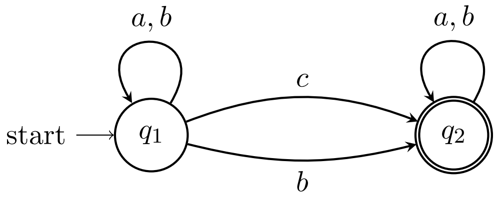
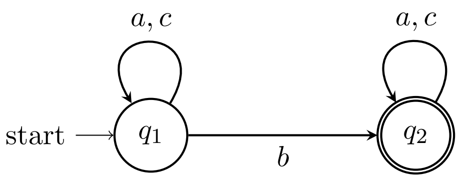
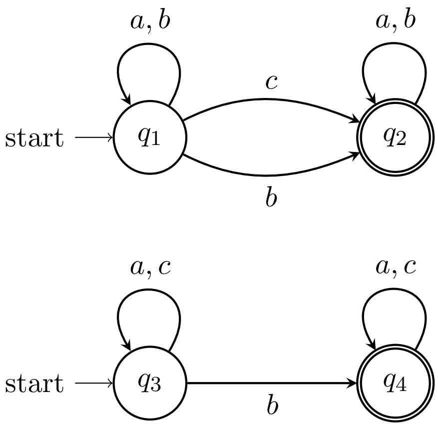
1.a) Draw an NFA with 2 states for the following language: $\mathbf{A} = (\{\mathbf{a}, \mathbf{b}\}^*)(\{\mathbf{b}, \mathbf{c}\})(\{\mathbf{a}, \mathbf{b}\}^*)$



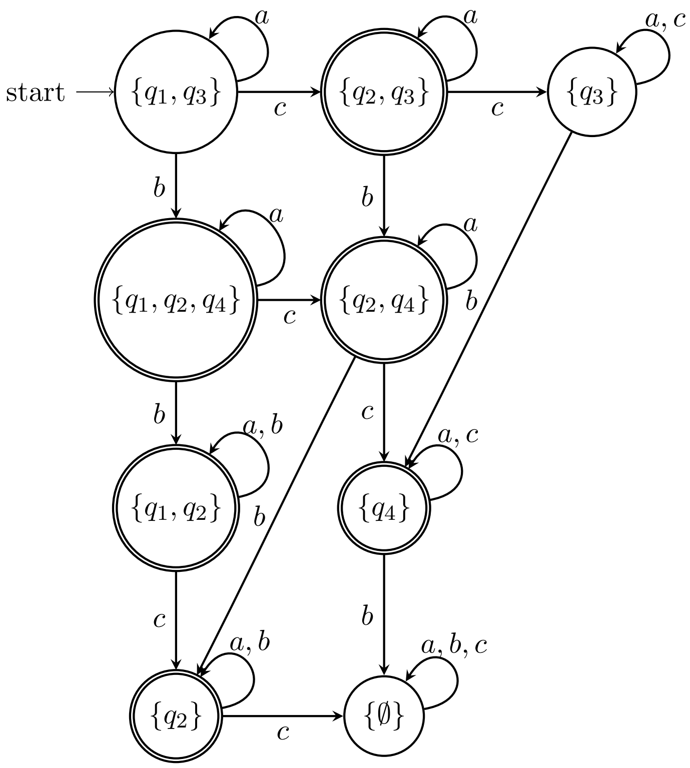
1.b) Draw an NFA with 2 states for the following language: $\mathbf{B} = (\{\mathbf{a}, \mathbf{c}\}^*)(\{\mathbf{b}\})(\{\mathbf{a}, \mathbf{c}\}^*)$



