Math 210A Lecture 6 Notes

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1 Inverse Limits, Direct Limits, and Adjoint Functors

1.1 Inverse and direct limits

Example 1.1. Consider the colimit of this diagram in Ab:

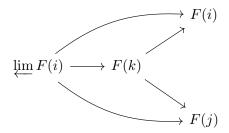
$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot p} \cdots \xrightarrow{\cdot p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\cdot p} \cdots$$

Then $\varinjlim_{p} \mathbb{Z}/p^{n}\mathbb{Z} \cong \mathbb{Q}_{p}/\mathbb{Z}_{p} \subseteq \mathbb{Q}/\mathbb{Z}$, where \mathbb{Q}_{p} is the free field of \mathbb{Z}_{p} . We can also show that $\mathbb{Q}_{p}/\mathbb{Z}_{p}: \{a \in \mathbb{Q}/\mathbb{Z} : p^{n}a = 0 \text{ for some } n \geq 0\}.$

Definition 1.1. A directed set I is a set with a partial ordering such that for all $i, j \in I$, there is a $k \in I$ such that $i \leq k, j \leq k$.

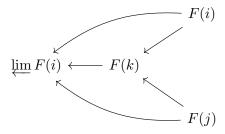
Definition 1.2. A **directed category** is a category where the objects are elements of a directed set I, and there are morphisms $i \to j$ iff $i \le j$. A **codirected category** \mathcal{I} is a category where \mathcal{C}^{op} is directed.

Definition 1.3. Suppose \mathcal{I} is codirected with $\operatorname{Obj}(\mathcal{I}) = I$ and $F : \mathcal{I} \to \mathbb{C}$. A limit of F is called the **inverse limit** of the F(i) for all $i \in I$. We write $\lim F = \varprojlim_{i \in I} F(i)$.



If \mathcal{I} is directed with $\mathrm{Obj}(\mathcal{I}) = I$ and $F : \mathcal{I} \to \mathcal{C}$. A colimit of F is called the **direct limit**

of the F(i) for all $i \in I$. We write $\operatorname{colim} F = \underline{\lim}_{i \in I} \operatorname{colim} F$.



Definition 1.4. A small category \mathcal{I} is filtered if

- 1. for all $i, j \in I$, there exists $k \in I$ such that there exist morphisms $i \to k, j \to k$,
- 2. for all $\kappa, \kappa' : i \to j$ in I there exists a morphism $\lambda : j \to k$ such that $\lambda \circ \kappa = \lambda \circ \kappa'$ A category it **cofiltered** if the opposite category is filtered.

Cofiltered limits and diltered limits generalize inverse and direct limits, respectively.

Example 1.2. Let I be cofiltered with an initial object c. Then if $F: I \to \mathcal{C}$, $\lim F = F(e)$.

1.2 Adjoint functors

Definition 1.5. A functor $F: \mathcal{C} \to \mathcal{D}$ is **left adjoint** to a functor $G: \mathcal{D} \to \mathcal{C}$ if for each $C \in \mathcal{C}$, $D \in \mathcal{D}$, there exist bijections $\eta_{C,D}: \operatorname{Hom}_{\mathcal{D}}(F(C),D) \to \operatorname{Hom}_{\mathcal{C}}(C,G(D))$ such that η is a natural transformation between functors $\mathcal{C}^{op} \times \mathcal{D} \to \operatorname{Sets}$. That is,

$$\operatorname{Hom}_{\mathcal{D}}(F(C),D) \xrightarrow{\eta_{C,D}} \operatorname{Hom}_{\mathcal{C}}(C,G(D))$$

$$\downarrow^{h \mapsto g \circ h \circ F(f)} \qquad \downarrow^{h \mapsto G(g) \circ h \circ f}$$

$$\operatorname{Hom}_{\mathcal{D}}(F(C'),D') \xrightarrow{\eta_{C',D'}} \operatorname{Hom}_{\mathcal{C}}(C',G(D'))$$

G is **right adjoint** to F if F is left adjoint to G.

Remark 1.1. If $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ are quasi-inverses and $\eta: \mathrm{id}_{\mathcal{C}} \to G \circ F$ is a natural isomorphism, then we can define $\phi_{C,D}: \mathrm{Hom}_{\mathcal{D}}(F(C),D) \to \mathrm{Hom}_{\mathcal{C}}(C,G(D))$ given by $h \mapsto G(h) \circ \eta_C$. Check that $\phi_{C,D}$ is a bijection. So F is left-adjoint to G. Similarly, G is left-adjoint to F.

Proposition 1.1. Suppose S is a set, and consider h_S : Set \to Set given by $h_S(T) = \operatorname{Maps}(S,T)$ and $h_S(f:T\to T') = g \mapsto f \circ g$. Then h_S is right adjoint to t_S : Set \to Set given by $t_S(T) = T \times S$ and $t_S(f) = (f, \operatorname{id}_S) : T \times S \to T' \times S$.

Proof. We need to find a bijection $\tau_{T,U}$: Maps $(T \times S, U) \to \text{Maps}(T, \text{Maps}(S, U))$. We can send $f \mapsto (t \mapsto (s \mapsto f(s,t)))$. To show that this is a bijection, we can go backward by sending $\varphi \mapsto ((t,s) \mapsto \varphi(t)(s))$. Check that these maps are inverses of each other and that this is a natural transformation.

Proposition 1.2. Suppose all limits $F: I \to \mathcal{C}$ exist. Then the functor $\lim : \operatorname{Fun}(I, \mathcal{C}) \to \mathcal{C}$ given by $F \mapsto \lim F$ and $(\eta: F \to F') \mapsto (\lim F \mapsto \lim F')$ has a left adjoint $\Delta: \mathcal{C} \to \operatorname{Fun}(I, \mathcal{C})$ such that $\Delta(A) = c_A$ is the constant functor $I \to \mathcal{C}$ with value A.

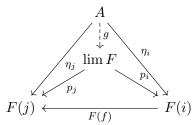
Proof. We want a bijection $\eta: \operatorname{Hom}_{\operatorname{Fun}(I,\mathcal{C})}(c_A,F) \to \operatorname{Hom}_{\mathcal{C}}(A,\lim F)$. Let $\eta: c_A \to F$ be $\eta_i: \underbrace{c_A(i)}_{=A} \to F(i)$ such that

$$A \xrightarrow{\eta_i} F(i) \qquad A \xrightarrow{\eta_i} F(i)$$

$$id_A = c_A(f) \downarrow \qquad \downarrow F(f) \qquad \downarrow F(f)$$

$$A \xrightarrow{\eta_j} F(j) \qquad F(j)$$

for all $f: i \to j$. So $\eta_j = F(f) \circ \eta_i$ for all $f: i \to j$. There exists a unique morphism $g: A \to \lim F$ such that



Send η to g. Conversely if we have $g: A \to \lim F$, $\eta_i = p_i \circ g$ is a morphism from $A \to F(i)$. So we get $\eta \in \operatorname{Hom}_{\operatorname{Fun}(I,\mathcal{C})}(c_A,F)$.

Definition 1.6. A contravariant functor $F: \mathcal{C} \to \operatorname{Set}$ is **representable** if there exists an object $B \in \mathcal{C}$ and a natural isomorphism $h^B \to F$, where $h^B = \operatorname{Hom}_{\mathcal{C}}(\cdot, B)$. We say that B **represents** F.

Example 1.3. The functor $P : \text{Set} \to \text{Set}$ given by $S \mapsto \mathcal{P}(S)$ and $(f : S \to T) \mapsto (V \mapsto f^{-1}(V))$ is representable by $\{0,1\}$.