Math 255A' Lecture 6 Notes

Daniel Raban

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1 Applications of The Hahn-Banach Theorem

The last application we will go over is not in Conway's textbook.

1.1 Banach limits

Let $\mathbb{F} = \mathbb{R}$ and $\ell^\infty = \ell^\infty(\mathbb{N})$. Then $c = \{(x_n) \in \ell^\infty : \lim_n x_n \text{ exists}\}$ is a closed subspace. lim is a bounded linear functional on c, so we can extend it to all of ℓ^∞.

Theorem 1.1. *There exists an $L \in (\ell^\infty)^*$ such that*

1. *$L(x) = \lim_n x_n$ for all $x \in c$,*
2. *$L(x) \geq 0$ if $x_n \geq 0$ for all n,*
3. *$L(\sigma(x)) = L(x)$, where $\sigma(x) = (x_2, x_3, \dots)$.*

Proof. Let $M = \{x - \sigma(x) : x \in \ell^\infty\}$, which is a linear subspace of ℓ^∞. We will apply a corollary of Hahn-Banach to get an L that kills M and $L((1, 1, \dots)) = 1$.

Claim: $\operatorname{dist}(\mathbf{1} = (1, 1, \dots), M) = 1$. Let $x - \sigma(x) \in M$. Then

$$\operatorname{dist}(\mathbf{1}, x - \sigma(x)) = \sup_n |1 - (x_n - x_{n+1})|.$$

Since $(x_n) \in \ell^\infty$, the right hand side gets arbitrarily close to 1 when x_n is close to $\inf_m x_m$. So there exists an $L \in (\ell^\infty)^*$ such that $L(M) = 0$, $L(\mathbf{1}) = 1$ and $\|L\| = 1$. This covers property 3.

For property 2, use $\|L\| = 1$ and $L(\mathbf{1}) = 1$. It's similar to the fact that if μ is a signed measure, then $|\mu|(X) = |\mu(X)| \implies \mu = 0$.

For property 1, suppose $x \in c$ and let $\alpha := \lim_n x_n$. We claim that $\|\sigma^n(x)\| - \alpha\mathbf{1}\| \to 0$ as $n \to \infty$; this is a rewording of $\alpha := \lim_n x_n$. So $L(x) = L(\sigma^n(x)) \to \alpha L(\mathbf{1}) = \alpha$. $\square$

Corollary 1.1. *$c_0 \subseteq \ker L$.*

1.2 Dual of quotients by subspaces

Let X be a normed space, and let $M \leq X$ (i.e. M is a closed subspace).

Definition 1.1. The **annihilator** of M is $M^{\perp} := \{L \in X^* : L|_M = 0\}$.

Theorem 1.2. *Let X be a normed space, and let $M \leq X$. Then the map $X^*/M^{\perp} \to M^*$ sending $f + M^{\perp} \mapsto f|_M$ is an isometric isomorphism.*

Proof. This map is linear. We need to show that it is surjective and that $\|f|_M\| = \|f + M^{\perp}\|$. We have the inequality $\leq$. The rest of the proof follows from the following claim: Given $g \in M^*$, there exists some $f \in X^*$ such that $f|_M = g$ and $\|f\| = \|g\|$. This is just Hahn-Banach. □

Theorem 1.3. *Let $O : X \to X/M$ be the quotient map. Then the map $(X/M)^* \to M^{\perp} \subseteq X^*$ given by $g \mapsto g \circ Q$ is an isometric isomorphism.*

Proof. Any $v\ in M^{\perp}$ defines a linear functional on X/M. We want it to be bounded with the same norm:

$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1\} \implies f(x) \leq \|f\| \cdot \|x + M^{\perp}\|$$

for all x. And Q is surjective. □

1.3 The double dual

We can keep taking the dual spaces of dual spaces to get $X^*, X^{**}, X^{***}, \ldots$.

Definition 1.2. The **natural map** $X \to X^{**}$ is given by $x \mapsto \hat{x}$, where $\hat{x}(f) = f(x)$.

Lemma 1.1. *The natural map is isometric.*

Proof. Since $|\hat{x}(f)| \leq \|f\| \cdot \|x\|$, we have $\|\hat{x}\| \leq \|x\|$. Equality is by Hahn-Banach. □

Definition 1.3. X is **reflexive** if $\hat{X} = X^{**}$; i.e. the natural map is surjective.

