

Math 210A Lecture 10 Notes

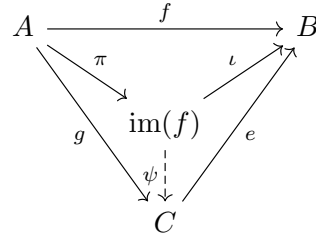
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1 Images, Coimages, and Generating Sets

1.1 Images

Definition 1.1. The **image** $\text{im}(f)$ of $f : A \rightarrow B$ is an object and a monomorphism $\iota : \text{im}(f) \rightarrow B$ such that there exists $\pi : A \rightarrow \text{im}(f)$ with $\pi \circ \iota$ and such that if $e : C \rightarrow B$ is a monomorphism and $g : A \rightarrow C$ is such that $e \circ g = f$, then there exists a unique morphism $\psi : \text{im}(f) \rightarrow C$ such that $g \circ \psi = \iota$.



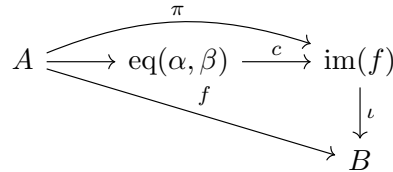
Example 1.1. In Set, $f(A) = \text{im}(f)$. Then $b \in F(A) \implies b = f(a)$ for some $a \in A$. Then $g(a) \in C$ is the unique element with $e(g(a)) = (a)$ because e is a monomorphism. So $\psi(f(a)) = g(a)$.

Proposition 1.1. If C has equalizers, then $\pi : A \rightarrow \text{im}(f)$ is an epimorphism.

Proof. Suppose

$$A \xrightarrow{\iota} \text{im}(f) \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} D$$

commutes. Then $\alpha \circ \pi = \beta \circ \pi$,



Then there is a unique $d : \text{im}(f) \rightarrow \text{eq}(\alpha, \beta)$, and $c \circ d = \text{id}$ and $d \circ c = \text{id}$ by uniqueness. So $(\text{im}(f), \text{id}_{\text{im}(f)})$ equalizes

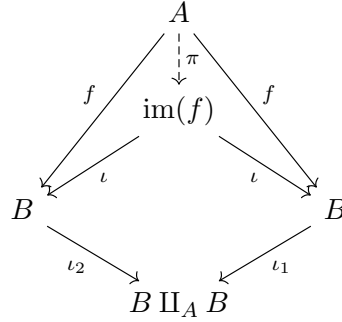
$$\text{im}(f) \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} D$$

so $\alpha = \beta$. □

Suppose that in \mathcal{C} , every morphism factors through an equalizer and the category has finite limits and colimits. Then $\text{im}(f)$ can be defined as the equalizer of the following diagram:

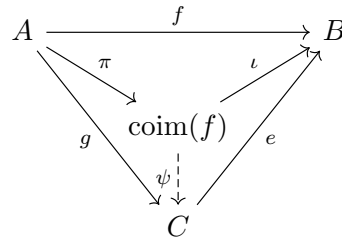
$$B \begin{array}{c} \xrightarrow{\iota_1} \\ \xrightarrow{\iota_2} \end{array} B \amalg_A B$$

We get the following diagram.

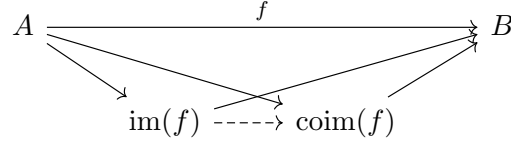


1.2 Coimages

Definition 1.2. The **coimage** $\text{coim}(f)$ of $f : A \rightarrow B$ is an object and a monomorphism $\pi : A \rightarrow \text{coim}(f)$ such that there exists $\iota : \text{coim}(f) \rightarrow B$ such that $\iota \circ \pi = f$ and such that if $g : A \rightarrow C$ is an epimorphism and $e : C \rightarrow B$ is such that $e \circ g = f$, then there exists a unique morphism $\theta : C \rightarrow \text{coim}(f)$ such that $\theta \circ g = \pi$.



