MODULE

Optimisation Theory and Applications

OPTIMISATION THEORY AND APPLICATIONS COURSEWORK

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Q1. Linear Programming

a) Dual Problem

In the dual problem the objective function takes its values from the b matrix of the original problem, the constraints and variables are switched such that the b matrix of the dual is the objective function of the original problem and the A matrix is transposed.

Minimise d such that:

$d = 8y_1 + 15y_2$

Subject to:

$2y_1 + y_2 \geq 3$

$2y_1 + 3y_2 \geq 2$

$y_1 + 3y_2 \geq 1$

$y_1, y_2 \geq 0$

b) Solving Problem P

objective $f = 3x + 2y + 1z$

so for linprog:

$f = -(3, 2, 1)^T$

subject to

$Ax \leq b$

shown below:

```
f = -[3 2 1];

A = [2 2 1; 1 3 3];
b = [8; 15];

lb = zeros(length(f),1);

[x, fval] = linprog(f, A, b, [],[],lb,[])
```

```
Optimal solution found.
x = 3×1
     4
     0
     0
fval = -12
```

```
% As this is a maximisation problem the true solution fval is *-1:
fval = -fval
```

```
fval = 12
```

c) Solving the Dual Using Graphical Methods

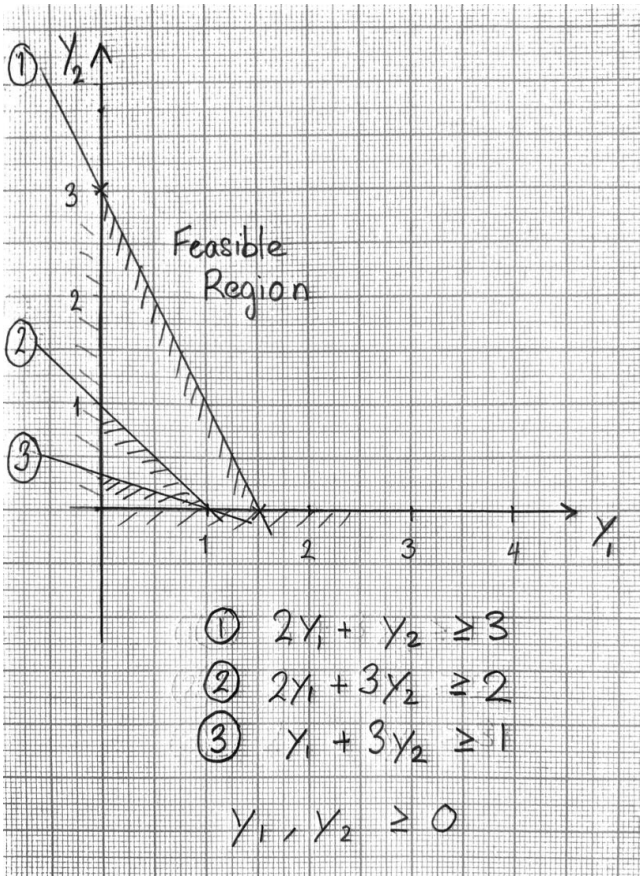


Figure 1: the constraints of the dual problem plotted. The constraints are numbered corresponding to the problem P variables that they are taken from.

From the figure there are two feasible solutions (vertices) that may be suitable for the minimisation problem:

$Y_{sol1} = (0, 3)$ or $Y_{sol2} = (1.5, 0)$

The optimal solution minimises $d = 8y_1 + 15y_2$ and so it clear that Y_{sol2} is optimal as it produces the smallest d value:

$Y^* = (1.5, 0)$ with a value of 12.

This confirms that the optimal objective function values of problems P and D are equal and the strong duality theorem holds.

d) Complementary Slackness Theorem

In the optimal solution of problem P, only the first variable is positive. The complementary slackness theorem states that the corresponding dual constraint (to the positive variable in P) is binding (not slack), whereas the constraints in D corresponding to the zero variables in the solution of P are slack.

From the graph in figure 1 it is clear that constraints 2 and 3 are slack, as they do not effect the feasible region, this corresponds to variables x_2, x_3 in the solution X^* of P which are zero.

Formally, the slackness theorem is such that:

$$y^T(Ax - b) = 0 \qquad (1)$$

$$x^T(A^T y - c) = 0 \qquad (2)$$

so for this problem Equation 1 is such that:

$$(1.5 \quad 0)\left(\begin{pmatrix}2 & 2 & 1\\ 1 & 3 & 3\end{pmatrix}\begin{pmatrix}4\\0\\0\end{pmatrix} - \begin{pmatrix}8\\15\end{pmatrix}\right)$$

Which simplifies to:

$$(1.5 \quad 0)\begin{pmatrix}0\\-11\end{pmatrix} = 0$$

For Equation 2:

$$(4 \quad 0 \quad 0)\left(\begin{pmatrix}2 & 1\\ 2 & 3\\ 1 & 3\end{pmatrix}\begin{pmatrix}1.5\\0\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix}\right)$$

$$(4 \quad 0 \quad 0)\begin{pmatrix}0\\1\\0.5\end{pmatrix} = 0$$

This therefore proves the complementary slackness theorem.

e) Discussion of Solving the Primal vs Dual

The primal problem:

2 constrains (2 slack variables)

n variables

The dual problem:

n constraints (*n* slack variables)

2 variables

- **Number of dimensions to be considered**

The dual problem will actually require more variables than solely the true variables as the simplex style method that is used by linprog for solutions will use slack variables for all constraints to turn them into equality

constraints with unknowns. From this aspect, the dual problem will likely become as more compex to solve as the total number of variables will remain the same (if the slack variables are included) and the number of constraints increases.

- **Redundancy in constraints**

As shown previously, the complementary slackness theorem shows that for any zero variables in the optimal solution of the primal problem, the corresponding constraints in the dual are slack. This means that although the number of variables gets transposed to the number of constraints in the dual, many of them are likely to be redundant in the feasible region in the solution space.

- **Method of linprog solving**

Linprog uses a simplex style method for solving LP problems by default. However there is an option for linprog to use a different solver, this is accessed for large scale problems: linprog can use a linear interior point solver to get the optimal solution [1]. This method traverses the feasible region instead of checking the vertices and so is more resistant to the number of constraints:

the big O complexity of the interior point solver is given as:

$$O\left(n^{3.5} log\left(\frac{1}{\epsilon}\right)\right) \quad [2]$$

Where n is the number of variables and ϵ is the target accuracy. In this description, the number of constraints does not impact the complexity and so it would be fair to conclude that the problem, P or D, with fewer true variables is the least complex from a computing standpoint.

References

[1] http://www.ece.northwestern.edu/local-apps/matlabhelp/toolbox/optim/linprog.html

[2] https://epubs.siam.org/doi/book/10.1137/1.9781611971453

Q2. Integer linear Programming

Introducing the problem cost matrix

```
clc; clear
c = readmatrix("ProblemRankData.csv");
[students, groups] = size(c)
```

```
students = 63
groups = 14
```

```
total = students*groups;
f = reshape(c',[],1);
```

a) Each student can only do one project

Make A matrices:

The A matrices must be constructed in a way that when multiplied with the solutions x it returns less than or equal to 1 (as all the x values will be limited to 1 or 0).

Suppose each of the costs (preferences) in the matrix are identified as $C_{i,j}$ where i is the student and j is the group. The maximum sum of C over i can only be 1 as this is the sum of how many groups that student is allocated to:

for all students i: $\sum_{j}^{groups} C_{i,j} \leq 1$ **(1)**

This translates to

$C_{1,1} + C_{1,2} + ... + C_{1,j} \leq 1$

$C_{2,1} + C_{2,2} + ... + C_{2,j} \leq 1$

...

Which is extracted from cost matrix:

$$C = \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{bmatrix}$$

using matrix multiplication of C and the a coefficient matrix A1 in the form (this is the same as the supply matrix in the transportation problem:

$$A1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Where a series of ones rotates by the number of groups between each row. In matlab this can be achieved by using the buitin circshift function:

```
% A for one student can only do one project
```

```
A_circ = zeros(1, total);
A_circ(1:groups) = 1;

A1 = A_circ;
for i = 1:students-1
    A_new = circshift(A_circ, i*groups);
    A1 = [A1; A_new];
end

% b for one student can only do one project
b1 = ones(1, students);
```

The b1 matrix is just a column vector of ones completing the equation 1

b) Ensuring group sizes

The max and min group sizes will be defined by:

$$g_{min} \leq \sum_{i}^{students} C_{i,j} \leq g_{max}$$

In other words, the sum of students in each group must be greater than the minimum group size and less than the maximum.

So for each group (sum over i - number of students):

$$C_{1,1} + C_{2,1} + ... + C_{i,1} \geq g_{min}$$

$$C_{1,1} + C_{2,1} + ... + C_{i,1} \leq g_{max}$$

That first quation will need to be rearranged into the normal less than or equal to form by multiplying everything by -1:

$$-C_{1,1} - C_{2,1} - ... - C_{i,1} \leq -g_{min}$$

Once again, to retrieve these equations from the cost matrix C, matrix multiplication of C and coefficient matrices A2 and A3 will be used:

Both A2 and A3 will take the form of identity matrices stacked horizontally in the form (with the matrix corresponding to the minimum group size being the negative):

$$A1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Where the number of rows is the number of groups so that when multiplied by the solution it sums the total values for each group. The number of columns is the number of groups times the number of students.

For the b vectors in this they are equal to vectors of the group sizes (with the minimum also being negative):

$$b2_{min} = -\begin{bmatrix} g_{min} \\ g_{min} \\ g_{min} \\ \dots \end{bmatrix},\ b3_{max} = \begin{bmatrix} g_{max} \\ g_{max} \\ g_{max} \\ \dots \end{bmatrix}$$

```
% A for max and min group size
A2 = eye(groups);
A2 = repmat(A2, 1, students);
A3 = A2;

% b for max and min group size
minGroup = 3;
maxGroup = 5;

b2_min = minGroup * ones(1,groups);
b3_max = maxGroup * ones(1,groups);
```

c) Calling intlinprog

intlinprog takes the following inputs: (f, intcon, A, b, A_eq, b_eq,lb, ub)

f is the cost matrix

A will contain all the inequality constraint coefficient matrices put together. In this case this is the group size constraints A2 and A3. A2 will correspond to the minimum group size and A3 to the maximum and so A2 will be negative. The corresponding b matrix will also be assembled in the same way:

$$A = \begin{bmatrix} -A2 \\ A3 \end{bmatrix},\ b = \begin{bmatrix} -b2_{min} \\ b3_{max} \end{bmatrix}$$

The equality constraints will be the ones that ensure that each student can only be assigned to one group.

