Math 250A Lecture 12 Notes

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1 More on Projective Modules

1.1 Projective modules as direct sums

Recall that P is a projective module if it satisfies the following commutative diagram for exact sequences of modules M and N:

$$\begin{tikzcd} M \arrow[r] & N \arrow[r] & 0 \\ & P \arrow[u] \arrow[ul, dashed] & \end{tikzcd}$$

We also showed that P projective iff $P \oplus Q$ is free for some Q. Are all submodules of free modules projective? The answer is no.

Example 1.1. Here is a non-projective submodule of a free module. Let $R = K[x, y]$, where K is a field, and let $I = (x, y)$, the ideal of polynomials with constant term 0. Look at $R \oplus R \xrightarrow{g} I \to 0$ where $(1, 0) \mapsto x$ and $(0, 1) \mapsto y$. If I is projective, then there exists some map $f : I \to R \oplus R$ such that $gf(a) = a$. Now suppose that $f(x) = (a, b)$ and $f(y) = (c, d)$. Then $ax + by = x$ and $cx + dy = y$. Then $y(a, b) = x(c, d))$, so $ya = xc$ and $yb = xd$. There are no polynomials satisfying this because $ax + by = x$ implies that $a = 1 + yp$ (where p is a polynomial), and $ya = xc$ implies that a cannot be $1 + yp$.

1.2 Eilberg-Mazur swindle

This is a technique useful for proving $1 = 0$. Here is a basic example.

Example 1.2. Start with $1 + (-1) = 0$. Then

$$0 = (1 + (-1)) + (1 + (-1)) + \cdots$$

$$1 = 1 + (-1 + 1) + (-1 + 1) + \cdots,$$

so we have shown that $1 = 0$.

We assumed two things in the above example:

1. 1 has an additive inverse -1.
2. All infinite sums make sense.

The second condition is violated in $\mathbb{Z}$, but we can use this technique to show that one of these two conditions does not hold.

Example 1.3. Knots have no inverse. Suppose we have a closed loop with a knot in it. Is there another knot we can put on the loop that will cancel out the first knot? The answer is no. Apply the swindle: add infinite numbers of knots, making each successive knot smaller so the knots all fit on the loop. Then the above contradiction would occur, so a knot must not have an additive inverse.

Example 1.4. Suppose P is projective. Then $P \oplus Q = F$, where F is free. Then Q is also projective. We can take Q to be free (in fact equal to F). Think of free modules as 0 in some sense. So $P \oplus Q$ is free means that Q is a sort of additive inverse of P (again, if we ignore free modules). So infinite sums are defined, and we can use the swindle to get that $P = 0$ if we ignore free modules. What we mean here is that $P \oplus Q$ is free for some free module Q. The catch is that this free module Q is not finitely generated.

2 Tensor Products

This is covered in Chapter XVI in Lang, but we will cover it here. This is something you really should know.

2.1 Construction and universal property

Definition 2.1. A *bilinear* map $f : X \times Y \to Z$ is a map such that $f(\cdot, y)$ is linear for fixed y and $f(x, \cdot)$ is linear for fixed x.

Definition 2.2. Suppose R is a commutative[1] ring, and suppose that M and N are R-modules. The *tensor product* $M \otimes N$ is the module such that if $f : M \times N \to P$ is bilinear, then there exists a linear map $\tilde{f} : M \otimes N \to P$ such that the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "f"'] & M \otimes N \arrow[d, dashed, "\tilde{f}"] \\ & P \end{tikzcd}$$

[1] This assumption is not necessary, but it simplifies things for now.

To construct $M \otimes N$, take the free module on elements $m \otimes n$ with $m \in M$ and $n \in N$. We get linear maps from this to $P : m \otimes n \mapsto f(m,n)$. Take the quotient by all elements of the form

$$(m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n$$

$$m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2$$

$$(rm) \otimes n - r(m \otimes n)$$

$$m \otimes (rn) - r(m \otimes n).$$

Taking the quotient by these elements enforces relations we want, such as

$$(rm) \otimes n = r(m \otimes n) = m \otimes (rn),$$

so the tensor product exists.

