Math 255A' Lecture 9 Notes

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1 Metrizibility and Normability of LCSs and The Geometric Hahn-Banach Theorem

1.1 Metrizable locally convex spaces

When is a LCS topology metrizable?

Theorem 1.1. *Let X be a LCS. Then X is metrizable (with a translation invariant metric) if and only if its topology can be generated by a countable family of seminorms.*

Proof. Suppose the topology is generated by $(p_n)_n$. Define

$$d(x,y) \sum_{n=1}^\infty 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}.$$

For every $\varepsilon > 0$ and $N \in \mathbb{N}$, there is a $\delta > 0$ such that

$$\{y : d(x,y) < \delta\} \subseteq \bigcap_{n=1}^N \{y : p_n(x-y) < \varepsilon\}.$$

Conversely, for any $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$\{y : d(x,y) < \delta\} \supseteq \bigcap_{n=1}^N \{y : p_n(x-y) < \varepsilon\}.$$

Now assume d is a translation invariant metric generating the topology of X. Then $\{x : d(0,x) < 1/n\}$ for $n \in \mathbb{N}$ form a neighborhood base at 0. Let $\mathcal{P}$ be any family of seminorms generating the topology. Then for any n, there exist seminorms $p_{n,1}, \dots, p_{n,N_n} \in \mathcal{P}$ and $\varepsilon_n > 0$ such that

$$\bigcap_{i=1}^n \{x : p_{n,i}(x) < \varepsilon_n\} \subseteq \{x : d(0,x) < 1/n\}.$$

Now $\mathcal{P}_0 = \bigcup_{n=1}^\infty \{p_{n,1}, \dots, p_{n,N_n}\}$ is countable and generates the same topology. $\square$

Example 1.1. $C(\mathbb{R}^n)$ has the metric

$$d(f,g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|f|_{B_n} - g|_{B_n}\|_\infty}{1 + \|f|_{B_n} - g|_{B_n}\|_\infty}.$$

Definition 1.1. A TVS is a **Fréchet space** if its topology can be generated by a complete, translation invariant metric.

1.2 Normable locally convex spaces

When does a LCS have a norm?

Definition 1.2. $A \subseteq X$ is **bounded** if for any neighborhood $U \ni 0$, there is an $\varepsilon > 0$ such that $U \supseteq \varepsilon A$.

Theorem 1.2. *A LCS X is normable if and only if it has a nonempty, open, bounded neighborhood of 0.*

Proof. Let B be a nonempty, open, bounded subset $B \ni 0$. By openness, there is a continuous seminorm p such that $B \supseteq \{p < \varepsilon\}$ for some ε. We can assume that $B \supseteq \{p < 1\}$. We must show that p generates the topology. Let q be another continuous seminorm on X, and consider $\{q < \delta\}$. By boundedness, there exists some $\varepsilon > 0$ such that $\varepsilon\{p < 1\} = \{p < \varepsilon\} \subseteq \{q < \delta\}$. So p generates the topology. Since an LCS must separate points, p must actually be a norm. $\square$

1.3 The geometric Hahn-Banach theorem

Since continuous linear functionals make sense for LCS spaces, we still denote the dual space as X^*. It will have a topology, but we will not discuss which topology yet.

Proposition 1.1. *Let $f : X \to \mathbb{F}$ be a linear functional. The following are equivalent:*

1. *f is continuous.*
2. *f is continuous at 0.*
3. *f is continuous at some point.*
4. $\ker f$ *is closed*
5. *$x \mapsto |f(x)|$ is a continuous seminorm.*
6. *There exist $p_1, \dots, p_n \in \mathcal{P}$ and $\alpha_1, \dots, \alpha_n \in [0, \infty)$ such that $|f| \le \sum_{i=1}^n \alpha_i p_i$.*

Proof. (5) $\implies$ (2): f is continuous at 0 iff for every $\varepsilon > 0$, the set $\{x : |f(x)| < \varepsilon\}$ is a neighborhood of 0.