So A1 and b1 from part b.

All of these constraints also require that the solution values for student allocations are either 1 or 0: either the student is or isn't in a group. This is enforced by intcon and the bounds. The integer constraint, intcon, is written in a way that every single solution element must be constrained as an integer and so each index of the solution must be passed in 1 to the total number of students times groups.

Finally upper and lower bounds must be added as the lower bound of any of the solution elements is 0 and the upper is 1 and so these are just vectors of the length of the number of students times the number of groups.

```
% intcon
intcon = [1:total];

% bounds
lb = zeros(total, 1);
ub = ones(total, 1);
```

```
% Arranging for intlinprog
A = [A3;-A2];
b = [ b3_max'; -b2_min'];

A_eq = A1;
b_eq = b1';
```

Once eveything is defined, intlinprog can be called.

```
% Run intlinprog
[x, fval] = intlinprog(f, intcon, A, b, A_eq, b_eq,lb, ub);
```

```
LP:               Optimal objective value is 83.000000.


Optimal solution found.

Intlinprog stopped at the root node because
the
objective value is within a gap tolerance
of the optimal value,
options.AbsoluteGapTolerance = 0 (the
default value). The intcon variables are
integer within tolerance,
options.IntegerTolerance = 1e-05 (the
default value).
```

```
fval
```

```
fval = 83
```

```
% and the average allocation is given by:
fval/students
```

```
ans = 1.3175
```

The solution can be visualized very simply by viewing which students got which choice

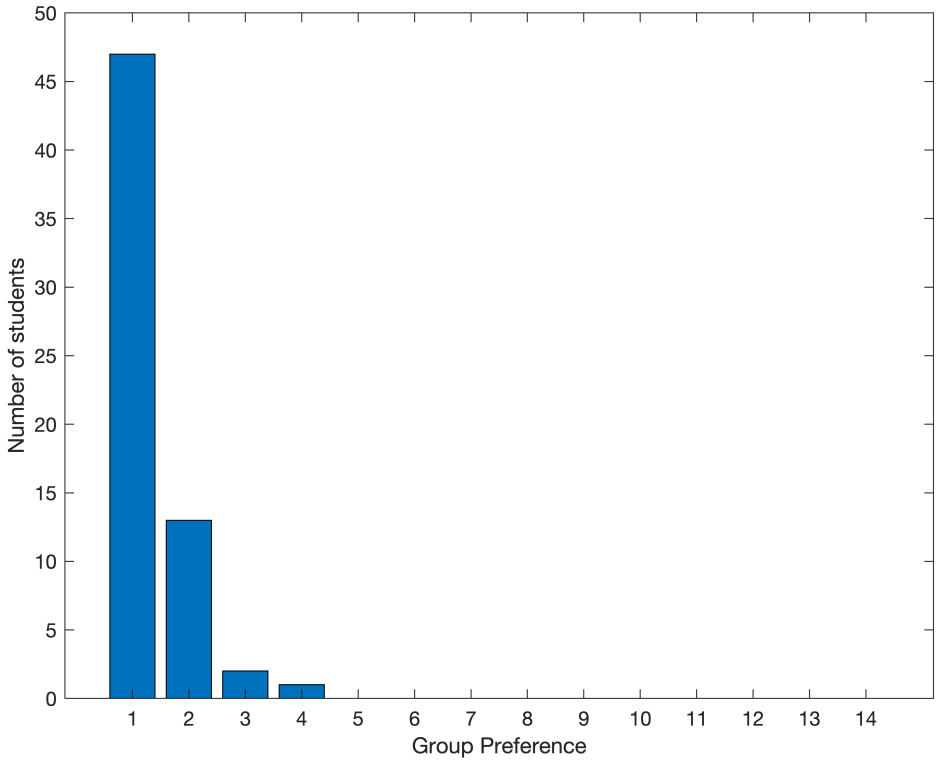
```
og_solutions = reshape(x, groups,students)';
% Show results
og_choiceAllocations = og_solutions.*c;
% worst choice value:
worst_choice_allocated = max(max(og_choiceAllocations))
```

```
worst_choice_allocated = 4
```

The worst choice is given above.

```
% plot how many of each choice:
choices = [];
for i = 1:14
    choices(i) = sum(og_choiceAllocations(:) == i);
end

bar(choices);
```

```
xlabel("Group Preference");
ylabel("Number of students");
```




```
choice1 = choices(1)
```

```
choice1 = 47
```

```
choice2 = choices(2)
```

```
choice2 = 13
```

```
choice3 = choices(3)
```

```
choice3 = 2
```

```
choice4 = choices(4)
```

```
choice4 = 1
```

d) Set a hard limit on the worst preference allowed:

To implement this the A_eq matrix should be copied (into A_hard) to contain the values of c instead of ones. This will mean that the limits imposed by a copy of b_eq (b_hard) will be used as hard limits on choice maximum, in the form of **A_hard*x <=b_hard**. This will be appended to the A inequality matrix: (everything else will stay the same.

```
A_hard = A1;
A_hard_circ = zeros(students,total);
A_hard_circ(1,1:groups) = c(1,:);
```

```
for i = 1:students-1
    A_hard_circ(i+1,(groups*i)+1:(groups*(i+1))) = c(i+1,:);
end

A_hard_circ
```

```
A_hard_circ = 63×882
    10     2     5     9     1    14     6     3    12     4    13     7    11 ···
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     0
     ⋮
```

b_hard will be just the maximum choice allocation

```
max_choice = 3;
b_hard = max_choice*ones(students,1);
```

New intlinprog call

```
[x, fval] = intlinprog(f, intcon, [A;A_hard_circ], [b;b_hard], A_eq, b_eq,lb, ub);
```

```
No feasible solution found.

Intlinprog stopped because no point
satisfies the constraints.
```

There should be no feasible solutions as now with a lower value of r. There is no integer solution that will satisfy a smaller maximum preferrence than 4 as that is already being optimised, and values of r greater than 4 also do not affect the result as that would be not optimal.

e) Exploration and Experimentation

Replacing the cost matrix with higher powers

```
all_choices = [];
for power = 1: 20
    c_power = c.^power;
    f_power = reshape(c_power',[],1);

    % Run intlinprog
    options = optimoptions("intlinprog","Display","none");
    [x, fval] = intlinprog(f_power, intcon, A, b, A_eq, b_eq,lb, ub,[],options);

    solutions = reshape(x, groups,students)';
    choiceAllocations = solutions.*c;
```

```
    % how many of each choice:
    choices = [];
    for i = 1:14
        choices(i) = sum(choiceAllocations(:) == i);
    end

    all_choices(power,:) = choices;

end
all_choices
```

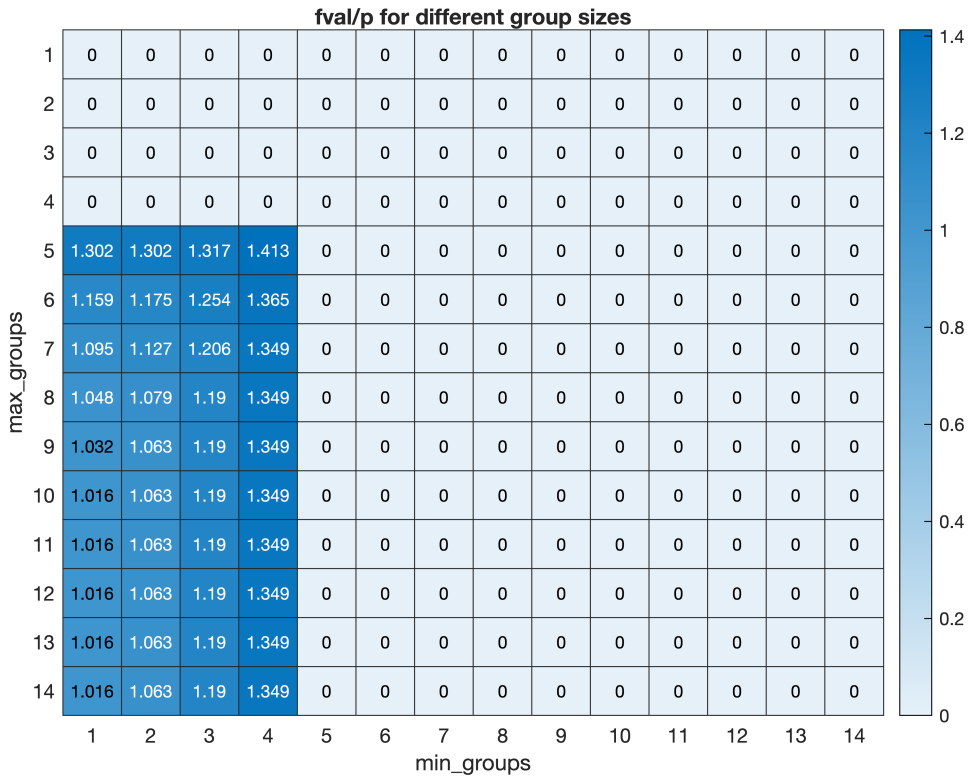
```
all_choices = 20×14
    47    13     2     1     0     0     0     0     0     0     0     0     0 ···
    47    13     2     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
    45    16     1     1     0     0     0     0     0     0     0     0     0
     ⋮
```

Some values have changed but not many, the worst allocation is still a 4th choice group. The power stops making a difference once the cost matrix is cubed (or greater power). Strangely this means that the number of second choice options increases at the cost of fewer first and third choice. This is perhaps because the 4th choice option has to be used otherwise the solution is very different.

How does varying group size impact the solution?

```
counter = 1;
all_choices = zeros(14*14,14);

for min_group = 1:14
    for max_group = 1:14
        b2_min = min_group * ones(1,groups);
        b3_max = max_group * ones(1,groups);
        b = [ b3_max'; -b2_min'];

        % Run intlinprog
        [x, fval,exitflag, output] = intlinprog(f, intcon, A, b, A_eq, b_eq,lb, ub,[],
        if(isempty(x))
        else
            solutions = reshape(x, groups,students)';
            choiceAllocations = solutions.*c;

            % how many of each choice:
            choices = [];
            for i = 1:14
                choices(i) = sum(choiceAllocations(:) == i);
```

```
            end
            all_choices(counter,:) = choices;
        end
        if(isempty(fval))
            fval = 0;
        end
        fval_table(counter,1) = fval;
        fval_table(counter,2) = min_group;
        fval_table(counter,3) = max_group;

        counter= counter + 1;
    end
end
fval_table;
% heatmap of all feasible solutions
fvals = fval_table(:,1)./students;
min_groups = fval_table(:,2);
max_groups = fval_table(:,3);
tbl = table(fvals,min_groups,max_groups);
h = heatmap(tbl,'min_groups','max_groups',ColorVariable='fvals',Title="fval/p for diff
```



The table above shows how the objective value varies with the group size limitation. As expected the lowest fval occurs when the min number is 1 and the max groups is large. Intlinprog return no solution (seen here as 0) when the minimum group size is greater than 63/14 = 4.5 or when the maximum group size is smaller than 4.5.

