Math 249 Lecture 13 Notes

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1 Schur Functions

1.1 Definition and connection to antisymmetric functions

Last lecture, we had the antisymmetric functions

$$a_{\lambda+\rho} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(x^{\lambda+\rho}) = \det \begin{bmatrix} x_1^{(\lambda+\rho)_1} & x_1^{(\lambda+\rho)_2} & \cdots & x_1^{(\lambda+\rho)_n} \\ x_2^{(\lambda+\rho)_1} & x_2^{(\lambda+\rho)_2} & \cdots & x_2^{(\lambda+\rho)_n} \\ \vdots & \vdots & & \vdots \\ x_n^{(\lambda+\rho)_1} & x_n^{(\lambda+\rho)_2} & \cdots & x_2^{(\lambda+\rho)_n} \end{bmatrix}$$

and the Vandermonde determinant

$$a_\rho = \det \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{bmatrix} = \prod_{i<j}(x_i - x_j).$$

We also had that a_ρ divides $a_{\lambda+\rho}$ for all λ.

Definition 1.1. The *Schur functions* are functions of the form

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho}.$$

We have an isomorphism of abelian groups $\Lambda(x_1, \dots, x_n) \to \mathbb{Z}[x_1, \dots, x_n]^\varepsilon$ given by $f \mapsto a_\rho f$. So the $\{s_\lambda\}$ form a $\mathbb{Z}$-basis of $\Lambda_{\mathbb{R}}(x_1, \dots, x_n)$. If $|\lambda| = d$, then $\deg(s_\lambda) = d$.

1.2 Independence of the number of variables

Remarkably, the Schur functions do not depend on the number of variables n.

Lemma 1.1. *Let $\ell(\lambda)$ be the number of parts in the partition λ. Then*

$$s_\lambda(x_1, \cdots, x_{n-1}, 0) = \begin{cases} s_\lambda(x_1, \ldots, x_{n-1}) & \ell(\lambda) < n \\ 0 & \ell(\lambda) = n \end{cases}$$

Proof. If $\ell(\lambda) = n$, then $x^{\lambda+\rho}$ is a multiple of $x_1 \cdots x_n$. So $a_{\lambda+\rho}$ is divisible by $x_1 \cdots x_n$, which makes $a_{\lambda+\rho}(x_1, \ldots, x_{n-1}, 0) = 0$.

If $\ell(\lambda) < n$,

$$\begin{aligned} a_{\lambda+\rho}(x_1, \ldots, x_{n-1}, 0) &= \det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ \vdots & \vdots & & \vdots \\ x_{n-1}^{\lambda_1+n-1} & x_{n-1}^{\lambda_2+n-2} & \cdots & x_{n-1}^{\lambda_n} \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= a_{\lambda+\rho_{n-1}+(1,\ldots,1)}(x_1, \ldots, x_{n-1}) \\ &= x_1 \cdots x_{n-1} a_{\lambda+\rho_{n-1}}(x_1, \ldots, x_{n-1}). \end{aligned}$$

Now note that

$$a_\rho(x_1, \ldots, x_{n-1}, 0) = \left(\prod_{i<j\le n-1} (x_i - x_j) \right) x_1 x_2 \cdots x_{n-1} = x_1 x_2 \cdots x_{n-1} a_{\rho_{n-1}},$$

so we get

$$\begin{aligned} s_\lambda(x_1, \cdots, x_{n-1}, 0) &= \frac{a_{\lambda+\rho}(x_1, \cdots, x_{n-1}, 0)}{a_\rho(x_1, \cdots, x_{n-1}, 0)} \\ &= \frac{x_1 \cdots x_{n-1} a_{\lambda+\rho_{n-1}}(x_1, \ldots, x_{n-1})}{x_1 x_2 \cdots x_{n-1} a_{\rho_{n-1}}} \\ &= \frac{a_{\lambda+\rho_{n-1}}(x_1, \ldots, x_{n-1})}{a_{\rho_{n-1}}} \\ &= s_\lambda(x_1, \ldots, x_{n-1}). \end{aligned}$$

