1.

4.1.1

a. Let $f = y - x^2$ and $g = z - x^3$. Then any point satisfying $f(\mathbf{x}) = g(\mathbf{x}) = 0$ also satisfies $[f(\mathbf{x})]^2 + [g(\mathbf{x})]^2 = 0$, which means $\mathbf{V}(f, g) \subseteq \mathbf{V}(f^2 + g^2)$.

Now suppose a point satisfies $[f^2 + g^2](\mathbf{x}) = 0$. Since $f^2, g^2 \geq 0$, it must be that $f(\mathbf{x}) = g(\mathbf{x}) = 0$, and therefore $\mathbf{V}(f^2 + g^2) \subseteq \mathbf{V}(f, g)$

b. Any variety $\mathbf{V}(I)$ is equivalent to

$$\begin{aligned}
\mathbf{V}(I) &= \mathbf{V}(\langle f_1, f_2, \ldots, f_n \rangle) && \text{Hilbert Basis Theorem} \\
&= \mathbf{V}(f_1, f_2, \ldots, f_n) \\
&= \mathbf{V}\left(\sum_{i=1}^{n} f_i^2\right). && \text{part (a)}
\end{aligned}$$

2.

4.1.2 The variety $\mathbf{V}(J)$ is the intersection of the unit circle with the line $y = 1$, so it consists of single point $(0, 1)$. Thus the polynomial $f = x$ is in $\mathbf{I}(\mathbf{V}(J))$, but it is not an element of J as seen by the Gröbner basis

$$J = \langle y - 1, x^2 \rangle.$$

3.

4.1.10 Clearly $f^2 + g^2 \in \langle f, g \rangle$, so $\langle f^2 + g^2 \rangle \subseteq \langle f, g \rangle$.

4.

Let $g \in \sqrt{\langle x^2, y^2 \rangle}$. Suppose that $f \notin \langle x, y \rangle$. Then f is either a constant polynomial, or a non-constant polynomial of another variable z. Thus all powers of f are likewise constant or a non-constant polynomial of z. This contradicts f being in $\sqrt{\langle x^2, y^2 \rangle}$.

Let $g \in \langle x, y \rangle$. By definition, $g = px + qy$ for some polynomials p, q. Thus

$$\begin{aligned} g^3 &= p^3x^3 + 3p^2x^2qy + 3pxq^2y^2 + q^3y^3 \\ &= (p^3x + 3p^2qy)x^2 + (3pxq^2 + q^3y)y^2 \\ &\in \langle x^2, y^2 \rangle, \end{aligned}$$

and $g \in \sqrt{\langle x^2, y^2 \rangle}$.

5.

4.2.3 Let f be a polynomial such that $f^m \in \langle x^2+1 \rangle$ for some integer m. Let $f = \prod q_k^{a_k}$ be the factorization of f into irreducibles. Then $f^m = \prod q_k^{m \cdot a_k}$. Since $f^m \in \langle x^2+1 \rangle$ and x^2+1 is irreducible, this means that some $q_i = x^2+1$. Therefore $f \in \langle x^2+1 \rangle$, and $\langle x^2+1 \rangle$ is a radical ideal.

6.

4.2.7 a. Yes, and the smallest power is 3

$$\begin{aligned}(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ &= x^3+y^3+3\left[xy(x+y)\right]\\ &\in \langle x^3, y^3, xy(x+y)\rangle.\end{aligned}$$

7.

4.2.11

$$
\begin{aligned}
&\sqrt{\langle x^5 - 2x^4 + 2x^2 - x, x^5 - x^4 - 2x^3 + 2x^2 + x - 1\rangle} \\
=&\sqrt{\langle \gcd(x^5 - 2x^4 + 2x^2 - x, x^5 - x^4 - 2x^3 + 2x^2 + x - 1)\rangle} \\
=&\sqrt{\langle (x-1)^3(x+1)\rangle} \\
=&\langle (x-1)(x+1)\rangle\,.
\end{aligned}
$$

Thus a basis is $\{(x-1)(x+1)\} = \{x^2 - 1\}$.

8.

4.2.12 hi