Math 222A Lecture 5 Notes

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1 Local Solutions for Linear, Semilinear, and Quasilinear Scalar PDEs

1.1 Local solutions for linear, scalar PDEs

Last time, we were studying linear, scalar PDEs of the form

$$\underbrace{A_j \partial_j u}_{\text{directional derivative}} + bu = f.$$

The **initial curves** (or **characteristics**) of A were the solutions to the ODE

$$\dot{x} = A(x), \qquad x(0) = x_0$$

Along the integral curves, the PDE looks like

$$\frac{d}{dt} u(x(t)) + b(x(t)) u(x(t)) = f(x(t)),$$

so solving the PDE is like solving two ODEs.

If we assume $A \in C^1$, then $x(t, x_0) \in C^1$. We want these characteristics to locally foliate $\mathbb{R}^n$; that is, we want them to cover the domain. One issue: what if $A(x_0) = 0$? Then $x(t) = x_0$ for all t!

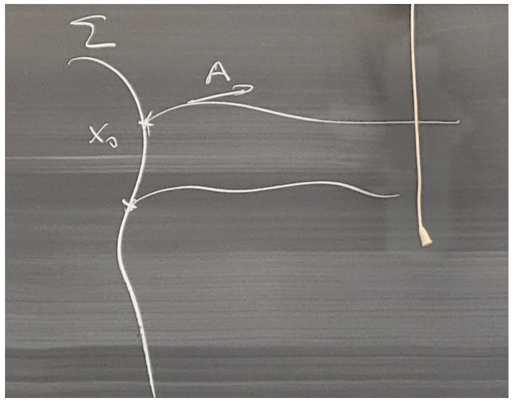
Example 1.1. Consider A that gives

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1.$$

Then the integral curves will be circles, so $A(0) = 0$.

The fix for this problem is to assume that $A(x) \neq 0$ for any x.

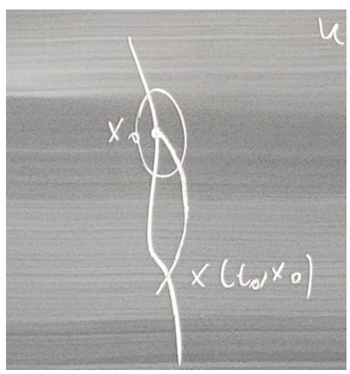
Now suppose we have initial data $u(x) = u_0(x)$ on a curve Σ. If we start at an x_0 on the curve or surface Σ, we can look at the integral curve starting from x_0.



From $x_0 \in \Sigma$ and $t \in [-\varepsilon, \varepsilon]$, we can construct $x(t, x_0)$. Once we know $u(x_0)$, we can solve the second ODE to get $u(x(t, x_0))$, where $x(t, x_0) \in C^1$. So by our ODE theorem, we will get $u \in C^1$.

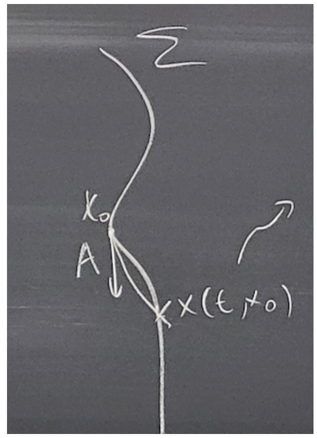
What are the bad cases?

- The integral curve may intersect Σ twice.



We might still get a local solution if we look at a smal enough neighborhood of x_0.

- A may be tangent to Σ, and re-intersection can happen arbitrarily close.



  Even if re-intersection is not arbitrarily close, there may be a more subtle issue with the solution not being C^1.

Here is how we avoid this issue.

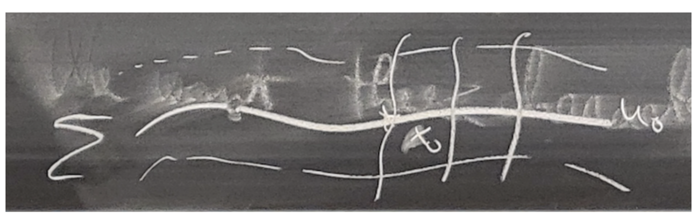
Definition 1.1. We say that Σ is **noncharacteristic** for our PDE if $A \cdot N \neq 0$ on Σ, where N is the normal to Σ.

