Math 247A Lecture 5 Notes

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1 Banach Space Properties of Lorentz Spaces

1.1 Proof of completeness, duality, and more

Theorem 1.1. *For $1 < p < \infty$ and $1 \leq q \leq \infty$,*

$$\|f\|^*_{L^{p,q}} \sim_{p,q} \sup\left\{ \left| \int f(x)g(x)\,dx \right| : \|g\|^*_{L^{p',q'}} \leq 1 \right\}.$$

Thus, $\|\cdot\|^*_{L^{p,q}}$ is equivalent to a norm, with respect to which $L^{p,q}(\mathbb{R}^d)$ is a Banach space. Moreover, for $q \neq \infty$, the dual of $L^{p,q}$ is $L^{p',q'}$, under the natural pairing.

Proof. Last time, we saw that it suffices to prove the equivalence for functions of the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n}$$

with F_n measurable, pairwise disjoint, and $\|f^*_{p,q} \sim \|2^n |F_n|^{1/p}\|_{\ell^q_n} \sim 1$. Last time, we showed that $RHS \lesssim LHS$ by testing it on $g = \sum 2^m \mathbb{1}_{E_m}$ with E_m measurable, pairwise disjoint, and $\|g\|^*_{L^{p'q'}} \sim \|2^m |E_m|^{1/p'}\|_{\ell^{q'}_m} \lesssim 1$. Let's show that $LHS \lesssim RHS$.

Compare: in the case of $L^q(\mathbb{R}^d)$, we take $g = \frac{|f|^{q-1} \operatorname{sgn} f}{\|f\|_q^{q-1}}$. Here, we take

$$g = \sum \left(2^n |F_n|^{1/p}\right)^{q-1} |F_n|^{-1/p'} \mathbb{1}_{F_n}$$

Check:

$$\begin{aligned}
\int f(x)g(x)\,dx &= \sum_n 2^n \left(2^n |F_n|^{1/p}\right)^{q-1} |F_n|^{-1/p'} |F_n| \\
&= \sum_n 2^{nq} |F_n|^{q/p} \\
&= \|2^n |F_n|^{1/p}\|^q_{\ell^q}
\end{aligned}$$

$$
\begin{aligned}
&\sim (\|f\|^*_{L^{p,q}})^q \\
&\sim 1.
\end{aligned}
$$

It remains to show that $\|g\|^*_{L^{p',q'}} \lesssim 1$. By the proposition which evaluates the norm as a dyadic sum,

$$
\left(\|g\|^*_{L^{p',q'}}\right)^{q'} \sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} |\{x : N \leq g(x) < 2N\}|^{q'/p'}
$$

Note that $\{x : g \sim N\} = \bigcup_{n \in S_N} F_n$, where $S_N = \{n \in \mathbb{Z} : 2^{n(q-1)}|F_n|^{q/p-1} \sim N\} = \{n \in \mathbb{Z} : |F_n| \sim [N2^{-n(q-1)}]^{p/(q-p)}\}$.

$$
\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \left(\sum_{n \in S_N} |F_n| \right)^{q'/p'}
$$

This is a dyadic sum, so we can pull the exponent inside (by our lemma).

$$
\begin{aligned}
&\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \sum_{n \in S_N} |F_n|^{q'/p'} \\
&\sim \sum_{n \in \mathbb{Z}} \left(2^{n(q-1)}|F_n|^{q/p-1}\right)^{q'} |F_n|^{q'/p'}
\end{aligned}
$$

Use $q' = q/(q-1)$.

$$
\begin{aligned}
&\sim \sum_{n \in \mathbb{Z}} 2^{nq} |F_n|^{q'(q/p-1/p)} \\
&\sim \sum_{n \in \mathbb{Z}} 2^{nq} |F_n|^{q/p} \\
&\sim (\|f\|^*_{L^{p,q}})^q \\
&\sim 1.
\end{aligned}
$$

The RHS defines a norm $|||\cdot|||$. To see that $L^{p,q}$ equiped with this norm is a Banach space, one uses the usual Riesz-Fischer argument.

Step 1: If $\|f_n\| \in L^{p,q}$ are such that $\sum |||\cdot||| < \infty$, then there exists a function $f \in L^{p,q}$ such that $f = \sum f_n$ in $|||\cdot|||$.

