Math 247A Lecture 3 Notes

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1 The Littlewood Principle and Lorentz Spaces

1.1 The Littlewood principle and optimality of the Hausdorff-Young inequality

Last time we were proving the following theorem.

Theorem 1.1. *If $\|\widehat{f}\|_{L^q} \leq \|f\|_{L^p}$ for some $1 \leq p, q \leq \infty$ and all $f \in \mathcal{S}(\mathbb{R}^d)$, then necessarily, $q = p'$ and $1 \leq p \leq 2$.*

We have already proven the first statement. To prove the second we will use the **Littlewood principle**: "A translation invariant operator does not improve decay." So if $T : L^p \to L^q$, then $q \geq p$. This is not a theorem but a general principle.

Say we have a bump function at 0 and we translate it far away. Keep doing this (N times), and let f be the superposition of all the bump functions. If we apply T to f, since T is translation invariant, we will get N translated copies of the modified bump. Then $\|f\|_{L^p} \sim N^{1/p}$, while $\|Tf\|_{L^q} \sim N^{1/q}$. Then we need $N^{1/q} \lesssim N^{1/p}$. Letting $N \to \infty$, we get $1/q \leq 1/p$, so $p \leq q$.

The Fourier transform is not translation invariant, however. And the Fourier transform of a compactly supported function no longer has compact support. However, we can use the fast decay of the Gaussian to achieve the same effect.

Proof. Let $\varphi(x) = e^{-\pi|x|^2}$. For $1 \leq k \leq N$ and $\alpha \gg 1$, define

$$\varphi_k(x) = e^{2\pi i x \cdot \alpha k e_1} \varphi(x - \alpha k e_1).$$

Then

$$\widehat{\varphi}_k(\xi) = e^{-2\pi i \alpha k \xi_1} \widehat{\varphi}(\xi - \alpha k e_1).$$

Let $f = \sum_{k=1}^N \varphi_k$ and $S = \bigcup_{j=1}^N \{x : |x - \alpha_j e_1| \leq \alpha/10\}$. Then

$$\|f\|_{L^p} = \|f\|_{L^p(S)} + \|f\|_{L^p(\mathbb{R}^d \setminus S)}.$$

We can bound each of these by

$$\|f\|_{L^p(\mathbb{R}^d\setminus S)} \le \sum_{k=1}^N \|\varphi_k\|_{L^p(\mathbb{R}^d\setminus S)} \lesssim N\alpha^{-100},$$

$$\|f\|_{L^p(S)}^p \sim \sum_{j=1}^N \underbrace{\left\|\sum_{k=1}^N \varphi_k\right\|_{L^p(|x-\alpha j e_1|\le \alpha/10)}}_{\sim 1+O(\sum_{k\ne j}(\alpha|k-j|)^{-100})} \sim N(1+O(\alpha^{-100}))$$

because $|x-\alpha k e_1| \ge |\alpha(j-k)e_1| - |x-\alpha j e_1| \ge \alpha|j-k| - \alpha/10 \ge (\alpha/2)|j-k|$.

Taking $\alpha \gg 1$, we get $\|f\|_{L^p} \sim N$. Similarly,

$$\|\widehat{f}\|_{L^{p'}} \sim N^{1/p'}$$

We need $N^{1/p'} \le N^{1/p}$ for all $N \ge 1$. This means that $1/p' \le 1/p$, so $p \le p'$. So $1 \le p \le 2$. $\square$

1.2 Weak L^p and Lorentz spaces

Definition 1.1. For $1 \le p < \infty$ and $f : \mathbb{R}^d \to \mathbb{C}$, define

$$\|f\|_{L^p_{\text{weak}}}^* = \sup_{\lambda>0} \lambda |\{x : |g(x)| > \lambda\}|^{1/p}.$$

The **weak L^p space** is the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ for which $\|f\|_{L^p_{\text{weak}}}^* < \infty$. We denote it by $L^p_{\text{weak}}(\mathbb{R}^d)$.

Example 1.1. $f(x) = |x|^{d/p}$ is in $L^p_{\text{weak}} \setminus L^p$. We have

$$\|f\|_{L^{\text{weak}}}^* = \sup_{\lambda>0} \lambda |\{x : |x|^{-d/p} > \lambda\}|^{1/p} \sim \sup_{\lambda>0} \lambda(\lambda^{-p})^{1/p} \sim 1.$$

Remark 1.1. We will show that the weak L^p "norm" is a quasinorm (not a norm) and that is why we append $*$ to the usual norm notation.