Example 1.1. Let $1 < p < \infty$. Then $L^p(\mu)$ is reflexive by Riesz representation.

Example 1.2. Id $\dim X < \infty$, then X is reflexive.

Example 1.3. $c_0 = C_0(\mathbb{N})$ is not reflexive. c_0^* is the collection of signed finite measures on $\mathbb{N}$, which is $\ell^1(\mathbb{N})$ by Riesz-representation. So c_0^{**} is $\ell^{\infty}(\mathbb{N})$, which is bigger than c_0.

1.4 Optimal transport

Let (X, ρ) be a metric, and let $\mu, \nu \in \mathrm{Prob}(X)$. We want to move the mass according to the distribution μ to that of ν. This is called the **transport problem**. For infinitesimal regions dx, dy, think of $\lambda(dx, dy)$ as how much mass moves from dx to dy. We interpret $\lambda \in \mathrm{Prob}(X \times X)$.

λ has to satisfy

$$\sum_{dy} \lambda(dx, dy) = \mu(dx), \qquad \sum_{dx} \lambda(dx, dy) = \nu(dy)$$

so we want

$$\lambda(A \times X) = \mu(A), \qquad \lambda(X \times B) = \nu(B)$$

for all measurable $A, B \subseteq X$.

Definition 1.4. A measure λ with these properties is called a **coupling** of μ, ν.

Call the collection of all such couplings C.

Example 1.4. The product measure $\lambda = \mu \times \nu$ is a coupling.

Let's suppose it costs us to move mass from dx to dy. Then we want to find λ and estimate

$$\min_{\lambda \in C} \int \rho(x, y)\, d\lambda(x, y).$$

This is not the inf because it is weak*-continuous.

Obstructions: Suppose $f \in L$, the collection of 1-Lipschitz functions $X \to \mathbb{R}$. Then for $\lambda \in C$,

$$\int f\, d\mu - \int f\, d\nu = \int (f(x) - f(y))\, d\lambda(x, y) \leq \int |f(x) - f(y)|\, d\lambda(x, y) \leq \int \rho\, d\lambda.$$

Theorem 1.4. *Let $D := \sup_{f \in L} |\int f\, d\mu - \int f\, d\nu|$. THen there exists a $\lambda \in C$ such that $\int \rho\, d\lambda = D$.*

Remark 1.1. This theorem says that these obstructions are the only ones.

Proof. Equivalently, by Riesz representation, we want $\phi \in C(X \times X)^*$ (which corresponds to λ) such that

1. (coupling) $\phi(f(x) \cdot 1(y)) = \int f\, d\mu$ and $\phi(1(x) \cdot g(y)) = \int g\, d\nu$ for all $f, g \in C(X)$,

2. (minimizer) $\phi(\rho) = D$,

3. (probability measure) $\|\phi\| = \phi(\mathbb{1}_{X \times X}) = 1$.

So define $M := \{f(x)+g(y)+a\rho(x,y); f,g \in C(X), a \in \mathbb{R}\} \subseteq C(X \times Y)$. Define ψ on M by

$$\psi(f+g+a\rho) = \int f\,d\mu + \int g\,d\nu + aD.$$

We need to check that ψ is well-defined and $\|\psi\|_{M^*} \leq 1$. Taking $b = 0, 1$ it follows from the claim: If $f+g+a\rho \leq b$, then $\psi(f+g+a\rho) \leq b$. It is equivalent to show that if $f+g \leq b + a\rho$ for all x, y then $\int f\,d\mu + \int g\,d\nu \leq b + aD$.

Case 1: $a \leq 0$. This is straightforward and is in the online notes.

Case 2: $a > 0$. We may assume $a = 1$. Rewrite this as $f(x) \leq \inf_y [b - g(y) + \rho(x,y)] =: h(x) \in L$. Then $f \leq h \leq b - g(x)$, so

$$\int f\,d\mu \leq \int h\,d\mu \leq \int d\,d\nu + D \leq b - \int g\,d\nu + D.$$

- Start with $b - g(x)$
- Draw cones at each point on the graph.
- Take h to be the minimum of the cones.

Then applying Hahn-Banach to ψ gives ϕ. $\square$