Step 2: For a Cauchy sequence $\{f_n\}_{n\geq 1} \subseteq L^{p,q}$, we pass to a subsequence so that $|||f_{k_{n+1}} - f_{k_n}||| < \frac{1}{2^n}$. So by Step 1,

$$
f_{k_n} = f_{k_1} + \sum_{j=2}^{n} f_{k_j} - f_{k_{j-1}} \xrightarrow{|||\cdot|||} f.
$$

For $1 \leq q < \infty$,we want to show that the dual of $L^{p,q}$ is $L^{p',q'}$. Let $\ell : L^{p,q} \to \mathbb{R}$ be a linear functional, so $\|\ell(f)\| \lesssim \|f\|^*_{L^{p,q}}$. For $f = \mathbb{1}_E$ with E of finite measure,

$$\ell(\mathbb{1}_E) \lesssim |E|^{1/p}.$$

So the measure $E \mapsto \ell(\mathbb{1}_E)$ is absolutely continuous with respect to Lebesgue measure. So there exists a $g \in L^1_{\text{loc}}$ such that

$$\ell(\mathbb{1}_E) = \int g(x)\mathbb{1}_E(x)\, dx.$$

As ℓ is linear, we get

$$\ell(f) = \int f(x)g(x)\, dx$$

for all simple functions $f \in L^{p,q}$.

- Claim 1: Boundedness of ℓ on simple functions yields $g \in L^{p',q'}$.
- Claim 2: Simple functions are dense in $L^{p,q}$ if $q \neq \infty$.

Given these two claims, we get $\ell(f) = \int f(x)g(x)\, dx$ for all $f \in L^{p,q}$. Thus, the dual of $L^{p,q}$ is $L^{p',q'}$.

Proof of Claim 1: It is enough to show that if $g = \sum 2^m \mathbb{1}_{E_m}$ with E_m measurable, pairwise disjoint, we have

$$\|2^m|E_m|^{1/p'}\|_{\ell^{q'}} \lesssim 1.$$

Choose $f = \sum_{|m|\leq M} \left(2^m|E|_m|^{1/p'}\right)^{q'-1} |E_m|^{-1/p}\mathbb{1}_{E_m}$. Then

$$\ell(f) = \int f(x)g(x)\, dx \sim \|2^m|E_m|^{1/p'}\|^{q'}_{\ell^{q'}_{|m|\leq M}}, \qquad \|f\|_{L^{p,q}} \sim \|2^m|E_m|^{1/p'}\|^{q'/q}_{\ell^{q'}_{|m|\leq M}}.$$

We have $\ell(f) \lesssim \|f\|^*_{L^{p,q}}$, so

$$\|2^m|E_m|^{1/p'}\|_{\ell^{q'}_{|m|\leq M}} \lesssim \|2^m|E_m|^{1/p'}\|^{1/(q-1)}_{\ell^{q'}_{|m|\leq M}}.$$

This gives

$$\|2^m|E_m|^{1/p'}\|_{\ell^{q'}_{|m|\leq M}} \lesssim 1$$

uniformly in M. So $g \in L^{p',q'}$.

Proof of Claim 2: Consider $f \geq 0$ and look at $f_m = f\mathbb{1}_{\{2^m \leq f < 2^{m+1}\}}$. For $2^{m+n} \leq k \leq 2^{m+1+n} - 1$, let

$$E^k_{m,n} = f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right), \qquad \varphi_{m,n} = \sum_{k=2}^{2^{m+n+1}-1} \frac{k}{2^n}\mathbb{1}_{E_{m,n}k}.$$

Then $0 \leq f_m - \varphi_{m,n} \leq \frac{1}{2^n}$. First choose $\varepsilon > 0$. If we look at $\|f\|^*_{L^{p,q}} \| \|f_m\|_{L^p} \|_{\ell^q}$, only finitely many terms matter, so we can truncate the series. This lets us estimate $\|f_m - \varphi_{m,n}\|^*_{p',q'}$, as any large numbers we get will be multiplied by our small step size, $\frac{1}{2^n}$. $\square$