Math 249 Lecture 34 Notes

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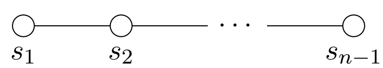
1 Dynkin Diagrams and the Classification of Coxeter Groups

1.1 Dynkin diagrams

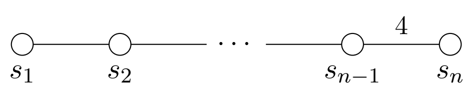
Last time we showed that the simple reflections generate a Coxeter group. This means that the simple reflections encode all the information about Coxeter groups.

We make diagrams of the simple reflections in a Coxeter group. We make each simple reflection a node and write the degree of $s_i s_j$ on the edge connecting s_i and s_j. The conventions are that an absent edge implies an edge with label 2, and an unlabeled edge is implicitly labeled with a 3.

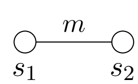
Example 1.1. Let $G = S_n$. The diagram is



Example 1.2. Let $G = B_n$. The diagram is



Example 1.3. Let $G = D_{2m}$. The diagram is



If you take two diagrams and place them together, you have sets of reflections that commute with each other; i.e. you get the product of Coxeter groups, which is indeed a Coxeter group.

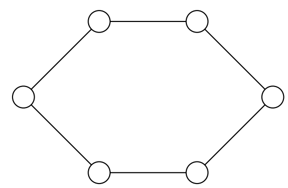
Definition 1.1. A *decomposable* Coxeter group is a Coxeter group that splits as a product of Coxeter groups.

A Coxeter group is decomposable iff its diagram has more than 1 connected component.

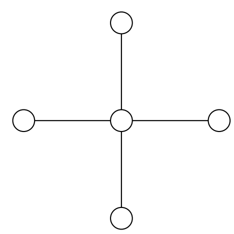
1.2 Restrictions on diagrams

Some possible diagrams do not correspond to Coxeter groups. These diagrams violate the condition that the inner product of normal vectors to each hyperplane must all be positive (or require the inner product to be not positive definite). Here are some forbidden diagrams for finite G.

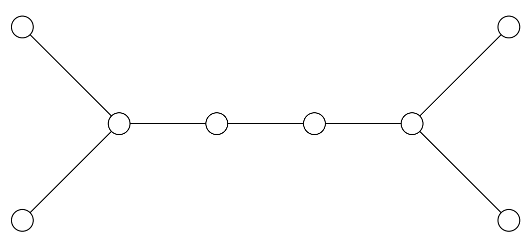
1. There cannot be loops in the diagram.



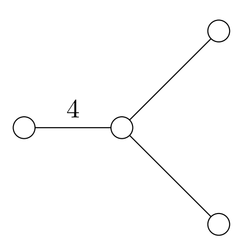
2. A vertex cannot have 4 neighbors.



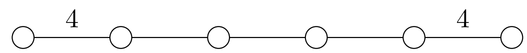
3. You cannot have 2 trivalent vertices linked by a chain of edges.



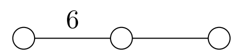
4. You cannot have a trivalent vertex with one edge labeled greater than 3.



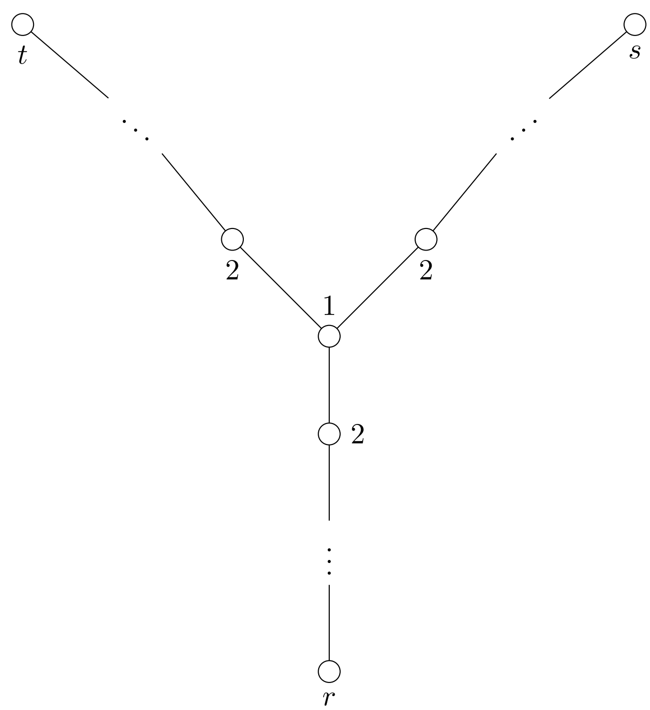
5. You cannot have two vertices linked by a chain starting and ending with labels greater than 3.



