

Math 259A Extra Note

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1 Student Presentations

In this class, every enrolled student gave a presentation on a topic. Here are notes I took for each presentation.

1.1 Kadison's transitivity theorem

Definition 1.1. If M is a C^* -algebra acting on a Hilbert space H , M is said to act **topologically irreducibly** if H has no proper, closed, invariant subspaces under M . M is said to act **algebraically irreducibly** if H has no proper, invariant subspaces under M .

From the definitions, we have that algebraically irreducible C^* -algebras are topologically irreducible.

Theorem 1.1 (Kadison's transitivity theorem). *If M is topologically irreducible, it is algebraically irreducible.*

Why is this called the transitivity theorem? We will show that M acts n -transitively on H ; i.e. for all linearly independent $x_1, \dots, x_n \in H$ and any $y_1, \dots, y_n \in H$, there is an $A \in M$ such that $Ax_i = y_i$ for all $1 \leq i \leq n$.

Lemma 1.1. *Let $x_1, \dots, x_n \in H$ be orthonormal, and let $z_1, \dots, z_n \in H$ with $\|z_i\| \leq r$. Then there exists an operator $B \in \mathcal{B}(H)$ such that $Bx_i = z_i$ for all i and $\|B\| \leq \sqrt{2nr}$. If there is a selfadjoint T with $Tx_i = z_i$, then we can take B to be self-adjoint.*

Proof. Extend $x_1, \dots, x_n, x_{n+1}, \dots, x_m$ to an orthonormal basis for $\mathbb{C}\{x_1, \dots, x_n, z_1, \dots, z_n\}$ ($m < 2n$). Let \tilde{B} be the matrix induced by splitting up the z_i according to this basis. Then

$$[\tilde{B}] = \sqrt{\sum |\alpha_{i,j}|^2} \leq (2n \cdot r^2)^{1/2} = \sqrt{2nr}.$$

Extend it by making it 0 on the orthogonal complement. □

Proof. Assume x_1, \dots, x_n are orthonormal, so $x_1, \dots, x_n \xrightarrow{B} y_1, \dots, y_n$. By changing basis and conjugating by change of basis operators, we can get this result for arbitrary sets. Choose B_0 such that $B_0 x_i = y_i$. Take $A_0 \in M$ such that $\|A_0 x_i - y_i\| \leq \frac{1}{2\sqrt{2n}}$; this is possible because M is topologically irreducible. Choose B_1 such that $B_1 x_i = y_i - A_0 x_i$ and $\|B_1\| \leq \frac{1}{2}$. By Kaplansky's density theorem, choose $A_1 \in M$ such that $\|A_1\| \leq \frac{1}{2}$ and $\|A_1 x_i - B_1 x_i\| \leq \frac{1}{4\sqrt{2n}}$.

Continue recursively: Suppose we have defined B_k such that $\|B_k\| \leq \frac{1}{2^k}$ and $B_k x_i = y_i - A_0 x_i - A_1 x_i - \dots - A_{k-1} x_i$. Choose $A_k \in M$ such that $\|A_k\| \leq \frac{1}{2^k}$, $\|A_k x_i - B_k x_i\| \leq \frac{1}{2^{k+1}\sqrt{2n}}$. Choose $\|B_{k+1}\| \leq \frac{1}{2^{k+1}}$ with $B_{k+1} x_i = y_i - A_0 x_i - A_1 x_i - \dots - A_k x_i$. If $T x_i = y_i$, we can choose the B_k and thus the A_k to be self-adjoint by Kaplansky's theorem. Let $A = \sum_{k=0}^{\infty} A_k$, This converges in norm to an element of M . Moreover,

$$y_i - A x_i = y_i - \sum_{k=0}^{\infty} A_k x_i = \lim_k (y_i - A_0 x_i - A_1 x_i - \dots - A_k x_i) = \lim_k (B_{k+1} x_i) = 0$$

because $\|x_i\| = 1$ and $\|B_{k+1}\| \leq 1/2^{k+1}$. This proves n -transitivity and thus Kadison's theorem. \square

1.2 Dixmier's averaging theorem

Theorem 1.2 (Dixmier's averaging theorem). *Let M be a von Neumann algebra with center $Z(M)$. For each $x \in M$, denote by $\overline{K(x)}$ the norm closure of the convex hull of $\{uxu^* : u \in U(M)\}$. Then $\overline{K(x)} \cap Z(M) \neq \emptyset$.*

The bulk of the proof is in the following lemma.

