2 Mathematical model

Several approximations and simplifications have been adopted/assumed to develop/derive a mathematical model of the plant:

1. Eddy currents in a laminated electrical steel magnetic core of an electromagned and in a steel sheet ball wall can be neglected.
2. Skin and proximity effects in the enamel-coated copper solid wire of an electric winding of the electromagnet can be neglected.
3. Magnetic and electric properties of the core and electric properties of the winding does not depend on temperature and do not change with time.
4. Magnetisation curves of an electromagnet core and ball walls are linear and unique. No magnetic core saturation, remanence, coercivity, nor histeresis occur.
5. The optoelectronic position measurement system is infinitely fast (its dynamics is negligible compared to the dynamics of the electromechanical plant).
6. Ambient light (direct or dissipated sunlight/daylight, artificial electric lighting) has no effect on the optoelectronic ball position sensor.
7. The power stage of an electronic circuit supplying/driving/energising/feeding the winding of the electromagnet is infinitelly fast (compared to the electromechanical system) and does not suffer from saturation or nonlinearities.
8. The current sourced or sinked by the power supply and the power stage (power amplifier) is unlimited – there is no overcurrent protection circuit in the system.
9. All mechanical elements of the plant (especially the ball) can be considered rigid bodies (hence not deformable/flexible/elastic).
10. The ball has/exercises only one degree of freedom – it can travel/translate along a vertical axis – towards or away from the face of a central column of the E-shaped magnetic core.
11. The drag force acting on the ball due to air inertia, compressibility, and viscosity (pressure and friction drag) can be neglected.

Neglected state space variables:

1. Several parameters of the plant depend on the temperature: resistivity of the copper wire, permeability of the electrical steel laminated core. Temperatures of the winding and the magnetic core change during plant operation due to the heat released in electric and magnetic energy dissipation processes (copper/ohmic and iron/core losses). These temperatures should be considered to be additional state space variables of the system.
2. Eddy currents induced in the laminated core of the electromagnet and in the mild steel shell of the ball interact with magnetic fields and produce mechanical force and electromotive forces.

Sources of nonlinearities in the mathematical model of the plant:

1. Optical ball position sensor with the ball modulating a beam of light travelling from a light source (an incadescend bulb or a set of LED diods) to a light detector (a phototransistor or a photodiode with a focusing screen/ground glass or a parabolic reflector) gives/exhibits/reveals a nonlinear current/voltage versus position characteristic.

2. The silicon electrical/transformer/relay steel used for laminated magnetic core of the electromagnet and steel sheet forming the wall of the ball exhibit nonlinearity, magnetic saturation, and hysteresis phenomena.
3. Nonlinear relationship between the air gap width and the inductance of the electric coil.
4. The inductance of the coil depends nonlinearly of the air gap width. In the electrical equation of the plant mathematical model the gap-dependent inductance is multipied with a winding current time differential/rate of change.
5. The force developed by the electromagnet and acting on the ball depends on a product of the squared winding current and a nonlinear function of the air gap width (the distance between the core face and the ball wall).
6. The back emf force induced in the coil depends on a product of the ball velocity and a nonlinear function of the air gap width.
7. The electronic power stage feeding the coil suffers from saturation phenomenom.
8. The drag experienced by the ball moving through the air depends on the ball speed/velocity in a nonliner fashion.

Euler-Lagrange formalism/equations of the second kind (no constraint equations, judicious selection of generalised coordinates). Generalised coordinates. External generalised forces. Generalised potential energy. Generalised kinetic co-energy. Lagrange function, Lagrangian. Rayleigh dissipation function.

Generalised coordinates (mechanical and electrical):
z – air gaph width/thickness – position of a steel-walled ball – a distance between an external surface of a ball and a face of a central limb of a E-shaped laminated magnetic core of an electromagnet
q – electrical charge flowing/passing through an electromagnet winding

External generalised force (electrical and mechanical, translational):
$g\,m$ – terrestial gravity force acting on/attracting the ball
u – electric voltage applied to terminals of a winding/coil/solenoid of an electromagnet

