# Stat 206B Lecture 1 Notes

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## 1 Kolmogorov Consistency and Weiner's Theorem

**Definition 1.1.** A *filtration* is a family $(\mathcal{F}_t, t \in I)$ of $\sigma$-fields such that

1. $\mathcal{F}_t \subseteq \mathcal{F}$.
2. $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

**Definition 1.2.** Given a state space $(S, \mathcal{S})$, a *transition kernel* $P_S(x, \cdot)$ is defined by:

1. $P_S(x, \cdot)$ is, for fixed $S$, a probability measure on $(S, \mathcal{S})$.
2. $x \mapsto P_S(x, B)$ is $\mathcal{S}$-measurable for $B \in \mathcal{S}$.
3. We have some niceness in $S$ (Chapman-Kolmogorov equation).

How do we specify a stochastic process? We specify finite dimensional distributions and glue them together. These distributions must be consistent.

**Definition 1.3.** GIven an index set $I$, finite subsets $F$ of $I$, and a "nice"[1] state space $(S, \mathcal{S})$, denote $S^F$ as the product space of $S$ indexed by $F$. Each $S^F$ has a probability measure $\pi_F$. The system of finite dimensional distributions is *consistent* if for all $F \subseteq G$, the image of $\pi_G$ by the projection from $S^G \to S^F$ is $\pi_F$.

**Theorem 1.1** (Kolmogorov consistency)**.** *Let $I$ be an index set whose finite dimensional distributions are consistent. Then there exists a process $(x_i, i \in I)$ whose finite dimensional distributions are the $\pi_F$. Here, we have $(S^I, \mathcal{S}^I)$, where $\mathcal{S}^I$ is generated by the projections $X : S^I \to S$.*

**Remark 1.1.** There is an issue with $\mathcal{S}^I$. Every event in $\mathcal{S}^I$ is determined by some countable number of coordinates (Proof: Consider the collection of all events of this kind, and prove it is a $\sigma$-field).

[1] Look at Durrett to see more about what "nice" means.

**Example 1.1.** Take Brownian motion as an example. $I = \mathbb{R}_+$, and the finite dimensional distributions are $N(0,t)$ for $(B_t, t \geq 0)$.

What is $P$(BM has continuous paths)? This cannot be determined by countably many points, so we must reframe the question.

**Theorem 1.2** (Weiner)**.** *On* $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, Leb)$*, there exists a stochastic process* $(B(t,\omega), t \geq 0, \omega \in \Omega)$ *such that*

1. *For* $0 \leq t_1 < t_2 < \cdots, < t_n$, $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent with mean 0 and variance* $t_1, t_2, -t_1, \ldots, t_n - t_{n-1}$ *(this is consistent because* $N(0,s) * N(0,t) \overset{d}{=} N(0, s+t)$*).*

2. $P(\{\omega \in \Omega : t \to B(t,\omega)$ *is a continuous function*$\}) = 1$.

This implies that we can discuss a law of the path of Brownian motion on $C[0,\infty)$, where $C[0,\infty)$ is the canonical space of continuous paths. It has Borel sets and can be made a metric space where convergence is uniform convergence on all compact sets. It is what is called a Polish space, which means that it is separable and complete. We get that $\omega \to (B(t,\omega), t \geq 0)$. The image of the map $\Omega \to C[0,\infty)$, is called *Wiener measure.*

*Proof.* (sketch) Let $D := \{k/2^n : k, n \in \mathbb{N}\}$. We first want to create $(B_t : t \in D)$. Even before that, we need to create infinitely many independent standard normal random variables. Given $x \in [0,1]$, we can take the binary expansion of $x$; we can turn this into two numbers by taking the even and odd places in the expansion respectively and interpreting them as separate binary expansion of two numbers. In this way, we can turn a $U[0,1]$ into a pair of independent uniform $[0,1]$ random variables. Similarly, we can get three independent $U[0,1]$ random variables by taking each third place in the binary expansion of $x$. There is a cute way to get infinitely many independent $U[0,1]$; do the same process except taking distinct multiples of distinct primes. Now, apply $\Phi^{-1}$ to each uniform $[0,1]$ random variable to get $(Z_1, Z_2, Z_3, \ldots)$, where the $Z_i$ are IID standard normal.

It is enough to construct Brownian motion on $[0,1]$ because we can just concatenate independent copies of paths on $[0,1]$ together. This is Levy's construction for Brownian motion on $[0,1]$. First, let $B_0 = 0$, and then let $B_1 = Z_1$. What is $B_{1/2}$? We know that $E[B_{1/2} \mid B_1] = (1/2)B_1$ and $\mathrm{Var}(B_{1/2} \mid B_1) = \sigma^2$ for some constant $\sigma$. The distribution of $B_{1/2}$ given $B_1$ is $N(B_1/2, \sigma^2)$. So $B_{1/2} := (1/2)B_1 + \sigma Z_2$. Check that $B_{1/2}$ and $B_1 - B_{1/2}$ are IID $N(0, 1/2)$. Do the same for $B_{1/4}$ and $B_{3/4}$; i.e. $B_{1/4} := (1/2)B_{1/2} + (6/\sqrt{2})Z_3$ and $B_{3/4} := (1/2)B_{1/2} + (\sigma/\sqrt{2})Z_4$. Continue until we have defined Brownian motion on all of $D$, the dyadic rationals. $\square$