Lemma 1.2. *If $x = x^* \in M$, there is a $u \in U(M)$ and $y = y^* \in Z(M)$ such that*

$$\left\| \frac{1}{2}(x + u^* x u) - y \right\| \leq \frac{3}{4} \|x\|.$$

Proof. Suppose $\|x\| = 1$. Define projections $p = \mathbb{1}_{[0,1]}(x)$ and $q = \mathbb{1}_{[-1,0]}(x)$. By the comparison theorem, there exists some $z \in P(Z(M))$ such that $zq \prec zp$ and $(1-z)p \prec (1-z)q$. Take p_1, p_2, q_1, q_2 such that $zq \sim p_1 \leq p_1 + p_2 = 2p$ and $(1-zp) \sim q - 1 \leq q_1 + q_2 = (1-z)q$.

Take two partial isometries $v, w \in M$ with $c^*c = w$ and $vv^* = p$, $w^*w = (1-z)p$, $vv^* = q$. Define $u = v + v^* + w + w^* + p_2 + q_2$. Then

$$\begin{aligned} u &= v^*v + vv^* + w^8w + ww^* + q_2 + p_2 \\ &= zq + p_2 + (1-z)p + q_1 + q_2 + p_2 \\ &= p + q \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} u^* p_1 u &= zq, & u^* q_1 u &= (1-z)p & u^* p_2 u &= p_2, \\ u^* zqu &= p_1, & u^* (1-z)pu &= q_1, & u^* q_2 u &= q_2. \end{aligned}$$

We have $-zq \leq zx \leq zp = p_1 + p_2$. So

$$\begin{aligned} &\implies -p_1 \leq zu^*xu \leq zq + p_2 \\ &\implies -\frac{1}{2}(zq + p_1) \leq \frac{1}{2}(zx + zu^*xu) \leq \frac{1}{2}zq + p_1 + p_2 \\ &\implies \frac{1}{2}z \leq \frac{1}{2}(2x + zu^*xu) \leq z \\ &\implies -\frac{3}{4} \leq \frac{1}{2}(2x - zu^*xu) - \frac{1}{4}z \leq \frac{3}{4}z. \end{aligned}$$

Similarly, repeating this with $1-z$ gives

$$-\frac{3}{4}(1-z) \leq \frac{1}{2}((1-z)x + (1-z)u^*xu) + \frac{1}{4}(1-z) \leq \frac{3}{4}(1-z).$$

If we add these together, we get

$$\left\| \frac{1}{2}(z + u^*xu) - \frac{2z-1}{4} \right\| \leq \frac{3}{4}. \quad \square$$

Proof. Let K denote the set of maps $\alpha : M \rightarrow M$ of the form $\alpha(x) = \sum_{i=1}^n c_i u_i^* x u_i$ with $u_i \in U(M)$, $\sum_i c_i = 1$ and $c_i \geq 0$. For general $x \in M$ denote $a_0 = \operatorname{Re}(x)$ and $b_0 = \operatorname{Im}(x)$. By the lemma, there exist some $u \in U(M)$ and $y_1 = y_1^* \in Z(M)$ with

$$\left\| \frac{1}{2}(a_0 + u^* a_0 u) - y_1 \right\| \leq \frac{3}{4} \|a_0\|.$$

Denote $\alpha_1(x) = \frac{1}{2}(x + u^*xu)$ and $a_1 = \alpha_1(a_0)$. Use the lemma again on $a_1 - y_1$. Continue inductively.

Given any $\varepsilon > 0$, we can find $\alpha \in K$ and $y \in Z(M)$ for which $\|\alpha(a_0) - y\| < \varepsilon$. Similarly, given this α , we can find $\beta \in K$ and $z \in Z(M)$ for which $\|\beta(\alpha(b_0)) - z\| < \varepsilon$. Thus,

$$\|\beta(\alpha(a_0)) - y\| = \|\beta(\alpha(a_0) - y)\| \leq \|\alpha(a_0) - y\| < \varepsilon.$$

Therefore,

$$\|\beta(\alpha(x)) - (y + iz)\| < 2\varepsilon$$

The problem is that $y + iz$ might be dependent on ε . To fix that, we define a sequence $(\Gamma_n) \subseteq K$ and $(z_n) \subseteq Z(M)$ such that if $x_0 = x$ and $x_n = \gamma_n(x_{n-1})$, we have $\|x_n - z_n\| \leq \frac{1}{2^n}$. Thus,

$$\|x_{n+1} - x_n\| = \|\gamma_{n+1}(x_n - z_n) - (x_n - z_n)\| \leq \|\gamma_{n+1}(x_n - z_n)\| + \|x_n - z_n\| < \frac{1}{2^{n-1}}.$$

Thus, $x_n \rightarrow x$ and $z_n \rightarrow x$, so $x \in \overline{K(x)} \cap Z(M)$. \square