Math 210A Lecture 1 Notes

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1 Introduction to Category Theory

1.1 Categories and subcategories

Definition 1.1. A **category** $\mathcal{C}$ is

1. a class[1] $\mathrm{Obj}(\mathcal{C})$ of **objects**,
2. for each $A, B \in \mathrm{Obj}\,\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of **morphisms** from A to B (we write $f : A \to B$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$),
3. a composition map $\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ for all $A, B, C \in \mathrm{Obj}(\mathcal{C})$ (we write this as $(f, g) \mapsto g \circ f$),

such that

1. for each $A \in \mathrm{Obj}(\mathcal{C})$, we have an **identity morphism** $\mathrm{id}_A : A \to A$ such that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$ for all $f : A \to B, g : B \to A$ and $B \in \mathrm{Obj}(\mathcal{C})$.
2. $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B, g : B \to C, h : C \to D$ with $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$.

Notation: we usually say $A \in \mathcal{C}$ to mean $A \in \mathrm{Obj}(\mathcal{C})$.

Definition 1.2. A category is **small** if $\mathrm{Obj}(\mathcal{C})$ is a set.

Example 1.1. Set is the category of sets. $\mathrm{Obj}(\mathrm{Set}) = \{\text{sets}\}$. $\mathrm{Hom}_{\mathrm{Set}}(A, B) = \{\text{functions } f : A \to B\}$.

Definition 1.3. A **semigroup** S is a pair $(S, \cdot)$ of a set S and a binary operation $\cdot : S \times S \to S$ on S that is associative. A **homomorphism of semigroups** is a function $f : S \to T$ of semigroups such that $f(a \cdot_S b) = f(a) \cdot_T f(b)$ for all $a, b, \in S$.

[1] We cannot use sets here because, for example, there is no set of all sets.

The idea of a homomorphism is that the function “respects” the operations on S and T. Sometimes, we write ab when we mean $a \cdot b$.

Example 1.2. The category Semi is the category with objects being semigroups and morphisms being homomorphisms of semigroups.

Definition 1.4. A **subcategory** $\mathcal{D}$ of a category $\mathcal{C}$ is a category with

1. $\mathrm{Obj}(\mathcal{D})$ a subclass of $\mathrm{Obj}(\mathcal{C})$,
2. $\mathrm{Hom}_{\mathcal{D}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$ for all $A, B \in \mathcal{D}$,
3. the composition in $\mathcal{D}$ agrees with the composition in $\mathcal{C}$,
4. the identity $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ for $A \in \mathcal{D}$ is the identity in $\mathrm{Hom}_{\mathcal{D}}(A,A)$.

Example 1.3. Here is a nonexample. Semi is not a subcategory of Set.

1.2 Monoids and groups

Definition 1.5. A **monoid** S is a semigroup with an identity element $e \in S$ such that $ex = x = xe$ for all $x \in S$. A **homorphism of monoids** is a function $f : S \to T$ of monoids such that $f(ab) = f(a)f(b)$ for all $a, b \in S$ and $f(e_S) = e_T$.

Example 1.4. The category Mon is the category with objects being monoids and morphisms being homomorphisms of monoids. Mon is a subcategory of Semi.

Example 1.5. A monoid G gives a category $\mathbb{G}$ with $\mathrm{Obj}(\mathbb{G}) = \{G\}$ and $\mathrm{Hom}_{\mathbb{G}}(G,G) = \{\text{elements of } G\} = G$. For all $g, h \in G$, we define $g \circ h = g \cdot h$.

This goes the other way, as well. If you have a category with one object, then its morphisms form a monoid.

Definition 1.6. A **group** G is a monoid in which every element has an inverse; i.e. for every $g \in G$, there exists a $g^{-1} \in G$ such that $g \cdot g^{-1} = e = g^{-1} \cdot g$.

Example 1.6. Grp is the category of groups. The objects are groups, and the morphisms are homomorphisms of semigroups between groups (“group homomorphisms”). These are also monoid homomorphisms because $f(g) = f(eg) = f(e)f(g)$ implies that $e = f(e)$ by multiplication by $f(g)^{-1}$. Also, $e = f(gg^{-1}) = f(g)f(g^{-1})$ implies that $f(g^{-1}) = f(g)^{-1}$.

Definition 1.7. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is **full** if $\mathrm{Hom}_{\mathcal{D}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$ for all $A, B \in \mathcal{D}$.

Example 1.7. Grp is a full subcategory of Semi.

Definition 1.8. A group G is **abelian** if its operation is commutative; i.e. $gh = hg$ for all $g, h \in G$.

Example 1.8. Ab is the category of abelian groups. This is a full subcategory of Grp. with objects the abelian groups.

Notation: If the operation on a group is $+$, then the group is assumed to be abelian. The identity element is denoted 0, and the inverse of a is denoted $-a$.

Definition 1.9. **Cyclic groups** are the groups $\langle x \rangle$ consisting of powers

$$x^n = \begin{cases} x \cdots x & n > 0 \\ e & n = 0 \\ (x^{-n})^{-1} & n < 0 \end{cases}$$

of a single element.

Example 1.9. $\mathbb{Z} = \langle 1 \rangle$, and $\mathbb{Z}/n\mathbb{Z} = \langle 1 \pmod{n} \rangle = \{\text{integers} \pmod{n}\}$.

Definition 1.10. A **ring** R is a triple $(R, +, \cdot)$ of an abelian group $(R, +)$ and an associative operation $\cdot$ on R with identity denoted 1 such that the distributive laws $a(b+c) = ab + ac$ and $(a+b)c = ac + b$ hold. A **ring homomorphism** is a function $f : R \to R'$ of rings such that $f(x+y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$, and $f(1) = 1$ for all $x, y \in R$.

1.3 Rings, fields, and modules

Definition 1.11. A **commutative ring** is a ring for which $\cdot$ is commutative. A **division ring** (or skew field) is a ring such that $R \setminus \{0\}$ is a group under $\cdot$. A **field** is a commutative division ring.

Example 1.10. Ring is the category of rings. It has the full subcategories CRing of commutative rings and Fld of fields.

Definition 1.12. A (left) **module** A for a ring R is a triple $(A, +, \cdot)$, where $(A, +)$ is an abelian group and $\cdot : R \times A \to A$

1. is associative ($(rs)a = r(sa)$ for all $r, s \in R$ and $a \in A$)
2. satisfies $1 \cdot a = a$ for all $a \in A$
3. is distributive ($(r+s)a = ra + sa$ and $r(a+b) = ra + rb$ for all $r, s \in R$ and $a, b \in A$).