Math 254A Lecture 24 Notes

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1 Thermodynamic Limits for Counting Empirical Measures

1.1 Recap + rest of proof of the thermodynamic limit

In our lattice models, we have an alphabet $|A| < \infty$ of local states. If $W \subseteq \mathbb{Z}^d$ is finite, then an observable is a function $\psi : A^W \to \mathbb{R}^r$. FOr a box B and $\omega \in A^B$,

$$\Psi_B(\omega) = \sum_{v+W \subseteq B} \psi(\omega_{v+W}).$$

We wanted to measure the size of

$$\Omega_B(\psi, U) = \{\omega \in A^B : \tfrac{1}{|B|}\Psi_B(\omega) \in U\}.$$

We were trying to prove the existence of the thermodynamic limit in this situation:

Theorem 1.1. *There exists a concave, upper semicontinuous function $s : \mathbb{R}^r \to [-\infty, \infty)$ such that*

(a) $\max_x s(x) = \log |A|$.

(b) If either $U \cap \{s > -\infty\} \neq \varnothing$ or $\overline{U} \cap \overline{\{s > -\infty\}} = \varnothing$, then

$$|\Omega_B(\psi, U)| = \exp\left(|B| \cdot \sup_{x \in U} s(x) + o(|B|)\right).$$

Last time, we showed that there is a function boxes $\to (0, \infty)$ sending $B \mapsto \varepsilon(B)$ such that $\varepsilon(B) \to 0$ as $B \uparrow \mathbb{Z}^d$ and

$$|\Omega_B(\psi, U_{2\varepsilon(B)})| = \exp(|B| \cdot s(U) + o(|B|))$$

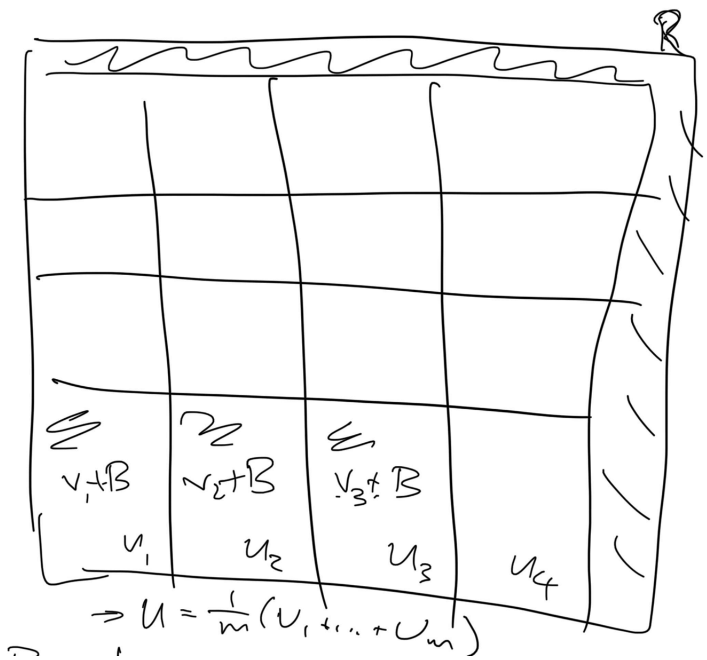
for some $s(U) \in [-\infty, \infty)$, where $U_{2\varepsilon(B)} := \{x : \overline{B_{2\varepsilon(B)}}(x) \subseteq U\}$. We can define

$$s(x) = \inf\{s(U) : U \ni x \text{ is open, convex}\}.$$

This s is automatically upper semicontinuous.

Last time, we showed the estimate that if R is big enough, then

$$|\Omega_R(\psi, U_{2\varepsilon(R)})| \geq \prod_{i=1}^{m} |\Omega_B(\psi, (U_i)_{2\varepsilon(B)})|.$$



Here is the rest of the proof of the theorem:

Proof. If $1/2 + O(1/m)$ of the U_is are U and $1/2 + O(1/m)$ of them are U',then this inequality gives

$$|\Omega_R(\psi, (\tfrac{1}{2}U + \tfrac{1}{2}U')_{2\varepsilon(R)+O(1/n)})| \geq |\Omega_B(\psi, U_{2\varepsilon(B)})|^{m/2+o(1)}| \cdot |\Omega_B(\psi, U'_{2\varepsilon(B)})|^{m/2+o(1)}|.$$

Let $R \uparrow \mathbb{Z}^d$ and then $B \uparrow \mathbb{Z}^d$, so we get

$$s\left(\frac{1}{2}U + \frac{1}{2}U'\right) \geq \frac{1}{2}(s(U) + s(U')).$$

Next we show that $s(U) = \sup_{x \in U} s(x)$. As before, this follows if $s(U) = \sup\{s(K) : K \subseteq U \text{ compact, convex}\}$. This works the same as in the non-interacting case because

$$\lim_{B \uparrow \mathbb{Z}^d} \frac{1}{|B|} \log |\Omega_B(\psi, U_{2\varepsilon(B)})| = \sup_{B} \frac{1}{|B|} \log |\Omega_B(\psi, U_{2\varepsilon(B)})|.$$

So if c is $<$ this, then there is a box B such that $\frac{1}{|B|}\log|\Omega_B(\psi, U_{2\varepsilon(B)})| \geq c$. There exists a compact set K such that $\Omega_B(\psi, U_{2\varepsilon(B)}) = \Omega_B(\psi, K)$. Take $\frac{1}{|B|}\log|\cdot|$, let $B \uparrow \mathbb{Z}^d$ and use superadditivity to get $s(K) \geq c$.