Randomising row order of cost matrix

```
% run random perm but keep track of how i've shuffled it
indices = randperm(students);

random_c = zeros(students,groups);
for i = 1:length(indices)
    random_c(i,:) = c(indices(i),:);
end

f = reshape(random_c',[],1);
% A for max and min group size
A2 = eye(groups);
A2 = repmat(A2, 1, students);
A3 = A2;
% b for max and min group size
minGroup = 3;
maxGroup = 5;
b2_min = minGroup * ones(1,groups);
b3_max = maxGroup * ones(1,groups);
% intcon
intcon = [1:total];
% bounds
lb = zeros(total, 1);
ub = ones(total, 1);
% Arranging for intlinprog
A = [A3;-A2];
b = [ b3_max'; -b2_min'];
A_eq = A1;
b_eq = b1';

% Run intlinprog
[x, fval] = intlinprog(f, intcon, A, b, A_eq, b_eq,lb, ub);
```

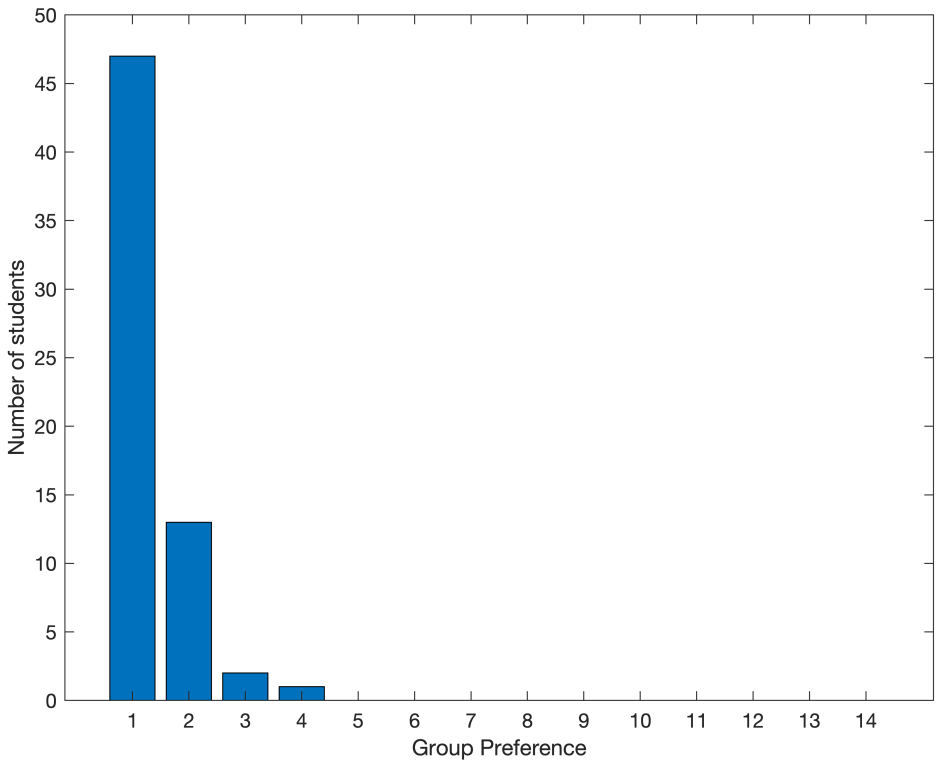
```
LP:               Optimal objective value is 83.000000.


Optimal solution found.

Intlinprog stopped at the root node because
the
objective value is within a gap tolerance
of the optimal value,
options.AbsoluteGapTolerance = 0 (the
default value). The intcon variables are
integer within tolerance,
options.IntegerTolerance = 1e-05 (the
default value).
```

```
solutions = reshape(x, groups,students)';

% shuffle back to original order
unrandom_solutions = zeros(students,groups);
for i = 1:length(indices)
    unrandom_solutions(indices(i)) = solutions(i);
```

```
end

% Show results
rand_choiceAllocations = solutions.*random_c;

% worst choice value:
worst_choice_allocated = max(max(rand_choiceAllocations))
```

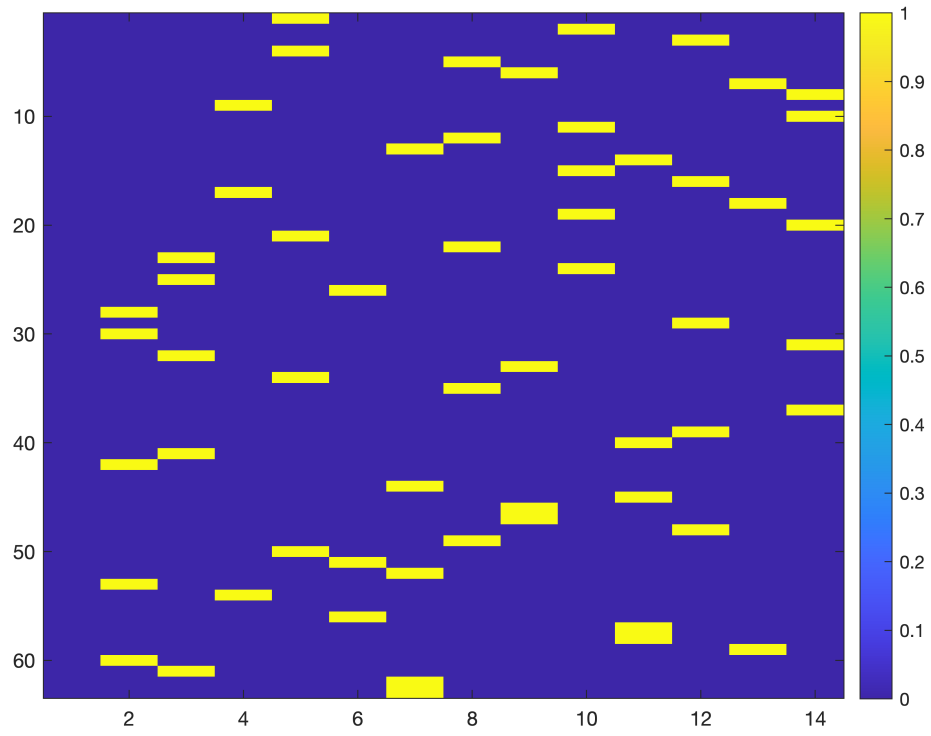
```
worst_choice_allocated = 4
```

```
% plot how many of each choice:
choices = [];
for i = 1:14
    choices(i) = sum(rand_choiceAllocations(:) == i);
end

bar(choices);
xlabel("Group Preference");
ylabel("Number of students");
```



The random permutation of the rows in the cost matrix does not appear to effect the actual fval however they do impact the allocation of groups:

```
difference = og_solutions - unrandom_solutions;
imagesc(difference); colorbar;
```



The plot above shows that a lot of the allocations have been moved around, each colour that is not equal to zero shows that the allocation for that student has changed. This means that the order of students inputted into this program will cause the which groups each student get to change. This solution also gives the same fval and same worst choice which means that the overall average happiness will not change but how the students are distributed does. For larger problems one can suspect that there may be more variations of allocations.

Q3. Non Linear Optimisation

Non linear optimisation.

$f(x, y) = 2x^2 + 5y^2 - 2xy - 2x - 8y$

a.

i) Freeze x=0 to calculate y*, freeze y=y* to calculate x*:

$f(0, y) = 5y^2 - 8y$

$f'(0, y) = 10y - 8$ differentiate

$y^* : 10y - 8 = 0, \ y^* = 0.8$ calculate minimum

$f(x, 0.8) = 2x^2 - 1.6x - 2x$

$f'(x, 0.8) = 4x - 3.6$ differentiate

$x^* : 4x - 3.6 = 0, \ x = 0.9$ calculate minimum

ii) Steepest descent algorithm:

Gradient of f:

$$\nabla f = \begin{pmatrix} 4x - 2y - 2 \\ 10y - 2x - 8 \end{pmatrix}$$

With starting point $\underline{X}^0 = (0, 0)$

The starting direction is:

$d = -\nabla f|_{X=(0,0)}$ it is negative because we are finding the minimum.

$d = (2, 8)$

For one iteration of the steepest descent algorithm, the distance to travel along the steepest direction must be found. This is done by minimising the resulting value of f as a function of λ.

$\underline{X}^1 = \underline{X}^0 + \lambda \nabla f(\underline{X}^0)$ Here to find the minimum X^1 a value of λ must be found that minimises $\underline{X}^1$ as a function of λ.

