



Security Games Involving Search and Patrolling

Steve Alpern
UNIVERSITY OF WARWICK

04/28/2017
Final Report

DISTRIBUTION A: Distribution approved for public release.

Air Force Research Laboratory
AF Office Of Scientific Research (AFOSR)/ IOE
Arlington, Virginia 22203
Air Force Materiel Command

REPORT DOCUMENTATION PAGE				Form Approved OMB No. 0704-0188	
<p>The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Department of Defense, Executive Services, Directorate (0704-0188). Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.</p> <p>PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ORGANIZATION.</p>					
1. REPORT DATE (DD-MM-YYYY) 29-04-2017		2. REPORT TYPE Final		3. DATES COVERED (From - To) 01 Feb 2014 to 31 Jan 2017	
4. TITLE AND SUBTITLE Security Games Involving Search and Patrolling				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER FA9550-14-1-0049	
				5c. PROGRAM ELEMENT NUMBER 61102F	
6. AUTHOR(S) Steve Alpern				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) UNIVERSITY OF WARWICK COVENTRY WEST MIDLANDS, CV4 8UW GB				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) EOARD Unit 4515 APO AE 09421-4515				10. SPONSOR/MONITOR'S ACRONYM(S) AFRL/AFOSR IOE	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S) AFRL-AFOSR-UK-TR-2017-0032	
12. DISTRIBUTION/AVAILABILITY STATEMENT A DISTRIBUTION UNLIMITED: PB Public Release					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT <p>This report covers work carried out on six topics where significant progress has been made under the grant. All are related to the problem of searching for an unknown Hider. The Hider can be stationary, such as an IED (improvised explosive device), or mobile (a terrorist or prey animal), or an abstract concept like Innocent or Guilty in a jury situation. In addition the work on search for a small object, carried out recently, has been published in Alpern and Lidbetter (2015). Related work of Alpern and Baston (2017) on searching for the best candidate among applicants for a job who interview sequentially has been recently published. Work of Alpern and Howard (2016) solves a class of winner-take-all games which include the search game where two searchers try to be the first one to find their target. In all, twelve papers have been published under the grant, including four in Operations Research, two in European Journal of Operational Research, and papers in Management Science, Journal of the Royal Society Interface and Mathematics of Operations Research.</p>					
15. SUBJECT TERMS <p>search games, networks, differential games, operations research, patrolling games, EOARD</p>					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT SAR	18. NUMBER OF PAGES 26	19a. NAME OF RESPONSIBLE PERSON PETERSON, JESSE
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NUMBER (Include area code) 314 235 6292

Network Security Games
Involving Search, Rendezvous or Patrolling
(FA9550-14-1-0049)

Steve Alpern Final Report: February 2013 to January 2016

April 28, 2017

Contents

1	Summary	2
2	Introduction	4
3	Methods, Assumptions and Procedures	4
4	Results and Discussion	4
4.1	Patrolling Games	4
4.2	Accumulation Games	8
4.3	Predator Search for Prey	9
4.3.1	Review of basic game $G(k, 0)$	10
4.3.2	Framework	12
4.3.3	Type II solution for general β and $k \geq M$	13
4.3.4	The ranges of Type I and II solutions	14
4.3.5	Discussion	14
4.4	Search for a Hider who can Escape the Search Region	15
4.4.1	The Limited Time Game $\Gamma(T)$	15
4.4.2	Indifference Search Strategies s_k	16
4.4.3	A Stochastic Game Approach to $\Gamma(T)$	17
4.5	Network Search Using Combinatorial Paths	20
4.6	Optimal Voting Order in Search for Truth	20
4.6.1	Roll-call voting	21
4.6.2	The casting-vote system	22
5	Conclusions	22
6	References	22

List of Figures

1. Attack Strategy $n = 11, m = 4$, taking $q = 1, r = 7$.
2. Attack Strategy, case C, $n = 13$ and $m = 5$.
3. Attack Strategy, case D, $n = 12$ and $m = 4$.
4. Attacks for $n = 10$ and $m = 4$.
5. Flowchart of game dynamics.
6. Optimal Searcher strategies $s_{\bar{k}(T)}$, $T = .5, 1, 2, 3$.
7. Outcome probabilities at equilibrium.
8. In optimal search either L or R must be traversed three times.

1 Summary

This report covers work carried out on six topics where significant progress has been made under the grant. All are related to the problem of searching for an unknown Hider. The hider can be stationary, such as an IED (improvised explosive device), or mobile (a terrorist or prey animal), or an abstract concept like Innocent or Guilty in a jury situation. In addition the work on search for a small object, carried out recently, has been published in Alpern and Lidbetter (2015). Related work of Alpern and Baston (2017) on searching for the best candidate among applicants for a job who interview sequentially has been recently published. Work of Alpern and Howard (2016) solves a class of winner-take-all games which include the search game where two searchers try to be the first one to find their target. In all, twelve papers have been published under the grant, including four in *Operations Research*, two in *European Journal of Operational Research*, and papers in *Management Science*, *Journal of the Royal Society Interface* and *Mathematics of Operations Research*. The work on six topics will be discussed in detail in this report.

- (i) **Patrolling Games** In Alpern, Morton and Papadaki (2011), patrolling games on general graphs were defined. The Attacker chooses a node and attacks it for m consecutive periods, the time needed to disable it or disrupt the network. The Patroller chooses a walk on the network. The Patroller wins the game if his walk is at the attacked node during the attack period, in which case we say he has *intercepted* the attack. With one additional author, T. Lidbetter, the original investigators have applied the general ideas of the first paper to the specific case of line graphs, which can be considered the case of guarding a channel or a border with a moving patrol. This complements earlier work, some of the original problems to be solved in the early work in operations research, of guarding a channel with fixed (but randomized) guards. This work is complete and ready for journal submission in 2015 as Papadaki et al (2015).
- (ii) **Accumulation Games** Accumulation games on discrete locations were introduced by Ruckle and Kikuta. The Hider secretly distributes his total wealth $h > 1$ over locations $1, 2, \dots, n$. The Searcher confiscates the material from any r of these locations. The Hider wins if the wealth remaining at the $n - r$ unsearched locations sums to at least 1; otherwise the Searcher wins. Their game models problems in which the Hider needs to have, after confiscation (or loss by natural causes), a sufficient amount of material (food, wealth, arms) to carry out some objective (survive the winter, buy a house, start an insurrection). The conjecture of Kikuta and Ruckle says that there is always an optimal Hider strategy which places equal amounts of material on certain locations (and nothing on the rest) is still open and known to be hard. In Alpern and Fokkink (2014) we take the hiding locations to be the nodes of a graph and restrict the node sets which the Searcher can remove

to be drawn from a given family: the edges, the connected r -sets, or some other given sets of nodes. This models the case where the pilferer, or storm, is known to act only on a set of close locations. Unlike the original game, our game requires mixed strategies. We give a complete solution for certain classes of graphs. The grant is acknowledged in this publication.

