

# Math 222A Lecture 12 Notes

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## 1 Operations on Distributions and Homogeneous Distributions

### 1.1 Operations on distributions

Last time, we introduced distributions. We had the set  $\mathcal{D} = C_0^\infty$  of **test functions** and the set  $\mathcal{D}'$  of **distributions**, continuous linear maps  $\mathcal{F} : \mathcal{D} \rightarrow \mathbb{R}$ . If  $u$  is a function, we interpreted it as a distribution via

$$u(\phi) = \int u\phi \, dx.$$

So we can think of distributions as generalized functions. We also saw distributions as a limit of functions, in this weak sense.

Now, we want to see distributions as solutions to PDEs, so we need to think about operations with distributions.

#### 1.1.1 Differentiation

We want to define  $u \mapsto \partial_j u$  for distributions. First suppose  $u$  is a function. Then  $\partial_j u$  is a function with

$$\begin{aligned}\partial_j u(\phi) &= \int \partial_j u \phi \, dx \\ &= - \int u \cdot \partial_j \phi \, dx \\ &= -u(\partial_j \phi).\end{aligned}$$

We can take this as a definition.

**Definition 1.1.** If  $u \in \mathcal{D}'$ , define  $\partial_j u$  by  $\partial_j u(\phi) = -u(\partial_j \phi)$ .

**Remark 1.1.** If  $u \in C^1$ , then  $u$  is the same classically and as a distribution.

**Example 1.1.** Consider the Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

in 1 dimension. Then  $\partial_x H = 0$  away from 0, in the classical sense. We can check that

$$\begin{aligned} \partial_x H(\phi) &= -H(\partial_x \phi) \\ &= -\int H(x) \partial_x \phi \, dx \\ &= -\int_0^\infty -\partial_x \phi(x) \, dx \\ &= -\phi|_0^\infty \\ &= \phi(0) \\ &= \delta_0(\phi), \end{aligned}$$

so  $\partial_x H = \delta_0$  as a distribution. The idea is that when we have a jump discontinuity, differentiating gives us a Dirac mass.

**Example 1.2.** What is the derivative of the Dirac mass?

$$\begin{aligned} \partial_x \delta_0(\phi) &= -\delta_0(\partial_x \phi) \\ &= -\delta_x \phi(0) \\ &= \delta'_0(0). \end{aligned}$$

So the derivative of  $\delta_0$  is what we previously called  $\delta'_0$ . Similarly, we can have  $\partial^\alpha \delta_0 = \delta_0^{(\alpha)}$  for a multi-index  $\alpha$ .

### 1.1.2 Multiplication by smooth functions

Suppose  $\psi \in \mathcal{E}$  and  $u$  is a function. Then  $\psi u$  is a function. What if  $u \in \mathcal{D}'$ ? If  $u$  is a function, then

$$\begin{aligned} \psi u(\phi) &= \int \psi u \phi \, dx \\ &= \int u \underbrace{\psi \phi}_{\in \mathcal{D}} \, dx \\ &= u(\psi \phi). \end{aligned}$$

We can again take this as a definition.

**Definition 1.2.** If  $u \in \mathcal{D}'$  and  $\psi \in \mathcal{E}$ , define  $\psi u$  by  $\psi u(\phi) = u(\psi \phi)$ .

The Leibniz rule for derivatives says

$$\partial(\psi u) = \partial\psi \cdot u + \psi \cdot \partial u.$$

Using these definitions, this rule also holds for  $u \in \mathcal{D}'$  and  $\psi \in \mathcal{E}$ .

If we have the equation  $P(x, \partial)u = f$  with  $P(x, \partial) = \sum c_\alpha(x)\partial^\alpha$ , then all these operations are well-defined for distributions, so we can think of distribution solutions to PDEs.

