l; t_1 + l/v_1)$ is not covered by a_1, because $(t_1 + l/v_1) - (t_1 - l/v_1) > 1$. It is not covered by a_2 either, because

$$\left(t_1 + \frac{l}{v_1}\right) - \left(t_2 - \frac{l}{v_2}\right) > t_1 + \frac{v_1 + v_2}{2v_1} - t_2 + \frac{v_1 + v_2}{2v_2} = (t_1 - t_2) + 1 + \frac{1}{2}\left(\frac{v_2}{v_1} + \frac{v_1}{v_2}\right) \geq 1.$$

Hence, it must be covered by a_3, i.e., a_3 visits l at some time $t' \in [t_1 + \frac{l}{v_1} - 1, t_1 + \frac{l}{v_1}) \subseteq [t_1 - \frac{l}{v_1}, t_1 + \frac{l}{v_1}]$. If we assume that $a_2(t) = l$ for some time $t \in [t_1 - \frac{l}{v_1}, t_1 + \frac{l}{v_1}]$, then, by Lemma 7, $a_1(t'') = l$ for some $t'' \in (t, t') \subseteq (t_1 - \frac{l}{v_1}, t_1 + \frac{l}{v_1})$ (or $t'' \in (t', t) \subseteq (t_1 - \frac{l}{v_1}, t_1 + \frac{l}{v_1})$). This contradicts that $a_1(t_1) = 0$ and $|t_1 - t''| < \frac{l}{v_1}$. Therefore, we conclude that a_2 cannot visit l during $[t_1 - \frac{l}{v_1}, t_1 + \frac{l}{v_1}]$. ☐

Lemma 9. *$a_3(t) \neq l_1 + l_2$ for any $t \in [t_1 - \frac{l_2+l_3}{v_1}, t_1 + \frac{l_2+l_3}{v_1}]$ (see Fig. 7).*

Proof. Assume that $a_3(t) = l_1 + l_2$ for some $t \in [t_1 - \frac{l_2+l_3}{v_1}, t_1 + \frac{l_2+l_3}{v_1}]$. Then, since $[t - \frac{1}{2}, t + \frac{1}{2}] \subseteq [t_1 - \frac{l}{v_1}, t_1 + \frac{l}{v_1}]$, neither a_1 nor a_3 covers $(l; t + \frac{1}{2})$. Furthermore, by Lemma 8, $(l; t + \frac{1}{2})$ is not covered by a_2 either. This is a contradiction. ☐

Lemma 10. *$a_3(t) = l_1 + l_2$ for some t such that $t_1 + \frac{l_2+l_3}{v_1} < t < t_1 + \frac{l_1+l_2}{v_1}$ (see Fig. 7).*

Proof. The pair $(l_1 + l_2; t_1 + \frac{l_1+l_2}{v_1})$ is not covered by a_1, because $\left(t_1 + \frac{l_1+l_2}{v_1}\right) - \left(t_1 - \frac{l_1+l_2}{v_1}\right) > 1$. It is not covered by a_2 either, because

$$\left(t_1 + \frac{l_1 + l_2}{v_1}\right) - \left(t_2 - \frac{l_1 + l_2}{v_2}\right) > t_1 + \frac{v_1 + v_2}{2v_1} - t_2 + \frac{v_1 + v_2}{2v_2} = (t_1 - t_2) + 1 + \frac{1}{2}\left(\frac{v_2}{v_1} + \frac{v_1}{v_2}\right) \geq 1.$$

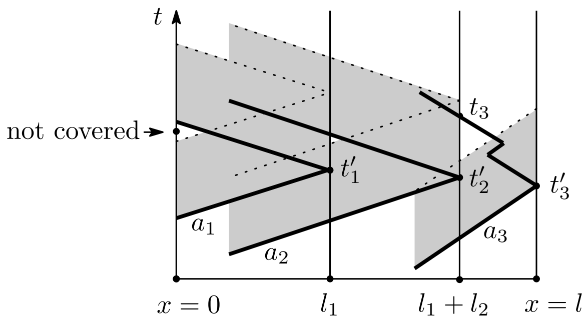

Figure 8: Construction of t_3', t_2', and t_1'

Hence, it must be covered by a_3, which means that $a_3(t) = l_1 + l_2$ for some $t \in [t_1 + \frac{l_1+l_2}{v_1} - 1, t_1 + \frac{l_1+l_2}{v_1})$.

