

# Math 247A Lecture 13 Notes

Daniel Raban

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## 1 The Hardy-Littlewood-Sobolev Inequality

### 1.1 Failure of bounds for $L^1$ vector-valued maximal function

Let  $f : \mathbb{R}^d \rightarrow \ell^1$ ,  $f = \{f_n\}_{n \geq 1}$  with  $|f(x)| = \sum_{n \geq 1} |f_n(x)|$ . We define the  $L^1$  vector-valued maximal function as  $\overline{M}_1 f(x) = \sum_{n \geq 1} M f_n(x)$ . The the following claims fail:

1.  $|\{x : \overline{M}_1 f(x) > \lambda\}| \leq \frac{1}{\lambda} \|f\|_{L^1}$  uniformly for  $\lambda > 0, f \in L^1$ .
2. For  $1 < p < \infty$ ,  $\|\overline{M}_1 f\|_{L^p} \lesssim \|f\|_{L^p}$  uniformly for  $f \in L^p$ .

Fix  $d = 1$ . Take  $[0, 1]$  and subdivide it into intervals  $I_1, \dots, I_N$  of equal length. Let

$$f_n = \begin{cases} \mathbb{1}_{I_n} & 1 \leq n \leq N \\ 0 & n > N. \end{cases}$$

Let  $f = \{f_n\}_{n \geq 1}$ . Then

$$|f(x)| = \sum_{n \geq 1} |f_n(x)| = \mathbb{1}_{[0,1]}(x) \in L^p \quad \forall 1 \leq p \leq \infty,$$

$$\|f\|_{L^p} = 1 \quad \forall 1 \leq p \leq \infty.$$

On the other hand,

$$\begin{aligned} \overline{M}_1 f(x) &= \sum_{n=1}^N M f_n(x) \\ &= \sum_{n=1}^N \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \mathbb{1}_{I_n}(y) dy \end{aligned}$$

For  $x \in [0, 1]$ ,  $\overline{M}_1 f(x) \geq \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{2(n/N)} \cdot \frac{1}{N} \gtrsim \log(N)$ . This tells us that

$$|\{x : \overline{M}_1 f(x) > \frac{1}{10} \log N\}| \geq 1,$$

$$\|\overline{M}_1 f\|_{L^p} \gtrsim \log N.$$

## 1.2 The Hardy-Littlewood-Sobolev inequality

**Theorem 1.1** (Hardy-Littlewood-Sobolev). *Fix  $1 < p < r < \infty$  and  $1 < q < \infty$  such that  $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ . Then*

$$\|f * g\|_{L^r} \lesssim \|f\|_p \|g\|_{L^{q,\infty}}^*,$$

*uniformly for  $f \in L^p, g \in L^{q,\infty}$ . In particular, for  $0 < \alpha < d$ ,*

$$\left\| f * \frac{1}{|x|^\alpha} \right\|_{L^r} \lesssim \|f\|_{L^p},$$

*provided  $1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{d}$ .*

*Proof.* Fix  $g \in L^{q,\infty}$ . We may assume that  $\|g\|_{L^{q,\infty}}^* = 1$ . We want to show that the sublinear operator  $f \xrightarrow{T} f * g$  is of strong type  $(p, r)$ . By the Marcinkiewicz interpolation theorem, it suffices to show  $T$  is of weak type  $(p, r)$  for all  $1 < p < r < \infty$  such that  $1 + 1/r = 1/p + 1/q$ . Say the target is strong-type  $(p_0, r_0)$ . Then choose  $1 < p_1 < p_0 < p_2 < \infty$  and write  $\frac{1}{p_0} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ . By Marcinkiewicz, if  $T$  is of weak-type  $(p_1, r_1)$  and  $(p_2, r_2)$ , then  $T$  is of strong-type  $(p_0, \tilde{r})$ , where

$$\frac{1}{\tilde{r}} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} = \theta \left[ \frac{1}{p_1} + \frac{1}{q} + -1 \right] = (1-\theta) \left[ \frac{1}{p_2} + \frac{1}{q} - 1 \right] = \frac{1}{p_0} + \frac{1}{q} + 1 = \frac{1}{r_0}.$$

Let's show  $T$  is of weak-type  $(p, r)$ :

$$|\{x : |(f * g)(x)| > \lambda\}| \lesssim \left( \frac{\|f\|_p}{\lambda} \right)^r$$

We may rescale so that  $\|f\|_p = 1$ . Write  $g = g_1 + g_2 = f \mathbb{1}_{\{|g| \leq R\}} + g \mathbb{1}_{\{|g| > R\}}$ . Then

$$|\{x : |(f * g)(x)| > \lambda\}| \lesssim |\{x : |(f * g_1)(x)| > \lambda/2\}| + |\{x : |(f * g_2)(x)| > \lambda/2\}|$$