## 1.2 The support of a distribution

Recall that if  $u$  is a function, its **support** is the largest closed set “where  $u$  is nonzero.” In particular,

$$x_0 \notin \text{supp } u \iff u = 0 \text{ in } B(x_0, r) \text{ for some } r > 0.$$

**Definition 1.3.** If  $u \in \mathcal{D}'$ , its **support** is the closed set defined by

$$x_0 \notin \text{supp } u(\phi) \iff u(\phi) = 0 \text{ for all } \phi \in \mathcal{D} \text{ with } \text{supp } \phi \subseteq B(x_0, r)$$

**Example 1.3.** The support of the Dirac mass is  $\text{supp } \delta_0 = \{0\}$ : If  $x_0 \neq 0$ , then there is a ball  $B(x_0, r) \not\ni 0$ . Then if we let  $\phi \in \mathcal{D}$  have  $\text{supp } \phi \subseteq B(x_0, r)$ , then  $\delta_0(\phi) = \phi(0) = 0$ .

Let  $\mathcal{E}'$  denote the **compactly supported distributions**.

**Proposition 1.1.** If  $f \in \mathcal{E}'$ , then  $f$  extends “naturally” to a continuous linear function on  $\mathcal{E}$ .

*Proof.* We know  $f(\phi)$  when  $\phi \in \mathcal{D}$ . Because  $\text{supp } f \in B(0, R)$ ,  $f(\phi) = 0$  if  $\phi$  is supported outside  $B(0, R)$ . We can truncate  $\phi$  outside  $B$  as follows: Replace  $\phi$  by  $\chi\phi$ , where  $\chi$  is a **cutoff function** with compact support,  $\text{supp } \chi \subseteq B(0, 2R)$ , and  $\chi = 1$  in  $B(0, R)$ . Then

$$\begin{aligned} f(\phi) &= f(\chi\phi) + f((1 - \chi)\phi) \\ &= f(\chi\phi). \end{aligned}$$

So for  $\phi \in \mathcal{E}$ , define  $f(\phi) := f(\chi\phi)$ . □

We have the following picture:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\text{dual}} & \mathcal{D}' \\ \downarrow \subseteq & & \uparrow \subseteq \\ \mathcal{E} & \xrightarrow{\text{dual}} & \mathcal{E}' \end{array}$$

We will extend this picture later when we learn about the Fourier transform.

### 1.3 Homogeneous distributions

**Example 1.4.** The polynomial  $f(x) = x^n$  is a homogeneous polynomial. We can express this homogeneity by

$$f(\lambda x) = \lambda^n f(x),$$

where  $n$  is the homogeneity index.

**Example 1.5.** The homogeneity index does not have to be an integer. If we have  $f(x) = |x|^\alpha$ , then

$$f(\lambda x) = \lambda^\alpha f(x)$$

for  $\lambda > 0$ . If  $\alpha$  is not an integer, this is not smooth at 0. Is  $|x|^\alpha$  a distribution? This is related to the question of whether  $|x|^\alpha$  is integrable (away from infinity). In 1 dimension,  $\int |x|^\alpha dx$  exists if  $\alpha > -1$ . In  $n$  dimensions, we can use polar coordinates:

$$\int |x|^\alpha dx = c_n \int r^\alpha r^{n-1} dr,$$

where  $c_n$  is the volume of the unit ball in  $n$ -dimensions. Here, we need  $\alpha + n - 1 > -1$ , i.e.  $\alpha > -n$ . So  $\frac{1}{|x|^n}$  is borderline.

**Example 1.6.** The heaviside function is homogeneous of index 0:

$$H(\lambda x) = \lambda^0 H(x)$$

for  $\lambda > 0$ .

**Example 1.7.** In 2 dimensions (expressed in polar coordinates  $(r, \theta)$ ), the function

$$f(x) = r^\alpha g(\theta)$$

is homogeneous of index  $\alpha$ .

For functions, the homogeneity condition  $f(\lambda x) = \lambda^\alpha f(x)$  has a distributional interpretation:

$$\int f(\lambda x) \phi(x) dx = \lambda^\alpha \int f(x) \phi(x) dx$$

Applying a change of variables on the left,

$$\int f(y) \phi(y/\lambda) \frac{1}{\lambda^n} dy = \lambda^\alpha \int f(x) \phi(x) dx.$$

Denoting  $\phi_\lambda(x) = \lambda^{-n} \phi(x/\lambda)$ , we get the relation

$$f(\phi_\lambda) = \lambda^\alpha f(\phi),$$

which is meaningful for distributions.

