Math 206A Lecture Notes

Professor: Igor Pak
Scribe: Daniel Raban

Contents

1 Overview of Course Topics

1.1 The Borsuk conjecture

Theorem 1.1 (Borsuk conjecture)**.** *Let $P \subseteq \mathbb{R}^d$ be a compact, convex body. Then* $P = \bigcup_{i=1}^{d+1} P_i$ *such that* $\operatorname{diam}(P_i) < \operatorname{diam}(P)$.

Why does this make sense?

Theorem 1.2 (Borsuk)**.** *This is true for $d = 2$. Moreover, there exists some $\varepsilon > 0$ such that* $\operatorname{diam}(P_i) < (1 - \varepsilon) \operatorname{diam}(P)$.

Try this out with a hexagon, and split it in three parts to get some intuition. We will actually prove this. However, we will not prove the following:

Theorem 1.3. *The Borsuk conjecture is true for $d = 3$.*

The Borsul conjecture is actually not always true. We will prove the following.

Theorem 1.4 (Kahn-Kalai,1990)**.** *The Borsuk conjecture is false for $d > 2200$.*

The proof uses linear algebra methods in extremal combinatorics.

1.2 Convex polytopes

Let $P \subseteq \mathbb{R}^d$ be a convex polytope, and let $f_i(P)$ be the number of i-dimensional faces. What can be said about $(f_0, f_1, f_2, \dots, f_{d-1})$?

Example 1.1. For $d = 2$, a pentagon has vector $(5, 5)$.

Example 1.2. For $d = 3$, if we have 5 vertices, what vectors can we have? We can have $(5, 9, 6)$ (for a slice of cake shape) and $(5, 8, 5)$ (for a square pyramid shape).

In dimensions $d \geq 4$, we do not have a full picture of what is going on.

Theorem 1.5 (conjecture)**.** *There does not exist $P \subseteq \mathbb{R}^4$ with f-vector $(n, 10n, 10n, n)$.*

Definition 1.1. P is **simplicial** if every face is a simplex.

Theorem 1.6 (D-S)**.** *There exist $\lfloor n/2 \rfloor$ linear relations on f-vectors of simplicial polytopes in $\mathbb{R}^n$.*

Later, we will prove an inequality relating f_2, f_1, and f_0.

1.3 Rigidity

Here is a question. Let E be the edges of an icosahedron, and suppose $f : E \to \mathbb{R}_+$ such that $|f(e) - 1| < 1/100$. Does there exist a "perturbed icosahedron" with edge lengths $\{f(e)\}$? The answer is yes, due to a theorem of Dehn[1] from about 1912. In fact, this is true for every simplicial polytope.

1.4 Combinatorial geometry of curves

Let Q be an equilateral convex polygon (all sides have the same unit length).

Example 1.3. For quadrilaterals, we can have a rhombus or a square.

Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be angles of Q. We know that $\sum \alpha_i = (n-2)\pi$.

Theorem 1.7 (4 vertex theorem for polygonal curves)**.** *There exist at least 4 sign changes in* $(\alpha_{i+1} - \alpha_i)$.

We will see a geometric proof of this, and we will provide a combinatorial proof for the result in 3 dimensions. It actually gets simpler!

This actually implies the following theorem about smooth curves:

Theorem 1.8. *The curvature of a smooth, closed curve changes sign at least 4 times.*

[1] Dehn was a student of Hilbert.

2 Helly's Theorem

2.1 Proof of Helly's theorem

Theorem 2.1 (Helly)**.** *Suppose $X_1, \ldots, X_n \subseteq \mathbb{R}^d$ are convex sets such that $X_I \neq \varnothing$ for all $|I| = d+1$, where $X_I = \bigcap_{i \in I} X_i$. Then $X_1 \cap \cdots \cap X_n \neq \varnothing$.*

When $d = 1$, we have a collection of intervals where every pair of intervals intersect; then all intervals intersect. In this case the proof is elementary. Take the largest left endpoint a^* and the smallest right endpoint b^* of one of the intervals. Then $a^* < b^*$, so a point between a^* and b^* is contained in all the intervals.

However, when $d = 2$, the result is a little less obvious.

Proof. Let's prove the theorem for $d = 2$, $n = 4$. Let $J = \{1, \ldots, n\}$. Let $y_i \in X_{J \setminus \{i\}}$. Either one of the y_i lies in the triangle formed by the three others or the y_i form a convex shape. In the first case, without loss of generality, $y_4 \in X_1 \cap X_2 \cap X_3$. But if $y_1, y_2, y_3 \in X_4$, then $y_4 \in X_4$. In the second case, find the point z at the intersection of the line segments connecting y_1 to y_3 and y_2 to y_4. Then $z \in X_2 \cap X_4$, and $z \in X_1 \cap X_3$. So $z \in X_J$.

Now proceed by induction on n. Why does n imply $n+1$? The proof is the same, except we just include the points $y_i \in X_5, X_6, \ldots$. So in the first case, we just ignore the extra points, we get

$$z \in (X_2 \cap X_4 \cap X_5 \cap \cdots \cap X_{n+1}) \cap (X_1 \cap X_3 \cap X_5 \cap \cdots \cap X_{n+1}) = X_J$$

for the second case.

Before we prove the general case, we will state a lemma. □

Lemma 2.1 (Radon)**.** *Let $y_1, \ldots, y_m \in \mathbb{R}^d$, where $m \geq d+2$. Then there exist $I, I' \neq \varnothing$ such that $I \cap I' = \varnothing$ and the convex hull of $\{y_i : i \in I\}$ intersects the convex hull of $\{y_j : j \in I'\}$.*

Proof. Let $y_i = (y_{i_1}, \ldots, y_{i,d}) \in \mathbb{R}^d$ with $i = 1, \ldots, m$, $m \geq d+2$. Consider the system of equations $\sum_{i=1}^m \tau_i = 0$ and $\sum_{i=1}^m \tau_i y_{i,j} = 0$ for $j \in \{1, \ldots, d\}$. These are $d+1$ equations. So there exist $(\tau_1, \ldots, \tau_m) \neq 0$ which satisfies the system. Let $I = \{i : \tau_i > 0\}$ and $I' = \{i : \tau_i < 0\}$. Then

$$\frac{\sum_{i \in I} \tau_i y_i}{c} = \frac{\sum_{j \in I'} (-\tau_j) y_j}{c},$$

where $c = \sum_{i \in I} \tau_i = \sum_{j \in I'} -\tau_j$. □

Now we can prove the general case of Helly's theorem.

Proof. For general d, we induct on n. The base case is $n = d+1$. By the lemma, we get $z \in X_r$, where $r \notin I$, and $z \in X_s$, where $s \notin I'$. So $z \in X_J$. □

2.2 Applications of Helly's theorem

Corollary 2.1. *Let $R_1, \ldots, R_n \subseteq \mathbb{R}^2$ be axis-parallel rectangles. Suppose $R_i \cap R_j \neq \varnothing$ for all i, j. Then $R_1 \cap \cdots \cap R_n \neq \varnothing$.*

We could have proved this like we proved the case of $d = 1$ because the intersection of rectangles is the pair of intersections of the corresponding intervals.

Corollary 2.2. *Let $A \subseteq \mathbb{R}^2$ be a fixed convex set, and let $X_1, \ldots, X_n \subseteq \mathbb{R}^2$ be convex sets such that for $|I| = 3$, there exists some $c \in \mathbb{R}^2$ such that $X_i \cap (A + c) \neq \varnothing$ for all $i \in I$. Then there exists some $c \in \mathbb{R}^2$ such that $X_i \cap (A + c) \neq \varnothing$ for all $i \in \{1, \ldots, n\}$.*

This says that if there is some translation where A intersects some of the X_i there is some translation where A intersects all of them.

Proof. Pick some point in $a \in A$, and look at all A translated by the extreme points of X_i. Let $\hat{X}_i$ be the convex hull of the translated copies of a. Then $\hat{X}_i$ is convex, so $\hat{X}_I \neq \varnothing$ for all $|I| = 3$. By Helly's theorem, $\hat{X}_J \neq \varnothing$, which completes the proof. $\square$

Remark 2.1. If we take A to be a point, we get the original statement of Helly's theorem for $d = 2$.

3 Generalized Helly's Theorem and Borsuk's Theorem

3.1 Corollaries of generalized Helly's theorem

We will show that Helly's theorem implies Borsuk's theorem in 2 dimensions.

Theorem 3.1 (generalized Helly)**.** *Suppose $X_1, \ldots, X_n, A$ are convex such that for all $i, j, k \in [n]$, X_i, X_j, X_k intersect some parallel translation of A. Then all X_i intersect some parallel translation of A.*

Corollary 3.1. *Let $z_1, \ldots, z_n \in \mathbb{R}^2$ be such that for all i, j, k, z_i, z_j, z_k lie in some circle of radius 1. Then all z_i lie in some circle of radius 1.*

Proof. In generalized Helly, let X_i be $\{z_i\}$ and A be the circle of radius of 1. □

Lemma 3.1. *Let $x, y, z \in \mathbb{R}^2$ be such that $|xy|, |xz|, |yz| \leq 1$. Then there exists a circle of radius $1/\sqrt{3}$ which covers them.*

Proof. There are 2 cases: ether the triangle xyz is acute or it is right or obtuse. If xyz is right or obtuse, then take R to be the midpoint of the longest edge. Then the points are contained in the circle of radius $1/2$ centered at R. If the triangle xyz is acute, let R be the center of the circle circumscribing the triangle. Let α be the angle xRz. Then $\alpha \geq 2\pi/3$. Then, since $|xz| \leq 1$, $|Rx| = |Rz| \leq 1/\sqrt{3}$. □

Corollary 3.2. *For every $Z = \{z_1, \ldots, z_n\} \subseteq \mathbb{R}^2$, if $|z_i z_j| \leq 1$ for all i, j, then there exists a circle of radius $1/\sqrt{3}$ which covers Z.*

Proof. Use the previous corollary and lemma. □

3.2 Borsuk's theorem in 2 dimensions

Theorem 3.2 (Borsuk, d=2)**.** *Suppose $X \subseteq \mathbb{R}^2$ is convex of diameter 1. Then $X = X_1 \cup X_2 \cup X_3$ such that* $\operatorname{diam}(X_i) < 1$.

We will start with an incorrect proof and then fix it in 3 places.

Proof. Put X in a circle of radius $1/\sqrt{3}$, and split the circle into 3 parts. Then the diameter of the circle is less than 1.

Error 1: Split a triangle into 3 parts. In this case, we have pieces with diameter ≤ 1, not < 1.

Error 2: The lemma needed a finite set, and X is infinite.

To fix error 1, take the circle and mark out two small regions on opposite sides of the circle. Then X cannot contain points of both sides at once. So X can still be covered by the truncated circle where we remove one of these regions. Now alter the partition by moving it $\varepsilon = 1/100$ away from the removed region. Then the diameters of the 3 parts of the circle are now all < 1.

To fix error 2, place X inside a polygon Q with diameter $1 + \delta$.

Here is error 3: We could have an issue in the fix of error 2 where we end up with pieces of diameter ≥ 1 because of the δ we added. However, if we take $\delta = 1/200$, we can avoid this situation because the pieces are not too big. In other words, the fixes of error 1 and error 2 must interact in some way. $\square$

3.3 Hadwiger's theorem

Borsuk's conjecture is false in general, but here is a result which says it is morally true.

Theorem 3.3 (Hadwiger)**.** *Borsuk's conjecture holds for smooth convex bodies.*

Proof. Step 1: (in $\mathbb{R}^3$ for simplicity) This is true when X is a ball of radius $1/2$. Inscribe a regular tetrahedron into the ball, and take a cone over each face. We can then partition the sphere into 4 cones with diameter < 1. So $X = C_1 \cup C_2 \cup C_3 \cup C_4$, which are cones from 0 over facets of the simplex.

Step 2: Since X is smooth, there is a tangent plane at every point on the boundary, giving us a normal vector at every point. Here is a lemma: If $\operatorname{diam}(X) = 1$, then for $|xy| = 1$, n_x and n_y are parallel. Now define $\gamma : \partial X \to S^{d-1}$ which takes $x \mapsto n_x$, the normal vector. Now let $Y_i = \gamma^{-1}(X_i)$. Then the partition from step 1 gives us a suitable partition of X. $\square$

What this theorem gives us is that for Borsuk's theorem to fail, we should really be looking at polytopes. So we only care about finitely many points, the extreme points of a convex polytope.

4 Bárány's Theorem and Equipartition

4.1 Statement of Bárány's theorem

Theorem 4.1 (Bárány)**.** *For every d, there exists a constant $\alpha_d > 0$ such that for every $Z = \{z_1, \ldots, z_n\} \subseteq \mathbb{R}^d$, there exists $x \in \mathbb{R}^d$ such that $x \in \mathrm{Con}(Z_I)$, $|I| = d+1$ for at least $\alpha_d \binom{n}{d+1}$ subsets I.*

What is this saying? In $d = 2$, there is some point that lies in a constant proportion of all the subsets you can make as the convex hulls of 3 points. In $d = 1$, we can do this by picking the middle of the z_i. Then x is contained in $(n/2)^2 \sim \binom{n}{2}/2$ of the Z_I.