□

Corollary 1.1. *The coefficient of m_μ in s_λ, $K_{\lambda,\mu} = \langle m_\mu \rangle s_\lambda$, is independent of the number of variables n (provided that $n \ge \ell(\lambda), \ell(\mu)$).*

Proof. Let $s_\lambda(x_1, x_2, \ldots) = \sum_\mu K_{\lambda,\mu} m_\mu(x_1, x_2, \ldots)$. Then

$$s_\lambda(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \begin{cases} s_\lambda(x_1, \ldots, x_n) & \ell(\lambda) \le n \\ 0 & \ell(\lambda) > n. \end{cases}$$

□

1.3 Relationship between e_k and s_λ

If f is a symmetric function, $\langle s_\lambda \rangle f = \left\langle x^{\rho+\lambda} \right\rangle f a_\rho$.

What is the coefficient of s_λ in e_k?

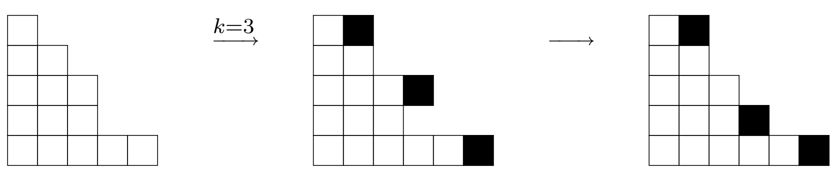
$$\begin{aligned} e_k a_\rho &= e_k \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(x_1^{n-1} x_2^{n-2} \cdots x_n^0) \\ &= \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(e_k x_1^{n-1} x_2^{n-2} \cdots x_n^0) \\ &= a_{(1^k)+\rho} \end{aligned}$$

because this multiplication adds 1 to each exponent of k variables in each monomial of the right term. However, if we make any two variables have the same exponent, then the coefficient of that term will be 0 because the product of a symmetric and an antisymmetric function must be antisymmetric; so we can only get the term where we added 1 to the exponents of $x_1, \ldots, x_k$. Since $e_k a_\rho = a_{(1^k)+\rho}$, we have that $e_k = s_{(1^k)}$.

If we want to express $e_k s_\lambda$ in terms of a sum of s_μ terms, we can show that $e_k a_{\lambda+\rho} = \sum_\mu c_\mu a_{\mu+\rho}$ and then divide by a_ρ on both sides. We have

$$e_k a_{\lambda+\rho} = e_k \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}) = \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(e_k x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}) = \sum a_{\mu+\rho},$$

where $\mu - \lambda$ is a sum of k distinct unit vectors. We call this a *vertical k-strip.* To get one of these μ, we take the Young diagram of λ and add k boxes, at most 1 to each row.



The only terms that survive are terms where we add boxes consecutively from the bottom, i.e. terms such that μ is still a partition. You can imagine gravity pulling the black boxes down. So we get that

$$e_k s_\lambda = \sum_\mu s_\mu,$$

where the sum is over paritions μ such that μ/λ is a vertical k-strip.

Example 1.1.

$$e_2 = s_{(1,1)}$$

$$e_{(1,1)} = e_1 e_1 = e_1 s_1 = s_{(2)} + s_{(1,1)}$$

1.4 Relationship to continuous characters

Let $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ be a continuous representation; that is, $\mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathbb{C}^N$. Then the character χ is continuous because the trace map is continuous. Diagonalizable matrices are dense in $\mathrm{GL}_n(\mathbb{C})$, so the values of χ are determined by its values on diagonal matrices of eigenvalues

$$D = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}.$$

The value of χ on these matrices is invariant under permutations of the diagonal entries, so $\chi \in \Lambda(x_1, \ldots, x_n)$. We will not prove this here, but the theory of Lie groups gives us that

$$\begin{aligned} \chi(D) &= \sum_{\sigma \in S_n} \sigma \left(\frac{x^\lambda}{\prod_{i<j}(1 - x_j/x_i)} \right) = \sum_{\sigma \in S_n} \sigma \left(\frac{x^{\lambda+\rho}}{\prod_{i<j}(x_i - x_j)} \right) = \frac{1}{a_\rho} \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma(x^{\lambda+\rho}) \\ &= s_\lambda. \end{aligned}$$