Math 279 Lecture 7 Notes

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1 Gubinelli's Sewing Lemma and Tensor Algebra for Increments

1.1 Gubinelli's sewing lemma

The method we have used so far can be used to show that if $x(t) = (f(t), g(t))$ with $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$, then there exists a unique candidate for $\int_0^t f\, dg$ (Young's theorem), provided that $\alpha + \beta > 1$. (The general case $\alpha + \beta \leq 1$ with $f, g : \mathbb{R}^d \to \mathbb{R}$ will be treated later.) More precisely, we can find a β-Hölder $h : [0, T] \to \mathbb{R}$ such that $h(0) = 0$, and

$$|h(t) - h(s) - \underbrace{f(s)(g(t) - g(s))}_{=:A(s,t)}| \leq c_0 |t - s|^{\alpha + \beta}.$$

In fact, Gubinelli's sewing lemma gives the sufficient (even necessary) conditions on A that would guarantee the existence of such an h

Definition 1.1. Given $A : [0, T]^2 \to \mathbb{R}$ and $\gamma > 0$, we say A is γ-**coherent** if

$$|A(s, t) - A(s, u) - A(u, t)| \leq c_0 |t - s|^{1+\gamma}$$

forall s, u, t satisfying $0 \leq s \leq u \leq t \leq T$.

Lemma 1.1 (Sewing lemma, Gubinelli)**.** *If A is γ-coherent, then*

$$h(t) = \lim_{|\pi| \to 0} \sum_{i=1}^n A(t_i, t_{i+1})$$

exists, where $\pi = \{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t\}$ is a partition of the interval $[0, t]$.

Proof. If $\pi = \{s = t_0 < t_1 < \cdots < t_n < t_{n+1} = t\}$ is a partition of $[s, t]$ and if $I(\pi) = \sum_{i=1}^n A(t_i, t_{i+1})$, then

$$I(\pi) - I(\pi \setminus \{t_i\}) = |A(t_{i-1}, t_i) + A(t_i, t_{i+1}) - A(t_{i-1}, t_{i-1})| \leq c_0 |t_{i+1} - t_{i-1}|^{1+\gamma}.$$

We may choose t_i so that $|t_{i+1} - t_i| \le \frac{2}{n}|t-s|$. We can repeat our previous argument to show that the limit exists and that

$$|h(t) - h(s) - A(s,t)| \le c|t-s|^{1+\gamma}, \quad \text{where} \quad c = \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^{1+\gamma}. \qquad \square$$

Remark 1.1. Observe that if $A(s,t) = f(s)(g(t) - g(s))$, then

$$\begin{aligned} |A(s,t) - A(s,u) - A(u,t)| &= |f(s)(g(t) - f(s)) - f(s)(g(u) - g(s)) - f(u)(g(t) - g(u))| \\ &= |(f(s) - f(u))(g(t) - g(u))| \\ &\le [f]_\alpha [g]_\beta |t-s|^{\alpha+\beta}. \end{aligned}$$

Note that our candidate h' represents fg', and we are comparing fg' with $f(s)g'$:

$$|h(t) - h(s) - f(s)(g(t) - g(s))| = |(h' - f(s)g')(\mathbb{1}_{[s,t]})| \le c_0 |t-s|^{1+\gamma}.$$

Perhaps we set $F_s = f(s)g'$, and the γ-coherence condition requires some kind of regularity of the map $s \mapsto F_s$.

$$\begin{aligned} A(s,t) - A(s,u) - A(u,t) &= \underbrace{F_s(\mathbb{1}_{[s,t]})}_{F(s)(\mathbb{1}_{[s,u]} + \mathbb{1}_{[u,t]}} - F_s(\mathbb{1}_{[s,u]}) - F_u(\mathbb{1}_{[u,t]}) \\ &= F_s(\mathbb{1}_{[u,t]}) - F_u(\mathbb{1}_{[u,t]}) \\ &= (F_s - F_u)(\mathbb{1}_{[u,t]}). \end{aligned}$$

Perhaps we should write $\varphi = \mathbb{1}_{[0,1]}$ and $\varphi_x^\lambda(\theta) := \lambda^{-1}\varphi(\frac{\theta - x}{\lambda}) = \lambda^{-1}\mathbb{1}_{[x,x+\lambda]}$, which approximates the δ distribution at x. Then $(F_s - F_u)(\mathbb{1}_{[u,t]}) = \lambda(F_s - F_u)(\varphi_u^\lambda)$, where $\lambda = t - u$. Gubinelli's condition means that

$$|(F_s - F_u)(\varphi_u^\lambda)| \le \lambda^{-1}(|s-u| + \lambda)^{1+\gamma}.$$

This condition is sharp.

