Math 250A Lecture 14 Notes

Daniel Raban

October 12, 2017

1 The Snake Lemma

1.1 Statement and proof of the snake lemma

Example 1.1. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "f"', "\times 2"] & \mathbb{Z} \arrow[r] \arrow[d, "g"', "\times 2"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "h"', "\times 2"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} f \arrow[r, "\times 2"] \arrow[d] & \operatorname{coker} g \arrow[r] \arrow[d] & \operatorname{coker} h \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The map $\ker g \to \ker h$ is not surjective, and $\operatorname{coker} f \to \operatorname{coker} g$ is not injective. The snake lemma says that these are the same problem.

Lemma 1.1 (Snake)**.** *Suppose we have the following commutative diagram with exact rows:*

$$\begin{tikzcd}
 & A_0 \arrow[r] \arrow[d, "f"] & B_0 \arrow[r] \arrow[d, "g"] & C_0 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & B_1 \arrow[r] & C_1 &
\end{tikzcd}$$

Then there is a map $\ker h \to \operatorname{coker} f$ *that makes the "snake sequence"*

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

exact. This yields the commutative diagram[1]

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners] & \\
 & A_0 \arrow[r] \arrow[d, "f"] & B_0 \arrow[r] \arrow[d, "g"] & C_0 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[d] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h &
\end{tikzcd}$$

Proof. We first construct the snake homomorphism by zigzaging through the diagram. Take $c \in \ker h$; then $c \in C$, so since $B_0 \to C_0$ is surjective, we can lift c to an element $b \in B_0$. Then we can map b to $b' \in B_1$. Since c was in $\ker h$ and the diagram is commutative, $B_1 \to C_1$ sends b' to 0. So $b' \in \ker(B_1 \to C_1) = \operatorname{im}(A_1 \to B_1)$, and we can lift b' to $a' \in A_1$. Note that a' is unique (given b) because $A_1 \to B_1$ is injective. Finally, let a'' be the image of a' under the map $(A_1 \to \operatorname{coker} f)$. So we map $c \mapsto a''$.

Is this well-defined? We have a choice of possibly different b. Suppose we picked some b_0 instead of b, and let a'_1 be the corresponding element of A_1 we get. Note that $B_0 \to C_0$ sends $b - b_0$ to 0, so there exists some $a \in A_0$ such that $A_0 \to B_0$ maps a to $b - b_0$. Since the diagram is commutative, the map $A_1 \to B_1$ should send $f(a)$ to $g(b - b_0)$. Then since f is injective and $A_1 \to B_1$ sends $a' - a'_0$ to $g(b - b_0)$, we have that $a' - a'_0 = f(a)$; then we have $a' - a'_0 \in \operatorname{im}(f)$, so a' and a'_0 have the same image in $\operatorname{coker} f = A_1 / \operatorname{im} f$.

We claim that the snake sequence is exact. The hard part is exactness at $\ker h$ and $\operatorname{coker} f$. Suppose we want to prove exactness at $\operatorname{coker} f$. Suppose $a'' \in \operatorname{coker} f$ and is in the kernel of the map $\operatorname{coker} f \to \operatorname{coker} g$. Lift it to $a' \in A_1$, and let $b' \in B_1$ be the image of a'. b' maps to 0 in $\operatorname{coker} g$ by the definition of a'' (and because the diagram commutes), so lift it to $b \in B_0$. Map b to $c \in C_0$. Now note that $h(c) = 0$ because $g(b) = b' \in \operatorname{im}(A_1 \to B_1) = \ker(B_1 \to C_1)$. So $c \in \ker f$, and the snake homomorphism

[1]The code for this diagram was modified from an answer on this StackExchange post.

takes c to a'', so the sequence is exact at $\operatorname{coker} f$. The similar proof for $\ker h$ is left as an exercise. $\square$

1.2 Applications of the snake lemma

1.2.1 Exact sequences of tensor products of modules

Recall that if $0 \to A \to B \to C \to 0$ is exact, then so is

$$A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

However, $A \otimes M \to B \otimes M$ is not always injective. What is the kernel? Choose free modules F_i, H_i so that

