Math 255A Lecture 15 Notes

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1 Perturbation of Fredholm Operators and the Logarithmic Law

1.1 Perturbation of Fredholm Operators

Last time, we were showing that Fredholm operators are stable under small perturbations. Let's finish the proof.

Theorem 1.1. *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator. If $S \in \mathcal{L}(B_1, B_2)$ is such that $\|S\|$ is sufficiently small, then $T + S$ is Fredholm and* $\operatorname{ind}(T+S) = \operatorname{ind}(T)$.

Proof. We take a "Grushin approach." Let $\mathcal{P} : B_1 \otimes \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}$ be

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix},$$

where $n_+ = \dim(\ker(T))$, $n_- = \dim(\operatorname{coker}(T))$, $R_- : \mathbb{C}^{n_-} \to B_2$ is injective, and $R_+ : B_1 \to \mathbb{C}^{n_+}$ is surjective. Then $\mathcal{P}$ is invertible, so

$$\tilde{\mathcal{P}} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix}$$

is also invertible with the inverse $\mathcal{E} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus C^{n_-}$ given by

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

We have

$$\tilde{\mathcal{P}}\mathcal{E} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_+E_+ \end{bmatrix},$$

so R_+E_+ is the identity on $\mathbb{C}^{n_+}$. So E_+ is injective. Similarly,

$$\mathcal{E}\tilde{\mathcal{P}} = \begin{bmatrix} * & * \\ * & E_-R_- \end{bmatrix},$$

so E_- is surjective.

We now show that $T+S$ is Fredholm.

$$\begin{aligned} x \in \ker(T+S) &\iff (T+S)x = 0 \\ &\iff \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_+ \end{bmatrix} \\ &\iff \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} 0 \\ a_+ \end{bmatrix} = \begin{bmatrix} E_+a_+ \\ E_{-+}a_+ \end{bmatrix}, \end{aligned}$$

where $a_+ = R_+x \in \mathbb{C}^{n_+}$. We get that $x \in \ker(T+S)$ if and only if $x = E_+a_+$, were $a_+ \in \ker(E_{+-})$. Thus, $E_+ : \ker(E_{-+}) \to \ker(T+S)$ is surjective. So it is injective, since $\ker(E_{-+})$ is finite dimensional. So $\dim(\ker(T+S)) = \dim(\ker(E_{-+})) \leq n_+$. In particular, we get that $\dim(\ker(T+S)) \leq \dim(\ker(T))$.

Also,

$$\begin{aligned} (T+S)x = y &\iff \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ a_+ \end{bmatrix} \\ &\iff \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} y \\ a_+ \end{bmatrix} \\ &\iff x = Ey + e_+a_+, 0 = E_-y + E_{-+}a_+. \end{aligned}$$

Thus, $\operatorname{im}(T+S) = \{y \in B_2 : \exists a_+ \in \mathbb{C}^{n_+} \text{ s.t.} E_-y = -E_{-+}a_+\}$. We get a map from $B_2/\operatorname{im}(T+S) \to \mathbb{C}^{n_-}/\operatorname{im}(E_{-+})$ given by $y + \operatorname{im}(T+S) \mapsto E_-y + \operatorname{im}(E_{-+})$. The map is injective and surjective since E_- is surjective. We get $\dim(\operatorname{coker}(T+S)) = \dim(\operatorname{coker}(E_{-+})) < \infty$. Thus, $T+S$ is Fredholm and

$$\begin{aligned} \operatorname{ind}(T+S) &= \operatorname{ind}(E_{-+}) = \dim(\ker(E_{-+})) - \dim(\mathbb{C}^n/\operatorname{im}(E_{-+})) \\ &= n_+ - n_- = \dim(\ker(T)) - \dim(\operatorname{im}(T)) = \operatorname{ind}(T). \end{aligned}$$

□

Corollary 1.1. *The set $\{T \in \mathcal{L}(B_1, B_2) : T \text{ is Fredholm}\}$ is open in $\mathcal{L}(B_1, B_2)$, and the index is constant on each component of this set. Moreover,* $\dim(\ker(T))$ *is upper semicontinuous.*

1.2 The logarithmic law

Proposition 1.1. *Let $T_1 \in \mathcal{L}(B_1, B_2)$ and $T_2 \in \mathcal{L}(B_2, B_3)$ be Fredholm. Then $T_2T_1 \in \mathcal{L}(B_1, B_3)$ is also Fredholm, and we have "the logarithmic law"*

$$\operatorname{ind}(T_2T_1) = \operatorname{ind} T_2) + \operatorname{ind}(T_1).$$

Proof. Consider $T_1 : \ker(T_2T_1) \to \ker(T_2)$ sending $x \mapsto T_1x$. From linear algebra, we have $\dim(\ker(T_2T_2)/\ker(T_1)) \leq \dim(\ker(T_2)$. So

$$\dim(\ker(T_2T_1)) \leq \dim(\ker(T_1)) + \dim(\ker(T_1)) + \dim(\ker(T_2)).$$

Also, we have the exact sequence

$$B_2/\operatorname{im}(T_1) \xrightarrow{T_2'} B_3/\operatorname{im}(T_2T_1) \xrightarrow{q} B_3/\operatorname{im}(T_2)$$

where T_2' sends $x + \operatorname{im}(T_2) \mapsto T_2x + \operatorname{im}(T_2T_1)$, and q sends $x + \operatorname{im}(T_2T_1) \mapsto x + \operatorname{im}(T_2)$. So we have $\operatorname{im}(T_2') = \ker(q)$. It follows that $\dim(B_3/\operatorname{im}(T_2T_1)) < \infty$. So T_2T_1 is Fredholm.

To prove the logarithmic law, consider the family of operators $B_1 \oplus B_2 \to B_2 \oplus B_3$ given by

$$L(t) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \cos(t)I_2 & \sin(t)I_2 \\ -\sin(t)I_2 & \cos(t)I_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix},$$

where I_2 is the identity on B_2, and $t \in \mathbb{R}$. Then $L(t)$ is a product of 3 Fredholm operators and is Fredholm for each t.

The map $t \mapsto L(t)$ is continuous (w.r.t. the operator norm on $\mathcal{L}(B_1 \oplus B_2, B_2 \oplus B_3)$). Then $\operatorname{ind}(L(t))$ is locally constant, so it is constant. If $t = 0$, we get

$$L(0) = \begin{bmatrix} T_1 & 0 \\ 0 & 2 \end{bmatrix},$$

so $\operatorname{ind}(L(0)) = \operatorname{ind}(T_1) + \operatorname{ind}(T_2)$. If $t = -\pi/2$,

$$L(-\pi/2) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0 & -I_2 \\ T_2T_1 & 0 \end{bmatrix}.$$

That is,

$$L(-\pi/2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ T_2T_1x \end{bmatrix}.$$

Since $\operatorname{ind}(L(-\pi/2)) = \operatorname{ind}(T_2T_1)$, we get the logarithmic law. $\square$

1.3 Introduction to compact operators

Definition 1.1. A linear operator $T : B_1 \to B_2$ between Banach spaces is called **compact** if the closure of the image of the unit ball in B_1 is compact in B_2: $\overline{T(\{\|x\| \leq 1\})}$ is compact in B_2.

In other words, T is compact if and only if for $\|x_n\| \leq 1$, $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Also, compact operators are continuous.