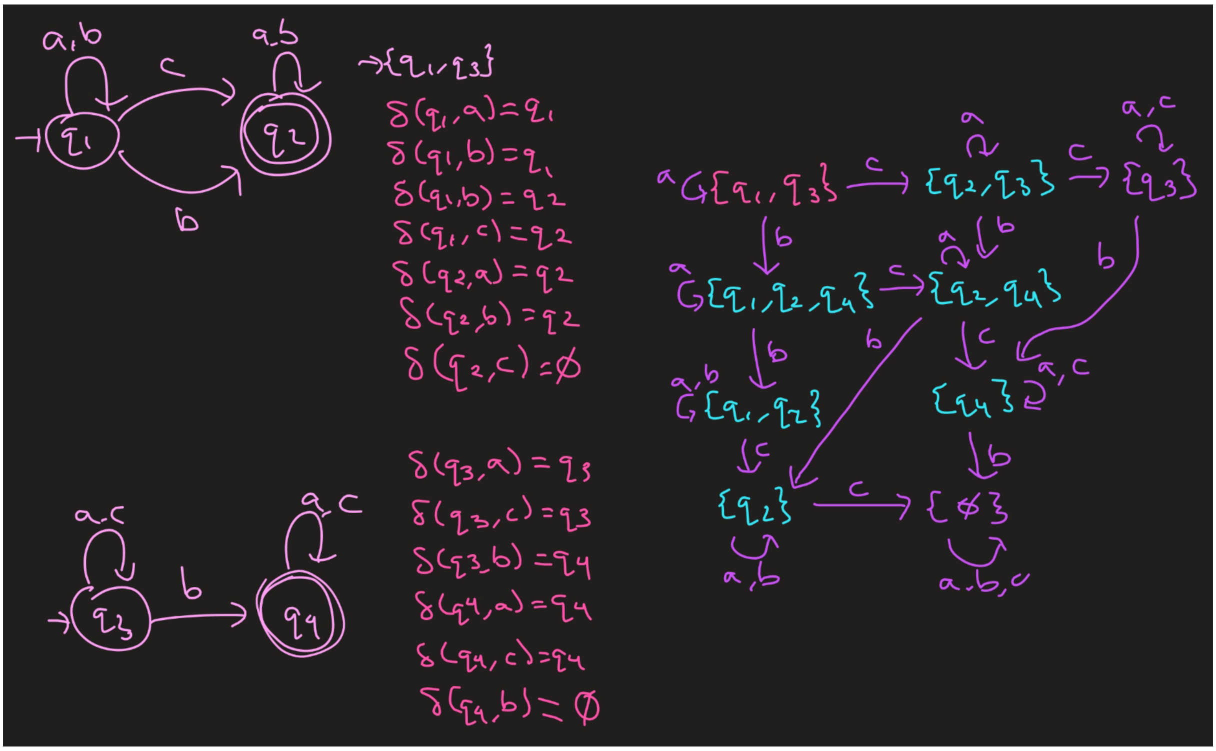
1.c) Draw an NFA for $\mathbf{C} = \mathbf{A} \cup \mathbf{B}$ with 4 states



1.d) Use the subset construction game to create a DFA for C out of its NFA. Label each state with subsets of C



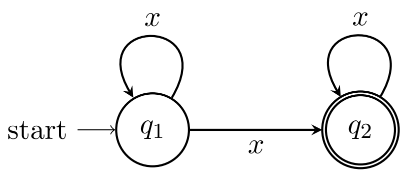
Work:



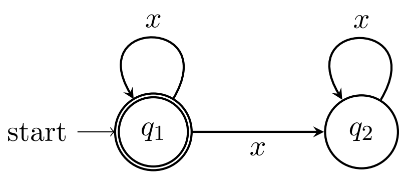
2.) Is the following statement true? "Let $\mathbf{N_1} = (\mathbf{Q}, \mathbf{\Sigma}, \mathbf{\Delta}, \mathbf{S}, \mathbf{F})$ be any non-deterministic finite state machine. Let $\mathbf{N_2} = (\mathbf{Q}, \mathbf{\Sigma}, \mathbf{\Delta}, \mathbf{S}, \mathbf{Q} \setminus \mathbf{F})$. Then $\mathbf{L(N_1)} = \sim \mathbf{L(N_2)}$". If you think it is true, then prove it. Otherwise, provide a counterexample.

Intuitively, this is false. Take the following counter-example:

Say N_1 has the NFA:



The states, alphabet, transitions, and start states of N_2 would need to be the same as N_1, and the final states would have to be all states of N_1 minus the final states, according to the definition of N_2. This would mean that the NFA for N_2 would be:



The language of N_1 can be described as $L(N_1) = \{x\}^+$. The language of N_2 can be described as $L(N_2) = \{x\}^*$. This means that $\sim L(N_2) = \{\emptyset\}$. Since $\{x\}^+ \neq \{\emptyset\}$, $L(N_1) \neq\sim L(N_2)$, meaning the statement is false.

