Math 246A Lecture 1 Notes

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1 Complex Numbers, Elementary Mappings, and the Fundamental Theorem of Algebra

1.1 The complex numbers

Consider the set $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. We can express this set differently, as the complex numbers.

Definition 1.1. The set of **complex numbers** is $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$.

Definition 1.2. **Addition** of complex numbers is defined component-wise (usual vector addition):

$$(x + iy) + (u + iv) := (x + u) + i(y + v).$$

Multiplication of complex numbers is defined as

$$(x + iy)(u + iv) := (xu - yv) + i(xv + yu).$$

With these operations, $\mathbb{C}$ is a **field**.

Definition 1.3. Let $z = x + iy$. The **complex conjugate** of z is $\overline{z} = x - iy$.

The map taking $z \mapsto \overline{z}$ is reflection about the real axis in the complex plane. Note that z is real ($y = 0$) if and only if $z = \overline{z}$.

Definition 1.4. The **absolute value** (or **modulus**) of z is $|z| = \sqrt{x^2 + y^2}$.

Note that $z\overline{z} = |z|^2$. So if $z \neq 0$, then $\overline{z}/|z|^2 = z^{-1}$. If $z \neq 0$, then $\left|\frac{z}{|z|}\right| = 1$. So there exists some $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ such that $z/|z| = \cos(\alpha) + i\sin(\alpha)$.

Definition 1.5. Let $z = |z|(\cos(\alpha) + i\sin(\alpha))$. Then α is called the **argument** of z (sometimes denoted $\arg(z)$).

Proposition 1.1. *Let $z, w \in \mathbb{C}$. Then*

$$|zw| = |z| \cdot |w|, \qquad \arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}.$$

Proof. The first assertion follows from an algebraic calculation. For the second assertion, recall the trigonometric identities

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha).$$

Combining these, we get that

$$(\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) = \cos(\alpha + \beta) + i\sin(\alpha + \beta).$$ □

So the geometric picture of multiplication of complex numbers is that the lengths $|z|$ and $|w|$ multiply, and the angles $\arg(z)$ and $\arg(w)$ add.

1.2 Elementary mappings

Let's look at what certain maps $f : \mathbb{C} \to \mathbb{C}$ look like.

1. $z \mapsto \overline{z}$: This is a reflection over the real axis in the complex plane. Note that $\overline{(\overline{z})}$, so this map is an involution.

2. $z \mapsto z^2$: This takes a circle of radius R about 0 to a circle of radius R^2 about zero. If $R > 1$, then the radius grows; if $R < 1$, then the radius shrinks. Additionally, if $\arg(z) = \alpha$, then $\arg(z^2) = 2\alpha$, so this map takes a circle and wraps it around the image circle twice.

3. $z \mapsto z^m$ for $m = 2, 3, 4, \dots$: Since $(\cos(\alpha) + i\sin(\alpha))^m = \cos(m\alpha) + i\sin(m\alpha)$, this wraps a circle around the corresponding image circle m times.

4. $z \mapsto 1/z$: $\left|\frac{1}{z}\right| = \frac{1}{|z|}$, so this map sends a circle of radius R to a circle of radius $1/R$. This map sends $\cos(\alpha) + i\sin(\alpha) \mapsto \cos(\alpha) - i\sin(\alpha)$ (think $1/i = -i$), so a circle being traversed counterclockwise gets mapped to a circle being traversed clockwise.

5. $z \mapsto 1/\overline{z}$: $\frac{1}{z} = \frac{x-iy}{x^2+y^2}$, so $\overline{\left(\frac{1}{z}\right)} = \frac{x+iy}{x^2+y^2} = \frac{z}{|z|^2}$. So this map inverts the modulus of z but doesn't change the angle.

1.3 The fundamental theorem of algebra

Theorem 1.1. *If $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ with $a_i \in \mathbb{C}$, $a_n \neq 0$, $n \geq 1$, then there exists some $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Before we prove this, we need a lemma.

Lemma 1.1. *There exists $R_0 > 0$ such that if $|z| > R_0$, then $|p(z)| > \frac{|a_n|}{2}|z|^n > a_0$.*

Proof. First, take $R_0 > \left(\frac{2|a_0|}{|a_n|}\right)^{1/n}$ and assume $R_0 > 1$. Now, also take R_0 to satisfy

$$\frac{|a_0| + \cdots + |a_{n-1}|}{|a_n|} < \frac{R_0}{2}.$$

Now, for $|z| > R_0$, $|z^j| > R_0^j > 1$. So

$$\begin{aligned} |p(z)| &\geq |a_n||z^n| - (|a_0| + \cdots + |a_{n-1}|)|z|^{n-1} \\ &\geq |a_n||z^n| - \frac{R_0}{2}|a_n||z|^{n-1} \\ &\geq |a_n||z|^n - \frac{|a_n|}{2}|z|^n \\ &= \frac{|a_n|}{2}|z|^n. \end{aligned}$$

□

Now, we prove the fundamental theorem of algebra. This proof, unlike proofs we will be able to give later, is very elementary.

Proof. We can assume that $a_0 \neq 0$ and $n \geq 2$. Let R_0 be as in the lemma, and let $K = \{z \in \mathbb{C} : |z| \leq R_0\}$. Outside of K, $|z| > R_0$, so $|p(z)| > |a_0| > 0$. So all zeros of p must be inside K. Since K is compact and $p : K \to \mathbb{C}$ is continuous, there exists some $z_0 \in K$ such that $|p(z_0)| = \inf\{|p(z)| : z \in K\}$.

For contradiction, assume there is no zero of p in K, and let $C_\delta = \{z : |z - z_0| = \delta\} \subseteq K$ be the circle of radius δ around z_0 for some small $\delta > 0$; we will choose δ later. Let $b_0 = p(z_0) \neq 0$. Then $p(z) = b_0 + b_1(z - z_0) + \cdots + b_n(z - z_0)^k$, where $b_n = a_n$. Take $1 \leq k \leq n$ least such that $b_k \neq 0$, so

$$p(z) = b_0 + \underbrace{b_k(z - z_0)^k}_{Q(z)} + \underbrace{\sum_{j=k+1}^{n} b_j(z - z_0)^j}_{R(z)}$$

Now observe that $z \mapsto b_0 + Q(z)$ maps C_δ to a circle, and wraps around it k times. If $\delta < 1$, then

$$|R(z)| \leq \sum_{j=k+1}^{n} |b_j|\delta^{k+1} \leq \sum_{j=k+1}^{n} |b_j| \frac{\delta}{|b_k|}|Q(z)|.$$

Now pick δ small enough that $(\sum_{j=k+1}^{n} |b_j|)\delta \leq |b_k|/2$. Then $|R(z)| \leq |Q(z)|/2$. Since $z \mapsto b_0 + Q(z)$ wraps around the circle $b_0 + Q(C_\delta)$ $k \geq 1$ times, we can pick the z that gets mapped closest to the origin (making δ small enough that this image circle does not contain the origin in its interior). This will give us a contradiction because we will get a point in the image of p that has smaller magnitude than z_0, contradicting the minimality of z_0. We will make this precise next lecture. □