E – emf (electromotive force) of a regulated DC power supply ($E = 12.19\,\mathrm{V} = \mathrm{const}$)
w – normalised voltage applied to terminals of the electromagnet winding – a control input signal for the plant ($w(t) \in [0,\, 1]$, $t \in [0,\, \infty]$)

$$u(t) = E\,w(t), \quad w(t) = \frac{u(t)}{E} \tag{1}$$

Plant/system parameters:
m – ball mass
L – inductance of the electromagnet winding (dependent on the air gap width – the distance between tha ball and the electromagnet core)
R – equivalent/net/resultant/total/overall resistance of the electromagnet winding/coil and components of an electric circuit that supplies/energises/drives the solenoid: resistance of the winding, leads, connectors, amplifier power stage (an integrated circuit MOSFET H-bridge controlled with a PWM signal), PCB circuit copper traces and tin-based solder, internal resistance of a power supply
b – coefficient of the viscous linear friction experienced by the ball moving through the stagnant air medium (we are going to neglect that phenomenon)
g – gravity of Earth, net acceleration on the Earth's surface due to gravitation and centrifugal force

Inductance of the electromagnet coil/solenoid/winding depends on the ball position/distance which in turn depends on time:

$$L = L(z) = L\left(z\left(t\right)\right)$$

T – total/overall/net generalised kinetic co-energy stored in the system
V – total/overall generalised potential energy (including terms representing generalised external forces) stored in the system
$\mathcal{R}$ – Rayleigh dissipation function

$$T = \frac{1}{2}\, m\, \dot{z}^2 + \frac{1}{2}\, L\, \dot{q}^2 \tag{2}$$

$$V = -m\, g\, z - u\, q \tag{3}$$

$$\mathcal{R} = \frac{1}{2}\, R\, \dot{q}^2 + \frac{1}{2}\, b\, \dot{z}^2 \tag{4}$$

$\mathcal{L}$ – Lagrange function (Lagrangian):

$$\mathcal{L} = T - V \tag{5}$$

$$\mathcal{L} = \frac{1}{2}\, m\, \dot{z}^2 + \frac{1}{2}\, L\, \dot{q}^2 + m\, g\, z + u\, q \tag{6}$$

Eulera-Lagrange equations of the second kind for both generalised coordinates (z and q):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) - \frac{\partial \mathcal{L}}{\partial z} + \frac{\partial \mathcal{R}}{\partial \dot{z}} = 0 \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{R}}{\partial \dot{q}} = 0 \tag{8}$$

Derivations/manipulations/transformations for the mechanical generalized coordinate z:

$$\frac{\partial \mathcal{L}}{\partial \dot{z}} = m\, \dot{z} \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = m\, \ddot{z} \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{1}{2}\, \frac{\mathrm{d}L}{\mathrm{d}z}\, \dot{q}^2 + m\, g \tag{11}$$

$$\frac{\partial \mathcal{R}}{\partial \dot{z}} = b\, \dot{z} \tag{12}$$

$$m\, \ddot{z} - \frac{1}{2}\, \frac{\mathrm{d}L}{\mathrm{d}z}\, \dot{q}^2 - m\, g + b\, \dot{z} = 0$$

$$m\, \ddot{z} = \frac{1}{2}\, \frac{\mathrm{d}L}{\mathrm{d}z}\, \dot{q}^2 + m\, g - b\, \dot{z}$$

$$\ddot{z} = \frac{1}{2\, m}\, \frac{\mathrm{d}L}{\mathrm{d}z}\, \dot{q}^2 - \frac{b}{m}\, \dot{z} + g$$

$$v = \dot{z}, \quad \dot{v} = \ddot{v}, \quad i = \dot{q}$$

$$\dot{v} = \frac{1}{2\, m}\, \frac{\mathrm{d}L}{\mathrm{d}z}\, i^2 - \frac{b}{m}\, v + g$$

Derivations/manipulations/transformations for electrical generalised coordinate q:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = L\,\dot{q} \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(L\,\dot{q}\right) = \frac{\mathrm{d}L}{\mathrm{d}z}\,\dot{z}\,\dot{q} + L\,\ddot{q} \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial q} = u \tag{15}$$

$$\frac{\partial \mathcal{R}}{\partial \dot{q}} = R\,\dot{q} \tag{16}$$

$$\frac{\mathrm{d}L}{\mathrm{d}z}\,\dot{z}\,\dot{q} + L\,\ddot{q} - u + R\,\dot{q} = 0$$

$$L\,\ddot{q} = -\frac{\mathrm{d}L}{\mathrm{d}z}\,\dot{z}\,\dot{q} - R\,\dot{q} + u$$

$$\ddot{q} = -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,\dot{z}\,\dot{q} - \frac{R}{L}\,\dot{q} + \frac{1}{L}\,u$$

$$v = \dot{z}, \quad i = \dot{q}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,v\,i - \frac{R}{L}\,i + \frac{1}{L}\,u$$

