Math 279 Lecture 4 Notes

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September 7, 2021

1 Final Overview of Stochastic PDEs

1.1 The KPZ equation

Last time, we argued that by Itô calculus, we can make sense of the SPDE

$$Z_t = Z_{xx} + Z\xi$$

when $d = 1$. We want to use this solution to come up with a candidate of a solution to the KPZ equation

$$h_t = h_{xx} + |h_x|^2 + \xi.$$

We may use the Hopf-Cole transform to get a solution for this equation utilizing the previous SPDE. To achieve this, we smoothize ξ in the first SPDE by replacing ξ with $\xi^\varepsilon *_x \chi^\varepsilon$, which is white in time and smooth in space. Here, $\chi^\varepsilon(x) = \frac{1}{\varepsilon}\chi(\frac{x}{\varepsilon})$ with χ a smooth function of compact support and total integral 1. Then

$$Z_t^\varepsilon = Z_{xx}^\varepsilon + Z^\varepsilon \xi^\varepsilon.$$

As we saw last time, for fixed x, $\xi(x,t)$ is a multiple of standard white noise with

$$\mathbb{E}[\xi(x,t)\xi(x,s)] = \delta_0(t-s),$$

$$\int (\xi^\varepsilon)^2(y)\,dy = \delta_0(t-s)\varepsilon^{-1} \underbrace{\int \chi^2(y)\,dy}_{\overline{C}} =: \delta_0(t-s)C^\varepsilon.$$

In other words, if B represents a standard Brownian motion, we can represent

$$\xi^\varepsilon(x,t) \overset{d}{=} \sqrt{C^\varepsilon}\dot{B}(t).$$

Writing $z(t) = Z^\varepsilon(x,t)$, we can write the smoothized equation as

$$dz = \underbrace{b(t)}_{Z_{xx}^\varepsilon(x,t)}\ dt + Z^\varepsilon(x,t)\sqrt{C^\varepsilon}\,dB.$$

We now apply Hopf-Cole:

$$d(\underbrace{\log z}_{h^\varepsilon}) = \frac{dz}{z} - \frac{(Z^\varepsilon)^2 C^\varepsilon}{z^2}\, dt$$

(using $(dB)^2 = dt$). Simplifying, we get

$$dh^\varepsilon = \left(\frac{Z^\varepsilon_{xx}}{Z^\varepsilon} - \frac{C^\varepsilon}{2} \right) dt + \sqrt{C^\varepsilon}\, dB.$$

Here,

$$h^\varepsilon = \log Z^\varepsilon, \qquad h^\varepsilon_x = \frac{Z^\varepsilon_x}{Z^\varepsilon}, \qquad h^\varepsilon_{xx} = \frac{Z^\varepsilon_{xx}}{Z^\varepsilon} - (h^\varepsilon_x)^2.$$

Hence,

$$h^\varepsilon_t = h^\varepsilon_{xx} + \left[(h^\varepsilon_x)^2 - \frac{C^\varepsilon}{2} \right] + \xi^\varepsilon.$$

Thus, we can renormalize the KPZ equation by subtracting a constant multiple of $1/\varepsilon$ from the right hand side:

$$h_t = h_{xx} + (h_x^2 - \infty) + \xi$$

1.2 Stochastic quantization

In Euclidean Quantum Field Theory, we need to make sense of probability measures that are formally expressed as

$$\frac{1}{Z} e^{-\mathscr{H}(\phi)}\, D\phi,$$

where ϕ is a field, i.e. $\phi : \mathbb{R}^d \to \mathbb{R}$, and $D\phi$ is a Lebesgue-like measure on the space of ϕs. This may be compared with the following finite dimensional model: $H : \mathbb{R}^N \to \mathbb{R}$ and the minimizer of H correspond to the equilibrium states. If we take into account the thermal fluctuations, we would have equilibrium measures of the form

$$\frac{1}{Z} e^{-H(x)} \underbrace{dx}_{\text{Leb in } \mathbb{R}^N}.$$

Observe that a gradient ODE would allow us to give a dynamical approximation to our equilibrium states. For example, $\dot{x} = -\nabla H(x)$ would allow us to approximate the minimizer of H. As for $\frac{1}{Z} e^{-H(x)}\, dx$, we need to solve

$$\dot{x} = -\nabla H(x) + \dot{B}(t).$$

Then the law of $x(t)$ as $t \to \infty$ is exactly $\frac{1}{Z} e^{-H(x)}\, dx$.

