1.

- 1.1: Polynomials as a formal ring over an arbitrary field. They also define functions.
- 1.2: Definition of an affine variety. Lots and lots of examples. What about consistency, finiteness, dimension?
- 1.3: Parameterization and Implicitization of affine varieties. Something about my favorite Bezier curves.
- 1.4: Definition of an ideal and their bases. How ideals relate to affine varieties.

$$f_1, \dots, f_s \to \mathbf{V}(f_1, \dots, f_s) \to \mathbf{I}(\mathbf{V}(f_1, \dots, f_s))$$

- 1.5: The division and Euclidean algorithm abstracted and applied to polynomials of a single variable.

2.

(a) Using Maple to factor the polynomial, we see that the square-free part is $(x+1)(x^3+x+1)(x-1) = x^5 + x^2 - x - 1$.

> $factor(x^{11} - x^{10} + 2x^8 - 4x^7 + 3x^5 - 3x^4 + x^3 + 3x^2 - x - 1)$

$$(x+1)^2 (x^3+x+1)^2 (x-1)^3 \quad \textbf{(1)}$$

> $expand((x+1)\cdot(x^3+x+1)\cdot(x-1))$

$$x^5 + x^2 - x - 1 \quad \textbf{(2)}$$

(b) Exercise 1.5.15 shows that the square-free part is also given by.

$$f_{\text{red}} = \frac{f}{\text{GCD}(f, f')} = x^5 + x^2 - x - 1$$

> $f := x^{11} - x^{10} + 2x^8 - 4x^7 + 3x^5 - 3x^4 + x^3 + 3x^2 - x - 1$

$$f := x^{11} - x^{10} + 2x^8 - 4x^7 + 3x^5 - 3x^4 + x^3 + 3x^2 - x - 1 \quad \textbf{(1)}$$

> $simplify\left(\frac{f}{gcd(f, f')}\right)$

$$x^5 + x^2 - x - 1 \quad \textbf{(2)}$$

3.

For each $f_1, \ldots, f_s$, compute

$$g_i = \frac{f_i}{\mathrm{GCD}(f_i, f_i')}$$

(by applying the Euclidean algorithm followed by the division algorithm). Then $\langle g_1, \ldots, g_s \rangle$ is a basis.

4.

1.a. $x^2 - 3x + 2 = (x-1)(x-2) + 0$. Thus $f(x) \in I$.

1.d. Note that $\gcd(x^9 - 1, x^5 + x^3 - x^2 - 1) = x^3 - 1$, so $I = \langle x^3 - 1 \rangle$, and $f(x) \in I$.

5.

2.a. Placing the system into reduced row echelon form gives

$$\begin{pmatrix} 1 & 0 & -\frac{1}{5} & \frac{12}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which gives the parameterization

$$\begin{aligned} x &= \tfrac{12}{5} + \tfrac{1}{5}t \\ y &= \tfrac{7}{5} + \tfrac{1}{5}t \\ z &= t. \end{aligned}$$

2.c.

$$\begin{aligned} x &= t \\ y &= t^3 \\ z &= t^5. \end{aligned}$$

6.

3.a.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 5 \\ 2 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & -6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 13 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_3 - 1 &= 0, \\ x_2 + 2x_3 - 13 &= 0. \end{aligned}$$

3.b.

$$\begin{pmatrix} 2 & -5 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & \frac{11}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{3}{4} & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + \tfrac{11}{4}x_3 + \tfrac{3}{4}x_4 &= 0, \\ x_2 + \tfrac{1}{4}x_3 - \tfrac{3}{4}x_4 &= 0. \end{aligned}$$

7.

1.a.

	f	$\mathrm{LM}(f)$	$\mathrm{LT}(f)$	$\mathrm{multideg}(f)$
lex	$x^3 + x^2 + 2x + 3y - z^2 + z$	x^3	x^3	$(3, 0, 0)$
grlex	$x^3 + x^2 - z^2 + 2x + 3y + z$	x^3	x^3	$(3, 0, 0)$
grevlex	$x^3 + x^2 - z^2 + 2x + 3y + z$	x^3	x^3	$(3, 0, 0)$

8.

2.b. grevlex order.

9.

3.b. $f(x) = zy^3x + z^2y^2x + z^3x^2$. None of the three orders.