Math 255A' Lecture 22 Notes

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1 Extension of Functional Calculus and Proof of The Spectral Theorem

1.1 Proof of the spectral theorem for compact operators

So far, we've constructed continuous functional calculus: a map $C([a,b]) \to \mathcal{B}_{\mathrm{sa}}(H)$ sending $f \mapsto f(T)$ which is

- linear,
- $f(g)(T) = f(T)g(T)$,
- $f \geq g \implies f(T) \geq g(T)$,
- $1(T) = I$,
- $\|f(T)\| \leq \|f\|_{\sup}$.

If $(f_n)_n$ is a sequence in $C([a,b])$ with $f_n \geq 0$ and $f_n(x) \downarrow g(x)$ for all $x \in [a,b]$, then we want to define $g(T)$ by $\langle g(T)x, y\rangle = \lim_n \langle f_n(T)x, y\rangle$. Last time, we showed that this limit exists (as a weak operator topology limit).

Lemma 1.1. *Suppose $f_n, f_n' \downarrow g$. Then the limit, $g(T)$, is the same.*

Proof. Let $f_n, f_n' \downarrow g$. For every x, $\varepsilon > 0$, and $n \in \mathbb{N}$, there exists an $n'(x,\varepsilon)$ such that $f_{n'}'(x) < g(x) + \varepsilon \leq f_n(x) + \varepsilon$. Then for each $n \in \mathbb{N}$, $\varepsilon > 0$ and x, we get $n'(n,x,\varepsilon)$ and a neighborhood $U(n,x,\varepsilon)$ of x such that $f_{u'}'|_{U_{n,x,\varepsilon}} < (f_n + \varepsilon)|_{U(n,x,\varepsilon)}$. Choose $x_1, \dots, x_t$ such that $\bigcup_{i=1}^t U(n, x_i, \varepsilon) = [a,b]$. Let $n'' = \max(n'(n,x_1,\varepsilon), \dots, n'(n,x_t,\varepsilon))$. Now $f_{n''}' < f_n + \varepsilon$ on $[a,b]$. Then $f_{n''}'(T) \leq f_n(T) + \varepsilon I$, so $\lim_{n''} f_{n''}'(T) \leq f_n(T) + \varepsilon$ for all n, ε. Since ε is arbitrary, and by symmetry, we get that $\lim f_n'(T) \leq \lim_n f_n(T)$ and $\lim f_n'(T) \geq \lim_n f_n(T)$. So the limits are equal. $\square$

Now, if we have $f_n \downarrow g \geq 0$, we get $g(T) \geq 0$. This is

- still additive: If $f_n \downarrow g$, $f'_n \downarrow g$;, then $f_n + f'_n \downarrow g + g'$. We have

$$(g+g')(T) = \mathrm{WO}\lim_n (f_n(T) + f'_n(T)) = g(T) + g'(T).$$

Lemma 1.2. *If $f_n \downarrow g \geq 0$, and $f'_n \downarrow g'_n \geq 0$, then*

$$(gg')(T) = g(T)g'(T).$$

Proof. We have $f_n f'_n \downarrow gg'$, so $(gg')(T) = \mathrm{WO}\lim_n (f_n f'_n)(T)$. We want to show that the is the product of the limits of $f_n(T)$ and $f'_n(T)$. By polarization, it is enough to show that $\lim_n \langle (f_n f'_n)Tx, x\rangle = \lim_n \lim_m \langle f_n(T) f'_m(T)x, x\rangle$. The limit of the diagonal terms is the same as $\lim_n \lim_m$ because the array is decreasing in n, m (a basic real analysis fact). $\square$

Given $\lambda\ in [a,b]$ and $n \in \mathbb{N}$, define

$$\varphi_n^\lambda(t) = \begin{cases} 1 & t \leq \lambda \\ -n(x - (\lambda + 1/n)) & \lambda < t \leq \lambda + 1/n \\ 0 & t > \lambda + 1/n \end{cases}$$

Then $\varphi_n^\lambda \downarrow \mathbb{1}_{(-\infty,\lambda]}$ as $n \to \infty$. Define $E(\lambda) := \lim_n \varphi_n^\lambda(T)$.

Here are the properties of $E(\lambda)$:

1. $E(\lambda)$ is self adjoint (as a WO limit of self-adjoints).
2. $E(\lambda) = \mathbb{1}_{(-\infty,\lambda]}(T) = \mathbb{1}^2_{(-\infty,\lambda]}(T) = E(\lambda)^2$.
3. If $\lambda \geq \mu$, then

   $$E(\mu)E(\lambda) = E(\lambda)E(\mu) = (\mathbb{1}_{(-\infty,\lambda]}\mathbb{1}_{(-\infty,\mu]})(T) = E(\mu).$$

4. Declare $E(\lambda) = 0$ if $\lambda < a$ and $E(b) = \lim_n 1(T) = I$.
5. Fix $\lambda \in [a,b]$. Then $E(\mu)x \to E(\lambda)x$ as $\mu \downarrow \lambda$ for all $x \in H$. Equivalently, $\langle (E(\mu) - E(\lambda))x, x\rangle \to 0$.

