

Math 247A Lecture 6 Notes

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1 Hunt's Interpolation Theorem

1.1 Strong type and weak type

Definition 1.1. We say that a map T on some measurable class of functions is **sublinear** if

1. $|T(cf)| \leq |c||Tf|$,
2. $|T(f, g)| \leq |T(f)| + |T(g)|$

for all constant $c \in \mathbb{C}$ and f, g in the domain of T .

Example 1.1. If T is linear, it is sublinear.

Example 1.2. If $\{T_t\}_{t \in S}$ is a family of linear maps, then

$$(Tf)(x) = \|(T_t f)(x)\|_{L_t^2}$$

is a sublinear map.

Definition 1.2. Let $1 \leq p, q \leq \infty$, and let T be a sublinear map.

1. We say that T is of **(strong) type** (p, q) if there exists a constant $C > 0$ such that

$$\|Tf\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^d).$$

2. If $q < \infty$, we say that T is of **weak-type** (p, q) if there exists a constant $C > 0$ such that

$$\|Tf\|_{L^{q,\infty}}^* \leq C\|f\|_{L^p(\mathbb{R}^d)} \quad \forall f \in L^p(\mathbb{R}^d).$$

If $q = \infty$, we say that T is of weak-type (p, q) if it is of strong type (p, q) .

3. If $p, q < \infty$, we say that T is of **restricted weak-type** (p, q) if there exists a constant $C > 0$ such that

$$\|T\mathbb{1}_F\|_{L^{q,\infty}}^* \leq C|F|^{1/p} (\leq C\|\mathbb{1}_F\|_{L^{p,1}}^*) \quad \forall F \subseteq \mathbb{R}^d, |F| < \infty.$$

Remark 1.1.

Strong type $(p, q) \implies$ weak-type $(p, q) \implies$ restricted weak-type (p, q) .

For the first implication, we have $\|Tf\|_{L^{q,\infty}}^* \lesssim \|Tf\|_{L^{q,q}}^* \lesssim \|f\|_{L^p}$. For the second implication,

$$\|T\mathbb{1}_F\|_{L^{q,\infty}}^* \lesssim \|\mathbb{1}_F\|_{L^p} = \|\mathbb{1}_F\|_{L^{p,p}}^* \lesssim \|\mathbb{1}_F\|_{L^{p,1}}^* \lesssim |F|^{1/p}.$$

Exercise 1.1. For $1 < p, q < \infty$, let T be defined on functions on $(0, \infty)$ via

$$(Tf)(x) = |x|^{-1/q} \int_0^\infty |y|^{-1/p'} f(y) dy.$$

Then T is of restricted weak-type (p, q) but not of weak type (p, q) .

Remark 1.2. Fix $1 < p, q < \infty$. If T is of restricted weak-type (p, q) , then for any finite-measure sets $E, F \subseteq \mathbb{R}^d$,

$$\int |(T\mathbb{1}_F)(x)| \cdot |\mathbb{1}_E(x)| dx \lesssim \|T\mathbb{1}_F\|_{L^{q,\infty}}^* \|\mathbb{1}_E\|_{L^{q',1}}^* \lesssim |F|^{1/p} |E|^{1/q'}.$$

Conversely, if this condition holds for all finite measure sets $E, F \subseteq \mathbb{R}^d$, then T is of restricted weak-type (p, q) . Indeed,

$$\|T\mathbb{1}_F\|_{L^{q,\infty}}^* \sim \sup_{\|g\|_{L^{q',1}}^* \leq 1} \left| \int T\mathbb{1}_F(x) g(x) dx \right|.$$

Take $g = \sum 2^m \mathbb{1}_{E_m}$ with E_m measurable and pairwise disjoint. Then

$$\begin{aligned} \left| \int T\mathbb{1}_F(x) g(x) dx \right| &\leq \sum 2^m \int |T\mathbb{1}_F(x)| \cdot |\mathbb{1}_{E_m}(x)| dx \\ &\lesssim \sum 2^m |F|^{1/p} |E_m|^{1/q'} \\ &\lesssim |F|^{1/p} \|g\|_{L^{q',1}}^* \\ &\lesssim |F|^{1/p}. \end{aligned}$$