$min \ \ f(\underline{X}^0 + \lambda \nabla f(\underline{X}^0))$

$min \ \ f((0, 0) + \lambda(2, 8))$

$min \ \ f(2\lambda, 8\lambda)$

Substituting these values into the original function f:

$min\ \ (8\lambda^2 + 320\lambda^2 - 32\lambda^2 - 4\lambda - 64\lambda)$

The minimum occurs when the derivative w.r.t λ is 0:

$$\frac{d}{d\lambda}(296\lambda^2 - 68\lambda) = 0$$

$$\lambda^1 = \frac{68}{592}$$

Now the first iterative step can be found

$$\underline{X}^1 = \underline{X}^0 + \lambda^1 d$$

$$\underline{X}^1 = (0,0) + \frac{68}{592}(2,8)$$

$$\underline{X}^1 = (0.2297..., 0.9189...)$$

```
nonlin_fun = @(x,y)(2.*x.^2 + 5*y.^2 - 2.*x.*y - 2.*x - 8.*y);

nonlin_fun(0,0)
```

```
ans = 0
```

```
nonlin_fun(136/592,544/592)
```

```
ans = -3.9054
```

The above verifies that we do move towards the minimum.

iii) Quadprog

Set up the Hessian matrix H

```
% the hessian can be calculated in matlab with the symbolic toolbox and the
% following code.
syms x y
f = symfun(2*x.^2 + 5*y.^2 - 2*x.*y - 2*x - 8*y,[x y]);
H = double(hessian(f,[x,y]))
```

```
H = 2×2
     4    -2
    -2    10
```

Quadprog attemps to minimise: $0.5(x'Hx + f'x)$ subject to: A*x <= b

In this example there are no constraints, so it is simply H and f_obj that are used

```
% the f_obj values are just the linear coefficients of the function f
f_obj = [-2 -8]
```

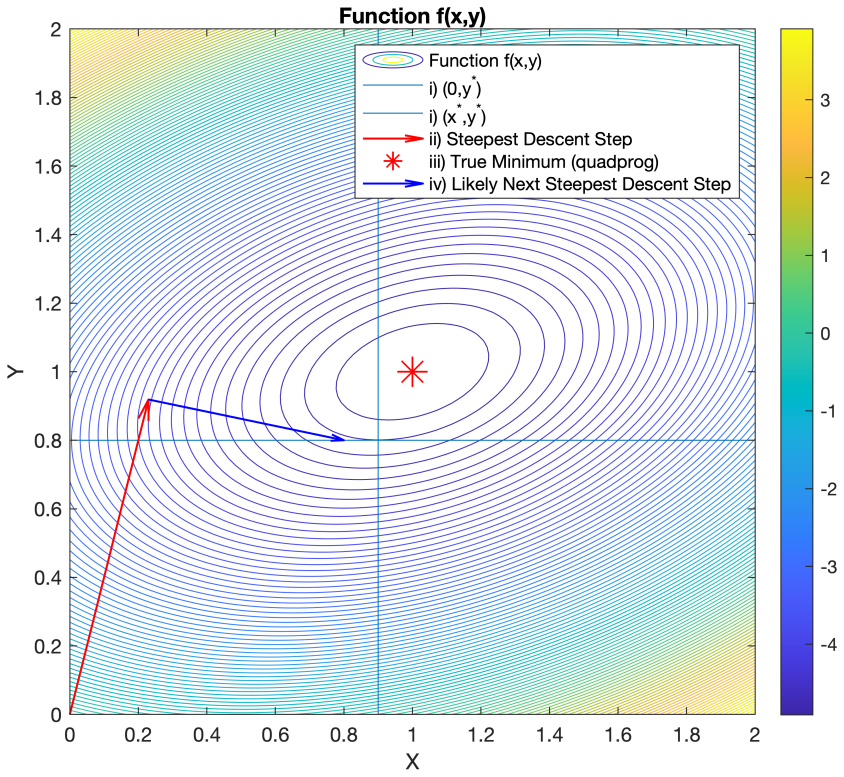
```
f_obj = 1×2
    -2    -8
```

```
[solution, fval] = quadprog(H, f_obj)
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
solution = 2×1
    1.0000
    1.0000
fval = -5
```

iv) Plot f and above computations

```
x=linspace(0,2,100);
y=linspace(0,2,100);
[X,Y]=meshgrid(x,y);

Z = nonlin_fun(X,Y);

contour(X,Y,Z,100); axis equal; colorbar; title('Function f(x,y)')
hold on

% overlay direction of gradient from starting point
p1 = [0 0];
p2 = [136/592 544/592];
dp = p2-p1;

% likely next steepest descent step (guess from graph)
p3 = p2;
p4 = [0.8 0.8];
dp2 = p4-p3;


% overlay parts i,ii,iii
line([0 2],[0.8 0.8])
line([0.9 0.9],[0 2])
quiver(p1(1),p1(2),dp(1),dp(2),0,LineWidth=1,Color='red')
plot([solution(1)],'o','Marker','*','Color','red','MarkerSize',15,'LineWidth',2)
quiver(p3(1),p3(2),dp2(1),dp2(2),0,LineWidth=1,Color='blue')


legend('Function f(x,y)','i) (0,y^*)','i) (x^*,y^*)','ii) Steepest Descent Step','iii)
xlabel('X')
ylabel('Y')

hold off
```



Discussion

Parts i) demonstrate the two lines where at first x is frozen at x=0 to find optimal y, then y is frozen at that value to find optimal x. You can see from the contours that both lines appear at a 1D minimum along the other variable (first vertically minimum then horizontally). This is not a very accurate method of obtaining a true local minimum but does provide an estimate that is not too far off.

ii) shows the first step of the gradient descent algorithm where the steepest downhill at (0,0) is followed to its minimum (when it starts going up hill) that is the end of the first step. As shown in iv) I have estimated the next step of the algorithm as a movement along the steepest downhill from the previous point to a minimum in 1D. This will continue until the true minimum is reached. Gradient descent will continue to make steps along the steepest gradient and will eventually find the true minimum as there are no false local minima in the viscinity of this function.

iii) shows the true solution as given by quadprog where X = (1,1) and the function objective value is -5.

b) Kirchoff Solver and Network Preparation

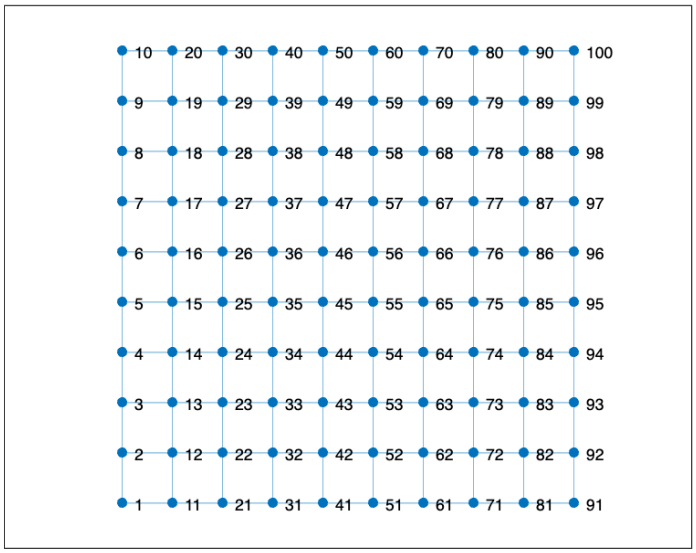
Clear the workspace, set input parameters

```
clear; close all; more off

% define the size of the grid (the number of nodes in one direction)
n = 10;
```

numgrid numbers the grid points in a square with n+2 being the number of nodes of each side to make a side length n.

delsq creates a grid from the numbered grid points using a 5 point stencil: i.e. up right down left and centre to connect each vertex to the next.

```
grid = delsq(numgrid('S',n+2));

% the graph function then creates a graph object (with nodes and edges)
% from the grid
ngraph = graph(grid,'omitselfloops');

% plot the graph
h = plot(ngraph);

% the data points need to be rounded to nearest integer so that the grid is
% straight
h.XData = floor((0:n^2-1)/n);
h.YData = rem((0:n^2-1), n);

axis equal
```



Calculate the positions of all nodes and the edges

```
% position of the nodes
x = h.XData;
y = h.YData;

% define edges st (graph object Edges outputs a table of edges and weights
% - we don't need the weights
st = table2array(ngraph.Edges);
st(:,3) = [];
```

Set Edge weights/resistances all as 1 since they are the same length

```
r = ones(length(st),1);
```

Create listAnode and listCnode (as starting node and ending node)

```
listAnode = [1];
listCnode = [n^2];
```

Call the solver, record solve time

```
tic;
[X,V]=KirchoffSolve(st,r,listAnode,listCnode);
```

```
_____________________________________________________________
   Diagnostic Information

Number of variables: 182


Number of linear inequality constraints:    0
Number of linear equality constraints:      102
Number of lower bound constraints:          0
Number of upper bound constraints:          0

Algorithm selected
   interior-point-convex


_____________________________________________________________
   End diagnostic information

 Iter            Fval  Primal Infeas    Dual Infeas  Complementarity
    0    9.000000e+01   1.000000e+00   9.778500e-01     0.000000e+00
    1    1.505835e+00   4.440892e-16   1.665335e-16     0.000000e+00

Optimization completed: The relative dual feasibility, 1.665335e-16,
is less than options.OptimalityTolerance = 1.000000e-08, the complementarity measure,
0.000000e+00, is less than options.OptimalityTolerance, and the relative maximum constraint
violation, 4.440892e-16, is less than options.ConstraintTolerance = 1.000000e-08.
```

```
t = toc
```

```
t = 0.2015
```

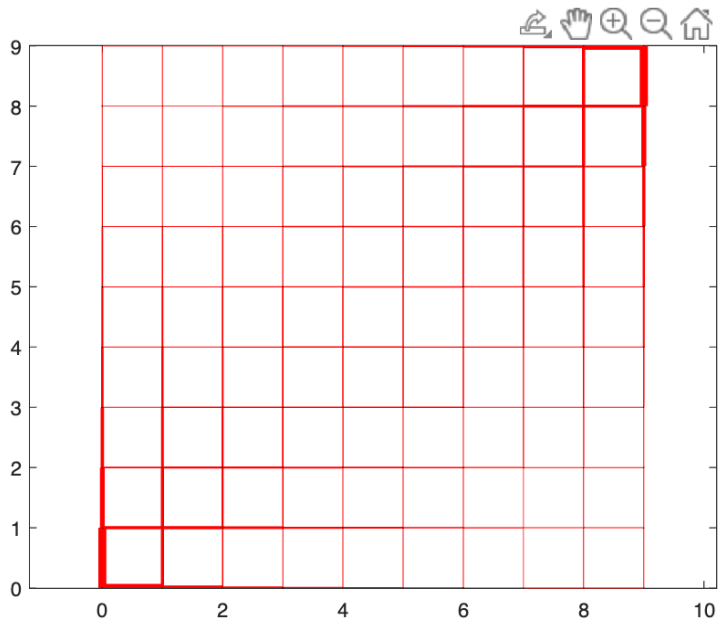
The voltage drop is the effective resistance of the network as the input current is 1. (V = IR)

```
Vdrop = V(listCnode(1))-V(listAnode(1))
```

```
Vdrop = 3.0117
```

Plot what it looks like

```
figure;
hp=plot(x(st)',y(st)','r-');
axis equal

for i=1:length(hp)
   hp(i).LineWidth = max(1e-3,4*abs(X(i))/(max(abs(X))));
end
hold on
for i=1:max(max(st)) % ie over nodes
  hr(i)=rectangle('Position',[x(i)-0.005,y(i)-0.005,0.01,0.01],'Curvature',[1 1], ...
      'FaceColor',[0 1 0]*(V(i)-min(V))/(max(V)-min(V)));
end
```



Exploring the size of N before my computer crashes

Package all the above into a callable function. Function defined with outputs of X,V as above, Vdrop for comparison, and t = time taken to run KirchoffSolve

The following while loop is run calling Kirch_Run() which is the packaged function and the result is plotted with an exponential line of best fit.

```
% N_plot = [];
% t_plot = [];
```

```
% counter = 1;
% N = 4;
% t = 0;
% while t < 10
%     [X,V,t,Vdrop] = Kirch_Run(N);
%     N_plot(counter) = N;
%     t_plot(counter) = t;
%     counter=counter+1;
%     N = N*2;
% end
```

The t values are plotted against the N values and an exponential fit is calculated:

```
% g = fittype('a-b*exp(-c*x)');
% f0 = fit(N_plot',t_plot',g,'StartPoint',[0 0 0]);
% xx = linspace(1,max(N_plot),50);
% plot(N_plot,t_plot,'o','DisplayName','Calculated Points');
% hold on
% plot(xx,f0(xx),'r-','DisplayName','Exponential line of Best Fit');
% xlabel("Dimension N")
% ylabel("Time to Compute (s)")
% title("Time Complexity")
% legend()
```

The resulting graph of time complexity for a maximum run time of 120s per iteration

NB: I plotted this in a separate file using all the code above just so I don't run it everytime this script is run.

