Math 210A Lecture 13 Notes

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1 Krull-Schmidt, Structure of Finitely Generated Abelian Groups, and Group Actions

1.1 The Krull-Schmidt theorem

Theorem 1.1 (Krull-Schmidt). *Suppose G has normal subgroups $N_i \trianglelefteq G$ for $1 \leq i \leq r$. Then $G \cong N_1 \times \cdots \times N_r$ iff $N_i \cap \prod_{\substack{j=1 \\ j \neq i}}^r N_j = \{e\}$ and $N_1 \cdots n_r = G$.*

Proof. For $r = 2$, $N_1 \cap N_2 = \{e\}$ and $N_1 N_2 = G$. Then if $n_i \in N_i$, $n_1 n_2 n_1^{-1} = n_2' \in N_2$. Then $n_2 n_1^{-1} n_2^{-1} = n_1^{-1} n_2' n_2^{-1} \in N_1$. But this is the product of something in N_1 and something in N_2, and $N_1 \cap N_2 = \{e\}$, so $n_2' n_2^{-1} = e$. So $n_2' = n_2$, which gives us that n_1 and n_2 commute. So $G = N \rtimes N_2 = N_1 \times N_2$.

Now induct on r. Suppose this is true for r. Then $N_1 \cdots N_4 \cap N_{r+1} = \{e\}$ and $N_1 \cdots N_{r+1} = G$. By induction, $N_1 \cdots N_r = N_1 \times \cdots \times N_r$. Applying the $r = 2$ case, we get $G = N_1 \times \cdots \times N_4 \times N_{r+1}$. $\square$

Corollary 1.1. *Let $n = p_1^{r_1} \cdots p_k^{r_k}$ with p_i distinct primes and $r_i \geq 1$. Then*

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{r_k}\mathbb{Z}.$$

Corollary 1.2. *If $\gcd(m, n) = 1$, then*

$$\mathbb{Z}/mn\mathbb{Z} \cong n\mathbb{Z}/mn\mathbb{Z} \times m\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

1.2 The structure theorem for finitely generated abelian groups

Definition 1.1. An abelian group is **torsion-free** if for all $a \in A \setminus \{0\}$ and $n \geq 1$, $na \neq 0$.

Definition 1.2. The **torsion subgroup** B of A is the subgroup of elements of A of finite order.

Theorem 1.2 (structure theorem for finitely generated abelian groups)**.** *Let A be a finitely generated abelian group. Then there exists a unique $r, k \geq 0$ and positive integers $n_i \geq 1$ with $n_k \mid n_{k=1} \mid \cdots \mid n_1$ such that*

$$A \cong \mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \mathbb{Z}/n_k\mathbb{Z}.$$

Proof. We claim that torsion-free finitely generated abelian groups are free. Here is a sketch: Choose $a_1, \ldots, a_r \in A$ giving a minimal set of generators. We get $\pi : \mathbb{Z}^r \to A$ sending $e_i \mapsto a_i$, where e_i is the i-th coordiate unit element. Suppose $x = \sum_{i=1}^r b_i e_i \in \ker(\pi)$. Let $d = \gcd(b_1, \ldots, b_r)$. If $d \neq 1$, there exists a $y \in \mathbb{Z}^r$ with $dy = x$. Then $y \in \ker(\pi)$. So we may assume $d = 1$. There exists $\phi \in \mathrm{Aut}(\mathbb{Z}^r) = \mathrm{GL}_n(\mathbb{Z})$ such that $\phi(e_1) = x$. Then $\mathbb{Z}^r \xrightarrow{\phi} \mathbb{Z}^r \xrightarrow{\pi} A$ sends $e_1 \mapsto x \mapsto 0$. But then $\pi \circ \phi(e_1)$ for $2 \leq i \leq r$ generate A, contradicting minimality. So $A \cong \mathbb{Z}^r$. For uniqueness, if $A \cong \mathbb{Z}^r \cong \mathbb{Z}^s$, then $A/2A \cong \mathbb{F}_2^r \cong F_2^s$, so $r = s$.