- (iii) **Predator Search for Prey** This work applies the theory of repeated games to the problem of finding a prey which can change locations during the search. An ambush mode (e.g. roadblocks in criminal applications) will catch a moving prey. The choice of the searchers lies in how to spread the effort, over time, between exhaustive search and ambush. This is published in Gal, Alpern and Casas (2015).
- (iv) **Search for a Hider who can Escape the Search Region** Sometime the Hider has the possibility of escaping the search region completely, thus 'winning the game'. An example is the escape of Bin Laden from the Tora Bora Caves, or one of the Paris terrorists from France. This is published in Alpern, Fokkink and Simanjuntak (2016).
- (v) **Network Search Using Combinatorial Paths** The classical problem of searching for an immobile hider on a network, dating from the 1965 book of Isaacs on Differential Games, has traditionally been studied under the assumption that the Searcher can move about the network, at unit speed, using paths that may turn around inside some edge. The new work requires that search paths are the paths more usually studied in computer science and graph theory, namely sequences of adjacent edges, traversed from end to end at unit speed. A useful method of searching semi Eulerian networks is given and applied to determine the optimal search method for certain networks. An unintended success of this work is finding a counterexample to a conjecture of S. Gal by exhibiting a network where optimal search requires that one of the edges must be traversed three times. This work is in the preprint Alpern (2016).
- (vi) **Search for true state of Nature through sequential voting** With my colleague Bo Chen, I have been determining the ideal voting order when experts (jurors) of varying abilities must sequentially give their opinion on a topic relating to their expertise. We find that for small groups of experts, or juries, neither seniority (most able vote first) or anti-seniority (most able vote last) is the ideal order. We use both numerical and algebraic techniques for this ongoing research.

2 Introduction

We cover the work done under the grant, which concerns the application of game theoretic concepts and techniques to the area of search theory, broadly defined. The notion of a search game, or hide-and-seek game, was introduced in the classic text of Rufus Isaacs, *Differential Games*. Most of that text concerns perfect information differential games, like pursuit-evasion, where perhaps the payoff is still capture time, but both players know the position and speed vector of the other. In a final, speculative section, Isaacs proposed the topic of differential games with imperfect information. The basic example was that of a search game, where neither player knew the location of the other (until they came close). The grantholder was the first to solve one of Isaacs' problems, and has worked in the field of search games (among other areas) ever since. Many of the initial results in the field are contained in the first text of Gal, published in 1980. More recent are in the book of Alpern and Gal, *The Theory of Search Games and Rendezvous*, published in 2003.

We will discuss in detail the progress made in carrying out the grant in six sections: Patrolling games, accumulation games, predator search for prey, search for a hider that can escape the search region, network search with combinatorial paths and search for the truth through sequential voting.

3 Methods, Assumptions and Procedures

Most of the problems studied under the grant involve the optimal search for a hidden target on a network. The main method is that of a *game against Nature*, in which the Searcher may assume that the Hider chooses his location antagonistically, that is, to maximize the capture time T . Our method is deductive reasoning and we present our results as theorems with proofs. These establish the optimality of certain methods of search.

4 Results and Discussion

We now present the main results obtained in each of the topics that were studied while carrying out the grant. The six subsections that describe these results are independent.

4.1 Patrolling Games

Patrolling games were introduced in Alpern, Morton and Papadaki (2011). They are based on four parameters. A graph G with n nodes; a positive integer m which denotes the number of time periods required for an attack (for example, how long does it take to unscrew a painting in a museum from a wall); a discrete time interval $\{1, 2, \dots, T\}$ of T

periods within which the game is played. This is a zero sum game with two players, the Patroller (minimizer) and Attacker (maximizer). The Attacker picks a node i to attack and a time interval of m consecutive periods in which to attack it. The Patroller picks $w(t)$, $t = 1, 2, \dots, T$, with $w(t)$ and $w(t+1)$ the same or adjacent, that is, a walk w on G . The Patroller wins (payoff 1) if he intercepts the attack – if $w(t) = i$ for some period t within the m attack times; otherwise the Attacker wins (payoff 0). The original paper presented general results on the value and optimal mixed strategies for patrolling games on arbitrary graphs and for some particular classes.

The work carried out this year, with the additional author T. Lidbetter, concerns the case of Line graphs L_n , n nodes arranged in linear fashion. It turns out that the time parameter T is not very important, as long as it is large enough. In particular our new results require that $T \geq 2m$. This graph models the problem of patrolling a channel or a border, which is like the classical problem of guarding a channel with fixed guards, except of course our guards are moving patrollers.

It turns out that optimal patrolling strategies are mixtures of random oscillations, sometimes overlapping and sometimes disjoint. A random oscillation is an oscillation (back and forth walk) on a discrete interval, which starts at a random (uniformly chosen) node. The problem with a single ‘grand oscillation’ is that the ends are not sufficiently covered and provide a good place to attack. The main result for large n is as follows.

Theorem 1 *Consider the patrolling game on the line graph L_n , $n \geq m+2$, with $T \geq 2m$. The value of the game is given by*

$$V = m / (n + m - 1).$$

The following mixed patrolling strategy, called the end-augmented oscillation, is optimal: Choose the grand oscillation, the random oscillation on L_n , with probability $(n-1)/(m+n-1)$; choose each of the random oscillations on the end intervals of size $\lfloor (m+2)/2 \rfloor$ with probability $(m/2)/(m+n-1)$. The optimal mixed attack strategies come in five classes: the independent attack (A), the horizontal attack (B), the vertical attack (C), the zig-zag attack (D) and the extended zig-zag attack (E). Each of these features $n+m-1$ equiprobable attacks, of which no more than m can be intercepted by a single patrol (walk). The optimal attack class depends on the two parameters m and ρ , where $\rho = (n-1) \bmod m$, as follows.

$m \setminus \rho$	0	1	even, > 0	odd, > 1
even	A	E	B	D
odd	A	B	C	B

Table 1. Optimal Attack Classes for ρ, m .

It is useful to describe the five attack types. The easiest is the independent attack (A), which is optimal when $\rho = 0$, or equivalently, when $n = qm + 1$. In this case equiprobable simultaneous (for example at times from 1 to m) attacks at the $q + 1$ nodes $1, m + 1, 2m + 1, \dots, qm + 1 = n$ prevent the Patroller from intercepting more than one out of q of them, as they are spaced at a distance of m from the nearest one. So the payoff, the probability the attack is intercepted, is at most $1/(q + 1) = m/(n + m - 1)$.

The other four types (B,C,D,E) of optimal mixed attacks are much more complicated and require the proper timing as well as location of the constituent pure attacks. In all cases we begin by writing, for some q and r , $n = qm + r$ and we decompose the line graph L_n into $L_{qm+1} = \{1, 2, qm + 1\}$ and L_r with its nodes labeled as $\{qm + 1, qm + 2, \dots, n\}$. The two line graphs L_r and L_{qm+1} overlap at the node $qm + 1$. It is easy to place $(q + 1)m$ attacks on L_{qm+1} in such a way that at most m of them can be intercepted by a single patrol. For this reason we only take a positive value for q in the first example below.

