Math 222A Lecture 22 Notes

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1 Properties of Harmonic Functions

1.1 Elliptic regularity

Recall that if we have the Laplace equation

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n,$$

then we have the fundamental solution

$$K(x) = \begin{cases} \frac{c_n}{|x|^{2-n}} = \frac{c_n}{|x|^{2-n}|} & n \geq 3 \\ \frac{1}{2\pi} \ln |x| & n = 2, \end{cases}$$

and we can get a solution $u = K * f$. However, there are a number of questions we have not answered, such as uniqueness of solutions.

Definition 1.1. A function u such that $-\Delta u = 0$ is called **harmonic**.

Theorem 1.1 (Elliptic regularity)**.** *Harmonic functions are smooth.*

That is, if we have a local solution $u \in \mathcal{D}'$, we want to show that $u \in C^\infty$. Why should harmonic functions be smooth? This is because the fundamental solution K is smooth away from 0. Let's see how the reasoning goes.

Proof. Let Ω be the domain where u lives. Choose a point $x_0 \in \Omega$, and we want to show that u is smooth around x_0. Draw a ball B around x_0 and a larger ball $2B$ around B. To use the fundamental solution, chop off u by using a cutoff function

$$\chi(x) = \begin{cases} 1 & x \in B \\ \text{smooth} & x \in 2B \setminus B \\ 0 & x \in 2B^c \end{cases}$$

[insert picture 1]

If we let $v = \chi u$, then

$$-\Delta u = \underbrace{-\chi\Delta u}_{=0} - u\Delta\chi - 2\nabla u \cdot \nabla\chi.$$

This gives us the new problem

$$-\Delta = f, \qquad f \in \mathcal{D}', \qquad \operatorname{supp} f \subseteq 2B \setminus B.$$

Then

$$\begin{aligned} v(x) &= (K * f)(x) \\ &= \int K(x-y) f(y)\, dy. \end{aligned}$$

Suppose we want a local solution in, say, $B/2$, where B has radius R. If $x \in B/2$ and $y \in 2B \setminus B$, then $|x-y| \geq r/2$. Now $K(z)$ is smooth where $|z| \geq r/2$, which means this convolution is smooth for $x \in B/2$. □

Remark 1.1. We didn't use much about the Laplace equation itself here. We only used the fact that K is smooth away from 0.

Remark 1.2. This is not all there is to elliptic regularity. K is analytic away from 0, which tells us that u is analytic.

Remark 1.3. More generally, we may want to make statements about what kind of regularity u has if f has a certain degree of regularity. This is what elliptic regularity really is, and this is only the tip of the iceberg.

1.2 The maximum principle

Definition 1.2. A function u such that $-\Delta u \leq 0$ is called **subharmonic**.

Definition 1.3. A function u such that $-\Delta u \geq 0$ is called **superharmonic**.

We will prove results for harmonic functions and claim that they hold for sub and superharmonic functions, as well.

Suppose $-\Delta u = 0$ in Ω. Where is the max/min of u? The first step to answering this question is to look at the **mean value property**.

Theorem 1.2 (Mean value property)**.** *Suppose $-\Delta u = 0$ in $B(x_0, a)$. Then*

$$\begin{aligned} u(x_0) &= \frac{1}{|B|}\int_B u(x)\, dx \\ &= \frac{1}{|\partial B|}\int_{\partial B} u(x) d\sigma, \end{aligned}$$

where σ is surface measure on the sphere ∂B.

Remark 1.4. If we assume u is subharmonic, i.e. $-\nabla u \leq 0$, then we get $\leq$ instead of equalities. The reverse inequality holds for superharmonic functions.

Lemma 1.1 (Green's theorem)**.** *Suppose $u : \Omega \to \mathbb{R}$. Then*

$$\int_\Omega \partial_j u \, dx = \int_{\partial\Omega} u \cdot \nu_j \, d\sigma,$$

where ν_j is the outward pointing normal to $\partial\Omega$. Equivalently,

$$\int \underbrace{\partial_j u_j}_{\operatorname{div} u} \, dx = \int_{\partial\Omega} u \cdot \nu \, d\sigma.$$

Here's how we can use this: Integrating by parts twice in the following integral keeps the sign the same and introducing 2 boundary terms:

$$\int -\Delta u \cdot v \, dx - \int_\Omega u \cdot (-\Delta n) \, dx = \int_{\partial\Omega} \underbrace{\partial_j u \nu_j}_{\frac{\partial u}{\partial v}} \cdot v - u \cdot \underbrace{\nu_j \partial_j v}_{\frac{\partial v}{\partial \nu}} \, d\sigma,$$

where these are normal derivatives. Now let's prove the mean value property:

Proof. Suppose $B = B(0, r)$, and apply Green's theorem with a well-chosen v. Looking at our equation, it would be nice if we could make $v = 0$ on the boundary. So we can try

$$v = K(|x|) - K(r).$$

We get

$$u(0) = c \int_{\partial B} u \, d\sigma.$$

This holds for all harmonic functions. If we set $u = 1$, then we get $c = \frac{1}{|\partial B|}$, so $u = \frac{1}{|\partial B|} \int_{\partial B} u$. $\square$

Corollary 1.1. *If $u(x_0) = \max u$ for $x_0 \in B$, then u is constant in B.*

Remark 1.5. If u is subharmonic, the same holds. But if u is superharmonic, then we need to replace the maximum with the minimum in this property.

