Math 222A Lecture 9 Notes

Daniel Raban

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1 Solutions to Hamilton-Jacobi Equations via Calculus of Variations

1.1 Recap: Connecting Hamilton-Jacobi equations to calculus of variations using the Lengende transform

Last time, we wanted to compare Hamilton-Jacobi equations to calculus of variations. The Hamilton-Jacobi equations are of the form

$$\begin{cases} u_t + H(x, \partial u) = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n \\ u(0) = u_0 & \text{in } \mathbb{R}. \end{cases}$$

The characteristics given to this equation are

$$\begin{cases} \dot{x} = H_p \\ \dot{p} = -H_x \\ \dot{z} = p \cdot H_p - H, \end{cases}$$

with initial data $x(0) = x_0$ and $p(0) = \partial u_0$. The first two equations are called the **Hamilton flow**.

In calculus of variations, we have a **Lagrangian** $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and we want to minimize an **action functional**

$$\min_{x \in \mathcal{A}} \underbrace{\int_0^T L(x, \dot{x})\, dt}_{\mathcal{L}(x)},$$

where $\mathcal{A} = \{x : [0, T] \to \mathbb{R} \text{ Lipschitz} \mid x(0) = x_0, x(T) = x_T\}$. Minimizers satisfy the **Euler-Lagrange equation**

$$L_x(x, \dot{x}) - \frac{d}{dt} L_q(x, \dot{x}) = 0.$$

Last time, we connected these two setups. We saw that

- L is strictly convex and coercive if and only if H is strictly convex and coercive.
- $$H(x,p) = \max_{q \in \mathbb{R}^n} -L(x,q) + p \cdot q,$$
  which is maximized at $p = L_q(x,q)$. This relation gives
  $$H(x,p) + L(x,q) \geq p \cdot q$$
  with equality when $p = L_q(x,q)$. This expression is symmetric in p and q, so it allows us to cast q in terms of p: $q = H_p(x,p)$. This relationship is known as the **Legendre transform**.

Remark 1.1. The Legendre transform well-defined and is an involution, only assuming convexity.

Example 1.1. If we remove strict convexity and coercivity, we can get functions which are not defined everywhere. For example, take

$$\begin{cases} L(0) = 0 \\ L(q) = \infty \quad q \neq 0. \end{cases}$$

What is H in this case?

We have not incorporated the initial data of the Hamilton-Jacobi equations into our calculus of variations. We will do this by adding $u_0(x_0)$ to the minimization problem (so when $T = 0$, we get $u_0(x_0)$) and removing the condition $x(0) = x_0$ from our set $\mathcal{A}$. So we are minimizing

$$\min_{x \in \mathcal{A}} \int_0^T L(x, \dot{x})\, dt + u_0(x_0) = u(T, x_T),$$

with $\mathcal{A} = \{x : [0,T] \to \mathbb{R} \text{ Lipschitz} \mid x(T) = x_T\}$.

1.2 Existence of minimizers for the Euler-Lagrange equation

We want to prove the following:

Theorem 1.1. *The minimal value function $u(T, x_T)$ is the calculus of variations is the solution to the Hamilton-Jacobi equations.*

First, we should ask: Does a minimum solution to the Euler-Lagrange equation exist? The answer is yes, as long as L is convex, coercive, and Lipschitz in x and if $u_0 \in \text{Lip}$. However, there is no guarantee of uniqueness. We will not prove this, but here is some intuition:

Here is the trivial case:

Proposition 1.1. *Suppose we have a continuous function $F : K \to \mathbb{R}$ with K compact. Then $\min F$ is attained.*

Proof. Let x_n be a minimizing sequence: $F(x_n) \to \inf F$. Then $x_n \to x_0$ along a subsequence. Then $F(x_n) \to F(x_0)$, so x_0 is the minimizer. □

