Math 246A Lecture 27 Notes

Daniel Raban

November 30, 2018

1 Schwartz's Reflection Principle and Harnack's Principle

1.1 Schwartz's reflection principle

Lemma 1.1 (Schwartz's reflection principle). *Let $v(z)$ be harmonic in $\mathbb{D} \cap \{z : y > 0\}$, and suppose that $\lim_{y\ to0} v(x+iy) = 0$. Then v extends to be harmonic on all of $\mathbb{D}$ by reflection:*

$$V(z) = \begin{cases} v(z) & \mathrm{Im}(z) > 0 \\ 0 & \mathrm{Im}(z) = 0 \\ -v(\overline{z}) & \mathrm{Im}(z) < 0 \end{cases}$$

is harmonic on $\mathbb{D}$.

Proof. The mean value property implies V is harmonic; for points on the boundary, the contribution of the upper and lower arcs to the integral cancel out. $\square$

Definition 1.1. Let $|Omega$ be a domain in $\mathbb{C}$ and let $\gamma \subseteq \partial\Omega$. γ is a **free boundary arc** if

1. $\gamma = \{\psi(t) : t \in (0,1)\}$, where $\psi : \mathbb{D} \to \Omega$ is a homeomorphism.
2. $\gamma \subseteq \partial\Omega$, and
$$p \in \gamma \implies \exists \delta > 0 \text{ s.t. } B(p(\delta) \cap \partial\Omega = \gamma \cap B(p,\delta).$$

Definition 1.2. γ is an **analytic arc** if for open $U \subseteq \mathbb{C}$ with $(0,1) \subseteq U$, $\psi : U \to \mathbb{C}$, $\gamma = \psi(0,1)$, where ψ is analytic, 1-1 and onto $\psi(U)$.

Assume γ is a free boundary arc on a domain Ω. Now split $U = U^+ \cup U^- \cup (0,1)$, where $U^\pm$ are connected components of $U \setminus (0,1)$.

If $\psi(U) \cap \Omega$ is $\psi(U^+)$ or $\psi(U^-)$, we say that γ is a **one-sided** analytic boundary arc. Otherwise, if $\psi(U) \cap \Omega = \psi(U^+) \cup \psi(U^-)$, we say that γ is a **two–sided** analytic boundary arc.

Definition 1.3. Let $\varphi : \Omega \to \Omega'$. Then φ is **proper** if for $z_n \to \partial\Omega$, $\varphi(z_n) \to \partial\Omega'$.

Lemma 1.2. *Suppose $\varphi : \Omega \to \Omega$ is a homeomorphism. Then φ is proper.*

Proof. If not, for some subsequence, $\varphi(z_{n_j}) \to w \in \Omega$. Then $z_{n_j} \to \zeta \in U^{-1}(U)$. $\square$

Theorem 1.1. *Assume Ω is a simply connected domain and γ is a one-sided analytic arc on $\partial\Omega$. Let $U, \psi(U)$ be as above. Let $\varphi : \Omega \to \mathbb{D}$ be conformal. Then φ extends to a 1-1 holomorphic map $\tilde{\varphi} : \Omega \cup \psi(U) \to \mathbb{C}$ such that $|\tilde{\varphi}(z)| = 1$ on γ, and*

$$|\tilde{\varphi}(z)| > 1 \iff z \in \psi(u) \setminus \overline{\Omega}.$$

Proof. Without loss of generality,$\Omega \cap \psi(U) = \psi(U^+)$ and $\varphi^{-1}(0) \notin \psi(u)$. Consider $v(u) = -\log|\varphi(\psi(z))|$ for $z \in U^+$. Then $v(z) \to 0$ as $z \to (0,1)$ since φ is proper. So there exists V harmonic on U, and $V = \mathrm{Re}(F)$,where $F \in H(U)$. We can say that $F(z_0) = e^{\varphi\circ\psi(z_0)}$ for some $z_0 \in U^+$ (since F is defined up to a constant). Let

$$\tilde{\varphi} = \begin{cases} \varphi & \text{on } \Omega \\ F \circ \psi^{-1} & \text{on } \psi(U). \end{cases}$$

Then $\tilde{\varphi}$ is analytic and has the desired properties. $\square$

1.2 Harnack's principle

Lemma 1.3. *Let u be a positive harmonic function on an open set $U \supseteq \overline{B(z_0, R)}$. Let $|z - z_0| = r < \mathbb{R}$. Then*

$$\frac{R-r}{R+r}u(z_0) \le U(z) \le \frac{R+r}{R-r}u(z_0).$$

Proof. By a change of variables and the Poisson integral formula,

$$u(z) = \frac{1}{2\pi}\int \frac{R^2 - r^2}{|Re^{i\theta} - (z - z_0)|^2} u(z_0 + Re^{i\theta})d\theta.$$

Use the M-L estimate. The maximum and the minimum of the kernel $(R^2 - r^2)/|Re^{i\theta} - (z - z_0)|^2$ are $(R + r)/(R - r)$ and $(R - r)/(R + r)$. $\square$

Theorem 1.2 (Harnack's prinicple)**.** *Let Ω_n be a sequence of domains and let $u_n : \Omega_n \to \mathbb{R}$ be harmonic. Let $\Omega \supseteq \bigcup_{n=1}^{\infty} \Omega_n$. Assume that:*

1. *If $K \subseteq \Omega$ is compact, then there exists n_K such that $K \subseteq \Omega_n$ for $n > n_k$.*
2. *If $K \subseteq \Omega$ is compact, there exists $n_1(K)$ such that if $n > n_1(K)$, then $u_{n+1} \ge u_n$ on K.*

Then either

1. *$u_n \to +\infty$ uniformly on K for all compact $K \subseteq \Omega$.*
2. *there exists a harmonic function u on Ω such that $u_n \to u$ uniformly on all compact $K \subseteq \Omega$.*