Math 245B Lecture 13 Notes

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1 The Hahn-Banach theorem and Dual spaces

1.1 Reflexive spaces and dual spaces

Last time, we showed a consequence of the Hahn-Banach theorem. Here is a special case, where we separate $x \in \mathcal{X}$ from the closed subspace $\mathcal{M} = \{0\}$.

Proposition 1.1. *Let $x \in \mathcal{X} \setminus \{0\}$. Then there exists some $f \in \mathcal{X}^*$ such that $\|f\| = 1$,and $|f(x)| = \|x\|$. Moreover, $\mathcal{X}^*$ separates points of $\mathcal{X}$.*

Proposition 1.2. *If $x \in \mathcal{X}$, define $\hat{x} : \mathcal{X}^* :\to K$ by $\hat{x}(f) := f(x)$. Then $x \mapsto \hat{x}$ is a linear isometry $\mathcal{X} \to \mathcal{X}^{**}$.*

This is called the **canonical embedding**.

Proof. For linearity,

$$\widehat{x+y}(f) = f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f).$$

Multiplication by constants is the same. To show that it is an isometry,

$$\begin{aligned} \|\hat{x}\|_{**} &= \sup\{\hat{x}(f) : \|f\|_* \leq 1\} \\ &= \sup\{f(x) : \|f\|_* \leq 1\} \\ &\leq \|x\|. \end{aligned}$$

We can achieve equality by the above corollary of Hahn-Banach. $\square$

Remark 1.1. Recall that if $\mathcal{Y}$ is complete, then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is complete. Then, since $\mathcal{X}^{**} = \mathcal{L}(\mathcal{X}^*, K)$, and $K = \mathbb{R}$ or $\mathbb{C}$ is complete, $\mathcal{X}^{**}$ is complete. So $\hat{\mathcal{X}} = \{\hat{x} : x \in \mathcal{X}\}$ is a canonical way of completing $\mathcal{X}$.

Definition 1.1. $\mathcal{X}$ is **reflexive** if $\hat{\mathcal{X}} = \mathcal{X}^{**}$.

Finite dimensional vector spaces are always reflexive. This is not always the case for infinite dimensional spaces.

Example 1.1. $C[0,1]$ is not reflexive. We will see this later, but its dual is not separable, and neither is its double dual.

Definition 1.2. The **adjoint** (or **transpose**) of $T : \mathcal{X} \to \mathcal{Y}$ is $T^* : \mathcal{Y}^* \to \mathcal{X}^*$ defined by $T^*(f) = f \circ T$.

$\mathcal{T}^*$ is linear, and it satisfies $\|T^*\| \leq \|T\|$.

Proposition 1.3. *Let $T : \mathcal{X} \to \mathcal{Y}$ be bounded and linear.*

1. $\|T^*\| = \|T\|$.
2. *Let $T^{**} : \mathcal{X}^{**} \to \mathcal{Y}^{**}$ be $T^{**} = (T^*)^*$. Then $T^{**}|_{\hat{\mathcal{X}}} = T$.*
3. *T^* is injective if and only if $T[\mathcal{X}]$ is dense in $\mathcal{Y}$.*
4. *If $T^*[\mathcal{Y}^*]$ is dense in $\mathcal{X}^*$, then T is injective.*

Proof. The verification of these either follows quickly from the definitions or is an application of Hahn-Banach. $\square$

1.2 The Hahn-Banach theorem

For now, $\mathcal{X}$ will be a real vector space.

Definition 1.3. A **sublinear functional** is a function $p : \mathcal{X} \to \mathbb{R}$ such that

1. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathcal{X}$
2. $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$ and $x \in \mathcal{X}$.

Example 1.2. Any seminorm is a sublinear functional.

Theorem 1.1 (Hahn-Banach, general form). *Let $\mathcal{X}$ be a real normed space, let p be a sublinear functional, let $\mathcal{M} \subseteq \mathcal{X}$ be a subspace, and let $f : \mathcal{M} \to \mathbb{R}$ be linear and such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional $F : \mathcal{X} \to \mathbb{R}$ such taht $F|_{\mathcal{M}} = f$ and $F \leq p$.*

Proof. Step 1: Suppose $\mathcal{X} = \mathcal{M} + \mathbb{R}x$, where $x \notin \mathcal{M}$. Then any element of $\mathcal{X}$ is $m + \lambda x$ for some unique $m \in \mathcal{M}$ and $\lambda \in \mathbb{R}$. We want to find $\alpha \in \mathbb{R}$ such that if we set $F(m + \lambda x) := f(m) + \lambda\alpha$, then $F \leq p$; i.e. $f(m) + \lambda\alpha \leq p(m + \lambda x)$ for all $m \in \mathcal{M}$ and $\lambda \in \mathbb{R}$. Equivalently, we want $\lambda\alpha \leq p(m + \lambda x) - f(m)$. We have two cases:

1. If $\lambda > 0$, then this is equivalent to $\alpha \leq p(m + \lambda x)/\lambda - f(m)/\lambda$ for all $m \in M$. That is, $\alpha \leq p(m/\lambda + x) - f(m/\lambda)$. This is equivalent to $\alpha \leq p(m + x) - f(m)$ for all $m \in M$.

2. If $\lambda < 0$, divide by $-\lambda$ and rearrange similarly. We want $-\alpha \leq p(m-x) - f(m)$ for all $m \in \mathcal{M}$. This is $\alpha \geq f(m') - p(m'-x)$ for all $m' \in \mathcal{M}$.

So it remains to show that $f(m') - p(m'-x) \leq p(m+x) - f(m)$ for all $m, m' \in \mathcal{M}$; if we have this, then we can pick any α between these upper and lower bounds. We can rearrange to get $f(m) + f(m') \leq p(m'-x) + p(m+x)$ for all $m, m' \in \mathcal{M}$. But this is

$$\begin{aligned} f(m) + f(m') &= f(m+m') \\ &\leq p(m+m') \\ &= p((m+x) + (m'-x)) \\ &\leq p(m'-x) + p(m+x), \end{aligned}$$

so we can have the desired α.

Step 2: Here is the general case. Let $\mathcal{E}$ be the collection of pairs $(\mathcal{N}, g)$ such that $\mathcal{N}$ is a subspace of $\mathcal{X}$ containing $\mathcal{M}$,and $g : \mathcal{N} \to \mathbb{R}$ is a linear functional such that $g|_{\mathcal{M}} = f$ and $g \leq p$. Define the partial order $(N, g) \leq (\mathcal{N}', g')$ if $N \subseteq \mathcal{N}'$, and $g'|_{\mathcal{N}} = g$. We will use Zorn's lemma. We wnt to show that every chain $((N_\alpha, g_\alpha))_\alpha$ has an upper bound. Let $\mathcal{N} = \bigcup_\alpha N_\alpha$, and let $g(x) = g_\alpha(x)$ for all $x \in \mathcal{N}_\alpha$. Then $g \leq p$, and $g|_{\mathcal{M}} = g_\alpha|_{\mathcal{M}} = f$. We must show that $\mathcal{N}$ and g are linear. If $x, y \in \mathcal{N}$, then $x \in N_\alpha$ and $y \in \mathcal{N}_\beta$, so $x, y \in N_\alpha$, where $N_\alpha \supseteq N_\beta$. So $x + y \in \mathcal{N}_\alpha < N$. So by Zorn's lemma there exists a maximal element $(\mathcal{N}, g) \in \mathcal{E}$. We must have $\mathcal{N} = \mathcal{X}$ otherwise step 1 contradicts the maximality of $\mathcal{N}$. $\square$