Math 255B Lecture 7 Notes

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January 22, 2020

1 Introduction to Unbounded Operators

1.1 Motivation from quantum mechanics

In this part of the course, we will discuss spectral theory of self-adjoint operators. We are most interested in unbounded operators, the background of which comes from quantum mechanics.

Classical mechanics: The classical phase space is $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, where x is position and ξ is momentum. Classical observables are, for example, $C^\infty(\mathbb{R}^{2n})$ functions.

Example 1.1. The Hamiltonian is

$$p(x, \xi) = |\xi|^2 + V(x),$$

where $V(x)$ is a potential.

In classical dynamics, we have the Hamilton equations

$$\begin{cases} x(t) = p'_\xi(x, \xi) \\ \xi(t) = -p'_x(x, \xi) \end{cases}$$

Quantum mechanics: We have a Hilbert space $H = L^2(\mathbb{R}^n)$. Quantum observables are self-adjoint operators on H.

Example 1.2. Quantum observables corresponding to x_j and ξ_j are M_{x_j}, multiplication by x_j, and $D_{x_j} = \frac{1}{i}\partial_{x_j}$. These can not be defined on the whole space, so they will come with their own domains. Associated to p is the **Schödinger operator**

$$P = -\Delta + V(x).$$

Quantum dynamics is given by the Schrödinger equation

$$i\frac{\partial u}{\partial t} = Pu, \qquad u|_{t=0} = u_0 \in L^2.$$

Formally,

$$u(t) = e^{-itP}u.$$

Interpreting what it means to exponentiate an unbounded operator will be one of the points of our theory.

1.2 Unbounded operators

Let H be a complex, separable Hilbert space, let $S \subseteq H$ be a linear subspace, and let $T : D \to H$ be a linear map. Then $D = D(T)$ is the **domain** of T. We shall always assume that T is **densely defined**, so that $D(T)$ is dense in H. Associated to T is the **graph**[1] of T: $G(T) = \{(x, Tx) : x \in D(T)\} \subseteq H \times H$.

Definition 1.1. We say that T is **closed** if $G(T)$ is closed subspace of $H \times H$.

Definition 1.2. The operator T is **closable** if $\overline{G(T)}$ is the graph of an linear operator $\overline{T} : D(T) \to H$, called the **closure** of T.

Note that

$$D(\overline{T}) = \{x \in H : \exists x_j \in D(T) \text{ s.t. } x_j \to x, Tx_j \text{ conv. in } H, \overline{T}x = \lim Tx_j\}.$$

So

$$T \text{ is closed } \iff \text{if } x_n \in D(T), x_n \to X, \text{ and } Tx_n \to y, \text{ then } x \in D(T) \text{ and } Tx = y.$$

On the other hand,

$$\begin{aligned} T \text{ is closable } &\iff G(T) \text{ contains no element of the form } (0, y) \text{ with } y \neq 0 \\ &\iff \text{if } x_n \in D(T),\ x_n \to 0, \text{ and } Tx_n \to y, \text{ then } y = 0. \end{aligned}$$

Example 1.3. Let $T = -\Delta$ on $L^2(\mathbb{R}^n)$, with $D(T) = C_0^\infty(\mathbb{R}^n)$. Then T is densely defined and closable: If $\varphi_n \in C_0^\infty$ are s.t. $\varphi \to 0$ in L^2 and $\Delta\varphi_n \to \psi \in L^2$, we want $\psi = 0$. For any $f \in C_0^\infty$, $\int \varphi_n f \to 0$, and integrating by parts gives $\int \Delta\varphi_n f = \int \varphi_n \Delta f \to 0$. On the other hand, $\int \Delta\varphi_n f \to \int \psi f$. We get that $\int \psi f = 0$ for all $f \in C_0^\infty$. So $\psi = 0$. In the language of distributions, $\varphi_n \to 0$ in $D'(\mathbb{R}^n)$, so $\Delta\varphi_n \to 0$ in D'. So $\psi = 0$.

We claim that $\overline{T} = -\Delta$ with $D(\overline{T}) = H^2(\mathbb{R}^n) = \{u \in L^2 : \partial^\alpha u \in L^2, |\alpha| \leq 2\}$, a **Sobolev space**. Here, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, and $|\alpha| = \sum_{j=1}^n \alpha_j$. Here, $\partial^\alpha u \in L^2$ means that there exists some $f_\alpha \in L^2$ such that $\partial^\alpha u = f_\alpha$; that is,

$$(-1)^{|\alpha|} \int u \partial^\alpha \varphi = \int f_\alpha \varphi \qquad \forall \varphi \in C_0^\infty.$$

[1] The idea of thinking about unbounded operators in terms of their graphs goes back to von Neumann.

We have

$$D(\overline{T}) = \{u \in L^2 : \exists \varphi_n \in C_0^\infty \text{ s.t. } \varphi_n \xrightarrow{L^2} u, \Delta\varphi_n \text{conv. in } L^2\}, \qquad \overline{T}u = \lim(-\Delta\varphi_n).$$

Hence, if $u \in D(\overline{T})$, then $\Delta u = \lim_n \Delta\varphi_n \in L^2$. Then

$$D(\overline{T}) \subseteq \{u \in L^2 : \Delta u \in L^2\} = H^2(\mathbb{R}^n),$$

as taking the Fourier transform, this is

$$D(\overline{T}) \subseteq \{u \in L^2 : (|\xi|^2 + 1)\widehat{u} \in L^2\}.$$

We also have $H^2(\mathbb{R}^n) \subseteq D(\overline{T})$, as $C_0^\infty(\mathbb{R}^n)$ is dense in $H^2(\mathbb{R}^n)$ (the norm on H^2 is given by $\|u\|_{H^2} = \sum_{|\alpha|\leq 2} \|\partial^\alpha u\|_{L^2}$.) This is the same proof that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

We get that $T = -\Delta$ with $D(T) = H^2(\mathbb{R}^n)$ is closed and densely defined.

Next time, we will define what it means for a densely defined operator to be self-adjoint, and we will see that this operator is indeed self-adjoint.