This says that A is not tangent to Σ at any point.

Theorem 1.1. *Assume $A, b, f, \Sigma, u_0 \in C^1$, and suppose that Σ is noncharacteristic. Then the equation*

$$A_j \partial_j u + bu = f$$

with initial data u_0 has a unique C^1 local solution.



Proof. Step 1: For $x_0 \in \Sigma$, solve for the characteristic $\Sigma \times [-\varepsilon, \varepsilon] \ni (x_0, t) \mapsto x(x_0, t)$.

Step 2: Solve the ODE

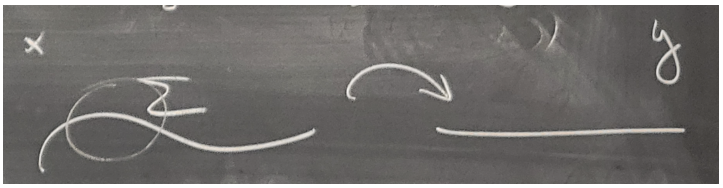
$$\frac{d}{dt} u(x(t)) + b(x(t)) u(x(t)) = f(x(t))$$

along the characteristics to get $u(x(t,x_0))$, which is C^1 in t and x_0.

Step 3: Show that our characteristics foliate a neighborhood of Σ. What does this mean? Looking at the map $(x_0,t) \mapsto x(t,x_0)$. We want this to be a local diffeomorphism, i.e. a C^1 map with a C^1 inverse. Recall the following theorem from real analysis:

Theorem 1.2 (Local inversion theorem)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n \in C^1$. If $\det dF(x_0) \neq 0$, Then F is a local diffeomorphism.*

We would like to change coordinates so that Σ is a hyperplane.



Since Σ is C^1, locally, Σ is the graph of a C^1 function, $x_n = f(x')$, $x' = (x_1, \dots, x_{n-1})$ with $f \in C^1$. The new coordinates are $y = (x', x_n - f(x'))$. To check that this is a local diffeomorphism, the theorem says we should look at

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y'}{\partial x'} & \frac{\partial y'}{\partial x_n} \\ \frac{\partial y_n}{\partial x'} & \frac{\partial y_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ df & 1 \end{bmatrix},$$

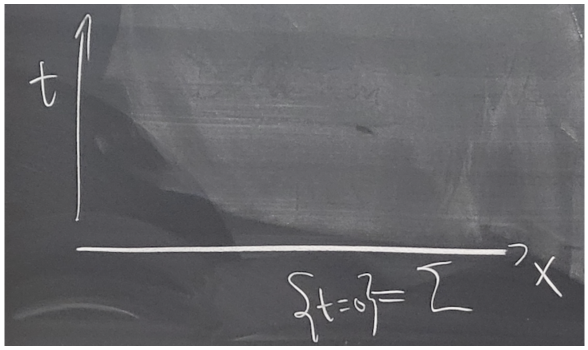
which has determinant 1. Check that the coefficients remain C^1 after changing coordinates.

In the new coordinates, $\Sigma = \{y_n = 0\}$, $y' = (y_1, \dots, y_{n-1})$ are coordinate on Σ, and we are looking at the equation $\dot{y} = A(y)$. Here, $y = y(t, y_0')$. Look at $\frac{\partial y}{\partial (y_0', t)}$ at $t = 0$. When $t = 0$, $y(y_0', 0) = (y_0', 0)$. So

$$\frac{\partial (y', y_n)}{(y_0', t)} = \begin{bmatrix} \frac{\partial y'}{\partial y_0'} & \frac{\partial y'}{\partial t} \\ \frac{\partial y_n}{\partial y_0'} & \frac{\partial y_n}{\partial t} \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ A' & A_n \end{bmatrix}.$$

So $\det \frac{\partial y}{\partial (y_0', t)} = A_n \neq 0$, precisely from our noncharacteristic surface property. $\square$

Remark 1.1. In the above proof, we reduced the situation to the case where Σ is a hyperplane. Let's use this to model the noncharacteristic case. Using coordinates (x,t), we can write $\Sigma = \{t = 0\}$.



Our equation looks like

$$A_t \cdot \partial_t u + A_1 \cdot \partial_1 u + \cdots + A_n \cdot \partial_n u + bu = f.$$

where $A_t \neq 0$ by the noncharacteristic assumption. So we may divide by it and just look at equations of the form

$$\partial_t u + A_1 \cdot \partial_1 u + \cdots + A_n \cdot \partial_n u + bu = f.$$

This is only a local modelling, however, not necessarily a global one.

