Math 210A Lecture 14 Notes

Daniel Raban

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1 Orbit-Stabilizer and Symmetric Groups

1.1 The orbit-stabilizer theorem

Theorem 1.1. *Let X be a G-set. For each x, there is a bijection $\psi_x : G/G_x \to G \cdot x$ given by $gG_x \mapsto g \cdot x$ for $g \in G$.*

Proof. Exercise. $\square$

Corollary 1.1.
$$[G : G_x] = |G \cdot x|.$$

Proposition 1.1 (class equation)**.** *Let T be the set of representatives of conjugacy classes in G. If G is finite,*
$$|G| = \sum_{x \in T} [G : Z_x] = |Z(G)| + \sum_{x \in G \setminus Z(G)} [G : Z_x|.$$

Proof. G acts on itself by conjugation, and the stabilizer of $x \in G¡$ is Z_x. The orbit of x is C_x, the conjugacy class of x. Then
$$|G| = \sum_{x \in T} |C_x| = \sum_{x \in T} [G : Z_x]. \qquad \square$$

1.2 Action of symmetric groups

Let $\sigma \in S_n$. An element σ acts on $X_n = \{1, \dots, n\}$.
$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

Definition 1.1. A **k-cycle** ($k \leq n$) is the permutation
$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix}(i) = \begin{cases} a_{j+!} & i = a_j, i \leq j \leq k-1 \\ a_1 * i = a_k & \\ i & \text{otherwise.} \end{cases}$$

Every permutation is a product of disjoint cycles, which commute.

Example 1.1.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 6 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix}$$

Definition 1.2. A **transposition** is a 2-cycle.

Proposition 1.2. *Every cycle can be written as a product of transpositions.*

Proof. Prove the following relationship by induction on n:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} a_2 & a_3 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & a_{n-2} \end{pmatrix}.$$ □

How does conjugation work?

$$\sigma \begin{pmatrix} a_1 & a_2 & \cdots a_k \end{pmatrix} \sigma^{-1} = \begin{pmatrix} \sigma(a_1) & \sigma(a_2) & \cdots & \sigma(a_k). \end{pmatrix}$$

Example 1.2. What is the centralizer of $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in S_5$? This is $\langle \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 5 \end{pmatrix} \rangle$.

Theorem 1.2. *If $\sigma = \tau_1 \cdots \tau_r = \rho_1 \cdots \rho_s$ for transpositions τ_i and ρ_i, then $r \equiv s \pmod 2$.*

Proof. Let $S_n \circlearrowright \mathbb{Z}[x_1, \ldots, x_n]$ by $\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Let

$$p(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Then $\tau \cdot p = \prod_{1 \leq i < j \leq n} (x_{\tau(i)} - x_{\tau(j)})$. If $\tau = \begin{pmatrix} k & \ell \end{pmatrix}$ with $k < \ell$, then $x_{\tau(i)} x_{\tau(j)}$ occurs with the sign in the product unless $i = k, j \leq \ell$ or $i \geq k, j = \ell$. So $\tau \cdot p = (-1)^{2(\ell - k) - 1} p = -p$.

In general, $\sigma \cdot p = \operatorname{sgn}(\sigma) p$, where $\operatorname{sgn} : S_n \to \{\pm 1\}$ is a homomorphism, and $\operatorname{sgn}(\tau) = -1$ for any transpposition τ. So $\operatorname{sgn}(\sigma) = (-1)^r = (-1)^s$, so $r \equiv s \pmod 2$. □

1.3 Alternating groups

In the above proof, we defined the **sign** of a permutation, which is ± 1.

Definition 1.3. A permutation is **even/odd** if its sign is $1/-1$.

Example 1.3. What is the sign of a cycle? $\operatorname{sgn} \begin{pmatrix} 1 & \cdots & k \end{pmatrix} = (-1)^{k+1}$

Definition 1.4. The **alternating group** is $A_n = \ker(\operatorname{sgn}) = \{\sigma \in S_n : \sigma \text{ is even}\} \trianglelefteq S_n$.

Note that $|A_n| = n!/2$ for $n \geq 2$.

Definition 1.5. A group is **simple** if it has no proper, nontrivial normal subgroups (and is nontrivial).

Example 1.4. A_4 is not simple. $\{\begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}c & d\end{pmatrix} : \{a,b,c,d\} = \{1,2,3,4\}\} \cup \{e\} \trianglelefteq A_4$

Theorem 1.3. *A_5 is simple.*

Proof. An element in A_5 must be e, a three cycle, a product of two two-cycles, or a five cycle. The centralizer of $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$ in $A_5 = \left\langle \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}4 & 5\end{pmatrix} \right\rangle \cap A_5 = \left\langle \begin{pmatrix}1 & 2 & 3\end{pmatrix} \right\rangle$. So $C_{(1\,2\,3)}$, the set of 3-cycles, has size 20. Similarly number of products of two 2-cycles is 15, and the number of five cycles is 12.

The conjugacy classes have order 1, 12, 12, 15, and 20. Every normal subgroup N is a union of conjugacy classes (including $\{e\}$) and has order dividing $|A_n| = 60$. The only way is to take $N = A_5$ or $N = e$. $\square$

Remark 1.1. An action $G \circlearrowright X$ can be thought of as a homomorphism $\rho : G \to S_X$. Then $\ker(\rho) = \bigcap_{x \in X} G_x$ is trivial if and only if the aciton is faithful. G acting on G by left multiplication gives us that $\rho : G \to S_G$ is injective. This is Cayley's theorem.