So $\iota \circ \theta \circ g = \iota \circ \pi = f = e \circ g$. Since g is an epimorphism, $\iota \circ \theta = e$. How are the image and coimage related?



Definition 1.3. A morphism $f : A \rightarrow B$ is **strict** if $\text{im}(f) \rightarrow \text{coim}(f)$ is an isomorphism.

In Grp, Ring, Rmod, Set, and Top, $\text{im}(f)$ is the set theoretic image. The coimages are quotient objects (of A).

Example 1.2. In Set, $\text{coim}(f) = A / \sim$, where $a \sim a'$ if $f(a) \sim f(a')$. All the morphisms are strict.

Example 1.3. In Gp, $\text{coim}(f : C \rightarrow C') = G / \ker(f)$. $\text{im}(f) \subseteq f(G) \leq G'$. So the image and coimage are isomorphic, which is the first isomorphism theorem.

Example 1.4. In Ring, let $\ker(f)$ be the category theoretic kernel. Then $\text{coim}(f) = R / \ker(f) \xrightarrow{\sim} \text{im}(f)$ by the first isomorphism theorem.

Example 1.5. In the category of left R -modules, morphisms are also strict.

1.3 Generating sets

Definition 1.4. Let $\Phi : \mathcal{C} \rightarrow \text{Set}$ be a faithful functor, and let F be a left adjoint to Φ . Let $F_X = F(X)$ be the free object on X . If $X \xrightarrow{f} \Phi(A)$ for $A \in \mathcal{C}$, we get $\phi : F_X \rightarrow A$. Suppose $\text{im}(\phi)$ exists. Then $\text{im}(\phi)$ is called the **subobject of A generated by X** .

Example 1.6. In Gp, let $X \subseteq G$. Then $\langle X \rangle$ is the subgroup of G generated by X . This is $\text{im}(\phi : T_X \rightarrow G)$, where $\phi(x_1^{n_1} \cdots x_r^{n_r}) = x_1^{n_1} \cdots x_r^{n_r}$. So this is $\{x_1^{n_1} \cdots x_r^{n_r} : x_i \in X, n_i \in \mathbb{Z}, 1 \leq i \leq r, r \geq 0\}$. We claim that $\langle X \rangle$ is the smallest subgroup of G containing X , or equivalently, the intersection of all subgroups of G containing X . Indeed, this is a subgroup of G containing X , and any subgroup of G containing X must contain these words, since it must be closed under products.

Example 1.7. In Rmod, if $X \subseteq A$, $R \cdot X = \{\sum_{i=1}^n r_i x_i : r_i \in R, x_i \in X, 1 \leq i \leq n, n \geq 0\}$. So $F_X = \bigoplus_{x \in X} R x \xrightarrow{\phi} A$, where $\phi(r \cdot x) = rx \in A$.

Example 1.8. In the category of (R, S) -bimodules, $RXS = \{\sum_{i=1}^n r_i x_i : r_i \in R, s_i \in S, x_i \in X, 1 \leq i \leq n, n \geq 0\}$. If we have the set of formal sums $RXS = \{\sum_{i=1}^n r_i x s_i : r_i \in R, s_i \in S, 1 \leq i \leq n, n \geq 0\}$ with $(r + r')xs = rxs + r'xs$ and $rx(s + s') = rxs + rxs'$, then the free object is $\bigoplus_{x \in X} RxS$.

Ideals uses (R, R) -subbimodules of R generated by $X \subseteq R$.

Definition 1.5. The **ideal generated by** X is $(X) = \{\sum_{i=1}^n r_i x_i r'_i : r_i, r'_i \in R, x_i \in X\}$. If $X = \{x_1, \dots, x_n\}$, then we write (x_1, \dots, x_n) .

Remark 1.1. Even if $X = \{x\}$, we still need to take sums to get (x) .