This is very clearly O(exp(N)) time complexity and so it will very quickly ruin my computer.

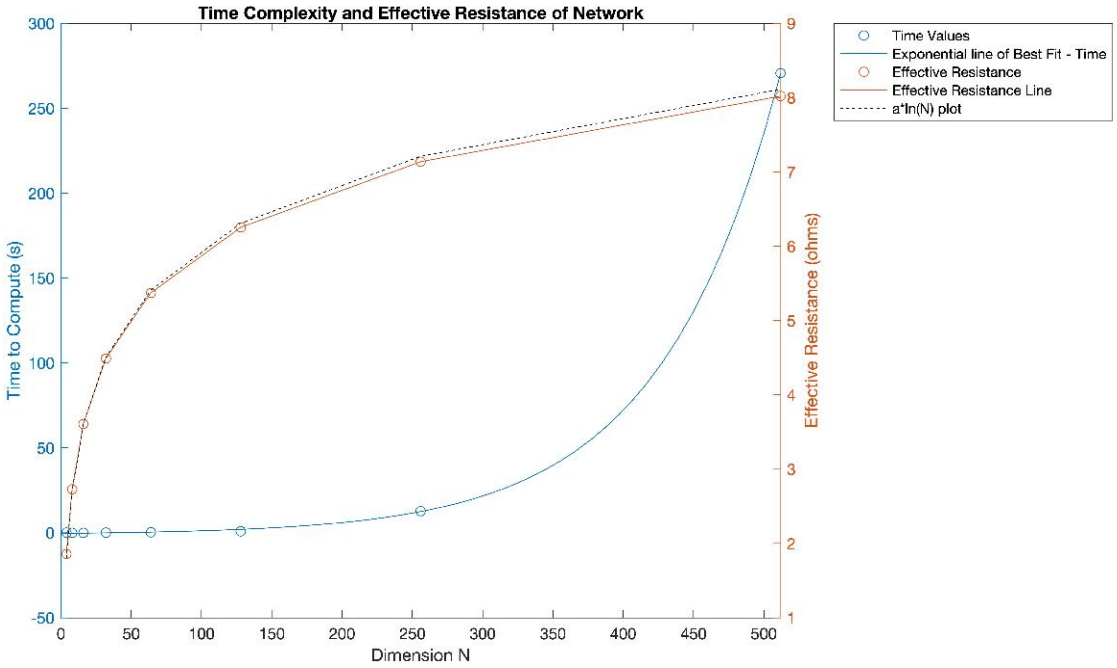
```
val =

     General model:
     val(x) = a-b*exp(-c*x)
     Coefficients (with 95% confidence bounds):
       a =    -0.05508  (-0.2013, 0.09115)
       b =    -0.08459  (-0.1367, -0.03245)
       c =    -0.01911  (-0.02148, -0.01674)
```

The coefficient of the line of best fit of time are shown above.

So the time complexity is more accurately: $O(0.08\exp(0.02N))$

For my computer (2.9Ghz i5 with 8gb ram) to take 1 hour to solve a calculation would only require around 540 variables.

The resistance is plotted as joined up scatter.



The resistance curve looks suspiciously like an log(N) plot and so I have added a plot of a*ln(N) for comparison with a being a mean of the resistance values/ln(N) for the values plotted.

This suggests that the effective resistance of this sort of network tends towards a value proportional to *a*ln(N)*. With a value of *a* of just under 1.3. This value is likely to be a constant related to the shape of the grid.

The ln(N) shape also suggests that expanding the grid after a certain point produces limited increases in resistance, while the computation increases significantly. By using a predictive formula calculated from a smaller number of dimensions a fairly accurate estimate can be produced for large number N. It is likely logarithmic because as the size of the grid increases the current remains the same and small in the majority of the grid. It is only the area of higher currrent in the start and end corners that slightly increases in size.

Q4. Global Optimisation

a. Particle Swarm Methods

a..1 Summary

The original Particle Swarm Optimisation (PSO) was originally introduced by Dr. Kennedy and Dr. Eberhart [1] in 1995 as a method of finding global optima, at the time for use in neural network optimisation. The algorithm was built on the basis of swarm intelligence, as is found in bird flocks or similar. The main attributes of a swarm that were also found to be the main attributes of PSO algorithms are:

1. The swarm must have the ability to carry out simple space and time calculations. In the case of PSO this means that the particles must be able to change direction and velocity in response to stimuli.
2. The swarm must be able to respond to its surrounding environmental quality. A particle in PSO must be able to assess its objective function value.
3. The swarm has some diversity in its response to stimuli and so there is some variation in movement.
4. The swarm must have some stability and therefore not respond to all stimuli equally/ maintain some sort of its own motion.
5. The opposite to 4, is that the swarm must also adapt its behaviour when the gains made by this change are considered worthwhile.

These concepts are all expressed in the general PSO algorithm.

a..2 Method

The general idea of the algorithm is that a population of particles begins at random positions with random velocities in the space. At each iteration the position of each particle is updated by calculating the displacement of the particle due to a calculated velocity. The velocity of the particle is a function of time and is influenced by three parameters: ω the inertia weight constant, C_1 the cognitive coefficient, and C_1 the social coefficient. The tuning of these parameters impacts the speed and accuracy of the PSO algorithm [2].

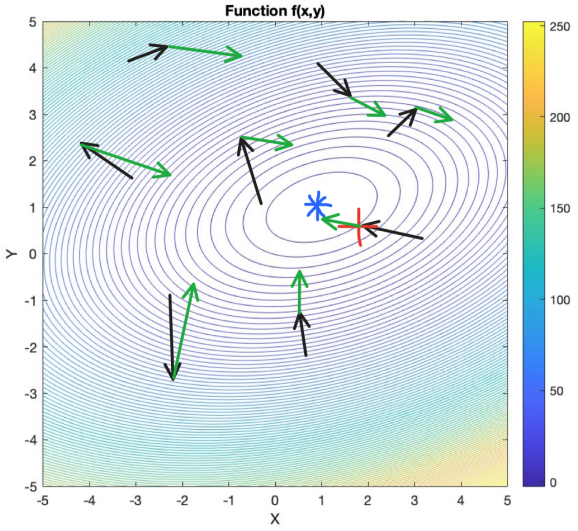


Figure 1: A rough demonstration of how PSO would act on a simple optimisation problem.

At each iteration, each particle updates its best value of the objective function pbest, and the global best gbest is also updated from all the particles. These variables give the swarm a 'memory' and allow locations of the optimal values (so far) to be stored. This allows the swarm intelligence to slowly discover new local optima while centering around a global optima. As shown in Figure 1, the black arrows indicate the start of the algorithm and the green show the actions of the particles on the second iteration. The red cross is the first iteration global best and the blue star is the true optimum. The particles velocities are impacted by their previous velocity (ω inertia) and both the location of their own optimal and the swarm optimal from the previous iteration. This relationship is described in the following:

$$X^i(t+1) = X^i(t) + V^i(t+1) \tag{1}$$

$$V^i(t+1) = \omega V^i(t) + c_1 r_1(pbest^i - X^i(t)) + c_2 r_2(gbest - X^i(t)) \tag{2}$$

These two equations govern the actions of all the particles in the algorithm [3]. Equation 1 simply determines new positions at each iteration in the same way as physics particles. Equation 2 determines the direction and magnitude of the velocity through comparison of the current location to the local best and the global best.

r_1, r_2 are random numbers given from a distribution giving the algorithm a stochastic quality. The influence of the local versus the global best is determined by the C_1 the cognitive coefficient and C_2 the social coefficient as mentioned above. A greater C_1 increases the exploration of new space by the particles whereas a greater C_2 increases exploitation, i.e. how much each particle is influenced by the discoveries of the rest of the swarm. ω is used to vary the inertia of a particle.

A greater inertia causes PSO to take more iterations to find a global optimum (if found) but also increases the likelihood it is not found. Whereas a smaller ω means that the PSO algorithm finds a global optimum quicker (if found) but also increases the likelihood that it is not found.

$$W_i = 1.1 - \left(\frac{gbest_i}{pbest_i}\right) \tag{3}$$

The solution to this is varying the inertia with time [3]. It is found that varying it non-linearly by method of Global Local best Inertia Weight (GLbestIW), in equation 3 has the best results. Generally this means the inertia increases the closer each particle is to the global optimum meaning that they are kept in that area.

a..3 Pros

1. The PSO method,first of all, is a simple algorithm with very basic calculation at each iteration. This means that it is computationally inexpensive and is therefore generally insensitive to the scaling of variables [4]. Similarly this means that the memory and processor requirements for implementing the method are minimal. In terms of computational efficiency the method can also be parallelized meaning each particle can be calculated simultaneously meaning that the algorithms are generally quick to run.

2. Another highly important feature is that the PSO method is derivative free and can therefore be applied to real world non-smooth objective functions [4].

3. By using the PSO method, the algorithm benefits from few parameters which can be tuned for optimal performance [5], as described in section a..2.

a..4 Cons

1. One of the main drawbacks of the PSO method is that it has a tendency to result in early convergence on sub optimal points. In other words, the method will likely converge in proximity to the true global optimum quickly [4]. For some applications this makes this optimisation method not suitable.

2. The convergence of this method is also not guaranteed for some functions [4].

3. For use in refined search areas, the PSO method is often slower when finding a true optimum in a mid optimal area.

Many of these drawbacks can be limited and controlled through the use of the method parameters, allowing PSO to be used in many different applications successfully.

