Formal Proofs for Rect as a Complete Semidistributive Lattice

Definitions

Let `Rect` be the set of all axis-aligned rectangles with coordinates `(left, top, right, bottom)` where `left ≤ right` and `top ≤ bottom`, plus the empty rectangle `⊠`.

Notation Convention

Throughout these proofs, let R_1, R_2, R_3 denote arbitrary rectangles where each is either:

- `⊠` (the empty rectangle), or
- A non-empty rectangle with coordinates, where:
  - $R_n = (l_n, t_n, r_n, b_n)$ with $l_n \leq r_n$ and $t_n \leq b_n$

When specific rectangles are needed, we use $R_1 = (l_1, t_1, r_1, b_1)$, $R_2 = (l_2, t_2, r_2, b_2)$, $R_3 = (l_3, t_3, r_3, b_3)$, etc.

Operations

Join Operation (`|`):

- If $R_1 = ⊠$, then $R_1 \mid R_2 = R_2$
- If $R_2 = ⊠$, then $R_1 \mid R_2 = R_1$
- Otherwise: $R_1 \mid R_2 = (\min(l_1, l_2), \min(t_1, t_2), \max(r_1, r_2), \max(b_1, b_2))$

Meet Operation (`&`):

- If $R_1 = ⊠$ or $R_2 = ⊠$, then $R_1 \& R_2 = ⊠$
- Otherwise: $R_1 \& R_2 = (\max(l_1, l_2), \max(t_1, t_2), \min(r_1, r_2), \min(b_1, b_2))$
  - If the result has `left > right` or `top > bottom`, return `⊠`

Identity Elements:

- `EMPTY = ⊠` (the empty rectangle)
- `PLANE = ⬚ = (-∞, -∞, +∞, +∞)` (the infinite plane)

Partial Order: $R_1 \leq R_2$ iff $R_1 \& R_2 = R_1$ (equivalently, $R_1 \mid R_2 = R_2$)

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1. Identity Element Proofs

Theorem 1.1: $R_1 \mid ⊠ = R_1$

Proof: By definition of join:

- If $R_1 = ⧄$: then $⧄ \mid ⧄ = ⧄$ ✓
- If $R_1 \neq ⧄$: then $R_1 \mid ⧄ = R_1$ ✓

∎

Theorem 1.2: $\mathbf{R_1 \;\&\; ⬚ = R_1}$

Proof: Case 1: $R_1 = ⧄$

- By definition, $⧄ \;\&\; ⬚ = ⧄$ ✓

Case 2: $R_1 \neq ⧄$

$$
\begin{aligned}
R_1 \;\&\; ⬚ &= (l_1, t_1, r_1, b_1) \;\&\; (-\infty, -\infty, +\infty, +\infty) \\
&= (\max(l_1, -\infty), \max(t_1, -\infty), \min(r_1, +\infty), \min(b_1, +\infty)) \\
&= (l_1, t_1, r_1, b_1) \\
&= R_1 \ ✓
\end{aligned}
$$

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2. Absorbing Element Proofs

Theorem 2.1: $\mathbf{R_1 \mid ⬚ = ⬚}$

Proof: Case 1: $R_1 = ⧄$

- $⧄ \mid ⬚ = ⬚$ by join definition ✓

Case 2: $R_1 \neq ⧄$

$$
\begin{aligned}
R_1 \mid ⬚ &= (l_1, t_1, r_1, b_1) \mid (-\infty, -\infty, +\infty, +\infty) \\
&= (\min(l_1, -\infty), \min(t_1, -\infty), \max(r_1, +\infty), \max(b_1, +\infty)) \\
&= (-\infty, -\infty, +\infty, +\infty) \\
&= ⬚ \ ✓
\end{aligned}
$$

