MAT 150C: MODERN ALGEBRA
Homework 5

Instructions. Please write the answer to each problem, including the computational ones, in connected sentences and explain your work. Just the answer (correct or not) is not enough. Write your name in every page and upload to Gradescope *with the correct orientation.* Make sure to indicate to Gradescope which pages correspond to each problem. **Finally, if you used another sources or discussed the problem with classmates, be sure to acknowledge it in your homework.**

1. Recall that, if $d \in \mathbb{Z}$, $d \neq 1$ is a square-free number, then $\mathcal{O}_{\sqrt{d}}$ denotes the domain of algebraic integers in $\mathbb{Q}[\sqrt{d}]$. Show that $\mathcal{O}_{\sqrt{-11}}$ is a UFD.

2. Recall that an ideal $\mathfrak{p} \subseteq R$ is called *prime* if $\mathfrak{p} \neq R$ and, for any two ideals $I, J \subseteq R$, $IJ \subseteq \mathfrak{p}$ implies that either $I \subseteq \mathfrak{p}$ or $J \in \mathfrak{p}$. Let $f : R \to R'$ be a ring homomorphism and let $\mathfrak{p}' \subseteq R'$ be a prime ideal. Show that $f^{-1}(\mathfrak{p}') \subseteq R$ is a prime ideal.

3. Let us define

$$\operatorname{Spec}(R) := \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal}\}$$

This set is called the *spectrum* of R. For an ideal $I \subseteq R$, define $V(I) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$.

   (a) Show that $V(0) = \operatorname{Spec}(R)$ and that $V(R) = \emptyset$.

   (b) Show that $V(IJ) = V(I) \cup V(J)$.

   (c) Let I_λ, $\lambda \in \Lambda$ be a collection of ideals (note that Λ does not have to be finite). Define

$$J := \left\{ \sum_{\lambda \in \Lambda} r_\lambda f_\lambda \mid f_\lambda \in I_\lambda, r_\lambda \in R, r_\lambda = 0 \text{ for all but a finite number of } \lambda \right\}.$$

   In other words, J is the minimal ideal that contains all of the I_λ. Show that

$$V(J) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$$

If you have taken MAT 147, you may recognize that (a), (b) and (c) say that the sets of the form $V(I)$ form the closed sets of a topological space. In this way, every commutative ring defines a topological space (however, it is a nasty one: it is not Hausdorff, and in most cases not even T_1!). Moreover, thanks to Exercise 2 of this homework, if $f : R \to R'$ is a ring homomorphism, there is a function $f^\sharp : \operatorname{Spec}(R') \to \operatorname{Spec}(R)$. You can try to show this function is continuous, if you know what this means.