Math 222A Lecture 2 Notes

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1 Function Spaces and Ordinary Differential Equations

We are interested in studying first order nonlinear scalar PDEs, equations of the form $F(x, u, \partial u) = 0$. Here is the battle plan: First, we will need to discuss function spaces and an provide an introduction to ODEs. Then we will be able to study nonlinear PDEs. We will study linear PDEs, then semilinear PDEs, and then work our way up to studying nonlinear PDEs

1.1 Function spaces

What functions could be solutions to a PDE? How do we verify that a function is a solution? We need functions that are differetiable, but this is far from the only thing we will consider.

Suppose we have a function $u : \mathbb{R}^n \to \mathbb{R}$.

Definition 1.1. The set of **(bounded) continuous functions** are denoted $C(\mathbb{R}^n)$. It has the norm $\|u\|_{C(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |u(x)|$, $C(\mathbb{R}^n)$.

For now, we will assume these functions are bounded, but we may not do so later. If $\Omega \subseteq \mathbb{R}^n$, we can similarly define $C(\Omega)$.

Definition 1.2. A **normed space** is a vector space equipped with a **norm** $u \mapsto \|u\| \geq 0$, which satisfies

1. $\|u + v\| \leq \|u\| + \|v\|$
2. $\|\lambda u\| = |\lambda| \|u\|$ for $\lambda \in \mathbb{R}$.
3. $\|u\| = 0 \implies u = 0$.

A **Banach space** is a normed space is a normed space which is **complete**, i.e. any Cauchy sequence is convergent.

That is, if $u_n \in X$ and $\lim_{n,m \to \infty} \|u_n - u_m\| = 0$, the sequence u_n must have a limit.

Example 1.1. $\mathbb{R}$ and $\mathbb{C}$ are complete.

Example 1.2. Equipped with the norm $\|u\|_{C(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |u(x)|$, $C(\mathbb{R}^n)$ is a Banach space.

Example 1.3. $C^1 = \{u \in C : u \text{ differentiable everywhere}, \partial u \in C\}$ is the space of **continuously differentiable functions**. This space has the norm $\|u\|_{C^1} = \|u\|_C + \|\partial u\|_C$.

More generally, we may consider $C^m(\mathbb{R}^n)$. The set $\bigcap_{m=1}^\infty C^m(\mathbb{R}^n) =: C^\infty(\mathbb{R}^n)$ is the set of **smooth functions**. In general, the smooth functions is too small a class of functions to be the only focus of study in PDEs.

Here are examples of functions.

Example 1.4. Observe that

$$u(x) = \frac{1}{1+x^2} \in C,$$

while

$$v(x) = x^2 \notin C(\mathbb{R})$$

because it is not bounded.

Definition 1.3. $C_{\text{loc}}(\mathbb{R})$ is the space of **continuous but not necessarily bounded functions**.

Example 1.5. If $I_N = [-N, N]$, we can try to use $\|u\|_{C(I_n)} = \sup_{x\in I_N} |u(x)|$. We would be able to get countably many of these to measure convergence of functions. But this is not a norm on all of $\mathbb{R}$, since it assigns 0 to nonzero functions. This is a *seminorm*.

Definition 1.4. A **seminorm** is a norm without the property that $\|u\| = 0 \implies u = 0$.

What does convergence look like with respect to seminorms? What happens is that $u_n \to u$ in C_{loc} if $\|u_n - u\|_{C(I_N)} \to 0$ for each N. So we extend the concept of a normed space to a **locally convex space**, where instead of a norm, we may have infinitely many seminorms.

Why is this called locally convex? In $\mathbb{R}^n$, we can specify convergence by a a fundamental system of neighborhoods, balls around each point. Another property of balls is that they are convex. If we want to talk about convergence in a locally convex space, we can also do it using by specifying convex balls. We could have many different types of balls around any point defined by different seminorms.

From this point on, we will use C to refer to C_{loc}. So our functions may be unbounded.

Example 1.6. The seminorms for $C^m(\mathbb{R}^n)$ look like

$$p_{K,N} \sup_{x\in K} \sup_{|\alpha|\le N} |\partial^\alpha u(x)|,$$

where $K \subseteq \mathbb{R}^n$ is compact.

Later, we will study more function spaces, such as Sobolev spaces.

1.2 Ordinary differential equations and Lipschitz functions

A (nonlinear) ODE regards a function $u : \mathbb{R} \to \mathbb{R}$ which solves an equation of the form

$$\begin{cases} u' = F(x, u(x)) \\ u(0) = u_0. \end{cases}$$

If we let the codomain be $\mathbb{R}^n$, we get a system of equations.