$$0 \to F_1 \to F_0 \to A \to 0, \qquad 0 \to H_1 \to H_0 \to C \to 0.$$

Extend this to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d, "f"] & F_1 + H_1 \arrow[r] \arrow[d, "g"] & H_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & F_0 \arrow[r, "\times 2"] \arrow[d] & F_0 + H_0 \arrow[r] \arrow[d] & H_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Tensor every row with M and put in the kernels to get the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] & \\
0 \arrow[r] & F_1 \otimes M \arrow[r] \arrow[d, "f"] & (F_1 \otimes M) + (H_1 \otimes M) \arrow[r] \arrow[d, "g"] & H_1 \otimes M \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & F_0 \otimes M \arrow[r] \arrow[d] & (F_0 \otimes M) + (H_0 \otimes M) \arrow[r] \arrow[d] & H_0 \otimes M \arrow[r] \arrow[d] & 0 \\
 & A \otimes M \arrow[r] \arrow[d] & B \otimes M \arrow[r] \arrow[d] & C \otimes M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that the bottom row is the row of cokernels of the vertical maps f, g, h, so by the snake lemma, we get an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

we can also call these

$$0 \to \mathrm{Tor}(A, M) \to \mathrm{Tor}(B, M) \to \mathrm{Tor}(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

Is $\mathrm{Tor}(A, M)$ well-defined? It seems to depend on the choice of $0 \to F_1 \to F_0 \to A \to 0$. It is, in fact, well-defined.

Let's calculate $\mathrm{Tor}(M, N)$ for finitely generated abelian groups M, N. First, we have $\mathrm{Tor}(M_1 \oplus M_2, N) \cong \mathrm{Tor}(M_1, N) \oplus \mathrm{Tor}(M_2, N)$, so it is enough to do the case where M, N are cyclic. If $M = N = \mathbb{Z}$, take the resolution $0 \to F_1 \to F_0 \to M \to 0$. If $M = \mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
\\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0
\end{array}$$

So $\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$.

If $m = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
\\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times m} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & 0 \\
\\
0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\times m} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \cdots & \longrightarrow & 0
\end{array}$$

Then $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \ker(\mathbb{Z}/n\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/(m, n)\mathbb{Z}$.

So Tor(M,N) depends only on the torsion subgroups of M, N. In fact, if M, N are finite, $M \otimes N \cong Tor(M, N)$, although this isomorphism is not natrual.

Example 1.2. Here is a historical example from algebraic topology. This is where the idea of Tor came from. The universal coefficient theorem states that

$$H_i(M, G) = (H_i(M, \mathbb{Z}) \otimes G) \oplus \mathrm{Tor}(H_{i-1}(M, \mathbb{Z}), G),$$

where $H_i(M, G)$ is the homology of the manifold M with coefficients in G.

Example 1.3. As a specific case of the previous example, let $M = P^2$ (2-dimensional projective space). This is S^2, where we identify opposite points. Suppose we know $H_0(M,\mathbb{Z}) = \mathbb{Z}$, $H_1(M,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and $H_i(M,\mathbb{Z}) = 0$ for $i > 1$. Then

$$H_0(M,\mathbb{Z}/2\mathbb{Z}) = H_0(M,\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$$

$$H_1(M,\mathbb{Z}/2\mathbb{Z}) = H_1(M,\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_0(M,\mathbb{Z}),\mathbb{Z},2\mathbb{Z})$$
$$H_2(M,\mathbb{Z}/2\mathbb{Z}) = H_2(M,\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_1(M,\mathbb{Z}),\mathbb{Z},2\mathbb{Z}),$$

which allows us to compute the homology[2] $H_2(M,\mathbb{Z}/2\mathbb{Z})$.

1.2.2 The Mitag-Leffler condition

Look at $\cdots \to A_3 \to A_2 \to A_1 \to A_0$. Does the sequence of images stabilize? In other words, does $ImA_i = ImA_{i+1} = \cdots$ for some i?

Definition 1.1. Let $\cdots \to A_3 \to A_2 \to A_1 \to A_0$. The *Mitag-Leffler condition* is that the sequence of images stabilizes for all A_n; that is, for each $n \in \mathbb{N}$, ther exists some $i \geq n$ such that $\mathrm{im}\, A_i = \mathrm{im}\, A_{i+1} = \cdots$.

Example 1.4. The Mitag-Leffler condition holds if all A_i are finite.

Theorem 1.1. *Suppose we have*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & B_{i+1} \arrow[r] \arrow[d] & C_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_i \arrow[r] \arrow[d] & B_i \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

If the Mitag-Leffler condition is satisfied, then

$$0 \to \lim A_i \to \lim B_i \to \lim C_i \to 0.$$

[2] In the first edition of Lang's book, there was an infamous exercise that said, "Take any book on homological algebra, and prove all the theorems without looking at the proofs given in that book." Professor Borcherds seemed dismayed that the exercise was removed in a later edition of the book.