4.2 Equipartition

Theorem 4.2. *Suppose $Q \subseteq \mathbb{R}^2$ is a convex polgyon. Then there exist perpendicular lines ℓ_1, ℓ_2 that partition Q into 4 parts of equal area.*

Proof. Fix a line ℓ in the plane, and consider ℓ_1 parallel to ℓ such that the area of Q_+ and Q_- are the same. Do the same with ℓ_2 perpendicular to ℓ. The diagonal pieces (when Q is split into 4) have the same area, but we may have adjacent areas $a \neq b$. Take this construction, and rotate ℓ up to $\pi/2$. There exists a rotation θ such that $a_\theta = b_\theta$. $\square$

Theorem 4.3. *Let $Q \subseteq \mathbb{R}^2$ be a convex polygon. Then there exist ℓ_1, ℓ_2, ℓ_3 that intersect at 1 point such that Q is partitioned into 6 parts of equal area.*

Proof. Fix $\ell \subseteq \mathbb{R}^2$ be a line that splits Q into two parts of equal area. Pick x on the line, and let 4 rays pass out of it. We rotate ℓ and the rays separately. Let ℓ_θ be the rotation of ℓ be θ, where $\theta \in [0, \pi]$. Let β_θ be the angle between the actual ray and the extension of the opposite ray. By convexity, the point x is uniquely determined by the rays. $\square$

Corollary 4.1. *For all $Z = \{z_1, \ldots, z_{6k}\} \subseteq \mathbb{R}^2$ with no 3 points on the same line, there exist lines ℓ_1, ℓ_2, ℓ_3 which separate Z into 6 groups of equal size.*

Proof. The same proof works. $\square$

Theorem 4.4 (Boros-Füredi)**.** *For every $Z = \{z_1, \ldots, z_{6k}\} \subseteq \mathbb{R}^2$ with no 3 points on the same line, there exists $x \in \mathbb{R}^2$ such that x is in at least $8k^3$ triangles $z_i z_j z_r$.*

Note that $\binom{6k}{3} \sim 36k^3$, so $\alpha_2 \geq 8/36 = 2/9$.

Proof. Let $x, \ell_1, \ell_2, \ell_3$ be as given by the previous corollary. Note that if you take 3 points from every other portion of the 6 portions of the plane, x is in their convex hull (a triangle). This gives us $2k^3$ triangles. Now, if we pick two points in opposite portions, there are 2 portions (on the side) where picking a point in them will make x in the convex hull of the 3 points. So we get $3k^2 \cdot (2k) = 6k^3$ more triangles. $\square$

These authors claimed that 2/9 was optimal, but their proof had a mistake in it. The result was true, but this was not corected until about 30 years later by Bukh.

5 Caratheòdory's Theorems and Weak Tverberg's Theorem

5.1 Geometric theorems of Caratheòdory

Theorem 5.1 (Bárány). *For every d, there exists a constant $\alpha_d > 0$ such that for every $Z = \{z_1, \ldots, z_n\} \subseteq \mathbb{R}^d$, there exists $x \in \mathbb{R}^d$ such that $x \in \mathrm{Conv}(Z_I)$, $|I| = d+1$ for at least $\alpha_d \binom{n}{d+1}$ subsets I.*

To prove this, we'll need some lemmas, all of which are interesting in their own right.

Theorem 5.2 (Caratheòdory). *Let $Z = \{z_1, \ldots, z_n\} \subseteq \mathbb{R}^d$ with $x \in \mathrm{Conv}(Z)$. Then there exists $I \subseteq [n]$ with $|I| = d+1$ such that $x \in \mathrm{Conv}(Z_I)$.*

Proof. By induction. Fix a vertex v, and use induction to triangulate all facets. Take cones over all simplices in the facets. □

Here is a result which uses an analogue of infinite descent, but in geometry.

Theorem 5.3 (Galloi-Sylvester). *For all $X = \{x_1, \ldots, x_n\}$ with the x_i not all on a line, there exist i, j such that the line $(x_i x_j)$ has no other x_r.*

Proof. Let

$$\gamma := \min_{(r,i,j) \text{ distinct}} \mathrm{dist}(x_r, (x_i x_j)).$$

Proceed by contradiction. □

Theorem 5.4 (colorful Caratheòdory). *Let $X_1, \ldots, X_{d+1} \subseteq \mathbb{R}^d$ be finite sets with $0 \in \mathrm{Conv}(X_i)$ for all i. Then there exist $x_1 \in X_1, x_2 \in X_2, \ldots, x_{d+1} \in X_{d+1}$ such that $0 \in \mathrm{Conv}(\{x_1, \ldots, x_{d+1}\})$.*

Proof. By contradiction. Let γ be the minimum distance between a colorful simplex and the origin, where the colorful simplexes are the ones formed by x_i. Note that $\gamma > 0$. Let u minimize this distance. The hyperplane H which contains u contains all the x_i except x_1 (wlog). Then there exists $x' \in X_1$ on the other side of H from x_1, otherwise we could not have $0 \in \mathrm{Conv}(X_1)$. Then the distance between 0 and x'_1 is smaller than γ, which is a contradiction.

If $u = x_2$ (instead of lying on a facet, it lies on a corner), then there exist $x'_i, i \neq 2$ on the other side of the perpendicular hyperplane separating 0 and x_2. Then the distance to the convex hull of $\{x_2\} \cup \{x'_i : i \neq 2\}$ is smaller than γ, which is a contradiction. □

5.2 Weak Tverberg's theorem

Theorem 5.5 (weak Tverberg). *Let $r, d \in \mathbb{N}$. For every $n \geq (r-1)(d+1)^2 + 1$ and $x_1, \ldots, x_n \in \mathbb{R}^d$, there exist $I_1, \ldots, I_r \subseteq [n]$ with $I_i \cap I_j = \varnothing$ such that $\bigcap_{i=1}^r \mathrm{Conv}(X_{I_i}) \neq \varnothing$.*

Proof. Let $k := (r-1)(d+1)$ and $s := n-k$. Observe that every $(d+1)$ subsets of size s have a common point; this is because $k(d+1) < k(d+1)+1 = n$. By Helly's theorem, there exists $z \in \mathbb{R}^d$ such that $z \in \operatorname{Conv}(X_I)$ for all $|I| \geq s$. So $z \in \operatorname{Conv}(X)$, so by Carathèodory's theorem, there is some $Y_1 \subseteq X$ with $|Y_1| = d+1$ such that $z \in \operatorname{Conv}(Y_1)$. Then $z \in \operatorname{Conv}(X \setminus Y_1)$, so we can get $Y_2 \subseteq X \setminus Y_1$ such that $|Y_2| = d+1$ and $z \in \operatorname{Conv}(Y_2)$. Continue this to get $I_1 = Y_1, \ldots, I_r = Y_r$, which is what we wanted. $\square$

Remark 5.1. The actual Tverberg's theorem is the same but without the power of 2 on the $d+1$ term.

6 Bárány's Theorem and Fractional Helly's Theorem

6.1 Proof of Bárány's theorem

We are now ready to prove Bárány's theorem.

Theorem 6.1 (Bárány). *For every d, there exists a constant $\alpha_d > 0$ such that for every $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$, there exists $z \in \mathbb{R}^d$ such that $z \in \mathrm{Con}(Z_I)$, $|I| = d+1$ for at least $\alpha_d \binom{n}{d+1}$ subsets I.*

Recall the two theorems we proved last time.

Theorem 6.2 (colorful Carathèodory). *Let $X_1, \ldots, X_{d+1} \subseteq \mathbb{R}^d$ be finite sets with $0 \in \mathrm{Conv}(X_i)$ for all i. Then there exist $x_1 \in X_1, x_2 \in X_2, \ldots, x_{d+1} \in X_{d+1}$ such that $0 \in \mathrm{Conv}(\{x_1, \ldots, x_{d+1}\})$.*

Theorem 6.3 (weak Tverberg). *Let $r, d \in \mathbb{N}$. For every $n \geq (r-1)(d+1)^2 + 1$ and $x_1, \ldots, x_n \in \mathbb{R}^d$, there exist $I_1, \ldots, I_r \subseteq [n]$ with $I_i \cap I_j = \varnothing$ such that $\bigcap_{i=1}^r \mathrm{Conv}(X_{I_i}) \neq \varnothing$.*

We will show these two imply Bárány's theorem.

Proof. Choose $r = \lfloor n/(d+1)^2 \rfloor$. By weak Tverberg, there exist $X_1, \ldots, X_r \subseteq X$ such that $\bigcap \mathrm{Conv}(X_i) \neq \varnothing$. Let $z \in \bigcap \mathrm{Conv}(X_i) \neq \varnothing$. By colorful Carathéodory, for all $(d+1)$-subsets of $[r]$, there exists a colorful simplex Δ which contains z. The number of such simplices is

$$\#\Delta = \binom{r}{d+1} = \binom{n/(d+1)^2}{d+1}.$$

Use the fact that $\binom{n}{k} > \frac{(n-k)!}{k!}$. Then

$$\#\Delta > \alpha_d \binom{n}{d+1}.$$

You can check that $\alpha_d \approx 1/d^d$. □

6.2 Fractional Helly's theorem

Theorem 6.4 (fractional Helly). *Fix $d, \alpha > 0$. Let $X_1, \ldots, X_n \subseteq \mathbb{R}^d$ be convex sets such that at least $\alpha\binom{n}{d+1}$ of $(d+1)$-element sets $I \subseteq [n]$ have nonempty X_I. Then there exists $J \subseteq [n]$ such that $|J| > \alpha n/(d+1)$ and $X_J \neq \varnothing$.*

Lemma 6.1. *Without loss of generality, one can assume all X_i are convex polytopes.*

Proof. Replace each X_i with Y_i, where $Y_i = \mathrm{Conv}(\{y_I : i \in I\})$, where $y_I \in \bigcap_{i \in I} X_i$. This does not change any of the desired properties of the X_i. □

Definition 6.1. A **Morse function** $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a linear function which is nonconstant on edges of the Y_i.

Lemma 6.2. *Let $I \subseteq [n]$, $Y_I \neq \varnothing$, and $v = \min_\varphi(Y_I)$. Then there exists $J \subseteq I$ such that $|J| \leq d$ and $v = \min_\varphi(Y_J)$.*

Proof. Apply the contrapositive of Helly's theorem where one of the subsets is the half space $H_- = \phi^{-1}((-\infty, \phi(v))$ and the other subsets are Y_i with $i \in I$. Then $\bigcap Y_i \cap H_- = \varnothing$, so the contrapositive of Helly's theorem gives $J \subseteq I$ such that $|J| \leq d$ and $\bigcap_{j \in J} Y_j \cap H_- = \varnothing$. □

We can now prove the theorem.

Proof. We have $\gamma : I \mapsto J$. Consider $I \subseteq [n]$ with $|I| = d + 1$. From lemma 2, there exists some $J_0 \subseteq [n]$ with $|J_0| = d$ such that $J_0 = \gamma(I)$ for at least $\alpha\binom{n}{d+1}/\binom{n}{d} = \alpha\frac{n-d}{d+1}$ different I. Let $v = \min_\varphi(Y_{J_0})$. Thus, there exist at least $\alpha\frac{n-d}{d+1}$ $i \in I \setminus J_0$ such that $v \in Y_i$. So v is in at least $|J_0| + \alpha\frac{n-d}{d+1} = d + \alpha\frac{n-d}{d+1} > \alpha n/(d+1)$ convex subsets Y_i. □

Remark 6.1. The optimal bound is $1 - (1-\alpha)^{1/(d+1)}$ instead of $\alpha n/(d+1)$.

7 Borsuk's Conjecture and the Kahn-Kalai Theorem

7.1 Borsuk's conjecture

Here is Borsuk's conjecture.

Theorem 7.1 (Borsuk)**.** *For all convex $X \subseteq \mathbb{R}^d$, there exists a decomposition $X = \bigcup_{i=1}^{d+1} X_i$ such that* $\text{diam}(X_i) < \text{diam}(X)$.

Borsuk showed that this holds for $d = 2$, and it was later shown that this holds in $d = 3$. However, the conjecture is false.

Theorem 7.2 (Kahn-Kalai,1993)**.** *For all $d > 2000$, there exists $X \subseteq \mathbb{R}^d$ such that for all $X = \bigcup_{i=1}^{N} X_i$, $\text{diam}(X_i) < \text{diam}(X) \implies N > c^{\sqrt{d}}$ for some $c > 1$.*

We will prove this. First, let us prove a theorem.

Theorem 7.3 (Pál)**.** *Let X be the unit ball. Then the minimum number of compact sets in the decomposition is $d + 1$.*

Proof. We have already shown that $N \leq d + 1$. We need to show that $N > d$. Look at proposition 3.4 in the textbook. The general proof uses the Borsuk-Ulam theorem from topology. □

7.2 Proof of the Kahn-Kalai theorem

Let's now prove the Kahn-Kalai theorem, which refutes Borsuk's conjecture in general. There have a sequence of simplifications by K-K, Alon[2], Aigner-Zieglar, then Skopenkov. We will see the Skopenkov version of the proof.

Proof. Let $M = \{(x_1, \ldots, x_n) \in \mathbb{R}^N : x_i \in \{\pm 1\}, x_1 = 1, x_2 \cdots x_n = 1\}$. Then $|M| = 2^{n-2}$. Let $f : M \to \mathbb{R}^{n^2}$ be $F(x_1, \ldots, x_n) = (x_i \cdots x_j)_{1 \leq i,j \leq n}$. So we take a vector and get a matrix. For example,

$$F(1, -1, -1) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

The construction is $F(M) \to X$. The idea is we can't separate these 2^{n-2} points in X. We need a few lemmas. □

Lemma 7.1. *For $x_i, y_i \in M$,*

$$(x_i x_j - y_i y_j)^2 = (1 - x_i x_j y_i y_j)^2$$

[2]Alon published on a pseudoynm: Nilli, the name of his daughter.