1.2 Semilinear PDEs

Now we move on to solving semilinear PDEs, of the form

$$\begin{cases} A_j(x)\partial_j u + b(u, x) = 0 \\ u|_\Sigma = 0 \end{cases}$$

The characteristics are still $\dot{x} = A(x)$ (so $x = x(x_0, t)$), and our noncharacteristic initial surface condition is still $A \cdot N \neq 0$ on Σ. The evolution along the characteristics is

$$\frac{d}{dt} u(x(x_0, t)) = -b(u(x(x_0, t)), x(x_0, t)).$$

The difference from before is that our second equation is a nonlinear ODE, so it may have finite time blow-up. So local well-posedness is identical to the linear case, but global well-posedness may fail because the second ODE blows up.

1.3 Quasilinear PDEs

Now we look at the quasilinear problem

$$\begin{cases} A_j(x,u)\partial_j(u) + b(x,u) = 0 \\ u|_\Sigma = u_0. \end{cases}$$

Our characteristics now look like $\dot{x} = A(x,u)$. We cannot solve this because we do not know what u is outside of Σ. The second equation would read $\dot{u} = b(x,u)$. These two ODEs would be true if we already had a solution, but we cannot solve them. What if we put these two equations together into a system?

$$\begin{cases} \dot{x} = A(x,u) \\ \dot{u} = b(x,u) \end{cases}$$

We call this a **characteristic system**.

The initial data for the characteristic system is

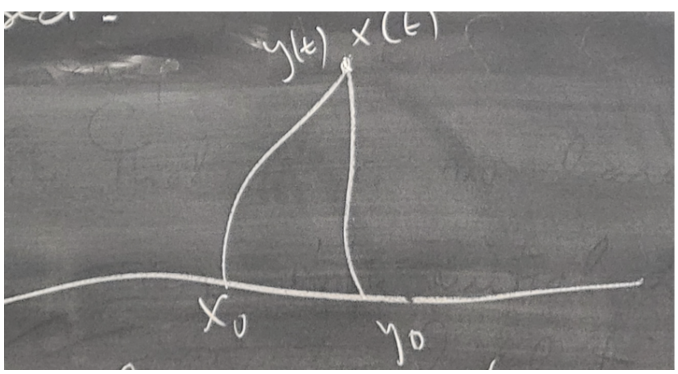
$$\begin{cases} x(0) = x_0 \in \Sigma \\ u(0) = u(x(0)) = u_0(x_0), \end{cases}$$

where the second initial condition depends on u_0. In this situation, our noncharacteristic Σ condition is

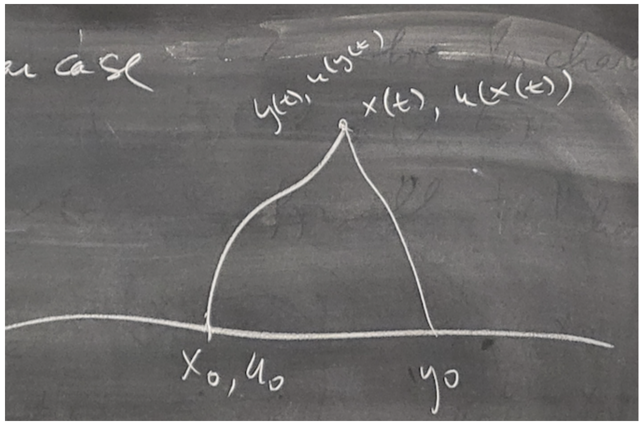
$$A(x_0, u_0(x_0)) \cdot N \neq 0.$$

Our local well-posedness theorem is identical: If Σ is noncharacteristic and $u_0 \in C^1$, then there exists a unique local C^1 solution u.

The key difference is that the characteristics may now intersect. In the semilinear case, suppose two characteristics were to intersect. Then the characteristic equation would have the same data, so the two characteristics must be the same.



In the quasilinear case, the initial data is both the location and the value of the function. Intersection means that $x(t) = y(t)$, but it does not necessarily mean $u(x(t)) = u(y(t))$. So we cannot say that the two characteristics must be the same.



Next time, we will talk about what might make characteristics intersect and what to do about it.