∎

Theorem 2.2: $\mathbf{R_1 \;\&\; ⧄ = ⧄}$

Proof: By definition of meet: $R_1 \;\&\; ⧄ = ⧄$ ✓

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3. Idempotency Proofs

Theorem 3.1: $R_1 \mid R_1 = R_1$

Proof: Case 1: $R_1 = \boxslash$

- $\boxslash \mid \boxslash = \boxslash$ by definition ✓

Case 2: $R_1 \neq \boxslash$

$$\begin{aligned} R_1 \mid R_1 &= (l_1, t_1, r_1, b_1) \mid (l_1, t_1, r_1, b_1) \\ &= (\min(l_1, l_1), \min(t_1, t_1), \max(r_1, r_1), \max(b_1, b_1)) \\ &= (l_1, t_1, r_1, b_1) \\ &= R_1 \checkmark \end{aligned}$$

∎

Theorem 3.2: $R_1 \,\&\, R_1 = R_1$

Proof: Case 1: $R_1 = \boxslash$

- $\boxslash \,\&\, \boxslash = \boxslash$ by definition ✓

Case 2: $R_1 \neq \boxslash$

$$\begin{aligned} R_1 \,\&\, R_1 &= (l_1, t_1, r_1, b_1) \,\&\, (l_1, t_1, r_1, b_1) \\ &= (\max(l_1, l_1), \max(t_1, t_1), \min(r_1, r_1), \min(b_1, b_1)) \\ &= (l_1, t_1, r_1, b_1) \\ &= R_1 \checkmark \end{aligned}$$

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4. Commutativity Proofs

Theorem 4.1: $R_1 \mid R_2 = R_2 \mid R_1$

Proof: Case 1: Either $R_1 = \boxslash$ or $R_2 = \boxslash$

- If $R_1 = \boxslash$: $\boxslash \mid R_2 = R_2 = R_2 \mid \boxslash$ ✓
- If $R_2 = \boxslash$: $R_1 \mid \boxslash = R_1 = \boxslash \mid R_1$ ✓

Case 2: Both $R_1 \neq \boxslash$ and $R_2 \neq \boxslash$

$$R_1 \mid R_2 = (\min(l_1, l_2), \min(t_1, t_2), \max(r_1, r_2), \max(b_1, b_2))$$

Since `min` and `max` are commutative, this equals:

$$(\min(l_2, l_1), \min(t_2, t_1), \max(r_2, r_1), \max(b_2, b_1)) = R_2 \mid R_1 \checkmark$$

∎

Theorem 4.2: $R_1 \,\&\, R_2 = R_2 \,\&\, R_1$

Proof: Case 1: Either $R_1 = ⊘$ or $R_2 = ⊘$

- $⊘ \,\&\, R_2 = ⊘ = R_2 \,\&\, ⊘$ ✓

Case 2: Both $R_1 \neq ⊘$ and $R_2 \neq ⊘$

$$R_1 \,\&\, R_2 = (\max(l_1, l_2), \max(t_1, t_2), \min(r_1, r_2), \min(b_1, b_2))$$

Since min and max are commutative, this equals:

$$(\max(l_2, l_1), \max(t_2, t_1), \min(r_2, r_1), \min(b_2, b_1)) = R_2 \,\&\, R_1 \checkmark$$

∎

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5. Associativity Proofs

Theorem 5.1: $(R_1 \mid R_2) \mid R_3 = R_1 \mid (R_2 \mid R_3)$

Proof: Case 1: Any of R_1, R_2, or R_3 is $⊘$

- If $R_1 = ⊘$: $(⊘ \mid R_2) \mid R_3 = R_2 \mid R_3 = ⊘ \mid (R_2 \mid R_3)$ ✓
- If $R_2 = ⊘$: $(R_1 \mid ⊘) \mid R_3 = R_1 \mid R_3 = R_1 \mid (⊘ \mid R_3)$ ✓
- If $R_3 = ⊘$: $(R_1 \mid R_2) \mid ⊘ = R_1 \mid R_2 = R_1 \mid (R_2 \mid ⊘)$ ✓

Case 2: All non-empty

Left side:

$$R_1 \mid R_2 = (\min(l_1, l_2), \min(t_1, t_2), \max(r_1, r_2), \max(b_1, b_2))$$

$$\begin{aligned}(R_1 \mid R_2) \mid R_3 = (&\min(\min(l_1, l_2), l_3),\\ &\min(\min(t_1, t_2), t_3),\\ &\max(\max(r_1, r_2), r_3),\\ &\max(\max(b_1, b_2), b_3))\end{aligned}$$