Proof.
$$(x_i x_j - y_i y_j)^2 = (x_i x_j)^2 (1 - x_i^{-1} x_j^{-1} y_i y_j)^2 = (1 - x_i x_j y_i y_j)^2. \qquad \square$$

Let's continue with our proof of the Kahn-Kalai theorem.

Proof. Let $n - a$ be the Hamming distance $(\overline{x}, \overline{y})$ i.e. a is the number of i such that $x_i = y_i$. This is the number of $x_i y_i$ that equal 1. So

$$\begin{aligned} d(f(\overline{x})f(\overline{y}))^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i x_j - y_i y_j)^2 \\ &= \sum_{i} \sum_{j} (1 - x_i y_i x_j y_j) \\ &= 8a(n-a) \end{aligned}$$

This is maximized at $a = n/2$, which is equivalent to $\overline{xy} = 0$. We will continue this next time. $\square$

8 Linear Algebra Methods and the Kahn-Kalai Theorem

8.1 Linear algebra methods

In the proof of the Kahn-Kalai theorem, we have $M \subseteq \{\pm 1\}^n \subseteq \mathbb{R}^n$ with $|M| = 2^{n-2}$. We want the maximal subset $|A|$ such that $a \cdot a' \neq 0$ for all $a, a' \in A$. We will show that this is less than c^n, where $c < 2$. We get that the number of parts in the Borsuk part of $M \otimes M > 2^{n-2}/c^n$.

Theorem 8.1 (odd town theorem)**.** *Suppose $\mathcal{A} = \{A_1, \dots, A_N\} \subseteq \mathscr{P}(\{1, \dots, n\})$ is a collection such that $|A_i|$ is odd for all i, and $|A_i \cap A_j|$ is even for all $i < j$. Then $|\mathcal{A}| \leq n$.*

Proof. Let v_i be the characteristic vector of A_i in $\mathbb{Q}^n$. For example, if $A_1 = \{1, 4, 5\}$ and $n = 5$, then $v_1 = (1, 0, 0, 1, 1)^\top$. Then $\|v_i\|^2 = 1 \pmod 2$, and $v_i \cdot b_j = 0 \pmod 2$ if $i \neq j$. We claim that the v_i are linearly independent as vectors in F_2^n. Assume $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. We can take the λ_i to be integers, and without loss of generality, λ_1 is odd. Then $\lambda_1 = \lambda_1 \|v_1\|^2 + \cdots + \lambda_n \langle v_n, v_1 \rangle = 0 \pmod 2$, which is a contradiction. $\square$

Theorem 8.2 (2-distance theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be such that $d(x, x') \in \{a, b\}$ for all $x \neq x'$ and $x, x' \in X$. Then $|X| = O(n^2)$.*

When the number of possible distances is 1 instead of 2, we get that $|X| \leq n + 1$, since X must be the vertices of a simplex.

Proof. Let $X = \{z_1, \dots, z_N\}$ and $F(x, y) := (|x - y|^2 - a^2)(|x - y|^2 - b^2)$. Then

$$F(z_i, z_j) = \begin{cases} a^2 b^2 & i = j \\ 0 & i \neq j. \end{cases}$$

Define $f_i(y) := F(z_i, y)$. Then the f_i are linearly independent. Indeed, suppose $\lambda_1 f_1 + \cdots + \lambda_N f_N = 0$. Then $\lambda_1 f_1(z_1) = 0$, so $\lambda_1 = 0$. This is true for all i. So the number of f_i is at most the dimension of the space containing the f_i. So $N = O(n^2)$. $\square$

8.2 Kahn-Kalai using linear algebra methods

Let's continue with the proof of the Kahn-Kalai theorem. Let $M = \{x_1 = 1, x_2, \dots, x_n \in \{\pm 1\}, x_2 \cdots x_n = 1\}$. We also had $n = 4p$, where p is prime.

Lemma 8.1. *Let $A \subset M$ be such that $a \cdot a' \neq 0$ for $a, a' \in A$. Then $|A| \leq 2^{n/2}$.*

Proof. Define $G(t) = (t - 1)(t - 2) \cdots (t - p + 1)$. Let $V \subseteq \mathbb{Q}[x_2, \dots, x_n]$ be the subspace of squarefree polynomials with $\deg \leq n/4 = p$; that is, the monomials generating V have no x_i to a square or higher power. We will show that $W \subseteq V \implies \dim(W) \leq 2^{n/4}(n/4)$. Note that $\dim V = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n/4} < 2^{n/2}$.

Let $F_a = G(a \cdot (1, z_2, \ldots, z_n))$ for $a \in A$; this is really a polynomial in $z_2, \ldots, z_n$. By definition, $F_a \in V$. Next time, we will show that the F_a are linearly independent, which will produce a bound on $|A|$. $\square$

9 Borsuk's Conjecture: The Final Chapter

9.1 Remaining lemmas

Let's finish the proof of the Kahn-Kalai theorem. Our main lemma is the following.

Lemma 9.1. *Let $M = \{(x_1, \ldots, x_n) \subseteq \{\pm 1\}^n : x_1 = 1, x_2 \cdots x_n = 1\}$, and let $A \subseteq M$ be such that $a \cdot a' \neq 0$ for all $a, a' \in A$. Then $|A| < c^n$ for some $c < 2$.*

Proof. Last time we had $G(t) = (t-1)(t-2)\cdots(t-p+1)$ with $t \in \mathbb{N}$. For $\overline{a} \in A$, $\overline{z} = (1, z_2, \ldots, z_n)$, think of $G(\overline{a} \cdot \overline{z})$ as a polynomial in the z_i of degree $p - 1 < n/4$. Let F_a be the square-free part of $G(\overline{a} \cdot \overline{z})$. For example, if $\overline{a} = (1, 1, -1, -1, 1)$, $n = 5$ and $p = 3$, then

$$\begin{aligned} F(\overline{z} \cdot \overline{z}) &= (1 + z_2 - z_3 - z_4 + z_5 - 1)(1 + z_2 - z_3 - z_4 + z_5 - 2) \\ &= 1 + z_2^2 + x_3^2 - x_2 z_3 + z_4^2 - z_2 z_4 + z_3 z_4 + \cdots . \end{aligned}$$

Then $F_a = 1 - z_2 z_4 - z_2 z_4 + z_3 z_4 + \cdots$. □

We need a lemma for our lemma.

Lemma 9.2 (independence lemma)**.** *The set $\{F_a : a \in A\}$ are linearly independent.*

Proof. Note that $t \neq 0 \pmod p \iff G(t) = 0 \mod p$. Proceed by contradiction, assuming $\lambda_1 F_{\overline{a}_1} + \lambda_2 F_{\overline{a}_2} + \cdots = 0$ with $\lambda_1 \neq 0 \pmod p$. Then $G(\overline{a}_1 \cdot \overline{a}_1) = G(n) = G(4p) \neq 0 \pmod p$. So $F_{\overline{a}_1} \neq 0 \pmod p$. Also note that $G(a \cdot a') = F_a(a')$ for all $a, a' \in M$. We also have that $F_{\overline{a}}(\overline{a}') = 0 \pmod p$ for $a' \neq a$. Together, these two imply the independence lemma. Indeed, substitute $z = \overline{a}_1$ into the linear combination to get $\lambda_1 F_{\overline{a}_1}(\overline{a}_1) + 0 + \cdots + 0 = 0 \pmod p$. Since $F_{\overline{a}_1}(\overline{a}_1) \neq 0 \pmod p$, $\lambda_1 = 0 \pmod p$. We claim that $a \cdot a' = 0 \pmod 4$. Do this as an exercise. This means that $a \cdot a' = 0 \pmod 4 \implies F_{\overline{a}_1}(\overline{a}_1) = 0 \pmod p$. Combining these results proves the lemma. □

We return to the main lemma.

Proof. So $|A| < \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n/4} < (n/4 + 1)\binom{n}{n/4} < c^n$. This implies the main lemma. □

So we can finish the proof of the theorem:

Proof. The lemma implies that M cannot be partitioned into fewer than $2^{n-2}/c^n \gg (n^2+1)$ parts with no $a \cdot a' = 0$. This implies that, for large enough n, $M \otimes M$ cannot be partitioned into fewer than $(n^2 + 1)$ parts of smaller diameter. The $n^2 + 1$ comes from the fact that $\dim(M \otimes M) = n^2$. □

9.2 Aftermath

Professor Pak believes that Borsuk's conjucture probably fails for $n = 4$ or $n = 5$. There is no reason why we need the large construction in the Kahn-Kalai proof. It is known (from 2016) that Borsuk's conjecture fails in dimension 64.

One can ask about the chromatic number $\chi_1(\mathbb{R}^d)$ of the unit distance graph. It is known that $5 \leq \chi_1(\mathbb{R}^2) \leq 7$. How does this behave asymptotically?

Theorem 9.1 (Franklin-Wilson). *$c^d \leq \chi_1(\mathbb{R}^d) \leq d^d$for some constant c.*

10 $\mathcal{F}$-Vectors of Polytopes

10.1 Types of polytopes

There are two types of convex polytopes in $\mathbb{R}^d$.

1. **simplicial polytopes** (all faces are simplices),
2. **simple polytopes** (degree of every vertex $= d$, $\dim(P) = d$).

There is a duality between these two types. Basically, a point on each face, and take the convex hull to get the dual polytope. P is simple iff P^* is simplicial.

Definition 10.1. Let $P \subseteq \mathbb{R}^d$ be a convex polytope with $\dim(P) = d$. Let $f_i(P)$ be the number of i dimensional faces of P. This is called the **$\mathcal{F}$-vector** of P.

Proposition 10.1. *$f_i(P) = f_{d-i-1}(P^*)$.*

Topologists like simplicial polytopes, but combinatorialists like simple polytopes. We will focus on simple polytopes, but the previous proposition tells us that this is really the same story.

10.2 Dehn-Sommerville equations

Theorem 10.1 (Dehn-Sommerville equations[3])**.** *Let $P \subseteq \mathbb{R}^d$ be simple. Then*

$$\sum_{i=k}^{d} (-1)^i \binom{i}{k} f_i = \sum_{i=d-k}^{d} (-1)^{d-i} \binom{i}{d-k} f_i.$$

for all $0 \leq k \leq d$.

Remark 10.1. When $k = 0$, this becomes

$$\sum_{i=0}^{d} (-1)^i f_i = 1.$$

This is Euler's formula. When $d = 3$, we get $f_0 - f_1 + f_2 = 2$, where $f_3 = 1$.

Example 10.1. Let P be a simplex in $\mathbb{R}^d$. Then $f_0 = d + 1$, and $f_i = \binom{d+i}{i+1}$.

[3]The case $d = 3$ was proved by Euler. The cases $d = 4, 5$ were proved by Dehn, and $d > 5$ was proved by Sommerville.

Example 10.2. Let $Q \subseteq \mathbb{R}^d$ be a d-cube. Then $f_0 = 2^d$, $f_d = 1$, and $f_{d-1} = 2d$. In general, $f_i = \binom{d}{i} 2^{d-i}$, because we have $\binom{d}{i}$ ways to choose a face and 2^{d-i} coordinates left. We get that

$$\sum_{i=0}^{d} f_i = 3^d,$$

which could be otherwise proven as an elementary exercise.

Proposition 10.2. *Let $\mathcal{F}(t) = \sum_{i=0}^d f_i t^i$. Define $\mathcal{G}(t) := \mathcal{F}(t-1) = \sum_{i=0}^d g_i t^i$. Then $g_k = \sum_{i=1}^d (-1)^i \binom{i}{k} f_i$.*

Proof.

$$\mathcal{G}(t+1) = \sum_{i=0}^{d} g_i (t+1)^i = \sum_{i=0}^{d} g_i \sum_{k=0}^{i} \binom{i}{k} t^k = \sum_{k=0}^{d} t^k \left[\sum_{i=k}^{d} g_i \binom{i}{k} \right] = \sum_{i=0}^{d} t^k f_k = \mathcal{F}(t). \quad \square$$

So the Dehn-Sommerville equations say that $g_i = g_{d-i}$.

Example 10.3. For a simplex, $\mathcal{F}(t) = (1+t)^{d+1}$. Then $\mathcal{G}(t) = t^{d+1}$.

Example 10.4. For the d-cube, $\mathcal{F}(t) = (2+t)^d$, and $\mathcal{G}(t) = (1+t)^d$.

Let's prove the theorem.

Proof. Fix $\varphi : \mathbb{R}^d \to \mathbb{R}$ a Morse function (a linear function that is nonconstant on edges of the polytope). For a vertex v, define the index $\mathrm{ind}_\varphi(v)$ to be the number of edges increasing by φ. Observe that $0 \le \mathrm{ind}_\varphi(v) \le d$. Define $h_i^{(\varphi)}$ to be the number of vertices $v \in V(P)$ such that $\mathrm{ind}_\varphi(v) = i$.

We claim that $f_k = \sum_{i=k}^d \binom{i}{k} h_i^{(\varphi)}$. Take any k-face Q, and let v be the minimum vertex with respect to φ. Let $i = \mathrm{ind}_\varphi(v)$. The number of k-faces Q is $\sum_{i=k}^d \binom{i}{k} h_i^{(\varphi)}$, which is the number of ways to choose Q with minimum vertex v times the index.

