MAT 150C - Homework 5

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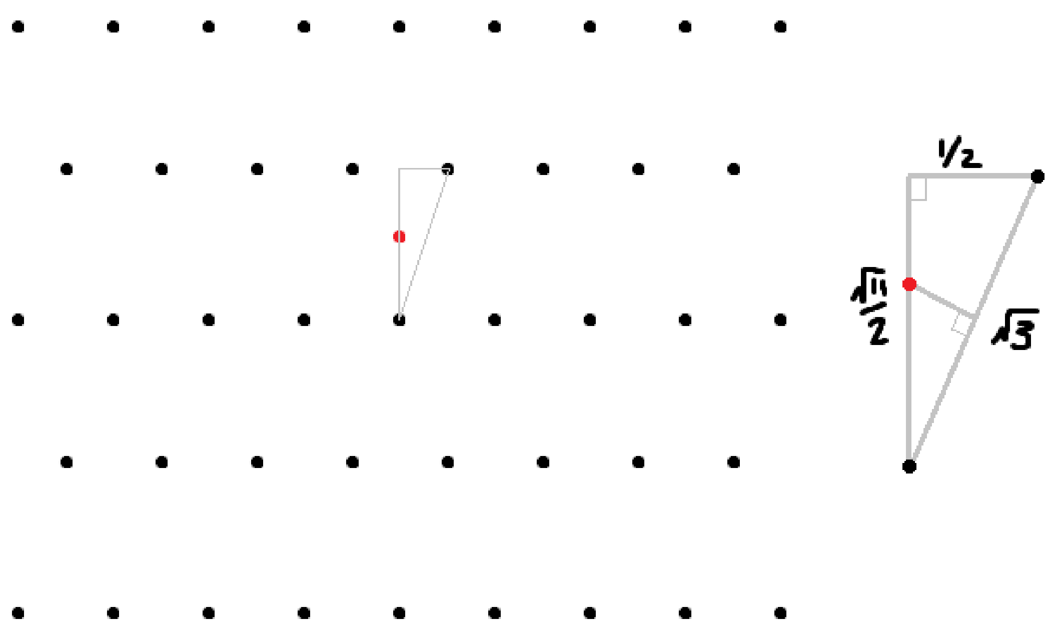
1. Let $\delta = \sqrt{d} = \sqrt{-11}$. Note that $d \equiv 1 \pmod 4$ so $\mathcal{O}_\delta$ is the lattice of integers and half-integers as pictured.



The lattice $\mathcal{O}_\delta$

Furthermore, let $a \in \mathbb{C}$ be a complex number. In the worst case, a is $\frac{3}{\sqrt{11}} < 1$ from an element of $\mathcal{O}_\delta$ as depicted by the red dot above.

$$\frac{x}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{\frac{\sqrt{11}}{2}} \implies x = \frac{3}{\sqrt{11}}.$$

Let $\sigma(a) = |a|^2$ be a size function over $\mathbb{C}$ which has been shown to be multiplicative. Let $a, b \in \mathcal{O}_\delta$ be two arbitrary elements with $b \neq 0$. Then let $q' = a/b \in \mathbb{C}$. If $q' \in \mathcal{O}_\delta$, then the division algorithm holds with $a = bq'$. Otherwise, there is an element $q \in \mathcal{O}_\delta$ within a distance of 1 from q'. Let $r = a - bq$ so that $a = bq + r$ where $q, r \in \mathcal{O}_\delta$. It follows that

$$\frac{\sigma(r)}{\sigma(b)} = \sigma\left(\frac{r}{b}\right) = \sigma\left(q' - q\right) < 1.$$

This implies that $\sigma(r) < \sigma(b)$. Let $E : (\mathcal{O}_\delta \setminus \{0\}) \to \mathbb{Z}^+$ be defined as $E(a) = 4\sigma(a)$ which is a Euclidean function for $\mathcal{O}_\delta$. Therefore $\mathcal{O}_\delta$ is a Euclidean domain and therefore also a unique factorization domain.