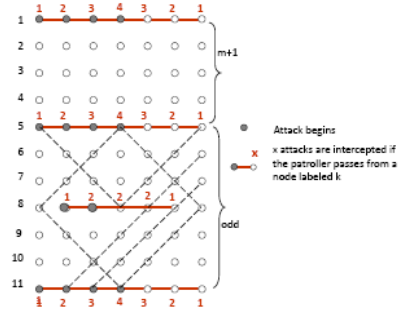


Figure 1. Attack Strategy $n = 11, m = 4$, taking $q = 1, r = 7$.

Consider for example the case $n = 11$ and $m = 4$ which is illustrated in Figure 1. The vertical axis shows the line graph L_{11} and the horizontal axis is the time axis, starting on the left with time 1. We have $\rho = (11 - 1) \bmod 4 = 2$, so we are in row 1, column 3 of Table 1, giving case B (horizontal). The word horizontal comes from the two attacks starting at times 2 and 3 in Figure 1, which look horizontal. The Figure indicates the locations (vertically) and starting times (horizontally) of the $4 + 4 + 2 + 4 = 14 = n + m - 1$ attacks. The dotted lines indicate potential walks of the Patroller. Note that such a walk intercepts an attack if it passes not more than $m - 1 = 3$ places to the right of the starting point of the attack. For example consider the walk that stays at node 11 up to time 3 and then moves up the figure at slope 1. It reaches node 8 at time 6, just in time to intercept one of the two attacks there, the one starting at time $3 = 6 - 3$ but not the one starting at time 2. (The times start at 1 on the left side of the space-time picture.) The red numbers at some points in space-time indicate the number of attacks to the left that would be intercepted by a walk passing through that space-time point. In the case considered above, this is 1, as one of the two attacks at node 8 would be intercepted. It is easy to check that no walk intercepts more than $m = 4$ attacks, though a general

proof takes a little work. The general pattern for case B is an extension of that given in Figure 1.

Case C (vertical) is illustrated in Figure 2 for the example $n = 13$ and $m = 5$, and so $\rho = (13 - 1) \bmod 5 = 2$. As m is odd and ρ is even, this is case C in Table 1. We take $q = 1$ and $r = m + \rho + 1 = 8$ (nodes 6 through 13). The pattern of attacks is shown in the space-time diagram of Figure 2, with five attacks each at nodes 1, 6 and 13, and two attacks at the middle time 3 of times 1 through 5, at the two middle nodes of $L_r = L_8$. That is, of the 8 nodes $\{6, \dots, 13\}$, the attacks are at the nodes 9 and 10.

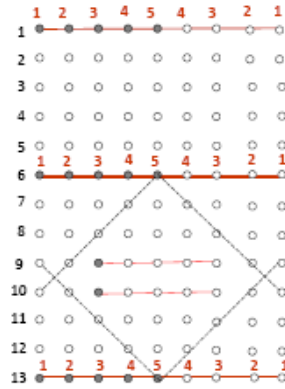


Figure 2. Attack Strategy, case C, $n = 13$ and $m = 5$.

To illustrate case D (zig-zag), we consider the parameters $n = 12$ and $m = 4$. Thus $\rho = 11 \bmod 4 = 3$ and Table 1 gives case D. We take $r = m + \rho + 1 = 8$, with nodes $\{6, 7, \dots, 13\}$ and so $q = 1$. The attack pattern is illustrated in the space-time diagram of Figure 3.

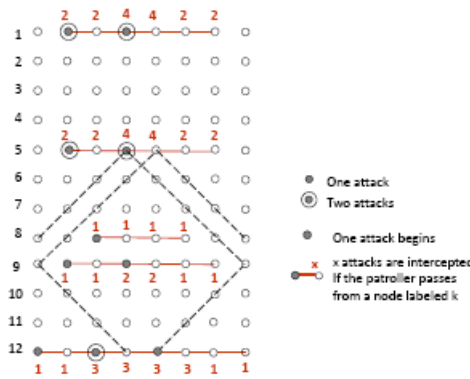


Figure 3. Attack Strategy, case D, $n = 12$ and $m = 4$.

The notation is slightly different from the earlier figures as now sometimes we need to start two attacks at the same time and place, which is indicated by the disk in a circle.

For larger q , the top pattern, with two attacks starting at times 2 and 4, is simply repeated $q + 1$ times at the top.

Finally, we illustrate the case E by the parameters $n = 10$ and $m = 4$, with $\rho = 1$. Thus Table 1 puts us in case E . We take a larger value of r compared with the previous cases, namely $r = 2m + \rho + 1 = 10 = n$, with $q = 0$. Figure 4 illustrates the $m + n - 1 = 13$ equiprobable attacks, of which no more than $m = 4$ can be intercepted by any walk.

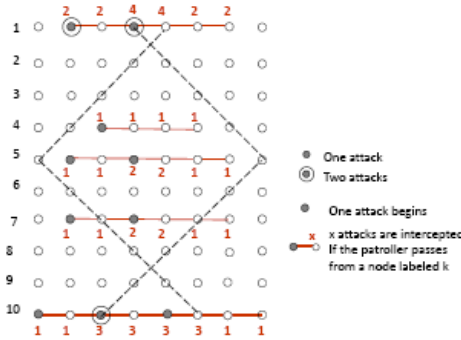


Figure 4. Attacks for $n = 10$ and $m = 4$.

The general description of the four types of attacks, along with the proofs of their optimality, can be found in Papadaki, Alpern, Lidbetter and Morton (2015).

4.2 Accumulation Games

The contribution to the theory of accumulation games is the extension to graphs. The game is played on a graph $G = (\mathcal{N}, \mathcal{E})$ consisting of nodes and edges. The Hider partitions a given quantity $h > 1$ of infinitely divisible material over the nodes. More precisely, he picks a *weighting* function w where $w(S)$ is the total material placed at nodes of the node set S , and $w(\mathcal{N}) = h$. The Searcher can confiscate all the material belonging to a chosen set S of nodes, where S belongs to a set (or hypergraph) $\mathcal{S} \subset 2^{\mathcal{N}}$. In particular, the search sets \mathcal{S} can be taken to be the set of edges or connected r sets. In other words, the Searcher can confiscate the material from r *close* locations. The Hider wins (payoff 1) if the remaining material located at nodes outside of S (unconfiscated) sums to at least 1, that is, $w(\mathcal{N} - S) \geq 1$. Otherwise the Searcher wins (payoff 0). The value of the game is the probability that the Hider wins. There are many interpretations of this game. The hidden material might be armaments (or nuclear fuel) and the normalized sum 1 might represent the minimum amount needed to start a resurrection (or make a bomb). In biological applications, the material could represent nuts hidden by a squirrel and the normalized amount 1 might represent the amount needed to survive the winter (See Alpern, Fokkink, Lidbetter and Clayton (2012)). This works appears in Alpern and Fokkink (2014). We mention some of the main results here. First we need an important definition.