Formally, a proof by contradiction:

Let $N_1 = (Q, \Sigma, \Delta, S, F)$ be any non-deterministic finite state machine. Let $N_2 = (Q, \Sigma, \Delta, S, Q \setminus F)$. Then, we assume that $L(N_1) = \sim L(N_2)$ is true.

Take $N_1 = (Q, \Sigma, \Delta, S, F)$ to be an arbitrary NFA where:

$$\begin{aligned}
Q &= \{q_1, q_2\} \\
\Sigma &= \{x\} \\
\Delta &= \Delta(q_1, x) = \{q_1, q_2\}, \Delta(q_2, x) = q_2 \\
S &= \{q_1\} \\
F &= \{q_2\}
\end{aligned}$$

Then, by definition of N_2, $N_2 = (Q, \Sigma, \Delta, S, Q \setminus F)$ where:

$$\begin{aligned}
Q &= \{q_1, q_2\} \\
\Sigma &= \{x\} \\
\Delta &= \Delta(q_1, x) = \{q_1, q_2\}, \Delta(q_2, x) = q_2 \\
S &= \{q_1\} \\
F' &= Q \setminus F = \{q_1, q_2\} \setminus \{q_2\} = \{q_1\}
\end{aligned}$$

Based on the definition of complement and our assumed statement, we can say that if a string $y \in L(N_1) \implies y \notin L(N_2)$.

Let $y \in \Sigma^* \mid y = x$

$$\begin{aligned}
& y \in L(N_1) \\
\iff & \hat{\Delta}_{N_1}(S_{N_1}, y) \cap F \neq \emptyset && \text{definition of acceptance} \\
\iff & \hat{\Delta}_{N_1}(q_1, x) \cap F \neq \emptyset && \text{definition of } N_1 \text{ acceptance} \\
\iff & \{q_1, q_2\} \cap \{q_2\} \neq \emptyset && \text{definition of } \Delta_{N_1}(q_1, x) \text{ and } F \\
\iff & \{q_2\} \neq \emptyset && \text{definition of } \cap \\
\iff & True
\end{aligned}$$

$$\begin{aligned}
& y \in L(N_2) \\
\iff & \hat{\Delta}_{N_2}(S_{N_2}, y) \cap F' \neq \emptyset && \text{definition of acceptance} \\
\iff & \hat{\Delta}_{N_2}(q_1, x) \cap F' \neq \emptyset && \text{definition of } N_2 \text{ acceptance} \\
\iff & \{q_1, q_2\} \cap \{q_1\} \neq \emptyset && \text{definition of } \Delta_{N_2}(q_1, x) \text{ and } F' \\
\iff & \{q_1\} \neq \emptyset && \text{definition of } \cap \\
\iff & True
\end{aligned}$$

We now have a contradiction because when $y \in \Sigma^* \mid y = x$, $y \in L(N_1)$ and $y \in L(N_2)$. However, in our assumed statement we said that $y \in L(N_1) \implies y \notin L(N_2)$. Therefore the statement is false.

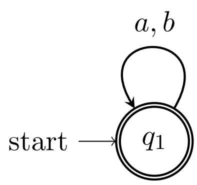
□

3.) "Prove" that the following language is regular: $\mathbf{A} = \{\mathbf{xy} \mid \mathbf{x}, \mathbf{y} \in \{\mathbf{a}, \mathbf{b}\}^*, \#\mathbf{a}(\mathbf{x}) = \#\mathbf{b}(\mathbf{y})\} \subseteq \{\mathbf{a}, \mathbf{b}\}^*$

Intuitively, we want to prove that for any string $w \in \{a, b\}^*$, it can be represented by the concatenation of x and y such that the number of a's in x = the number of b's in y. We can then say that $\{a, b\}^* \subseteq A$, and through double inclusion, say that $\{a, b\}^* = A$, meaning that A must be regular because $\{a, b\}^*$ is regular.

First, we must prove that $\{a, b\}^*$ is regular. A set $B \subseteq \Sigma^*$ is regular if there exists a DFA such that $B = L(M)$. Let $B = \{a, b\}^*$. By definition (given in the question), $\{a, b\}^*$ is a language, therefore $L(M) = \{a, b\}^*$, so $B = L(M)$.

Here is a corresponding DFA for B:



Therefore $\{a, b\}^*$ is regular, and we also claim that $A = \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\} \subseteq \{a, b\}^*$ is regular.

Now, using a proof by induction on the length of an arbitrary string $w \in \{a, b\}^*$, we will prove that $\{a, b\}^* \subseteq A = \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\}$.