Remark 1.3. If $1 < p, q < \infty$, then T is of restricted weak-type (p, q) if and only if there is a constant $C > 0$ such that

$$\|Tf\|_{L^{q,\infty}}^* \leq C \|f\|_{L^{p,1}}^* \quad \forall f \in L^{p,1}(\mathbb{R}^d).$$

1.2 Hunt's interpolation theorem

Theorem 1.1 (Hunt's interpolation theorem). *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. Assume that T is a sublinear map satisfying $\|Tf\|_{L^{q_j}, \infty} \lesssim \|f\|_{L^{p_j, 1}}^*$ for $j = 1, 2$. Then, for any $1 \leq r \leq \infty$ and $\theta \in (0, 1)$, we have*

$$\|Tf\|_{L^{q_\theta, r}}^* \lesssim \|f\|_{L^{p_\theta, r}}^*, \quad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Remark 1.4. 1. If $p_\theta \leq q_\theta$, then T is of strong type (p_θ, q_θ) . Indeed, taking $r = q_\theta$, we get

$$\|Tf\|_{L^{q_\theta}} \lesssim \|f\|_{L^{p_\theta, q_\theta}}^* \lesssim \|f\|_{L^{p_\theta}}.$$

2. The condition $p_\theta \leq q_\theta$ is needed to obtain the strong-type conclusion. For example, let $(Tf)(x) = f(x)|x|^{-1/2}$. Then $T : L^p(0, \infty) \rightarrow L^{2p/(p+2)}(0, \infty)$ boundedly for any $2 \leq p < \infty$. But T is not bounded from L^p to $L^{2p/(p+2)}$ for all $2 < p < \infty$. To see that $T : L^p \rightarrow L^{2p/(p+2), \infty}$ is bounded, we use the Hölder inequality in Lorentz spaces (which we will prove later): If $1 \leq p_1, p_2, p < \infty$ and $1 \leq q_1, q_2, q \leq \infty$, then

$$\|f_1 f_2\|_{L^{p, q}}^* \lesssim \|f_1\|_{L^{p_1, q_1}}^* \|f_2\|_{L^{p_2, q_2}}^*, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|Tf\|_{L^{2p/(p+2), \infty}}^* \lesssim \| |x|^{-1/2} \|_{L^{2, \infty}}^* \|f\|_{L^{p, \infty}}^* \lesssim \|f\|_{L^p}.$$

Take

$$f(x) = |x|^{-1/p} |\log(x + 1/x)|^{-(p+2)/(2p)}.$$

We get

$$\begin{aligned} \|f\|_{L^p}^p &= \int_0^\infty |\log(x + 1/x)|^{-(p+2)/p} \frac{dx}{x} \\ &= \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} 2^{n+1} |\log(x + 1/x)|^{-1-p/2} \frac{dx}{x} \\ &\sim \sum_{n \in \mathbb{Z}} |n|^{-1-p/2} \\ &< \infty \end{aligned}$$

On the other hand,

$$\begin{aligned} \|Tf\|_{L^{2p/(p+2)}}^{2p/(p+2)} &= \int_0^\infty |\log(x + 1/x)|^{-1} \frac{dx}{x} \\ &\sim \sum_{n \in \mathbb{Z}} |n|^{-1} \end{aligned}$$

$$= \infty$$

We know that $T : L^{p_1, p_1} \rightarrow L^{2p_1/(p_1+2), \infty}$ and $T : L^{p_2, p_2} \rightarrow L^{2p_2/(p_2+2), \infty}$ for $2 \leq p_1 < p_2 < \infty$. Hunt's interpolation theorem gives that for all $1 \leq r \leq \infty$, $T : L^{p_\theta, r} \rightarrow L^{2p_\theta/(p_\theta+2), r}$. Note that $\frac{2p_\theta}{p_\theta+2} < p_\theta$.

Next time, we will prove the following as a consequence of Hunt's interpolation theorem.

Corollary 1.1 (Marcinkiewicz interpolation theorem). *Let $1 \leq p_1 \leq q_1 \leq \infty$ and $1 \leq p_2 \leq q_2 \leq \infty$ with $p_1 \leq p_2$ and $q_1 \neq q_2$. Let T be a sublinear map that satisfies*

$$\|Tf\|_{L^{q_j, \infty}}^* \lesssim \|f\|_{L^{p_j}}, \quad j = 1, 2.$$

Then for any $\theta \in (0, 1)$, T is of strong type (p_θ, q_θ) , where

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

We will also prove Hunt's theorem next time.