By Chebyshev,

$$\begin{aligned} |\{x : |(f * g_1)(x)| > \lambda/2\}| &\lesssim \frac{\|f * g_1\|_s^s}{\lambda^s} \\ &\lesssim \frac{\|f\|_p^s \|g_1\|_{ps/(ps+p-s)}^s}{\lambda} \\ &\lesssim \lambda^{-s} \int_0^\infty \left( \int_0^\infty \alpha^{ps/(ps+p-s)} \mathbb{1}_{\{|g_1| > \alpha\}} \frac{d\alpha}{\alpha} \right)^{(ps+p-s)/p} \\ &= \lambda^{-s} \int_0^\infty \left( \int_0^R \alpha^{ps/(ps+p-s)} \mathbb{1}_{\{|g| > \alpha\}} \frac{d\alpha}{\alpha} \right)^{(ps+p-s)/p} \\ &= \lambda^{-s} (\|g\|_{L^{q,\infty}}^*)^{q \cdot (ps+p-s)/p} \left( \int_0^R \alpha^{ps/(ps+p-s)-q} \frac{d\alpha}{\alpha} \right)^{(ps+p-s)/p} \end{aligned}$$

$$\lesssim \lambda^{-s} R^{s-q \cdot (ps+p-s)/p-s},$$

Provided  $\frac{1}{q} > 1 + \frac{1}{s} - \frac{1}{p}$ . Since  $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}$ , this means that  $s > r$ .

On the other hand, by Chebyshev,

$$\begin{aligned} |\{x : |(f * g_1)(x)| > \lambda/2\}| &\lesssim \frac{\|f * g_2\|_p^p}{\lambda^p} \\ &\lesssim \frac{\|f\|_p^p \|g_2\|_1^p}{\lambda^p} \\ &\lesssim \lambda^{-p} \left( \int_0^\infty |\{x : |g_2|(x) > \alpha\}| d\alpha \right)^p \\ &\lesssim \lambda^{-p} (\|g\|_{L^{q,\infty}}^*)^{pq} \left( \int_R^\infty \alpha^{-q} dx \right)^p \\ &\lesssim \lambda^{-p} R^{(1-q)p}. \end{aligned}$$

Optimize in  $R$ :

$$\begin{aligned} \lambda^{-s} R^{s-q \cdot (1+1/s-1/p)} &= \lambda^{-q} R^{(1-q)p} \\ \lambda^{p-s} &= R^{(1-q)(p-s)} R^{q/p \cdot (p-s)} \\ \lambda &= R^{1-q+q/p} = R^{q(q/q+q/p-1)} = R^{q/r}. \end{aligned}$$

So we optimize at  $R = \lambda^{r/q}$ .

So

$$\begin{aligned} |\{x : |f * g|(x) > \lambda\}| &\lesssim \lambda^{-p} \lambda^{r/q \cdot p(1-q)} \\ &\lesssim \lambda^{-p(1-r/q+r)} \\ &\lesssim \lambda^{-pr(1/r-1/q+1)} \\ &\lesssim \lambda^{-pr/p} \\ &\lesssim \lambda^{-r}. \end{aligned}$$

□

Although we have just proven this claim, here is Hedberg's proof of  $\|f * \frac{1}{|x|^\alpha}\|_r \lesssim \|f\|_p$  whenever  $0 < \alpha < d$  and  $1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{d}$ .

*Proof.* Fix  $x \in \mathbb{R}^d$ . Then

$$\left( f * \frac{1}{|x|^\alpha} \right) (x) = \int \frac{f(y)}{|x-y|^\alpha} dy = \int_{|x-y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy + \int_{|x-y| > R} \frac{f(y)}{|x-y|^\alpha} dy.$$

$$\begin{aligned}
\left| \int_{|x-y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \right| &\leq \sum_{\substack{r \in 2^{\mathbb{Z}} \\ r \leq R}} \int_{R \leq |x-y| \leq 2r} \frac{f(y)}{|x-y|^\alpha} dy \\
&\lesssim \sum_{r \leq R} r^{-\alpha} r^d \frac{1}{|B(x, 2r)|} \int_{B(x, 2r)} |f(y)| dy \\
&\lesssim R^{d-\alpha} Mf(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left| \int_{|x-y| > R} \frac{f(y)}{|x-y|^\alpha} dy \right| &= \left| f * \frac{\mathbb{1}_{\{|x| > R\}}}{|x|^\alpha} \right| (x) \\
&\lesssim \|f\|_p \left\| \frac{\mathbb{1}_{\{|x| > R\}}}{|x|^\alpha} \right\|_{p'} \\
&\lesssim \|f\|_p \int_R^\infty \frac{r^{d-1}}{r^{\alpha p'}} dr \\
&\lesssim \|f\|_p R^{d/p' - \alpha} \\
&\lesssim \|f\|_p R^{d(1-1/p - \alpha/d)} \\
&\lesssim \|f\|_p R^{-d/r}.
\end{aligned}$$

Optimize in  $R$ : choose

$$\begin{aligned}
R^{d-\alpha} Mf(x) &= \|f\|_p R^{-d/r} \\
R^{d/p} &= R^{d(1-\alpha/d+1/r)} = \frac{\|f\|_p}{Mf(x)}.
\end{aligned}$$

So

$$\left| \left( f * \frac{1}{|x|^\alpha} \right) (x) \right| \lesssim \|f\|_p \left( \frac{\|f\|_p}{Mf(x)} \right)^{-p/r} \lesssim Mf(x)^{p/r} \|f\|_p^{1-p/r}.$$

So

$$\begin{aligned}
\left\| f * \frac{1}{|x|^\alpha} \right\|_r &\lesssim \|f\|_p^{1-p/r} \|(Mf)^{p/r}\|_r \\
&\lesssim \|f\|_p^{1-p/r} \|Mf\|_p^{p/r} \\
&\lesssim \|f\|_p.
\end{aligned}$$

□