Math 249 Lecture 2 Notes

Daniel Raban

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1 Multisets

Definition 1.1. A *multiset* is a set that allows repeated elements. Formally, we can define it as a function from a set to the natural numbers, where $f(a) = n$ means that the element a is contained n times in the multiset. Then a set is a multiset where the range of f is a subset of $\{0, 1\}$.

Denote multisets by $\left[\!\left[a_1^{k_1}, \dots, a_\ell^{k_\ell}\right]\!\right]$, and let $n = k_1 + \cdots + k_\ell$.

Example 1.1. The multiset of the letters in the word "MISSISSIPPI" is $[\![M^1, I^4, S^4, P^2]\!]$.

If we want to count multisets, we can use the same method as before with ordinary binomial coefficients. The stabilizer of a multiset is $S_{k_1} \times \cdots \times S_{k_\ell}$, a product of Young subgroups. Then the number of permutations of a multiset is

$$\binom{n}{k_1, \dots, k_\ell} := \frac{n!}{k_1! \cdots k_\ell!}.$$

With this notation, the usual binomial coefficient is just $\binom{n}{k} = \binom{n}{k, n-k}$.

If we have n-letter words $(\omega_1, \dots, \omega_n)$ with $\omega_i \in A$, they can be expressed as a map $[n] \xrightarrow{\omega} A$. Then S_n acts on these words by $\sigma \cdot \omega = \omega \circ \sigma^{-1}$; we precompose by the permutation because $[n]$ is the domain, not the codomain, and we use σ^{-1} to make the associative law work out correctly.

1.1 Counting k-element multisets of $[n]$

Definition 1.2. The multiset coefficient $\left\langle {n \atop k} \right\rangle$ is the number of k-element multisets of $[n]$.

A multiset of this type is $[[1^{r_1}, \dots, n^{r_n}]]$, where $r_1 + \cdots + r_n = k$ and $r_i \geq 0$. This is counting *compositions* of k with n parts (different from partitions). Note that the number of these compositions where $r_i > 0$ (strict inequality) is the same as the number of compositions where the sum is $k - n$ ($\left\langle {n \atop k} \right\rangle$). We will look at 3 solutions to this problem.

1.1.1 Counting multisets using generating functions

We try to find $\sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle x^k$. Think of it as a weighted count of all multisubsets of $[n]$. Given a finite set A, $|A| = \sum_{a\in A} 1$. A weighted sum will put weights instead of the 1s here. This is useful because

$$\sum_{\substack{\rho=(r_1\dots,r_n)\\ r_i\geq 0}} x^{|\rho|} = \sum_{\rho} x^{r_1+\cdots+r_n}.$$

In fact, consider the fact that $|A\times B| = |A|\,|B|$. Using the concept of the sum above, we can see that this is true because the weights (all wights being 1) multiply. In fact, it makes sense to make arbitrary weights multiply here.

Our set here is $\mathbb{N}^n$, since $(r_1,\dots,r_n)\in\mathbb{N}^n$. So weight $\mathbb{N}$ with $r\mapsto x^r$. Then

$$\sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle x^k = F(x)^n, \qquad \text{where } F(s) = \sum_{r\geq 0} x^r = \frac{1}{1-x}.$$

$$\sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle x^k = \left(\frac{1}{1-x}\right)^n = (1-x)^{-n}.$$

Note the similarity to the binomial theorem, which gives us that $\sum_{k=0}^{\infty}\binom{n}{k}x^k = (1+x)^n$.

Recall Newton's binomial coefficients $\binom{\alpha}{k} := \frac{[\alpha]_k}{k!}$ from last lecture. Analogously, we have

Theorem 1.1 (Newton's Binomial theorem)**.** *For any $\alpha\in\mathbb{R}$,*

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k.$$

Proof. To find the coefficient of the x^k term of the Maclaurin series of $(1+x)^{\alpha}$, differentiate both sides k times and evaluate at $x=0$. We get that the coefficient of x^k is equal to $\alpha(\alpha-1)\cdots(\alpha-k+1)/k! = [\alpha]_k/k! = \binom{\alpha}{k}$. □

We may then write

$$(1-x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k,$$

which implies that

$$\begin{aligned}\left\langle {n \atop k} \right\rangle &= (-1)^k\binom{-n}{k} = (-1)^k\frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}\\ &= \frac{n(n+1)\cdots(n+k-1)}{k!} = \frac{[n+k-1]_k}{k!}\\ &= \binom{n+k-1}{k}.\end{aligned}$$

Example 1.2. $\left\langle {n \atop k} \right\rangle$ is the number of polynomials $x_1^{r_1} \cdots x_n^{r_n}$ of degree k in n variables, which is $\dim \mathbb{Q}[x_1, \ldots, x_n]_{(k)}$. So $\dim \mathbb{Q}[x_1, x_2, x_3]_{(4)} = \left\langle {3 \atop 4} \right\rangle = \binom{3+4-1}{3} = \binom{6}{3} = 20$.

Remark 1.1. What do we mean when we use generating functions? In general, we don't want to think about it analytically, bothering with radii of convergence, which in some cases may be 0 (such as for the power series of factorials). Instead, consider these as *formal power series*, in the algebraic sense. In this case, the $F(x) = 1/(1-x)$ we defined before can be interpreted in the sense that $(1-x)\sum_{r\geq 0} x^r = 1$. In this case, Newton's Binomial Theorem still holds; we just have to define $(1+x)^\alpha := \exp(\alpha \log(1+x))$, where $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ and $\log(1+x) = \sum_{n=1}^{\infty} x^n/n$.