Math 206A Lecture 8 Notes

Daniel Raban

October 15, 2018

1 Linear Algebra Methods and the Kahn-Kalai Theorem

1.1 Linear algebra methods

In the proof of the Kahn-Kalai theorem, we have $M \subseteq \{\pm 1\}^n \subseteq \mathbb{R}^n$ with $|M| = 2^{n-2}$. We want the maximal subset $|A|$ such that $a \cdot a' \neq 0$ for all $a, a' \in A$. We will show that this is less than c^n, where $c < 2$. We get that the number of parts in the Borsuk part of $M \otimes M > 2^{n-2}/c^n$.

Theorem 1.1 (odd town theorem)**.** *Suppose $\mathcal{A} = \{A_1, \dots, A_N\} \subseteq \mathscr{P}(\{1, \dots, n\})$ is a collection such that $|A_i|$ is odd for all i, and $|A_i \cap A_j|$ is even for all $i < j$. Then $|\mathcal{A}| \leq n$.*

Proof. Let v_i be the characteristic vector of A_i in $\mathbb{Q}^n$. For example, if $A_1 = \{1, 4, 5\}$ and $n = 5$, then $v_1 = (1, 0, 0, 1, 1)^\top$. Then $\|v_i\|^2 = 1 \pmod 2$, and $v_i \cdot b_j = 0 \pmod 2$ if $i \neq j$. We claim that the v_i are linearly independent as vectors in F_2^n. Assume $\lambda_1 v_1 + \cdots \lambda_n v_n = 0$. We can take the λ_i to be integers, and without loss of generality, λ_1 is odd. Then $\lambda_1 = \lambda_1 \|v_1\|^2 + \cdots + \lambda_n \langle v_n, v_1 \rangle = 0 \pmod 2$, which is a contradiction. $\square$

Theorem 1.2 (2-distance theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be such that $d(x, x') \in \{a, b\}$ for all $x \neq x'$ and $x, x' \in X$. Then $|X| = O(n^2)$.*

When the number of possible distances is 1 instead of 2, we get that $|X| \leq n + 1$, since X must be the vertices of a simplex.

Proof. Let $X = \{z_1, \dots, z_N\}$ and $F(x, y) := (|x - y|^2 - a^2)(|x - y|^2 - b^2)$. Then

$$F(z_i, z_j) = \begin{cases} a^2 b^2 & i = j \\ 0 & i \neq j. \end{cases}$$

Define $f_i(y) := F(z_i, y)$. Then the f_i are linearly independent. Indeed, suppose $\lambda_1 f_1 + \cdots + \lambda_N f_N = 0$. Then $\lambda_1 f_1(z_i) = 0$, so $\lambda_1 = 0$. This is true for all i. So the number of f_i is at most the dimension of the space containing the f_i. So $N = O(n^2)$. $\square$

1.2 Kahn-Kalai using linear algebra methods

Let's continue with the proof of the Kahn-Kalai theorem. Let $M = \{x_1 = 1, x_2, \ldots, , x_n \in \{\pm 1\}, x_2 \cdots x_n = 1\}$. We also had $n = 4p$, where p is prime.

Lemma 1.1. *Let $A \subset M$ be such that $a \cdot a \neq 0$ for $a, a \in A$. Then $|A| \leq 2^{n/2}$.*

Proof. Define $G(t) = (t-1)(t-2)\cdots(t-p+1)$. Let $V \subseteq \mathbb{Q}[x_2, \ldots, x_n]$ be the subspace of squarefree polynomials with deg $\leq n/4 = p$; that is, the monomials generating V have no x_i to a square or higher power. We will show that $W \subseteq V \implies \dim(W) \leq 2^{n/4}(n/4)$. Note that $\dim V = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n/4} < 2^{n/2}$.

Let $F_a = G(a \cdot (1, z_2, \ldots, z_n))$ for $a \in A$; this is really a polynomial in $z_2, \ldots, z_n$. By definition, $F_a \in V$. Next time, we will show that the F_a are linearly independent, which will produce a bound on $|A|$. $\square$