MAT 150C: MODERN ALGEBRA
Homework 6

Instructions. Please write the answer to each problem, including the computational ones, in connected sentences and explain your work. Just the answer (correct or not) is not enough. Write your name in every page and upload to Gradescope *with the correct orientation.* Make sure to indicate to Gradescope which pages correspond to each problem. **Finally, if you used another sources or discussed the problem with classmates, be sure to acknowledge it in your homework.**

1. Let R be a finite (meaning that it has a finite number of elements) integral domain. Show that R is a field.

2. (a) Let $f(x) \in \mathbb{R}[x]$. Assume $z \in \mathbb{C}$ is a root of f. Show that $\overline{z}$ is a root of f as well, where $\overline{z}$ denotes the complex conjugate.
   (b) Deduce that every irreducible polynomial in $\mathbb{R}[x]$ has degree 1 or 2. Give a necessary and sufficient condition for the polynomial $ax^2 + bx + c \in \mathbb{R}[x]$ $(a \neq 0)$ to be irreducible in $\mathbb{R}[x]$.
   (c) Let $f(x) \in \mathbb{R}[x]$ be a polynomial of odd degree. Show, *without using calculus*, that f has a root in $\mathbb{R}$.

3. Let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension, and let $\alpha \in \mathbb{K}$ be such that $[\mathbb{F}(\alpha) : \mathbb{F}] = 5$. Show that $\mathbb{F}(\alpha^2) = \mathbb{F}(\alpha)$.

   For the next exercises, we denote $\zeta_n := e^{2\pi\sqrt{-1}/n}$, a primitive n-rooth of 1 in $\mathbb{C}$.

4. Assume p is prime and $r > 0$. Show that $[\mathbb{Q}(\zeta_{p^r}) : \mathbb{Q}] = (p-1)p^{r-1}$. (Hint: you can refer to previous homeworks). [1]

5. Show that $\zeta_5 \notin \mathbb{Q}(\zeta_7)$.

[1] In general, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \mathrm{Tot}(n)$, where Tot is Euler's totient function