Stat 155 Lecture 20 Notes

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1 Shapley Value

1.1 Shapley's axioms

Here are Shapley's[1] axioms for allocation functions.

1. Efficiency: $\sum_{i=1}^{n} \psi_i(v) = v(\{1, \ldots, n\})$.

2. Symmetry: If, for all $S \subseteq \{1, \ldots, n\}$ and $i, j \notin S$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\psi_i(v) = \psi_j(v)$.

3. No freeloaders: For all i, if for all $S \subseteq \{1, \ldots, n\}$, $v(S \cup \{i\}) = v(S)$, then $\psi_i(v) = 0$.

4. Additivity: $\psi_i(v + u) = \psi_i(v) + \psi_i(u)$.

Shapley's theorem says that Shapley's axioms uniquely determine the allocation ψ. We call the unique allocation $\psi(v)$ the Shapley value of the players in the game defined by the characteristic function v.

Theorem 1.1 (Shapley). *The following allocation uniquely satisfies Shapley's axioms:*

$$\psi_i(v) = \mathbb{E}_\pi \phi_i(v, \pi),$$

where the expectation is over uniformly chosen permutations π on $\{1, \ldots, n\}$ and

$$\phi_i(v, \pi) = v(\pi(\{1, \ldots, k\})) - v(\pi(\{1, \ldots, k-1\})),$$

where $k = \pi^{-1}(i)$.

Example 1.1. For the identity permutation, $\pi(i) = i$,

$$\phi_i(v, \pi) = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\}),$$

which is how much value i adds to $\{1, \ldots, i-1\}$. And for a random π, $\phi_i(v, \pi)$ is how much value i adds to the random set $\pi(\{1, \ldots, \pi^{-1}(i) - 1\})$.

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[1]Lloyd Shapley was a professor of mathematics at UCLA. He won the 2012 Nobel Prize for Economics.

1.2 Junta games

Example 1.2. A Junta[2] game (J-veto game) is a game where there is a set $J \subseteq \{1, \ldots, n\}$ with all the power:

$$w_J(S) = \mathbb{1}_{(J \subseteq S)} = \begin{cases} 1 & J \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

For any permutation π,

$$\begin{aligned}
\psi_i(w_J, \pi) &= w_J(\pi(\{1, \ldots, \pi^{-1}(i)\})) - w_J(\pi(\{1, \ldots, \pi^{-1}(i) - 1\})) \\
&= \mathbb{1}_{(i \in J, J \subseteq \pi(\{1, \ldots, \pi^{-1}(i)\}))} \\
&= \mathbb{1}_{(i \in J, \pi^{-1}(J) \subseteq \{1, \ldots, \pi^{-1}(i)\})} \\
&= \mathbb{1}_{(i \in J, \pi^{-1}(i) \in \pi^{-1}(J), \pi^{-1}(J) \subseteq \{1, \ldots, \pi^{-1}(i)\})},
\end{aligned}$$

so

$$\begin{aligned}
\psi_i(w_J) &= \mathbb{E}_\pi[\phi_i(w_J, \pi)] \\
&= \mathbb{1}_{(i \in J)} \mathbb{P}(\pi^{-1}(i) = \max_{j \in J} \pi^{-1}(j)) \\
&= \mathbb{1}_{(i \in J)} \frac{1}{|J|}.
\end{aligned}$$

Check that this agrees with the axioms:

1. Efficiency: $\sum_{i=1}^n \psi_i(w_J) = 1 = w_J(\{1, \ldots, n\})$.

2. Symmetry: If for all $S \subseteq \{1, \ldots, n\}$ not containing i and j, $w_J(S \cup \{i\}) = w_J(S \cup \{j\})$ (and this is true for $i, j \in J$ and for $i, j \notin J$), then $\psi_i(w_J) = \psi_j(w_J)$.

3. Dummy: If for all $S \subseteq \{1, \ldots, n\}$, $w_J(S \cup \{i\}) = w_J(S)$ (and this is true for $i \notin J$), then $\psi_i(w_J) = 0$.

Lemma 1.1 (Characteristic functions as Junta games). *We can write any v as a unique linear combination of w_J.*

Proof. Write v as a vector, with one coordinate for each subset $S \subseteq \{1, \ldots, n\}$. Write a matrix W, with rows indexed by $S \subseteq \{1, \ldots, n\}$, columns indexed by $J \subseteq \{1, \ldots, n\}$, and entries $w_J(S)$. If we make sure these subsets are ordered by cardinality, then this matrix is lower triangular, with 1s on its diagonal. Since W is invertible, we can solve the equation $v = Wc$ to obtain a unique c, with one entry c_J for each $J \subseteq \{1, \ldots, n\}$, and then we have

$$v(S) = \sum_J w_J(S) c_J. \qquad \square$$

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[2] In Spanish, a "junta" is a small group with all the power. In Latin America, this has historically occurred many times.

1.3 Shapley's theorem

Let's prove Shapley's theorem.

Proof. First, we want to show that the allocation $\psi_i(v) = \mathbb{E}_\pi[\phi_i(v, \pi)]$ satisfies Shapley?s axioms. For any π, $\psi_i(v, \pi)$ satisfies the efficiency, dummy, and additivity axioms. These axioms all involve linear expressions in i, so they are preserved under expectation. Symmetry follows from the randomization.

1. Efficiency:

$$\sum_{i=1}^{n} \psi_i(v, \pi) = \sum_{i=\pi(1)}^{\pi(n)} [v(\pi(\{1, \ldots, i\})) - v(\pi(\{1, \ldots, i-1\}))]$$
$$= \sum_{j=1}^{n} [v(\pi(\{1, \ldots, j\})) - v(\pi(\{1, \ldots, j-1\}))]$$
$$= v(\{1, \ldots, n\})$$

2. Dummy:
$$\pi(\{1, \ldots, \pi^{-1}(i)\}) = \pi(\{1, \ldots, \pi^{-1}(i) - 1\}) \cup \{i\},$$

so

$$\psi_i(v, \pi) = v(\pi(\{1, \ldots, i\})) - v(\pi(\{1, \ldots, i-1\})) = 0.$$

To prove uniqueness, represent v as a unique linear combination of Junta game characteristic functions $w_J(S) = \mathbb{1}_{(J \subseteq S)}$. Then $\psi_i(w_J) = \mathbb{1}_{(i \in J)}/|J|$ is the unique allocation satisfying the Shapley axioms for the Junta games. Additivity implies that $\psi_i(v)$ is unique. $\square$

This proof actually gives us a nice way to compute the characteristic function. Solve for the coefficients c_J in $v(S) = \sum_J c_J w_J(S)$ by solving the linear system mentioned in the Junta game lemma.

Example 1.3. Consider a glove game like before, with characteristic function

$$v(\{1, 2, 3\}) = v(\{1, 2\}) = v(\{1, 3\}) = 100,$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = v(\varnothing) = 0.$$

Solving the linear system, we get

$$v(S) = 100w_{\{1,2\}}(S) + 100w_{\{1,3\}}(S) - 100w_{\{1,2,3\}}(S),$$

and hence

$$\psi_1(v) = 100 \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{3} \right) = 100 \cdot \frac{2}{3},$$
$$\psi_2(v) = \psi_3(v) = 100 \left(\frac{1}{2} - \frac{1}{3} \right) = 100 \cdot \frac{1}{6}.$$