Let B be the torsion subgroup of A. Note that A/B is torsion-free. We get an exact sequence

$$0 \to B \to A \to \mathbb{Z}^r \to 0.$$

If we want to go back from $\mathbb{Z}^r \to A$. Then for $e_i \in \mathbb{Z}^r$, there exists some $a_i \in A$ that maps to e_i. Since $\mathbb{Z}^r$ is free in Ab, there exists $\iota : \mathbb{Z}^r \to A$ such that $\iota(e_i) = a_i$ for all o. Then $A \cong B \oplus \mathbb{Z}^r$. Let n_1 be the exponent of B (lcm of orders is the highest order in this case). Choose $b_1 \in B$ of order n_1; then $A \cong \langle n_1 \rangle \oplus A/\langle n_1 \rangle \cong \mathbb{Z}/n_1\mathbb{Z} \oplus A/\langle n_1 \rangle$. Repeat with n_2, etc. We get $A \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$. Uniqueness follows from the uniqueness of the exponent of a group. $\square$

Example 1.1. Here is an example of this decomposition.

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/360\mathbb{Z} \oplus \mathbb{Z}/36\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

1.3 Group actions

Definition 1.3. A **group action** is a map $\cdot : G \times X \to X$ such that

1. $e \cdot x = x$,
2. $g \cdot (h \cdot x) = (gh) \cdot x$.

The pair of G with the action on X is called a G**-set**.

Remark 1.1. These are left G-sets. We can define right G-sets in a similar way.

Example 1.2. S_X acts on X by $\sigma \cdot x = \sigma(x)$.

Example 1.3. D_n acts on the vertices of a regular n-gon by rotating and reflecting them.

Example 1.4. $\mathrm{GL}_n(R)$ for a ring R acts on R^n viewed as column vectors.

Definition 1.4. G-set is the category with objects a set X with a G-action $G \times X \to X$ and morphisms $f : X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and $g \in G$.

Definition 1.5. Te **orbit** of $x \in X$ is $G \cdot x = \{g \cdot x : g \in G\} \subseteq X$.

Remark 1.2. Being in the same orbit gives an equivalence relation on X.

Definition 1.6. The **stabilizer** is $G_x = \{g \in G : g \cdot x = x\} \subseteq G$.

Definition 1.7. G acts **transitively** on X if it has just one orbit ($G \cdot x = X$ for all $x \in X$). G acts **faithfully** if no element of $G \setminus \{e\}$ fixes all $x \in X$; i.e. $\bigcap_{x \in X} G_x = \{e\}$.

Example 1.5. S_X acts transitively and fairhfully on X. The stabilizer of $x \in X$ is $S_{X \setminus \{x\}}$, viewed as a subgroup of S_X.

Example 1.6. D_n acts faithfully and transitively on vertices/edges. The stabilizer of the vertex is the subgroup generated by reflection across the axis through 0 and the vertex.

Example 1.7. G acts faithfully and transitively on G by left multiplication but not necessarily by conjugation if $G \neq \{e\}$. With the action of conjugation, the orbits are conjugacy classes $C_x = \{gxg^{-1} : g \in G\}$. $Z(G) = \bigcap_{x \in X} Z_x \neq \{e\}$, where $Z_x = \{g \in G : gxg^{-1} = x\}$, so if $Z(G) \neq \{e\}$, then this is nontrivial.

Example 1.8. G acts on subsets $S \subseteq G$ by conjugation. The orbits are conjugate subsets. The stabilizer of S is $N_G(S)$, the **normalizer** of S. $N_G(S) = \{g \in G : gSg^{-1} = S\}$. Note that $N_G(S)$ acts on S by conjugation. So $\bigcap_{x \in S} Z_x = Z_G(S) = \{g \in G : gs = sg \; \forall x \in S\}$, which is called the **centralizer** of S.