Then $\sum_{i=0}^d h_i^{(\varphi)} t^i = \mathcal{F}(t-1)$. So for all φ, $h_i^{(\varphi)} = g_i$. If we replace φ with $-\varphi$, we get $h_i^{(\varphi)} = h_{d-i}^{-\varphi}$. The left hand side is g, and the right hand side is g_{d-i}. $\square$

11 Polytopes and Permutahedra

11.1 Polytopes

Definition 11.1. A polytope $P \subseteq \mathbb{R}^d$ is either of the following two equivalent things:

1. $P = \operatorname{conv}(X)$, $|X| < \infty$, $X \subseteq \mathbb{R}^d$.
2. $P = \bigcap H_i$ such that P is compact, where the H_i are half spaces.

Definition 11.2. The **dimension** of P is $\dim(P) = \dim_{\mathbb{R}} \langle P \rangle$, the affine subspace of $\mathbb{R}^n$ spanned by P.

Definition 11.3. A **face** $F \subseteq P$ is a subset of P such that there exists an affine subspace $W \subseteq \mathbb{R}^d$ such that

1. $F = P \cap W$,
2. there exists a half-space H such that $P \subseteq H$, $W \subseteq \partial H$, and $P \cap \partial H = F$.[4]

Example 11.1. Let C_3 be the cube in $\mathbb{R}^3$. Then $C_3 = \operatorname{conv}(\{(\pm 1, \pm 1, \pm 1)\})$. On the other hand, $C_3 = \bigcap_{i=1}^3 \{x : x_i \leq 1\} \cap \bigcap_{i=1}^3 \{x : x_i \geq -1\}$. The faces are $C_3 \cap \{x : x_i = \pm 1\}$.

Definition 11.4. An **edge** is a 1-dimensional face. A **vertex** is a 0-dimensional face. A **facet** is a $(d-1)$-dimensional face.

Definition 11.5. The **graph** $\Gamma(P)$ of a polytope is a graph $\Gamma = (V, E)$, where $V(P)$ is the set of vertices of P and $E(P)$ is the set of edges of P.

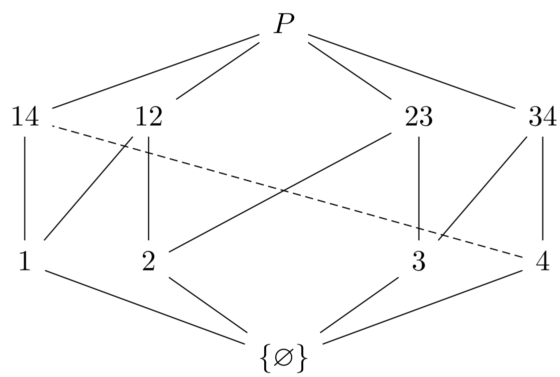
Definition 11.6. The **face lattice** $\alpha(P)$ is the partially ordered set of faces of P, ordered by inclusion.

This is a lattice because the meet of F and F' is $F \cap F'$, and the join of F and F' is $\langle F \cup F' \rangle \cap P$.

Example 11.2. Let P be a square in $\mathbb{R}^2$. Then $\Gamma(P)$ is the graph of the boundary of the

[4] This second condition implies the first, so you should really think of it as a clarification of the previous condition.

square, and $\alpha(P)$ is



Here is a theorem we will prove later.

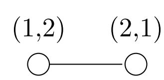
Theorem 11.1 (Blind-Mani)**.** *If $P \subseteq \mathbb{R}^d$ is simple, then Γ, the graph of P determines the face lattice of P.*

11.2 Permutahedra

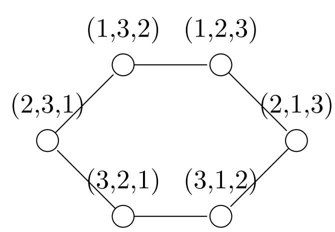
Definition 11.7. This **permutahedron** is $P = \operatorname{conv}(\{(\sigma(1), \ldots, \sigma(n)) : \sigma \in S_n\})$.

Observe that $\dim(P) = n - 1$.

Example 11.3. For $n = 2$, the permutahedron is



For $n = 3$, we have



Proposition 11.1. *$\Gamma(P_n) \cong \operatorname{Cay}(S_n, R_n)$, where R_n is the set of transpositions $(i\ j)$ with $1 \leq i, j \leq n$ with a left action.*

Observe that P_n is simple. In particular, we can figure out the $\mathcal{F}$-vector. Consider a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ nonconstant on edges where

$$\varphi(x_1, \ldots, x_n) = x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \cdots + \varepsilon^{n-1} x_n.$$

Then $\operatorname{ind}(\sigma)$ is the number of $i \in \{1, \ldots, n-1\}$ such that $\varphi((i\ \ i+1)\sigma) > \varphi(\sigma)$. This is the number of i such that $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. So $g_k = h_k^{(\varphi)}$ is the number of $\sigma \in S_N$ such that σ^{-1} has k ascents.

Let $A(n, K)$ be the number of $\sigma \in S_n$ with exactly k ascents.[5] We can prove the following proposition.

Proposition 11.2.

$$A(n,k) = (n-K)A(n-1,k-1) + (k+1)A(n-1,k)$$

Example 11.4. This is called the **Birkhoff polytope**. It is the set of matrices of non-negative entries such that the sum of the rows and columns are all 1. Formally, this is $B_n = \bigcap_{i=1}^n \bigcap_{j=1}^n \{x : x_{i,j} \geq 0\} \cap \bigcap_{j=1}^n \{x : \sum_{i=1}^n x_{i,j} = 1\} \cap \bigcap_{i=1}^n \{x : \sum_{j=1}^n x_{i,j} = 1\}$.

Theorem 11.2. $V(B_n) = \{\mathrm{Mat}(\sigma) : \sigma \in S_n\}$. $E(B_n) = \{(\sigma, w\sigma) : w \in S_n \textit{ is a cycle}\}$

Corollary 11.1. $\deg_\Gamma(v)$ *is the number of cycles in* S_n*, and* $\dim(B_n) = (n-1)^2$.

Question: Is $f_i(B)$ computable in polynomial time?

Theorem 11.3 (Pak)**.** *Let Q_n be the set of such matrices but with dimension $n \times (n+1)$. Then $f_i(Q_n)$ can be computed in polynomial time.*

[5] The bivariate generating function for $A(n,k)$ has a nice form.

12 The Blind-Mani Theorem

12.1 Acyclic orientations

Let's prove the Blind-Mani theorem.

Theorem 12.1 (Blind-Mani)**.** *Let $P \subseteq \mathbb{R}^d$ be a simple, convex polytope. Then the face lattice $\alpha(P)$ is determined by the graph $\Gamma(P)$ of the polytope.*

Example 12.1. Here are non-simple convex polytopes with don't satisfy this theorem. Let $\Gamma = K_6$ be the complete graph on 6 vertices. Then the simplex Δ^5 has graph Γ. But there also exists a polytope $Q \subseteq \mathbb{R}^4$ such that $f_0 = 6$ and $\Gamma(Q) = K_6$. To construct Q, think of $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$. Take two triangles, one in each copy of $\mathbb{R}^2$, and connect them together. So $Q = \Delta^2 \times \Delta^2$. Note that $\alpha(Q) \ncong \alpha(\Delta^5)$. This is an example in a large family of polytopes called **neighborly polytopes**, which have $\Gamma(P) \cong K^n$.

Proof. (Kalai[6]) Let $\Gamma = \Gamma(P)$. This is connected. Let $d = \deg(\Gamma)$. Γ is d-regular. Let O be the acyclic orientation of the edges E (so the edges all receive an orientation such that no cycles form). Now define h_i^O be the number of vertices $v \in V$ with out degree equal to i. This is to take the place of Morse functions in our proof. □

Define O to be good if $T \in \alpha(P)$ has a unique source. How do we know if an orientation is good?

Lemma 12.1. *Let $\alpha(O) := h_0^O + 2h_1^O + 4h_2^O + \cdots + 2^d h_d^O$. Then $\alpha(O) \geq f_0 + f_1 + \cdots + f_d =: \beta(P)$. Moreover, $\alpha(O) = \beta(P)$ if and only if O is good.*

This is Theorem 8.6 in Professor Pak's textbook. Let's prove the lemma.

Proof. Suppose O is an acyclic orientation coming from a Morse function φ on $P \subseteq \mathbb{R}^d$. Then $h_i^O = h_i^\varphi$. Then from the Dehn-Sommerville equations, $f_k = \sum_{i=k}^d \binom{i}{k} h_i^O$. Then $\beta(P) = \sum_{k=0}^d f_k = \mathcal{F}_P(1) = \mathcal{G}_P(2) = \sum_{i=0}^d h_i^O 2^i$. If O is good, then, we have the same equality ($\alpha(O) = \beta(P)$) because our proof of the Dehn-Sommerville equations only relied on the fact that each face had a unique source.

If O is any orientation, we write the same thing, except $f_k \leq \sum_{i=0}^d h_i^O \binom{i}{k}$. So $\alpha(O) \geq \beta(P)$. Then the only way to get an exact equality is if we never count a face twice. This is only if every face has a unique source. □

Now we need to use this characterization to find out when a subgraph of $\Gamma(P)$ is the graph of a face.

[6] The original proof was "plain boring," according to Professor Pak. But this proof is more interesting than the theorem itself.

12.2 The face criterion

Let $\Gamma = \Gamma(P)$ be the graph of a simple d-dimensional polytope, and let O be a good acyclic orientation of Γ. Think of a face as $\Gamma(F) \subseteq \Gamma$, where $V(F) \subseteq V(P)$. Suppose $\deg(\Gamma(F)) = k$.

Proposition 12.1. *$H \subseteq \Gamma(F)$ is a graph of a face if and only if the following two conditions are satisfied:*

1. *$\Gamma(F)$ is k-regular.*
2. *There exists a good orientation O such that $V(F)$ is final (no edges from outside $V(F)$ are oriented into $V(F)$).*

Proof. Suppose $F \in \alpha(P)$ is a k-dimensional face. Then $H = \Gamma(F)$ is k-regular. There also exists a final O on H; take a hyperplane containing that face, perturb it a little, and take a Morse function that defines O.

For the opposite direction, take the minimum point (since O is final). Create 2 graphs, one spanned by $\Gamma(F)$ and one containing everything you can reach from the minimum vertex. They are both k-regular and one contains the other, so they are equal. $\square$

13 Balinski's Theorem and Associahedra

13.1 Balinski's theorem

Definition 13.1. A graph $G = (V, E)$ is called k**-connected** if for every $(k-1)$ vertices $v_1, \dots, v_{k-1}$, $G \setminus \{v_1, \dots, v_{k-1}\}$ is connected.

Theorem 13.1 (Balinski)**.** *For every convex polytope $P \subseteq \mathbb{R}^d$ with $\dim(P) = d$, $\Gamma = \Gamma(P)$ is d-connected.*

For $d = 2$, the graph is a cycle, so removing a vertex does not disconnect the graph.

Proof. Suppose $X = \{v_1, \dots, v_{d-1}\} \subseteq V(P)$. Choose any vertex $z \in V \setminus X$. Let H be a hyperplane spanned by $X \cup \{z\}$. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a linear function such that $\psi(v_i) = \psi(z) = 0$, and let ψ' to be a small perturbation of ψ which is nonconstant on H. Let u be the vertex maximizing φ and w be the vertex minimizing φ. Also, let $H_- = \{x \in V : \psi'(x) < 0\}$ and $H_+ = \{y \in V : \psi'(y) > 0\}$.

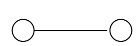
If we start at $y \in H_+$ and travel along edges where ψ' is increasing, we end up at u. If we start at $x \in H_-$ and travel along edges where ψ' is decreasing, we end up at w. So we know that H^+ and H_- are connected. We claim that z is connected to both u and w. Depending on our choice of perturbation φ, $\varphi(z) > 0$, in which case z is connected to H_+, or $\varphi(z) < 0$, in which case z is connected to H_-. $\square$

13.2 Associahedra

Fix $n \geq 3$, and construct the graph $\Gamma = (V, E)$, where V is the set of triangulations of an n-gon ($|V| = \binom{2n}{n}/(n+1)$, the n-th Catalan number) and E is the set of triangulations that differ by a flip. Here, a flip means removing an edge in the triangulation and replacing it with the opposite diagonal of the resulting quadrilateral. Then Γ is $n-3$ regular because an n-gon has $n-3$ diagonals.

Is Γ the graph of a simple polytope in $\mathbb{R}^{n-3}$?

Example 13.1. For $n = 4$, we get



For $n = 5$, we get the graph of a pentagon. For $n = 6$, the graph has 14 vertices; try to come up with it yourself![7]

Theorem 13.2. *Let $\Gamma = (V, E)$ be the above graph. It is a graph of a simple polytope P_n.*

[7] There's no way I'm making a diagram for this one.

Stasheff said that $\alpha(P_n)$ is the set of subdivisions of the n-gon by non-crossing diagonals, ordered by inclusion. K. Lee showed that yes, there exists such a polytope P_n.

Here is the Gelfan-Zelevinsky-Kapranov construction.[8] For each triangulation T of a fixed n-gon Q, let $f_T : V(Q) \to \mathbb{R}_+$ be

$$f(v) = \sum_{\triangle \ni v} \mathbf{area}(\triangle)$$

Theorem 13.3 (GZK,c.1990)**.** *For every Q, the set of f_T for all triangulations of A is the set of vertices of the associahedron P_n; i.e. $P = \operatorname{conv}(\{f_T\})$.*

P_n sits in $\mathbb{R}^n$. What linear equations does it satisfy that makes the dimension $n-3$? One equation is

$$\sum_{v \in V(Q)} f_T(v) = 3\,\mathrm{area}(Q).$$

Theorem 13.4 (TTQ)**.** *For $n > 20$, $\operatorname{diam}(\Gamma_n) = 2n - 10$.*

Proving that $\operatorname{diam}(\Gamma_n) \geq 2n - 10$ is the easier part, but $\operatorname{diam}(\Gamma_n) \leq 2n - 10$ is hard.

