Math 246A Lecture 4 Notes

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1 The Complex Exponential, Logarithm, and Differentials

1.1 Exponentials and logarithms

Lat time, we defined

$$E(z) = \sum_{n=0}^\infty \frac{z^n}{n!}.$$

We had $E(z+w) = E(z)E(w)$, $E'(z) = E(z)$, and $E(\overline{z}) = \overline{E(z)}$. Suppose that $z = i\theta$ with θ real. Then $|E(i\theta)| = 1$. Define $\cos(\theta)$, $\sin(\theta)$ by $E(i\theta) = \cos(\theta) + i\sin(\theta)$. That is,

$$\cos(\theta) = \frac{E(i\theta) + E(-i\theta)}{2} = \sum_{k=0}^\infty (-1)^k \frac{\theta^2 k}{(2k)!},$$

$$\sin(\theta) = \frac{E(i\theta) - E(-i\theta)}{2i} = \sum_{k=0}^\infty (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}.$$

We can then obtain the identities

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta), \qquad \frac{d}{d\theta}\sin(\theta) = \cos(\theta).$$

Lemma 1.1.

$$\cos(2) < 0.$$

Proof.

$$\cos(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} \mp + \cdots$$

By the alternating series theorem,

$$\cos(2) < 1 - 2 + \frac{2}{3} < 0. \qquad \square$$

Define $\pi/2 = \inf\{t > 0 : \cos(t) = 0\}$. Cosine maps $[0, \pi/2]$ to the quarter of the unit circle that lies in the first quadrant.

Definition 1.1. The **exponential function** is $e^z := E(z)$.

Then $E(z + 2\pi ni) = E(z)$ for all $n \in \mathbb{Z}$, $E(q) = Q(z) \iff q = z + 2\pi ni$ for some $n \in \mathbb{Z}$, and $E(t + i\theta) = E(t)(\cos(\theta) + i\sin(\theta))$.

What does E do to the horizontal strip S given by $-\pi \leq \operatorname{im}(z) \leq \pi$? It maps vertical lines on this strip to circles centered at the origin. If we translate the stripvertically by some multiple of 2π, we get the same thing. Each strip maps injectively onto $\mathbb{C} \setminus 0$ (paying attention to only use 1 boundary of the strip for the set $(-\infty, 0]$. So if we stitch together infinitely many copies of the complex plane along that set, we get a spiral-like version of the complex numbers. So if we go backward like this, we get an inverse function for the exponential, the logarithm.

Definition 1.2. $\operatorname{Log}(w) = z$ if $z \in S \cup \{z : \operatorname{Im}(z) = \pi\}$ and $E(z) = w$.

So logarithm is a **multivalued** function. The real logarithm is $\log(w) = \operatorname{Re}(\operatorname{Log}(w)) = \operatorname{Log}(|w|)$.

1.2 $\partial/\partial z$ and $\partial/\partial \overline{z}$

Let $f : \Omega \to \mathbb{R}^2$, and write

$$f(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}.$$

Recall that if f is differentiable at (x_0, y_0), then

$$f(x, y) = f(x, y) + A \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + o\sqrt{(x - x_0)^2 + (y - y_0)^2},$$

$$\text{where} \quad A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} (x_0, y_0).$$

Call this matrix df. What are the derivatives of important functions?

1. Let $z : \mathbb{R}^2 \to \mathbb{R}^2$ send $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $dz = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
2. Let $\overline{z} : \mathbb{R}^2 \to \mathbb{R}^2$ send $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$. Then $d\overline{z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
3. Let $x : \mathbb{R}^2 \to \mathbb{R}^2$ send $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. Then $dx = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

4. Let $y : \mathbb{R}^2 \to \mathbb{R}^2$ send $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$. Then $dy = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
5. Let $iy : \mathbb{R}^2 \to \mathbb{R}^2$ send $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$. Then $d(iy) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

The matrix $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is supposed to take the place of i. Check that $Jdy = d(iy)$. We can also check that since $z = x + iy$,

$$dx + Jdy = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = dz.$$

We can also check that

$$dx = \frac{1}{2}(dz - d\overline{z}), \qquad dy = -\frac{1}{2}J(dz + d\overline{z}).$$

Definition 1.3. Let A be a 2×2 real matrix. Then A is **complex antilinear** if $JA = -AJ$.

Remark 1.1. This definition is equivalent to $A = JB$ for some complex linear B.

Lemma 1.2. *Suppose $A : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Then there exists a unique complex linear T_1 and a unique complex antilinear T_2 such that $A = T_1 + T_2$.*

Proof. For existence, let $T_1 = (A - JAJ)/2$ and $T_2 = (A + JAJ)/2$. For uniqueness, suppose $A = T_1 + T_2 = S_1 + S_2$. Then $S_1 - T_1 = T_2 - S_2$, which means a complex linear matrix is equal to a complex antilinear matrix, so both are zero. $\square$