   To show this, we know $\langle E(\lambda)x, x\rangle = \lim_n \left\langle \varphi_n^\lambda(T)x, x\right\rangle$. Pick n large enough so that $\left\langle \varphi_n^\lambda(T)x, x\right\rangle < \langle E(\lambda)x, x\rangle + \varepsilon$. This is also $\lim_{\mu\downarrow\lambda} \langle \varphi_n^\mu(T)x, x\rangle$. So for μ close enough to λ, we get

   $$\langle E(\mu)x, x\rangle \leq \langle \varphi_n^\mu(T)x, x\rangle < \langle \varphi_n^\lambda(T)x, x\rangle + \varepsilon < \langle E(\lambda)x, x\rangle + 2\varepsilon.$$

This gives us a spectral family for T. If $a \leq \mu \leq \lambda \leq b$, then

$$E(\mu, \lambda] := E(\lambda) - E(\mu) = \mathrm{WO}\lim_n [\varphi_n^\lambda(T) - \varphi_n^\mu(T)].$$

This gives us

$$TE(\mu,\lambda] = \mathrm{WO}\lim_n T[\varphi_n^\lambda(T) - \varphi_n^\mu(T)] = \mathrm{WO}\lim_n [(t\cdot(\varphi_n^\lambda(t) - \varphi_n^\mu(t)))(T)].$$

Now

$$\mu \mathbb{1}_{(\mu+1/n,\lambda]} \leq t(\varphi_n^\lambda(t) - \varphi_n^\mu(\lambda)) \leq \lambda \mathbb{1}_{(\mu,\lambda+1/n]}$$

Taking the weak operator limit, we get

$$\mu E(\mu,\lambda) \leq TE(\mu,\lambda) \leq \lambda E(\mu,\lambda).$$

Now let $a = \lambda_0 < \lambda_1 < \cdots < \lambda_m = b$. Then

$$\begin{aligned} I &= E(B) \\ &= (E(\lambda_n) - E(\lambda_{n-1})) + \cdots + (E(\lambda_1) - E(\lambda_0)) \\ &= E(a,\lambda_1] + E(\lambda_1,\lambda_2] + \cdots + E(\lambda_{n-1}, b]. \end{aligned}$$

Multiplying by T, we get

$$T = TE(a,\lambda_1] + TE(\lambda_1,\lambda_2] + \cdots + TE(\lambda_{n-1}, b].$$

So we get

$$\sum_{i=1}^m \lambda_{i-1} E(\lambda_{i-1},\lambda_i] \leq T \leq \sum_{i=1}^n \lambda_i E(\lambda_{i-1},\lambda_i].$$

This gives

$$\sum_{i=1}^m \lambda_{i-1} \langle E(\lambda_{i-1},\lambda_i]x, x\rangle \leq \langle Tx, x\rangle \leq \sum_{i=1}^n \lambda_i \langle E(\lambda_{i-1},\lambda_i]x, x\rangle .$$

These are partial sums in the definition of the Riemann-Stieltjes integral. So taking the limit as $\max_i |\lambda_i - \lambda_{i-1}| \to 0$, we get

$$\langle Tx, x\rangle = \int \lambda \, d\langle E(\lambda)x, x\rangle .$$

This completes the proof of the spectral theorem.

1.2 Borel functional calculus and spectral measure

How far can we take this functional calculus? Here is another method which allows us to extend to all Borel functions. Assume we have a continuous functional calculus: $f \mapsto f(T)$ for all $f \in C([a,b])$. Given $x, y \in H$, consider

$$f \mapsto \langle f(T)x, y\rangle .$$

This is bounded by $|\langle f(T), x, y\rangle| \leq \|f\|_{\sup}\|x\|\|y\|$. So there exists some $\mu_{x,y} \in M([a,b])$ such that $\|\mu_{x,y}\| \leq \|x\|\|y\|$ and $\langle f(T)x, y\rangle = \int f\, d\mu_{x,y}$. So given g bounded and Borel, define

$$Q_g(x,y) := \int g\, d\mu_{x,y}.$$

This is bilinear in x, y and bounded: $|Q_g(x,y)| \leq \|g\|_\infty \|x\|\|y\|$. At each step, our construction is symmetric in x, y, so $Q_g(x,y)$ is symmetric in x, y. Now define $g(T)$ by $\langle g(T)x, y\rangle = Q_g(x,y)$. We can now define, as before, $\mathbb{1}_{(-\infty,\lambda]}(T)$.

The advantage of this method is that we can also define $E(A) := \mathbb{1}_A(T)$ for all $A \in \mathcal{B}([a,b])$. We can now show that

- Every $E(A)$ is a projection.
- $E(A \cap B) = E(A)E(B)$.
- $E(\varnothing) = 0$, and $E([a,b]) = I$.
- $E(\bigcup_n A_n) = \sum_n E(A_n)$.

This gives a **spectral measure**, which has the properties of a measure but takes values in projections. More advanced versions of the spectral theorem use this approach.