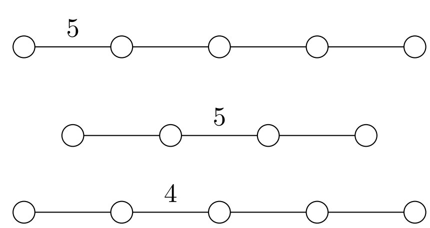
6. If you have more than two vertices linked together, you cannot have any number greater than 5.



7. If you have a trivalent tree with branch lengths r, s, and t (including the shared vertex), then you cannot have $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} \leq 1$.



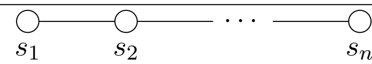
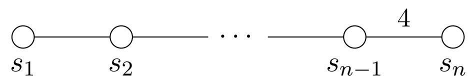
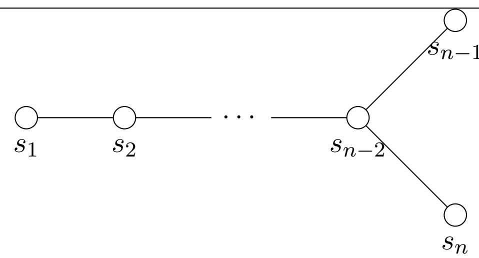
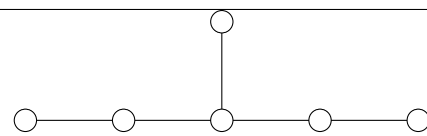
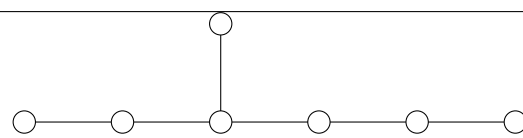
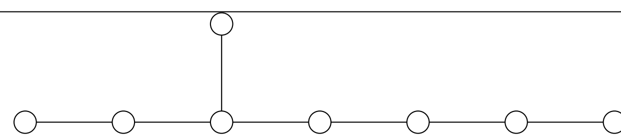
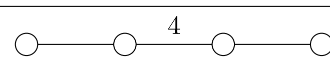
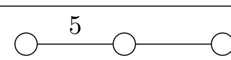
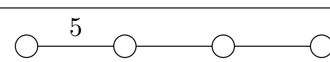
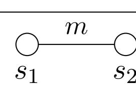
8. You also cannot have the following:



1.3 Classification of Coxeter groups

These turn out to be all the restrictions, and the remaining diagrams all correspond to Coxeter groups.

Theorem 1.1 (Classification of Coxeter groups)**.** *All indecomposable Coxeter groups are of one of these types:*

Type	*Dynkin Diagram*
A_n	
B_n (*or* C_n)	
D_n	
E_6	
E_7	
E_8	
F_4	
H_3	
H_4	
$I_2^{(m)}$	

2 Invariant functions

2.1 Homogeneous invariant functions and the covariant ring

Consider any finite group $G \circlearrowleft K^n$, where K has characteristic 0. Let $R = K[x_1, \ldots, x_n]$; we can think of R as polynomial functions on K^n. This induces an action $G \circlearrowleft R$ given by $(g \cdot f)(x) = f(g \cdot x)$. We can then consider R^G, the ring of invariant functions.

Example 2.1. If $G = S_n$ with the usual action, then we get the symmetric functions.

Remark 2.1. If you have an invariant function f, then f will have constant value over an orbit of the action of G on K^n. You can then think of invariant functions as functions on the set of orbits of the action, K^n/G.

Let $I_G \subseteq R$ be the ideal generated by $(R^G)_+$, the homogeneous invariant functions with $\deg(f) > 0$. R/I_G is called the *covariant ring*.

Proposition 2.1. *The homogeneous elements $f \in (R^G)_+$ generate I_g as an ideal iff they generate R^G as a K-algebra.*

Proof. $\Longleftarrow$: $f_{i_1}, \ldots, f_{i_k}$ span $(R^G)_+$, so the products generate I_G. Then the f_i generate I_G because the ideal generated by the products is contained in the ideal the f_i generate.

$\Longrightarrow$: Assume that $f_1, \ldots, f_k$ generate I_G. Let $R' = K[f_1, \ldots, f_k] \subseteq R^G$. We want to show that $R^G \subseteq R'$. Pick $f \in R^G$ with $\deg(f) = d > 0$. Then $f = \sum a_i f_i$, and without loss of generality, $\deg(a_i) = d - d_i$ and $a_i \in R^G$; this is because we can apply the reynolds operator to both sides, getting $f = \sum_i (Ra_i) f_i$. So by induction, we can assume that $a_i \in R'$. So $f \in R'$. $\square$