Statistics 210A Lecture 2 Notes

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1 Estimation and Introduction to Exponential Families

1.1 Review of measure theory

Last time, we introduced some ideas from measure theory. Let's review: A measure μ assigns a "weight" to subsets $A \subseteq \mathcal{X}$ (for $A \in \mathcal{F}$).

Example 1.1. The **counting measure** is $\#(A) = \operatorname{card}(A)$.

Example 1.2. Lebesgue measure gives $\lambda(A) = \text{vol}(A)$ (in \mathbb{R}^n).

Example 1.3. The Gaussian distribution gives $P(A) = \int_A \phi(x) dx$.

Measures give rise to integrals:

$$\int f(x) \, d\mu(x) = \begin{cases} \mu(A) & f(x) = \mathbbm{1}_{\{x \in A\}} \\ \sum_i c_i \mu(A_i) & f(x) = \sum_i c_i \mathbbm{1}_{\{x \in A_i\}} \\ \text{limit} & f(x) \text{ nice enough.} \end{cases}$$

If $P \ll \mu$ (meaning $\mu(A) = 0 \implies P(A) = 0$), there is a **density** p(x) with $p: \mathcal{X} \to [0, \infty)$ such that $\int f dP = \int f p d\mu$ for all (nice) f.

The **outcome space** Ω containing outcomes ω is equipped with a measure \mathbb{P} . Random variables are functions with $X(\omega) \in \mathcal{X}$ (e.g. $\mathcal{X} = \mathbb{R}$). You can think of X "decoding" the randomness ω to tell you what the value in our nicer space \mathcal{X} is. We write $X \sim Q$ if $\mathbb{P}(X \in B) = Q(B)$.

1.2 Estimation

In statistics, there are multiple possible distributions that could have generated the data.

Definition 1.1. A statistical model is a family of candidate probability distributions $\mathcal{P} = \{P_{\theta} : \theta \in \Theta\}$ for a random variable $X \sim P_{\theta}$. X is called the **data**, and θ is called the **parameter**.

The data X is observed by the statistical analyst, whereas θ is unobserved by the analyst. For now, θ is fixed and unknown.¹ The goal of estimation is to observe $X \sim P_{\theta}$ and guess the value of some estimand $g(\theta)$.

Example 1.4. Flip a biased coin n times. The parameter $\theta \in [0,1]$ is the probability of heads, and $X \sim \text{Binom}(n,\theta)$ is the number of heads after n flips. X has a density $p_{\theta}(x) = \theta^{x}(1-x)^{n-x}\binom{n}{x}$ for $x = 0, 1, \ldots, n$ (this is a density with respect to counting measure on $\{0, 1, \ldots, n\}$).

Definition 1.2. A statistic is any function T(X) of X.

In particular, a statistic is not a function of θ . It is something the statistical analyst can calculate.

Definition 1.3. An **estimator** $\delta(X)$ of $g(\theta)$ is a statistic intended to guess $g(\theta)$.

Example 1.5. In our coin flipping example, the natural estimator for θ is $\delta_0(X) = X/n$.

1.3 Loss and risk

How can we tell if an estimator is good?

Definition 1.4. The loss function $L(\theta, d)$ measures how badly an estimate is.

Example 1.6. One important loss function is the **squared error loss** $L(\theta, d) = (d - g(\theta))^2$.

Usually, $L(\theta, d) \ge 0$ for all θ, d with $L(\theta, g(\theta)) = 0$.

Definition 1.5. The **risk function** $R(\theta; \delta(\cdot)) = \mathbb{E}_{\theta}[L(\theta, \delta(X))]$ is the expected loss as a function of θ .

Remark 1.1. The \mathbb{E}_{θ} notation refers to the expectation with respect to X, where θ is the true parameter. This is in contrast to other disciplines which use the notation \mathbb{E}_X to denote what variables we are conditioning on in the expectation. We will use the notation $\mathbb{E}[f(X,X')\mid X']$ when we want to only integrate over certain random variables.