References

[1] J. Kennedy and R. Eberhart. Particle swarm optimization. In *Proceedings of ICNN'95 - International Conference on Neural Networks*, volume 4, pages 1942–1948 vol.4, 1995.

[2] Y. Shi and R. Eberhart. A modified particle swarm optimizer. In *1998 IEEE International Conference on Evolutionary Computation Proceedings. IEEE World Congress on Computational Intelligence (Cat. No.98TH8360)*, pages 69–73, 1998.

[3] M Senthil Arumugam and M V C Rao. On the performance of the particle swarm optimization algorithm with various inertia weight variants for computing optimal control of a class of hybrid systems. *Discrete Dynamics in Nature and Society*, 2006:79295, 2006.

[4] Mohd Nadhir Ab Wahab, Samia Nefti-Meziani, and Adham Atyabi. A comprehensive review of swarm optimization algorithms. *PloS one*, 10(5):e0122827, 2015.

[5] Ahmad Rezaee Jordehi and Jasronita Jasni. Particle swarm optimisation for discrete optimisation problems: a review. *Artificial Intelligence Review*, 43(2):243–258, 2015.

b. Developing PSO code

```
hold off; clear; close all; clc;
```

Define the eggholder function

```
fun = @(x)(-(x(2)+47).*sin(sqrt(abs((x(1)./2) + (x(2) + 47)))) + ...
    - x(1).*sin(sqrt(abs(x(1) - (x(2) +47)))))
```

```
fun = function_handle with value:
    @(x)(-(x(2)+47).*sin(sqrt(abs((x(1)./2)+(x(2)+47))))+-x(1).*sin(sqrt(abs(x(1)-(x(2)+47)))))
```

```
% True value of global optimum (check function works) and for comparison
% later
true = fun([512, 404.2319]);
```

Run `x = particleswarm(fun,nvars)` where nvars is the number of dimensions.

```
nvars = 2;
```

The bounds are the search domain as given in the wikipedia page

```
lb = [-512,-512];
ub = [512,512];
```

The options as with all optimisation can be defined. The swarm size is simply the number of particles, and the number of iterations that are carried out is MaxIterations, after attempting with default parameters, I set the options to those shown below as it provided a correct solution quickly.

Generally a greater number of iterations and a greater swarm size will result in better results at the cost of computational price.

```
options = optimoptions('particleswarm','SwarmSize',100,MaxIterations=100);

[x, fval] = particleswarm(fun,nvars,lb,ub,options)
```

```
Optimization ended: relative change in the objective value
over the last OPTIONS.MaxStallIterations iterations is less than OPTIONS.FunctionTolerance.
x = 1×2
  512.0000  404.2318
fval = -959.6407
```

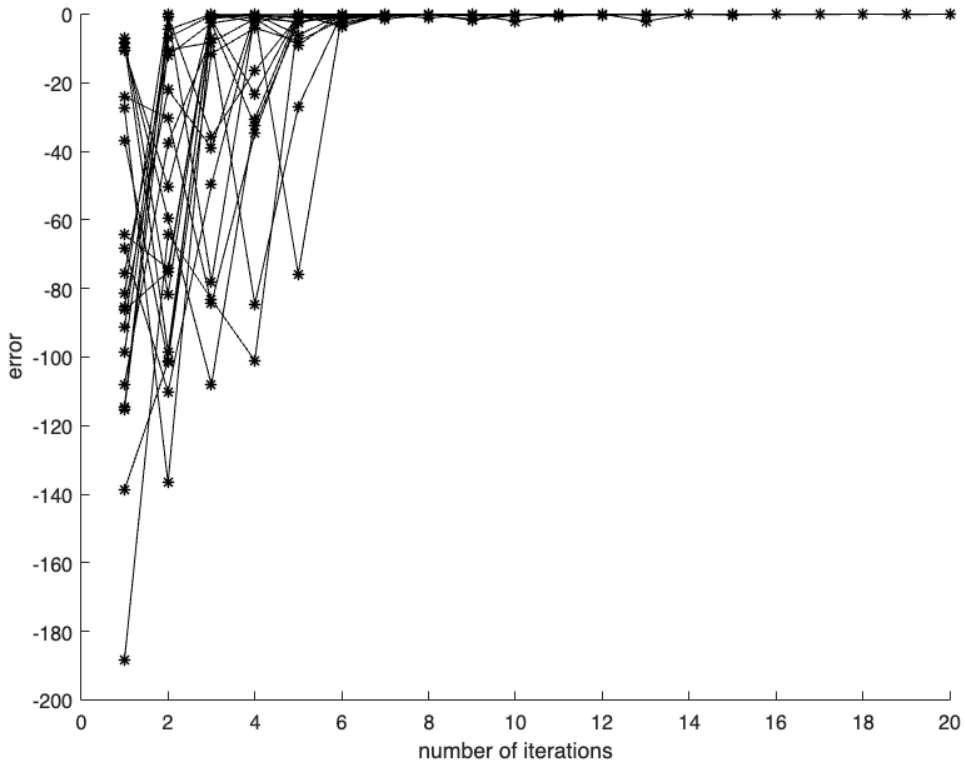
```
error = true - fval
```

```
error = 1.0235e-08
```

Test for many attempts

```
hold on;
for repeat = 1:20
    fvals = [];
    for i = 1: 20
```

```
        options = optimoptions('particleswarm','SwarmSize',200,MaxIterations=i,Display=
        [x, fval] = particleswarm(fun,nvars,lb,ub,options);
        fvals(i) = true - fval;
    end
    plot(1:i,fvals,'k-',LineWidth=0.1);
    plot(1:i,fvals,'k*',LineWidth=1);
    ylabel('error');
    xlabel('number of iterations');
end
hold off;
```



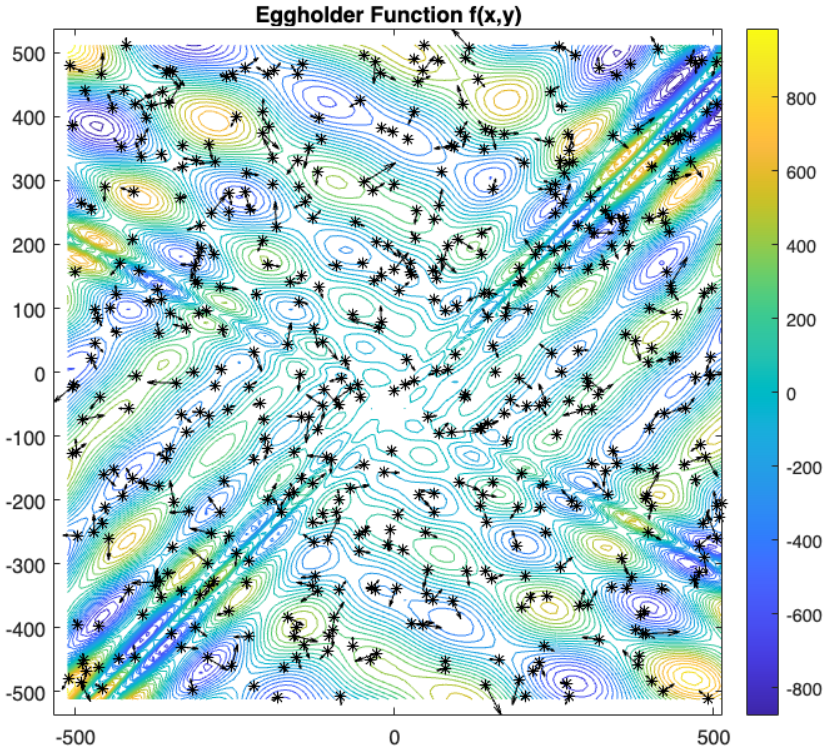
After plotting the optimisation for 20 iterations at the same problem, you can see the stochastic property of the algorithm and also that is it is not always guaranteed to converge in the set time steps. A better measure of convergence would be the change on previous fval estimates which would stop the algorithm from diverging after attaining a zero error.

Writing particle swarm optimisation algorithm from scratch

Because the maths seems so simple it would be interesting to compare performance of the toolbox PSO against a home made one:

```
% define the inputs
num_iter = 50;
swarm_size = 500;
```

```
% plot surface
x=linspace(-512,512,100);
y=linspace(-512,512,100);
[X ,Y] = meshgrid(x,y);
egg = (-(Y+47).*sin(sqrt(abs((X./2) + (Y + 47)))) + ...
    - X.*sin(sqrt(abs(X - (Y +47)))));
contour(X,Y,egg,30); axis equal; colorbar; title('Eggholder Function f(x,y)')
hold on


% give each particle a random position in the within the bounds
X_i =(rand(2,swarm_size)*512*2)-512;

% give each particle a random velocity
V_i = randn(2,swarm_size)*0.1;

% plot points with velocities
scatter(X_i(1,:),X_i(2,:),'k*');
quiver(X_i(1,:),X_i(2,:),V_i(1,:),V_i(2,:),LineWidth=0.5,Color='black');

hold off;
```



This is the initial layout of particles and their respective velocities. At each iteration of the algorithm both of these properties are updated and new values stored.

```
% Set coefficients
```

