Math 255A Lecture 3 Notes

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1 Proof of the Geometric Hahn-Banach Theorem

1.1 Gauges and the real geometric Hahn-Banach theorem

Theorem 1.1 (geometric Hahn-Banach)**.** *Let V be a real normed vector space with $A, B \subseteq V$ convex, nonempty and disjoint. Also assume A is open. Then there exists a closed affine hyperplane separating A and B.*

Before we prove this, we need a bit of background.

Definition 1.1. Let $C \subseteq V$ be convex and open such that $0 \in C$. Define the **gauge** of C as

$$p(x) = \inf\{t > 0 : x/t \in C\}.$$

Lemma 1.1. *The gauge of C satisfies the following properties:*

1. *$p(\lambda x) = \lambda p(x)$ for $\lambda > 0$ and $x \in V$*
2. *$p(x + y) \leq p(x) + p(y)$ for $x, y \in V$*
3. *there exists $M > 0$ such that $p(x) \leq M\|x\|$ for all $x \in V$ ($\implies$ p is continuous at 0).*
4. *$C = \{x \in V : p(x) < 1\}$*

Proof. (i) is clear.

(iii) Let $r > 0$ be such that $\{x : \|x\| \leq r\} \subseteq C$. Then for all x with $\|x\| = 1$, $rx \in C$, so $p(x) \leq 1/r$. So $p(x) \leq \|x\|/r$ for all $x \in V$.

(iv) We first show $C \subseteq \{x : p(x) < 1\}$. If $x \in C$, then $(1 + \varepsilon)x \in C$ for ε small. So $p(x) \leq 1/(1 + \varepsilon) < 1$. On the other hand, if $p(x) < 1$, then $x/t \in C$ for some $0 < t < 1$. So $x = t(x/t) + (1 - t)0 \in C$ (by convexity of C).

(ii) Let $x, y \in V$ and $\varepsilon > 0$. Then $x/(p(x) + \varepsilon), y/(p(y) + \varepsilon) \in C$, and their convex combination

$$t\frac{x}{p(x) + \varepsilon} + (1 - t)\frac{y}{p(y) + \varepsilon}$$

is also in C for $0 \le t \le 1$. Take $t = (p(x)+\varepsilon)/(p(x)+p(y)+2\varepsilon)$. So

$$\frac{x+y}{p(x)+p(y)+2\varepsilon} \in C$$

which gives us that $p(x+y) < p(x)+p(y)+2\varepsilon$. So p is subadditive. $\square$

Lemma 1.2. *Let $C \subseteq V$ be open, convex, and nonempty, and let $x_0 \notin C$. Then there exists a continuous linear form $f : V \to \mathbb{R}$ such that $f(x) < f(x_0)$ for all $x \in C$. In particular, the closed affine hyperplane $H = f^{-1}(f(x_0))$ separates x_0 and C.*

Proof. By translation, we may assume that $0 \in C$. Let $g : \mathbb{R}x_0 \to \mathbb{R}$ send $tx_0 \mapsto t$.Then $g(tx_0) \le p(tx_0)$ for any $t \in \mathbb{R}$, where p is the gauge of C; indeed, for $t \le 0$, this is ok, and if $t > 0$, this is also ok, as $p(x_0) \ge 1$. By the analytic version of the Hahn-Banach theorem, g extends to a linear form $f : V \to \mathbb{R}$ such that $f(x_0) = 1$ and $f(x) \le p(x)$ for any $x \in V$. In particular, $f(x) < 1 = f(x_0)$ for $x \in C$. The function f is continuous as $f(x) \le p(x) \le M\|x\|$ for all $x \in V$. $\square$

We are now ready to prove the geometric Hahn-Banach theorem.

Proof. Let $C = A - B = \{x - y : x \in A, y \in B\}$. Then C is convex because A, B are convex, $0 \notin C$, and C is open (because $C = \bigcup_{y \in B}(A - y)$, which is a union of open sets). By the previous lemma, there exists a linear continuous form f such that $f < 0$ on C. Then $f(x) < f(y)$ for $x \in A$ and $y \in B$. If $\sup_{x \in A} f(x) \le \alpha \le \inf_{y \in B} f(y)$, then $f^{-1}(\alpha)$ separates A and B. $\square$

1.2 The complex geometric Hahn-Banach theorem

Definition 1.2. Let V be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. We say that $M \subseteq V$ is **balanced** if $\lambda x \in M$ for all $x \in M$ and $\lambda \in K$ with $|\lambda| \le 1$.

Proposition 1.1. *Let V be a normed vector space over $\mathbb{C}$, and let $C \subseteq V$ be open, convex, nonempty, and balanced. Let $x_0 \notin C$. Then there exists a complex linear continuous map $f : V \to \mathbb{C}$ such that $f(x_0) \neq f(x)$ for all $x \in C$. In particular, the closed affine hyperplane $H = f^{-1}(f(x_0))$ contains x_0 and does not meet C.*

Proof. Since C is balanced, $0 \in C$. Let p be the gauge of C. Then $C = \{x : p(x) < 1\}$, and p is a seminorm; i.e. $p(\lambda x) = |\lambda| p(x)$ and $p(x+y) \le p(x)+p(y)$. We can now conclude that there is a continuous linear form $f : V \to \mathbb{C}$ such that $f(x_0) = 1$ and $|f| \le p$ on V. Then $|f| < 1$ on C, so f is continuous. $\square$

Remark 1.1. The gauge p of C (convex, open, balanced, contains 0) satisfies the following inequality:

$$|p(x+y) - p(y)| \le p(x) \le M\|x\|.$$

So p is Lipschitz continuous on V.

Corollary 1.1. *Let V be a normed vector space over $\mathbb{C}$, and let $A \subseteq V$ be a closed, convex, nonempty, and balanced. Let $x \notin A$. We can find a continuous linear form f on V such that* $\inf_{y \in A} |f(y) - f(x)| > 0$.

Proof. Let $\varepsilon > 0$ be so small that $(x + B(0, \varepsilon)) \cap A = \varnothing$. The set $B(0, \varepsilon) + A$ is open, convex, balanced, and does not contain x, so by the previous lemma, there is a continuous linear form f such that $f(x) \neq f(y) + f(z)$, where $y \in A$ and $z \in B(0, \varepsilon)$. Here, $f(B(0, \varepsilon)) \neq \{0\}$ is a balanced subset of $\mathbb{C}$, so it contains a neighborhood of 0. $\square$