Example 1.7. The **mean squared error** is the risk function $MSE(\theta, \delta_0(\cdot)) = \mathbb{E}_{\theta}[(\delta(x) - \theta)^2].$

Example 1.8. In our coin flipping example, we have the estimator $\delta_0(X) = X/n$ with $\mathbb{E}_{\theta}[X/n] = \theta$ (this is an **unbiased estimator**). The loss is

$$MSE(\theta, \delta_0(\cdot)) = \mathbb{E}_{\theta}[(\delta(x) - \theta)^2]$$

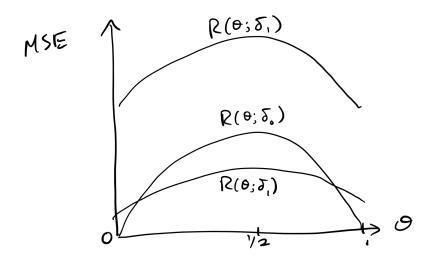
This is a frequentist perspective. With a Bayesian perspective, we may assume that θ follows some distribution.

$$= \operatorname{Var}_{\theta}(X/n)$$
$$= \frac{\theta(1-\theta)}{n}.$$

Here are other choices of estimators. We could take

$$\delta_1(X) = \frac{X+3}{n}.$$

$$\delta_2(X) = \frac{X+3}{n+6}$$



There is no estimator which is always the best; if $\theta = 3/4$, then the constant estimator $\delta(X) = 3/4$ would be better than any estimator which has a chance of suggesting anything other than 3/4.

1.4 Comparing estimators

Definition 1.6. An estimator $\delta(X)$ is **inadmissible** if there exists another estimator $\delta^*(X)$ such that

- (a) $R(\theta; \delta^*) \leq R(\theta; \delta)$ for all θ ,
- (b) $R(\theta; \delta^*) < R(\theta; \delta)$ for some θ .

In our previous example, δ_0 rendered δ_1 inadmissible. Here are some strategies to resolve the ambiguity:

1. Summarize $R(\theta)$ by a scalar:

- (a) Average-case risk: Minimize $\int_{\Theta} R(\theta; \delta) d\Lambda(\theta)$. The minimizer δ is called the Bayes estimator.
- (b) Worse-case risk: Minimize $\sup_{\theta \in \Theta} R(\theta; \delta)$. The minimizer δ is called the minimax estimator.
- 2. Constrain the choice of estimator:
 - (a) Only consider **unbiased** $\delta(X)$ ($\mathbb{E}_{\theta}[\delta(X)] = g(\theta)$).

1.5 Exponential families

Definition 1.7. An s-parameter exponential family is a family $\mathcal{P} = \{P_{\eta} : \eta \in \Xi\}$ with densities $p_{\eta}(x)$ with respect to a common dominating measure μ on \mathcal{X} of the form

$$p_{\eta}(x) = e^{\eta^{\top} T(x) - A(\eta)} h(x),$$

where

- $T: \mathcal{X} \to \mathbb{R}^s$ is called the **sufficient statistic**,
- $h: \mathcal{X} \to [0, \infty)$ is called the **carrier/base density**,
- $\eta \in \Xi \subseteq \mathbb{R}^s$ is called the **natural parameter**,
- $A: \mathbb{R}^s \to \mathbb{R}$ is called the **cumulant generating function** (or the **normalizing constant**).

Remark 1.2. $A(\eta)$ is totally determined by h, T, since we always must have $\int_{\mathcal{X}} p_{\eta} d\mu = 1$ for all η . So we can solve

$$A(\eta) = \log \left[\int_{\eta} e^{\eta^{\top} T(x)} h(x) d\mu(x) \right] \le \infty.$$

Definition 1.8. p_{η} is normalizable if $A(\eta) < \infty$. The natural parameter space is $\Xi_1 = \{\eta : A(\eta) < \infty\}$. We say \mathcal{P} is in canonical form if $\Xi = \Xi_1$.

Remark 1.3. $A(\eta)$ is a convex function, so Ξ_1 is a convex set.

In general, you can think of an s-parameter exponential family as describing an s-dimensional hyperplane in the space of log-densities.