Theorem 1.3 (Strong maximal principle)**.** *Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Moreover, if $\max u$ is attained inside Ω, then u is constant.

The hypotheses here are much stronger than they need to be.

Remark 1.6. If u is subharmonic, the same holds. But if u is superharmonic, then we need to replace the maximum with the minimum.

Proof. If $\max u$ is only attained on $\partial\Omega$, then we ar edone. What if $\max u$ is attained at $x_0 \in \Omega$? Here is a proof by picture. Put a ball around x_0. By the corollary, u is constant in B. Then the other points in this ball are maximum points, and we can get to any other point via a sequence of balls.

[insert picture 2]

If you want to write down a proof, you can use path-connectedness, or you can use an argument like this: Let $A = \{x \in \Omega : u(x) = u(x_0)\}$. Since u is continuous, A is closed. But the corollary says that if $x_0 \in A$, then $B(X_0, r) \subseteq A$. So A is open. Thus, $A \subseteq \Omega$ is open and closed, and if Ω is connected, we get $A = \Omega$. □

The maximal principle is much more general than the proof we have given here. Here is a restatement of this property:

Corollary 1.2 (Comparison principle)**.** *Let u be subharmonic, i.e., $-\Delta u \leq 0$, and let v be subharmonic, i.e., $-\Delta v \geq 0$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\partial}\Omega$.*

Proof. Apply the maximal principle to $u - v$. □

This comparison principle is the correct statement for nonlinear elliptic stuff and also for the Hamilton-Jacobi equations. There is a simpler proof of the maximum principle without the use of the fundamental solution where we drop the strong part.

Proof. Suppose first that $-\Delta u < 0$. Let x_0 be a maximum point inside Ω. Then $\nabla u(x_0) = 0$, and $Hu(x_0) \prec 0$, where $H = \frac{\partial^2 u}{\partial x_i \partial x_j}$ is the Hessian matrix. Observe that

$$\Delta u = \sum_j \partial_j \partial_j u = \operatorname{tr} Hu \leq 0.$$

Then $\Delta u(x_0) \leq 0$, so $-\Delta u(x_0) \geq 0$. But this contradicts our assumption that $-\Delta u < 0$.

Now if $-\Delta u \leq 0$, then we penalize u by replacing u by $u_\varepsilon = u + \varepsilon x^2$. Then

$$-\Delta u_\varepsilon = -\Delta u - 2u\varepsilon < 0.$$

This tells us that

$$\max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon.$$

If we let $\varepsilon \to 0$, both sides converge uniformly to $\max_{\overline{\Omega}} u$ and $\max_{\partial\Omega} u$, respectively. □

1.3 Liouville's theorem

We have been looking at harmonic functions in a domain Ω. What if we are looking at harmonic functions in all of $\mathbb{R}^n$? If you allow exponential growth, then the sky is the limit as to what you can get. But what if we only want polynomial growth. Further yet, what if u is bounded?

Theorem 1.4 (Liouville)**.** *Let u be harmonic in $\mathbb{R}^n$. If u is bounded, then u is constant.*

Proof. If u is harmonic, so are its derivatives. Then

$$\begin{aligned} u(x_0) &\overset{\text{MVP}}{=} ⨍_\Omega \partial_j u(x)\, dx \\ &= \frac{1}{|B_R|} \int_{\partial B_R} u \cdot \nu_j \, d\sigma(x). \end{aligned}$$

If $|u| \leq M$, we can estimate this by

$$\begin{aligned} |\partial_j u(x_0)| &\leq \underbrace{\frac{1}{B_R}}_{R^n} M \underbrace{|\partial B_R|}_{R^{n-1}} \\ &\lesssim \frac{M}{R} \\ &\xrightarrow{R \to \infty} 0. \end{aligned}$$

So $\nabla u(x_0) = 0$, which means that u is constant. $\square$

Remark 1.7. If u is temperate, then $\widehat{u}||\xi|^2 = 0$, so $\widehat{u}$ is supported at 0. Then $\widehat{u} = \sum_\alpha c_\alpha \partial_0^{(\alpha)}$, which implies that u is a polynomial. Thus all temperate harmonic functions are polynomials. This also serves as a proof of Liouville's theorem, since the only bounded polynomials are constant.

1.4 Boundary value problems

Let $\Omega \subseteq \mathbb{R}^n$, and suppose that

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

This give us uniqueness: Suppose u_1, u_2 are solutions. If $u_1 - u_2 = v$, then v is harmonic. The maximum and minimum principles give

$$\max_\Omega v \leq \max_{\partial\Omega} v = 0,$$

$$\min_{\Omega} v \geq \min_{\partial\Omega} v = 0.$$

So $v = 0$.

There is also a proof of existence using hte maximum principle. Consider a subsolution v^- satisfying

$$\begin{cases} -\Delta v^- \leq f \\ v \leq g \end{cases}$$

and a supersolution satisfying

$$\begin{cases} -\Delta v^+ \geq f \\ v \geq g \end{cases}$$

The maximum principle $v^* \geq v^-$. Taking the maximum over all supersolutions and subsolutions gives the largest subsolution and the smallest supersolution. This is called **Perron's method**. We can also find a fundamental solution in Ω, called a **Green function**.