Math 247A Lecture 2 Notes

Daniel Raban

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1 Fourier Inversion and Plancherel's Theorem

1.1 Fourier inversion

Theorem 1.1 (Fourier inversion). *For $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$[(\mathcal{F} \circ \mathcal{F})f](-x) = f(x),$$

or equivalently,

$$f(x) = \int e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\, d\xi.$$

We can think of this as decomposing f into a linear combination of characters with Fourier coefficients.

Proof. We can't use Fubini like we want to because the integrand is not necessarily absolutely integrable. The (standard) trick is to force a Gaussian in there. For $\varepsilon > 0$, let

$$I_\varepsilon(x) = \int e^{-\pi \varepsilon^2 |\xi|^2} e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\, d\xi.$$

Then the dominated convergence theorem tells us that $I_\varepsilon(x) \to \int e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\, d\xi$ as $\varepsilon \to 0$. On the other hand,

$$\begin{aligned} I_\varepsilon(x) &= \iint e^{-\pi \varepsilon^2 |\xi|^2} e^{2\pi i x \cdot \xi} e^{-2\pi i y \cdot \xi} f(y)\, dy\, d\xi \\ &= \int f(y) \int e^{-\pi \varepsilon^2 |\xi|^2} e^{-2\pi i (y-x) \cdot \xi}\, d\xi\, dy \end{aligned}$$

Use our lemma from last time with the linear transformation $A = \pi \varepsilon^2 I$:

$$\begin{aligned} &= \int f(y) (\pi \varepsilon^2)^{-d/2} \pi^{d/2} e^{-\pi^2 (y-x) \frac{1}{\pi \varepsilon^2} (y-x)}\, dy \\ &= \int \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2} |x-y|^2} f(y)\, dy. \end{aligned}$$

Note that $\int \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|x|^2}\,dx = \int e^{-\pi|x|^2}\,dx$.

$$\xrightarrow{\varepsilon\to 0} f(x).$$

To show this convergence, we have $\int \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|x-y|^2} f(y)\,dy - f(x) = \int \varepsilon^{d} e^{-\frac{\pi}{\varepsilon^2}|x-y|^2}\,dx[f(y) - f(x)]\,dy$. For $\eta > 0$, there is a $\delta(\eta) > 0$ such that $|f(y) - f(x)| < \eta$ whenever $|x-y| < \delta$. Then

$$\left| \int_{|x-y|<\delta} \varepsilon^{-d} e^{\frac{\pi}{\varepsilon^2}|x-y|^2} [f(y) - f(x)]\,dy \right| \le \eta \int_{|x-y|<\delta} \varepsilon^{-d} e^{\frac{\pi}{\varepsilon^2}|x-y|^2}\,dy = \eta,$$

$$\begin{aligned}
\left| \int_{|x-y|>\delta} \varepsilon^{-d} e^{\frac{\pi}{\varepsilon^2}|x-y|^2} [f(y) - f(x)]\,dy \right| &\le 2\|f\|_{L^\infty} \int_{|y|>\delta} \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|y|^2}\,dy \\
&\le 2\|f\|_{L^\infty} \int_{|y|>\delta/2} e^{-\pi|y|^2}\,dy \\
&\lesssim \|f\|_{L^\infty} e^{-\pi\frac{\delta^2}{2\varepsilon^2}} \\
&\xrightarrow{\varepsilon\to 0} 0.
\end{aligned}$$

First pick $\eta \ll 1$. Then choose $\varepsilon = \varepsilon(\delta) = \varepsilon(\eta) \ll 1$. $\square$

Corollary 1.1. *The Fourier transform is a homeomorphism on* $\mathcal{S}(\mathbb{R}^d)$.

1.2 Plancherel's theorem

Lemma 1.1. *For* $f, g \in \mathcal{S}(\mathbb{R}^d)$, *we have*

$$\int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi = \int f(x)\overline{g(x)}\,dx.$$

In particular,

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}.$$

so $\mathcal{F}$ *is an isometry in* L^2 *on* $\mathcal{S}(\mathbb{R}^d)$.

Proof. For $h \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\int \widehat{f}(\xi)h(\zeta)\,d\zeta &= \iint e^{-2\pi i x\cdot\xi} f(x)h(\zeta)\,dx\,d\xi \\
&= \int f(x)\widehat{h}(x)\,dx.
\end{aligned}$$

Now let $h = \overline{\hat{g}}$. Then $(\mathcal{F}h)(x) = \overline{\mathcal{F}(\widehat{g})(-x)} = \overline{g(x)}$. $\square$

Theorem 1.2 (Plancherel). *The Fourier transform extends from $\mathcal{S}(\mathbb{R}^d)$ to a unitary map on $L^2(\mathbb{R}^d)$.*

Proof. Fix $f \in L^2(\mathbb{R}^d)$. To define the Fourier transform on $\mathcal{F}$, let $f_n \in \mathcal{S}(\mathbb{R}^d)$ be such that $f_n \xrightarrow{L^2} f$. Since $\mathcal{F}$ is an isometry in L^2 on $\mathcal{S}(\mathbb{R}^d)$, $\|\widehat{f}_n - \widehat{f}_m\|_{L^2} = \|f_n - f_m\|_{L^2} \xrightarrow{n,m\to\infty} 0$. So $\{\widehat{f}_n\}_{n\geq 1}$ is Cauchy and hence convergent in $L^2(\mathbb{R}^d)$. Let $\widehat{f}$ be the L^2 limit of the $\widehat{f}_n$.

