Math 246B Lecture 1 Notes

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1 Harmonic Functions

1.1 Relationship to holomorphic functions

We will denote the complex plane as both $\mathbb{R}^2$ with coordinates x_1, x_2 and as $\mathbb{C}$ with complex coordinate $z = x_1 + ix_2$.

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ be open. We say that $u \in C^2(\Omega)$ is **harmonic** if $\Delta u = 0$ in Ω. Here, $\Omega = \partial_{x_1}^2 + \partial_{x_2}^2 = 4\partial_z \partial_{\overline{z}}$, where

$$\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}).$$

Proposition 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be simply connected, and let u be real and harmonic. Then $u = \operatorname{Re}(f)$, where $f \in \operatorname{Hol}(\Omega)$, the set of functions $f : \Omega \to \mathbb{C}$ that are holomorphic.*

Proof. Observe that $2\partial_z u$ is holomorphic. So there exists a $g \in \operatorname{Hol}(\Omega)$ such that $g' = \partial_z g = 2\partial_z u$. Then $\partial_z(g + \overline{g}) = 2\partial_z u$. Then $\partial_z(2\operatorname{Re}(g)) = 2\partial_z(2u)$, so $2\operatorname{Re}(g) = 2u + c$ with $c \in \mathbb{R}$. So $u = \operatorname{Re}(g - c)$. $\square$

Remark 1.1. It follows that $u \in C^\infty(\Omega)$ and even real analytic. That is, for any $a \in \Omega$, we have in a neighborhood of a that

$$u(x) = \sum_{j,k=0}^{\infty} c_{j,k}(x_1 - a_1)^j (x_2 - a_2)^k.$$

This is an absolutely convergent power series.

1.2 The Poisson formula and Poisson kernel

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be open with u harmonic in Ω. If the disc $\{x : |x-a| \leq R\} \subseteq \Omega$, then we have the Poisson formula:*

$$u(x) = \frac{1}{2\pi R} \int_{|y|=R} P_R(x - a, y) u(a + y)\, ds(y), \qquad |x - a| < R.$$

Here, $ds(y)$ is the arc length element along $|y| = R$, and

$$P_R(x,y) = \frac{R^2 - |x|^2}{|x-y|^2}, \qquad |x| < R, |y| = R.$$

Proof. We may assume $a = 0$. Now u is harmonic in $\{|x| < R_1\}$ for some $R_1 > R$. So $u = \mathrm{Re}(f)$, where f is holomorphic in $|z| < R$. Let $|z| < R$, $|w| = R$, and compute:

$$P_R(z,w) = \mathrm{Re}\left(\frac{w+z}{w-z}\right) = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{\overline{w+z}}{\overline{w} - \overline{z}}\right) = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2 + w\overline{z}}{R^2 - w\overline{z}}\right).$$

Set

$$\varphi_z(w) = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2 + w\overline{z}}{R^2 - w\overline{z}}\right).$$

If $0 < |z| < R$, then $\varphi_z(0) = 0$. Consider the function $\psi_z(w)$ sending $w \mapsto \varphi_z(w)f(w)/w$ for $|z| < R$.

1. If $0 < |z| < R$, then the singularity at $w = 0$ is removable and the only other singularity in the disc $|w| \leq R$ occurs when $w = z$. It is a simple pole with the residue equals $f(z)/z(1/2)2z = f(z)$.
2. If $z = 0$, $\psi_z(w) = f(w)/w$ has a simple pole at 0, and the residue equals $f(0)$.

For $|z| < R$ and $w = Re^{i\varphi}$, we get $ds(w) = |dw| = R\frac{dw}{iw}$. So we may write

$$\frac{1}{2\pi i}\int_{|w|=R} P_R(z,w)f(w)\,ds(w) = \frac{1}{2\pi i}\int_{|w|=R} \underbrace{P_R(z,w)\frac{f(w)}{w}}_{\psi_z(w)}\,dw = f(z)$$

by the residue theorem. Taking the real part, we get the result. □

Remark 1.2. We can write the Poisson formula as follows:

$$u(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{|Re^{i\tau} - re^{it}|^2} u(Re^{i\tau}\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} \tilde{P}_{R,r}(t-\tau)u(Re^{i\tau})\,d\tau,$$

where

$$\tilde{P}_{R,r}(t) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(t) + r^2}.$$

This is a convolution with the kernel $P_{R,r}(t)$. This function tends as $1/(R-r)$.

Proposition 1.2. *The Poisson kernel $P_R(x,y)$ has the following properties:*

1. $P_R(x,y) \geq 0$.
2. *$x \mapsto P_R(x,y)$ is harmonic for $|x| < R$, $|y| = R$.*

3. *For* $|x| < R$*,*

$$\frac{1}{2\pi R}\int_{|y|=R} P_R(x,y)\,ds(y) = 1.$$

4. *For all* $\varepsilon > 0$ *and* $\delta > 0$*, there exists* $R_1 < R$ *such that if* $|x-y| \geq \delta$ *and* $R_1 < |x| < R$*, then* $P_R(x,y) \leq \varepsilon$*.*

Proof. For the second property, observe that we expressed the Poisson kernel as the real part of a holomorphic function. For the third, apply the Poisson formula to the harmonic function 1. □

1.3 The Dirichlet problem in the disc

Using the Poisson kernel, we can solve the Dirichlet problem in the disc.

Theorem 1.2. *Let* $f \in C(\{x : |x| = R\}; \mathbb{R})$*. Then there exists a unique* $u \in C(\{|x| \leq \mathbb{R}\})$ *such that* $u = f$ *on* $|x| = R$ *and* u *is harmonic in* $|x| < R$*. The function* u *is given by*

$$u(x) = \frac{1}{2\pi R}\int_{|y|=R} P_R(x,y)f(y)\,ds(y), |x| < R.$$

Proof. Uniqueness: If u solves the problem, consider $u_\rho(x) = u(\rho(x)$ for $0 < \rho < 1$. The scaled function u_ρ is harmonic near $\{|x| \leq R\}$, so

$$u_\rho = \frac{1}{2\pi R}\int_{|y|=R} P_R(x,y)u_\rho(y)\,ds(y)$$

for $|x| < R$. Keep x fixed and let $\rho \to 1$. We get that

$$u(x) = \frac{1}{2\pi R}\int_{|y|=R} P_R(x,y)f(y)\,ds(y).$$

For existence, define

$$u(x) = \begin{cases} \frac{1}{2\pi R}\int_{|y|=R} P_R(x,y)f(y)\,ds(y) & |x| < R \\ f & x \in \partial D_R. \end{cases}$$

We will give more detail for this part of the proof next time. □

Remark 1.3. We can replace this continuous function f by many things, such as a measure.