Math 246A Lecture 5 Notes

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1 Stereographic Projection and Introduction to Möbius Transformations

1.1 Power series for the complex logarithm

The exponential map $E : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is onto. Fix $z_0 \in \mathbb{C} \setminus \{0\}$, and let c_0 be such that $E(c_0) = z_0$. Define

$$L(z) = c_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{z - z_0}{z_0} \right)^{n+1}.$$

This converges if $|z - z_0| < |z_0|$ and has the property that $E(L(z)) = z$.

$$L'(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z - z_0)^n}{{z_0}^{n+1}} = \frac{1}{z_0} \frac{1}{1 + (z - z_0)/z_0} = \frac{1}{z}.$$

$$\frac{d}{dz} z e^{-L(z)} = e^{-L(z)} - \frac{z}{z} e^{-L(z)} = 0.$$

Since $L(z_0) = c_0$, we get that $z_0 e^{-L(z_0)} = 1$. So $\log(z) = L(z) + 2\pi n i$.

1.2 Stereographic projection

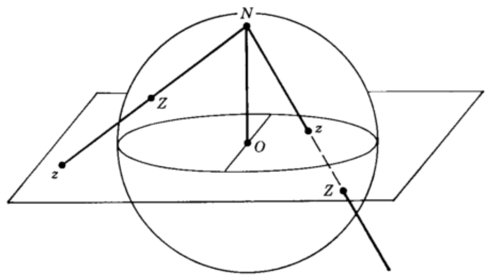
Let $\mathbb{C}^* = \cup \{\infty\}$ be the **one point compactification** of $\mathbb{C}$.

Definition 1.1. Let Ω be a neighborhood of $z_0 \in \mathbb{C}$, and let $f \Omega \to \mathbb{C}^*$ be such that $f(z_0) = \infty$. Then f is **meromorphic** at z_0 if $1/f$ is analytic in Ω.

Example 1.1. Let $U = \{\infty\} \cup \{z : |z| > R\}$ and $f : U \to \mathbb{C}^*$. The f is analytic if $f(1/z)$ is analytic on $\{w : |w| < 1/R\}$.

Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^3 = 1\}$. Let $N = (0, 0, 1)$ be the north pole. Let Z be a point on the sphere, and draw the line connecting X and Z. Then let $z = T(Z)$

be the point where this line intersects the xy plane. View this as a point on the complex plane. Here $T(Z) = tN + (1-t)Z$ for some $t > 0$. Here is a picture:[1]



Definition 1.2. The map $T : S^2 \setminus \{N\} \to \mathbb{C}$ is called **stereographic projection**.

Observe that $T(x_1, x_2, 0) = x_1 + ix_2$, so T sends the equator of S^2 to itself.

Lemma 1.1. *The map $T : S^2 \setminus \{\mathbb{N}\} \to \mathbb{C}$ is a homeomorphism.*

Proof. Let $z = T(Z)$, where $Z = (x_1, x_2, x_3)$. Then (verify yourself that)

$$T(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}.$$

Note that

$$|z|^2 = \frac{x_1^2 + x_2^2}{1 - x_3^2} = \frac{x_1^2 + x_2^2}{1 - x_3} = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}.$$

So

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}, \qquad x_1 = \frac{z + \overline{z}}{1 + |z|^2}, \qquad x_2 = \frac{z - \overline{z}}{i(1 + z|)^2}.$$ □

We can extend T to a map $T : S^2 \to \mathbb{C}^*$ by setting $T(N) = \infty$. The homeomorphism property still holds.

Theorem 1.1. *Let Γ be a circle on S^2, so $\Gamma = S^2 \cap \{X : |X - A| = R\}$. Then $T(\Gamma) \cap \mathbb{C}$ is*

$$\begin{cases} \textit{a line in } \mathbb{C} & N \in P \\ \textit{a circle in } \mathbb{C} & N \notin P. \end{cases}$$

Proof. $\Gamma = S^2 \cap \{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0 : \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \alpha_0 > 0\}$. Then $z \in T(\Gamma)$ iff

$$\alpha_1 \frac{z + \overline{z}}{1 + |z|^2} + \alpha_2 \frac{z - \overline{z}}{i(1 + z|)^2} + \alpha_3 \frac{|z|^2 - 1}{|z|^2 + 1} = \alpha_0$$

[1] I did not create this picture; I found it on Google.

$$\iff (\alpha_3 - \alpha_0)(x^2 + y^2) + 2\alpha_1 x + 2\alpha_2 y - (\alpha_0 - \alpha_3) = 0.$$

If $\alpha_3 = \alpha_0$, we get a line. Otherwise, we can complete the square.

$$x^2 + y^2 + \frac{2\alpha_1}{\alpha_3 - \alpha_0} x + \frac{2\alpha_2}{\alpha_3 - \alpha_0} y = \frac{\alpha_3 + \alpha_0}{\alpha_3 - \alpha_0},$$

which gives a circle. □

Conversely, every circle or line in $\mathbb{C}$ has the form $T(\Gamma)$.

Corollary 1.1.

$$|T^{-1}(z) - T^{-1}(z')| = \frac{2|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}},$$

$$|T^{-1}(z) - T^{-1}(\infty)| = \frac{2}{\sqrt{1 + |z|^2}}.$$

Proof. Homework. □

1.3 Möbius transformations

Let $S : \mathbb{C}^* \to \mathbb{C}^2$ be

$$S(z) = \frac{az + b}{cz + d} = w,$$

where $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$. This is invertible because

$$z = \frac{dw - b}{-cw + a} = S^{-1}(w).$$

So S, S^{-1} are bijections from $\mathbb{C}^*$ to $\mathbb{C}^*$. These are analytic because we can expand the denominator to a convergent power series around a point. Also define $S(\infty) = a/c$ and $S^{-1}(\infty) = -d/c$.

Definition 1.3. The **projective special linear group** is the group of matrices

$$PSL(2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C}, \det(A) = ab - cd = 1 \right\}$$

Theorem 1.2. *The group of Möbius transformations (with group operation composition) is isomorphic to* $\mathrm{PSL}(2, \mathbb{C})$.

Proof. Let $D : \mathrm{PSL}(2, \mathbb{C}) \to MT$ be

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)(z) = \frac{az + b}{cz + d}.$$

Check yourself that $F(AB) = F(A) \circ F(B)$ and that F is 1 to 1 and onto. □

Example 1.2. Here are some important examples of Möbius transformations.

1. translation: $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ corresponds to $z \mapsto z + \alpha$.
2. rotation: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ with $|k| = 1$
3. dilation: $\begin{bmatrix} \sqrt{k} & 0 \\ 0 & 1/\sqrt{k} \end{bmatrix}$ with $k > 0$ corresponds to $z \mapsto kz$
4. inversion: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ corresponds to $z \mapsto 1/z$

Theorem 1.3. *Translation, rotation, dilation, and inversion generate MT.*