Math 275D Lecture 16 Notes

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November 6, 2019

1 Itô's Formula

1.1 Integrating with respect to Brownian motion

Let $f \in C^2([0,1])$. We want to say something like

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_s.$$

Itô's formula tells us that there is actually an extra term:

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_t + \frac{1}{2}\int_0^t f''(B_s)\, ds.$$

If we change a $f(B_t)$ a little bit, how can we measure this change? The answer is

$$df(B_t) = f'(B_t)\, dB_t + \frac{1}{2} f''(B_t)\, dt.$$

Some people also write

$$\Delta f(B_t) = (d \to \Delta)$$

for this in physics and finanical contexts.

There is something that is not clear in the above equation. What does the integral $\int_0^t f'(B_s)\, dB_s$ mean?

Definition 1.1. We can define an integral with respect to Brownian motion by

$$\int_0^s g(B_t)\, dB_t := \lim_{0=t_1<\cdots<t_n=s} \sum_{k=1}^n g(B(t_k))(B(t_{k+1}) - B(t_k))$$

This is very similar to a Riemann sum, except a Riemann sum has $g(t_k^*)$, where $t_k^* \in [t_k, t_{k+1}]$. In our definition, we explicitly pick t_k, instead. In fact, the limit will change if we replace t_k with $st_k + (1-s)t_{k+1}$ for some $s < 1$.

Example 1.1. Here is an example to show that the limit can be different if we change t_k to t_{k+1}.

$$\sum_k [g(B(t_{k+1})) - g(B(t_k))](B(t_{k+1}) - B(t_k)) \approx \sum_k g'(B_{t_k})(B(t_{k+1}) - B(t_k))^2$$

Suppose $g' \approx 2$. Then this is

$$\begin{aligned} &\approx 2\sum_k (B(t_{k+1}) - B(t_k))^2 \\ &\approx 2\sum_k t_{k+1} - t_k \\ &= 2s. \end{aligned}$$

1.2 Examples and applications

Example 1.2. Say you buy stocks every day. Let A_n be the number of stocks you have on day n, and let $\Delta B_n = (B_{n+1} - B_n)$ be the change of stock price on day n. Then we want to calculate

$$\sum_{n=1}^{365} A_n \cdot \Delta B_n$$

We have that $A_n \in \mathcal{F}_n$, where $\mathcal{F}_n = \sigma(B_1, \ldots, B_n)$. We also assume that as $n \to \infty$, B_n looks like a Brownian motion. In the limit, we get

$$\int A_t \, dB_t,$$

where A_t is $\mathcal{F}_t$-measurable ($\mathcal{F}_t$ is our σ-field for Brownian motion).

Example 1.3. Itô's formula implies

$$\sin(B_t) = \int \cos(B_s)\, dB_t - \frac{1}{2}\int \sin(B_s)\, ds.$$

Often, we use Itô's formula backwards, to find the value of the integral.

Example 1.4. We have

$$B_t^2 = \int_0^t B_s \, dB_s + t,$$

so we can solve to get

$$\int_0^t B_s \, dB_s = B_t^2 - t.$$

In fact, we have the following theorem:

Theorem 1.1. *Let $g \in C^2$. Then*

$$F(t) = \int_0^t g(B_s)\, dB_s$$

is a martingale.

We will prove this later.

1.3 Proof of Itô's formula

Theorem 1.2 (Itô's formula)**.** *Let $f \in C^2([0,1])$. Then*

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_t + \frac{1}{2}\int_0^t f''(B_s)\, ds.$$

Here is a heuristic argument:

Proof. We have

$$f(B_t) = f(B_{t_0}) + f'(B_{t_0}) \underbrace{(B_t - B_{t_0})}_{\approx(\Delta t)^{1/2}} + \frac{1}{2} f''(B_{t_0}) \underbrace{(B_t - B_{t_0})}_{\approx \Delta t} + \cdots ,$$

where $\Delta t = t - t_0$. We can get rid of the terms with order > 2, since they will be small when Δt is small. Then

$$f(B_t) - f(B_0) = \sum_k f'(t_k) B(\Delta_k) + \frac{1}{2}\sum_k f''(B_{t_k})(B(\Delta_k))^2,$$

$$B(\Delta_k) := B(t_{k+1}) - B(t_k).$$

When we take the limit, since the $B(\Delta_k)$ are independent of each other, so we get

$$\int f(B_s)\, dB_s + \lim \frac{1}{2}\sum_k f''(B_{t_k}) A_k,$$

where the A_k are independent $\mathcal{N}(0, \Delta_k)^2$. The law of large numbers makes the right hand side approximately $\frac{1}{2}\sum_k f''(B_{t_k})\, \mathbb{E}[A_k]$, so the right hand side converges to $\int f''(B_s)\, ds$. $\square$

What if we want to prove this for $f \in C^2(\mathbb{R})$? We prove it for when $\|f'\|_\infty, \|f''\|_\infty$ are bounded. Then with high probability, $\|B_t\|_{L^\infty(0,s)} < \infty$, so we can extend to the general case.