Adelson-Velsky-Landis[9] trees: If you have a binary tree with too much depth on one side of the root, you might want to choose a different root so the tree is more balanced. This is related to triangulations of an n-gon because the dual graph of a triangulation is a binary tree.

[8]The names are in this order because alphabetic order in Russian is different from alphabetic order in English.

[9]Adelson-Velsky is one person.

14 $\mathcal{F}$-Vectors of Associahedra

14.1 $\mathcal{F}$-vectors of associahedra

Recall the GZK construction of the associahedron. Let $Q \subseteq R^2$ be a fixed convex n-gon, and let $\tau = \tau(Q)$ be the set of triangulations of Q. For every $T \in \tau$, let $f_T : V \to \mathbb{R}$ be

$$f_T(v) = \sum_{\substack{\triangle \subseteq T \\ v \in \triangle}} \operatorname{area}(\triangle)$$

Then $P_n = \operatorname{conv}(\{f_T : T \in \tau\})$.

Theorem 14.1 (GZK). *P_n has lattice $\alpha(P_n)$ isomorphic to the graph of all diagonal subdivisions of Q.*

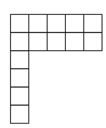
Proof. Here is a sketch. First, show $\dim(P_n) = n - 3$. Then show that f_T are in convex positions. Finally show that $(f_T, f_{T'}) \in E(P_n) \iff T, T'$ differ by a flip. Then, use the Blind-Mani theorem. □

We want to compute the $\mathcal{F}$-vector of P_n. Note that f_k is the number of ways to place $n - 3 - k$ non-crossing diagonals in Q. This is sort of a generalization of the Catalan numbers because $f_0 = \binom{2n}{n}/(n+1)$. We can also see that $f_1 = (n-3)f_0/2$.

Theorem 14.2.

$$f_k = \frac{1}{n-k-2}\binom{n-3}{n-k-3}\binom{2n-k-4}{n-k-3}.$$

Remark 14.1. This is equal to the dimension of the representation of the symmetric group corresponding to the Young diagram



where the first 2 rows have k boxes, and there are $2n$ rows.

Things are nicer with g-vectors, so let's work with those instead.

Theorem 14.3.

$$g_k = \frac{1}{n-2}\binom{n}{k}\binom{n}{k-1}.$$

Proof. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Morse function such that $\varphi(x_1, \ldots, x_n) = x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \cdots$. Look at the binary tree dual to T. Imagine entering from outside Q and turning either left to right to travel along each edge of the binary tree. We can then denote the edges of the tree as left or right. Then flipping the edge corresponding to a left edge in the tree will increase the value of the Morse function φ because it will change a diagonal connected to a larger labeled vertex to a smaller one. So $\mathrm{ind}_\varphi(T)$ is the number of left edges in the binary tree. So $g_n = h_k^\varphi$ is the number of binary trees with $n-2$ vertices.

Denote by $b(n,k)$ the number of binary trees on n vertices with k left edges. How can we count this? Start with a binary tree. There are $2n-(n-1)=n+1$ places to add an edge to increase the size of the tree. The number of left open places to put an edge is $(n-k)$, and the number of right open places is $(n+1)-(n-k)$. These relations give us a sort of Pascal's triangle for $b(n,k)$; we get $b(n+1,k) = kb(n,k-1)+(n-k)b(n,k)$. We can check the recurrence against the expression in the theorem. $\square$

Remark 14.2. In this specific example, the Dehn-Sommerville equations $g_k = g_{n-3-k}$ are just a consequence of the face that we can flip every edge of the triangulation to get another triangulation.

14.2 Narayana numbers

These numbers actually come up in a lot of places in combinatorics. They have a name.

Definition 14.1. The **Narayana numbers** are

$$N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Proposition 14.1.

$$\sum_{k=0}^{n-1} N(n,k) = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Each term in the sum on the left is the number of binary trees on n vertices with k left edges. The n-th Catalan number is the number of binary trees on n vertices. $\square$

15 Simplex Methods and Klee-Minty Cubes

15.1 Simplex methods

Let $P \subseteq \mathbb{R}^d$ be a convex polytope and $\Gamma = \Gamma(P) = (V, E)$ be a graph. Suppose $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a linear function non constant on edges of P. We want to find $v \in V$ such that φ is maximized at v. The idea is to start at some $s \in V$, and walk along the graph edges in increasing direction with respect to φ. How do we know which edge to take if there are multiple increasing edges?

Definition 15.1. A **pivot rule** is a method of determining how to choose which up edges to walk along.

Example 15.1. We can choose the edge with steepest ascent, max value of φ at the endpoint, lexicographically first, random, or a different pivot rule.

In reality, people use a pivot rule different from all of these.

15.2 Klee-Minty Cubes

Theorem 15.1 (Klee-Minty,1972)**.** *There exists a simple $P \subseteq \mathbb{R}^d$ with $|V| = 2^d$, $f_{d-1} = 2d$, and $\alpha(P) \cong \alpha(C_d)$. such that the length of the simplex method can be 2^{d-1}.*

We will prove a weaker theorem, which does not rely on a rule. The construction is the same, and you can modify it to work with a given pivot rule.

Theorem 15.2. *There exists such a $P \subseteq \mathbb{R}^d$ with a maximum increasing path of length $2^d - 1$.*

Proof. First is a sketch of the intuitive idea. Proceed by induction on d. We can do this for $d = 2$, by creating an isosceles trapezoid. Given the construction for d, place a small copy of the construction parallel to the construction, and connect them with edges to get the construction for $d + 1$.

Explicitly, let

$$x_1 \leq 5$$

$$4x_1 + x_2 \leq 5^2$$

$$8x_1 + 4x_2 + x_3 \leq 5^3$$

$$\vdots$$

$$2^d x_1 + 2^{d-1} x_2 + \cdots + 4x_{d-1} + x_d \leq 5^d$$

and $x_i \geq 0$ for $i = 1, \ldots, d$. Let $\varphi = 2^{d-1}x_1 + 2^{d-2}x_2 + \cdots + 2x_{d-1} + x_d$. The vertices are when we have exactly d equalities.

The function φ has a minimum at $(0,\ldots,0)$ and a maximum at $(0,\ldots,0,5^d)$. Why? Let's construct a Hamiltonian path.

vertex	φ
$v_0 = (0,\ldots,0)$	0
$v_1 = (5,0,\ldots,0)$	$5 \cdot 2^{d-1}$
$v_2 = (5,5,0,\ldots,0)$	$5 \cdot 2^{d-1} + 5 \cdot 2^{d-2}$
$v_3 = (0,25,0,\ldots,0)$	$25 \cdot 2^{d-2}$
$v_4 = (0,25,25,0,\ldots,0)$	
$\vdots$	

The idea is that we have an iterative procedure for finding the path, and we can prove that it works by induction. The Klee-Minty cube is designed to have the bounds grow exponentially faster than the Morse function φ. $\square$

The moral of the story is that simplex methods can be exponentially slow. But in practice, people still use them. One reason is that they use randomized pivot rules. Moreover, since computers only calculate things up to finite precision, you eventually don't even see the extra paths after a certain point.

Theorem 15.3 (Spielman-Teng). *For "random" constraints, the simplex method runs in polynomial time.*

16 Cauchy's Arm Lemma

16.1 The arm lemma

Lemma 16.1. *Let $Q = [x_1, \ldots, x_n], Q' = [x'_1, \ldots, x'_n]$ be two noncongruent convex polygons with equal corresponding lengths $|x_i - x_{i+1}| = |x'_1 - x'_{i+1}|$ for $i = 1, \ldots, n \pmod n$. Then ere exist at least 4 sign changes in $\delta_i = \angle x_{i-1}x_ix_{i+1} - \angle x'_{i-1}x'_ix'_{i+1}$.*

Example 16.1. Suppose $n = 4$, let Q be a square, and let Q' be a rhombus with angles α and $\pi - \alpha$. Then $\overline{\delta} = \{+, -, +, -\}$. The idea is that the diagonal length increases, which is impossible. Let's make this more rigorous.

Proof. Proceed by contradiction. Then $\overline{\delta} = \{+, +, \ldots, +, -, -, \ldots, -\}$. Then on one side of the polygon, the angles are increasing. □

Lemma 16.2 (arm lemma)**.** *Let $P = [y_1, \ldots, y_k]$, $P' = [y'_1, \ldots, y'_k]$ be convex polygons with $|y_i - y_{i+1}| = |y'_i - y'_{i+1}|$ for $i = 1, \ldots, k - 1$. If for $i = 1, \ldots, k - 2$*

$$\angle y_iy_{i+1}y_{i+2} \leq \angle y'_iy'_{i+1}y'_{i+2},$$

then $|y_1 - y_k| \leq |y'_1 - y'_k|$.

16.2 Cauchy and Zaremba's proofs

Cauchy proved this in 1813 but incorrectly.[10] Let's go though Cauchy's proof.

Proof. Proceed by induction. When $n = 3$, we use the law of cosines. For the inductive step, increase all the angles except 1. Then, applying the law of cosines to the triangle formed by the triangle $x_1x_nx_{n+1}$, we get that the length $x_1 - x_{n+1}$ increases. □

Where does this proof fail? It does not use convexity, and this theorem is not true for nonconvex polygons. There are cases where the inductive step does not work.

Proof. This is a proof by Zaremba.[11] In a case where Cauchy's proof doesn't work, first, increase the angle $\angle x_3x_2x_1$ until x_n lies on the segment connecting x_1 and x_{n-1} This is as far as we can expand the angle without losing convexity. Let the x'_1 be the new point where x_1 is at. By the inductive hypothesis, the polygon $[x'_1, \ldots, x_{n-1}]$ has the desired property, that is the line x'_1 to x_{n-1} has gotten bigger (compared to x_1x_n). So if you append a triangle onto this side to get a polygon with 1 more vertex, called x'_n, the length of the segment connecting x_1 and x_n is smaller than the length of the segment connecting x'_1 and x'_n. □

[10] He was about 19 at the time. Legendre gave him this as a project.

[11] Zaremba and Schonberg corresponded, coming up with iterative constructions for this proof. Eventually, Zaremba came up with this proof. They published all three of their proofs. Basically, they jsut published their correspondence. But everyone only cares about the last proof.

17 Cauchy's Rigidity Theorem

17.1 Cauchy's Rigidity Theorem

Theorem 17.1 (Cauchy)**.** *Let $P, P' \subseteq \mathbb{R}^3$ be convex polytopes such that $\alpha(P) \simeq_\pi \alpha P'$, and for every 2-dimensional face F, $F \cong F'$ (up to rigid motion by $O_3(\mathbb{R}) \ltimes \mathbb{R}^3$). where $F' = \pi(F)$. Then $P \cong P'$.*

Example 17.1. Let P be a cube. If we have P' with 6 unit squares as faces, then $P \cong P'$. This is expected, since there is only 1 way to make a corner using 3 squares. But if P is an icosahedron, this is much less obvious.

We will make two mistakes in our proof, a small one and a medium-sized one.

Proof. Let $\Gamma = \Gamma(P) = \Gamma(P')$ be the graph of P. Γ is planar. Put pluses and minuses on the edges of the graph to indicate whether the dihedral angles of the faces meeting at those edges increase or decrease when going from P to P'. We know that the number of sign changes around each vertex is ≥ 4. This is our first mistake; since Cauchy's lemma's only applies to polygons, we cannot quite use it here. But a version of Cauchy's lemma for spherical polygons is true (and only relies on the spherical law of cosines).[12] Think of the cone extending from the vertex, and intersect it with a small sphere.

Let $M := \sum_{v \in V(P)} m_v$, where m_v is the number of sign changes around v. Then $M \geq 4|V|$. Let f_k be the number of k-sided faces in P. Then $|\mathcal{F}| = f_3 + f_4 + f_5 + \cdots$. We also know that $2|E| = 3f_3 + 4f_4 + 5f_5 + \cdots$. Subtracting these, we get

$$4|E| - 4|\mathcal{F}| = 2f_3 + 4f_4 + 6f_5 + \cdots,$$

and Euler's formula gives us $4|V| - 8 = 4|E| - 4|\mathcal{F}|$.

We claim that $M \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + \cdots$. In an n-gon, you can have at most n sign changes. We must also have an even number of sign changes. The trick is that we can count the sign changes around faces instead of vertices because if two edges are adjacent, then they bound the same face.

We now have that

$$2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + 8f_8 + \cdots \geq M > 4|V| - 8 = 2f_3 + 4f_4 + 6f_5 + \cdots,$$

which is a contradiction. But we have made a medium-sized mistake; what if some angles have no sign change? In this case, draw the graphs but omit the edges with no sign changes. This still gives you a planar graph. What if the graph becomes disconnected? The 2 in Euler's formula ($|V| - |E| + |\mathcal{F}| = 2$) grows, which only works in favor of our inequality. □

[12] Back in 1900, spherical geometry was taught in high school. You might think that Professor Pak should know enough spherical geometry to know the spherical law of cosines, but he was, in fact, born after that time.

17.2 The story

Around the time of the French revolution, Legendre wanted to translate Euclid's Elements into French. It took him years, but he managed to translate it successfully. However, he discovered that there was a proof in the last section which was incorrect! He assigned Cauchy, his student, to prove the theorem and generalize it, and Cauchy, 19 at the time, did just that. There was a mistake in his proof of the arm lemma, but otherwise, Cauchy solved the problem. So in a sense, this theorem was proved as a consequence of the French revolution.