Right side:

$$R_2 \mid R_3 = (\min(l_2, l_3), \min(t_2, t_3), \max(r_2, r_3), \max(b_2, b_3))$$

$$\begin{aligned} R_1 \mid (R_2 \mid R_3) = (&\min(l_1, \min(l_2, l_3)), \\ &\min(t_1, \min(t_2, t_3)), \\ &\max(r_1, \max(r_2, r_3)), \\ &\max(b_1, \max(b_2, b_3))) \end{aligned}$$

By associativity of `min` and `max`, both sides are equal ✓

∎

Theorem 5.2: $(R_1 \,\&\, R_2) \,\&\, R_3 = R_1 \,\&\, (R_2 \,\&\, R_3)$

Proof: Case 1: Any of R_1, R_2, or R_3 is ⧄

- $(⧄ \,\&\, R_2) \,\&\, R_3 = ⧄ \,\&\, R_3 = ⧄ = ⧄ \,\&\, (R_2 \,\&\, R_3)$ ✓
- Similar for $R_2 = ⧄$ or $R_3 = ⧄$ ✓

Case 2: All non-empty

Left side:

$$R_1 \,\&\, R_2 = (\max(l_1, l_2), \max(t_1, t_2), \min(r_1, r_2), \min(b_1, b_2))$$

$$\begin{aligned} (R_1 \,\&\, R_2) \,\&\, R_3 = (&\max(\max(l_1, l_2), l_3), \\ &\max(\max(t_1, t_2), t_3), \\ &\min(\min(r_1, r_2), r_3), \\ &\min(\min(b_1, b_2), b_3)) \end{aligned}$$

Right side:

$$R_2 \,\&\, R_3 = (\max(l_2, l_3), \max(t_2, t_3), \min(r_2, r_3), \min(b_2, b_3))$$

$$\begin{aligned} R_1 \,\&\, (R_2 \,\&\, R_3) = (&\max(l_1, \max(l_2, l_3)), \\ &\max(t_1, \max(t_2, t_3)), \\ &\min(r_1, \min(r_2, r_3)), \\ &\min(b_1, \min(b_2, b_3))) \end{aligned}$$

By associativity of `min` and `max`, both sides are equal ✓

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6. Absorption Laws

Theorem 6.1: $\boxed{R_1 \mid (R_1 \mathbin{\&} R_2) = R_1}$

Proof: Case 1: $\boxed{R_1 = ⊠}$

$$⊠ \mid (⊠ \mathbin{\&} R_2) = ⊠ \mid ⊠ = ⊠ \checkmark$$

Case 2: $\boxed{R_2 = ⊠}$

$$R_1 \mid (R_1 \mathbin{\&} ⊠) = R_1 \mid ⊠ = R_1 \checkmark$$

Case 3: Both $\boxed{R_1 \neq ⊠}$ and $\boxed{R_2 \neq ⊠}$

$$R_1 \mathbin{\&} R_2 = (\max(l_1, l_2), \max(t_1, t_2), \min(r_1, r_2), \min(b_1, b_2))$$

If $\boxed{R_1 \mathbin{\&} R_2 = ⊠}$ (non-intersecting), then $\boxed{R_1 \mid (R_1 \mathbin{\&} R_2) = R_1 \mid ⊠ = R_1}$ ✓

If $\boxed{R_1 \mathbin{\&} R_2 \neq ⊠}$, then:

$$\begin{aligned} R_1 \mid (R_1 \mathbin{\&} R_2) = (&\min(l_1, \max(l_1, l_2)),\\ &\min(t_1, \max(t_1, t_2)),\\ &\max(r_1, \min(r_1, r_2)),\\ &\max(b_1, \min(b_1, b_2))) \end{aligned}$$

Since $\boxed{l_1 \leq \max(l_1, l_2)}$: $\boxed{\min(l_1, \max(l_1, l_2)) = l_1}$ Since $\boxed{t_1 \leq \max(t_1, t_2)}$: $\boxed{\min(t_1, \max(t_1, t_2)) = t_1}$ Since $\boxed{r_1 \geq \min(r_1, r_2)}$: $\boxed{\max(r_1, \min(r_1, r_2)) = r_1}$ Since $\boxed{b_1 \geq \min(b_1, b_2)}$: $\boxed{\max(b_1, \min(b_1, b_2)) = b_1}$