Now that we have constructed the tensor product, what does it look like? We have the identity

$$(M_1 \oplus M_2) \otimes N \cong (M_1 \otimes N) \oplus (M_2 \otimes N),$$

which says that a bilinear map $(M_1 \oplus M_2) \otimes N \to P$ is the same as a pair of bilinear maps from $(M_1 \otimes N) \to P$ and $(M_2 \otimes N) \to P$. Similarly, we have the identity

$$R \otimes M \cong M,$$

which says that bilinear maps $R \times M \to P$ are the same as linear maps from $M \to P$.

Example 2.1.

$$R^m \otimes R^n \cong R^{m+n}$$

If V, W are vector spaces with bases $\{v_i\}$ and $\{w_j\}$, then $V \otimes W$ has basis $v_i \otimes w_j$.

2.2 Exact sequences and the tensor product

Proposition 2.1. *Suppose $A \to B \to C \to 0$ is exact. Then so is*

$$A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

Remark 2.1. This does not hold if we put a $0 \to$ before both of these sequences. We say that $\otimes M$ is *right exact.*

Proof. To prove things about the tensor product, forget the construction of the tensor product using relations and instead use the universal property.

Homomorphisms $A \otimes B \to C$ are bilinear maps $A \times B \to C$, which are linear maps $A \to \mathrm{Hom}_R(B, C)$. Think of this as an analogue of the fact that functions $R \times S \to T$ are the same as functions from R to the set of functions from S to T.

The key point of this proof is that $A \to B \to C \to 0$ is exact if and only if

$$\mathrm{Hom}(A, M) \leftarrow \mathrm{Hom}(B, M) \leftarrow \mathrm{Hom}(C, M) \leftarrow 0$$

is exact. We leave this as an exercise.[2]

We want to show that $A \otimes N \to B \otimes N \to C \otimes N \to 0$ is exact. Then this is equivalent to the following sequence being exact:

$$\mathrm{Hom}(A \otimes N, M) \leftarrow \mathrm{Hom}(B \otimes N, M) \leftarrow \mathrm{Hom}(C \otimes N, M) \leftarrow 0.$$

Then, using our identification of homomorphisms $A \otimes N \to M$ with linear maps $A \to \mathrm{Hom}_R(N, M)$, this is equivalent to the following sequence being exact:

$$\mathrm{Hom}(A, \mathrm{Hom}(N, M)) \leftarrow \mathrm{Hom}(B, \mathrm{Hom}(N, M)) \leftarrow \mathrm{Hom}(C, \mathrm{Hom}(N, M)) \leftarrow 0.$$

And this is exact by applying the key point again. $\square$

We can now calculate $M \otimes N$. Pick $R^a \to R^b \to M \to 0$, where R^a, R^b are free. Pick relations generating $\ker(R^b \to M)$ and pick a set of b generators of M. Tensoring with N gives us that

$$R^a \otimes N \to R^b \otimes N \to M \otimes N \to 0$$

is exact. So we get

$$N^a \to N^b \to M \otimes N \to 0,$$

which makes $M \otimes N = N^b / \mathrm{im}(N^a \to N^b)$.

Example 2.2. We can find $M \otimes N$ for finitely generated abelian groups M, N. Recall that finitely generated abelian groups are direct sums of copies of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$. Since $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$, it is enough to work out a few cases:

1. $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$
2. $\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}.m\mathbb{Z}$
3. $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}/m\mathbb{Z}$
4. $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/(\gcd(m, n)\mathbb{Z})$.

To obtain this last result, take the exact sequence

$$\mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0.$$

Then the sequence

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}/n\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}) \to 0$$

is exact, so

$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/n\mathbb{Z})/m(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(\gcd(m, n)\mathbb{Z}).$$

[2] This was an exercise from last lecture, but Professor Borcherds suspects that no one actually does them.

Example 2.3.

$$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$$
$$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = 0$$
$$\mathbb{Z}/9\mathbb{Z} \otimes \mathbb{Z}/12\mathbb{Z} = \mathbb{Z}/3\mathbb{Z}$$

The tensor product is not left exact. Look at $0 \to \mathbb{Z} \xrightarrow{\times 2} Z \to \mathbb{Z}/2\mathbb{Z} \to 0$. The following sequence is not exact:

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

2.3 More examples and properties

Definition 2.3. An *algebra* S over a ring R is a commutative ring with a homomorphism $R \to S$ that makes S an R-module.