18 Meditations on Cauchy's Theorem

18.1 Alexandor's theorem and Stoker's Conjecture

Theorem 18.1 (Alexandor, 1920s)**.** *Let $P, P' \subseteq \mathbb{R}^3$ are convex polytopes with $\Phi : \alpha(P) \to \alpha(P')$ is such that for a;; $F \in \alpha(P)$ with $\dim(F) = 2$, $\{\angle \text{ in } F\} \simeq \{\angle \text{ in } \Phi(F)\}$. Then P and P' have equal corresponding dihedral angles.*

This is really a corollary of our proof of Cauchy's theorem. We basically proved this as a lemma to get Cauchy's theorem.

Here is a related conjecture.

Theorem 18.2 (Stoker's conjecture, 1960s)**.** *If you know all face angles, you know all dihedral angles and vice versa.*

People believe this to be true, but the conjecture is still open.

18.2 Non-examples to Cauchy's theorem

Here are some non-examples of Cauchy's theorem.

Example 18.1. Take a triangular prism, and remove a triangular pyramid from one of the sides. This is not convex, so Cauchy's theorem doesn't apply, even though it has the same lattice as the triangular prism with with a triangular pyramid on top. But we can get from one to another by continuously deforming.

Corollary 18.1 (Cauchy)**.** *Let $\{P_t : t \in [0,1]\}$ be a continuous family of 3-dimensional convex polytopes such that $\alpha(P_t) \cong\simeq \alpha(P_0)$ and 2-faces in P_t are congruent. Then $P_0 \simeq P_1$.*

Example 18.2 (Bricard's octahedron)**.** Draw four chords on a circle, with 2 intersecting. Now, in the z direction, put a vertex above and below the center of the circle. Now connect the vertices with edges to form 8 faces that intersect each other. If you push the north pole and the south pole towards each other, the polygon is flexible. So this is a non-example to Cauchy's theorem because it is self-intersecting.

Are all non-examples self intersecting?

Theorem 18.3 (Conelly, 1977)**.** *There exists a flexible polyhedral sphere embedded into $\mathbb{R}^3$.*

Scientific American used to publish paper cutouts of these kinds of things, where you could build your own flexible polyhedron. Probably dozens of kids made their own flexible polyhedra.[13]

[13] According to Professor Pak, you have to be a very special kid to enjoy this sort of thing.

18.3 Spherical Cauchy and high-dimensional Cauchy

Theorem 18.4 (spherical Cauchy's theorem). *For all $P, P' \subseteq S^3_+$, the conclusions of Cauchy's theorem hold.*

Proof. The part in our proof where we used a property of Euclidean space was that intersecting a small sphere with a cone gives us a spherical polygon. This is even more clear for spherical polygons. $\square$

Why do we care about spherical polytopes?

Theorem 18.5 (high-dimensional Cauchy). *For all convex polytopes $P, P' \subseteq \mathbb{R}^d$ with $d \geq 3$, $\dim(F) = d - 1$.*

Proof. Prove high-dimensional spherical Cauchy by induction. Then we get this theorem by reduction to the non-spherical case. $\square$

18.4 Rigidity

If you've ever been to a construction site, you know that the rigidity of a building is only dependent on the beams holding up the building.[14] These are the edges. If we have n vertices of a polytope, and we triangulate it, we get $3n - 6$ edges. We want to say that the lengths of these edges should really determine the polytope. Next time, we will prove Dehn's theorem, which talks about this.

[14] Who knew that discrete geometry would be interesting to engineers?

19 Bar and Joint Frameworks and Rigidity

19.1 Frameworks and static rigidity

Definition 19.1. A **bar and joint framework** is a pair (G, L) where $G = (V, E)$ is a graph, and L is a length function such that $L(e)$ is the length of $e_{i,j}$.

Definition 19.2. A **realization** of a bar and joint framework is a map $f : V \to \mathbb{R}^d$ such that for all edges $e = (i, j)$, $|f(i) - f(j)| = L(e)$.

This gives us a new interpretation of Cauchy's theorem.

Theorem 19.1. *If $P \subseteq \mathbb{R}^3$ is a simplicial polytope and (G, L) is a corresponding framework, then there exists a unique convex realization of (G, L).*

Here is a corollary of the 4-color theorem.

Theorem 19.2. *Suppose G is a planar graph, and $L(e) = 1$ for all e. Then there exists a realization $f : G \to \mathbb{R}^3$.*

Definition 19.3. A realization is **statically rigid** if there does not exist a nonzero function $\lambda : E \to \mathbb{R}$ such that for every $v \in V$, $\sum_{(v,w)\in E} \lambda((v, w)) \cdot (vw) = 0$, where vw denotes the vector from v to w in $\mathbb{R}^3$.

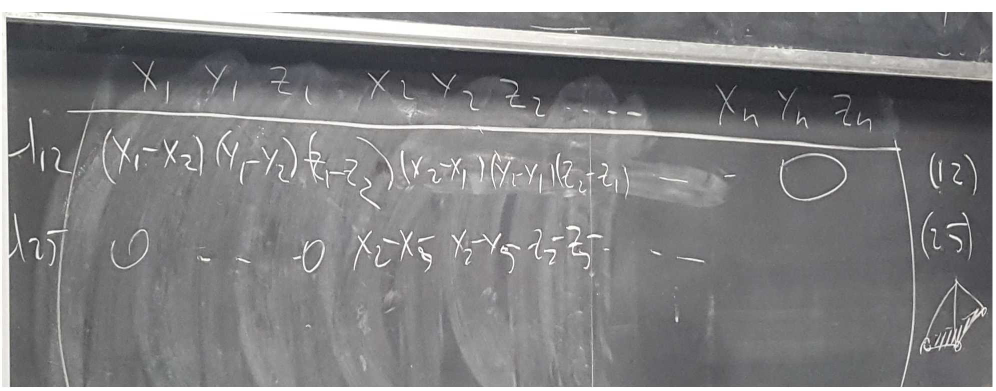
The function λ basically allows us to change the length slightly to have a little flexibility in our realizations.

19.2 Dehn's rigidity theorem

Theorem 19.3 (Dehn)**.** *Let (G, L) be a framework of a simplicial polytope in $\mathbb{R}^3$. Then it is statically rigid.*

B. Fuller came up with an architectural design for a dome which is statically rigid. It needs to pillars to stand up. You can prove that it is statically rigid using Dehn's theorem.

Definition 19.4. Let each vertex be $v_i = (x_i, y_i, z_i)$. Construct a $3n \times 3n - 6$ matrix as follows:



The **rigidity matrix** R_G is the $(3n-6) \times (3n-6)$ where we delete 6 of the columns.

Static rigidity means that the rigidity matrix R_G has full rank. The idea of deleting the 6 columns is that we are grounding a triangular face of the polytope.

Lemma 19.1. *Dehn's theorem is equivalent to* $\det(R_G) = 0$.

Proof. Here is the idea. The determinant of R_G is the sum of terms times (-1) to the something. We show that there is at least 1 nonzero term, and then we show that all terms have the same sign. $\square$

20 Proof of Dehn's Rigidity Theorem

20.1 Determinant of the rigidity matrix

Last time, we were proving this theorem.

Theorem 20.1 (Dehn's rigidity theorem)**.** *Let $P \subseteq \mathbb{R}^3$ be a simplicial convex polytope with graph $\Gamma = (V, E)$. Let $L : E \to R_+$, and let (Γ, L) be a framework. Then (Γ, L) is statically rigid.*

We had the following lemma. Let R be the rigidity matrix (formally $3n \times 3n - 6$). If the rank of R is $3n - 6$, then (Γ, L) is statically rigid.

Lemma 20.1. *Let R' be the square submatrix obtained by removing 9 rows and 3 columns (removing 3 vertices and the edges between them). Then* $\det(R') \neq 0$*.*

Proof. Let $a, b, c, d \in V$ be such that $(a, b), (a, c), (a, d) \in E$. Then consider the minor

$$\begin{bmatrix} x_a - x_b & x_a - x_c & x_a - x_d \\ y_a - y_b & y_a - y_c & y_a - y_d \\ z_a - z_b & z_a - z_c & z_a - z_d \end{bmatrix}.$$

The determinant of the matrix will be the product of determinants of the minors.

The proof is in 2 parts.[15]

1. There exists a permutation σ such that $\prod R'_{i,\sigma(i)} \neq 0$. This is equivalent to every triangulation having a claw partition (a partition into $K_{1,3}$ bipartite graphs). Proceed by induction. Γ is a triangulation, so there exists a vertex of deg ≤ 5. If v has degree 3, we can find a claw connecting v all its neighbors. If v has degree 4, then pick 3 of the neighbors to get a claw, and then make another claw with the vertex of the remaining neighbor. The $\deg(v) = 5$ case can be split up into various cases we can similarly solve.

2. For all permutations σ, $\prod R'_{i,\sigma(i)}$ have the same sign. This is equivalent to all claw partitions having the same sign, where the order of the edges in the graph determines the sign (depending on whether it is clockwise or counterclockwise). We claim that every 2 claw partitions are connected by a sequence of triangle moves, where we take a triangle in the graph and reverse the orientation of the triangle's edges in the claw partition. Let Π, Π' be claw partitions of Γ, and let v, v' be vertices with an edge between them that differs in Π, Π'. Flip this edge, and keep doing this until you form a cycle. Using Euler's formula, we can show that there is a path through the interior of the cycle. Reverse the two halves of the cycle, one at a time. Then we have reduced to smaller cycles, and we can continue doing this until we get triangles. Then we apply our triangle moves. $\square$

[15] This proof has as many holes as a colander.

21 Applications of Dehn's Rigidity Theorem

21.1 Gluck's theorem

Theorem 21.1 (Gluck)**.** *Almost all simplicial polyhedra are rigid.*

By a polyhedron, we mean a 2-dimensional polyhedral surface in $\mathbb{R}^3$ homeomorphic to S^2. The "almost all" is a statement about measure.

Example 21.1. Suppose G is the graph of an octahedron, and let $L : E \to \mathbb{R}_+$ be the edge length function. In the Bricard octahedron, we have restrictions on the lengths of the sides (opposing sides have to have the same length. The idea is that flexible polyhedra are like this; they have 0 measure.

Corollary 21.1 (of Dehn's theorem)**.** *Let $P \subseteq \mathbb{R}^3$ be a simplicial polytope with graph $G = (V, E)$ and length function $L : E \to \mathbb{R}_+$. Then there exists $\varepsilon > 0$ such that for every $L' : E \to \mathbb{R}_+$ with $|L'(e) - L(e)| < \varepsilon$, there exists a convex polytope $P' \subseteq$ combinatorially equivalent to P with length function L'.*

Proof. Let $\mathcal{X}_G$ be the space of all possible length functions $L : E \to \mathbb{R}_+$. Let $n = |V|$. Then $\dim(\mathcal{X}) = 3n - 6$. Let $f_{i,j} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$. We claim that the $f_{i,j}$ are algebraically independent. Let J be the matrix of partial derivatives of $f_{i,j}$. If you look at the partial derivatives, we get that $J = 2R'$, where R' is our rigidity matrix. Then use the inverse function theorem. So there exists an open set around the realization L. □

Here is an incorrect "proof" of Gluck's theorem.

Proof. Let (G, L) be a framework corresponding to the simplicial polyhedron. Take $(i, j) \notin E$. Then $g = f_{i,j} \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n]/\{\text{rigid motions}\}$. Then there exist $c_i \in \mathbb{C}[fe, e \in E]$ such that $c_0 g^N + c_1 g^{N+1} + \cdots + c_N = 0$.

What is the mistake? It is possible that all the coefficients are 0. The idea is that this is a measure zero set. □

There was another issue. We don't know that the a framework for a simplicial polyhedron corresponds to the graph of a simplicial polytope.

Lemma 21.1 (Steinitz theorem for triangulations)**.** *For all $G = (V, E)$ plane triangulations, there exists a convex simplicial polytope $P \subseteq \mathbb{R}^3$ with graph G.*

We will prove this next time.

21.2 Area of polygons given side lengths

Here is an application due to Robbins.

Suppose we have a triangle with side lengths a, b, c. We can find the area of the triangle using Heron's formula. If we have a quadrilateral, we can use another formula (Brahmagupta's formula) to find out the area given the side lengths. Can we do this for pentagons, hexagons, etc.?

How is this related to what we have been talking about? If we have a polygon inscribed in a circle, connect the edges of it via a double suspension (above and below) to a polyhedron (like when we constructed Bricard's octahedron). We can then get a polytope.

22 Weak Steinitz Theorem and Robbins' Conjecture

22.1 Weak Steinitz theorem

This is the last ingredient we need to prove Gluck's theorem.

Lemma 22.1 (weak Steinitz theorem)**.** *Let $G = (V, E)$ be a plane triangulation.*[16] *Then there exists a polytope $P \subseteq \mathbb{R}^3$ with graph $G(P) \simeq G$.*

You might think this is obvious, since you can just pretend the graph is you looking at the polytope from above. But there are actually counterexamples to that approach.