What if we try to apply this to calculus of variations? Suppose we have a minimizing sequence $x_n : [0,T] \to \mathbb{R}^n$. Then $\mathcal{L}(x_n) \to u(T, x_T)$ but in what topology? Is x_n in a bounded set? We know that $\mathcal{L}(x_n)$ is bounded. If $L(x,q) = q^2$, for example, we cold conclude that $\int_0^T (\dot{x}_n)^2 \leq c$. Then $\dot{x}_n$ is bounded in $L^2([0,T])$. This would imply that x_n is bounded in $C^{1/2}$ using Hölder's inequality: $(|x_n(t) - x_n(s)| \leq c|t-s|^{1/2})$. This implies that x_n is equicontinuous (and equibounded by the $x(T) = x_T$ assumption). So the Arzelà Ascoli theorem says that $x_n \to x$ uniformly. Then

$$\lim_{n\to\infty} \mathcal{L}(x_n) = \lim_{n\to\infty} \int_0^T L(x_n, , \dot{x}_n)\, dt + \underbrace{u(x_{n,0})}_{\to u_0(x_0)}$$

We can pass to the limit without a problem for x, but convergence with respect to $\dot{x}$ is trouble.

The limit of the integral may not exist, but maybe we can hope for

$$\int_0^T L(x, \dot{x})\, dt \leq \liminf_{n\to\infty} \int_0^T L(x_n, \dot{x}_n)\, dt.$$

This is lower semicontinuity for the map $x \mapsto \mathcal{L}(X)$. The key observation is that convexity of $\mathcal{L}$ implies lower semicontinuity of $\mathcal{L}$:

Proof. The convexity inequality tells us that

$$L(\dot{x}_n) \geq L(\dot{x}) + L_q(\dot{x})(\dot{x}_n - \dot{x}).$$

Integrating gives us

$$\int_0^T L(\dot{x}_n)\, dt \geq \int_0^T L(\dot{x})\, dt + \int_0^T L_q(\dot{x})(\dot{x} - \dot{x}_n)\, dt$$

We are done if $\lim_{n\to\infty} \int L_q(\dot{x})(\dot{x}_n - \dot{x}) = 0$. We have replaced our nonlinear dependence on $\dot{x}_n - \dot{x}$ by a linear property.

Since $\dot{x} \in L^2$, we can approximate $L_q(\dot{x})$ by smooth functions. Suppose $y_k \in C^\infty$ with $y_k \to L(\dot{x})$ in L^2. It is enough to see that

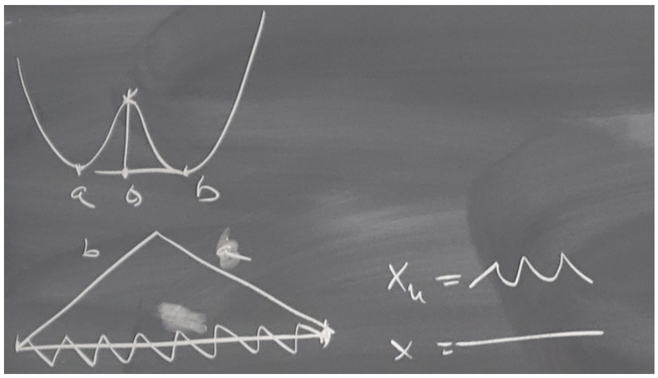
$$\lim_{n\to\infty} \int_0^T y_k(\dot{x}_n - \dot{x})\, dt = 0$$

In this context, we can integrate by parts. The integral equals

$$\int_0^T y_k(\dot{x}_n - x)\,dt = \int_0^T \dot{y}_k(x_n - x)\,dt + y_k(x_n - x)|_0^T \xrightarrow{n\to\infty} 0$$

by uniform convergence of $x_n \to x$. $\square$

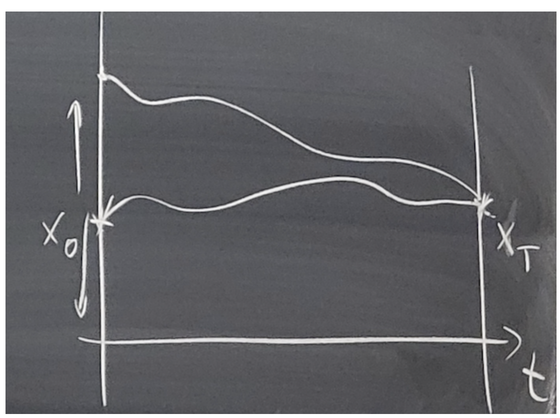
Example 1.2. Recall our double well potential.