Definition 2 Let a (hypergraph) $\mathcal{S} \subset 2^{\mathcal{N}}$ be given. A weighting w is called heavy if $w(S) \geq 1$ for every search set $S \in \mathcal{S}$. A measure y on \mathcal{S} is called a light weight if for every $i \in \mathcal{N}$ we have $y(\mathcal{S}_i) \leq 1$, where $\mathcal{S}_i = \{S \in \mathcal{S} : i \in S\}$ denotes the collection of all search sets containing node i . A heavy weight w minimizing the total weight $w(\mathcal{N})$ is called a minimal heavy weight; a light weight y maximizing the total weight $y(\mathcal{S})$ is called a maximal light weight.

These definitions are essentially linear programs.

Theorem 3 Suppose that every node in the graph lies in exactly d search sets $S \in \mathcal{S}$ and every search set S contains exactly k nodes. If G is a regular graph of degree d , then the minimal heavy weight problem is solved by putting weight $1/2$ at all nodes.

It turns out that when the Searcher can search any edge of the graph ($\mathcal{S} = \mathcal{E}$) the game has a simple solution.

Theorem 4 When $\mathcal{S} = \mathcal{E}$, there is an optimal Hider mixed strategy which uses only weightings w where either all the weight is at a single node i with $w(i) = h$ or there are at most two nonzero weights, with one twice the other.

In other words either all the hidden material is at a single node or all the nodes have weights $0, a$ or $2a$ for some fixed value of a . This is an analog of the "0,1/2,1 Theorem" of fractional graph theory.

4.3 Predator Search for Prey

Consider a predator-prey scenario where both populations are sparse and where a single predator enters the home range of a single prey. We assume that this territory, called the search region, has a fixed number n of hiding places and that in each bout the predator can search a fixed number k of these before tiring. If he searches the place where the hider has located, there follows a pursuit phase. Each location i has its own probability p_i that the pursuit is successful. If the predator (searcher) exhausts his k searches without finding the prey (hider), he gives up and the prey wins the game. Thus far, this is the model analyzed in earlier by Gal and Casas. In that paper, an unsuccessful pursuit after the predator finds the prey was considered to be a permanent escape for the prey. Here, we extend that model to allow for a more persistent predator. After an unsuccessful pursuit there is a probability β , called the *persistence probability*, that the 'stage game' will be repeated; that is, the prey relocates in one of the hiding places and the predator repeats his attempts to find and pursue the prey. The persistence probability might be a measure of the urge of the predator to feed, or it might represent the probability that the prey fails to find a way out of the predator's hunting range. Our main result is that when the persistence probability is high, the prey should hide more randomly rather

than more concentrated on the best sites for fleeing. Although phrased in a biological (animal behaviour) setting, the model applies equally to a hunt for a fleeing terrorist who has only a limited number of places to hide.

4.3.1 Review of basic game $G(k, 0)$

All the games we study are zero-sum two person games. The payoff function is always the probability that the Searcher eventually finds and captures the Hider. Thus the Searcher is the maximizer and the Hider is the minimizer. Such games have a value, denoted by v , and optimal mixed strategies for each player. The optimal strategy for the Searcher will guarantee that the capture probability (payoff) is at least v and the optimal strategy for the Hider will guarantee that the capture probability is at most v . The pair of optimal strategies form a Nash equilibrium, though of a special type.

Before describing the general game $G(k, \beta)$ with persistence probability β , we review what we call the basic game. There is no persistence of attack, the persistence probability β is 0, which is why the second parameter (for β) describing the game is 0. There are n locations $i \in \{1, 2, \dots, n\}$ for the Hider to hide in. If the hider stays at location i , then the probability of capture, in case that the searcher looks at this location is p_i , where $p = (p_1, \dots, p_n)$ is a known vector capture probabilities, with the locations ordered so that $p_1 \leq p_2 \leq \dots \leq p_n$. Thus location 1 is always the location where it is hardest to catch the hider. The Searcher is limited to looking in k out of the n possible locations of the Hider. Since the time that he finds the Hider is not important in our model, a pure strategy for the Searcher is simple a subset $S \subset \{1, 2, \dots, n\}$ of cardinality k . If the Hider's location i belongs to the searched set S , the Hider is captured with probability p_i ; otherwise he is not captured. The payoff to the Searcher (Hider) is 1(0) if the Hider is captured and 0(1) otherwise.

In terms of pure strategies, where the Hider chooses location i and the Searcher looks at each location in the set S , the payoff (probability of capture) is given by

$$\text{Payoff}(S, i) = \begin{cases} 0, & \text{if } i \in S, \text{ and} \\ p_i, & \text{if } i \in S. \end{cases}$$

A mixed strategy for the Hider is a probability distribution $h = (h_1, \dots, h_n)$ over the locations, so that $0 \leq h_i \leq 1$ and $h_1 + \dots + h_n = 1$. For the Searcher, it is a probability distribution over the subsets of $\{1, 2, \dots, n\}$ of cardinality k . A simpler representation of a search strategy is an n -vector $r = (r_1, \dots, r_n)$ satisfying

$$\sum_{i=1}^n r_i = k, \quad 0 \leq r_i \leq 1$$

where r_i is the probability that $i \in S$. (Note that the extreme points of the set all such points r have k coordinates equal to 1 and the rest equal to 0, and thus can be identified with particular k -sets S .)

It is useful to observe that in the basic game $G(k, 0)$ (and similarly in the more general games defined later), pure strategies are generally not very good for either player. A pure strategy S is never any good for the Searcher, as the Hider can easily win (avoid capture) by simply choosing a location i which does not belong to S (which will not be searched). For large k (see Theorem 1) it is true that the Hider can optimally adopt the pure strategy of hiding at the location (namely $i = 1$) from which he can most likely escape if he is found. However for small k he must randomize.

In terms of mixed strategies r (for the Searcher) and h (for the Hider), the capture probability (payoff) is given by the following formula, which is however never explicitly used in finding the solution.

$$\text{Payoff } (r, h) = \sum_{i=1}^n h_i r_i p_i.$$

That is, the Hider will be captured if for some location i : the Hider hides at location i , the Searcher looks in location i and the Search successfully catches the Hider in the pursuit stage at location i .

The solution to the basic game $G(k, 0)$, can be summarized in the following result , where

$$\lambda = \lambda(p) = \frac{1}{\sum \frac{1}{p_i}}.$$

Theorem 5 (Gal-Casas) *The value of the game $G(k, 0)$ is given by*

$$v = \min(k\lambda, p_1).$$

The optimal strategies come in two types, mixed (Type I) and pure (Type II).

Type I solution *If $k < p_1/\lambda$ (that is, $v = k\lambda$) then the unique optimal search strategy satisfies $r_i = k\lambda/p_i$, and the unique optimal hiding strategy satisfies $h_i = \lambda/p_i$, $i = 1, \dots, n$ (the Hider makes all locations equally attractive for the Searcher).*

Type II solution *If $k \geq p_1/\lambda$ (that is, $v = p_1$) then any optimal search strategy satisfies $r_1 = 1$ ($1 \in S$) and $r_i \geq k\lambda/p_i$ for $i = 1, 2, \dots, n$ and the uniquely optimal hiding strategy is to hide at location 1, that is, $h_1 = 1$.*

The game with persistence of attack, $G(k, \beta)$ This section covers the more general game, $G(k, \beta)$, with an arbitrary persistence probability β .