1.2 Tensor algebra structure for increments

So far, for a rough path, we need a vector $x(s,t) = x(t) - x(s)$ and a matrix $\mathbb{X}(s,t)$. For $\alpha > 1/k$, we are dealing with a tensor algebra that is truncated at order k. For $k = 3$, we cut it at 3 and only deal with 1 and 2 tensors. Consider the vector space $V = \mathbb{R} \oplus \mathbb{R}^\ell \oplus \mathbb{R}^{\ell \times \ell}$ with elements (λ, v, A) (which we may write as $\lambda + v + A$). We equip V with a multiplication (tensor product)

$$(\lambda + v + A) \otimes (\lambda' + v' + A') = (\lambda a') + (\lambda v' + \lambda' v) + (\lambda A' + \lambda' A + v \otimes v').$$

Note that if $G = \{1 + v + A : v \in \mathbb{R}^\ell, A \in \mathbb{R}^{\ell\times\ell}\}$, then G is closed with respect to $\otimes$. In fact G is a group. Indeed,

$$\begin{aligned}(1+v+A)^{-1} &= 1-(v+A)+(v+A)\otimes(v+A)+\cdots \\ &= 1-(v+A)+v\otimes v \\ &= 1-v+(v\otimes v-A).\end{aligned}$$

Let's take a rough path: $x : [0,T] \to \mathbb{R}^\ell$, $:[0,T]^2 \to \mathbb{R}^{\ell\times\ell}$. Given such a path, set

$$\mathbf{x}(s,t) = 1 + \underbrace{x(t)-x(s)}_{x(s,t)} + \mathbb{X}(s,t).$$

so $\mathbf{x} : [0,T]^2 \to G$. Recall Chen's relation,

$$\mathbb{X}(s,t)) = \mathbb{X}(s,u) + \mathbb{X}(u,t) + x(s,u)\otimes x(u,t).$$

Observe that

$$\begin{aligned}\mathbf{x}(s,u)\otimes\mathbf{x}(u,t) &= (1+x(s,u)+\mathbb{X}(s,u))\otimes(1+x(u,t)+\mathbb{X}(u,t)) \\ &= 1+x(s,t)+(\mathbb{X}(s,t)+\mathbb{X}(s,u))+(x(s,u)\otimes x(u,t)) \\ &= \mathbf{x}(s,t).\end{aligned}$$

Thus, Chen's relation is equivalent to saying that $\mathbf{x}(s,t) = \mathbf{x}(s,u)\otimes\mathbf{x}(u,t)$. This says that with respect to $\otimes$, $\mathbf{x}(s,t)$ is an increment. We can also see that $\mathbf{x}(s,t) = \mathbf{x}(0,s)\otimes\mathbf{x}(s,t)$, which gives

$$\mathbf{x}(s,t) = \mathbf{x}(0,s)^{-1}\otimes\underbrace{\mathbf{x}(0,t)}_{\mathbf{x}(t)}.$$

In summary, $\mathscr{R}^\alpha$ is isomorphic to the set of paths $\mathbf{x} : [0,T] \to G$ and, by choosing a suitable metric on G, that are left-invariant with $\mathbf{x}$ being α-Hölder with respect to this metric:

$$[\![x]\!]_\alpha = \sup_{s\neq t}\frac{d_G(x(t),x(s))}{|t-s|^\alpha} < \infty.$$

How about $\mathscr{R}_g^\alpha$? Even in this case, we get the set of α-Hölder paths $\mathbf{x} : [0,T] \to \widehat{G}$, where $\widehat{G}$ is a subgroup of G. Remember that

$$\begin{aligned}\mathscr{R}_g^\alpha &= \{(x,\mathbb{X})\in\mathscr{R}^\alpha : \mathbb{X}(s,t)+\mathbb{X}^*(s,t) = x(s,t)\otimes x(s,t)\} \\ &= \left\{(x,\mathbb{X})\in\mathscr{R}^\alpha : \mathbb{X}(s,t) = \frac{1}{2}x(s,t)\otimes x(s,t)+C(s,d), C^*+C=0\right\}.\end{aligned}$$

This suggests that

$$\widehat{G} = \{1+v+(\tfrac{1}{2}v\otimes v+C) : C^*+C=0\}.$$