Math 250A Lecture 6 Notes

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1 Category Theory

1.1 Categories

The purpose of category theory is to generalize common properties of existing structures so we do not need to refer to the internal structure of our objects at all.

Definition 1.1. A *category* is a collection of *objects* and a set of *morphisms* such that

1. Each morphism has a domain and a range, both of which are objects
2. For each object a, there is an identity morphism 1_a
3. For morphisms $X : a \to b$ and $Y : b \to c$, there is a composite morphism $Y \circ X$
4. $(X \circ Y) \circ Z = X \circ (Y \circ Z)$ if both are defined
5. $1_b \circ X = X \circ 1_a = X$ if $X : a \to b$

Example 1.1. In the category of sets, the objects are sets, and the morphisms/arrows are functions.

Example 1.2. In the category of groups, the objects are groups, and the morphisms/arrows are group homomorphisms.

Example 1.3. In the category of topological spaces, the objects are topological spaces, and the morphisms/arrows are continuous functions.

Example 1.4. Take a category with a single object, and let the morphisms be the elements of a group G, where composition of the morphisms is the group operation. This is a group.

Example 1.5. Let S be a partially ordered set with $\leq$. We can make a category with objects equal to the elements of S and morphisms from $a \to b$ such that there is 1 morphism if $a \leq b$ and 0 otherwise.

1.2 Functors

Definition 1.2. A *(covariant) functor* F from a category $\mathscr{C}$ to a category $\mathscr{D}$ is defined by the properties

1. F is a function from objects of $\mathscr{C}$ to objects of $\mathscr{D}$
2. F is a function[1] from morphisms of $\mathscr{C}$ to morphisms of $\mathscr{D}$
3. $F(1_A) = 1_{F(A)}$
4. $F(f \circ g) = F(f) \circ F(g)$.

Let $f : A \to B$ be a morphism. The fourth condition makes it so $F(f)$ is a morphism from $F(A) \to F(B)$; this is because we can set $g = 1_A$.

Example 1.6. We can define a functor F from the category of groups to the category of sets by $F(G) =$ the underlying set of G. F sends group homomorphisms to themselves as functions.

Example 1.7. The reason why functors were introduced was to study homology groups H_i. H_i is a functor from topological spaces to abelian groups.

Example 1.8 (Abelianization of a group)**.** Suppose G is a group. We can make G abelian by quotienting out $G/\left\langle \left\{ghg^{-1}h^{-1} : g, h \in G\right\}\right\rangle$ to get an Abelian group G^{ab}. This is a functor from Groups to Abelian groups. If $f : G \to H$, we get a map $G^{\text{ab}} \to H^{\text{ab}}$ (exercise).

Example 1.9. We have a functor from sets to abelian groups given by $F(S) = F_{\text{ab}}(S)$, the free abelian group on S. This is the set of elements $n_1s_1+n_2s_2+\cdots$ such that all but finitely many $n_i = 0$. If $f : S \to S'$ is a function, $F(f') : S \to S'$ sends $\sum_\alpha n_\alpha s_\alpha \mapsto \sum_\alpha n_\alpha s'_\alpha$.

Example 1.10. Take a group G, viewed as a category with 1 object. A functor from the group to sets will send the 1 object to some set and each $g \in G$ to some function $S \to S$. So we get the action of G on a set S, the permutations of S.

Definition 1.3. A *contravariant functor* is a functor where $F(f \circ g) = F(g) \circ F(f)$.

Similarly to the note we made above, this property implies that if $f : A \to B$ is a morphism, $F(f)$ is a morphism from $F(B) \to F(A)$.

Example 1.11. Let both categories be vector spaces over the same field K. We can define a functor $F(V) = \text{Hom}(V, K)$; this is V^*, the dual of V. Suppose $f : V \to W$ is a morphism; we must map it to some morphism $F(f) : W^* \to V^*$. We get the morphism $\lambda \mapsto \lambda \circ f$.

[1] Really, these are two separate functions, but we refer to them together as one function, the functor F.

Example 1.12. Suppose $\mathscr{C}$ is the category of a abelian groups. Look at $\mathrm{Hom}(A, B)$ for abelian groups A, B. This is a *bifunctor* in 2 variables form $C \times C \to C$. It is covariant in B and contravariant in A. If $f : B_1 \to B_2$, we get a map $F(f) : \mathrm{Hom}(A, B_1) \to \mathrm{Hom}(A, B_2)$. If $g : A_1 \to A_2$, we get a map $F(g) : \mathrm{Hom}(A_2, B) \to \mathrm{Hom}(A_1, B)$.

Example 1.13. We can have the category of categories, where the objects are categories and the morphisms are functors.

Remark 1.1. This does not actually exist because there is no set of all sets. Let $R = \{x : x \notin x\}$; then $R \in R \iff R \notin R$. Similarly, the category of all groups does not exist, either. We have a few possible solutions:

1. Only work with groups whose elements are in some fixed large set
2. Work in set theory with "classes"
3. Grothendieck universes
4. Ignore it

We will adopt the 4th solution.

