Problem 0

I affirm the code of student conduct. I, the student enrolled in this course, am taking this exam by myself without collaborating with or using the help of other persons, external resources, or services.

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Problem 1

$$
\begin{array}{llrrrcl}
\text{Maximize} & & & z = 11x_1 + 5x_2 & & & \\
\\
\text{such that} & (12-9) & -2x_1 & +10x_2 & \geq & 0 \\
 & & 2x_1 & +10x_2 & \leq & 33-1 \\
 & 18 & +8x_1 & -6x_2 & \geq & 0 \\
\\
 & & & x_1, x_2 \geq 0 & & &
\end{array}
$$

(i) Moving constants to the right-hand side puts the LP into standard inequality form.

$$
\begin{array}{lrrcl}
\text{Maximize} & & z = 11x_1 + 5x_2 & & \\
\\
\text{such that} & 2x_1 & -10x_2 & \leq & 3 \\
 & 2x_1 & +10x_2 & \leq & 32 \\
 & -8x_1 & +6x_2 & \leq & 18 \\
\\
 & & x_1, x_2 \geq 0 & &
\end{array}
$$

(ii) Add slack variables to put the LP into standard equality form.

$$
\begin{array}{lrrrrrcl}
\text{Maximize} & & z = 11x_1 + 5x_2 & & & & & \\
\\
\text{such that} & 2x_1 & -10x_2 & +x_3 & & & = & 3 \\
 & 2x_1 & +10x_2 & & +x_4 & & = & 32 \\
 & -8x_1 & +6x_2 & & & +x_5 & = & 18 \\
\\
 & & x_1, x_2, x_3, x_4, x_5 \geq 0 & & & & &
\end{array}
$$

Step 1

- BFS: $\mathbf{x} = (0, 0, 3, 32, 18)$.
- Objective value: $z(\mathbf{x}) = 0$.
- Dictionary, for basis of basic variables $\{x_3, x_4, x_5\}$ with non-basic variables $\{x_1, x_2\}$:

$$
\begin{array}{ll}
\text{Maximize} & z = 11x_1 + 5x_2 \\
\\
\text{such that} & \begin{array}{rcrrr} x_3 & = & 3 & -2x_1 & +10x_2 \\ x_4 & = & 32 & -2x_1 & -10x_2 \\ x_5 & = & 18 & +8x_1 & -6x_2 \end{array} \\
\\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

- Ratios, for entering variable $x_1 = \lambda \geq 0$:

$$
\begin{array}{lcll}
x_3 = 3 - 2\lambda & \Longrightarrow & \lambda \leq 3/2 & \leftarrow \text{minimum ratio} \\
x_4 = 32 - 2\lambda & \Longrightarrow & \lambda \leq 16 & \\
x_5 = 18 + 8\lambda & \Longrightarrow & \lambda \leq \infty &
\end{array}
$$

- Leaving variable: x_3.

Step 2

- BFS: $\mathbf{x} = (\frac{3}{2}, 0, 0, 29, 30)$.
- Objective value: $z(\mathbf{x}) = \frac{33}{2}$.
- Dictionary, for basis of basic variables $\{x_1, x_4, x_5\}$ with non-basic variables $\{x_3, x_2\}$:

$$
\begin{array}{ll}
\text{Maximize} & z = \frac{33}{2} - \frac{11}{2}x_3 + 60x_2 \\
\\
\text{such that} & \begin{array}{rcrrr} x_1 & = & \frac{3}{2} & -\frac{1}{2}x_3 & +5x_2 \\ x_4 & = & 29 & +x_3 & -20x_2 \\ x_5 & = & 30 & -4x_3 & +34x_2 \end{array} \\
\\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

- Ratios, for entering variable $x_2 = \lambda \geq 0$:

$$
\begin{array}{lcll}
x_1 = 3/2 + 5\lambda & \Longrightarrow & \lambda \leq \infty & \\
x_4 = 29 - 20\lambda & \Longrightarrow & \lambda \leq 29/20 & \leftarrow \text{minimum ratio} \\
x_5 = 30 + 34\lambda & \Longrightarrow & \lambda \leq \infty &
\end{array}
$$

- Leaving variable: x_4.

Step 3

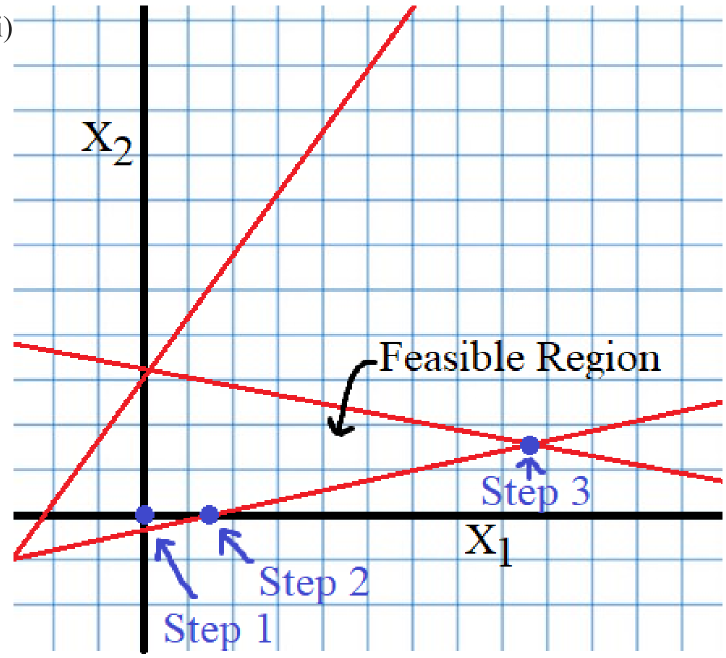
- BFS: $\mathbf{x} = (\frac{35}{4}, \frac{29}{20}, 0, 0, \frac{793}{10})$.
- Objective value: $z(\mathbf{x}) = \frac{207}{2}$.

