MAT 150C - Homework 2

Markus Tran

Computational Exercises

1. By **Hilbert's Nullstellensatz**, there is a bijection between maximal ideals containing f and the points satisfying $f(x,y) = 0$. Therefore we can count all the solutions to the system of equations

$$\begin{aligned} x^2 + y^2 - 5 &= 0 \\ xy - 2 &= 0. \end{aligned}$$

The second equation asserts $y = 2/x$. Plugging that into the first equation gives

$$x^2 - 5 + \frac{4}{x^2} = 0 \implies \left(x - \frac{1}{x}\right)\left(x - \frac{4}{x}\right) = 0.$$

There are 4 solutions:

$$(1,2), \quad (-1,-2), \quad (2,1), \quad (-2,-1).$$

2. (a)

$$\begin{aligned} x^3 + x^2 + x + 1 &= x^3 + 3x^2 + 3x + 1 \\ &= (x+1)(x+1)(x+1). \end{aligned}$$

(b)

$$\begin{aligned} x^2 - 3x - 3 &= x^2 - 3x + 2 \\ &= (x-2)(x-1) \\ &= (x+3)(x+4). \end{aligned}$$

Theoretical Exercises

1. It has been shown that $V(f_1, \ldots, f_n) \cap V(f_{n+1}, \ldots, f_m) = V(f_1, \ldots, f_{n+m})$ is an algebraic variety. Therefore

$$\begin{aligned} V(f_1, \ldots, f_k) \cup V(g_1, \ldots, g_m) &= \left[\bigcap_i V(f_i)\right] \cup \left[\bigcap_j V(g_j)\right] \\ &= \bigcap_{i,j} \left[V(f_i) \cup V(g_j)\right]. \end{aligned}$$

Thus it suffices to show that $V(f) \cup V(g)$ is an algebraic variety for any $f, g \in \mathbb{C}[x_1, \ldots, x_n]$. Let x be in the algebraic variety $V(fg)$. This means $[fg](x) = 0$, so either $f(x) = 0$ or $g(x) = 0$. Thus $x \in V(f) \cup V(g)$. And if $x \in V(f) \cup V(g)$, then either $f(x) = 0$ or $g(x) = 0$, which means $[fg](x) = 0$ and $x \in V(fg)$. Therefore $V(f) \cup V(g) = V(fg)$ is an algebraic variety.

2. Let $(x, y) \in V(f) \cap L$. Then (x, y) must satisfy $ax + by + c = 0$. If $a = b = 0$, then $L = \mathbb{C}^2$ or $L = \emptyset$, and this theorem does not apply. Thus either x or y must have a non-zero coefficient, and we can solve for it. Assume y has a non-zero coefficient, so $y = -ab^{-1}x - cb^{-1}$, a polynomial of degree at most 1. Also (x, y) must satisfy $f(x, y) = 0$. Substituting $y = -ab^{-1}x - cb^{-1}$ into

$f(x,y)$ gives another polynomial $f'(x) \in \mathbb{C}[x]$ with degree at most d or it is the zero polynomial. If f' is not zero, it has at most d roots, and thus there are at most d such points that satisfy both L and $V(f)$. Otherwise there are infinitely many points because $L \subseteq V(f)$.

3. Define the *small-degree* of a polynomial to be the degree of its first non-zero term (the zero polynomial does not have a small-degree). For example, $x-2$ has small-degree 0, and $x^2+x^4+x^6+\cdots$ has small-degree 2. Note that if f has small-degree n and g has small-degree m, then fg has small-degree $n+m$.

   The units in $\mathbb{C}[[x]]$ are precisely the polynomials with small-degree 0 (the polynomials with non-zero constant term). Let $f(x)$ be a polynomial of small-degree $n \geq 2$. Then $f(x)$ can be written as $f(x) = x \cdot g(x)$ where x and $g(x)$ have small-degrees greater than 0. Thus f is not irreducible, and the irreducible polynomials in $\mathbb{C}[[x]]$ are precisely the polynomials with small-degree 1. Furthermore, f can be written as

$$f(x) = \underbrace{xx\cdots x}_{n-1 \text{ times}} \cdot h(x)$$

   where $h(x)$ is a polynomial of small-degree 1. Every polynomial that is not a unit, is either irreducible or can be written as a product of irreducibles.

   Suppose $f \in \mathbb{C}[[x]]$ has two factorizations

$$f(x) = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s.$$

Since the small-degrees of p_i and q_j are all 1, $r = s$. Every irreducible $r(x)$ can be written as $r(x) = x \cdot u(x)$ where $u(x)$ has small-degree 0 and is invertible. Let $f = xu$ and $g = xv$ be two irreducible. Then $f = gv^{-1}u$ where $v^{-1}u$ is a unit, and all irreducibles are associates. Thus $\mathbb{C}[[x]]$ is a unique factorization domain.