You can think of algebras as modules with multiplication.

Example 2.4. Let S, T be algebras over R. Then $S \otimes_R T$ is a push-out of S, T over R.

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \otimes_R T \end{array}$$

Check that $S \otimes_R T$ is a commutative ring. We need a bilinear map $(S \otimes T) \times (S \otimes T) \to (S \otimes T)$. This is a linear map from $S \otimes T \otimes S \otimes T \to S \otimes T$. This relies on associativity of the tensor product; $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ because maps from each to M are trilinear maps $A \times B \times C \to M$. We have a map $S \otimes S \to S$ given by the product on S. Same for $T \otimes T \to T$. So we get a map $S \otimes T \otimes S \otimes T \to S \otimes S \otimes T \otimes T \to S \otimes T$ by sending $(s_1 \otimes t_1) \times (s_2 \otimes t_2) \to s_1 s_2 \otimes t_1 t_2$. We leave verification of the pushout property as an exercise.

Example 2.5. $S = K[x]$ and $T = K[y]$ with bases $\{x^m\}$ and $\{y^n\}$, respectively. $S \otimes_R T$ has a basis $x^m \otimes y^n$. This can be identified as as the polynomial ring $K[x, y]$ via the map $x^m \otimes y^n \to x^m y^n$.

Example 2.6. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a ring. $\mathbb{C}$ has basis $\{1, i\}$, so $\mathbb{C} \otimes \mathbb{C}$ has basis $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\}$. Calculating a few products, we get

$$(i \otimes i)(i \otimes i) = i^2 \otimes i^2 = -1 \otimes -1 = 1 \otimes 1$$
$$(1 \otimes 1)(a \otimes b) = (a \otimes b)$$
$$(1 \otimes 1 + i \otimes i)^2 = 2(1 \otimes 1 + i \otimes i).$$

Call $e = (1 \otimes 1 + i \otimes i)/2$. Then $e^2 = e$, so e is idempotent. Then this ring splits as a product, so $\mathbb{C} \otimes \mathbb{C} = e(\mathbb{C} \otimes \mathbb{C}) \times (1 - e)\mathbb{C}(\mathbb{C} \otimes \mathbb{C}) \cong \mathbb{C} \times \mathbb{C}$.

Example 2.7. The tensor product satisfies all the axioms of a commutative *semiring* (a ring but without subtraction)

1. $(A \otimes B) \otimes C$
2. $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$
3. $A \otimes B \cong B \otimes A$
4. $A \oplus B \cong B \oplus A$
5. $(A \oplus B) \oplus C \cong A \oplus (B \oplus C)$
6. $R \otimes A \cong A$.

If we want to construct a ring out of this structure, we have a few problems:

1. The set of all modules is not a set.
2. There is no subtraction.

   This can be circumvented by constructing the set of all pairs $M - N$ for M, N modules under some equivalence relation.
3. By the swindle, $M = 0$ for any M.

We circumvent problems 1 and 3 by only considering finitely generated modules.[3]

Example 2.8. Take $R = \mathbb{Z}$, the integers. The finitely generated modules are all of the form $\mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{n_2} \oplus (\mathbb{Z}/4\mathbb{Z})^{n_4} \oplus (\mathbb{Z}/8\mathbb{Z})^{n_8} \oplus \cdots \oplus (\mathbb{Z}/3\mathbb{Z})^{n_3} \oplus \cdots$. So we get a basis $\{n_i b_i\}$, where we allow the n_i to be positive or negative. The product is $b_0 \times b_n = b_n$ and $b_{p^a} \times b_{p^b} = b_{p^{\min(a,b)}}$.

2.4 Tensor products of noncommutative rings

When R is a noncommutative ring, $M \otimes_R N$ is only defined for M a right module and N a left module. This is because we need

$$(mr) \otimes n = m \otimes (rn).$$

Secondly, $M \otimes_R N$ is only an abelian group, not an R-module. We have that

$$mr \otimes n = m \otimes rn,$$

but multiplying by s gives us

$$mrs \otimes n = m \otimes srn,$$

even though we want $m(rs) \otimes n = m \otimes (rs)n$.

[3]This leads into K-theory, where you consider the ring of finitely generated modules over a ring R.