Stat 155 Lecture 10 Notes

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1 Two Player General-Sum Games

1.1 General-sum games and Nash equilibria

Definition 1.1. A two-person *general-sum* game is sepcified by two payoff matrices $A, B \in \mathbb{R}^{m \times n}$. Simultaneously, Player 1 chooses $i \in \{1, \dots, m\}$, and Player 2 chooses $j \in \{1, \dots, n\}$. Player 1 receives payoff $a_{i,j}$, and Player 2 receives payoff $b_{i,j}$.

Because it is easier to view, we will often write a single bimatrix, that is a matrix with ordered pair entries $(a_{i,j}, b_{i,j})$.

Example 1.1. A zero-sum game is the case when $B = -A$.

Definition 1.2. A pure strategy e_i for Player 1 is *dominated* by $e_{i'}$ in the payoff matrix A if, for all $j \in \{1, \dots, n\}$, $a_{i,j} \leq a_{i',j}$. Similarly, a pure strategy e_j for Player 2 is dominated by $e_{j'}$ in the payoff matrix B if, for all $i \in \{1, \dots, n\}$, $b_{i,j} \leq b_{i,j'}$.

Definition 1.3. A *safety strategy* for Player 1 is an $x^* \in \Delta_m$ that satisfies

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^\top A y = \min_{y \in \Delta_n} (x^*)^\top A y.$$

A safety strategy for Player 2 is a $y^* \in \Delta_n$ that satisfies

$$\max_{y \in \Delta_n} \min_{x \in \Delta_m} x^\top B y = \min_{x \in \Delta_m} x^\top B y^*.$$

So x^* and y^* maximize the worst case expected gain for Player 1 and Player 2, respectively. Recall that for zero-sum games, the safety strategy for Player 2 was defined using A (because in that case, $B = -A$):

$$\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^\top A y = \max_{x \in \Delta_m} x^\top A y^*.$$

These definitions coincide because taking out the negative switches the max to a min (and vice versa).

Definition 1.4. A pair $(x^*, y^*) \in \mathbb{R}^{m\times n}$ is a *Nash equilibruim* for payoff matrices $A, B \in \mathbb{R}^{m\times n}$ if

$$\max_{x\in\Delta_m} x^\top Ay^* = (x^*)^\top Ay^*,$$

$$\max_{y\in\Delta_n} (x^*)^\top Ay = (x^*)^\top By^*.$$

This is a strategy where if Player 1 plays x^* and Player 2 plays y^*, neither player has an incentive to unilaterally deviate. In other words, x^* is a best response to y^*, and y^* is a best response to x^*. For zero-sum games, we saw that Nash equilibria were safety strategies, and the payoff from playing them was the value of the game. However, in general-sum games, there might be many Nash equilibria, with different payoffs.

1.2 Examples of general-sum games

Example 1.2. Here is the Prisoners' Dilemma. Two suspects are imprisoned by the police, who ask each of them to confess. The charge is serious, but there is not enough evidence to convict the suspects. Separately (in different rooms), each prisoner is offered the following plea deal:

- If one prisoner confesses, and the other prisoner remains silent, the confessor goes free, and their confession is used to sentence the other prisoner to ten years of jail.
- If both confess, they will both spend eight years in jail.
- If both remain silent, the sentence is one year to each for the minor crime that can be proved without additional evidence.

The payoff bimatrix for this game is

	silent	confess
silent	$(-1, -1)$	$(-10, 0)$
confess	$(0, -10)$	$(-8, -8)$.

If each player solves their own payoff matrix, then they will each choose to confess with probability 1.

Example 1.3. Two hunters are following a stag, when a hare runs by. Each hunter has to make a split-second decision: to chase the hare or to continue tracking the stag. The hunters must cooperate to catch the stag, but each hunter can catch the hare on his own. If they both go for the hare, they share it.

The payoff bimatrix for this game is

	stag	hare
stag	$(4, 4)$	$(0, 2)$
hare	$(2, 0)$	$(1, 1)$.

For each player, a safety strategy is to go for the hare. So (hare, hare) is a pure Nash equilibrium with payoff $(1, 1)$. Another pure Nash equilibruim is (stag, stag).

Let's find a mixed Nash equilibrium. For $((x, 1-x), (y, 1-y))$ to be a Nash equilibrium, the players don't want to shift to a different mixture. If Player 2 plays first and plays $(1-y, y)$, the the payoff for Player 1 is

$$(x, 1-x)\begin{pmatrix}4 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}y\\ 1-y\end{pmatrix} = (x, 1-x)\begin{pmatrix}4y\\ 2y+1-y\end{pmatrix}.$$

So Player 1 will play e_1 if $4y > 2y + 1 - y$. Player 1 will play e_2 if $4y < 2y + 1 - y$. This means that Player 1 will play a mixed strategy $(x, 1-x)$ if and only if

$$4y - 2y + 1 - y.$$

Similarly, if Player 2 plays second, Player 2 will play a safety strategy if and only if

$$4x = 2x + 1 - x.$$

Solving this, we get that $((1/3, 2/3), (1/3, 2/3))$ is a mixed Nash equilibrium. The payoff is $(4/3, 4/3)$.

Example 1.4. Player 1 is choosing between parking in a convenient but illegal parking spot (payoff 10 if they are not caught) and parking in a legal but inconvenient spot (payoff 0). If Player 1 parks illegally and is caught, they will pay a hefty fine (payoff -90).

Player 2, the inspector representing the city, needs to decide whether to check for illegal parking. There is a small cost (payoff -1) to inspecting. However, there is a greater cost to the city if Player 1 has parked illegally since that can disrupt traffic (payoff -10). This cost is partially mitigated if the inspector catches the offender (payoff -6).

The payoff bimatrix for this game is

	inspect	chill
illegal	$(-90, -6)$	$(10, -10)$
legal	$(0, -1)$	$(0, 0)$.

Safety strategies are for Player 1 to park legally and for Player 2 to inspect the parking spot.[1] There are no pure Nash equilibria. What about mixed Nash equilibria? For (x, y) to be a Nash equilibrium (where we implicitly mean the strategies are $((x, 1-x), (y, 1-y))$), the players don't want to shift to a different mixture. The strategies need to satisfy

$$0 = 10(1-y) - 90y, \qquad -10x = -(1-x) - 6x.$$

So $(1/5, 1/10)$ is a Nash equilibrium. The expected payoff is $(0, -2)$.

[1] Let this be a lesson to you.