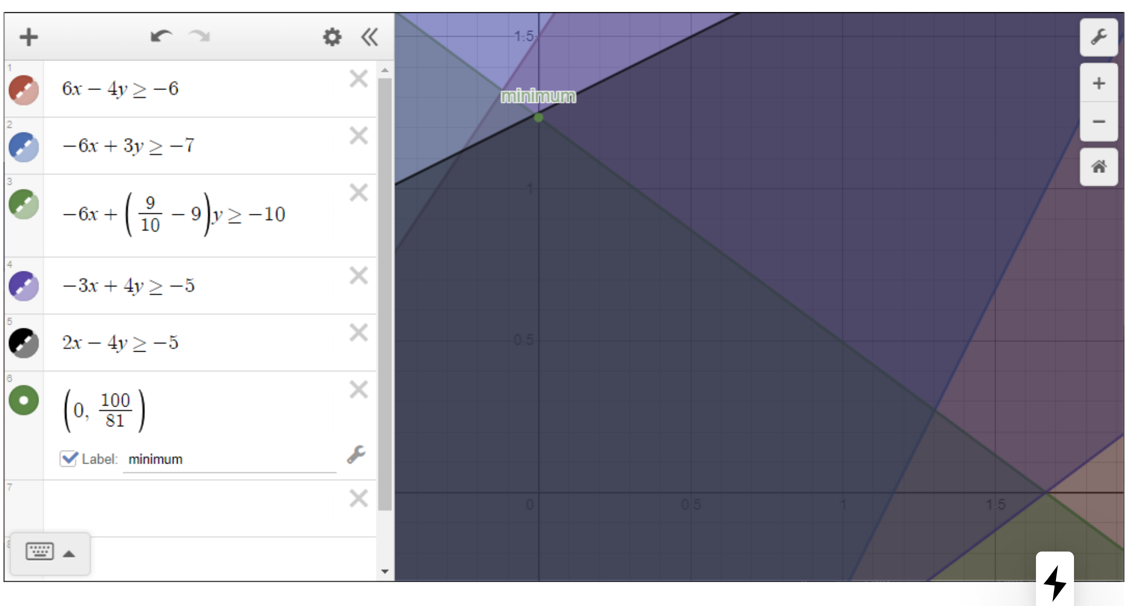
Problem 1 (Duality)

$$
\begin{array}{rrrrrrl}
\text{maximize} & -6x_1 & -7x_2 & - \quad 10x_3 & -5x_4 & -5x_5 & \\
\text{s.t.} & 6x_1 & -6x_2 & - \quad 6x_3 & -3x_4 & +2x_5 & \leq -19 \qquad \text{(P.I)} \\
 & -4x_1 & +3x_2 & +(\frac{9}{10}-9)x_3 & +4x_4 & -4x_5 & \leq \frac{1}{2} - 34 \quad \text{(P.II)} \\
 & \multicolumn{6}{l}{x_1, x_2, x_3, x_4, x_5 \geq 0}
\end{array}
$$

i. MIDDLE-OF-THE-QUARTER-D

$$
\begin{array}{rrrll}
\text{minimize} & -19y_1 & +(\frac{1}{2}-34)y_2 & & \\
\text{s.t.} & 6y_1 & - \quad 4y_2 & \geq -6 & \text{(D.I)} \\
 & -\ 6y_1 & + \quad 3y_2 & \geq -7 & \text{(D.II)} \\
 & -\ 6y_1 & +(\frac{9}{10}-9)y_2 & \geq -10 & \text{(D.III)} \\
 & -\ 3y_1 & + \quad 4y_2 & \geq -5 & \text{(D.IV)} \\
 & 2y_1 & - \quad 4y_2 & \geq -5 & \text{(D.V)} \\
 & \multicolumn{4}{l}{y_1, y_2 \geq 0}
\end{array}
$$

ii. Graphing MIDDLE-OF-THE-QUARTER-D with Desmos gives a minimum value of $\zeta_{\text{dual}}(0, \frac{100}{81}) = -\frac{3350}{81}$.