Therefore: $\boxed{R_1 \mid (R_1 \mathbin{\&} R_2) = (l_1, t_1, r_1, b_1) = R_1}$ ✓

∎

Theorem 6.2: $\boxed{R_1 \mathbin{\&} (R_1 \mid R_2) = R_1}$

Proof: Case 1: $\boxed{R_1 = ⊠}$

$$⊠ \mathbin{\&} (⊠ \mid R_2) = ⊠ \mathbin{\&} R_2 = ⊠ \checkmark$$

Case 2: $\boxed{R_2 = ⊠}$

$$R_1 \mathbin{\&} (R_1 \mid ⊠) = R_1 \mathbin{\&} R_1 = R_1 \checkmark$$

Case 3: Both $\boxed{R_1 \neq ⊠}$ and $\boxed{R_2 \neq ⊠}$

```
R₁ | R₂ = (min(l₁, l₂), min(t₁, t₂), max(r₁, r₂), max(b₁, b₂))

R₁ & (R₁ | R₂) = (max(l₁, min(l₁, l₂)),
                  max(t₁, min(t₁, t₂)),
                  min(r₁, max(r₁, r₂)),
                  min(b₁, max(b₁, b₂)))
```

Since $l_1 \geq \min(l_1, l_2)$: $\max(l_1, \min(l_1, l_2)) = l_1$ Since $t_1 \geq \min(t_1, t_2)$: $\max(t_1, \min(t_1, t_2)) = t_1$ Since $r_1 \leq \max(r_1, r_2)$: $\min(r_1, \max(r_1, r_2)) = r_1$ Since $b_1 \leq \max(b_1, b_2)$: $\min(b_1, \max(b_1, b_2)) = b_1$

Therefore: $R_1 \mathbin{\&} (R_1 \mid R_2) = (l_1, t_1, r_1, b_1) = R_1$ ✓

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7. Partial Order Properties

Theorem 7.1: Least Element ($\text{⧄} \leq R_1$)

Proof: We must show $\text{⧄} \mathbin{\&} R_1 = \text{⧄}$ for all R_1.

By definition of meet: $\text{⧄} \mathbin{\&} R_1 = \text{⧄}$ ✓

Therefore $\text{⧄} \leq R_1$ for all R_1 ✓

∎

Theorem 7.2: Greatest Element ($R_1 \leq \text{⬚}$)

Proof: We must show $R_1 \mathbin{\&} \text{⬚} = R_1$ for all R_1.

This was proven in Theorem 1.2 ✓

Therefore $R_1 \leq \text{⬚}$ for all R_1 ✓

∎

Theorem 7.3: Reflexivity ($R_1 \leq R_1$)

Proof: We must show $R_1 \mathbin{\&} R_1 = R_1$.

This was proven in Theorem 3.2 (idempotency of meet) ✓

∎

Theorem 7.4: Transitivity ($R_1 \leq R_2 \land R_2 \leq R_3 \Longrightarrow R_1 \leq R_3$)

Proof: Assume $R_1 \leq R_2$ and $R_2 \leq R_3$.

By definition: $R_1 \mathbin{\&} R_2 = R_1$ and $R_2 \mathbin{\&} R_3 = R_2$.

We must show $R_1 \mathbin{\&} R_3 = R_1$.

```
R₁ & R₃ = (R₁ & R₂) & R₃        [since R₁ & R₂ = R₁]
        = R₁ & (R₂ & R₃)        [by associativity, Theorem 5.2]
        = R₁ & R₂               [since R₂ & R₃ = R₂]
        = R₁                    [by assumption]
```

Therefore $R_1 \leq R_3$ ✓

∎

Theorem 7.5: Antisymmetry ($R_1 \leq R_2 \wedge R_2 \leq R_1 \Longrightarrow R_1 = R_2$)

Proof: Assume $R_1 \leq R_2$ and $R_2 \leq R_1$.