So $s(U) = \sup s(K)$, and so $s(U) = \sup_{x\in U} s(x)$. Now we have a concave upper semicontinuous function such that

$$|\Omega_B(\psi, U_{2\varepsilon(B)})| = \exp\left(|B| \cdot \sup_{x\in U} s(x) + o(|B|)\right).$$

for all open convex U. If we remove the ε, certainly

$$|\Omega_B(\psi, U)| \geq \exp\left(|B| \cdot \sup_{x\in U} s(x) + o(|B|)\right).$$

But if $U \cap \{s > -\infty\} \neq \varnothing$ or $\overline{U} \cap \overline{\{s > -\infty\}} = \varnothing$, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sup_{x\in B_\delta(U)} s(x) < \sup_{x\in U} s(x) + \varepsilon, \qquad \text{where } B_\delta(U) = \bigcup_{y\in U} B_\delta(y).$$

But then $U \subseteq (B_\delta(U))_{2\varepsilon(B)}$ for all large enough boxes B, and we have

$$\begin{aligned} |\Omega_B(\psi, U)| &\leq |\Omega_B(\psi, (B_\delta V)_{2\varepsilon(B)}| \\ &= \exp\left(|B| \cdot \sup_{x\in B_\delta(U)} s(x) + o(|B|)\right) \\ &\leq \exp\left(|B| \cdot (\sup_{x\in U} s(x) + \varepsilon) + o(|B|)\right). \end{aligned}$$

Therefore,

$$\limsup_{B\uparrow\mathbb{Z}^d} \frac{1}{|B|}\log|\Omega_B(\psi, U)| \leq \sup_{x\in U} s(x) + \varepsilon.$$

Here, ε is arbitrary, so in fact $\lim_{B\uparrow\mathbb{Z}^d} = \sup_U s(x)$.

Here is the last detail: Take $U = \mathbb{R}^n$ to get

$$|A|^{|B|} = |\Omega_B(\psi, U)| = e^{|B|\cdot\sup_x s(x) + o(|B|)}.$$

This gives

$$\sup_{\mathbb{R}^n} s = \log|A|. \qquad \square$$

1.2 The exponent function for measure-valued observables

In the non-interacting case, we described s:

(a) in general via s^*,

(b) explicitly in case ψ is measure-valued.

We will aim to do the same in this setting.

Let's try to approach (b). To set this up, fix again a finite window $W \subseteq \mathbb{Z}^d$ and define $\psi : A^W \to M(A^W) = \mathbb{R}^{A^W}$ sending $a \mapsto \delta_a$. Then we have

$$\Psi_B(\omega)(\{a\}) = \sum_{v+W\subseteq B} \psi(\omega_{v+W})(\{a\}) = |\{v : v+W \subseteq B, \omega_{v+W} = a\}|.$$

We now look at $\frac{1}{|B|}\Psi_B(\omega)$, but it would be cleaner to look at $\frac{1}{|\{v:v+W\subseteq B\}|}\sum_{v+W\subseteq B}\psi(\omega_{v+W})$ so this can be an average. Fortunately, these are asymptotically equivalent, as

$$|\{v : v+W \subseteq B\}| = |B| + o(|B|),$$

so both averages behave the same asymptotically.

Definition 1.1.

$$P_\omega^W = \frac{1}{|\{v : v+W \subseteq B\}|} \sum_{v+W\subseteq B} \delta_{\omega_{v+W}} \in P(A^W)$$

is called the **W-empirical measure** of $\omega \in A^B$.

What are the possible limits of empirical measures, and what is the exponent function s for those? We will answer this as $W \uparrow \mathbb{Z}^d$ (after everything else). Here is the first observation: Suppose $W \subseteq W'$ and $\pi : A^{W'} \to A^W$ is the projection. Consider $\omega \in A^B$ and

$$\begin{aligned}
\pi_* P_\omega^{W'} &= \frac{1}{|\{v : v+W \subseteq B\}|} \sum_{v+W'\subseteq B} \pi_*\delta_{\omega_{v+W'}} \\
&= \frac{1}{|\{v : v+W \subseteq B\}|} \sum_{v+W'\subseteq B} \delta_{\omega_{v+W}} \\
&= P_\omega^W + O\left(\frac{|W'|}{\text{min side length}(B)}\right),
\end{aligned}$$

where the big O term is a bound on the total variation $\|\pi_* P_\omega^{W'} - P_\omega^W\|$.

This is an "approximate compatibility" of empirical measures. This means that we can look at $\mu \in P(A^{\mathbb{Z}^d})$ and a weak*-neighborhood of the form $U = \{\nu : \|(\pi_W)_*\nu - (\pi_W)_*\mu\| < \varepsilon\}$ for some $\varepsilon > 0$ and finite $W \subseteq \mathbb{Z}^d$. Then consider

$$s(U) = \lim_{B\uparrow\mathbb{Z}^d} \frac{1}{|B|} \log |\{\omega \in A^B : \|P_\omega^W - (\pi_W)_*\mu\}| < \varepsilon\}|.$$

This lets us define $s(U)$ for any weak* open set U of this form for some W. These are a base for the weak* topology on $P(A^{\mathbb{Z}^d})$. This will let us find a concave, upper semicontinuous exponent function $s : P(A^{\mathbb{Z}^d}) \to [-\infty, \infty)$.