Math 246A Lecture 9 Notes

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1 Goursat's Theorem and Integration of 1-Forms

1.1 Goursat's theorem

Theorem 1.1 (Goursat)**.** *Let $f : \Omega \to \mathbb{C}$ be such that $f'(z)$ exists for all $z \in \Omega$. Let R be a rectangle such that $\overline{R} \subseteq \Omega$. Then $\int_{\partial R} f(z)\, dz = 0$.*

Corollary 1.1. *With f as above, $f(z) = F'(z)$ with $F \in H(\Omega)$. So f is continuous, and $f \in H(\Omega)$, as well.*

Proof. Assume $\oint_{\partial R} f(z)\, dz = \alpha \neq 0$. Split R into 4 rectangles. Then

$$\oint_{\partial R} f(z)\, fz = \sum_{j=1}^{4} \oint_{\partial R_j} f(z)\, dz$$

So there exists some j_1 such that

$$\left| \oint_{\partial R_{j_1}} f(z)\, dz \right| \geq \frac{|\alpha|}{4}.$$

Repeat this, spliiting R_{j_1} into 4 rectangles. We get a j_2 such that

$$\left| \oint_{\partial R_{j_1, j_2}} f(z)\, dz \right| \geq \frac{|\alpha|}{4^2}.$$

If we continue this, we get $R_{j_1, j_2, \dots, j_n}$ such that

$$\left| \oint_{\partial R_{j_1, j_2, \dots, j_n}} f(z)\, dz \right| \geq \frac{|\alpha|}{4^n},$$

and $\operatorname{diam}(R_{j_1, j_2, \dots, j_n}) = 2^{-n} \operatorname{diam}(\partial R)$.

Note that $\bigcap_{n=1}^{\infty} R_{j_1 j, j_2, \ldots, j_n} = \{z_0\}$ for some z_0. Then, since

$$\oint_{\partial R_{j_1, j_2, \ldots, j_n}} 1\, dz = \oint_{\partial R_{j_1, j_2, \ldots, j_n}} z\, dz = 0$$

and $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|)$, we get that

$$\begin{aligned}
|\alpha| &\le 4^n \left| \oint_{\partial R_{j_1, j_2, \ldots, j_n}} f(z)\, dz \right| \\
&= 4^n \left| \oint_{\partial R_{j_1, j_2, \ldots, j_n}} f(z) - f(z_0) - f'(z_0)(z - z_0)\, dz \right| \\
&= 4^n \left| \oint_{\partial R_{j_1, j_2, \ldots, j_n}} o(|z - z_0|)\, dz \right|
\end{aligned}$$

For any $\varepsilon > 0$, for large enough N, the $o(|z - z_0|)$ part is $< \varepsilon |z - z_0|$.

$$\begin{aligned}
&\le 4^n \operatorname{perim}(R_{j_1, j_2, \ldots, j_n}) \operatorname{diam}(R_{j_1, j_2, \ldots, j_n}) \varepsilon \\
&\le 4^n (2^{-n} \operatorname{perim}(R))(2^{-n} \operatorname{diam}(R)) \varepsilon \\
&= \operatorname{perim}(R))(2^{-n} \operatorname{diam}(R)) \varepsilon.
\end{aligned}$$

This goes to 0 for large enough n, so $\alpha = 0$. □

1.2 Integration of 1-forms

Let Ω be a domain, and let $\omega = P(x,y)dx = G(x,y)dy$ with P, Q complex continuous functions on Ω. Let $\gamma = \{z(t) : 0 \le t \le 1\}$, where $z(t) = z(t) + iy(t)$.

Definition 1.1. If ω is a 1-form, then the **integral of** ω **over** γ is

$$\int_\gamma \omega := \int_0^1 P(z(t))x'(t)\, dt + \int_0^1 z(t)y'(t)\, dt = \int P(z) \frac{dz + d\overline{z}}{2} + \int Q(z) \frac{dz - d\overline{z}}{2i}.$$

$$f(z)dz = f(z)dz + if(z)dy$$

Definition 1.2. ω is **exact** if there exists $f : \Omega \to \mathbb{C}$ such that f is C^1 in the real analysis sense, and $\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df$.

$$f(x,y) = f(x_0, y_0) + \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + o(\operatorname{dist}((x,y) - (x_0, y_0))).$$

So ω is exact iff there exists a C^1 (but not necessarily holomorphic) f such that $\omega = df$.

Theorem 1.2. *Let Ω be a domain and ω a 1-form on Ω. Then ω is exact if and only if*

$$\int_{\partial R} \omega = 0$$

for all rectangles R with $\partial R \subseteq \Omega$.

Proof. Assume $\omega = df = \frac{\partial f}{\partial x} dz + \frac{\partial t}{\partial y} dy$. Then

$$\int_\gamma \omega = \int_0^1 \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\, dt + \int_0^1 \frac{\partial f}{\partial y}\frac{\partial x}{\partial t}\, dt = \int_0^1 \frac{df}{dt} f(x(t), y(t))\, dt = f(\gamma(1)) - f(\gamma(0)).$$

Conversely, assume $\int_{\partial R} \omega = 0$. For $z, z_0 \in \Omega$, there exists a polygonal path from z_0 to z that always moves parallel to the real or imaginary axis. Let $f(z) = \int_{\gamma_{z_0,z}} \omega$. Then f is well-defined because it is independent of the path taken. Let $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. $\square$