Math 246B Lecture 2 Notes

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1 Mean Value Property and Maximum Principles of Harmonic Functions

1.1 Solving the Dirichlet problem

Last time, given $f \in C(|x| = R)$, we wanted to find a $u \in C^2(|x| < R) \cap C(|x| \leq R)$ such that $\delta = 0$ in $|x| < R$ and $u = f$ on $|x| = R$. We defined

$$u(x) = \frac{1}{2\pi R} \int_{|y|=R} P_R(x,y) f(y) \, ds(y), \qquad |x| < R.$$

Then u is harmonic in the disc $|x| < R$, and we need to show that $u \in C(\||x| \leq R)$. Let's finish this proof.

Proof. When $0 < \rho < 1$, we let $u_\rho = u(\rho x)$ and show that $u_\rho \to f$ uniformly on $|x| = R$ as $\rho \to 1$. Given $\varepsilon > 0$, let $\delta > 0$ be such that if $|y| = |\tilde{y}| = R$ and $|y - \tilde{y}| \leq \delta$, then $|f(y) - f(\tilde{y})| \leq \varepsilon$. Let $\rho_1 < 1$ be such that if $|x| = R$, $|y| = R$, and $|x - y| \geq \delta$, then $\rho_1 < \rho < 1 \implies P_R(\rho x, y) \leq \varepsilon$. We get

$$\begin{aligned} u_\rho(x) - f(x) &= \frac{1}{2\pi R} \int_{|y|=R} P_R(\rho x, y)(f(y) - f(x)) \, ds(y) \\ &= \frac{1}{2\pi R} \left(\int_{\substack{|y|=R \\ |y-x| \leq \delta}} + \int_{\substack{|y|=R \\ |y-x| \geq \delta}} \right) \\ &= I_1 + I_2. \end{aligned}$$

Note that $|I_1| \leq \varepsilon$. When $\rho_1 < \rho < 1$ we get

$$|I_2| \leq \frac{1}{2\pi R} \int_{\substack{|y|=R \\ |y-x| \geq \delta}} P_R(\rho x, y) |f(y) - f(x)| \, ds(y) \leq 2M\varepsilon,$$

where $M = \max_{|y|=R} |f(y)|$. We get that

$$|u_\rho(x) - f(x)| \leq (1 + 2M)\varepsilon$$

for $\rho_1 < \rho < 1$ and $|x| = R$. Next, if $|x| < R$,

$$|u_\rho(x) - u(x)| = \left| \frac{1}{2\pi R} \int_{|y|=R} P_R(x,y)(u_\rho(y) - f(y))\, ds(y) \right| \leq \max_{|y|=R} |u_\rho - f| \xrightarrow{\rho \to 1} 0.$$

We get that $u_\rho \to u$ uniformly on $|x| \leq R$, as $\rho \to 1$. The u_ρ are continuous on $|x| \leq R$, so $u \in C(|x| \leq R)$. □

1.2 Mean value property

Harmonic functions enjoy the following unique continuation principle:

Proposition 1.1. *If $\Omega \subseteq \mathbb{R}^2$ is a domain, $u \in H(\Omega) = \{$harmonic functions on $\Omega\}$, and $u|_\omega = 0$ for nonempty open $\omega \subseteq \Omega$, then u vanishes identically.*

Proposition 1.2 (Mean value property of harmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open, $u \in H(\Omega)$, and $\{|x-a| \leq R\} \subseteq \Omega$. Then*

$$u(a) = \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y).$$

Proof. Take $x = a$ in the Poisson formula. □

1.3 Maximum principles of harmonic functions

Theorem 1.1 (maximum principle)**.** *Let $\varnothing \neq \Omega \subseteq \mathbb{R}^2$ be open and bounded with $u \in H(\Omega) \cap C(\overline{\Omega})$. Then for every $x \in \overline{\Omega}$,*

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u.$$

Proof. It suffices to show the result for the maximum; then replace u by $-u$. Let $M = \max_{\overline{\Omega}} u$, and consider the compact set $E = \{x \in \overline{\Omega} : u(x) = M\}$. We have to show that $E \cap \partial\Omega \neq \varnothing$. If $E \cap \partial\Omega = \varnothing$, take $a \in E$ at the smallest positive distance to $\partial\Omega$; this distance exists because E and $\partial\Omega$ are disjoint compact sets. Take $R > 0$ such that $\{|x-a| \leq R\} \subseteq \Omega$. Then $u < M$ on an open arc contained in $\{|x-a| = R\}$. On the other hand, by the mean value property,

$$M = u(a) = \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y) < \frac{1}{2\pi R} \int_{|y|=R} M\, ds(y) = M.$$

This is a contradiction. □

There exists a local version of the maximum principle:

Theorem 1.2. *If $u \in H(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$, and u has a local maximum at $a \in \Omega$, then u is constant in the component of a.*

Theorem 1.3 (Hopf's maximum principle)**.** *Let $D = \{|x| < 1\}$ and let $u \in H(D) \cap C(\overline{D})$. Let $x \in \partial D$ be such that $u(x) = \max_{\overline{D}} u$. Then the normal derivative of u at x*

$$N_x = \lim_{t \to 0^-} \frac{u(x + tx) - u(x)}{t} = \lim_{t \to 1^-} \frac{u(tx) - u(x)}{t - 1}$$

exists (in the sense that $N_x \in [0, \infty]$), and

$$0 \leq u(x) - u(z) \leq 2\frac{1 + |z|}{1 - |z|} N_x$$

for $|z| < 1$.

Proof. For $0 < t < 1$, write

$$u(tx) = \frac{1}{2\pi} \int_{|y|=1} P(tx, y) u(y)\, ds(y).$$

So

$$\begin{aligned} u(tx) - u(x) &= \frac{1}{2\pi} \int_{|y|=1} P(tx, y)(u(y) - u(x))\, ds(y) \\ &= \frac{1}{2\pi} \int_{|y|=1} \frac{1 - t^2}{|tx - y|^2} (u(y) - u(x))\, ds(y). \end{aligned}$$

Then the difference quotient is

$$\frac{u(tx) - u(x)}{t - 1} = \frac{t + 1}{2\pi} \int_{|y|=1} \frac{u(x) - u(y)}{|tx - y|^2}\, ds(y).$$

Let $t \to 1$. The first case is when $\liminf_{t \to 1^-} \frac{u(tx) - u(x)}{t - 1} < \infty$. By Fatou's lemma,

$$\frac{t + a}{2\pi} \int \liminf_{t \to 1^-} \frac{u(x) - u(y)}{|tx - y|^2}\, ds < \infty.$$

It follows that $y \mapsto u(x) - u(y)/|x - y|^2 \in L^1(\partial D)$. Try to apply dominated convergence to the above:

$$|x - y| \leq |tx - y| + |(1 - t)x| = |tx - y| + 1 - t \leq 2|tx - y|.$$

We get that

$$\frac{u(x)-u(y)}{|tx-y|^2} \leq 4\frac{u(x)-u(y)}{|x-y|} \in L^1(y),$$

and by dominated convergence, we get

$$\frac{u(tx)-u(x)}{t-1} \to \frac{1}{\pi}\int_{|y|=1} \frac{u(x)-u(y)}{|x-y|^2}\, ds(y) < \infty.$$

Case 2 is when $\liminf_{t\to 1^-} \frac{u(tx)-u(x)}{t-1} = \infty$. In this case, $N_x = \infty$. We see also that $N_x > 0$ unless u is constant. $\square$

Remark 1.1. It follows that $N_x > 0$ unless u is constant.