In this case, if x_n is a wiggle approximating the 0 trajectory, we have $L(\dot{x}_n) = 0$ by $L(\dot{x}) = L(0) > 0$.

Remark 1.2. The Hamilton-Jacobi equation can be solved for a short time using characteristics. In calculus of variations, the analogue turns out to be that minimizers are unique for a short time.

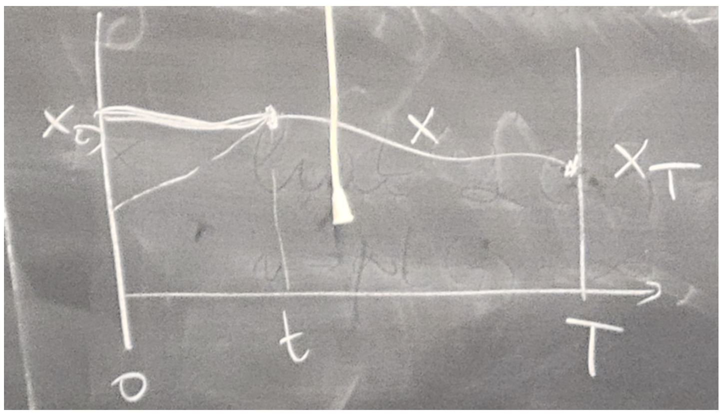
We want to think of two minimizers in calculus of variations as characteristics that intersect.



1.3 Proving that Euler-Lagrange equation minimizers solve Hamilton-Jacobi equations

Here is the "proof" of our theorem.

Proof. Suppose x is a minimizer for the action functional. We can choose a intermediate point t, and first minimize relative to the time t.



$$\min_x \int_0^T L(x, \dot{x})\, dt + u_0(x_0) = \min_x \int_0^t L(x, \dot{x})\, ds + u_0(x_0) + \int_t^T L(x, \dot{x})\, ds$$

If $x|_{[0,T]}$ is a minimizer, then $x|_{[0,t]}$ is also a minimizer. So

$$u(x_T, x_0) = \min u(x_t, x_0) + \int_t^T L(x, \dot{x})\, ds.$$

This is called the **dynamic programming principle**.[1] This principle tells us that for minimizers,

$$u(x_T, x_0) = u(x_t, x_0) + \int_t^T L(x, \dot{x})\, ds,$$

which we can differentiate with respect to t to get

$$\begin{aligned} \frac{d}{dt} u(x_t, x_0) &= L(x, \dot{x}) \\ &= p \cdot q - H(x, p) \\ &= p \cdot H_p - H. \end{aligned}$$

We conclude that $u(t, x_t)$ from the calculus of variations is the same as the $u(t, x_t)$ from the Hamilton-Jacobi equation because they solve the same equation with the same initial data at time 0. □

[1] This is discussed near the end of Evans' book.

Remark 1.3. This is not an entirely correct proof. How do we know that there is an optimal trajectory starting at x_0? If the time is short enough, we can guarantee a minimizer starting at x_0, but this is exactly the issue of uniqueness of minimizers. This proof can be made rigorous for short times.

Remark 1.4. More generally, this is related to **control theory**, where we try to find

$$u(x_0, T) = \min \int_0^T L(x, u)\, dt + u_0(x(0)), \qquad \dot{x} = h(x, f)$$

Here, we can choose some weight of influence by changing f, and we are trying to optimize some cost functional. The function $u(x_0, T)$ solves a Hamilton-Jacobi equation.

We can think of our calculus of variations problem as the case where the ODE for x is given by $\dot{x} = f$.

Remark 1.5. Calculus of variations allows us to obtain meaningful solutions for Hamilton-Jacobi equations after characteristics begin to intersect. Instead of picking which characteristic to continue, we can just look for a minimizer for a calculus of variations problem in longer time.