(5) $\implies$ (6): For any $\varepsilon > 0$, there exist $p_1, \ldots, p_n \in \mathcal{P}$ and $\beta_1, \ldots, \beta_n > 0$ such that $\{|f| < \varepsilon\} \supseteq \bigcap_{i=1}^n \{p_i < \beta_i\}$. So $|f| < \frac{\varepsilon}{\sum_i \beta_i} \sum_i p_i$. □

Proposition 1.2. *Let X be a TVS, and let $G \subseteq X$ be an open, convex neighborhood of 0. Then $q(x) := \inf\{t \geq 0 : tG \ni x\}$ is a nonnegative continuous sublinear functional (and $G = \{q < 1\}$).*

Theorem 1.3 (Geometric Hahn-Banach theorem)**.** *Let X be a TVS, and let $G \subseteq X$ be a nonempty, open, convex set with $G \not\ni 0$. Then there is a closed hyperplane $M \subseteq X$ such that $M \cap G = \varnothing$.*

Proof. Suppose $\mathbb{F} = \mathbb{R}$. Let $x_0 \in G$, and let $H := G - x_0$ be an open, convex neighborhood of 0. Then $0 \in H$, but $-x_0 \notin H$; as H is convex, $tH \not\ni -x_0$ for any $0 \leq t < 1$. Let $q(x) := \inf\{t \geq 0 : tH \ni x\}$ as in the proposition. Then $q(-x_0) \geq 1$. Now let $Y = \operatorname{span}\{-x_0\}$. Then $g : Y \to \mathbb{R}$ with $g(-x_0) = 1$ is a continuous linear functional, and Hahn-Banach gives a linear $f : X \to \mathbb{R}$ such that $f(-x_0) = 1$, $|f| \leq q$; so f is continuous. Now $\{f = 1\} \cap H = \varnothing$, so $\ker(f) \cap G = \varnothing$. So pick $M = \ker(f)$.

In the case $\mathbb{F} = \mathbb{C}$, applied the theorem to X (viewed as a vector space over $\mathbb{R}$). We get a continuous $\mathbb{R}$-linear $f : X \to \mathbb{R}$ such that $\ker(f) \cap G = \varnothing$. Construct $g(x) := f(x) - if(ix)$, which is a complex linear functional. Then $\ker g = (\ker f) \cap i(\ker f)$. □

Corollary 1.1. *Let X is a TVS, $Y \subseteq X$ be a closed affine subspace, and $G \neq 0$ be an open convex subset with $Y \cap G \neq \varnothing$. Then there is a closed affine hyperplane $M \supseteq Y$ such that $M \cap G = \varnothing$.*

Proof. Suppose $0 \in Y$. Consider the quotient map $Q : X \to X/Y$. Then $Q(G)$ is an open, convex subset of X/Y with $Q(G) \not\ni 0$. Find a hyperplane $\overline{M} \subseteq M/Y$ such that $\overline{M} \cap Q(G) = \varnothing$, and let $M := Q^{-1}[M]$.

If $0 \notin Y$, do the same with a translation. □

1.4 Half-spaces and separated sets

Definition 1.3. In a real TVS an open **half-space** is a is a set of the form $\{f > \alpha\}$ for some $f \in X^*$ and $\alpha \in \mathbb{R}$. A closed **half-space** is a is a set of the form $\{f \geq \alpha\}$ for some $f \in X^*$ and $\alpha \in \mathbb{R}$.

Definition 1.4. $A, B \subseteq X$ are **separated** of there exist closed half-spaces H, K such that $A \subseteq H$, $B \subseteq K$, and $H \cap K$ is an affine hyperplane. A and B are **strictly separated** if there are open half-spaces $H \supseteq A$ and $K \supseteq B$ with $H \cap K = \varnothing$.

Theorem 1.4. *Half-spaces and separated sets have the following properties:*

1. *The closure of an open half-space is a closed half-space.*
2. *The interior of a closed half-space is an open half-space.*
3. *If A, B are separated, then there exists an $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f|_A \leq \alpha$ and $f|_B \geq \alpha$.*
4. *If A, B are strictly separated, then there exists an $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f|_A < \alpha$ and $f|_B > \alpha$.*

Theorem 1.5. *Let X be a real TVS, and let A, B be disjoint, convex sets with A open. Then there exist an $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f|_A < \alpha$, $f|_B \geq \alpha$. If B is also open, then A and B are strictly separated.*

We will get this as a consequence of geometric Hahn-Banach next time.