1.3 Natural transformations

What does natural mean? Look at finite dimensional vector spaces. We know that $V \cong V^*$, but there is no *natural* isomorphism. However, $V \cong V^{**}$ with a "natural isomorphism"

$$v \mapsto f_v, \text{ where for each } w \in W,\ f_v(w) = w(v).$$

Definition 1.4. Suppose we have 2 categories C, D with functors $F : C \to D$ and $G : C \to D$. A *natural transformation* $\varphi : F \to G$ is a function φ such that

1. $\varphi(a)$ is a morphism from $F(a) \to G(a)$
2. if $f : a \to b$, $\varphi(b) \circ F(f) = G(f) \circ \varphi(a)$. That is, the following diagram commutes:

$$\begin{array}{ccc} F(a) & \xrightarrow{\varphi(a)} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow{\varphi(b)} & G(b) \end{array}$$

Example 1.14. Look at $C = D =$ vector spaces over a field K, and let F be the identity from an dlet G be the double dual. $G(V) = V^{**}$. Then there is a natural transformation from $F \to G$. For each vector space V, we have a morphism (in fact an isomorphism since it has an inverse) from $F(V) \to G(V)$ that satisfies the conditions above.

1.4 Products

The product is a typical construction in many spaces. Familiar examples include:

Example 1.15. The product of sets A, B is $A \times B = \{(a,b) : a \in A, b \in B\}$.

Example 1.16. The product of groups A, B is $A \times B = \{(a,b) : a \in A, b \in B\}$ with a group operation given by $(a,b)(c,d) = (ac, bd)$.

Example 1.17. The product of topological spaces A, B is $A \times B = \{(a,b) : a \in A, b \in B\}$ with the product topology.

Definition 1.5. Suppose X is any object with morphisms $f : X \to A$ and $f : X \to B$. Then a *product* $A \times B$ of A and B is an object with morphisms $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that there exists a unique map $\varphi : X \to A \times B$ such that $\varphi \circ \pi_1 = f$ and $\varphi \circ \pi_2 = g$.

$$\begin{tikzcd} & X \arrow[ddl, "f"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "g"] & \\ & A \times B \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ A & & B \end{tikzcd}$$

This property defines $A \times B$ up to canonical isomorphism (a morphism $f : A \to B$ such that we can find $g : B \to A$ with $f \circ g = 1_B$ and $g \circ f = 1_A$). Suppose X, Y are both products of A and B. Then the composition of the two maps φ, ψ between X and Y is the identity by the uniqueness of the map defined above.

So we can define products in any category, and this definition ignores the internal structure of the objects.

1.5 Equalizers

Definition 1.6. Let A and B be objects in a category. The *equalizer* of two morphisms $f, g : A \to B$ is an object X and a morphism $h : X \to A$ such that

1. $f \circ h = g \circ h$
2. If Y is an object with $i : Y \to A$ such that $f \circ i = g \circ i$, then Y factors uniquely through X.

That is, the followng diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Y \arrow[u, dashed] \arrow[ur] & & \end{tikzcd}$$

Suppose A, B are groups with $f : A \to B$. The kernel of f is the equalizer of f and $\mathbb{1}$, the trivial map from $A \to B$.

1.6 Initial and final objects

Definition 1.7. A is an *initial object* if there is a unique morphism from A to any other object in the category.

Initial objects are unique up to isomorphism (exercise).

Example 1.18. The empty set is an initial object in the category of sets.

Example 1.19. The trivial group is an initial object in the category of groups.

Definition 1.8. A is a *final object* if there is a unique morphism from any other object in the category to A.

Example 1.20. A 1-element set is a final object in the category of sets.

Example 1.21. The trivial group is a final object in the category of groups.

1.7 Limits and pull-backs

Definition 1.9. A *limit* of $\{A_\alpha\}$ is an object X with morphisms $f_\alpha : X \to A_\alpha$ is characterized by the following properties:

1. If $g_{\alpha,\alpha'} : A_\alpha \to A_{\alpha'}$ is a morphism, then $f_{\alpha'} = g_\alpha \circ f_\alpha$.
2. Any Y with this property factors through X.

$$\begin{tikzcd}
& Y \arrow[ddl] \arrow[dd, bend right] \arrow[ddr] \arrow[d, dashed, "\varphi"] & \\
& X \arrow[dl, "f_1"] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\
A_1 \arrow[r, "g_{1,2}"'] & A_2 \arrow[r, "g_{2,3}"'] & A_3 \arrow[ll, bend left, "g_{3,1}"]
\end{tikzcd}$$

Example 1.22. A product is a limit of A and B.

Example 1.23. The equalizer is a limit of A and B with morphisms $f, g : A \to B$.

Definition 1.10. The *pull-back* X is a limit of A and B with morphisms $f : A \to C$ and $g : B \to C$.

$$\begin{tikzcd}
& Y \arrow[d, dashed, "\varphi"] \arrow[ddl] \arrow[ddr] & \\
& X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

Example 1.24. The pull-back of sets A, B is $\{(a, b) \in A \times B : f(a) = b(y)\}$.

1.8 Coproducts

If we reverse the arrows in a product, we get a coproduct.

$$\begin{tikzcd}
& X & \\
& A \amalg B \arrow[u, dashed, "\varphi"'] & \\
A \arrow[uur, "f"] \arrow[ur, "\pi_1"'] & & B \arrow[uul, "g"'] \arrow[ul, "\pi_2"]
\end{tikzcd}$$

Example 1.25. In the category of sets, the coproduct is the disjoint union.

Example 1.26. In the category of abelian groups, the coproduct equals $A \times B$, so the coproduct equals the product.

In the category of groups, what is the coproduct of A and B? It is the *free group* on two generators. We will discuss this next lecture.

We can also take infinite products and coproducts. The infinite product of abelian groups is the usual infinite product, and the infinite coproduct of abelian groups is the subgroup of the infinite product such that all but finitely many of the coordinates vanish.