**Definition 1.4.** A distribution  $f \in \mathcal{D}'$  is **homogeneous of order  $\alpha$**  if

$$f(\phi_\lambda) = \lambda^\alpha f(\phi)$$

for  $\phi \in \mathcal{D}$ .

**Example 1.8.** Can we think of the Dirac mass  $\delta_0$  as a homogeneous distribution?

$$\delta_0(\phi_\lambda) = \phi_\lambda(0) = \lambda^{-n} \phi(0) = \lambda^{-n} \delta_0(\phi),$$

so  $\delta_0$  has homogeneity  $-n$ .

In calculus, we have  $\partial_x x^n = nx^{n-1}$ . That is, we differentiate something which is homogeneous of order  $n$  and get something which is homogeneous of order  $n - 1$ .

**Proposition 1.2.** *If  $f \in \mathcal{D}'$  is homogeneous of order  $\alpha$ , then  $\partial_x f$  is homogeneous of order  $\alpha - 1$ .*

*Proof.* The chain rule works for functions, so it also works using the definition for distributions by passing the derivative to the test function.  $\square$

**Example 1.9.** The Heaviside function is homogeneous of order 0, and  $\partial_x H = \delta_0$  is homogeneous of order  $-1$ . Similarly,  $\partial_x \delta_0 = \delta'_0$  is homogeneous of order  $-1$ .

In 1 dimension, we want to classify homogeneous distributions. Start with functions and  $\alpha > -1$ . We need to assign  $f(-1)$  and  $f(1)$ , so this is a linear space of dimension 2. Here is a basis:

$$x_+^\alpha = \begin{cases} 0 & x < 0 \\ x^\alpha & x > 0, \end{cases} \quad x_-^\alpha = \begin{cases} |x|^\alpha & x < 0 \\ 0 & x > 0. \end{cases}$$

Then  $|x|^\alpha = x_+^\alpha + x_-^\alpha$ , and

$$\partial_x x_+^\alpha = \alpha x_+^{\alpha-1}, \quad \partial_x x_-^\alpha = -\alpha x_-^{\alpha-1}.$$

Now look at when  $\alpha \in (-2, -1)$ . We can define

$$\partial_x x_+^{\alpha+1} := (\alpha + 1)x_+^\alpha.$$

If we repeat this, we can get homogeneous distributions to all noninteger negative  $\alpha$ s.

What about  $\alpha = -1$ ? We have  $\delta_0$ . At order 0, we have 2 homogeneous distributions:  $H$  and the constant 1 function. But differentiating these gives  $\delta_0$  and 0, which do not have a 2 dimensional span. Other candidates are  $\frac{1}{|x|}$  or  $\frac{1}{x}$ . We can look at the integrals

$$\int \frac{1}{|x|} \phi(x) dx \quad \int \frac{1}{x} \phi(x) dx.$$

On the left, there may be no cancelation at 0, but we may be able to get some cancelation at 0 for the right integral. We may try to define

$$\int_{\mathbb{R}} \frac{1}{x} \phi(x) dx := \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{1}{x} \phi(x) dx.$$

Does this limit exist? We can look at

$$\int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \frac{1}{x} \phi(x) dx = \int_{-1}^{-\varepsilon} \frac{1}{x} \phi(x) dx + \int_{\varepsilon}^1 \frac{1}{x} \phi(x) dx$$

Use the change of variables  $y = -x$  on the left integral to get

$$= \int_{\varepsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx.$$

$\phi(x) - \phi(-x)$  is  $o(x)$ , so this converges.

Thus, we can define the **principal value**  $\text{PV} \frac{1}{x}$  by

$$\text{PV} \frac{1}{x}(\phi) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{\phi(x)}{x} dx,$$

which is homogeneous of order  $-1$ .