Let us gather first order ODE equations derived above:

$$\dot{z} = v \tag{17}$$

$$\dot{v} = \frac{1}{2\,m}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,i^2 - \frac{b}{m}\,v + g \tag{18}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,v\,i - \frac{R}{L}\,i + \frac{1}{L}\,u \tag{19}$$

Let us introduce/define/select state space variables:

$$x_1 = z, \qquad x_2 = \dot{z} = v, \qquad x_3 = \dot{q} = i$$

Nonlinear continuous-time state-space equations written in terms of new denotations:

$$\dot{x}_1 = x_2 \tag{20}$$

$$\dot{x}_2 = \frac{1}{2\,m}\,\frac{\mathrm{d}L}{\mathrm{d}x_1}\,x_3^2 - \frac{b}{m}\,x_2 + g \tag{21}$$

$$\dot{x}_3 = -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}x_1}\,x_2\,x_3 - \frac{R}{L}\,x_3 + \frac{1}{L}\,u \tag{22}$$

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^{\mathsf{T}} \tag{23}$$

$$\dot{x}(t) = f(x(t),\ u(t)) \tag{24}$$

$$f(x,\,u) = \begin{bmatrix} f_1(x,\,u) & f_2(x,\,u) & f_3(x,\,u) \end{bmatrix}^{\mathrm{T}} \tag{25}$$

Looking for an equilibrium point corresponding to a given air gap width/tickness.

$$\dot{v} = \frac{1}{2\,m}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,i^2 - \frac{b}{m}\,v + g$$

$$0 = \frac{1}{2\,m}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,i^2 - \frac{b}{m}\cdot 0 + g$$

$$
\begin{aligned}
&\frac{1}{2\,m}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,i^2 = -g \\
&i = \left(-2\,m\,g\,\left(\frac{\mathrm{d}L}{\mathrm{d}z}\right)^{-1}\right)^{\frac{1}{2}} \\
&\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}z}\,v\,i - \frac{R}{L}\,i + \frac{1}{L}\,u \\
&0 = -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}z}\cdot 0\cdot i - \frac{R}{L}\,i + \frac{1}{L}\,u \\
&\frac{R}{L}\,i = \frac{1}{L}\,u \\
&u = R\,i
\end{aligned}
$$

Linearisation about an equilibrium point $(x_{\mathrm{ep}}, u_{\mathrm{ep}})$

$$
\begin{aligned}
&\mathrm{J}_x f(x,\,u) = \nabla_x f(x,\,u) = \frac{\partial f(x,\,u)}{\partial x}, \qquad \mathrm{J}_u f(x,\,u) = \nabla_u f(x,\,u) = \frac{\partial f(x,\,u)}{\partial u} \\
&f(x_{\mathrm{ep}},\,u_{\mathrm{ep}}) = 0 \\
&A = \mathrm{J}_x f(x_{\mathrm{ep}},\,u_{\mathrm{ep}}), \qquad B = \mathrm{J}_u f(x_{\mathrm{ep}},\,u_{\mathrm{ep}}) \\
&\triangle\dot{x} = A\,\triangle x + B\,\triangle u \\
&\triangle x(t) = x(t) - x_{\mathrm{ep}}, \qquad \triangle u(t) = u(t) - u_{\mathrm{ep}} \\
&A = \left.\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2\,m}\,\frac{\mathrm{d}^2 L}{\mathrm{d}x_1^2}\,x_3^2 & -\frac{b}{m} & \frac{1}{m}\,\frac{\mathrm{d}L}{\mathrm{d}x_1}\,x_3 \\ \left(\frac{1}{L}\,\left(\frac{\mathrm{d}L}{\mathrm{d}x}\right)^2 - \frac{1}{L^2}\,\frac{\mathrm{d}^2 L}{\mathrm{d}z^2}\right)\,x_2\,x_3 & -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}x_1}\,x_3 & -\frac{1}{L}\,\frac{\mathrm{d}L}{\mathrm{d}x_1}\,x_2 - \frac{R}{L} \end{bmatrix}\right|_{\substack{x_1 = x_{1\mathrm{ep}} \\ x_2 = x_{2\mathrm{ep}} \\ x_3 = x_{3\mathrm{ep}} \\ u = u_{\mathrm{ep}}}}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}
\end{aligned}
$$

The case of magnetically coupled coils/windings:

$$
T = \frac{1}{2}\,\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} L_1 & M_{12} \\ M_{12} & L_2 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} + \frac{1}{2}\,m\,\dot{z}^2
$$