Math 255B Lecture 17 Notes

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1 Quadratic Forms and the Friedrichs Extension Theorem

1.1 Obtaining self-adjoint operators from quadratic forms

Last time, we said that a nonnegative, symmetric quadratic form is closed if when $u_n \xrightarrow{q} u$ for $u_n \in D(q)$, then $u \in D(q)$ and $q(u_n - u) \to 0$. We checked that q is closable if and only if when $u_n \xrightarrow{w} 0$, $q(u_n) \to 0$.

Theorem 1.1. *Let q be a nonnegative, symmetric, quadratic form. Assume that $D(q)$ is dense and that q is closed. Then there exists a unique self-adjoint operator $\mathscr{A}$ such that $D(\mathcal{A}) \subseteq D(q)$ and $q(u,v) = \langle \mathscr{A}u, v\rangle$ for all $u \in \mathscr{A}, v \in D(q)$. Also, $D(\mathscr{A})$ is a **core** for q in the sense that $D(\mathscr{A})$ is dense in H_q.*

Example 1.1. Let $q(u) = \int |u'|^2 + V|u|^2\,dx$, where $V \in L^1(\mathbb{T};\mathbb{R})$ and $D(q) = H^1(\mathbb{T})$. Then there exists a unique self-adjoint operator $P = -\partial_x^2 + V$ such that $D(P) \subseteq H^1$ and $q(u,v) = \langle Pu, v\rangle$.

Proof. Let $\langle x, y\rangle_q := q(x,y) + \langle x, y\rangle$ for $x, y \in D(q)$. Then H_q is a Hilbert space with respect to this scalar product. Then $\|x\|_q \geq \|x\|$ for all $x \in H_q$, so for any $u \in H$, the linear form $H_q \to \mathbb{C}$ sending $v \mapsto \langle v, u\rangle$ is continuous. By the Riesz representation theorem, there is a unique $u^* \in H_q$ such that $\langle v, u\rangle = \langle v, u^*\rangle_{H_q}$. We get a linear map $K : H \to H_q$ sending $u \mapsto u^*$ such that $\langle v, u\rangle = \langle v, Ju\rangle_q$ for all $v \in H_q$ and $u \in H$.

We claim that J is a bounded, self-adjoint operator on H. If $y, x \in H$,

$$\langle Jy, x\rangle = \langle Jy, Jx\rangle_q = \overline{\langle Jx, Jy\rangle_q} = \overline{\langle J_x, y\rangle} = \langle y, Jx\rangle .$$

So J is symmetric. By the closed graph theorem, $J \in \mathcal{L}(H,H)$. Moreover, J is injective: If $Jx = 0$, then $\langle y, x\rangle = \langle y, Jx\rangle_w = 0$ for all $y \in H_q$. But H_q is dense in H, so $x = 0$.

Write

$$H = \ker J \oplus \overline{\operatorname{Ran} J^*} = \overline{\operatorname{Ran} J}.$$

So $\operatorname{Ran} J$ is dense and contained in H_q. Define $\mathscr{A} : D(\mathcal{A}) = \operatorname{Ran} J$ by $\mathscr{A}x = J^{-1}x - x$. We have J^{-1} is self-adjoint: J^{-1} is symmetric, and if (x, y) are such that $\left\langle J^{-1}z, x\right\rangle = \langle z, y\rangle$, where $z = Jw$, then $\langle w, x\rangle = \langle Jw, y\rangle = \langle w, Jy\rangle$. So $x \in D(\mathcal{A})$, and $y = J^{-1}x$. ($D(\mathscr{A})$ is dense in H_q.)

Finally, $\mathscr{A}$ corresponds to the quadratic form q:

$$\langle x, y\rangle_q = \left\langle x, J^{-1}y\right\rangle,$$

so

$$q(x, y) = \langle x, y\rangle_q - \langle x, y\rangle = \langle x, \mathscr{A}y\rangle, \qquad y \in D(\mathscr{A}), x \in D(q).$$

□

1.2 The Friedrichs extension theorem

In the previous theorem, we don't have much control over the domain of the self-adjoint operator $\mathscr{A}$. Here is a frequently encountered use of the theorem.

Theorem 1.2 (Friedrichs extension)**.** *Let S be a symmetric, densely defined operator $D(S) \to H$ such that S is bounded below: $\langle Su, u\rangle \geq -C\|u\|^2$ for every $u \in D(S)$. Let the quadratic form q be given by $D(q) = D(S)$, $q(u) = \langle Su, u\rangle$. Then q is closable. The self-adjoint operator associated to $\overline{q}$, the **Friedrichs extension** of S, is also bounded below.*

Remark 1.1. The Friedrichs extension theorem can give a different result compared to if we just closed the operator S.

Remark 1.2. Let $q \geq 0$ be closed. Then $q(u, v) = \langle u, \mathscr{A}v\rangle$ for $v \in D(\mathscr{A})$ and $u \in D(q)$, where $D(\mathscr{A}) = \{v \in H_q : \exists f \in H \text{ s.t. } q(u, v) = \langle u, f\rangle \ \forall u \in D(q)\}$. So in the theorem, $D(\mathscr{A}) \supseteq D(S)$.

Proof. We can assume that $S \geq 0$. We only need to show that q is closable. Let $u_n \in D(S)$ be such that $u_n \to 0$ in H and $q(u_n - u_m) \xrightarrow{n,m\to\infty} 0$. We want to show that $q(u_n) \to 0$. We have

$$\begin{aligned} q(u_n) &\leq |q(u_n - u_m, u_n)| + |q(u_m, u_n)| \\ &\overset{\text{C-S}}{\leq} q^{1/2}(u_n - q_m)q^{1/2}(u_n) + |\left\langle u_m, Su_n\right\rangle \end{aligned}$$

For all $\varepsilon > 0$, there exists an N such that $q(u_n - u_m) \leq \varepsilon$ when $n, m \geq N$. So

$$q(u_n) \leq \varepsilon^{1/2}q^{1/2}(u_n) + |\left\langle u_m, Su_n\right\rangle| \qquad \forall n, m \geq N.$$

Leting $m \to \infty$, we get

$$q(u_n) \leq \varepsilon \qquad \forall n \geq N.$$

So q is closable. □

Example 1.2 (The Dirichlet realization of $-\Delta$)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $S = -\Delta$ with $D(S) = C_0^\infty(\Omega)$ ($S \geq 0$). The Friedrichs extension is associated with the closure of the quadratic form

$$q(u) = \langle Su, u \rangle = \int_\Omega (-\Delta u)\overline{u} = \int_\Omega |\nabla u|^2 \, dx.$$

Next time, we will see that in this case, $D(\overline{q})$ is the closure of C_0^∞ in the topology of $H^1(\Omega)$. This is usually called $H_0^1(\Omega)$ (but is not all of $H^1(\Omega)$).