By comparison, for $1 \le p < \infty$,

$$\begin{aligned} \|f\|_{L^p}^p &= \int |f(x)|^p \, dx \\ &= \int \int_0^{|f(x)|} p\lambda^{p-1} \, d\lambda \, dx \\ &= \int_0^\infty p\lambda^{p-1} |\{x : |f(x)| > \lambda\}| \, d\lambda \end{aligned}$$

$$= p\int_0^\infty \lambda^p |\{x : |f(x)| > \lambda\}| \frac{1}{\lambda}\, d\lambda.$$

So we can write

$$\|f\|_{L^p} = p^{1/p} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^p((0,\infty), \frac{d\lambda}{\lambda})}.$$

With the convention that $p^{1/\infty} = 1$, we also have

$$\|f\|^*_{L^p_{\text{weak}}} = p^{1/\infty} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^\infty((0,\infty), \frac{d\lambda}{\lambda})}.$$

Can we do this to L^p spaces for other exponents?

Definition 1.2. For $1 \le p < \infty$ and $1 \le q \le \infty$, the **Lorentz space** $L^{p,q}(\mathbb{R}^d)$ is the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ for which

$$\|f\|^*_{L^{p,q}(\mathbb{R}^d)} = p^{1/q} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^q((0,\infty), \frac{d\lambda}{\lambda})} < \infty.$$

Note that $L^{p,p} = L^p$ and $L^{p,\infty} = L^p_{\text{weak}}$.

Lemma 1.1. *$\|f\|^*_{L^{p,q}(\mathbb{R}^d)}$ is a quasinorm.*

Proof. If $\|f\|^*_{L^{p,q}} = 0$, then $f = 0$ a.e. For $a \neq 0$,

$$\begin{aligned}
\|af\|^*_{L^{p,q}} &= p^{1/q} \|\lambda |\{x : |af(x)| > \lambda\}|^{1/p}\|_{L^q(\frac{d\lambda}{\lambda})} \\
&= p^{1/q}|a| \left\| \frac{\lambda}{|a|} |\{x : |f(x)| > \lambda/|a|\}|^{1/p} \right\|_{L^q(\frac{d\lambda}{\lambda})} \\
&= |a| \|f\|^*_{L^{p,q}}.
\end{aligned}$$

For the "triangle inequality," we have

$$\begin{aligned}
\|f+g\|^*_{L^{p,q}} &= p^{1/q} \|\lambda |\{x : |f(x)+g(x)| > \lambda\}|^{1/p}\|_{L^q(\frac{d\lambda}{\lambda})} \\
&\le p^{1/q} \|\lambda [|\{x : |f(x)| > \lambda/2\}| + |\{x : |f(x)| > \lambda/2\}|]^{1/p}\|_{L^q(\frac{d\lambda}{\lambda})}
\end{aligned}$$

By the concavity of fractional powers, we get

$$\begin{aligned}
&\le p^{1/q} \left[\left\| \frac{\lambda}{2} |\{x : |f(x)| > \lambda/2\}|^{1/p} \right\|_{L^q(\frac{d\lambda}{\lambda})} + \left\| \frac{\lambda}{2} |\{x : |f(x)| > \lambda/2\}|^{1/p} \right\|_{L^q(\frac{d\lambda}{\lambda})} \right] \\
&\le 2[\|f\|^*_{L^{p,q}} + \|g\|^*_{L^{p,q}}].
\end{aligned}$$

□

Remark 1.2. We will show that for $1 < p < \infty$ and $1 \le q \le \infty$, there exists a norm equivalent to this quasinorm. For $p = 1$ and $q \neq 1$, no such norm exists. Nonetheless, in this latter case, there is a metric that generates the same topology. In all cases, $L^{p,q}(\mathbb{R}^d)$ is complete.

Proposition 1.1. *For $f \in L^{p,q}(\mathbb{R}^d)$, decompose $f = \sum_{m\in\mathbb{Z}} f_m$ by defining $f_m(x) = f(x)\mathbb{1}_{\{2^m \le |f(x)| < 2^{m+1}\}}(x)$. Then*

$$\|f\|^*_{L^{p,q}} \sim \Big\| \|f_m\|_{L^p(\mathbb{R}^d)} \Big\|_{\ell^q_m(\mathbb{Z})}.$$

In particular, $L^{p,q_1} \subseteq L^{p,q_2}$ whenever $q_1 \le q_2$.