Math 210C Lecture 7 Notes

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April 15, 2019

1 Krull Dimension, Height, and Dedekind Domains

1.1 Krull dimension of polynomial rings

Definition 1.1. The **Krull dimension** of a ring R is the maximum length of an increasing chain of prime ideals.

Remark 1.1. If $R \to R/I$ is a surjection, then $\dim(R) \geq \dim(R/I)$.

Remark 1.2. If $S \subseteq R$ is multiplicatively closed, $\dim(S^{-1}R) \leq \dim(R)$.

Remark 1.3. If $\mathfrak{p}$ is not minimal, then $\dim(R) \geq \dim(R/\mathfrak{p}) + 1$. In particular, for a field F, $\dim(F[x_1, \dots, x_n]) \geq n$.

Proposition 1.1. *Let F be a field. Then* $\dim(F[x_1, \dots, x_n]) = n$.

Proof. Let $(\mathfrak{p}_i)_{i=0}^m$ be a chain of primes in $R = F[x_1, \dots, x_n]$. Assume $\mathfrak{p}_0 = (0)$ and $\mathfrak{p}_m = \mathfrak{m}$ is maximal. Let $E = R/\mathfrak{m}$ is a field extension of F. E is also a finitely-generated F-algebra. This means that E/F has to be algebraic (and hence finite). Then $x_i \mapsto \alpha_i \in E$, so each α_i has a minimal polynomial $\alpha_i \in F[x]$. Then $g_i = f(x_i) \in \mathfrak{m}$. Since $\mathfrak{p}_{m-1} \neq \mathfrak{m}$, $R/\mathfrak{p}_{m-1}$ is not a field. So there exists a k such that $g := g_k \notin \mathfrak{p}_{m-1}$ (since $\mathfrak{m} = (g_1, \dots, g_k)$); without loss of generality, $k = n$.

Let $S = R/(g) \cong F(\alpha_n)[x_1, \dots, x_{n-1}]$. By induction, $\dim(S) = n - 1$. Let $\overline{\mathfrak{p}_i}$ be the image of $\mathfrak{p}_i$ in S. Then $\overline{\mathfrak{p}_i} = \overline{\mathfrak{p}_{i+1}}$ iff $\mathfrak{p}_{i+1} = \mathfrak{p}_i + (g)$. But $g \notin \mathfrak{p}_{m-1}$, so $i \geq m-1$. So $i = m-1$. Then we have $\overline{\mathfrak{p}_0} \subsetneq \overline{\mathfrak{p}_1} \subsetneq \cdots \subsetneq \overline{\mathfrak{p}_{m-1}}$, which gives that $m - 1 \leq n - 1$. So we get $m \leq n$. ☐

Theorem 1.1. *Let R be a noetherian domain. Then* $\dim(R[x]) = \dim(R) + 1$.

1.2 Height of prime ideals

Definition 1.2. The **height** of a prime $\mathfrak{p}$ is $\max\{n : \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}, \mathfrak{p}_i \text{ prime}\}$.

Remark 1.4. The height of a prime is precisely $\dim(R_{\mathfrak{p}})$.

Example 1.1. Let $R = K[x_1, \ldots, x_n]$, where K is algebraically closed. If $V \subseteq \mathbb{A}_K^n$ is an algebraic set, then we can consider $K[V] = R/I(V)$. Then we can define $\dim(V) := \dim(K[v])$. This notion of dimension corresponds with your intuition for dimension.

If $V = V(\mathfrak{p})$, then $K[V] = R/\mathfrak{p}$, so $\dim(V(\mathfrak{p})) = \dim(R/\mathfrak{p})$. This is inversely related to the height of $\mathfrak{p}$. We have $n = \dim(R/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p})$, so the height can be thought as a notion of codimension.

Example 1.2. In $F[x_1, \ldots, x_n]$, $\mathrm{ht}((x_1, \ldots, x_k)) = k$.

Example 1.3. If R is a UFD, then $\mathrm{ht}(\mathfrak{p}) = 1$ iff $\mathfrak{p} = (f)$, where f is irreducible.

Example 1.4. If R is noetherian, then $\mathrm{ht}(\mathfrak{p}) = 0$ iff $\mathfrak{p}$ is an isolated prime of (0).

1.3 Krull dimension in integral extensions

Recall the going up theorem. If B/A is an integral extension of domains, then $\dim(B) \geq \dim(A)$.

Lemma 1.1. *Suppose B/A is an integral extension, where A is an integral domain. If $\mathfrak{b} \subseteq B$ contains a nonzero element that is not a zero divisor, then $\mathfrak{b} \cap A \neq (0)$.*

Proof. Let $\beta \in \mathfrak{B}$ be nonzero and not a zero divisor. There exists a monic $g \in A[x]$ such that $g(\beta) = 0$. Then $g = x^n f$, where $f(0) \neq 0$. Then $f(\beta) = 0$. Then $f(0) \in \mathfrak{b} \cap A$ because $f(0) = -\sum_{i=1}^n a_i \beta^i$ $\square$

Proposition 1.2. *If B/A is an integral extension of domains,* $\dim(A) = \dim(B)$.

Proof. Let $n = \dim(B)$. We already know $n \geq \dim(A)$. Let $(\mathfrak{q}_i)_{i=0}^n$ be a chain of primes in B of maximal length. Then $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. By the lemma, $\mathfrak{p}_i \neq (0)$ for all $i \geq 1$. So $\dim(B/\mathfrak{q}_1) = \dim(B) - 1$. Also, $\dim(A/\mathfrak{p}_1) \leq \dim(A) - 1$. On the other hand, $(B/\mathfrak{q}_1)/(A/\mathfrak{p}_1)$ is an integral extension of domains, and $\dim(B/\mathfrak{q}_1) = n - 1$. By induction, $n - 1 = \dim(A/\mathfrak{p}_1) \leq \dim(A) - 1$. So $n \leq \dim(A)$. $\square$

1.4 Dedekind domains

Definition 1.3. A **Dedekind domain** is a noetherian domain which is integrally closed and of Krull dimension ≤ 1.

Remark 1.5. Having Krull dimension 1 can be rephrase as the condition that every nonzero prime ideal is maximal.

Lemma 1.2. *Every PID is a Dedekind domain.*

Proof. PIDs are noetherian. They are integrally closed because PIDs are UFDs, and UFDs are integrally closed. They also have Krull dimension 1. $\square$

Proposition 1.3. *Let A be a Dedekind domain. Let $K = Q(A)$, and suppose that L/K is finite, separable. Let B be the integral closure of A in L. Then B is a Dedekind domain.*

Proof. B is finitely generated as an A-module, and A is noetherian, so B is noetherian. B is integrally closed by definition. And $\dim(B) = \dim(A) \leq 1$. $\square$

Corollary 1.1. *If K is a number field, let O_K be its ring of integers. Then O_K is a Dedekind domain.*