Math 245B Lecture 6 Notes

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1 Tychonoff's Theorem

1.1 Locally compact spaces

Sometimes, we want to generalize results for compact spaces to spaces that are not quite compact but can be broken up into compact pieces.

Definition 1.1. A topological space X is **locally compact** if for every $x \in X$ there exists a neighborhood U of x such that $\overline{U}$ is compact.

Example 1.1. $\mathbb{R}^n$ is not compact, but it is locally compact.

1.2 FIP closed families

Theorem 1.1 (Tychonoff). *Suppose $\langle x_\alpha \rangle_{\alpha \in A}$ is a collection of compact sets. Then $\prod_{\alpha \in A} X_\alpha$ is also compact.*

We will prove a special case of this theorem.[1]

Theorem 1.2. *Suppose $\langle X_\alpha \rangle_{\alpha \in A}$ is a collection of compact, Hausdorff sets. Then $\prod_{\alpha \in A} X_\alpha$ is also compact and Hausdorff.*

Recall that we showed last time that X is compact if any FIP family $\mathcal{F}$ of closed sets has $\bigcap \mathcal{F} \neq \varnothing$.

Lemma 1.1. *Let X be a topological space, and let $\mathcal{F}$ be an FIP closed family. Then there exists a maximal FIP closed family $\mathcal{G} \supseteq \mathcal{F}$.*

Proof. Let Γ be the collection of families $\mathcal{G} \subseteq \mathscr{P}(X)$ such that $\mathcal{G}$ consists of closed sets, is FIP, and $\mathcal{G} \subseteq \mathcal{F}$. We will use Zorn's lemma. Let's verify the conditions:

1. $\mathcal{F} \in \Gamma$, so $\Gamma \neq \varnothing$.

[1] Professor Austin has never seen an application of Tychonoff's theorem where the spaces were not Hausdorff.

2. Every chain $\Lambda \subseteq \Gamma$ has an upper bound. Check that $\bigcup \Lambda \in \Gamma$; the crucial property is that $\bigcup \Lambda$ is FIP. Let $C_1, \dots, C_m \in \bigcup \Lambda$ Then $C_i \in \mathcal{G}_i \in \Lambda$ for all $i = 1, \dots, m$. Then there exists $i_i \leq m$ such that $C_1, \dots, C_m \in \mathcal{G}_{i_0}$. There $C_1 \cap \dots \cap C_m \neq \varnothing$. $\square$

Remark 1.1. This theorem actually needs Zorn's lemma. It is a theorem from the 1970s that it is possible to choose topological spaces such that Tychonoff's theorem implies Zorn's lemma.

Corollary 1.1. *X is compact if and only if every maximal FIP closed family $\mathcal{F}$ has $\bigcap \mathcal{F} \neq \varnothing$.*

Proof. ($\Longleftarrow$): Let $\mathcal{F}$ be an arbitray FIP closed family. If $\mathcal{F} \subseteq \mathcal{G}$, then $\bigcap \mathcal{F} \supseteq \mathcal{G} \neq \varnothing$. $\square$

Lemma 1.2. *Let X be any topological space, and let $\mathcal{F}$ be a maximal FIP closed family.*

1. *If $C_1, \dots, C_m \in \mathcal{F}$, then $C_1 \cap \dots \cap C_m F$.*

2. *If $C \subseteq X$ is closed and $C \cap F \neq \varnothing$ for all $F \in \mathcal{F}$, then $C \in \mathcal{F}$.*

Proof. For the first statement, let $\mathcal{F}' = \{C_1 \cap \dots \cap C_m : m \in \mathbb{N}, C_1, \dots, C_m, \in \mathcal{F}\}$. Now $\mathcal{F} \subseteq \mathcal{F}'$, so $\mathcal{F} = \mathcal{F}'$.

For the second statement, let $\mathcal{F}'' = \mathcal{F} \cup \{C\}$. This is still FIP: if $D_1, \dots, D_m \in \mathcal{F}$, then $C \cap (D_1 \cap \dots \cap D_m) \neq \varnothing$, as $D_1 \cap \dots \cap D_m$ is in $\mathcal{F}$ by property 1. $\square$

Remark 1.2. If X is compact, then every maximal FIP closed family equals $\{C : C \text{ closed }, C \ni x\}$ for some $x \in X$.