2. Let $f : R \to R'$ be a ring homomorphism, and let $\mathfrak{p}' \subseteq R'$ be a prime ideal. Let $\mathfrak{p} = f^{-1}(\mathfrak{p}')$. Since $0 \in \mathfrak{p}'$ and $f(0) = 0$, it follows that $0 \in \mathfrak{p}$ and $\mathfrak{p}$ is not empty. Let $a, b \in \mathfrak{p}$ and $r \in R$. Then

$$\begin{aligned} f(a+b) = f(a) + f(b) \in \mathfrak{p}' \qquad & \implies a + b \in \mathfrak{p}, \\ f(ra) = f(r)f(a) \in \mathfrak{p}' \qquad & \implies ra \in \mathfrak{p}. \end{aligned}$$

Thus $\mathfrak{p}$ is an ideal in R. Now let $I, J \subseteq R$ be ideals such that $IJ \subseteq \mathfrak{p}$. Let f_X denote $f(X)$ to clear up parentheses. Then $(f_{IJ}) = (f_I)(f_J) \subseteq \mathfrak{p}'$.

$f_{IJ} \subseteq \mathfrak{p}'$ and (f_{IJ}) is the smallest ideal containing f_{IJ}, which means $(f_{IJ}) \subseteq \mathfrak{p}'$.

Let $x \in (f_{IJ})$ so that $x = r(f_i f_j)$ for some $r \in R', i \in I, j \in J$. Then $x = (rf_i)(f_j) \in (f_I)(f_J)$.

Similarly, let $x \in (f_I)(f_J)$ so that $x = (r_i f_i)(r_j f_j) = r_i r_j (f_i f_j) \in (f_{IJ})$.

Since $\mathfrak{p}'$ is prime, either $(f_I) \subseteq \mathfrak{p}'$ in which case $I \subseteq f^{-1}\big[(f_I)\big] \subseteq f^{-1}\big[\mathfrak{p}'\big] = \mathfrak{p}$, or $(f_J) \subseteq \mathfrak{p}'$ in which case $J \subseteq f^{-1}\big[(f_J)\big] \subseteq f^{-1}\big[\mathfrak{p}'\big] = \mathfrak{p}$.

3. (a) The zero ideal is a subset of every ideal, so by definition, $V(0) = \mathbf{Spec}(R)$. By definition of a prime ideal, R is not prime, and it is also not the subset of any other ideal. Thus $V(R) = \emptyset$.

   (b) Let $\mathfrak{p} \in V(IJ)$ which means $\mathfrak{p}$ is a prime ideal containing IJ. Since $\mathfrak{p}$ is prime, then either $I \subseteq \mathfrak{p}$ in which case $\mathfrak{p} \in V(I)$, or $J \subseteq \mathfrak{p}$ in which case $\mathfrak{p} \in V(J)$. Now suppose that $\mathfrak{p} \in V(I) \cup V(J)$ and assume that $\mathfrak{p} \in V(I)$ so that $I \subseteq \mathfrak{p}$ where $\mathfrak{p}$ is prime. Then $IJ \subseteq I \subseteq \mathfrak{p}$ which means $\mathfrak{p} \in V(IJ)$.

   (c) Let $\mathfrak{p} \in V(J)$ so that $\mathfrak{p}$ is a prime and $J \subseteq \mathfrak{p}$. In particular, $I_\lambda \subseteq J \subseteq \mathfrak{p}$ which means $\mathfrak{p} \in V(I_\lambda)$ and

$$V(J) \subseteq \bigcap_{\lambda \in \Lambda} V(I_\lambda).$$

Let $\mathfrak{p} \in \bigcap_{\lambda \in \Lambda} V(I_\lambda)$ so that $\mathfrak{p}$ is a prime and $I_\lambda \subseteq \mathfrak{p}$ for all $\lambda \in \Lambda$. Let $j = \sum_{\lambda \in \Lambda} r_\lambda f_\lambda \in J$. Since I_λ and $\mathfrak{p}$ are ideals, we have $r_\lambda f_\lambda \in I_\lambda \subseteq \mathfrak{p}$ and $j = \sum_{\lambda \in \Lambda} r_\lambda f_\lambda \in \mathfrak{p}$ (it is a finite sum). This means $J \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(J)$ and

$$\bigcap_{\lambda \in \Lambda} V(I_\lambda) \subseteq V(J).$$