iii. MIDDLE-OF-THE-QUARTER-D is rewritten in standard equation form with slack variables.

$$\begin{array}{lrlrl}
\text{maximize} & 19y_1 & -\ (\frac{1}{2}-34)y_2 & & \\
\text{s.t.} & -\ 6y_1 & +\ \ 4y_2 & +z_1 & = 6 \\
& 6y_1 & -\ \ 3y_2 & +z_2 & = 7 \\
& 6y_1 & -\ (\frac{9}{10}-9)y_2 & +z_3 & = 10 \\
& 3y_1 & -\ \ 4y_2 & +z_4 & = 5 \\
& -\ 2y_1 & +\ \ 4y_2 & +z_5 & = 5 \\
& \multicolumn{4}{l}{y_1, y_2, z_1, z_2, z_3, z_4, z_5 \geq 0}
\end{array}$$

The optimal solution $(0, \frac{100}{81})$ is plugged in to solve for the slack variables.

$$\begin{array}{rlrl}
-\ 6(0) & +\ \ 4(\frac{100}{81}) & +z_1 = 6 & \\
6(0) & -\ \ 3(\frac{100}{81}) & +z_2 = 7 & \\
6(0) & -\ (\frac{9}{10}-9)(\frac{100}{81}) & +z_3 = 10 & \\
3(0) & -\ \ 4(\frac{100}{81}) & +z_4 = 5 & \\
-\ 2(0) & +\ \ 4(\frac{100}{81}) & +z_5 = 5 &
\end{array}
\implies
\begin{array}{l}
z_1 = \frac{86}{81} \\
z_2 = \frac{289}{27} \\
z_3 = 0 \\
z_4 = \frac{805}{81} \\
z_5 = \frac{5}{81}
\end{array}$$

By the Complementary Slackness Theorem, $y_2 > 0$ implies equality for P.II, and $z_1, z_2, z_4, z_5 > 0$ implies $x_1, x_2, x_4, x_5 = 0$. Thus

$$\begin{aligned}
-4x_1 + 3x_2 + (\tfrac{9}{10} - 9)x_3 + 4x_4 - 4x_4 &= \tfrac{1}{2} - 34 \\
(\tfrac{9}{10} - 9)x_3 &= \tfrac{1}{2} - 34 \\
x_3 &= \tfrac{335}{81}
\end{aligned}$$

and the optimal solution for MIDDLE-OF-THE-QUARTER-P is $\zeta(0, 0, \frac{335}{81}, 0, 0) = -\frac{3350}{81}$.

Problem 2 (Dual Simplex Method)

$$\begin{array}{rrrrrl}
\text{maximize} & -\ 3x_1 & -20x_2 & -\ 9x_3 & -\ 12x_4 & \\
\text{s.t.} & 20x_1 & +16x_2 & +(9-16)x_3 & +(1-32)x_4 & \leq -40 \\
& & x_2 & -\ x_3 & & \leq -1 \\
& 4x_1 & +\ 8x_2 & +\ 2x_3 & -\ 2x_4 & \leq 17 \\
& \multicolumn{5}{l}{x_1, x_2, x_3, x_4 \geq 0}
\end{array}$$

$$\Big\downarrow \text{add slack variables}$$

$$\begin{array}{rrrrrrrl}
20x_1 & +16x_2 & -7x_3 & -31x_4 & +x_5 & & & = -40 \\
& x_2 & -\ x_3 & & & +x_6 & & = -1 \\
4x_1 & +\ 8x_2 & +2x_3 & -\ 2x_4 & & & +x_7 & = 17 \\
\multicolumn{8}{l}{x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0}
\end{array}$$

Step 1

- Dictionary: (select most negative to exit)

$$\begin{array}{ll}
\text{Basic variables:} & \{x_5, x_6, x_7\} \\
\text{Nonbasic variables:} & \{x_1, x_2, x_3, x_4\} \\
\text{Objective:} & \zeta(\mathbf{x}) = 0
\end{array}$$

$$\begin{array}{rrrrrrr}
& \zeta = & & -\ 3x_1 & -20x_2 & -9x_3 & -12x_4 \\
\text{exiting} \rightarrow & x_5 = & -40 & -20x_1 & -16x_2 & +7x_3 & +31x_4 \\
& x_6 = & -1 & & -\ x_2 & +\ x_3 & \\
& x_7 = & 17 & -\ 4x_1 & -\ 8x_2 & -2x_3 & +\ 2x_4
\end{array}$$