Proof. Proceed by induction. The basecase is $G = K_4$, which si a square pyramid. There exists a vertex in V with $\deg(v) \leq 5$. We have 3 cases:

1. $\deg(v) = 3$: If we remove v, we still have a triangulation. Now take the polytope with this graph, and add a small pyramid to a face.
2. $\deg(v) = 4$: If we remove v, add an edge to the resulting quadrilateral, take the polytope with thi graph, and take a vertex in the middle of the added edge. Raise it up ε and add edges to the remaining 2 vertices on the boundary of the face.
3. $\deg(v) = 5$: This case is more difficult than the other cases. It involves transforming the polytope using an affine transformation to get it to look nice. □

Here is a conjecture.[17]

Theorem 22.1. *For every triangulation $G = (V, E)$ with n vertices. There exists a convex polytope $P \subseteq \mathbb{R}^3$ with graph G and integer coordinates $\leq n^{10000}$.*

It is known that this is $\leq c^n$ for some constant c.

22.2 Robbins' conjecture

Theorem 22.2 (Robbins' conjecture[18])**.** *Let $A = A(a_1, \ldots, a_N)$ be the area of the inscribed convex polygon with sides $a_1, \ldots, a_n$. Then*

1. *There exists a polynomial $f_n(x) = c_0 x^N + c_1 x^{N-1} + \cdots + c_N$ such that $c_i \in \mathbb{Z}[a_1^2, \ldots, a_n^2]$ such that $f_n(A^2) = 0$.*
2. *If $n = 2k+1$, $N(n) = \frac{2k+1}{2}\binom{2k}{k} - e^{2k-1}$. If $n = 2k+2$, then $N(n) = 2N(n-1)$.*

Example 22.1. If $n = 4$, then $A^2 = (\rho - a)(\rho - b)(\rho - c)(\rho - d)$, where $\rho = (a+b+c+d)/2$.

[16] We can assume straight edges, since every planar triangulation can be written with straight edges.

[17] Professor Pak came up with the above proof to try to resolve this conjecture, but the method doesn't actually work. The issue is in the $n = 5$ case.

[18] Robbins figured out the first part, but he was diagnosed with a terminal illness and put out a request for someone to prove the second part before he died. Professor Pak and a student proved the second part, but when they tried to contact Robbins, they found that he had passed away a week prior.

23 Steinitz's Theorem

23.1 Steinitz's theorem

Theorem 23.1 (Steinitz, 1928). *For every 3-connected,planar graph $G = (V, E)$, there exists a convex polytope $P \subseteq \mathbb{R}^3$ such that G is a graph of P.*

Last time, we talked about when G is a plane triangulation. We also discussed the following open problem:

Theorem 23.2 (quantitative Steinitz problem). *Let $M(n)$ be the minimum m such that for all $G = (V, E)$ with $|V| = n$, there exists a polytope with nonnegative integer coordinates $\leq m$ with graph G. Then $M(n) < n^c$ for some constant c.*

It is currently known that $M(n) \leq 147.7^n$.

23.2 Ideas in the development of Steinitz's theorem

The idea is due to Maxwell (the physicist) in the 1860s. Suppose we have a polytope shaped like a dome, and we project it down to get a graph.

Lemma 23.1. *There exists a wight function $w : E \to \mathbb{R}_+$ such that for all $v \in V \setminus \partial V$, $\sum_{(v,v')\in E} w(v, v')vv' = 0$, where vv' is the vector from v to v'.*

If you think of each edge as a spring, this says that this is an equilibrium state of the springs. Suppose we have a vertex v; what does the equilibrium say? If we take the edge vectors connected to v, and take a perpendicular vector to each edge vector (oriented in circular fashion), we get a polygon enclosing the vertex v. How do we get these perpendicular vectors? This is left to the reader.[19]

Lemma 23.2 (Cremona). *Maxwell's map from polytopes to networks in equilibrium is invertible.*

Here is the next idea in the proof.

Theorem 23.3 (Tutte). *For every 3-connected, planar graph G, there exists an embedding/drawing of G in $\mathbb{R}^2$ such that all faces are convex polygons.*

Proof. Take $w = 1$ (but this works for all $w > 0$). There exists an equilibrium (spring) embedding of G. Pin down the boundary vertices. From a physics perspective, if we let the spring network go, the springs will move around until they reach an equilibrium. If we define the energy function $\mathcal{E} = \sum_{e\in E} w(e)|e|^2$, then Tuttle is saying that there exists a (unique) minimum of $\mathcal{E}$. $\square$

[19] When Maxwell gave his lecture, he just read the paper to the audience without any pictures. Everything was left to the imagination. These notes are like that, as well.

There is a hole the size of the Pacific ocean in the above proof. What is the issue? We need an equilibrium embedding, not just an equilibrium. We need to make sure the graph is planar. So we need to use Kuratowski's theorem, that every planar graph contains K_5 or $K_{3,3}$. We also need to use the 3-connectedness of the graph. The actual argument is very complicated.

Here is the last step in the proof of Steinitz's theorem. The algorithm is that we start with a graph, get the Tutte embedding and use Cremona's lemma to get a polytope.

Lemma 23.3. *The Tutte spring embedding can be realized in the box $[1, M]^2$, where M is proportional to the number of spanning trees in G.*

Proof. The expression $\sum_{(v,v')\in E} w(v, v')vv' = 0$ is the determinant of a matrix. □

Lemma 23.4. *The number of spanning trees in a planar graph $G = (V, E)$ is $\leq 5.3^n$*

This gives an idea of how to get bounds in the quantitative problem.

24 Perles' Theorem and Point and Line Configurations

24.1 Perles' theorem

Last time, we dealt with Steinitz's theorem, which we pretended to prove. Here is a corollary.

Corollary 24.1. *Let $P \subseteq \mathbb{R}^3$ be a convex polytope. Then there exists a $P' \subseteq \mathbb{Q}^3$ such that $\alpha(P) \simeq \alpha(P')$.*

This comes from the proof of Steinitz's theorem, not the theorem itself. The proof actually constructs a rational polytope.

Corollary 24.2. *Let $P \subseteq \mathbb{R}^3$ be a convex polytope. Then there exists a $P' \subseteq \mathbb{Q}^3$ such that $\|P - P'\| < \varepsilon$ and $\alpha(P) \simeq \alpha(P')$.*

Here the norm is the maximum distance between corresponding vertices of the polytopes.

Proof. If the polytope is simplicial, we can just perturb each vertex by some ε to make it rational.

What is nonobvious is that for all P, there exists a sequence of vertex-face perturbations with final polytope $P' \subseteq \mathbb{Q}^3$. □

Here is a conjecture: The first corollary generalizes to all $P \subseteq \mathbb{R}^d$ for $d \geq 4$. It is wrong, however.

Theorem 24.1 (Perles[20], 1960s)**.** *There exist $P \subseteq \mathbb{R}^d$ such that for all $P' \in \mathbb{Q}^d$, $\alpha(P) \not\simeq \alpha(P')$.*

How can we prove this?

24.2 Point and line configurations

Definition 24.1. A **point and line configuration** $K = (V, L)$ is a set of "points" $V = \{v_1, \dots, v_n\}$ and "lines" $L = \{\ell_1, \dots, \ell_m\}$, where $\ell_i \subseteq 2^V$.

This is an abstract set-theoretic object, like a graph.[21]

Definition 24.2. A **realization** of K in $\mathbb{F}^2$ is a map $f : V \to \mathbb{F}^2$ and a map $\tilde{f} : L \to \{\text{lines in } \mathbb{F}^2\}$ such that $v_i \in \ell_j$ iff $f(v_i) \in \tilde{f}(\ell_j)$.

Example 24.1. A **Fano plane** is a (triangular) configuration with $V = \{1, 2, 3, 4, 5, 6, 7\}$.[22] There exists a realization over $\mathbb{F}_2$ but not over $\mathbb{R}$.

[20] Perles was the advisor of Gil Kalai.
[21] Hilbert was interested in these as a possible foundation for geometry.
[22] Look up a picture online!

Example 24.2. The **Pappus configuration** looks like a graph of $K_{3,3}$ with vertices at the intersection points of the edges and two lines connecting the 3 vertices on each side.

Theorem 24.2 (Pappus)**.** *There does not exist a realizatio nof the Pappus configuration over* $\mathbb{R}$.

Proof. The idea is that these middle three vertices are always colinear, but there is no line containing them specified in the configuration. □

Example 24.3. Another example is the **Desargues configuration**.[23]

Theorem 24.3 (Desargues)**.** *The Desargues configuration cannot be realized over* $\mathbb{R}$.

Theorem 24.4. *There exists a configuration* $K = (V, L)$ *that is realizable over* $\mathbb{R}$ *but not over* $\mathbb{Q}$.

Remark 24.1. In fact, there exists a configuration which is realizable over the algebraic numbers $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$.

Proof. The proof is heavily pictoral, so you'll have to read about it in Professor Pak's book. The idea is universality theorems. Basically, we can encode algebraic equations using point and line-configurations. Construct an algebraic equation which does not have solutions over $\mathbb{Q}$. □

[23] I really can't draw this, so look it up online.

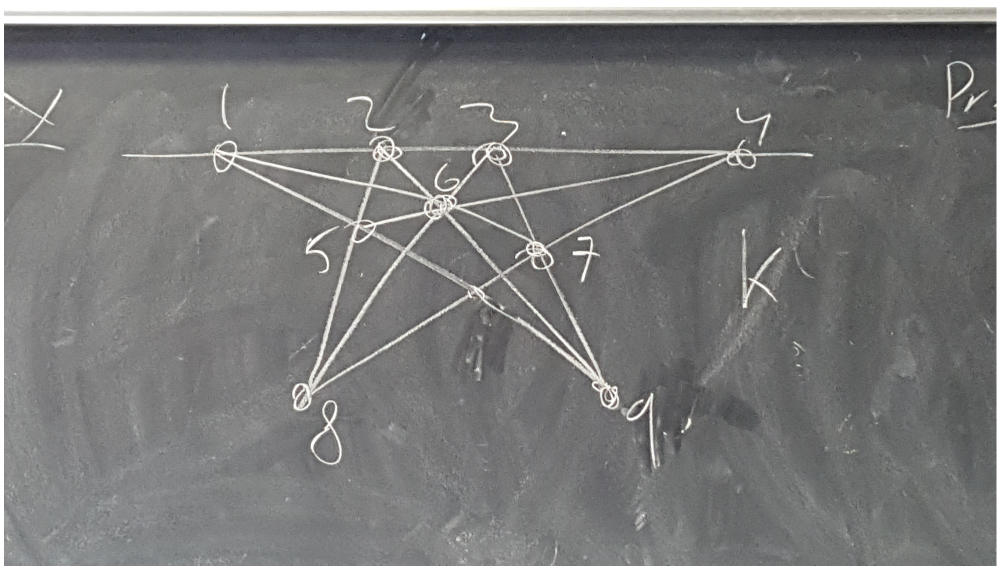
25 Irrational Point and Line Configurations, and Lawrence's Construction

25.1 Irrational point and line confingurations

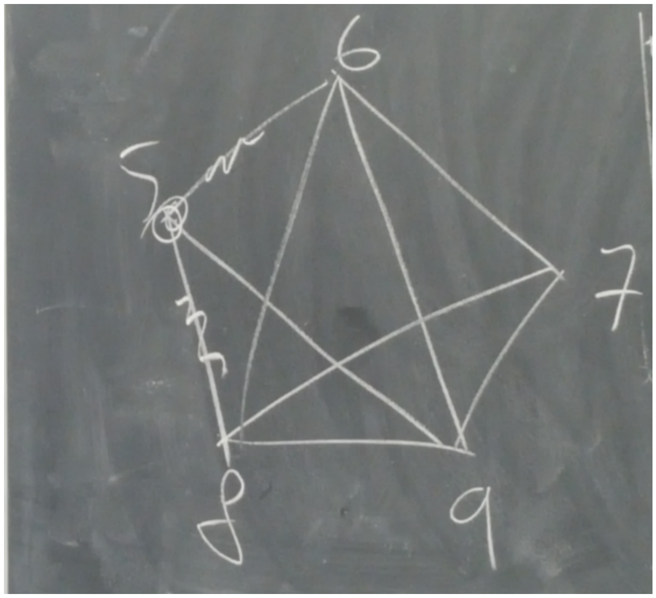
Theorem 25.1 (Perles, 1970s)**.** *There exists $d > 3$ and a convex polytope $P \subseteq \mathbb{R}^d$ such that for all $P \subseteq \mathbb{Q}^d$, $\alpha(P) \not\simeq \alpha(P')$.*

Theorem 25.2. *There exists a point and line configuration $K = (V, L)$ realizable over $\mathbb{R}$ but not over $\mathbb{Q}$.*

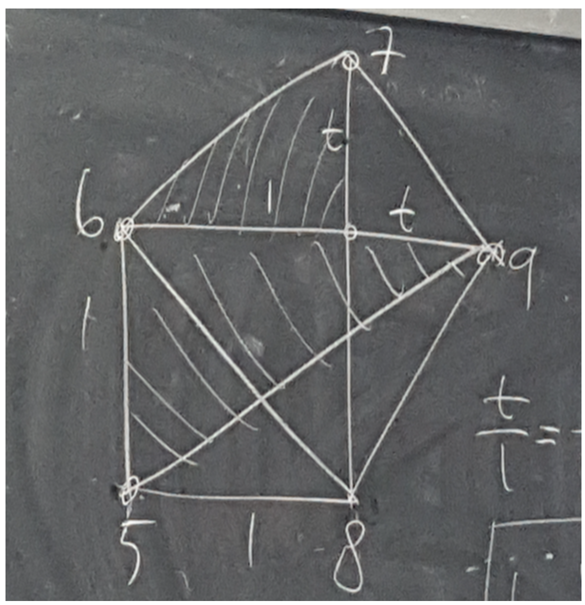
Example 25.1. Here is an actual example. Start with the configuration



Send the line passing through $1, 2, 3, 4$ to ∞. We get the following picture,



where 56 || 78, 69 || 58, 59 || 67, and 58 || 69. Specifically, we set the edge lengths as follows:



From similar triangles, we get $t/1 = 1/(1+t)$. So $t = (\sqrt{5}-1)/2$.

