MAT 150C: MODERN ALGEBRA
Homework 2

Recall that a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called *irreducible* if it cannot be factored into non-constant polynomials.

Computational exercises

1. How many maximal ideals in $\mathbb{C}[x, y]$ contain both $f(x, y) = x^2 + y^2 - 5$ and $g(x, y) = xy - 2$? (OPTIONAL: How many ideals in $\mathbb{C}[x, y]$, maximal or not, contain $f(x, y)$ and $g(x, y)$?)
2. Factor the following polynomials into irreducible factors in $\mathbb{F}_p[x]$:
   (a) $x^3 + x^2 + x + 1$, $p = 2$.
   (b) $x^2 - 3x - 3$, $p = 5$.

Theoretical exercises

1. Let $f_1, \ldots, f_k, g_1, \ldots, g_m \in \mathbb{C}[x_1, \ldots, x_n]$. We have the algebraic varieties $V(f_1, \ldots, f_k) \subseteq \mathbb{C}^n$ and $V(g_1, \ldots, g_m) \subseteq \mathbb{C}^n$. Show that $V(f_1, \ldots, f_k) \cup V(g_1, \ldots, g_m)$ is an algebraic variety.
2. (Special case of Bézout's theorem) Let $L = V(ax + by + c) \subseteq \mathbb{C}^2$ be a complex line, and let $f \in \mathbb{C}[x, y]$ be an irreducible degree d polynomial. Show that $V(f) \cap L$ has at most d points, unless $V(f)$ contains L.
3. Show that the ring of formal power series $\mathbb{C}[[x]]$ is a unique factorization domain.
4. (OPTIONAL) Let $f \in \mathbb{C}[x, y]$ be an irreducible polynomial. A point $p \in V(f)$ is called a *singular point* if
$$f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$$
   Show that $V(f)$ has finitely many singular points.