4.3.2 Framework

We now extend the game of Gal and Casas to multiple periods. When the persistence β is 1 we have a repeated game, and when $\beta < 1$ the game is mathematically equivalent to a discounted repeated game, where β plays the role of the discount factor.

The repeated game $G(k, \beta)$ has an unlimited number of stages played at the same n locations. In each stage the pure strategies are the same as in the basic game $G(K, 0)$ discussed in the previous section: the Hider choose a location to hide in and the Searcher chooses a subset S of cardinality k to inspect. There is no influence of previous play in earlier stages, except in the variation discussed in the final section, where the Hider cannot return to a location where he has previously hidden. There are three possible outcomes:

1. If the Searcher does not find the Hider, then the game ends with zero payoff for the Searcher and a payoff of one to the Hider. (Hider wins.)
2. If the Searcher finds the Hider and successfully pursues it (captures it), then the game ends with a payoff of one to the Searcher and a payoff of zero to the Hider. (Searcher wins.) If the Searcher finds the Hider but does not catch it, then there are two possibilities. With probability $1 - \beta$ the predator gives up and the game ends with a win for the Hider. With the persistence probability β the process restarts with the Hider finding a new location.

The value v ($0 \leq v \leq 1$) is the probability that the Searcher eventually captures the Hider, with best play on both sides. We will show that there exist optimal strategies for both players which are usually unique. The dynamics of the game can be seen in the figure below.

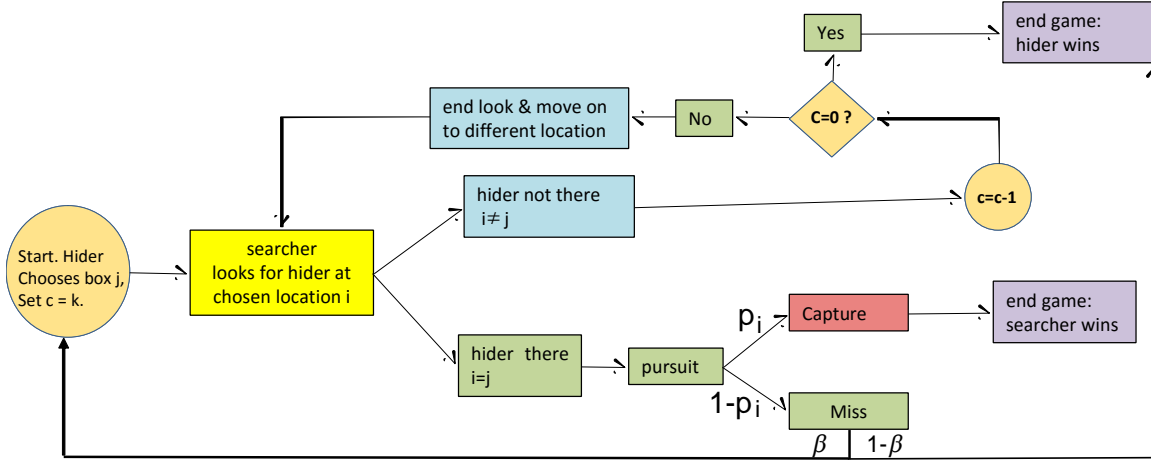


Figure 5. Flowchart of game dynamics.

Type I solutions of $G(k, \beta)$ A Type I solution for $G(k, \beta)$ is a pair (h^*, r^*) such that h^* is the hiding strategy that makes all locations equally attractive for the Searcher to visit for each stage **independently** of history and r^* is the search strategy that makes all locations equally attractive to hide at for each stage **independently** of history. We now show that these strategies are optimal if k is smaller than the threshold M .

Theorem 6 Consider the equation

$$\sum_1^n \frac{v}{p_i + (1 - p_i) \beta v} = k.$$

If $k \leq M$, then the solution of the game $G(k, \beta)$ is of Type I: v is the solution of the above equation and the optimal strategies, unique for $k < M$, are

$$h_i^* = \frac{v/k}{p_i + (1 - p_i) \beta v}$$

$$r_i^* = \frac{v}{p_i + (1 - p_i) \beta v}.$$

4.3.3 Type II solution for general β and $k \geq M$

Here we show that if $k \geq M$, then the optimal solution is Type II: always hide at the most favorable location i.e. at location 1 which has the smallest p_i .

Theorem 7 *if $k \geq M$, then $v = v_1$ and an optimal solution for the Hider is to hide at location 1 (Type II) and for the Searcher to use a r^1 vector satisfying $r_1^1 = 1$ and for $i \geq 2$:*

$$r_i^1 \geq \frac{v_1}{p_i + (1 - p_i) \beta v_1}$$

4.3.4 The ranges of Type I and II solutions

Here we show that in general there is a cutoff value of the persistence probability β above which the game $G(k, \beta)$ has only Type I (mixed) solutions and below which it has only Type II (pure) solutions. Sometimes there are only Type I solutions.

Theorem 8 *For a given $k \leq n$, Let β_k denote the solution of the equation*

$$k = w(p, \beta) \equiv \sum_1^n \frac{p_1}{p_i(1 - \beta) + p_1\beta}.$$

Then

1. *If $\beta > \beta_k$, then there are only Type I solutions to the game $G(k, \beta)$.*
2. *If $\beta < \beta_k$, then there are only Type II solutions to the game $G(k, \beta)$.*
3. *If the equation above has no solution, then there are only Type I solutions to the game $G(k, \beta)$.*

4.3.5 Discussion

We first address here the biological significance of our findings and end up the paper with an enhancement of our analysis by accounting for the possibility that prey do not return to locations where they were previously found but escaped capture.

Persistence in the search leads to more randomization in the prey choice of hiding locations. In the one stage game of Gal and Casas, a larger parameter k means more potential searches. It also means that prey is more likely to hide at best location, in other words there is less randomization in the hiding locations. The persistence game of this paper implies, for k fixed, more potential searches for bigger beta. Therefore bigger beta means prey is more likely to be found – but in this case he is less likely to hide at best location. This implies more randomization, so the effect of more searches due to persistence is the opposite.

Further details and full proofs for this material can be found in the published paper Gal, Alpern and Casas (2015).

4.4 Search for a Hider who can Escape the Search Region

Since their introduction in the classical text of Rufus Isaacs (1965), search games have proved a useful method of modeling optimal search for a mobile or immobile antagonistic hider who is confined to a bounded search region \mathcal{R} . Even for mobile hiders, it was shown by Gal for multidimensional regions and Alpern and Asic for finite length networks, that eventual capture is almost surely accomplished and moreover capture time has finite expectation. Hence such games have been traditionally solved by taking capture time (search time) as the payoff of a zero sum game.

However in many real life problems, the hider is able to at least attempt to leave the region in which he is initially known to be confined. For example, Osama bin Laden successfully escaped from the Tora Bora caves, where he was at one time known to be hiding. In the predator-prey context, it is possible for the hider (prey animal) to escape the hunting territory of the predator who is searching for it. In such a context of potential escape, a more reasonable searching aim is to maximize the probability of eventually finding the hider, placing less emphasis on the search time but more on the search outcome. Here we initiate the study such problems where the searcher has a limited time horizon T in which to capture the hider and the eventual outcome is uncertain.

