Math 279 Lecture 6 Notes

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1 Considerations for Integration Theories

1.1 Definition of Lyons' integral

We are interested in pairs of the form $\mathbf{x} = (x, \mathbb{X})$, where $x \in \mathcal{C}^\alpha$ (i.e. $x : [0,1] \to \mathbb{R}^\ell$ is Hölder of exponent α), and $\mathbb{X} : [0,T]^2 \to \mathbb{R}^{\ell \times \ell}$ (into the $\ell \times \ell$ real matrices) such that

$$x(s,t) = x(t) - x(s), \qquad \mathbb{X}(s,t) = \mathbb{X}(s,u)\mathbb{X}(u,t) + x(s,u) \otimes x(u,t),$$

which is **Chen's relation**. We write

$$\|\mathbf{x}\|_{\alpha,2\alpha} = |x(0)| + \underbrace{\sup_{s \neq t} \frac{|x(t) - x(s)|}{|t-s|}}_{[x]_\alpha} + \sup_{s \neq t} \frac{|\mathbb{X}(s,t)|}{|t-s|^{2\alpha}}.$$

We write $\mathscr{R}^\alpha = \{\mathbf{x} = (x, \mathbb{X}) : \|\mathbf{x}\|_{\alpha,2\alpha} < \infty, \text{Chen's relation holds}\}$. Last time, we proved the following theorem.

Theorem 1.1. *Assume $\alpha \in (1/3, 1/2]$. If $\mathbf{x} \in \mathscr{R}^\alpha$ and $F : \mathbb{R}^\ell \to \mathbb{R}^\ell \in \mathcal{C}^2$, then*

$$\int_0^t F(x) \cdot d\mathbf{x} := \lim_{\substack{|\pi| \to 0 \\ \pi = \{t_0 = 0 < t_1 < \cdots < t_{n+1} = t\}}} \sum_{i=0}^n [F(x(t_i)) \cdot x(t_i, t_{i+1}) + DF(x(t_i)) : \mathbb{X}(t_i, t_{i+1}),]$$

here if $A = [a_{i,j}]$ and $B = [b_{i,j}]$, then $A : B := \sum_{i,j} a_{i,j} b_{i,j}$, exists, and

$$\left| \int_s^t F(x) \cdot d\mathbf{x} - (F(x(s)) \cdot x(s,t) + DF(x(s)) : \mathbb{X}(s,t)) \right| \leq c_0(\alpha) \|F\|_{\mathcal{C}^2} \|\mathbf{x}\|_{\alpha,2\alpha}^2 |t-s|^{3\alpha}.$$

The way to think about $\mathbb{X}(s,t)$ is

$$(s,t) = \int_s^t \int_s^\theta dx(\theta') \otimes dx(\theta).$$

1.2 Remarks on integration theories

Remark 1.1. Write $\mathscr{R}^\alpha(x) = \{\mathbb{X} : (x, \mathbb{X}) \in \mathscr{R}^\alpha\}$, with $\alpha > 1/3$. Now if $\mathbb{X}, \mathbf{X}' \in \mathscr{R}^\alpha(x)$, then $W = \mathbb{X}' - \mathbb{X}$, and

$$W(s,u) + W(u,t) = W(s,t).$$

So if $W(t) := W(0,t)$, then we can write $W(s,t) = W(t) - W(s)$. Moreover,

$$\sup_{s \neq t} \frac{|W(t) - W(s)|}{|t-s|^{2\alpha}} < \infty.$$

So $W \in \mathcal{C}^{2\alpha}$. Thus, if $\mathbb{X}^0 \in \mathscr{R}^\alpha(x)$, then

$$\mathscr{R}^\alpha(x) = \{(\mathbb{X}^0(s,t) + W(t) - W(s) : s, t \in [0,T]) : W \in \mathcal{C}^{2\alpha}.$$

In particular, if $\alpha > 1/2$, $\mathscr{R}^\alpha(x)$ consists of one element.

Remark 1.2. To generalize our theorem, we define the following function space: Given $x \in \mathcal{C}^\alpha$, let $\mathscr{G}^\alpha(x)$ be the set of pairs $(y, \widehat{y})$ with the following properties:

- $y : [0,T] \to \mathbb{R}^{d \times \ell}$,
- $y \in \mathcal{C}^\alpha$ (could be $\mathcal{C}^\beta$ also),
- $\widehat{y} : [0,T] \to \mathbb{R}^{d \times \ell \times \ell}$,
- $\widehat{y} \in C^\alpha$,
- $$\|(y, \widehat{y})\|_{\alpha, 2\alpha} = [y]_\alpha + [\widehat{y}]_\alpha + \sup_{s \neq t} \frac{|y(t) - y(s) - \widehat{y}(s) : (x(t) - x(s))|}{|t-s|^{2\alpha}} < \infty.$$

For example, if $F : \mathbb{R}^\ell \to \mathbb{R}^\ell \in \mathcal{C}^2$ and $x : [0,T] \to \mathbb{R}^\ell$, then $(y, \widehat{y}) = (F(x), DF(x)) \in \mathscr{G}(x)$. We call $\mathscr{G}(x)$ the Gubinelli class of x.

