# Math 255B Lecture 21 Notes

Daniel Raban

February 26, 2020

## 1 Properties of Spectral Measures

### 1.1 Total mass of spectral measures

Let $\varphi \in (C \cap L^\infty)(\mathbb{R})$ and let $A : D(A) \to H$ be self-adjoint. Last time, we had

$$\langle \varphi(A)u, v\rangle = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(\lambda) \left\langle (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))u, v \right\rangle \, d\lambda.$$

This is similar to the finite dimensional case, where

$$\varphi(A) = \sum_{\lambda \in \operatorname{Spec}(A)} \varphi(\lambda)\Pi_\lambda,$$

where $\Pi_\lambda$ is the orthogonal projection on to $\ker(A - \lambda)$.

**Remark 1.1.** Observe that $\varphi(A)^* = \overline{\varphi}(A)$:

$$\begin{aligned}
\langle \varphi(A)^* u, v\rangle &= \langle u, \varphi(A)v\rangle \\
&= \overline{\langle \varphi(A)v, u\rangle} \\
&= -\frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int \overline{\varphi(\lambda)\langle (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))u, v\rangle} \, d\lambda
\end{aligned}$$

Using $R(\lambda \pm i\varepsilon) - R(\lambda \mp i\varepsilon)$,

$$= \langle \overline{\varphi}(A)u, v\rangle .$$

We also introduced the spectral measures $d\mu_{u,v}$ with $\langle \varphi(A)u, v\rangle = \int \varphi(\lambda) \, d\mu_{u,v}(\lambda)$.

**Proposition 1.1.** *The total mass of the spectral measure $d\mu_{u,v}$ is*

$$\int d\mu_{u,v} = \langle u, v\rangle .$$

*Equivalently, $1(A) = I \in \mathcal{L}(H, H)$.*

*Proof.* By continuity and density, we may assume $u, v$ are in a dense subset of $H$; assume $u, v \in D(A)$. By polarization, we may take $u = v$. If $u \in D(A)$, then

$$(A - z)u = Au - zu$$

If $\operatorname{Im} z > 0$, then we get

$$u = R(z)Au - zR(z)u,$$

so we get

$$R(z)u = -\frac{1}{z}u + \frac{1}{z}R(z)Au.$$

If $z \to \infty$ with $\operatorname{Re} z$ fixed, we get $R(z)u = -\frac{1}{z}u + O(1/|z|^2)$. Recall from Nevannlinna's theorem that

$$\int d\mu_u = \lim_{z\to\infty} (-z \langle R(z)u, u\rangle) = \|u\|^2. \qquad \square$$

## 1.2 Decay of spectral measures

**Proposition 1.2.** *Let $\varphi \in C_B$ For all $u \in D(A)$ and $v \in H$,*

$$\langle \varphi(A)Au, v\rangle = \langle \varphi_1(A)u, v\rangle ,$$

*where $\varphi_1(\lambda) = \lambda\varphi(\lambda) \in C_B$.*

*Proof.* The left hand side is

$$\text{LHS} = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int \varphi(\lambda) \langle (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))u, v\rangle$$

Note that $(R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))u = u + (\lambda + i\varepsilon)R(\lambda + i\varepsilon)u - (\lambda - i\varepsilon)R(\lambda - i\varepsilon)u$.

$$= \langle \varphi_1(A)u, v\rangle + \lim_{\varepsilon\to 0^+} \frac{i\varepsilon}{2\pi i} \int \varphi(\lambda) \langle (R(\lambda + i\varepsilon) + R(\lambda - i\varepsilon))u, v\rangle \, d\lambda$$

To show that the right term equals 0, we have

$$O(\varepsilon) \left| \int \varphi(\lambda) \langle R(\lambda \pm i\varepsilon u, v\rangle \, d\lambda \right| \le O(\varepsilon) \int |\varphi(\lambda)| \|R(\lambda \pm i\varepsilon)u\| \, d\lambda$$

By Cauchy-Schwarz,

$$\le O(\varepsilon) \left( \int \|R(\lambda \pm i\varepsilon u)\|^2 \, d\lambda \right)$$

Recall that $\operatorname{Im} \langle R(\lambda + i\varepsilon)u, u\rangle = \varepsilon \|R(\lambda + i\varepsilon)u\|^2$.

$$\le O(\varepsilon^{1/2}) \left( \int \operatorname{Im} \langle R(\lambda + i\varepsilon)u, u\rangle \, d\lambda \right)^{1/2}$$

$$\xrightarrow{\varepsilon\to 0} 0.$$

We get that $\langle \varphi(A)Au, v\rangle = \langle \varphi_1(A)u, v\rangle$. $\square$

In particular, if $\varphi \in C_0(\mathbb{R})$, we have

$$\langle \varphi(A)Au, Au\rangle = \langle \varphi_1(A)u, Au\rangle = \langle u, \overline{\varphi}_1(A)Au\rangle = \left\langle u, \overline{\psi}(A)u\right\rangle,$$

where $\psi(\lambda) = \lambda^2\varphi(\lambda)$. We get

$$\langle \varphi(A)Au, Au\rangle = \langle \psi(A)u, u\rangle\,.$$

On the level of spectral measures, we get

$$\int \varphi(\lambda)\, d\mu_{Au}(\lambda) = \int \lambda^2\varphi(\lambda)\, d\mu_u(\lambda).$$

If $0 \le \varphi \le 1$, then the left hand side is $\le \|Au\|^2$. Letting $\varphi \uparrow 1$, we get by Fatou's lemma:

$$\int \lambda^2\, d\mu_u(\lambda) < \infty.$$

By monotone convergence, we get

$$\int \lambda^2\, d\mu_u(\lambda) = \|Au\|^2 < \infty \qquad \forall u \in D(A).$$

### 1.3 Multiplicativity of the functional calculus

**Proposition 1.3.** *Let $\varphi, \psi \in C_0(\mathbb{R})$. Then $\varphi(A)\psi(A) = (\varphi\psi)(A)$.*

*Proof.* Let $\varphi_k(\lambda) = \lambda^k\varphi(\lambda)$ for $k = 1, 2, \ldots$. For $u \in H$ and $v \in D(A)$, we have:

$$\begin{aligned}\langle \varphi(A)u, Av\rangle &= \langle u, \overline{\varphi}(A)Av\rangle \\ &= \langle u, \overline{\varphi}_1(A)v\rangle \\ &= \langle \varphi_1(A)u, v\rangle\,.\end{aligned}$$

Thus, $\varphi(A)u \in D(A^*) = D(A)$ and $\varphi_1(A)u = A\varphi(A)u$. In particular, $\operatorname{im}\varphi(A) \subseteq D(A)$ for all $\varphi \in C_0$. So $\operatorname{im}\varphi_1(A) \subseteq D(A)$, so $\operatorname{im}\varphi(A) \subseteq D(A^2)$. Iterating this argument, we get that $\operatorname{im}\varphi(A) \subseteq D(A^j)$ for $j = 1, 2, \ldots$ and $\varphi_j(A) = A^j\varphi(A)$ for any $j$. When $p$ is a polynomial, we get $p(A)\varphi(A) = (p\varphi)(A)$.

The idea is to let $\chi \in C_0(\mathbb{R})$ with $0 \le \chi \le 1$ be such that $\chi = 1$ on $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi)$. Pick a sequence of polynomials $p_j$ such that $\overline{p}_j\chi \to \psi$ uniformly. Then $(p_j\chi)(A) \to \psi(A)$ in $\mathcal{L}(H, H)$. We will give the details next time. □