Thus we are led to a continuous time game, where the hider can stay still or choose any time m , $0 \leq m \leq T$, in which to attempt a flight from the search region. To counter this possibility, the searcher has an additional ‘ambush’ mode, in which he can counter an attempt at flight. In the predator-prey context, the ambush mode might involve sitting still and surveying the search region for a move of the prey, or an eagle circling above the region to spot the prey if it goes out of the vegetation cover in an attempt to leave the region. For law enforcement, an ambush mode might involve setting up road blocks to counter an attempted location change of an escaped prisoner (who has escaped prison but not the surrounding search region). In a search game context on a network, ambush strategies might consist of the searcher waiting at a node (e.g. the central node of a star network) to catch a mobile hider. The two outcomes where the hider wins are (i) where the hider never attempts to flee and is not found by time T , and (ii) where he successfully flees because the searcher is cruise-searching at the flight time m . The two outcomes where the searcher wins are when he (iii) finds the hider while cruise-searching, before any hider attempt to flee, and when he (iii) ambushes the hider when the latter attempts to flee, by adopting the ambush mode at time m .

4.4.1 The Limited Time Game $\Gamma(T)$

The alternation between searching and ambushing can be modeled very simply by a search strategy $s(t)$, $0 \leq t \leq T$ which measures the amount (or equivalently, fraction) of \mathcal{R} that the searcher has covered by time t , given that he covers area at unit rate while

cruising and at zero rate while ambushing. We have the restriction $s(t_2) - s(t_1) \leq t_2 - t_1$ as well as the initial condition $s(0) = 0$. We evaluate the payoff function $P = P(s, m)$ (the probability that the searcher wins) when the searcher adopts strategy $s = s(t)$ and the hider moves at time m as the sum of two terms. The first term $s(m)$ is the probability that the searcher finds him before he has moved, and the second is the probability that he has not been found before he moves $(1 - s(m))$ times the probability $1 - s'(m)$ that the searcher is ambushing when the hider moves. Thus for $m \leq T$ we have

$$\begin{aligned} P(s(t), m) &= s(m) + (1 - s(m))(1 - s'(m)) \\ &= 1 - (1 - s(m))s'(m). \end{aligned}$$

If the hider never flees, we say that $m = T^+$, and the payoff is $P(s, T^+) = s(T)$, the total amount searched by time T . The searcher's strategy function $s(t)$ can be considered as a continuous behavioral strategy and since this game has perfect recall such strategies should be sufficient.

4.4.2 Indifference Search Strategies s_k

A heuristic approach to this game is for the searcher to choose $s(t)$ so that he is indifferent to which moving (fleeing) time m the hider chooses. So if the payoff $P(s(t), m)$ does not depend on m , it must satisfy the ordinary differential equation

$$\frac{\partial P(s, m)}{\partial m} = -(1 - s(m))s''(m) + (s'(m))^2 = 0.$$

The general solution to the differential equation, for the boundary condition $s(0) = 0$, is given in terms of a parameter k as

$$\begin{aligned} s_k(t) &= 1 - \sqrt{1 - kt}, \text{ with constant payoff} \\ P(s_k, m) &= 1 - k/2. \end{aligned}$$

Now k can be evaluated using the searcher's unit speed constraint,

$$\begin{aligned} s'_k(t) &\equiv \frac{1}{2} \frac{k}{\sqrt{1 - kt}} \leq 1, \text{ for } t \leq T, \text{ giving the inequality} \\ 0 &\leq k \leq \bar{k}, \text{ where} \\ \bar{k} &= \bar{k}(T) = 2\sqrt{T^2 + 1} - 2T. \end{aligned}$$

(Here \bar{k} is the value of k so that $s'_{\bar{k}}(T) = 1$.) The searcher wishes to choose k satisfying $k \geq \bar{k}$ to maximize his payoff $1 - k/2$, so he takes $k = \bar{k}$ and $\bar{s}(t) = s_{\bar{k}}(t)$. This gives maximum payoff

$$V(T) = 1 - \bar{k}/2 = T + 1 - \sqrt{T^2 + 1},$$

which we will show is the value of the game.

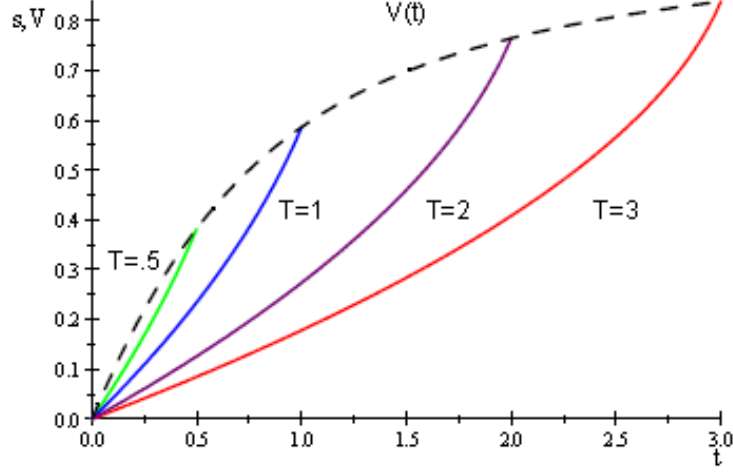


Figure 1: Figure 6. Optimal Searcher strategies $s_{\bar{k}(T)}$, $T = .5, 1, 2, 3$.

Proposition 9 Consider the game $\Gamma(T)$ where the searcher has time T in which to try to find the hider. By adopting the search strategy $\bar{s}(t) = 1 - \sqrt{1 - \bar{k}t}$, $0 \leq t \leq T$, the searcher guarantees that for any hider fleeing time m , he captures the hider with probability

$$P(\bar{s}, m) \geq V(T) \equiv T + 1 - \sqrt{T^2 + 1}.$$

The figure below plots the value $V(t)$ (dashed curve) and the optimal searcher strategies $\bar{s}(t) = s_{\bar{k}(T)}(t)$ for time horizons $T = .5, 1, 2, 3$, each drawn on the relevant interval $[0, T]$. Note that each curve ends at T at unit slope $s'(T) = 1$ when it has searched a proper fraction $\bar{s}(T) = V(T) < 1$ of the search region \mathcal{R} , that is, when it reaches the $V(t)$ curve. Thus the search strategies $\bar{s}(t)$, which we later show to be optimal, never complete an exhaustive search of the region \mathcal{R} , no matter how large the time horizon.