Base Case: $|w| = 0$

We know that $|w| = 0 \implies w = \epsilon$ based on the definition of $|\epsilon| = 0$

Let $x = \epsilon$ and $y = \epsilon$:

xy	concatenation of x and y
$\iff \epsilon\epsilon$	definition of x and y
$\iff \epsilon$	identity of null string for concatenation
$\iff w$	definition of w

As stated earlier, by definition, $|\epsilon| = 0$. We also know that by definition, $|xy| = |x| + |y|$. So we can say that:

$$|xy| = |x| + |y|$$
$$|xy| = 0 + 0$$
$$|xy| = 0$$
$$|xy| = |w|$$

Since we proved that $w = xy$ and $|w| = |xy|$, this means that w can be written in terms of xy. By definition of #, $\#a(x) = 0$ and $\#b(x) = 0$, which means that $(xy \mid x = \epsilon, y = \epsilon) \subseteq A$, but also $w \in A$, meaning that the base case holds true.

Inductive Hypothesis:

For any arbitrary string $w \in \{a, b\}^*$ where $|w| = n$, we assume that $w = xy \mid \#a(x) = \#b(y) \implies w \in A$.

Inductive Step:

Now, we use our inductive hypothesis to prove that for any arbitrary string $w \in \{a, b\}^*$ where $|w| = n + 1$, our hypothesis holds.

Based off our inductive hypothesis, we know that there exists an $xy = w$. For our inductive step, we concatenate another letter c to w such that $|c| = 1$, so that $|w| = n + 1$. Using these facts, we can say that w can be represented by xcy.

By definition, we know that $|\epsilon| = 0$, so the only possible letters that can be added to our string w in order for $|w| = n + 1$ and $w \in \{a, b\}^*$ to hold true is a or b because we also know that $|a| = 1$ and $|b| = 1$ by definition. We now have two cases.

Case 1: $\mathbf{c = a}$

If $c = a$, we can say that w can be represented as xay. Looking back at the definition of the language A, $xy \in A$ if $\#a(x) = \#b(y)$. We can use the fact that we only care about $\#b(y)$ to manipulate xay into xy.

Based on our induction hypothesis, we know that $xy \in A$. So $\#a(x) = \#b(y)$ holds true. c can either be concatenated with x or y. If we were to put c into x, $\#a(x)$ would change because $c = a$, causing $\#a(x) \neq \#b(y)$. However, if we were to put c into y, $\#a(x)$ would remain unchanged because x does not have any new letters, and $\#a(x) = \#b(y)$ would still hold true because we are not counting/do not care about $\#a(y)$.

This means that when $c = a$, we can concatenate c with y, so that it still follows that $xay \in A$ because $\#a(x) = \#b(y)$ holds true. Since we said that w can be represented using xay, it follows that $w \in A$.

Case 2: $\mathbf{c = b}$

Very similarly, if $c = b$, we can say that w can be represented as xby. Looking back at the definition of the language A, $xy \in A$ if $\#a(x) = \#b(y)$. We can use the fact that we only care about $\#a(x)$ to manipulate xby into xy.

Based on our induction hypothesis, we know that $xy \in A$. So $\#a(x) = \#b(y)$ holds true. c can either be concatenated with x or y. If we were to put c into y, $\#b(y)$ would change because $c = b$, causing $\#a(x) \neq \#b(y)$. However, if we were to put c into x, $\#b(y)$ would remain unchanged because y does not have any new letters, and $\#a(x) = \#b(y)$ would still hold true because we are not counting/do not care about $\#b(x)$.

This means that when $c = b$, we can concatenate c with x, so that it still follows that $xby \in A$ because $\#a(x) = \#b(y)$ holds true. Since we said that w can be represented using xby, it follows that $w \in A$.

Now we have proved our inductive step because when $c = a$ or $c = b$, $w \in A$. This means that $\forall w \in \{a, b\}^*$, $w \in A$. So $\{a, b\}^* \subseteq A$. In the very beginning we claimed that by definition $A \subseteq \{a, b\}^*$, so by double inclusion, we can say that $\{a, b\}^* = A$. Since we also proved that $\{a, b\}^*$ is regular, $A = \{xy \mid x, y \in \{a, b\}^*, \#a(x) = \#b(y)\} \subseteq \{a, b\}^*$ is regular.

$\square$