By definition: $R_1 \& R_2 = R_1$ and $R_2 \& R_1 = R_2$.

By commutativity (Theorem 4.2): $R_1 \& R_2 = R_2 \& R_1$.

Therefore: $R_1 = R_1 \& R_2 = R_2 \& R_1 = R_2$ ✓

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8. Monotonicity

Theorem 8.1: Join Monotonicity

Statement: If $R_1 \leq R_2$ and $R_3 \leq R_4$, then $R_1 \mid R_3 \leq R_2 \mid R_4$.

Proof: Assume $R_1 \leq R_2$ and $R_3 \leq R_4$.

By definition: $R_1 \& R_2 = R_1$ and $R_3 \& R_4 = R_3$.

For non-empty rectangles:

- If $R_1 = (l_1, t_1, r_1, b_1)$ and $R_2 = (l_2, t_2, r_2, b_2)$ with $R_1 \leq R_2$, then coordinate-wise: $l_2 \leq l_1 \leq r_1 \leq r_2$ and $t_2 \leq t_1 \leq b_1 \leq b_2$
- If $R_3 = (l_3, t_3, r_3, b_3)$ and $R_4 = (l_4, t_4, r_4, b_4)$ with $R_3 \leq R_4$, then coordinate-wise: $l_4 \leq l_3 \leq r_3 \leq r_4$ and $t_4 \leq t_3 \leq b_3 \leq b_4$

Then:

```
R₁ | R₃ = (min(l₁, l₃), min(t₁, t₃), max(r₁, r₃), max(b₁, b₃))
R₂ | R₄ = (min(l₂, l₄), min(t₂, t₄), max(r₂, r₄), max(b₂, b₄))
```

Since the inequalities hold coordinate-wise and `min`/`max` preserve order:

$$\min(l_2, l_4) \le \min(l_1, l_3),\ \min(t_2, t_4) \le \min(t_1, t_3)$$
$$\max(r_1, r_3) \le \max(r_2, r_4),\ \max(b_1, b_3) \le \max(b_2, b_4)$$

Therefore $R_1 \mid R_3 \le R_2 \mid R_4$ ✓

∎

Theorem 8.2: Meet Monotonicity

Statement: If $R_1 \le R_2$ and $R_3 \le R_4$, then $R_1 \,\&\, R_3 \le R_2 \,\&\, R_4$.

Proof: Similar coordinate-wise argument as Theorem 8.1.

For non-empty rectangles with $R_1 \le R_2$ and $R_3 \le R_4$:

$$R_1 \,\&\, R_3 = (\max(l_1, l_3), \max(t_1, t_3), \min(r_1, r_3), \min(b_1, b_3))$$
$$R_2 \,\&\, R_4 = (\max(l_2, l_4), \max(t_2, t_4), \min(r_2, r_4), \min(b_2, b_4))$$

Since $\min$/$\max$ preserve the coordinate inequalities:

$$\max(l_2, l_4) \le \max(l_1, l_3),\ \max(t_2, t_4) \le \max(t_1, t_3)$$
$$\min(r_1, r_3) \le \min(r_2, r_4),\ \min(b_1, b_3) \le \min(b_2, b_4)$$

Therefore $R_1 \,\&\, R_3 \le R_2 \,\&\, R_4$ ✓

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9. Completeness

Theorem 9.1: Arbitrary Joins Exist

Statement: For any $S \subseteq \mathrm{Rect}$, the supremum $\bigvee S$ exists.

Proof: If $S = \emptyset$, define $\bigvee S = ⧄$.

Otherwise, define:

$$L = \inf \{\, l \mid (l,t,r,b) \in S, (l,t,r,b) \ne ⧄ \,\}$$
$$T = \inf \{\, t \mid (l,t,r,b) \in S, (l,t,r,b) \ne ⧄ \,\}$$
$$R = \sup \{\, r \mid (l,t,r,b) \in S, (l,t,r,b) \ne ⧄ \,\}$$
$$B = \sup \{\, b \mid (l,t,r,b) \in S, (l,t,r,b) \ne ⧄ \,\}$$

and set:

$$\bigvee S = (L, T, R, B)$$

This rectangle is well-defined since $L \le R$ and $T \le B$.