The new objective function in terms of the non-basic variables $\{x_3, x_4\}$ is

$$z = \frac{207}{2} - \frac{5}{2}x_3 - 3x_4.$$

Since the coefficients are all negative, the solution is optimal, and we are done.

(iii)



(iv) In the optimal solution, the non-basic variables are x_3, x_4 (which are equal to 0), indicating that we are currently at the corner where the lines

$$\begin{array}{rrcl} 2x_1 & -10x_2 & = & 3 \qquad \text{and} \\ 2x_1 & +10x_2 & = & 32 \end{array}$$

intersect. The values of the basic variables x_1, x_2, x_5 are obtained by setting the non-basic variables to 0.

Problem 2

(i)

$$\begin{array}{llllll}
\text{Maximize} & z = \frac{2}{100}x_0 + \frac{6}{100}x_1 + \frac{4}{100}x_2 & & & & \\
\text{such that} & & x_0 & & & \leq 35\,353\,535 \\
& & x_0 & & & \geq -20\,201\,026 \\
& & x_0 & +x_1 & +x_2 & \leq 35\,353\,535 \\
& 35\,353\,535 & +\frac{2}{100}x_0 & -\frac{80}{100}x_1 & -\frac{50}{100}x_2 & \geq 9\,817\,234
\end{array}$$

$$x_0 \in \mathbb{R}, \quad x_1, x_2 \geq 0$$

(ii) Let $x_0 = x_3 - x_4$ with $x_3, x_4 \geq 0$ to obtain the standard inequality form

$$\begin{array}{llllll}
\text{Maximize} & z = \frac{2}{100}(x_3 - x_4) + \frac{6}{100}x_1 + \frac{4}{100}x_2 & & & & \\
\text{such that} & & (x_3 - x_4) & & & \leq 35\,353\,535 \\
& & (x_3 - x_4) & & & \geq -20\,201\,026 \\
& & (x_3 - x_4) & +x_1 & +x_2 & \leq 35\,353\,535 \\
& 35\,353\,535 & +\frac{2}{100}(x_3 - x_4) & -\frac{80}{100}x_1 & -\frac{50}{100}x_2 & \geq 9\,817\,234
\end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\Downarrow$$

$$\begin{array}{llllll}
\text{Maximize} & z = \frac{6}{100}x_1 + \frac{4}{100}x_2 + \frac{2}{100}x_3 - \frac{2}{100}x_4 & & & & \\
\text{such that} & & & x_3 & -x_4 & \leq 35\,353\,535 \\
& & & -x_3 & +x_4 & \leq 20\,201\,026 \\
& x_1 & +x_2 & +x_3 & -x_4 & \leq 35\,353\,535 \\
& \frac{80}{100}x_1 & +\frac{50}{100}x_2 & -\frac{2}{100}x_3 & +\frac{2}{100}x_4 & \leq 25\,536\,301
\end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

(iii)The solution

$$\begin{aligned}
x_0 &= 35\,353\,535 \\
x_1 &= 0 \\
x_2 &= 0
\end{aligned}$$

satisfies all constraints.

(iv) **INFAMOUSLP**

Suppose the solution lies between the two points $y = (y_0, y_1, y_2)$ and $z = (z_0, z_1, z_2)$. Since x_1 lies between y_1, z_1 and $y_1, z_1 \geq 0$, we have $y_1 = z_1 = 0$. Similarly, $y_2 = z_2 = 0$. And since x_0 lies between y_0, z_0 and $y_0, z_0 \leq 35\,353\,535$, then $y_0 = z_0 = 35\,353\,535$. Thus $y = z$, and x is an extreme point.

STANDARDLP

A corresponding solution is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 35\,353\,535 \\ x_4 &= 0 \end{aligned}$$

Suppose the solution lies between the two points $y = (y_1, y_2, y_3, y_4)$ and $z = (z_1, z_2, z_3, z_4)$. By a similar argument, $y_1 = y_2 = y_4 = z_1 = z_2 = z_4 = 0$. Since $y_4 = z_4 = 0$, the constraint

$$x_3 \quad -x_4 \quad \leq \quad 35\,353\,535$$

means that y_3, z_3 is bounded above by $35\,353\,535$. Thus $y = z$ and x is an extreme point.

(v) Yes. The STANDARDLP solution

$$x_1 = x_2 = x_3 = x_4 = 0$$

is an extreme point, but it corresponds to the INFAMOUSLP solution

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

which lies between the two feasible points

$$\begin{aligned} x_0 &= -1 \\ x_1 &= 0 \\ x_2 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_0 &= 1 \\ x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$