Math 255A Lecture 10 Notes

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1 Applications of Baire's Theorem II: The Closed Graph Theorem

1.1 Differential operators and the open mapping theorem

Last time, we had a differential operator $P(D)$ on $\mathbb{R}^n$ with constant coefficients and of order m such that if $u \in C^m(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open, then $Pu = 0 \implies u \in C^{m+1}(\Omega)$. Write $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$.

Proposition 1.1. *if* $|\operatorname{Im}(\zeta)| \to \infty$ *as* $|\zeta| \to \infty$*, then* $\zeta \in P^{-1}(0) \subseteq \mathbb{C}^n$*, where* $P(\zeta) = \sum_{|\alpha| \leq m} a_\alpha \zeta^\alpha$.

Example 1.1. If $P(D) = -\Delta = \sum_{j=1}^n D_{x_j}^2$, then $P(\zeta) = \sum_{j=1}^n \zeta_j^2 = \zeta \cdot \zeta$ for $\zeta \in \mathbb{C}^n$. We get $P^{-1}(0) = \{\zeta \in \mathbb{C}^n : |\operatorname{Re}(\zeta)| = |\operatorname{Im}(\zeta)|, \operatorname{Re}(\zeta) \cdot \operatorname{Im}(\zeta) = 0\}$. So $|\zeta| \to \infty$ along $P^{-1}(0) \iff |\operatorname{Im}(\zeta)| \to \infty$ along $P^{-1}(0)$.

Example 1.2. Consider also the Schrödinger equation: $i\partial_t u = -\Delta_x u$, where $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Then $P(D_x, D_t) = \sum D_{x_j}^2 + D_t$ gives us the polynomial $P(\xi, \tau) = \xi \cdot \xi + \tau$, where $\xi \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. If $|\xi| + |\tau| \to \infty$ along $P^{-1}(0)$, the Schrödinger equation has a solution in $C^2 \setminus C^3$.

Proof. Let $F_1 = \{x \in C^{m+1}(\Omega) : Pu = 0\}$ and $F_2 = \{x \in C^m(\Omega) : Pu = 0\}$. Then F_1 and F_2 are Fréchet spaces. Our assumption is that the inclusion map $F_1 \to F_2$ is surjective. By the open mapping theorem, the inverse $F_2 \to F_1$ is continuous. So for any compact set $K \subseteq \Omega$, there exists a compact set $K' \subseteq \Omega$ and $C > 0$ such that

$$\sum_{|\alpha| \leq m+1} \sup_K |\partial^\alpha u| \leq C \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha u|$$

for any $u \in F_1 = F_2$. If $\zeta \in \mathbb{C}^n$ is such that $P(\zeta) = 0$, then apply this inequality, where $u(x) = e^{ix\cdot\zeta}$. Then $P(e^{ix\cdot\zeta}) = P(\zeta)e^{ix\cdot\zeta} = 0$. So we get

$$\sum_{|\alpha| \leq m+1} \sup_K |\zeta^\alpha| e^{-x\cdot\operatorname{Im}(\zeta)} \leq C \sum_{|\alpha| \leq m} |\zeta^\alpha| \sup_K e^{-x\cdot\operatorname{Im}(\zeta)}.$$

So there exists $C > 0$ such that

$$\sum_{|\alpha| \le m+1} |\zeta^\alpha| \le Ce^{C|\operatorname{Im}(\zeta)|} \sum_{|\alpha| \le m} |\zeta^\alpha| = O((1 + |\zeta|)^m).$$

It follows that $|\operatorname{Im}(\zeta)| \to \infty$ when $|\zeta| \to \infty$ and $P(\zeta) = 0$. □

1.2 The closed graph theorem

Definition 1.1. Let $T : D(T) \to F_2$, where $D(T) \subseteq F_1$ and F_1, F_2 are Fréchet spaces. We say that T is **closed** if when $x_n \in D(T)$ with $x_n \to x \in F_1$ and $Tx_n \to y \in F_2$, then $x \in D(T)$ and $y = Tx$.

Note that T is closed iff the graph of T, $G(T) = \{(x, Tx) : x \in D(T)\}$ is closed in $F_1 \oplus F_2$. If T is linear and closed, then the graph of T is a Fréchet space (as a closed linear subspace of a Fréchet space).

Theorem 1.1 (closed graph theorem)**.** *Let $T : D(T) \to F_2$ be a closed linear map, where $D(T) \subseteq F_1$. Then either $D(T)$ is of the first category in F_1, or $D(T) = F_1$ and T is continuous. The range of T is either of the first category, or it is all of F_2.*

Proof. For the first statement, apply the open mapping theorem to the linear, continuous, injective map $G(T) \to F_1$ given by $(x, Tx) \mapsto x$. For the second statement, apply the open mapping theorem to the map $G(T) \to F_2$ given by $(x, Tx) \mapsto Tx$. □

Corollary 1.1. *Let H be a Hilbert space, and let $T : H \to H$ be linear such that $F(T) = H$ and T is symmetric ($\langle Tx, y\rangle = \langle x, Ty\rangle$). Then T is continuous.*

Proof. Check that T is closed. If $x_n \to x \in H$ and $Tx_n \to y \in H$, then $\langle Tx,_n, z\rangle = \langle x_n, Tz\rangle$ for all $x \in H$. Then $\langle y, z\rangle = \langle x, Tz\rangle = \langle Tx, z\rangle$ for all z, so $y = Tx$. □

Corollary 1.2. *Let B_0, B_1, B_2 be Banach spaces, and let T_j be closed linear maps $D(T_j) \to B_j$ with $D(T_j) \subseteq B_0$ for $j = 1, 2$. If $D(T_1) \subseteq D(T_2)$, then there exists some $C > 9$ such that $\|T_2 x\| \le C(\|T_1 x\|_{B_1} + \|x\|_{B_0})$ for any $x \in D(T_1)$.*

Proof. Consider the map $\hat{T} : G(T_1) \to B_2$ sending $(x, T_1 x) \mapsto T_2 x$. It suffices to show that $\hat{T}$ is closed. Suppose that (x_n, Tx_n) converges in $G(T_1)$ and $(T_2 x_n)$ converges in B_2. T_1 is closed, so $x_n \to x \in D(T_1)$, and $T_1 x_n \to T_1 x$. T_2 is closed, so $x \in D(T_2)$, and $T_2 x_n \to T_2 x$. □