We claim that $\widehat{f}$ does not depend on the sequence $\{f_n\}_{n\geq 1}$. Let $\{g_n\}_{n\geq 1} \subseteq \mathcal{S}(\mathbb{R}^d)$ be another sequence such that $g_n \xrightarrow{L^2} f$. Let

$$h_n = \begin{cases} f_k & n = 2k-1 \\ g_k & n = 2k. \end{cases}$$

We have that $\{h_n\} \subseteq \mathcal{S}(\mathbb{R}^d)$, and $h_n \xrightarrow{L^2} f$. By the same argument as before, $\{\widehat{h}_n\}_{n\geq 1}$ converges in L^2. This means that $\lim_n \widehat{h}_n = \lim_n \widehat{f}_n = \lim_n \widehat{g}_n$.

We now claim that $\|\widehat{f}\|_2 = \|f\|_2$ for all $f \in L^2(\mathbb{R}^d)$; i.e. $\mathcal{F}$ is an isometry on L^2. Indeed,

$$\|\widehat{f}\|_2 = \lim_n \|\widehat{f}_n\|_2 = \lim \|f_n\|_2 = \|f\|_2.$$

Remark 1.1. This is not yet enough to show that $\mathcal{F}$ is unitary. In infinite dimensions, isometries need not be unitary. For example, take $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. Then

$$\langle T(a_1, a_2, \dots), (b_1, b_2, \dots)\rangle = \sum_{n\geq 1} a_n b_{n+1} = \langle (a_1, a_2, \dots), (b_2, b_3, \dots)\rangle ,$$

so $T^*(a_1, a_2, \dots) = (a_2, a_3, \dots)$. So $TT^* = \mathrm{id}$, but $T^*T \neq \mathrm{id}$. What we need to get an isometry is surjectivity.

We claim that $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is onto. We will show that $\mathrm{Ran}(\mathcal{F})$ is closed in $L^2(\mathbb{R}^d)$. As $\mathrm{Ran}(\mathcal{F}) \supseteq \mathcal{S}(\mathbb{R}^d)$, this will give $L^2(\mathbb{R}^d) = \overline{\mathcal{S}(\mathbb{R}^d)}^{L^2} \subseteq \overline{\mathrm{Ran}(\mathcal{F})}^{L^2} = \mathrm{Ran}(\mathcal{F})$. Let $g \in \overline{\mathrm{Ran}(\mathcal{F})}^{L^2}$. Then there exist $f_n \in L^2$ such that $\widehat{f}_n \xrightarrow{L^2} g$. $\mathcal{F}$ is an isometry on $L^2(\mathbb{R}^d)$, so $\|f_n - f_m\|_2 = \|\widehat{f}_n - \widehat{f}_m\|_2 \xrightarrow{n,m\to\infty} 0$. So $\{f_n\}_{n\geq 1}$ converges in L^2 to some f. Then $g = \widehat{f}$ because

$$\|\widehat{f} - \widehat{f}_n\|_2 = \|f - f_n\|_2 \xrightarrow{n\to\infty} 0.$$

By the uniqueness of limits, we get $g = \widehat{f}$. So we get $g = \widehat{f} \in \mathrm{Ran}(\mathcal{F})$. $\square$

1.3 The Hausdorff-Young inequality

Theorem 1.3 (Hausdorff-Young). *For $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|\widehat{f}\|_p' \leq \|f\|_p, \qquad \forall 1 \leq p \leq 2,$$

where $1/p + 1/p' = 1$.

Proof. This follows from interpolation, as we have $\mathcal{F}: L^1 \to L^\infty$ with $\|\widehat{f}\|_{L^\infty} \le \|f\|_{L^1}$ and $\mathcal{F}: L^2 \to L^2$ with $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$. $\square$

Remark 1.2. As in the proof of Plancherel's theorem, we can use Hausdorff-Young to extend the Fourier transform from $\mathcal{S}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for any $1 \le p \le 2$.

Note that the Riemann-Lebesgue lemma gives that for $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \in C_0(\mathbb{R}^d)$. So we can think of evaluating the Fourier transform at a single point or on a measure 0 set, such as a plane in $\mathbb{R}^3$. The **restriction problem** asks: For which values of p can we make sense of the Fourier transform on measure 0 sets, such as a parabaloid or a cone? This is important in PDE, and it is very hard (still open!).

The next theorem says that the Hausdorff-Young inequality is the best we can do.

Theorem 1.4. *If $\|\widehat{f}\|_{L^2} \le \|f\|_{L^p}$ for some $1 \le p, q \le \infty$ and all $f \in \mathcal{S}(\mathbb{R}^d)$, then necessarily, $q = p'$ and $1 \le p \le 2$.*

Proof. For $f \in \mathcal{S}(\mathbb{R}^d)$ with $f \not\equiv 0$, define $f_\lambda(x) = f(x/\lambda)$ for $\lambda > 0$. Then $\|f_\lambda\|_p = \lambda^{d/p}\|f\|_p$. We also have

$$\widehat{f_\lambda}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x/\lambda)\, dx = \lambda^d \widehat{f}(\lambda\xi),$$

so $\|\widehat{f_\lambda}\|_q = \lambda^{d-d/q}\|\widehat{f}\|_q$. Then $\|\widehat{f_\lambda}\|_q \le \|f_\lambda\|_p$ if and only if $\lambda^{d-d/2}\|\widehat{f}\|_q \le \lambda^{d/p}\|f\|_p$, so $\lambda^{d(1-1/q-1/p)}\|\widehat{f}\|_q \le \|f\|_p$. Letting $\lambda \to 0$ or $\lambda \to \infty$, we conclude that $1 - 1/q - 1/p = 1$. So we get $q = p$. $\square$

Next time, we will prove the remaining portion of this theorem, that $1 \le p \le 2$.