Proof. We first do two easy cases:

1. Suppose all maps $A_{i+1} \to A_i$ are onto (so ML condition is satisfied). We want to show that $\lim B_i \to \lim C_i$ is onto. Pick some element of $\lim C_i$, which looks like $(c_0, c_1, \dots)$ for $c_i \in C_i$, where c_i is the image of c_{i+1}. We can lift the c_i to b_i. Is b_i the image of b_{i+1}? Pick $b_0 \in B_0$, and choose some $b_1 \in B_1$. Then $\mathrm{im}(b_1) - b_0 \in \ker(B_0 \to C_0) = \mathrm{im}(A_0 \to B_0)$, so let $a_0 \in A_0$ be its preimage. Then we can lift a_0 to $a_1 \in A_1$. Now replace b_1 by $b_1 + \mathrm{im}(a_1)$. Repeat this to find $b_2, b_3, \dots$. So b_i maps to c_i and b_{i-1}.

2. Suppose for each i, we can find j so that $A_j \to A_i$ is 0 (this is the extreme opposite condition to case 1). Then the ML condition holds. We want to show that $\lim B_i \to \lim C_i$ iis onto. Pick A_{i_0}. Pick A_{i_1} so $A_{i_1} \to A_{i_0}$ is 0. Do the same over and over to get $\to Ai_2 \to A_{i_1} \to A_{i_0}$. Take the inverse limits over B_0, B_{i_1}, B_{i_2}, etc.. So we can assume all maps $A_{i+1} \to A_i$ are 0. Pick $(c_0, c_1, c_2, \dots)$, and pick b_i mapping to c_i. Is $\mathrm{im}(b_i) = b_{i-1}$? The image of $\mathrm{im}(b_2)$ is $\mathrm{im}(b_1)$ because $\mathrm{im}(b_2) - b_1$ is in the image of A_1, which is 0 in A_0. So the sequence $\mathrm{im}(b_1), \mathrm{im}(b_2), \mathrm{im}(b_3), \dots$ is in $\lim B_i$, and has image $(c_0, c_1, c_2, \dots)$.

Now we combine the special cases 1 and 2. Suppose A_i satisfied the ML condition. Put $X_i = \bigcap_{j \geq i} \mathrm{im}(A_j \to A_i)$. So $X_i \subseteq A_i$, and we get exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_i & \longrightarrow & A_i & \longrightarrow & A_i/X_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{i-1} & \longrightarrow & A_{i-1} & \longrightarrow & A_{i-1}/X_{i-1} & \longrightarrow & 0
\end{array}$$

where the down maps for the X_i are surjective. For each i, we can find j so that $\mathrm{im}(A_j/X_i \to A_i/X_i) = 0$.

Use the snake lemma. Recall that $0 \to A \to B \to C \to 0$ is exact implies that

$$A \otimes M \to B \otimes M \to C \otimes M \to 0$$

is exact and

$$\mathrm{Tor}(A, M) \to \mathrm{Tor}(B, M) \to \mathrm{Tor}(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

is exact.

Copy this argument since the limit is left exact. We do this by flipping all the arrows. We constructed Tor by taking $0 \to G_1 \to F_0 \to A \to 0$; this works when F is free or projective. So we can flip the arrows by replacing the projective modules by injective modules $0 \to A \to I_0 \to I_1 \to 0$; this uses our fact that every module is contained in an injective module.

So the analogue of Tor is $\lim^1(A_i)$. We get a sequence

$$0 \to \lim A_i \to \lim B_i \to \lim C_i \to \lim^1 A_i \to \lim^1 B_i \to \lim^1 C_i.$$

For this to be exact, we want $\lim^1 A_i = 0$. The proofs above show that this is true if either of the special cases hold. Now look at $0 \to X_i \to A_i \to A_i/X_i \to 0$. We have

$$0 \to \lim X_i \to \lim A_i \to \lim A_i/X_i \to \lim^1 X_i \to \lim^1 A_i \to \lim^1 A_i/X_i \to 0. \qquad \square$$

1.3 Unrelated: Finitely generated modules over a PID

Theorem 1.2. *Any finitely generate modules over PID are sums of cyclic modules of the form R/I.*

Proof. We don't have time in class to prove the whole theorem, so we will cheat and just do the case of Euclidean domains. The proof is the same as the one we gave for $\mathbb{Z}$. If M is any submodule of $\mathbb{Z}^n$, we can find a basis $b_1, \ldots, b_n$ of $\mathbb{Z}^n$. So M is spanned by $d_1b_1, d_2b_2, \ldots, d_nb_n$ for some d_i Then the finitely generated module $\mathbb{Z}^n/m = \bigoplus \mathbb{Z}/d_i\mathbb{Z}$. □