With an identical proof we can show this: If $x = (x, \mathbb{X}) \in \mathscr{R}^\alpha$ and $y = (y, \widehat{y}) \in \mathcal{G}\alpha(x)$, then

$$\int_0^t \mathbf{y} \cdot d\mathbf{x} := \lim_{\substack{|\pi| \to 0 \\ \pi = \{t_0 = 0 < t_1 < \cdots < t_{n+1} = t\}}} \left[\sum_i y(t_i) x(t_i, t_{i+1}) + \widehat{y}(t_i) : \mathbb{X}(t_i, t_{i+1}) \right].$$

The analogue of the bound in the theorem also holds, provided that $\|F\|_{\mathcal{C}^2}$ is replaced with $\|\mathbf{y}\|_{\alpha, 2\alpha}$.

Remark 1.3. If $\alpha > 1/2$, then

$$\int_0^t \mathbf{y} \cdot d\mathbf{x} =: \int_0^t y \cdot dx$$

because in the above limit definition of the integral, the contribution from $\sum_i \widehat{y}(t_i) : (t_i, t_{i+1})$ is 0, so we can drop it. If this is the case, we refer to it as a **Young integral**.

Remark 1.4. Suppose $\mathbb{X}^0 \in \mathscr{R}^\alpha(x)$, and let $W \in \mathcal{C}^{2\alpha}$ with $\mathbb{X}(s,t) = \mathbb{X}^0(s,t) + W(t) - W(s)$. Now

$$\int_0^t (y, \widehat{y}) \cdot d(x, \mathbb{X}) = \int_0^t (y, \widehat{y}) \cdot d(x, \mathbb{X}^0) + \underbrace{\int_0^t \widehat{y} : dW}_{\text{Young integral}} .$$

Remark 1.5. We say $\mathbf{x} = (x, \mathbb{X})\imath\mathscr{R}_g^\alpha$, i.e. $\mathbf{x}$ is **(weakly) geometric**, if

$$\mathbb{X}(s,t) + \mathbb{X}^*(s,t) = x(s,t) \otimes x(s,t),$$

Equivalently, we can say

$$\mathbb{X}_{i,j}(s,t) + \mathbb{X}_{j,i}(s,t) = x_i(s,t) x_j(s,t),$$

where $\mathbb{X}_{i,j} = \int_s^t x_i \, dx_j - x_i(s)(x_j(t) - x_j(s))$. Hence, if $\mathscr{R}_g^\alpha(x) = \{\mathbb{X} : (x, \mathbb{X}) \in \mathscr{R}_g^\alpha\}$, then the symmetric part of $\mathbb{X}$ is uniquely determined. Hence if $\mathbb{X}^0 \in \mathscr{R}_g^\alpha$, then

$$\mathscr{R}_g^\alpha(x) = \{\mathbb{X}^0(s,t) + W(t) - W(s) : W \in \mathcal{C}^{2\alpha}, W^* = -W\}.$$

Now consider the corresponding integral:

$$\int_0^t F(x) \cdot d(x, \mathbb{X}) = \int_0^t F(x) \cdot d(x, \mathbb{X}^0) + \int_0^t DF(\mathbf{x}) : dW.$$

Example 1.1. Take any $\mathbf{x} = (x, \mathbb{X}) \in \mathscr{R}^\alpha$, and pick any 1-periodic function $f : [0,1] \to \mathbb{R}^\ell$. If $y_n(t) = n^{-1/2} f(nt)$, then $\mathbf{y} \to 0$. Now consider $x_n = x + y_n \xrightarrow{n \to \infty} x$. Then one can show that the norm is uniformly bounded. Define

$$\mathbb{X}_n = \mathbb{X} + \underbrace{\int_s^t (y_n(\theta) - y_n(s)) \otimes \, dy_n(\theta)}_{\text{classical integral}} \to \mathbb{X} + (t-s)C,$$

where C is an antisymmetric matrix.

Remark 1.6. Start from $(\mathscr{R}^\alpha, \| \cdot \|_{\alpha, 2\alpha})$. Let us define

$$\mathcal{C}_\infty = \left\{ (x, \mathbb{X}) : x \in C^1, \mathbb{X} \text{ is defined by } X(s,t) = \int_s^t (x(\theta) - x(s)) \otimes \, dx(\theta) \right\},$$

where $\int_s^t (x(\theta) - x(s)) \otimes dx(\theta)$ is a classical integral. Write $\mathscr{R}_{\text{sg}}^\alpha$ to be the closure of $\mathcal{C}_\infty$ with respect to $\|\cdot\|_{\alpha,2\alpha}$. It is not hard to see[1] $\mathscr{R}_{\text{sg}} \subsetneq \mathscr{R}_g^\alpha$. This has to do with the fact that $\mathcal{C}^\alpha$ is not topologically separable. In fact, what is the closure of the set of smooth functions with respect to $\|\cdot\|_\alpha$? The closure is exactly the set of $x : [0,T] \to \mathbb{R}^d$ such that

$$\lim_{\varepsilon \to 0} \underbrace{\sup_{\substack{|s-t|<\varepsilon \\ s \neq t}} \frac{|x(t) - x(s)|}{|t-s|^\alpha}}_{\psi(\varepsilon)} = 0.$$

[1] Not my words.