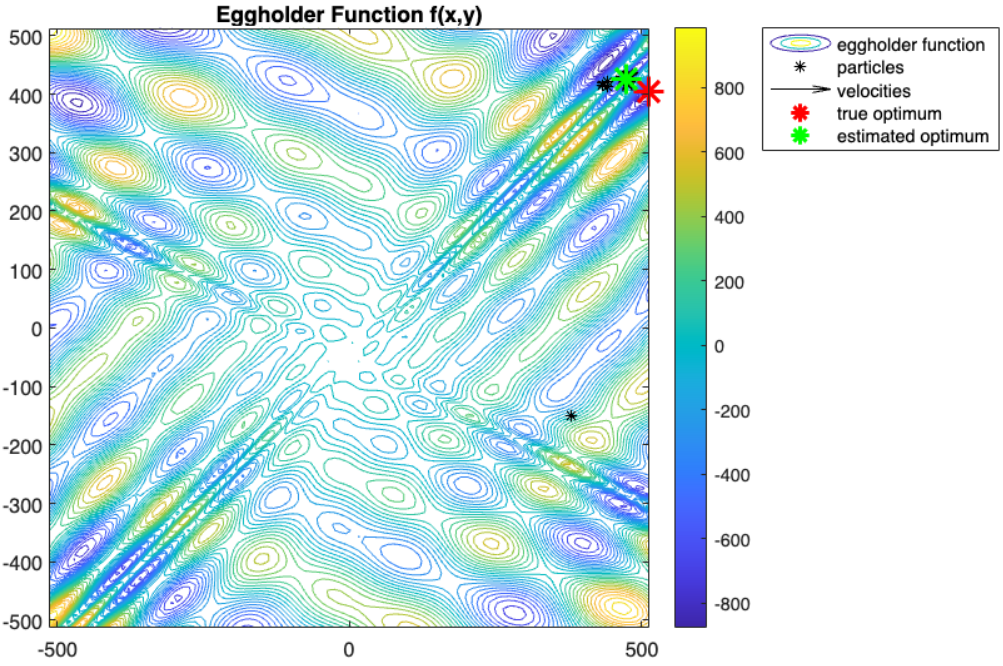
```
c1 = 1;
c2 = 1;

V_1 = zeros(2,swarm_size);

for iter = 1:num_iter
    for i = 1:swarm_size

        r = rand(1,2);
        % update the best values
        [pbest, pbest_val, gbest , gbest_val] = calc_bests(X_i,fun); %function found a
        % w as GLbestIW as described in section 4a
        w = 1.1 - (gbest_val./pbest_val(i));
        % velocities update
        V_1(:,i) = w.*V_i(:,i) + c1.*r(1).*(pbest(:,i) - X_i(:,i)) + c2.*r(2).*(gbest -

        % location update
        X_i(:,i) = X_i(:,i) + V_1(:,i);
        % I have added this line to stop the particles escaping during the
        % optimisation, if the new coordinate is out of bounds the velocity
        % is reversed.
        if X_i(1,i) > ub(1) || X_i(1,i) < lb(1) ||  X_i(2,i) > ub(2) || X_i(2,i) < lb(
            X_i(:,i) = X_i(:,i) -  V_1(:,i);
        end
    end
end

% final error
error = true - gbest_val
```

```
error = -14.4360
```

```
gbest_val, gbest
```

```
gbest_val = -945.2047
gbest = 2×1
  474.4765
  424.9815
```

```
contour(X,Y,egg,30); axis equal; colorbar; title('Eggholder Function f(x,y)');
hold on
% plot points with velocities
scatter(X_i(1,:),X_i(2,:),'k*');
quiver(X_i(1,:),X_i(2,:),V_i(1,:),V_i(2,:),LineWidth=0.5,Color='black');
% true optimum
plot(512, 404.2319,'o','Marker','*','Color','red','MarkerSize',15,'LineWidth',2);
% estimated optimum
plot(gbest(1),gbest(2),'o','Marker','*','Color','green','MarkerSize',15,'LineWidth',2)
legend('eggholder function', 'particles', 'velocities', 'true optimum','estimated optim
hold off;
```



Above is a plot of the final result after the iterations are complete.

Below is the function to calculate the best values and positions (used in the homemade function)

```
function [pbest, pbest_val, gbest , gbest_val] = calc_bests(X_i,fun)
% calculate global and local bests
    pbest = X_i;
    for i = 1:length(X_i)
        pbest_val(i) = fun([X_i(1,i),X_i(2,i)]);
    end
    [gbest_val, idx] = min(pbest_val);
    [gbest] = pbest(:,idx);
end
```

Discussion of results

With this homemade algorithm, its clear that it has a habit of falling into local optima and converging early. Having experimented with a constant inertia weight and GLbestIW, the latter performs better on average but still runs into the same issue of converging early on not the global optimum. A decrease in the c2 coefficient slows convergence and increases the variability of results with more spread - i.e. increase individual particle exploration.

I have also added a couple of lines to ensure that the particles do not escape the search area, I have done this by reversing the direction of the particles if a new location is about to be outside the search area. This is not

mentioned in my report as I didn't think this would be an issue. However, having attempted to build PSO from scratch it is clear that this needs to be a consideration.

Even with this fairly basic code a fairly reasonable estimate of the minimum function value can be achieved, however, relative to the toolbox function with the same input parameters the result is somewhat disappointing. Even with swarm size and iterations set to a much higher value the results are not as accurate.

Increasing the swarm size and number of iterations will also improve performance of the homemade algorithm - I do not want to run these large problems as the small scale still provides a relatively good estimate of the minimum. Similarly the coefficients can be tuned to the problem but this will likely make the algorithm generalise to other functions badly.

c) Eddies Secret Function

```
hold off; close;
eddieSecretFunction([-1,-2]')
```

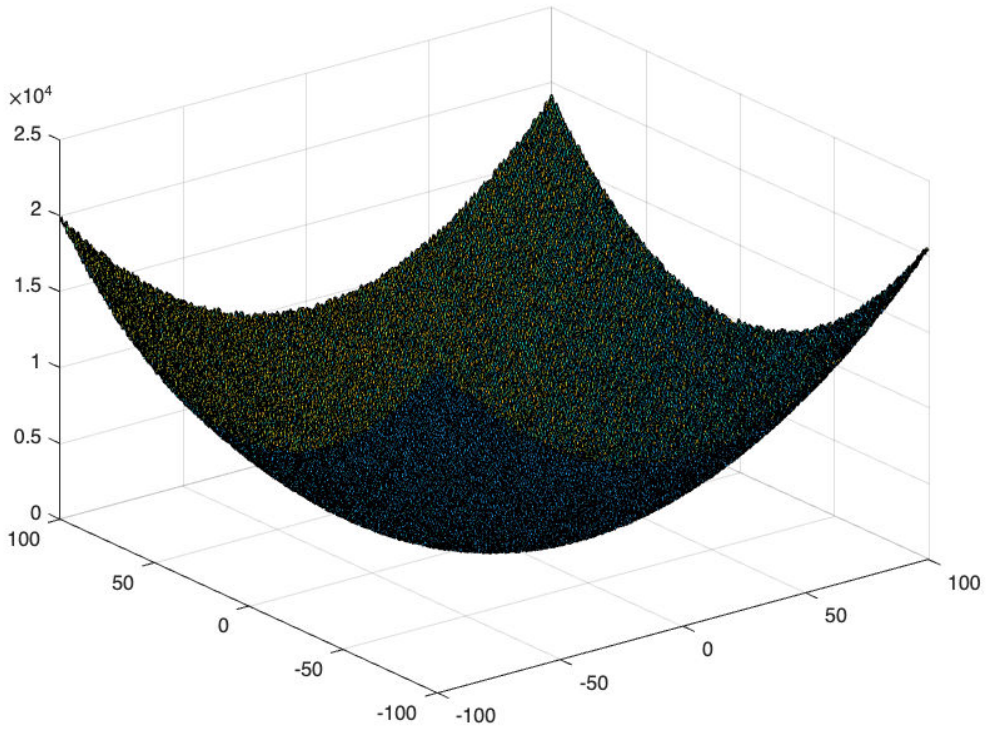
```
ans = 338.6427
```

```
eddieSecretFunction([-1,-2]')
```

```
ans = 93.2079
```

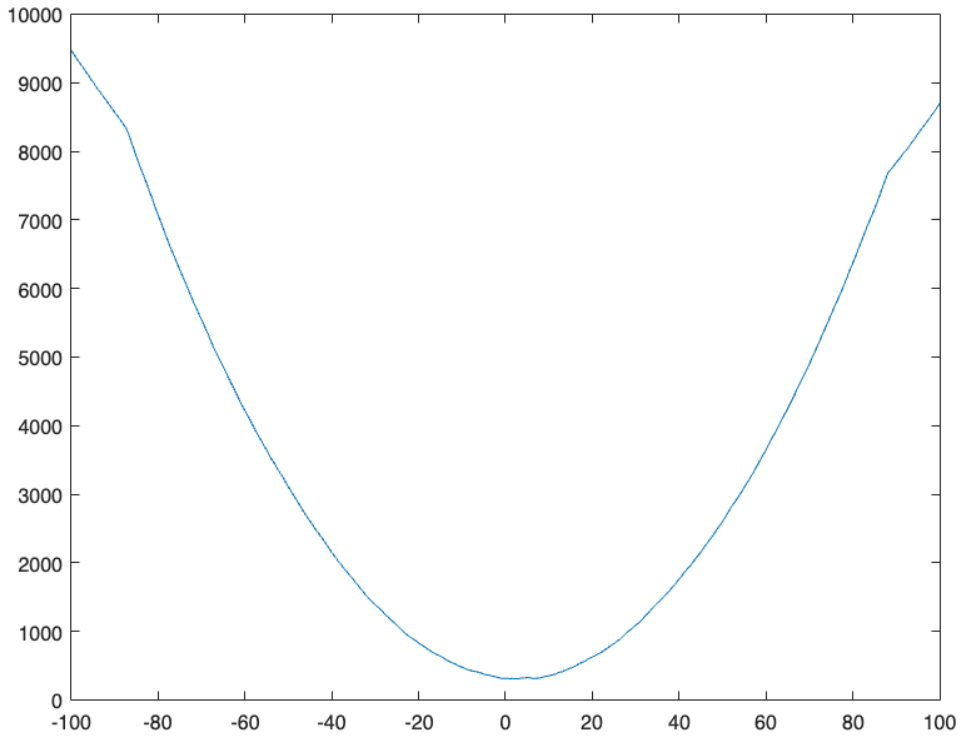
Can see that the function is noisy / randomised as the same input returns different inputs.

```
% package the function nicely
fun = @(x)(eddieSecretFunction([x(1),x(2)]'));

% plot the function
x = linspace(-100,100,300);
y = linspace(-100,100,300);
[X,Y] = meshgrid(x,y);

for i = 1:length(x)
    for ii = 1:length(y)
        Z(i,ii) = fun([X(i,ii),Y(i,ii)]);
    end
end
surfl(X,Y,Z)
```



It's a noisy bowl!

```
% look at cross section at y=0 as that seems close to the global minimum
y = 0;
for i = 1:length(x)
    Z_cross(i) = fun([x(i),0]);
end
% Smooth input data
Z_cross = smoothdata(Z_cross,"movmean","SmoothingFactor",0.15);
plot(x,Z_cross,'-')
```



When it's slightly smoothed you can see there is a minimum somewhere around the origin, possibly towards the positive side of the origin.

The function is clearly non-linear, stochastic(noisy) and non smooth meaning the global optimisation methods that can be used:

- Particle Swarm
- Genetic Algorithms
- Simulated Annealing
- and others

Since I have already looked at Particle Swarm optimisation I will first try optimisation using PSO.