Lemma 1.3. *Let X be a topological space. The following are equivalent:*

1. *X is T_3.*

2. *If $U \subseteq X$ is an open neighborhood of x, then there exists an open $V \ni x$ such that $\overline{V} \subseteq U$.*

Proof. $X \setminus U$ is a closed set not containing x. T_3 is the statement that there is a closed set containing x and an open set in U not intersecting the closed set $X \setminus U$. $\square$

1.3 Proof of Tychonoff's theorem

We are now ready to prove Tychonoff's theorem.

Proof. Suppose X_α is compact, Hausdorff for all $\alpha \in A$. To show that X is Hausdorff, let $x = \langle x_\alpha \rangle, y = \langle y_\alpha \rangle \in C$, with $x \neq y$. So there exists an α such that $x_\alpha \neq y_\alpha$. Then there exist disjoint $U_\alpha \ni x_\alpha$ and $v_\alpha \ni y_\alpha$ (because X_α is Hausdorff). Now $U = \pi_\alpha^{-1}[Y_\alpha] \ni x$, and $V = \pi_\alpha^{-1}[V_\alpha] \ni y$.

To show that X is compact, let $\mathcal{F}$ be a maximal FIP closed family in X.

- Step 1: Find a good point candidate to be in the intersection of elements of $\mathcal{F}$: For all $\alpha \in A$, define $\mathcal{F}_\alpha = \{\pi_\alpha(F) : F \in \mathcal{F}\}$. This collection is FIP;

$$\pi_\alpha(F_1) \cap \pi_\alpha(F_m) \supseteq \pi_\alpha(F_1 \cap \cdots \cap F_m) \neq \varnothing.$$

Let $G_\alpha := \{\overline{\pi_\alpha(F)} : F \in \mathcal{F}\}$. This is an FIP closed family. By the compactness of X_α, there exists $x_\alpha \in \bigcap_{\mathcal{G}_\alpha}$. Let $x = \langle x_\alpha \rangle_{\alpha \in A}$.

- Step 2: Let V_α be any open set containing x_α.

$$\begin{aligned} x_\alpha \in \bigcup \mathcal{G}_\alpha &\implies V_\alpha \cap \pi_\alpha(F) \neq \varnothing \quad \forall F \in \mathcal{F} \\ &\implies \overline{V_\alpha} \cap \pi_\alpha(F) \neq \varnothing \quad \forall F \in \mathcal{F} \\ &\implies \pi_\alpha^{-1}[\overline{V_\alpha}] \cap F \neq \varnothing \quad \forall F \in \mathcal{F}. \end{aligned}$$

So $\pi_\alpha^{-1}[\overline{V_\alpha}] \in \mathcal{F}$ for all α and for all open $V_\alpha \ni x_\alpha$.

- Step 3: Show that $x \in F$ for all $F \in \mathcal{F}$. It is enough to check that $U \cap F \neq \varnothing$ for all $F \in \mathcal{F}$ when U is an open neighborhood of x. It is enough to check for open sets in a neighborhood base. That is, we only need to check $U = \bigcap_{j=1}^m \pi_{\alpha_j}^{-1}[U_{\alpha_j}]$ for all open $U_\alpha \ni x_\alpha$. Because every x_{α_j} is Hausdorff, there exists an open set V_{α_j} such that $x_{\alpha_j} \in V_{\alpha_j} \subseteq \overline{V}_{\alpha_j} \subseteq U_{\alpha_j}$. Now, by step 2, $\pi_{\alpha_j}^{-1}[\overline{V}_{\alpha_j}] \in \mathcal{F}$ for all j. So

$$U \cap F = \left[\bigcap_{j=1}^n \pi_{\alpha_j}^{-1}[U_{\alpha_j}]\right] \cap F \supseteq \left[\bigcap_{j=1}^n \pi_{\alpha_j}^{-1}[\overline{V}_{\alpha_j}]\right] \cap F \neq \varnothing.$$

So $x \in \bigcap \mathcal{F}$. □