- Ratios: (select smallest to enter)

$$\begin{array}{rl}
& x_1 \implies \lambda < \infty \\
& x_2 \implies \lambda < \infty \\
& x_3 \implies \lambda < 9/7 \\
\text{entering} \rightarrow & x_4 \implies \lambda < 12/31
\end{array}$$

- Pivot:

$$\begin{aligned}
\zeta &= -\tfrac{180}{31} - \tfrac{333}{31}x_1 - \tfrac{812}{31}x_2 - \tfrac{195}{31}x_3 - \tfrac{12}{31}x_5 \\
x_4 &= \tfrac{40}{31} + \tfrac{20}{31}x_1 + \tfrac{16}{31}x_2 - \tfrac{7}{31}x_3 + \tfrac{1}{31}x_5 \\
x_6 &= -1 - x_2 + x_3 \\
x_7 &= \tfrac{607}{31} - \tfrac{84}{31}x_1 - \tfrac{216}{31}x_2 - \tfrac{76}{31}x_3 + \tfrac{2}{31}x_5
\end{aligned}$$

Step 2

- Dictionary: (select most negative to exit)

$$\begin{aligned}
&\text{Basic variables:} && \{x_4, x_6, x_7\} \\
&\text{Nonbasic variables:} && \{x_1, x_2, x_3, x_5\} \\
&\text{Objective:} && \zeta(\mathbf{x}) = -\tfrac{180}{31}
\end{aligned}$$

$$\begin{aligned}
\zeta &= -\tfrac{180}{31} - \tfrac{333}{31}x_1 - \tfrac{812}{31}x_2 - \tfrac{195}{31}x_3 - \tfrac{12}{31}x_5 \\
x_4 &= \tfrac{40}{31} + \tfrac{20}{31}x_1 + \tfrac{16}{31}x_2 - \tfrac{7}{31}x_3 + \tfrac{1}{31}x_5 \\
\text{exiting} \rightarrow \quad x_6 &= -1 - x_2 + x_3 \\
x_7 &= \tfrac{607}{31} - \tfrac{84}{31}x_1 - \tfrac{216}{31}x_2 - \tfrac{76}{31}x_3 + \tfrac{2}{31}x_5
\end{aligned}$$

- Ratios: (select smallest to enter)

$$\begin{aligned}
x_1 &\implies \lambda < \infty \\
x_2 &\implies \lambda < \infty \\
\text{entering} \rightarrow \quad x_3 &\implies \lambda < 195/31 \\
x_5 &\implies \lambda < \infty
\end{aligned}$$

- Pivot:

$$\begin{aligned}
\zeta &= -\tfrac{375}{31} - \tfrac{333}{31}x_1 - \tfrac{1007}{31}x_2 - \tfrac{195}{31}x_6 - \tfrac{12}{31}x_5 \\
x_4 &= \tfrac{33}{31} + \tfrac{20}{31}x_1 + \tfrac{9}{31}x_2 - \tfrac{7}{31}x_6 + \tfrac{1}{31}x_5 \\
x_3 &= 1 + x_2 + x_6 \\
x_7 &= \tfrac{531}{31} - \tfrac{84}{31}x_1 - \tfrac{292}{31}x_2 - \tfrac{76}{31}x_6 + \tfrac{2}{31}x_5
\end{aligned}$$

Step 3

We have reached a feasible solution which is optimal.

Basic variables:	$\{x_4, x_3, x_7\}$
Nonbasic variables:	$\{x_1, x_2, x_6, x_5\}$
Objective:	$\zeta(0, 0, 1, \frac{33}{31}, 0, 0, \frac{531}{31}) = -\frac{375}{31}$