Since $\frac{l_1+l_2}{v_1} + \frac{l_2+l_3}{v_1} = \frac{l_1+2l_2+l_3}{v_1} > \frac{v_1+2v_2+v_3}{2v_1} \geq 1$ by the assumption $v_1 \leq 2v_2 + v_3$, we have $t_1 + \frac{l_1+l_2}{v_1} - 1 > t_1 - \frac{l_2+l_3}{v_1}$. Hence, by Lemma 9, $a_3(t) = l_1 + l_2$ for some t such that $t_1 + \frac{l_2+l_3}{v_1} < t < t_1 + \frac{l_1+l_2}{v_1}$. □

Let t_3 be the minimum value such that $a_3(t_3) = l_1 + l_2$ and $t_1 + \frac{l_2+l_3}{v_1} < t_3 < t_1 + \frac{l_1+l_2}{v_1}$ (see Fig. 7).

Lemma 11. *$a_3(t) = l$ for some $t \in [t_1 + \frac{l}{v_1} - 1, t_3 - \frac{l_3}{v_3}]$.*

Proof. The pair $(l; t_1 + \frac{l}{v_1})$ is not covered by a_1, because $\left(t_1 + \frac{l}{v_1}\right) - \left(t_1 - \frac{l}{v_1}\right) > 1$. By Lemma 8, it is not covered by a_2 either. Hence, $a_3(t) = l$ for some $t \in [t_1 + \frac{l}{v_1} - 1, t_1 + \frac{l}{v_1})$. On the other hand, since $a_3(t_3) = l_1 + l_2$, we have $a_3(t) \neq l$ for any t such that $t_3 - \frac{l_3}{v_3} < t < t_3 + \frac{l_3}{v_3}$. By combining them, we obtain the claim. □

Let t_3' be the maximum value such that $a_3(t_3') = l$ and $t_3' \in [t_1 + \frac{l}{v_1} - 1, t_3 - \frac{l_3}{v_3}]$. Then, $(l_1 + l_2; t_3)$ is not covered by a_3, because $t_3 > (t_3' - \frac{l_3}{v_3}) + 1$ and $t_3' + \frac{l_3}{v_3} > t_1 - \frac{l_2+l_3}{v_1}$. It is not covered by a_1 either, because

$$t_3 > t_1 + \frac{l_2 + l_3}{v_1} > \left(t_1 - \frac{l_1 + l_2}{v_1}\right) + 1$$

by $v_1 \leq 2v_2 + v_3$. Hence, it is covered by a_2, which means that $a_2(t_2') = l_1 + l_2$ for some t_2' such that $t_3 - 1 \leq t_2' < t_3$ (see Fig. 8).

Since $(l_1; t_2' + \frac{l_2}{v_2})$ is not covered by a_2 or a_3, it is covered by a_1, which means that $a_1(t_1') = l_1$ for some t_1' such that $t_2' + \frac{l_2}{v_2} - 1 \leq t_1' < t_2' + \frac{l_2}{v_2}$. In this case, $(0; t_1' + \frac{l_1}{v_1})$ is not covered by any of a_1, a_2, and a_3, which is a contradiction. We have proved Theorem 4.

4 Final remarks

The partition-based strategy is widely used as part of multi-agent patrolling strategies. We studied its theoretical optimality in one of the simplest settings: the terrain is a line segment, and the agents are points with given maximal speeds.

The weighted setting. It may be natural to consider the *weighted* version of the problem where each agent has a different power of influence. That is, the idle time $T_i > 0$ depends on the agent a_i, and is called the *weight* of a_i. The setting we have been dealing with in the previous sections is the special case where $T_i = 1$ for all i. In the general setting, we say that a_i *covers* the pair $(x; t^*)$ if $a_i(t) = x$ for some $t \in [t^* - T_i, t^*)$. The agents $a_1, \ldots, a_k$ are said to *patrol* $[0, l]$ if for any $x \in [0, l]$ and $t^* \in [\max_i(T_i), \infty)$, the pair $(x; t^*)$ is covered by some a_i.

As in the unweighted case, we can consider the partition-based strategy. This time, each agent a_i is assigned a segment of length proportional to the weighted speed $v_i T_i$.