For all $R_i \in S$, we have $R_i \le \bigvee S$ by construction.

If u is any upper bound of S, then coordinate-wise:

$$l_u \le L,\quad t_u \le T,\quad R \le r_u,\quad B \le b_u$$

hence $\bigvee S \le u$.

Therefore $\bigvee S$ is the least upper bound of S ✓

∎

Theorem 9.2: Arbitrary Meets Exist

Statement: For any $S \subseteq \mathrm{Rect}$, the infimum $\bigwedge S$ exists.

Proof: If $S = \varnothing$, define $\bigwedge S = ⬚$.

Otherwise define:

$$\begin{aligned} L &= \sup \{ l \mid (l,t,r,b) \in S \} \\ T &= \sup \{ t \mid (l,t,r,b) \in S \} \\ R &= \inf \{ r \mid (l,t,r,b) \in S \} \\ B &= \inf \{ b \mid (l,t,r,b) \in S \} \end{aligned}$$

If $L > R$ or $T > B$, define $\bigwedge S = ⧄$; otherwise:

$$\bigwedge S = (L, T, R, B)$$

For all $R_i \in S$, we have $\bigwedge S \le R_i$ by construction.

If m is any lower bound of S, then coordinate-wise:

$$l_m \ge L,\quad t_m \ge T,\quad r_m \le R,\quad b_m \le B$$

hence $m \le \bigwedge S$.

Therefore $\bigwedge S$ is the greatest lower bound of S ✓

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Corollary 9.3: Rect is a Complete Lattice

Statement: $(\mathrm{Rect}, |, \&, ⧄, ⬚)$ is a complete lattice.

Proof: By Theorems 9.1 and 9.2, arbitrary joins and meets exist ✓

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10. Semidistributivity

Lemma 10.1: Meet Projections

Statement: For all R_1,R_2: $R_1 \,\&\, R_2 \leq R_1$ and $R_1 \,\&\, R_2 \leq R_2$.

Proof:

$$(R_1 \,\&\, R_2) \,\&\, R_1 = R_1 \,\&\, R_2 \quad \text{[by associativity and idempotency]}$$
$$(R_1 \,\&\, R_2) \,\&\, R_2 = R_1 \,\&\, R_2 \quad \text{[by associativity and idempotency]}$$

Therefore $R_1 \,\&\, R_2 \leq R_1$ and $R_1 \,\&\, R_2 \leq R_2$ ✓

∎

Lemma 10.2: Join Injections

Statement: For all R_1,R_2: $R_1 \leq R_1 \mid R_2$ and $R_2 \leq R_1 \mid R_2$.

Proof:

$$R_1 \,\&\, (R_1 \mid R_2) = R_1 \quad \text{[by absorption, Theorem 6.2]}$$
$$R_2 \,\&\, (R_1 \mid R_2) = R_2 \quad \text{[by absorption, Theorem 6.2]}$$

Therefore $R_1 \leq R_1 \mid R_2$ and $R_2 \leq R_1 \mid R_2$ ✓

∎

Theorem 10.3: Meet-Semidistributivity

Statement: $(R_1 \,\&\, R_2) \mid (R_1 \,\&\, R_3) \leq R_1 \,\&\, (R_2 \mid R_3)$ for all R_1, R_2, R_3.

Proof: By Lemma 10.1:

$$(R_1 \,\&\, R_2) \leq R_1, \quad (R_1 \,\&\, R_3) \leq R_1$$

hence by monotonicity (Theorem 8.1):

$$(R_1 \,\&\, R_2) \mid (R_1 \,\&\, R_3) \leq R_1$$

Also by Lemma 10.1:

$$(R_1 \mathbin{\&} R_2) \leq R_2, \quad (R_1 \mathbin{\&} R_3) \leq R_3$$

hence by monotonicity:

$$(R_1 \mathbin{\&} R_2) \mid (R_1 \mathbin{\&} R_3) \leq R_2 \mid R_3$$

Since $\boxed{(R_1 \mathbin{\&} R_2) \mid (R_1 \mathbin{\&} R_3) \leq R_1}$ and $\boxed{(R_1 \mathbin{\&} R_2) \mid (R_1 \mathbin{\&} R_3) \leq R_2 \mid R_3}$:

$$(R_1 \mathbin{\&} R_2) \mid (R_1 \mathbin{\&} R_3) \leq R_1 \mathbin{\&} (R_2 \mid R_3)$$