25.2 Lawrence's construction

Theorem 25.3 (Lawrence)**.** *Let $K = (V, L)$ be irrational with $|V| = n$. Then there exists a convex polytope $P \subseteq \mathbb{R}^d$ with $d = n + 3$ and $2n$ vertices such that P is irrational.*

Proof. Let $f : V \to \mathbb{R}^2$ be a realization of K. Let $w = (f(v_i), 1) \in \mathbb{R}^3$. Consider the $2n$ points $x_i = (w_i, e_i), y_i = (w_i, we_i)\mathbb{R}^{3+n}$, where $\mathbb{R}^n = \langle e_1, \ldots, e_n \rangle$. Now let $P = \mathrm{conv}(\{x_i, y_i : i = 1, \ldots, n\})$. Then P has all x_i, y_i as vertices. P is irrational.

To show that P is irrational, suppose that $\sum_{i=1}^n \alpha_i x_i + \beta_i y_i = 0$. Then $\alpha_i = -2\beta_i$ for all i. Then $\sum_{i=1}^n (-2\beta_i)w_i + (\beta_i)w_i = 0$, so $\sum_{i=1}^n \beta_i w_i = 0$. If some of the w_i lie on a line, then we get such a linear relation. Then P has some 5-dimensional faces (containing $x_i, y_i, x_j, y_j, x_k, y_k$), not just 6-dimensional faces. Then v_i, v_j, v_k lie on the same line. $\square$

25.3 Constructing regular n-gons

Gauss was interested in the following question: For which n does there exist a ruler and compass construction of the regular n-gon?

Theorem 25.4 (Gauss)**.** *If $n = 7$, there is no ruler and compass construction of the regular n-gon.*

Theorem 25.5 (Gauss)**.** *If $n = 17$, there is a ruler and compass construction of the regular n-gon.*

26 Mechanical Linkages

26.1 Watt's linkage

Say you want to convert forces into other forces. Out story starts about 200 years ago with James Watt, who invented trains. A steam engine in a train works by having steam pressure build up and create linear motion. Watt created a mechanical linkage that (approximately) converts this linear force into a rotational force that makes the train wheels turn.

For decades, people considered the following problem: Does there exist a mechanical linkage which transfers linear into rotational motion? Chebyshev traveled to England because he was obsessed with this question. For this reason, he studied polynomials which approximate straight lines. Cayley and Sylvester were also interested in this problem. They did not believe that such a linkage could exist.

26.2 Kempe's theorem

Theorem 26.1 (A. Kempe[24], 1880s)**.** *If such a linkage exists, then for any compact portion of an algebraic curve $C \subseteq \mathbb{R}^2$, there exists a linkage with realization space $= C$.*

Think of it like this. Create a bar and joint framework for your linkage, and ground some of the vertices. Then, take a pencil at one vertex and move the framework around (which moves the pencil and draws the curve) with those grounded vertices stationary. Think of making a stencil out of your mechanical linkage.

Proof. We construct polynomials using linkages step by step:[25]

1. Rigidifying linkages: If you have two bars, linked at a joint, you can rigidify them together by adding some extra bars and joints.

2. Coordinates: If we can make lines using mechanical linkages, then we can create coordinates for $\mathbb{R}^2$ with these lines

3. Transfer: We can create a linkage which sets $x = y$.

4. Addition by a constant: We can create a linkage where if we know x, then we can draw $x + c$ for a constant c.

5. Multiplication by a constant: Use a linkage like a pantograph.[26]

6. Addition of vectors: If we know x, y we can construct a linkage that lets us draw $x + y$.

[24] Kempe provided the first incorrect proof of the 4 color theorem.
[25] In effect, this shows how to construct mechanical computers.
[26] Like me, you might be too young to know what this is.

7. Inversion: If we have x, we can create a linkage that lets us draw $1/x$. Since the inversion of a circle is a line, and we can get both a line (by assumption) and a circle (by fixing one end of a bar, putting our pencil on the other vertex, and rotating around the fixed vertex), we can get inversions.

8. Multiplication of vectors: Note that $1/(z-1)+1/(z+1)=2/(z^2-1)$. So since we can do inversion, we can get squares. Then $(x+y)^2-(x-y)^2=4xy$ gives us multiplication. $\square$

There is in fact such a linkage that converts circular motion to linear motion, called the Paucillier linkage.[27] It was invented by Lipkin, but Paucillier took credit for the invention. Watt's linkage is still used in basically very car today. Why don't they use the Paucillier linkage? It has a few more moving parts, and Watt's linkage is good enough.

[27]Check out some videos of it online!

27 Purel's Theorem and Differential Analyzers

27.1 Purel's theorem

Look up a video of a planimeter online.

Proposition 27.1. *There exists a mechanical instrument which "computes" the area of a planar regions.*

Proof. It is sufficient to show that the instrument can measure the area of rectangles. Let's not actually prove this. □

This is a bit different from the mechanical linkages we atalked about last time. This instrument involves a cone and wheels.

Recall Kempe' theorem from last time.

Theorem 27.1 (Kempe, 1880s)**.** *For every semi-algebraic system, there exists a plane linkage which "solves" it.*

By a semi-algebraic system, we mean that we can draw the solution set of $F_1(x_1, x_2, \dots) = 0, F_2(x_1, x_2, \dots) = 0, \dots$, where the F_i are polynomials.

Theorem 27.2 (Purel[28], 1970s)**.** *For every differentiable semi-algebraic system, there exists a 3-dimensional mechanical instrument which "solves" it.*

This is the case where we find the solution to $\sum a_{i,j,k,\ell} x^i y^j \frac{\partial F}{\partial^k x \partial^\ell y} = 0$. Here, the $a_{i,j,k,\ell} \in \mathbb{R}$.

Proof. Integrate the system as many times as you need to get rid of all the derivatives. Then we get a system of integral equations. We can solve integrals using the planimeter. □

27.2 Differential analyzers

Around 1915, Vanniver Bush, an engineer at MIT, built a differential analyzer. This was a machine where you put gears in the right places, and if you operate a crank, then it draws the solution to a differential equation.[29] People came from all over the country to use the machine. Around 1925, this became motorized. The machine was able to solve differential equations of order 6.

Claude Shannon was a student at the time, and he was hired to operate the crank. He realized that this was applicable to boolean logic, and wrote his master's thesis on how to compute boolean logic using an electrical computer. This led to the birth of the modern computer.

In the 1940s UCLA bought a massive differential analyzer for \$250000. By the time it was made it was already obsolete. It never got used.[30]

[28] Purel was one of the first famous American female mathematicians.

[29] Apparently, for harder differential equations, it was a real job to turn the crank.

[30] You can find a hilarious advertisement about it on Professor Pak's website.

28 Isometric Deformation of Polytopes, Inflating Pillows, and Mylar Balloons

28.1 Isometric deformation of polytopes

Bending of polytopes

Definition 28.1. Let $P \subseteq \mathbb{R}^3$ be a convex polyope. An **isometric deformation** of P is a family $\{S_t, t \in [0,1]\}$ continuous in t such that $S_0 = \partial P$ and $S_t \simeq S_0$ for all t, where $\simeq$ means isometric as metric spaces.

Definition 28.2. $S \simeq S'$ if there exists a homeomorphism $\pi : S \to S'$ such that $|xy|_S = |\pi(x)\pi(y)|_{S'}$.

Example 28.1. Take the unit cube, and punch in one corner to make a cube-shaped indent. If the pinched in cube-shaped indent has side length t, then $\{S_t\}$ is an isometric deformation of the unit cube.

28.2 Inflating pillows

Theorem 28.1. *There are no polyhedral inflated pillows.*

What does this mean? Here is the real theorem.

Theorem 28.2. *For every $P \subseteq \mathbb{R}^3$, there exists a continuous isometric deformation $\{S_T, t \in [0,1]\}$ with $S_0 = \partial P$ such that $\{S_t\}$ is volume increasing.*

Example 28.2. In our punched in cube example, the volume of S_t is $1-t^3$. This is volume decreasing with t..

What we mean by there are no inflated pillows is that if you take a rectangle and fill it up so that there is as much stuffing as possible, then it cannot be a polyhedron. Why? It it were, we could fill it with more stuffing using a volume-increasing continuous isometric deformation.[31]

We won't prove the full theorem, but here is the main example.

Example 28.3. The main example is when Γ is a unit cube in $\mathbb{R}^3$. On each face, cut out a corner quare of side length t from the cube. We get a surface with 8 holes. Then push out the sides of this cube as much as possible (think of filling the cube with air so the sides puff out). To get a closed surface, we need to fill in the corner holes; we can do this by turning each one into a triangular pyramid shape.

[31] The teabag problem is to figure out what the maximum volume of such a filled up rectangular pillow is.

Proposition 28.1. *With the above deformation of the cube,* $\mathrm{Vol}(\hat{S}_t) = 1 + c_1 t + c_2 t^2 + c3t_6$, *where* $c_i \in |R$ *and* $c_1 > 0$.

Proof. S_t is contained in a cube of side length $1 + \alpha t^3$. The length of the sides of the square pyramid must be $\sqrt{2}t$. So $\mathrm{vol}(\hat{S}_t) = (1 + 2\sqrt{2}t)^3 - 12(ct) + O(t^2)$. For small t, $\mathrm{vol}(S_t) = 1 + g\sqrt{2}t + O(t^2)$. $\square$

So the deformation of the cube is volume increasing. This method of deformation is what we want to do in general, but the problem is how to make the corners work out. This is difficult in general but still possible.

28.3 Mylar balloons

When you go to the store, you can buy Mylar balloons. These might say happy birthday on them and have a stick to hold them or something. What these really are is a doubly covered circle, inflated with helium. You can actually calculate the shape and volume of such a shape.[32] A company in Minnesota that produces party balloons actually asked Professor Pak to figure out the shape of party balloons so they could manufacture them more efficiently.[33]

[32] People have published papers about this. Professor Pak has oulbished a related paper about the shape of a rectangular pillow.

[33] He declined.

29 Unfolding of Polytopes

29.1 Edge unfolding and Dürer's conejcture

You've probably seen an unfolding of a cube before. Think of the stencil you would need to make if you wanted to make an origami cube. How should we define this in general for other polytopes?

Definition 29.1. Let $P \subseteq \mathbb{R}^3$ be a polytope with $G(P) = (V, E)$. Let T be a spanning tree in G. An **edge unfolding** of P is $S = \partial P \setminus T$, isometric to a polygon in $\mathbb{R}^2$.

How do we know that when we unfold the polytope, it doesn't overlap with itself? In fact, it can.

Example 29.1. Take a cube, and remove a triangular pyramid from one of the corners (but close up the figure). Make an unfolding including cuts along two edges of the triangle left. When you unfold it, the triangle flap part will overlap with other faces of the polygon.

Here is a conjecture (which is still open).

Theorem 29.1 (Dürer, c. 1950)**.** *For all P, there exists a spanning tree $T \subseteq G(P)$ such that $\partial P \setminus T$ has a non-overlapping unfolding.*

Many people have worked on this problem, but no one has proved it yet. It may not be true![34] In some sense, this theorem says that our original definition makes sense.

Theorem 29.2 (M. Ghomi, c. 2012)**.** *For every polytope $P \subseteq \mathbb{R}^3$, there exists an affine transformation M such that MP has a non-overlapping unfolding.*

The idea of this proof is to use an affine transformation to stretch the affine polytope really thin.

29.2 The geodesic distance problem and source unfolding

Say we have 2 points on a cube. What is the shortest path on the cube from 1 point to the other? We can figure this out by looking at different unfoldings and taking the straight-line distance between the points. But this may be difficult if there are a lot of faces of your polytope; there could be a lot of possible unfoldings!

Theorem 29.3. *For all convex polytopes $P \subseteq \mathbb{R}^3$, if $S := \partial P$, then $|x, y|_S$ can be computed in polynomial time.*

[34] Maybe it's a good thing that this doesn't have a lot of practical applications.

The idea is called **source unfolding**. We'll discuss it using a cube. Fix $x \in S$, and let K be our **cut locus**. This is a set of points $K = \{z \in S : \exists \geq 2 \text{ shortest paths } z \to s\}$. Then no points on the face containing x are in K, and the edges with only 1 vertex touching this face containing x are completely contained in K. The idea is that no shortest path will intersect K. If we cut along K, we get an unfolding (but not necessarily an edge unfolding).

Here is the algorithm for this theorem:

1. Compute source unfolding at x (harder step)
2. Compute $|xy|_U$ (easier step)

How do we find the source unfolding? We use a continuous version of Dijkstra's algorithm.

Here is a conjecture:

Theorem 29.4. *For all d and convex $P \subseteq \mathbb{R}^d$, the number of cbonatorial shortest parths on ∂P is $n^{O(d^2)}$, where n is the number of facets of P.*

A combinatorial shortest path is a shortest path where we record the facets that the path passes through.

Theorem 29.5 (Miller-Pak)**.** *If the previous conjecture holds, then source unfolding for $P \subseteq \mathbb{R}^d$ for fixed d can be computed in polynomial time.*

This is a very technical result.[35]

[35] But it was hard to get published because everyone thought it should be trivial.