Math 255B Lecture 13 Notes

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1 Examples of Self-Adjoint Extensions

1.1 Self-adjoint extensions of differential operators

Let $S : D(S) \to H$ be symmetric, closed, and densely defined. Last time, we made the observation that S is self-adjoint $\iff$ $\operatorname{Im}(S \pm i) = H$ $\iff$ $\ker(S^* \mp i) = \{0\}$. We also saw that S has a self-adjoint extension $\iff$ $\dim \operatorname{Im}(S+i)^\perp = \dim \operatorname{Im}(S-i)$.

Example 1.1. Let $H = L^2(\mathbb{R}^n)$, and let $P = P(D)$ be a linear, differential operator with constant, real coefficients:

$$P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \qquad a_\alpha \in \mathbb{R}, D = \frac{1}{i}\partial.$$

Let $P_{\min}$ be the minimal realization of P: $P_{\min} = \overline{P|_{C_0^\infty}}$. Then $P_{\min}$ is closed, densely defined, and symmetric: if $u, v \in C_0^\infty$,

$$\langle Pu, v \rangle_{L^2} = \int Pu\overline{v}\,dx = \sum_{|\alpha| \leq m} \int a_\alpha D^\alpha u \overline{v}\,dx = \langle u, Pv \rangle_{L^2}.$$

We claim that $P_{\min}$ is self-adjoint. Check that $\ker(P_{\min}^* - \pm i) = \{0\}$: Here, $D(P_{\min}^*) = \{u \in L^2 : Pu \in L^2\}$. If $u \in D(P_{\min})$, then we get a differential equation:

$$(P_{\min}^* \pm i)u = 0 \iff (P(D) \pm i)u = 0$$

Take the Fourier transform:

$$\mathcal{F}[(P(D) \pm i)u] = 0 \iff \left(\sum_{|\alpha| \leq m} a_\alpha \xi^\alpha \pm i \right) \widehat{u}(\xi) = 0.$$

Then $\widehat{u} = 0$, so $u = 0$.

Since $P_{\max} = P_{\min}^*$, we get that $P_{\max} = P_{\min}$. So P has only one realization, which is self-adjoint. That is, it only has one self-adjoint extension.

Example 1.2. Let $H = L^2((0,\infty))$, and let $P(D) = D = \frac{1}{i}\frac{d}{dx}$. Let $P_{\min} = \overline{P|_{C_0^\infty}}$. Compute the deficiency indices: $(P(D) \pm i)i = 0$ for $u \in L^2((0,\infty))$, so

$$\left(\frac{1}{i}\frac{d}{dx} - i\right) u = 0 \iff u' + u = 0 \iff u(x) = Ce^{-x} \in L^2.$$

So $n_+ = 1$. For the $+$ case, we have

$$\left(\frac{1}{i}\frac{d}{dx} + i\right) u = 0 \iff u' - u = 0 \iff u(x) = Ce^{x}.$$

But such a $u \notin L^2((0,\infty))$, so $n_- = 0$.

Thus, $P_{\min}$ is maximal, symmetric, and has no self-adjoint extensions.

Remark 1.1. We have omitted the argument that these differential equations have no nonclassical solutions. We have

$$u' + u = 0 \iff (e^x u)' = 0,$$

where this derivative is in the distributional sense. We use the fact that if $u \in D'(\mathbb{R})$ with $u' = 0$, then u is constant.

Remark 1.2. In this example, $D(P^*_{\min}) = \{u \in L^2 : Pu \in L^2\} = H^1((0,\infty))$.

1.2 Essentially self-adjoint operators

Definition 1.1. Let $S : D(S) \to H$ be symmetric and densely defined. We say that S is **essentially self-adjoint** if $\overline{S}$ is self-adjoint.

Here is an example.

Theorem 1.1 (Essential self-adjointness of the Schrödinger operator with a semibounded potential)**.** *Let $P = P(x,D) = -\Delta + q(x)$, where $q \in C(\mathbb{R}^n;\mathbb{R})$. Let P_0 be the minimal realization of P: $P_0 = \overline{P|_{C_0^\infty}}$, which is closed, symmetric and densely defined. Assume that $q \geq -C$ on $\mathbb{R}^n$. Then P_0 is self-adjoint (i.e. $P(x,D)$ is essentially self-adjoint).*

Remark 1.3. $-\Delta \geq 0$: If $u \in C_0^\infty$, $\langle -\Delta u, u\rangle = \int -\Delta u\overline{u} = \int |\nabla u|^2 \geq 0$. We cannot let the operator tend to $-\infty$ unchecked, which is why we need this semiboundedness condition. This condition can be relaxed, but there needs to be some condition.

If q were actually bounded, this theorem is easier to prove. One can prove that a self adjoint operator plus a bounded self-adjoint operator is still self-adjoint (and with the same domain).

Proof. $D(P_0^*) = \{u \in L^2 : Pu = (-\Delta + q)u \in L^2\}$, and $P_0^* u = P_u$ for $u \in D(P_0^*)$. We shall show that P_0^* is symmetric; that is, $\langle u, P_0^* u \rangle_{L^2} \in \mathbb{R}$ for all $u \in D(P_0^*)$. First, if $u \in D(P_0^*)$, then $\Delta u \in L^2_{\text{loc}}$. So $u \in H^2_{\text{loc}} = \{u \in L^2_{\text{loc}} : \partial^\alpha u \in L^2_{\text{loc}}\ \forall |\alpha| \leq 2\}$. In particular, $\nabla u \in L^2_{\text{loc}}$.

We claim that if $u \in D(P_0^*)$, then $\nabla u \in L^2(\mathbb{R}^n)$. We may assume that $u \in D(P_0^*)$ is real (by considering real and imaginary parts separately). Consider

$$\int \psi_t(x) i P u \, dx = \int \psi_t(x) u (-\Delta + q) u \, dx,$$

where $\psi_t(x) = \psi(tx)$, $0 \leq \psi \in C_0^\infty(\mathbb{R}^n)$ is a cutoff which is 1 near 0. The idea is that once we introduce this cutoff, we can integrate by parts. We will get something like $\int \psi_t |\nabla u|^2$ and will try to control this uniformly in t to use Fatou's lemma. □

We will finish the proof next time.