∎

Theorem 10.4: Join-Semidistributivity

Statement: $\boxed{R_1 \mid (R_2 \mathbin{\&} R_3) \leq (R_1 \mid R_2) \mathbin{\&} (R_1 \mid R_3)}$ for all $\boxed{R_1, R_2, R_3}$.

Proof: By Lemma 10.2:

$$R_1 \leq R_1 \mid R_2, \quad R_1 \leq R_1 \mid R_3$$

hence by monotonicity (Theorem 8.2):

$$R_1 \leq (R_1 \mid R_2) \mathbin{\&} (R_1 \mid R_3)$$

Also by Lemma 10.1:

$$R_2 \mathbin{\&} R_3 \leq R_2, \quad R_2 \mathbin{\&} R_3 \leq R_3$$

hence by monotonicity:

$$R_2 \mathbin{\&} R_3 \leq R_1 \mid R_2 \quad \text{and} \quad R_2 \mathbin{\&} R_3 \leq R_1 \mid R_3$$

Therefore:

$$R_2 \mathbin{\&} R_3 \leq (R_1 \mid R_2) \mathbin{\&} (R_1 \mid R_3)$$

Since $\boxed{R_1 \leq (R_1 \mid R_2) \mathbin{\&} (R_1 \mid R_3)}$ and $\boxed{R_2 \mathbin{\&} R_3 \leq (R_1 \mid R_2) \mathbin{\&} (R_1 \mid R_3)}$:

$$R_1 \mid (R_2 \mathbin{\&} R_3) \leq (R_1 \mid R_2) \mathbin{\&} (R_1 \mid R_3)$$

11. Failure of Modularity

The modular law states:

> If $\boxed{R_1 \leq R_2}$, then $\boxed{R_1 \mid (R_3 \,\&\, R_2) = (R_1 \mid R_3) \,\&\, R_2}$

We demonstrate with an explicit counterexample that the Rect lattice is not modular.

Theorem 11.1: The Rect Lattice is Not Modular

Counterexample: Let:

```
R₁ = (0, 0, 1, 1)  [small 1×1 square]
R₂ = (0, 0, 1, 2)  [1×2 rectangle containing R₁]
R₃ = (2, 0, 3, 2)  [disjoint 1×2 rectangle]
```

Verify the precondition $\boxed{R_1 \leq R_2}$:

```
R₁ & R₂ = (max(0, 0), max(0, 0), min(1, 1), min(1, 2))
      = (0, 0, 1, 1)
      = R₁ ✓
```

So indeed $\boxed{R_1 \leq R_2}$.

Left-hand side: $\boxed{R_1 \mid (R_3 \,\&\, R_2)}$

```
R₃ & R₂ = (max(2, 0), max(0, 0), min(3, 1), min(2, 2))
      = (2, 0, 1, 2)
      = ⊘  [since 2 > 1]

R₁ | (R₃ & R₂) = R₁ | ⊘
           = (0, 0, 1, 1)
           = R₁
```

Right-hand side: $\boxed{(R_1 \mid R_3) \,\&\, R_2}$

```
R₁ | R₃ = (min(0, 2), min(0, 0), max(1, 3), max(1, 2))
       = (0, 0, 3, 2)

(R₁ | R₃) & R₂ = (max(0, 0), max(0, 0), min(3, 1), min(2, 2))
             = (0, 0, 1, 2)
             = R₂
```

Result:

```
R₁ | (R₃ & R₂) = (0, 0, 1, 1)  ≠  (0, 0, 1, 2) = (R₁ | R₃) & R₂
```

Therefore the modular law fails ✗

Geometric Intuition

The failure occurs because:

1. `R₃` and `R₂` are disjoint, so `R₃ & R₂ = ⊠`
2. The join `R₁ | R₃` creates a **bounding box** that spans from `R₁` to `R₃`, including the empty space between them: `(0, 0, 3, 2)`
3. When this bounding box is intersected with `R₂`, we get all of `R₂`: `(0, 0, 1, 2)`
4. But the left side just gives us `R₁`: `(0, 0, 1, 1)`

The key insight: **The join operation adds "virtual" space between disjoint rectangles.** This space persists through subsequent meet operations, violating the modular law's expectation that operations should be "well-behaved" when `R₁ ≤ R₂`.