4.4.3 A Stochastic Game Approach to $\Gamma(T)$

We now consider a stochastic game approach, taking as state variables the amount of time h left for the searcher, $h = T - t$ (if t represents elapsed time) and the unsearched area r remaining to be searched. This approach gives a subgame perfect optimal strategy for the hider. If the total area of the search region \mathcal{R} is normalized to 1, then $r = 1 - s(t)$. Given the stage game $G(r, h)$, we need to determine the probability $p(t) = s'(t)$ for the predator to cruise search and the probability $q(t) \Delta t$ that the prey flees in the next time interval $(t, t + \Delta t)$. Let $F(t) = \Pr(\tau \leq t)$ be the probability distribution of the prey's

fleeing time m , and let $f(t) = F'(t)$ be the probability density. The a priori probability that the prey flees in $(t, t + \Delta t)$ is equal to $f(t) \Delta t$ up to order $O(\Delta t^2)$. However, if the game reaches stage r, h then the prey has not fled up to time t . In the stage game, the probability of fleeing in the next time interval is a conditional probability:

$$q(t) \Delta t = \Pr(m \in (t, t + \Delta t) \mid m \geq t) = f(t) \Delta t / (1 - F(t)) + O(\Delta t^2).$$

In other words, q is the intensity of the probability distribution F . It is the negative logarithmic derivative of $1 - F$.

The problem in the stage game is represented by the following matrix, where the searcher chooses the rows with (unknown) optimal probabilities p and $1 - p$, while the hider chooses the columns with unknown optimal probabilities $q \Delta t$ and $1 - q \Delta t$. We determine the optimal fleeing intensity q by letting the time interval Δt go to zero.

	flee ($q \Delta t$)	wait ($1 - q \Delta t$)
cruise search (p)	0	$\frac{\Delta t}{r} + \left(1 - \frac{\Delta t}{r}\right) V\left(\frac{h - \Delta t}{r - \Delta t}\right)$
ambush ($1 - p$)	1	$V\left(\frac{h - \Delta t}{r}\right)$

Matrix for the stochastic game $G(r, h)$.

It is clear that there is no pure strategy saddle point as the optimal response to the searcher cruising is for the hider to flee; and the optimal response to a fleeing hider is for the searcher to ambush. We compute the optimal probabilities, and to compute the intensity we need to take limits $\Delta t \rightarrow 0$. The result therefore depends on the terms up to first order

	flee ($q \Delta t$)	wait ($1 - q \Delta t$)
cruise search (p)	0	$\frac{\Delta t}{r} + V\left(\frac{h}{r}\right) - V\left(\frac{h}{r}\right) \frac{\Delta t}{r} + V'\left(\frac{h}{r}\right) \frac{h}{r} \left(\frac{\Delta t}{r} - \frac{\Delta t}{h}\right)$
ambush ($1 - p$)	1	$V\left(\frac{h}{r}\right) - V'\left(\frac{h}{r}\right) \frac{\Delta t}{r}$

Matrix up to first order terms.

For the probability of search p only the terms which are independent of Δt matter, and the matrix reduces to

	flee ($q \Delta t$)	wait ($1 - q \Delta t$)
cruise search (p)	0	$V\left(\frac{h}{r}\right)$
ambush ($1 - p$)	1	$V\left(\frac{h}{r}\right)$

Matrix of constant terms.

The probability p of cruising that makes the hider indifferent between fleeing or waiting, is such that the sums of the columns in the matrix of constant terms, weighted by p and $1 - p$, are balanced: $1 - p = V\left(\frac{h}{r}\right)$. Solving for p gives

$$p = 1 - V\left(\frac{h}{r}\right) = \sqrt{(h/r)^2 + 1} - h/r.$$

Solving for the fleeing intensity q gives

$$q = \frac{1}{r} \left(1 - V \left(\frac{h}{r} \right) + V' \left(\frac{h}{r} \right) \frac{h}{r} \right), \text{ or}$$

$$q = \frac{1}{\sqrt{r^2 + h^2}}.$$

Next, we consider the equilibrium path of the game.

Theorem 10 *The unique subgame perfect equilibrium path in the game is characterized as follows, when the area of the search region is normalized to 1. As already established, the searcher by time t covers an area equal to $\bar{s}(t) = 1 - \sqrt{1 - \bar{k} t}$. By time t , the hider has fled with cumulative probability*

$$F(t) = \frac{t}{\sqrt{T^2 + 1}}.$$

That is, the hider flees with uniform density on the time horizon interval $[0, T]$, in such a way that he has fled with probability $T/\sqrt{T^2 + 1}$ by time T .

We plot below the probabilities of the four types of ending for our game, as a function of the time horizon T .

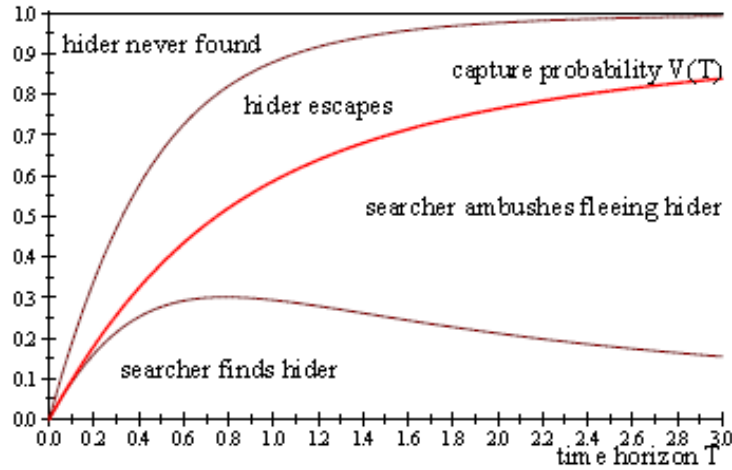


Figure 7. Outcome probabilities at equilibrium.

Further details and full proofs of this material can be found in Alpern, Fokkink and Simanjuntak (2016).

4.5 Network Search Using Combinatorial Paths

Hide-and-seek on networks has traditionally been studied under the assumption that the Searcher may turn around inside an arc. This new work takes the assumption that the Searcher must adopt what we call combinatorial paths, paths in the usual sense of computer science or graph theory, where a path is a sequence of adjacent arc, traversed from end to end at unit speed. Under this assumption we are able to solve the search game on some networks which have not previously been susceptible to analysis. For example networks consisting of two nodes connected by an odd number of arcs of unequal lengths. A useful result in this analysis is the following.

Lemma 11 *Let Q be a semi Eulerian network with O and Z its two nodes of odd degree. Let P be a path from O to Z of minimum length a such that $Q - P$ is connected. Then*

$$V(Q, O) \leq \bar{V} \equiv (a^2 + \mu^2) / 2\mu.$$

A strategy guaranteeing this expected search time is as follows: with probability $p = (a + \mu) / (2\mu)$, first traverse P from O to Z and then follow an Eulerian tour of $Q - P$ from Z , equiprobably in either direction; with probability $1 - p$, first follow an Eulerian tour of $Q - P$ from O , equiprobably in either direction, and then traverse P from O to Z . To obtain this expected time, the Hider must hide near Z when hiding on P .

This result is used to give a complete solution to some networks where the inequality holds exactly or with slack. An unexpected by-product of this analysis is a counter example to a conjecture of S. Gal asserting that optimal searching on a network (for an antagonistic stationary hider) never traverses the same arc more than twice. The counterexample is the following network, where the Searcher starts at the node labelled O .

