MAT 150C: MODERN ALGEBRA
Homework 1

COMPUTATIONAL EXERCISES

1. Consider the field $\mathbb{F}_5$ with 5 elements, and the field extension $\mathbb{F}_5[\alpha]$ where α is a square root of 3.
   (a) Find $x, y \in \mathbb{F}_5$ such that $(3+\alpha)(-1+2\alpha) = x + y\alpha$.
   (b) Find $x, y \in \mathbb{F}_5$ such that $(4+2\alpha)^{-1} = x + y\alpha$.
2. Find all $c \in \mathbb{F}_7$ such that $\mathbb{F}_7[x]/(x^2 - c)$ is a field.
3. Consider the ring $\mathbb{C}[[x]]$ of formal power series with coefficients in $\mathbb{C}$. Show that the following elements are units by explicitly finding a multiplicative inverse.
   (a) $1 + x + x^2$.
   (b) $\sum_{i=0}^{\infty} x^i$, where $x^0 = 1$.

THEORETICAL EXERCISES

1. Let I, J be ideals in a ring R. Define

$$IJ := \left\{ \sum_{i=1}^{n} f_i g_i \mid f_i \in I, g_i \in J, n \geq 0 \right\}.$$

   (a) Show that IJ is an ideal of R.
   (b) Show that $IJ \subseteq I \cap J$.
   (c) Give an example of ideals I and J such that $IJ \neq I \cap J$.
2. An element $a \in R$ is called *nilpotent* if $a^n = 0$ for some $n \geq 1$.
   (a) Show that the set of all nilpotent elements in R forms an ideal. This is known as the *nilradical* of R, and it is usually denoted by $\mathfrak{nil}(R)$ or $\sqrt{(0)}$.
   (b) Show that $\mathfrak{nil}(R/\mathfrak{nil}(R)) = 0$. In other words, the quotient ring $R/\mathfrak{nil}(R)$ does not contain nonzero nilpotent elements.
   (c) Show that if $u \in R$ is a unit and $a \in \mathfrak{nil}(R)$, $u + a$ is also a unit.
3. Let $\mathbb{F}_q$ be a finite field.
   (a) Let $a \in \mathbb{F}_q$ be a fixed element. Construct a polynomial $f(x) \in \mathbb{F}_q[x]$ such that $f(a) = 1$ and $f(b) = 0$ for $b \neq a$. (Hint: Use the fact that if $f(b) = 0$ then $(x - b)$ divides $f(x)$)
   (b) Show that for *any* function $\Phi : \mathbb{F}_q \to \mathbb{F}_q$, there exists a polynomial $f(x) \in \mathbb{F}_q[x]$ such that $f(a) = \Phi(a)$ for every $a \in \mathbb{F}_q$ (Hint: use (a))
   (c) Consider the evaluation map $\mathrm{ev} : \mathbb{F}_q[x] \to \mathrm{Fun}(\mathbb{F}_q, \mathbb{F}_q)$. Show that it is surjective and its kernel is generated by the polynomial $\prod_{a \in \mathbb{F}_q}(x - a)$. (Hint: Count dimensions over $\mathbb{F}_q$)