In a true modular lattice, the constraint `R₁ ≤ R₂` ensures that operations involving `R₁`, `R₂`, and any `R₃` maintain certain symmetries. But in the Rect lattice, the bounding box operation breaks this symmetry by including regions that aren't actually part of either input rectangle.

∎

---

12. Failure of Full Distributivity

Since distributivity implies modularity, and we have shown that modularity fails (Theorem 11.1), it follows that **distributivity must also fail**.

However, we provide an explicit counterexample to illustrate the specific manner in which distributivity fails in the Rect lattice.

Counterexample for `R₁ & (R₂ | R₃) = (R₁ & R₂) | (R₁ & R₃)`

Consider:

```
R1 = (0, 0, 4, 4)
R2 = (1, 1, 7, 3)
R3 = (5, 2, 6, 5)
```

Left side:

```
R2 | R3 = (1, 1, 7, 5)
R1 & (R2 | R3) = (0, 0, 4, 4) & (1, 1, 7, 5)
             = (max(0, 1), max(0, 1), min(4, 7), min(4, 5))
             = (1, 1, 4, 4)
```

Right side:

```
R1 & R2 = (0, 0, 4, 4) & (1, 1, 7, 3)
      = (max(0, 1), max(0, 1), min(4, 7), min(4, 3))
      = (1, 1, 4, 3)

R1 & R3 = (0, 0, 4, 4) & (5, 2, 6, 5)
      = (max(0, 5), max(0, 2), min(4, 6), min(4, 5))
      = (5, 2, 4, 4)
      = ∅  [since 5 > 4]

(R1 & R2) | (R1 & R3) = (1, 1, 4, 3) | ∅
                  = (1, 1, 4, 3)
```

Result:

```
R1 & (R2 | R3) = (1, 1, 4, 4)  ≠  (1, 1, 4, 3) = (R1 & R2) | (R1 & R3)
```

Therefore distributivity fails ✗

Why This Happens Geometrically

The key insight is:

- $R_2 \mid R_3$ is the **bounding box** of R_2 and R_3, which includes regions not in either R_2 or R_3
- When we intersect R_1 with this bounding box, we can include parts of R_1 that don't intersect either R_2 or R_3
- But $(R_1 \,\&\, R_2) \mid (R_1 \,\&\, R_3)$ only includes the parts of R_1 that actually intersect R_2 or R_3

In the example above:

- Rectangle R_3 doesn't intersect R_1 at all (R_1 & R_3 = ⧄)
- But R_2 | R_3 creates a bounding box that covers more area than just R_2 and R_3
- The bottom coordinate: R_1 & (R_2 | R_3) extends to 4 (from R_1), while R_1 & R_2 only extends to 3 (from R_2)
- Since R_1 & R_3 = ⧄, the join (R_1 & R_2) | (R_1 & R_3) is just R_1 & R_2 = (1, 1, 4, 3)

∎

Conclusion

We have proven that (Rect, |, &, ⧄, ⬚) forms a **complete semidistributive lattice** with the following properties:

Proven to hold:

- ✓ Identity elements (⧄ and ⬚)
- ✓ Absorbing elements
- ✓ Idempotency
- ✓ Commutativity
- ✓ Associativity
- ✓ Absorption laws
- ✓ Partial order properties (reflexivity, transitivity, antisymmetry, least and greatest elements)
- ✓ Monotonicity
- ✓ Completeness (arbitrary joins and meets exist)
- ✓ Semidistributivity (both inequalities)

Proven to fail:

- ✗ Full distributivity (equality fails in both laws)
- ✗ Modularity (fails even when precondition holds)

The precise classification is: **Complete Semidistributive Non-Modular Lattice**.