

Math 255B Lecture 6 Notes

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1 Consequences of Analytic Fredholm Theory

1.1 Analytic Fredholm theory

Last time, we were proving the analytic Fredholm theory.

Theorem 1.1 (analytic Fredholm theory). *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ be a holomorphic family of Fredholm operators. Assume that there exists a $z_0 \in \Omega$ such that $T(z_0) : B_1 \rightarrow B_2$ is bijective. Then the set*

$$\Sigma = \{z \in \Omega : T(z) \text{ is not bijective}\}$$

is discrete.

Proof. Let $z_1 \in \Omega$. Then there is a neighborhood $N(z_1)$ of z_1 such that for every $z \in N(z_1)$, the Grushin operator

$$\mathcal{P}_{z_1}(z) = \begin{bmatrix} T(z) & R_-(z) \\ R_+(z) & 0 \end{bmatrix}$$

is bijective with the inverse

$$\mathcal{E}_{z_1}(z) = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_0} \rightarrow B_1 \oplus \mathbb{C}^{n_0}.$$

We claim that for $z \in N(z_1)$, $T(z) : B_1 \rightarrow B_2$ is bijective $\iff E_{-+}(z) : \mathbb{C}^{n_0} \rightarrow \mathbb{C}^{n_0}$ is bijective.¹ Check:

$$\begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies TE + R_-E_- = 1, TE_+ + R_-E_{-+} = 0.$$

¹What we lose from this reduction is that if $T(z)$ has some simple dependence of z (e.g. polynomial), $E_{-+}(z)$ may not have a simple dependence. In some contexts, the operator E_{-+} is called the effective Hamiltonian.

If E_{-+}^{-1} exists, then $R_- = -TE_+E_{-+}^{-1}$, so

$$T(E - E_-E_{-+}^{-1}E_-) = 1.$$

So T^{-1} exists and

$$T^{-1}(z) = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z).$$

Using that $\mathcal{EP} = 1$, so $E_-R_- = 1$ and $E_-T + E_{-+}R_+ = 0$, we get T^{-1} exists $\implies E_{-+}$ exists.

We get for $z \in N(z_1)$ that $T(z)$ is invertible if and only if $\det E_{-+}(z) \neq 0$. The function $\det E_{-+}(z)$ is holomorphic on $N(z_1)$. So either $\det E_{-+}(z) \equiv 0$, or $\det E_{-+}(z) \neq 0$ in a punctured neighborhood of z_1 . Let $\Omega_1 = \{z \in \Omega : T(z') \text{ is invertible } \forall z' \neq z \text{ near } z\}$ and $\Omega_2 = \{z \in \Omega : T(z') \text{ is not invertible } \forall z' \neq z \text{ near } z\}$. Then each Ω_j is open, $\Omega_1 \cup \Omega_2 = \Omega$, and $\Omega_1 \neq \emptyset$ (as $z_0 \in \Omega_1$). Since Ω is connected, $\Omega_1 = \Omega$ and thus, $\Sigma = \{z \in \Omega : T(z) \text{ is not invertible}\}$ is discrete. \square

Remark 1.1. The map $\Omega \setminus \Sigma \rightarrow \mathcal{L}(B_2, B_1)$ sending $z \mapsto T(z)^{-1}$ is holomorphic. Consider $T(z)^{-1}$ for z in a punctured neighborhood of $w \in \Sigma$: We have

$$T^{-1}(z) = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z),$$

where E, E_+, E_- are all holomorphic in a neighborhood of w . We have that

$$E_{-+}(z)^{-1} = \frac{\text{holomorphic near } w}{\det E_{-+}(z)},$$

so we have a Laurent expansion

$$E_{-+}(z)^{-1} = \frac{R_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{R_{-1}}{z-w} + \text{Hol}(z),$$

where $1 \leq N_0 < \infty$ and the R_j are of finite rank. Combining these formulas, we get that $z \mapsto T(z)^{-1}$ has a pole of order N_0 at $z = w$:

$$T(z)^{-1} = \frac{A_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{A_{-1}}{z-w} + Q(z), \quad Q(z) \text{ holomorphic near } w,$$

where for $1 \leq j \leq N_0$, the $A_{-j} \in \mathcal{L}(B_2, B_1)$ can be expressed in terms of R_{-N_0}, \dots, R_{-1} and are therefore of finite rank.

1.2 Application: the residue of the resolvent

Here is an example/special case of the analytic Fredholm theory.

Assume that $B_1 \subseteq B_2$ with continuous inclusion, and let $T(z) = T - z$ for $z \in \Omega$, where T is some operator. Assume that $T(z)$ is Fredholm for each z and that $T(z_0)^{-1}$ exists for some $z_0 \in \Omega$. We get a Laurent expansion for the resolvent $(T - z)^{-1}$ at $w \in \Sigma$:

$$(z - T)^{-1} = \frac{A_{-N_0}}{(z - z_0)^{N_0}} + \cdots + \frac{A_{-1}}{z - w} + Q(z), \quad Q(z) \text{ holomorphic near } w.$$

for $0 < |z - w| \ll 1$.

Proposition 1.1. *The operator $\Pi := A_{-1}$ projection² on B_2 which commutes with T (on B_1).*

Proof. Integrate the Laurent expansion along $\gamma_r = \partial D(w, r)$ for $0 < r \ll 1$. Then

$$\Pi = \frac{1}{2\pi i} \int_{\gamma_r} (z - T)^{-1} dz.$$

We claim that $\Pi^2 = \Pi$: Let $0 < r_1 < r_2 \ll 1$, and write

$$\Pi^2 = \int_{\gamma_{r_2}} \int_{\gamma_{r_1}} (z - T)^{-1} (\tilde{z} - T)^{-1} \frac{dz}{2\pi i} \frac{d\tilde{z}}{2\pi i}$$

and use $(\tilde{z} - T)^{-1} - (z - T)^{-1} = (\tilde{z} - T)^{-1} (z - \tilde{z}) (z - T)^{-1}$ to finish the calculation (we will do this next time). We have $\Pi T u = T \Pi u$ for $u \in B$, so $T(\text{Ran } \Pi) \subseteq \text{Ran } T$. $(T - w)|_{\text{Ran } \Pi}$ is nilpotent, so $(T - w)^{N_0} \Pi = 0$. \square

²This is sometimes called the Riesz projection.