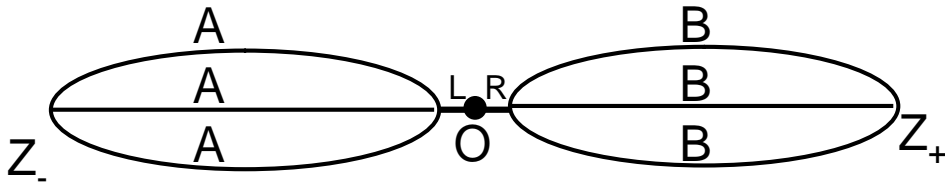


Figure 8. In optimal search either L or R must be traversed three times.

The full details of this work are in Alpern (2016).

4.6 Optimal Voting Order in Search for Truth

We extend the well known results on the reliability of juries, that is the probability that the majority verdict is the actual state of Nature, to the case where the experts (or

jurors) give their opinions sequentially. Later voters know the votes of earlier voters. This problem fits into the theory of search because the state of Nature (say Innocent or Guilty in a legal jury) can be viewed as an unknown Hider. Sequential voting can come in various forms, here we consider two of them. The first, roll-call voting, has the voters in simple sequence: first, second, third, and so on. The second format we consider we call the 'casting vote' system, where $n - 1$ jurors vote simultaneously, and then if there is a tie the last 'casting voter' breaks the tie. In both models the private information available to each juror comes in the form of a signal $s \in [-1, 1]$, which indicates the likelihood that Nature is in state A (indicated by high signals) or in state B (indicated by low signals). The *a priori* probability of state A is denoted by θ , but here for simplicity we will assume that the states are equally likely ($\theta = 1/2$). A signal of $s = 0$ leaves a juror still viewing the states as equally likely, but any positive signal makes A more likely and any negative signal makes B more likely. A strategy for a juror specifies a threshold τ , such that they vote A when their signal is greater than τ and B when it is not. When the jurors vote *honestly* they vote for the state they view as more likely, given their own signal and the votes (but not the signals) of prior voters. When the jurors vote *strategically* they employ the strategy profile that maximizes the probability Q that the majority verdict is correct. So the reliability of strategic voting is at least as large as the reliability of honest voting, and usually it is higher. The *ability* of a juror is determined by his signal distribution, and jurors of higher abilities are more likely to guess the correct state of Nature.

4.6.1 Roll-call voting

The paper of Alpern and Chen (2017a) presents a numerical analysis of a discrete model of roll-call juries. We have an odd number n of jurors, and each one gets signals in the set $\{-10, -9, \dots, +10\}$ and has an ability from the set $\{0, 1, 2, 3, 4\}$, with ability 0 jurors essentially not getting any signal. Suppose we have a jury with abilities say in the set $\{1, 3, 4\}$. In what order should they vote to maximize the reliability Q of the majority verdict? We find that whether they are voting strategically, with any value of θ , the best order is for the juror of median ability to vote first and that the order of the remaining two jurors does not affect the reliability. For honest jurors we have the following result.

Proposition 12 *Suppose that the two states of Nature are equiprobable, that is, $\theta = 1/2$. Suppose the jury consists of three honest jurors with distinct abilities $a < b < c$. Then voting in the ability order (b, c, a) gives the highest verdict reliability and this reliability is also the strategic optimum. That is, they could not do better by voting strategically.*

Some examples are given in the table below. For example when A is more likely ($\theta = 4/52$) and jurors with abilities 1, 2, 3 vote in the optimal ability order 2, 3, 1, their

verdict is correct with probability .827 when they vote strategically and with probability .815 when they vote honestly. But for equally likely alternatives ($\theta = 1/2$), and abilities 1, 2, 4, when voting in optimal ability order 2,41 they get the same optimal reliability .794 when voting either honestly or strategically, as indicated in the above proposition.

θ	Ability order	\bar{Q}	\tilde{Q}	θ	Ability order	\bar{Q}	\tilde{Q}
4/5	(2, 1, 3)	.827	.800	1/2	(2, 1, 4)	.794	.639
4/5	(2, 3, 1)	.827	.815	1/2	(2, 4, 1)	.794	.794
4/5	(1, 2, 3)	.825	.800	1/2	(1, 2, 4)	.781	.718
4/5	(1, 3, 2)	.825	.8169	1/2	(1, 4, 2)	.781	.781
4/5	(3, 1, 2)	.825	.815	1/2	(4, 1, 2)	.780	.778
4/5	(3, 2, 1)	.825	.8172	1/2	(4, 2, 1)	.780	.778

Table 2. Best reliability for strategic (\bar{Q}) or honest (\tilde{Q}) voting.

4.6.2 The casting-vote system

For the casting vote system with three jurors, in a continuous model of signals and abilities, we have the following.

Proposition 13 *Reliability is maximized for jurors of abilities $a < b < c$ when the median ability (b) juror has the casting vote.*

5 Conclusions

The work carried out under the grant has been going very well. Already the grant has been 'acknowledged' in papers published in *Operations Research* and *Networks*. In addition, many partial results have been obtained in various areas of search theory which will likely be grouped in articles later in the grant and discussed in future annual reports. Actually more new questions have been raised than problems solved.

6 References

The following journal articles have been supported by the grant.

S. Alpern and T. Lidbetter (2014). How to search a variable speed network. *Mathematics of Operations Research* 39: 697-711.

- S. Alpern and R. Fokkink (2014). Accumulation games on graphs. *Networks*, 64, 40-47.
- S. Alpern and T. Lidbetter (2015). Optimal trade-off between speed and acuity when searching for a small object. *Operations Research* 63: 122-133.
- S. Alpern, J. V. Howard (2015). "Winner-take-all games: the strategic optimization of rank" *Operations Research* (in press, 2017).
- S. Gal, S. Alpern, J. Casas (2015). Prey should hide more randomly when predator attacks more persistently *Journal of the Royal Society Interface* 12: 0150861.
<http://dx.doi.org/10.1098/rsif.2015.0861>
- K. Papadaki, S. Alpern, T. Lidbetter, A. Morton (2015). Patrolling the Line. *Operations Research* 64 (2016), 1256-1269.
- S. Alpern, B. Chen (2017a). The importance of voting order for decisions by sequential majority voting. *European Journal of Operational Research* 258 (2017), 1072-1081.
- S. Alpern, V. Baston (2016). The secretary problem with a selection committee: Do conformist committees hire better secretaries?" *Management Science* 63 (2017):4 , 1184-1197.
- S. Alpern, R. Fokkink, M. Simanjuntak (2016).. Optimal search and ambush for a hider who can escape the search region. *European Journal of Operational Research*, 251, Issue 3, 707–714.
- S. Alpern and B. Chen (2017b). Who should cast the casting vote? Using sequential voting to amalgamate information ", *Theory and Decision* Published online 7 March 2017.
- S. Alpern, K. Papadaki, T. Lidbetter, A. Morton (2016). Patrolling a pipeline. In *International Conference on Decision and Game Theory for Security*, pp. 129-138. Springer International Publishing, 2016.
- S. Alpern (2016). Hide-and-seek games on a network, using combinatorial search paths, *Operations Research* (in press 2017)