Math 206A Lecture 10 Notes

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1 $\mathcal{F}$-Vectors of Polytopes

1.1 Types of polytopes

There are two types of convex polytopes in $\mathbb{R}^d$.

1. **simplicial polytopes** (all faces are simplices),
2. **simple polytopes** (degree of every vertex $= d$, $\dim(P) = d$).

There is a duality between these two types. Basically, a point on each face, and take the convex hull to get the dual polytope. P is simple iff P^* is simplicial.

Definition 1.1. Let $P \subseteq \mathbb{R}^d$ be a convex polytope with $\dim(P) = d$. Let $f_i(P)$ be the number of i dimensional faces of P. This is called the **$\mathcal{F}$-vector** of P.

Proposition 1.1. $f_i(P) = f_{d-i-1}(P^*)$.

Topologists like simplicial polytopes, but combinatorialists like simple polytopes. We will focus on simple polytopes, but the previous proposition tells us that this is really the same story.

1.2 Dehn-Sommerville equations

Theorem 1.1 (Dehn-Sommerville equations[1])**.** *Let $P \subseteq \mathbb{R}^d$ be simple. Then*

$$\sum_{i=k}^{d} (-1)^i \binom{i}{k} f_i = \sum_{i=d-k}^{d} (-1)^{d-i} \binom{i}{d-k} f_i.$$

for all $0 \leq k \leq d$.

[1] The case $d = 3$ was proved by Euler. The cases $d = 4.5$ were proved by Dehn, and $d > 5$ was proved by Sommerville.

Remark 1.1. When $k = 0$, this becomes

$$\sum_{i=0}^{d} (-1)^i f_i = 1.$$

This is Euler's formula. When $d = 3$, we get $f_0 - f_1 + f_2 = 2$, where $f_3 = 1$.

Example 1.1. Let P be a simplex in $\mathbb{R}^d$. Then $f_0 = d + 1$, and $f_i = \binom{d+i}{i+1}$.

Example 1.2. Let $Q \subseteq \mathbb{R}^d$ be a d-cube. Then $f_0 = 2^d$, $f_d = 1$, and $f_{d-1} = 2d$. In general, $f_i = \binom{d}{i} 2^{d-i}$, because we have $\binom{d}{i}$ ways to choose a face and 2^{d-i} coordinates left. We get that

$$\sum_{i=0}^{d} f_i = 3^d,$$

which could be otherwise proven as an elementary exercise.

Proposition 1.2. *Let* $\mathcal{F}(t) = \sum_{i=0}^{d} f_i t^i$. *Define* $\mathcal{G}(t) := \mathcal{F}(t-1) = \sum_{i=-}^{d} g_i t^i$. *Then* $g_k = \sum_{i=1}^{d} (-1)^i \binom{i}{k} f_i$.

Proof.

$$\mathcal{G}(t+1) = \sum_{i=0}^{d} g_i (t+1)^i = \sum_{i=0}^{d} g_i \sum_{k=0}^{i} \binom{i}{k} t^k = \sum_{k=0}^{d} t^k \left[\sum_{i=k}^{d} g_i \binom{i}{k} \right] = \sum_{i=0}^{d} t^k f_k = \mathcal{F}(t). \quad \square$$

So the Dehn-Sommerville equations say that $g_i = g_{d-i}$.

Example 1.3. For a simplex, $\mathcal{F} = (1+t)^{d+1}$. Then $\mathcal{G}(t) = t^{d+1}$.

Example 1.4. For the d-cube, $\mathcal{F}(t) = (2+t)^d$, and $\mathcal{G}(t) = (1+t)^d$.

Let's prove the theorem.

Proof. Fix $\varphi : \mathbb{R}^d \to \mathbb{R}$ a Morse function (a linear function that is nonconstant on edges of the polytope). For a vertex v, define the index $\mathrm{ind}_\varphi(v)$ to be the number of edges increasing by φ. Observe that $0 \le \mathrm{ind}_\varphi(v) \le d$. Define $h_i^{(\varphi)}$ to be the number of vertices $v \in V(P)$ such that $\mathrm{ind}_\varphi(v) = i$.

We claim that $f_k = \sum_{i=k}^{d} \binom{i}{k} h_i^{(\varphi)}$. Take any k-face Q, and let v be the minimum vertex with respect to φ. Let $i = \mathrm{ind}_\varphi(v)$. The number of k-faces Q is $\sum_{i=k}^{d} \binom{i}{k} h_i^{(\varphi)}$, which is the number of ways to choose Q with minimum vertex v times the index.

Then $\sum_{i=0}^{d} h_i^{(\varphi)} t^i = \mathcal{F}(t-1)$. So for all φ, $h_i^{(\varphi)} = g_i$. If we replace φ with $-\varphi$, we get $h_i^{(\varphi)} = h_{d-i}^{-\varphi}$. The left hand side is g, and the right hand side is g_{d-i}. $\square$