Stat 155 Lecture 7 Notes

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1 Symmetry in Two Player Zero-sum Games

1.1 Submarine Salvo

Submarine Salvo is a game with a 3×3 grid.

7	8	9
4	5	6
1	2	3

One player picks two adjacent squares (vertically or horizontally) and hides a submarine on those squares. The other player picks a square and drops a bomb to blow up a submarine on that square, if it is there. The payoff matrix is

	1 2	1 4	2 3	2 5	3 6	4 5	4 7	5 6	5 8	6 9	7 8	8 9
1	1	1	0	0	0	0	0	0	0	0	0	0
2	1	0	1	1	0	0	0	0	0	0	0	0
3	0	0	1	0	1	0	0	0	0	0	0	0
4	0	1	0	0	0	1	1	0	0	0	0	0
5	0	0	0	1	0	1	0	1	1	0	0	0
6	0	0	0	0	1	0	0	1	0	1	0	0
7	0	0	0	0	0	0	1	0	0	0	1	0
8	0	0	0	0	0	0	0	0	1	0	1	1
9	0	0	0	0	0	0	0	0	0	1	0	1

Consider a transformation that flips the board from left to right. What happens to the payoff matrix? All we do is permute the rows and the columns of the matrix. Can we exploit this symmetry to help solve the game?

1.2 Invariant vectors and matrices

Definition 1.1. A game with payoff matrix $A \in \mathbb{R}^{m \times n}$ is *invariant* under a permutation π_x on $\{1, \dots, m\}$ if there is a permutation π_y on $\{1, \dots, n\}$ such that for all i and j, $a_{i,j} = a_{\pi_x(i), \pi_y(j)}$.

If A is invariant under π_1 and π_2, then A is invariant under $\pi_1 \circ \pi_2$. So if A is invariant under some set S of permutations, then it is invariant under the group G of permutations generated by S.

Definition 1.2. A mixed strategy $x \in \Delta_m$ is invariant under a permutation π_x on $\{1, \dots, m\}$ if for all i, $x_i = x_{\pi_x(i)}$.

Example 1.1. In Submarine Salvo, x is invariant for the permutation corresponding to a left-to-right flip if $x_1 = x_3$, $x_4 = x_6$, and $x_7 = x_9$.

Definition 1.3. An orbit of a group G of permutations is a set

$$O_i = \{\pi(i) : \pi \in G\}.$$

Example 1.2. For the group generated by horizontal, vertical, and diagonal flips in Submarine Salvo, a few orbits are

$$O_1 = \{1, 3, 7, 9\}, \quad O_2 = \{2, 4, 6, 8\}, \quad O_5 = \{5\}.$$

If a mixed strategy x is invariant under a group G of permutations, then for every orbit, x is constant on the orbit.

Theorem 1.1. *If A is invariant under a group G of permutations, then there are optimal strategies $\bar{x}$ and $\bar{y}$ that are invariant under G.*

Proof. Let x, y be an optimal strategies, and define the strategy $\bar{x}$ to have

$$\bar{x}_i = \frac{1}{|O_i|} \sum_{i' \in O_i} x_{i'},$$

$$\bar{y}_j = \frac{1}{|O_j|} \sum_{j' \in O_j} x_{j'},$$

where O_i is the unique orbit containing move i for Player 1, and O_j is the unique orbit containing move j for Player 2. As an exercise, show that these are optimal. $\square$

1.3 Using invariance to solve games

Using an optimal strategy that is symmetric across orbits, we can simplify a complicated payoff matrix. Let $\bar{x}$ and $\bar{y}$ be invariant optimal strategies. Let $O_1^1, \dots, O_{K_1}^1$ and $O_1^2, \dots, O_{K_2}^2$ be partitions of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. Let $\bar{x}^s$ be the value of x_i for $i \in O_s^1$, and let $\bar{y}^t$ be the value of y_j for $j \in O_t^2$ Then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \bar{x}_i a_{i,j} \bar{y}_j = \sum_{s=1}^{K_1} \sum_{t=1}^{K_2} \left[\sum_{i \in O_s^1} \sum_{j \in O_t^2} \bar{x}_i a_{i,j} \bar{y}_j \right]$$

$$= \sum_{s=1}^{K_1}\sum_{t=1}^{K_2} \bar{x}^s \left[\sum_{i\in O_s^1}\sum_{j\in O_t^2} a_{i,j}\right] \bar{y}^t$$

$$= \sum_{s=1}^{K_1}\sum_{t=1}^{K_2} (|O_s^1|\bar{x}^s) \left[\sum_{i\in O_s^1}\sum_{j\in O_t^2} a_{i,j}\right] (|O_t^2|\bar{y}^t).$$

Note also that

$$\sum_{s=1}^{K_1} |O_s^1|\bar{x}^s = \sum_{i=1}^{m} \bar{x}_i = 1, \qquad \sum_{t=1}^{K_2} |O_t^2|\bar{y}^t = \sum_{j=1}^{n} \bar{y}_j = 1,$$

so we can simplify the matrix to a smaller payoff matrix on the orbits of moves (instead of on each move). The entries of the new matrix are the averages of the original $a_{i,j}$ elements over the orbits containing move i and move j for Players 1 and 2, respectively.

Example 1.3. In Submarine Salvo, we get the payoff matrix over orbits of actions

	edge	center
corner	1/4	0
mid-edge	1/4	1/4
center	0	1

Solving this by finding dominated rows and columns, we get the optimal strategies

$$\hat{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In terms of the original game, this means that an optimal strategy is for the Bomber is to put weight 1/4 for each mid-edge move and for the Submarine to put weight 1/8 on each of 1 2, 1 4, 2 3, 3 6, 4 7, 6 9, 7 8, and 8 9.

Example 1.4. In Rock, Paper, Scissors, each player's moves fall into 1 orbit: $O = \{\text{Rock, Paper, Scissors}\}$. Then an optimal strategy for each player is

$$\bar{x} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \qquad \bar{y} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}.$$