In 1981, Parisi and Wu suggested that a dynamical approximation as in this previous equation would approximate the formal probability measures with a mathematically more tractable model. Indeed, if we have a candidate for an inner product on our function space, then

$$\phi_t = -\partial\mathscr{H}(\phi) + \xi(x,t),$$

which is called the **stochastic quantization**. Hopefully, $\phi(\cdot,t) \approx \frac{1}{Z}e^{-\mathscr{H}(\phi)}\,D\phi$ for large t.

Let's consider some examples:

Example 1.1. Consider

$$\mathscr{H}(\phi) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\phi|^2 + V(\phi)\right)\,dx,$$

where $V : \mathbb{R} \to \mathbb{R}$. We may replace $\mathbb{R}^d$ with a bounded domain with a suitable boundary condition. If we use the L^2 inner product, then

$$(\partial\mathscr{H})_\phi\psi = \int \left(-\Delta\phi + V'(\phi)\right)\psi.$$

Hence, the stochastic quantization equation becomes

$$\phi_t = \Delta_x\phi - V'(\phi) + \xi.$$

This is a perturbation of the SHE. The best we can hope for is a regularity of the form $\phi \in \mathcal{C}^{(1-d/2)-}$, which means that ϕ is a function only when $d = 1$. Hence, $V'(\phi)$ is the main challenge when V' is nonlinear.

1.3 The Gaussian Free Field

Here is a brief history of $\frac{1}{Z}e^{-\mathscr{H}(\phi)}\,D\phi$ and stochastic quantization. First consider the case $V = 0$ (or $V(\phi) = m^2\phi^2/2$). Then what we have for our formal probability measure is a Gaussian measure though in infinite dimension. Using the L^2 inner product and when $V = 0$, what we have is

$$\frac{1}{Z}e^{-\frac{1}{2}\langle(-\Delta)\phi,\phi\rangle}.$$

This is the celebrated **Gaussian Free Field (GFF)**. Its covariance is $(-\Delta)^{-1}$, which has a kernel known as Green's function. In a domain D, we write $G^D(x,y)$ for this kernel: Under GFF,

$$\mathbb{E}[\phi(x)\phi(y)] = G^D(x,y).$$

However, we expect $\phi \in \mathcal{C}^{(1-d/2)-}$, hence not a function when $d > 1$.

For example, when $d = 1$, $D = (0,\infty)$, and we have the boundary condition $\phi(0) = 0$, then

$$G^D(x,y) = \min(x,y).$$

This is the correlation of Brownian motion in $d = 1$. Similarly, for $D = (0, \ell)$ with 0 boundary condition, we get

$$G^D(x, y) = \min(x, y) - \frac{1}{\ell} xy,$$

which corresponds to a Brownian bridge in $(0, \ell)$.

More generally, we have Feynman-Kac

$$\frac{1}{Z} e^{-\int (\frac{1}{2}|\phi'(x)|^2 + V(\phi(x)))\,dx}\, D\phi = e^{-\int V(\phi(x))\,dx} \underbrace{\mu_0(d\phi)}_{\text{law of BM}} .$$

Next, consider $d = 2$. In this case, the GFF is "conformally invariant." This has to do with the fact that if $h : D \to D'$ is conformal, then $G^D(z, z') = G^{D'}(h(z), h(z'))$. In fact, ϕ in GFF can be used to study Schramm-Loewner Evolution in critical statistical mechanics ($\dot{z} = e^{\gamma\phi(z)}$). Also, there are models for randomly selected Riemannian metrics that can be expressed as $e^{\gamma\phi(x,y)}(dx^2 + dy^2)$, where ϕ is selected according to the GFF.

Finally, let us go back to the PDE

$$\phi_t = \Delta\phi - V'(\phi) + \xi$$

and examine the existence of a solution when V' is not linear. As a classical example, consider $V(\phi) = \phi^4/4$, so that $V'(\phi) = \phi^3$. Again, it is not clear how to make sense of ϕ^3 when $d \geq 2$, as ϕ is a distribution. To get a feel for this, first let us figure out when this equation is subcritical. Let ϕ solve this equation, and set $\widehat{\phi}(x, t) = \lambda^{d/2-1}\phi$. Then we can readily show

$$\widehat{\phi}_t = \Delta\widehat{\phi} - \lambda^{4-d}\widehat{\phi}^3 + \widehat{\xi}.$$

So the model is subcritical iff $d \leq 3$. The case $d = 2$ was solved back in the late 80s. The case $d = 3$ was solved in 2014 by Hairer. We need to renormalize the equation as

$$\phi^\varepsilon_t = \Delta\phi^\varepsilon - [(\phi^\varepsilon)^3 - c_\varepsilon\phi^\varepsilon] + \xi^\varepsilon$$

with $c_\varepsilon = O(\varepsilon^{-1})$.