If this equation solvable? We are asking about existence of solutions, uniqueness of solutions, dependence of solutions on initial data, and local vs global solutions. At the minimum, we require that F is continuous and look for a C^1 local solution.

Theorem 1.1 (Peano)**.** *If F is continuous, then a local C^1 solution exists.*

However, uniqueness can fail, as the following example shows.

Example 1.7. Consider the equation $u'(x) = \sqrt{u}$ with $u(0) = 0$. One solution is $u = 0$. Alternatively, $u = x^2/4$ is another solution for $x > 0$. We can extend this second solution to a global solution by making it 0 for $x \leq 0$. Moreover, we can translate this solution to the left or right to get another solution. So there are infinitely many solutions.

Example 1.8. Consider the equation $u' = |u|^\alpha$. If we check $u = x^\beta$, we get that $\beta = \frac{1}{1-\alpha}$. We can consider this with a range of α, up to any $\alpha < 1$. What happens when $\alpha = 1$? The function $|u|^\alpha$ becomes Lipschitz.

Definition 1.5. A function F is **Lipschitz continuous** with **Lipschitz constant** L if

$$|F(x) - F(y)| \leq L|x - y| \qquad \forall x, y.$$

The Lipschitz fucntions form a Banach space when equipped with the norm $\|F\|_{\text{Lip}} := \sup_{x,y} \frac{|F(x)-F(y)|}{|x-y|}$ which gives the "best" Lipschitz constant L.

Lipschitz functions have bounded slope, so it is reasonable to compare the spaces Lip and C^1. What is the relationship? We have $C^1 \subseteq \text{Lip}$. In 1 dimension, we can see this by the mean value theorem: $F(x) - F(y) = F'(c)(x - y)$ for some $x \in (x, y)$. For more than 1 dimension, we can still restrict the function to its values on a line connecting x, y to reduce to the 1 dimensional case.

However, $\text{Lip} \not\subseteq C^1$.

Example 1.9. The function $F(x) = |x|$ is 1-Lipschitz but not C^1.

Remark 1.1. It actually turns out that a Lipschitz function is differentiable outside a set of measure zero, but we will not use this.

This inclusion of Banach spaces is actually very nice because by the mean value theorem, we can use the same norm for both Lip and C^1.

1.3 Holder continuous functions and fixed point methods

Starting from the continuous functions C^0, we have the subspaces $C^0 \supseteq \mathrm{Lip} \supseteq C^1$. Is there anything in between C^0 and Lip?

Definition 1.6. The **s-Holder continuous functions** are $C^s(\mathbb{R}) = \{F : |F(x) - F(y)| \leq M|x-y|^s\}$ for $0 < s < 1$, equipped with the norm $\|F\|_{C^s} := \sup \frac{|F(x)-F(y)|}{|x-y|^\alpha}$.

Remark 1.2. If $s > 1$, the only functions that work are the constant functions.

Returning to our previous example, the function $|x|^\alpha$ is α-Holder continuous.

Theorem 1.2. *If G is locally Lipschitz, then a local solution exists and is unique.*

Here is the beginning of the proof:

Proof. Restate the problem using the fundamental theorem of calculus. Integrating the equation gives

$$u(x) = u(0) + \int_0^x F(y, u(y))\, dy.$$

This allows us to think of the problem as a fixed point problem. Define the map $C^1 \ni u \mapsto N(u)(x) := u(0) + \int_0^x F(y, u(y))\, dy$. Observe that u solves our ODE if and only if $N(u) = u$. That is, we want u to be a fixed point of N.

In 1-dimension, if we have $f : \mathbb{R} \to \mathbb{R}$, when do we have fixed points $f(x) = x$? We can look for the points where the graph of f intersects the line $y = x$. One thing we can do to get fixed points is ask that the function does not increase very fast: $|f'| < 1$. In this case, f will have a unique fixed point.

We have just stated the following theorem:

Theorem 1.3. *If $f : \mathbb{R} \to \mathbb{R}$ with $|f'| < 1$, then f has a unique fixed point.*

This fact extends to Banach spaces.

Theorem 1.4. *Let B be a Banach space. If $f : B \to B$ is Lipschitz with Lipschitz constant $L < 1$ ($\|f(x) - f(y)\| \leq L\|x - y\|$), then f has unique fixed point.*

This is not sufficient for us because we are not looking at the entire space of C^1 functions. We only want local solutions.

Theorem 1.5 (Banach contraction principle)**.** *If $f : D \subseteq B \to D$ with D closed is Lipschitz with constant < 1, then f has a unique fixed point.*

Example 1.10. We need the domain D to be closed. If $D = (0, 1)$ and $f(x) = x/2$, then f has no fixed points. But adding the endpoints of the interval rectifies this.

Next time, we will further discuss this fixed point theorem.