Main issues:

- input to function must be one variable, this is addressed in fun line.
- output from function must be a single value, inputting a vertical column vector outputs a single scalar value.

```
lb = [-100,-100];
ub = [100,100];

% Minimising using particle swarm
options = optimoptions("particleswarm","MaxIterations",600,"SwarmSize",1500);
[x, fval] = particleswarm(fun,2,lb,ub,options)
```

```
Optimization ended: relative change in the objective value
over the last OPTIONS.MaxStallIterations iterations is less than OPTIONS.FunctionTolerance.
x = 1×2
    1.9968    4.9909
fval = 1.0709
```

```
% minimising using GA
GAoptions = optimoptions("ga","MaxGenerations",600,"PopulationSize",1500);
[x,fval] = ga(fun,2,[],[],[],[],lb,ub,[],[],GAoptions)
```

```
Optimization terminated: maximum number of generations exceeded.
x = 1×2
    2.0160    3.9995
fval = 0.4926
```

After a few attempts with Particle Swarm and Genetic Algorithms with similar parameters, the algorithms tend to find a similar minimum value in a similar area, however they do not converge on the same point. This confirms what can be deduced visually, and can be used to make the search area smaller:

```
lb = [-10,-10];
ub = [10,10];

[x, fval] = particleswarm(fun,2,lb,ub,options)
```

```
Optimization ended: relative change in the objective value
over the last OPTIONS.MaxStallIterations iterations is less than OPTIONS.FunctionTolerance.
x = 1×2
    2.9851    3.9852
fval = 1.1480
```

```
[x,fval] = ga(fun,2,[],[],[],[],lb,ub,[],[],GAoptions)
```

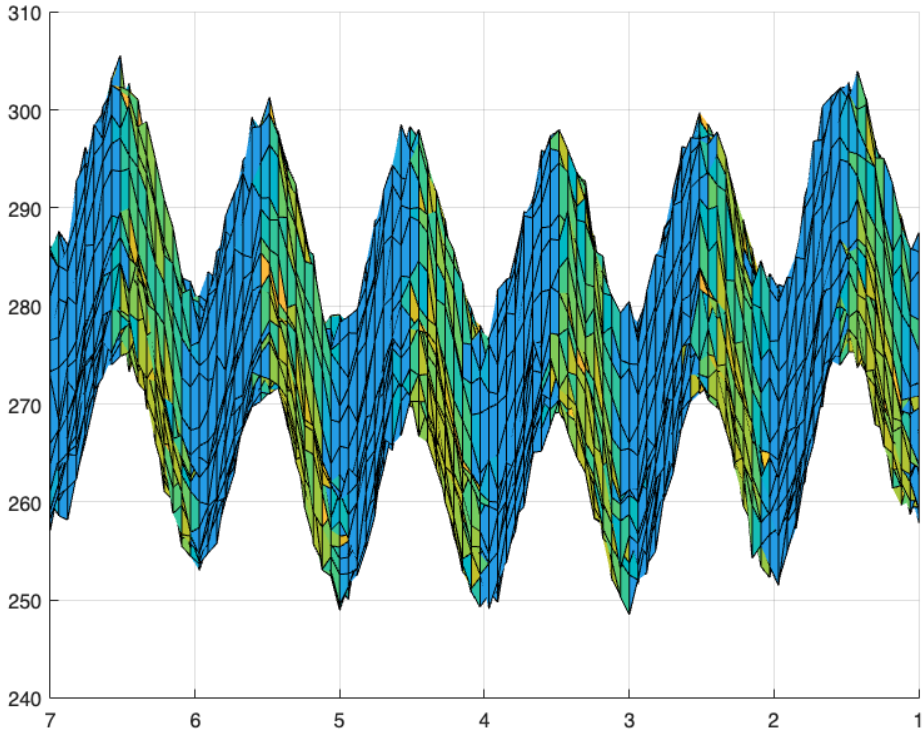
```
Optimization terminated: maximum number of generations exceeded.
x = 1×2
    1.9656    4.0358
fval = 2.0568
```

In this area the both algorithms cope fairly well, narrowing in around the (2,4) area (with an fval around in the range [0.08 0.19].

Can explore this closer:

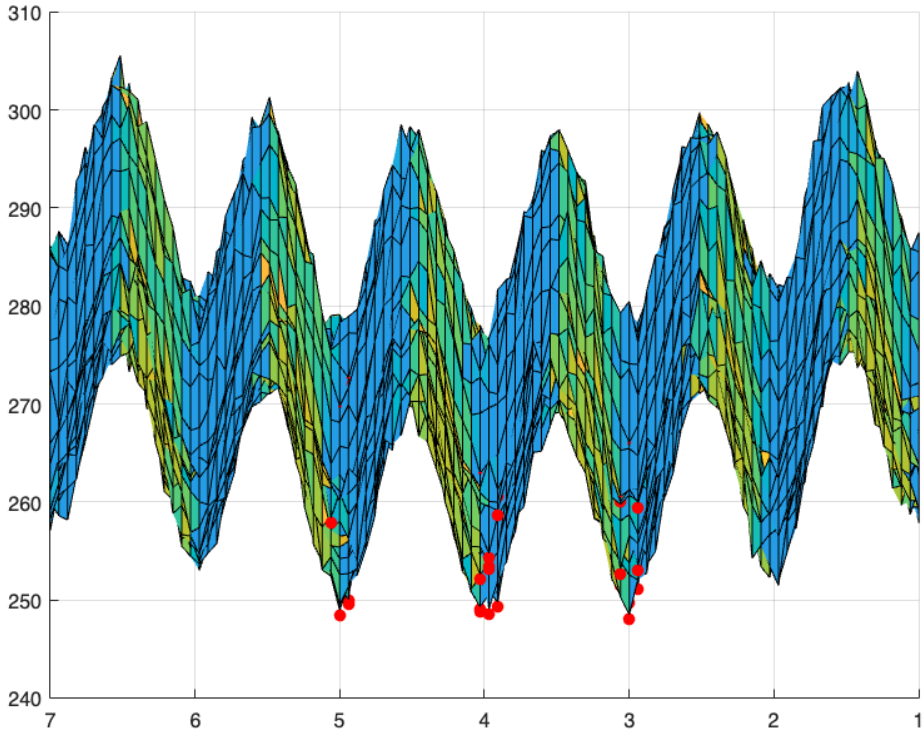
```
close;
```

```
x = linspace(-1,5,100);
y = linspace(1,7,100);
[X,Y] = meshgrid(x,y);
Z_zoomed = zeros(100,100);
for iter = 1:10000
    for xi = 1:length(x)
        for yi = 1:length(y)
            Z_zoomed(xi,yi) = Z_zoomed(xi,yi) + fun([x(xi),y(yi)]);
        end
    end
end
Z_zoomed = Z_zoomed./10000;

surfl(X,Y,Z_zoomed);
view([-90 -0]) % view in the yz plane
```



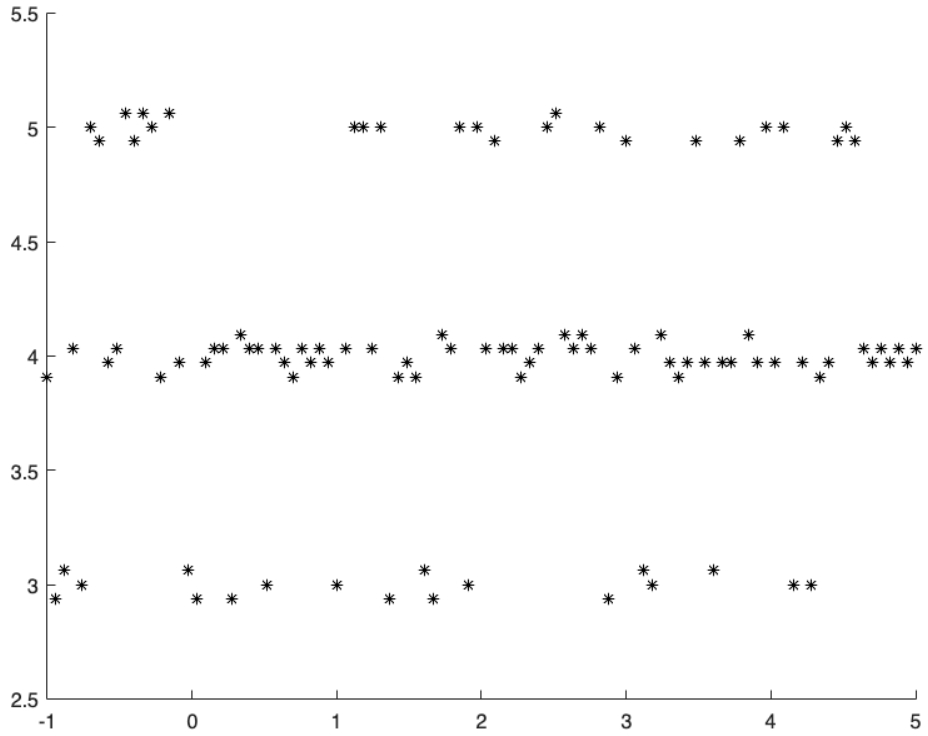

```
hold on;
threshold = min(Z_zoomed) + 0.0001; % find troughs with Z below this value
peakLogical = Z_zoomed < threshold;
xx = X(peakLogical);
yy = Y(peakLogical);
zz = Z_zoomed(peakLogical);
vOffset = 0.5; % shift a bit so points are clearly visible
zz = zz - vOffset;
% show peaks in surface plot
```

```
scatter3(xx, yy, zz, 'r', 'filled'); hold off; view([-90 -0]);
```



You can see from the plot of the averages (10000 iterations for each point and them averaged - removing noise) that although there are many local minima the true minimum generally falls on around (2,4)

```
% Plot the minima on x,y plane
scatter(xx,yy,'k*')
```



The plot above shows that there are multiple points in the original region that are close to the minimum with lowest values of z plotted. The horizontal lines suggest that there are multiple local minima, the central line has the greatest number of points and so must be slightly more optimal than the troughs above and below. Similarly the concentration of points increases around the centre (2,4) confirming (as per the solver algorithms) that the function objective value optimum must be there.

Whether this is a true global optimal cannot be said as this function likely acts over an infinite domain and so there may be a point somewhere where the function objective value falls below but nothing can be said for certain. The input coordinate value found will generally provide the minimum value, and given that the function appears to contain either an 'abs' or positive even index somewhere it is unlikely that there are any other global minima.

From all this I would guess that the function would be something similar to the form:

$$z = (x-2)^2 + (y-4)^2 + r$$

Where r is a random offset value to give the graph its stochastic form for each value of z. Whether the value r is a random value that goes down to 0 is unknown, if it does then the true optimal function value is likely to be 0 at (2,4).