Theorem 2 remains true in this general setting: the partition-based strategy is optimal when the agents have different weights T_i but the same speed v. To see this, suppose that we could patrol a fence of length $l = \alpha + \sum_{i=1}^{k} vT_i/2$ for some $\alpha > 0$. Let $\tau = 2\alpha/kv$. Since an agent of weight T_i can be simulated by $\lceil T_i/\tau \rceil$ agents of weight τ moving in parallel, this fence can be patrolled by $\kappa = \sum_{i=1}^{k} \lceil T_i/\tau \rceil$ agents, all with weight τ (and speed v). This contradicts (a suitably rescaled version of) Theorem 2, since $l = k\tau v/2 + \sum_{i=1}^{k} T_i v/2 = \sum_{i=1}^{k} (T_i/\tau + 1)v\tau/2 > \kappa v\tau/2$.

Theorem 3 (optimality of the partition-based strategy for two agents) also remains true for weighted agents: the proof goes through if we set $l_i = v_i T_i l/(v_1 T_1 + v_2 T_2)$ instead.

However, Theorem 4 (optimality of the partition-based strategy for three agents) fails for the weighted setting. To see this, consider our first example for Theorem 1 (Fig. 1), and regard the four agents in the left diagram as one agent with weight 4.

Summary of our results. Thus, our current knowledge can be summarized as follows.

- The partition-based strategy is optimal when all agents (possibly weighted) have the same speed (Theorem 2), but not when there are two distinct speeds (Fig. 2).

- The partition-based strategy is optimal when there are two agents with different speeds and weights (Theorem 3), but not when there are three (Fig. 1).

- The partition-based strategy is optimal when there are three agents with the same weight (Theorem 4), but not when there are six (Fig. 1).

The third part settles a conjecture of Czyzowicz et al. [5], but our proof for three agents is already quite involved and seems hard to generalize. It remains open whether the partition-based strategy is optimal for four and five (unweighted) agents.

Related work and generalizations. We considered the patrolling problem in one of its most basic forms: the terrain to be patrolled is a line segment, every point in the terrain must be visited, and each agent is a point with a maximum speed. The problem setting can be generalized in many ways. Another simple terrain that has been studied in Czyzowicz et al. [5] is a cycle, where again it turns out that simple strategies may not be optimal (see also Dumitrescu, Ghosh and Tóth [8]). Collins et al. [3] study the patrolling problem where only part of the fence needs to be visited frequently. Chen, Dumitrescu and Ghosh [1] and Czyzowicz et al. [7] discuss agents with some visibility. Czyzowicz et al. [4] study the setting where agents can move faster when walking without watching (although their problem is to cover the line segment just once, rather than patrolling perpetually).

For practical purposes, it is important to consider decentralized settings where agents need to cooperate with limited global knowledge or computational power [12]. The fact that the partition-based strategy is not always optimal may be bad news in this context, since it is one of the simplest strategies to be realized in a distributed way, using systems of self-stabilizing robots, e.g., in models of "bouncing robots" [6]. Thus a natural question to ask next is whether and how movements better than the partition-based strategy can be realized in various distributed settings.

A revised conjecture. Since the partition-based strategy covers each $(x;t) \in [0,l] \times [1,\infty)$ only doubly, it achieves a 2-approximation (for the problem of finding the longest possible fence that can be patrolled). That is, no strategy patrols a fence longer than $v_1 + \cdots + v_k$ (in the unweighted setting). Although we have shown that the partition-based strategy is not always optimal, it may still be somewhat close to being optimal, given that it is outperformed only slightly by our examples for Theorem 1. In other words, the following may be the case, with a constant c fairly close to $1/2$:

Conjecture. *There is a constant $c < 1$ such that for any k and any $v_1, \ldots, v_k$, no strategy can patrol a fence longer than $c(v_1 + \cdots + v_k)$.*

The partition-based strategy gives a lower bound of $1/2$ for such a constant c. Our first example (Fig. 1) gives $21/41 = 0.5121\ldots$. After a preliminary version of this paper [10] was presented, Chen, Dumitrescu and Ghosh [1] (see also Dumitrescu, Ghosh and Tóth [8]) improved this bound to $25/48 = 0.5208\ldots$. Determining the least c is an interesting question.

Note added for the arXiv version. Kawamura and Soejima [14] recently announced a lower bound of $2/3$. The above conjecture still remains open.

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