Math 255A Lecture Notes

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1 The Hahn-Banach Theorem

1.1 The real Hahn-Banach theorem

Theorem 1.1 (Hahn-Banach, analytic form)**.** *Let V be a vector space over $\mathbb{R}$, and let $p : V \to \mathbb{R}$ be a map which satisfies*

1. *positive homogeneity: $p(\lambda x) = \lambda p(x)$ for all $x \in V$, $\lambda > 0$,*
2. *subadditivity: $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.*

Let $W \subseteq V$ be a linear subspace and let $g : W \to \mathbb{R}$ be a linear form such that $g(x) \leq p(x)$ for all $x \in W$. Then there exists a linear form $f : V \to \mathbb{R}$ which agrees with g on W such that $f(x) \leq p(x)$ for all $x \in V$.

Proof. We will use Zorn's lemma to obtain f. For notation, we write $D(f)$ as the domain of f. Let us consider the set

$$\begin{aligned} P = \{h \mid\, & h : D(h) \to \mathbb{R}, D(h) \subseteq V \text{ is a linear subspace s.t. } W \subseteq D(h), \\ & h|_W = g, h(x) \leq p(x), x \in D(h)\}. \end{aligned}$$

$P \neq \varnothing$ because $g \in P$. P is equipped with the partial order relation $\leq$:

$$h_1 \leq h_2 \iff D(h_1) \subseteq D(h_2) \text{ and } h_2 \text{ extends } h_1.$$

Claim: The set P is inductive, in the sense that any totally ordered subset $Q \subseteq P$ has an upper bound; i.e. there exists $x \in P$ such that $a \leq x$ for all $a \in Q$. Write $Q = (h_j)_{j \in I}$. Let $D(h) = \bigcup_{j \in I} D(h_j)$, and define h by saying $x \in D(h_j) \implies h(x) = h_j(x)$. The function h is well defined, $h \in P$, and $h_j \leq h$ for all $j \in I$.

By Zorn's lemma, we conclude that P has a maximal element f, in the sense that if $f \leq h \in P$, then $h = f$. We have to check that $D(f) = V$; proceed by contradiction. If $D(f) \neq V$, let $x_0 \in V \setminus D(f)$, and define h by $D(h) = D(f) + \mathbb{R}x_0$ and for $x \in D(f)$, $h(x + tx_0) = f(x) + t\alpha$, where $\alpha \in \mathbb{R}$ is to be chosen such that $h \in P$ ($h(x) \leq p(x)$ for $x \in D(h)$).

We have to arrange: $f(x) + t\alpha \leq p(x + tx_0)$ for all $t \in \mathbb{R}$ and $x \in D(f)$. By the positive homogeneity of p, we need only check when $t = \pm 1$. So we need to satisfy:

$$f(x) + \alpha \leq p(x + x_0) \qquad f(x) - \alpha \leq p(x - x_0).$$

In other words, we have to choose α so that

$$\sup_{y \in D(f)} f(y) - p(y - x_0) \leq \alpha \leq \inf_{x \in D(f)} p(x + x_0) - f(x).$$

This is possible as $f(y) - p(y - x_0) \leq p(x + x_0) - f(x)$ for all $x, y \in D(f)$, which follows from $f(x + y) \leq p(y - x_0) + p(x + x_0)$ (by $p(x + y) \geq f(x + y)$). We conclude that $f \leq h$, $h \neq f$, which contradicts the maximality of f. $\square$

1.2 The complex Hahn-Banach theorem

Definition 1.1. Let V be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. A function $p : V \to [0, \infty)$ is a **seminorm** if

1. $p(\lambda x) = |\lambda| p(x)$ for all $x \in V$, $\lambda \in K$
2. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.

Theorem 1.2 (Hahn-Banach, complex version)**.** *Let V be a vector space over $\mathbb{C}$, $W \subseteq V$ a $\mathbb{C}$-linear subspace, and $p : V \to [0, \infty)$ a seminorm. Let $g : W \to \mathbb{C}$ be $\mathbb{C}$-linear such that $|g(x)| \leq p(x)$ for all $x \in W$. Then g can be extended to a $\mathbb{C}$-linear form $f : V \to \mathbb{C}$ such that $|f(x)| \leq p(x)$ for all $x \in V$.*

Proof. Let $g = g_1 + ig_2$, where $g_1(x) = \mathrm{Re}(g(x))$ and $g_2(x) = \mathrm{Im}(g(x))$; g_1, g_2 are $\mathbb{R}$-linear. and defined on W. Note that $g_1(iy) = \mathrm{Re}(g(iy)) = \mathrm{Re}(ig(y)) = -g_2(y)$, so we can recover g_2 from g_1. Now $g_1(y) \leq p(y)$ for all $y \in W$, so by the real version of the Hahn-Banach theorem, there exists an $\mathbb{R}$-linear $f_1 : V \to \mathbb{R}$ such that $f_1|_W = g_1$ and $f_1(x) \leq p(x)$ for all $x \in V$. Let $f(x) = f_1(x) - i(f_1(ix))$. Then, by our previous observation, $f|_W = g$. Note that f is $\mathbb{R}$-linear and $f(ix) = f_1(ix) - if_1(-x) = i(f_1(x) - if_1(ix)) = if(x)$, so f is $\mathbb{C}$-linear. Finally, we check that $|f(x)| \leq p(x)$ for all $x \in V$. If $f(x) \neq 0$, write $f(x) = |f(x)|e^{i\varphi}$ with $\varphi \in \mathbb{R}$. Then

$$|f(x)| = e^{-i\varphi} f(x) = f(e^{-i\varphi} x) = f_1(e^{-i\varphi} x) \leq p(e^{-i\varphi} x) = p(x). \qquad \square$$

1.3 Introduction to dual spaces

Definition 1.2. Let B be a complex Banach space. The **dual space** B^* is the space of linear continuous maps $\xi : B \to \mathbb{C}$.

The form on $B \times B^*$ given by $(x, \xi) \mapsto \xi(x) = \langle x, \xi \rangle$ is bilinear. There may exist linear forms in B^* which are not of the form $\xi \mapsto \langle x, \xi \rangle$.

2 Dual Spaces and the Geometric Hahn-Banach Theorem

2.1 The dual space

Last time, we established the analytic version of the Hahn-Banach theorem. Given Banach spaces B_1, B_2, let $\mathcal{L}(B_1, B_2)$ be the space of continuous linear maps $T : B_1 \to B_2$. Then $\mathcal{L}(B_1, B_2)$ is a Banach space when equipped with the norm

$$\|T\| = \sup_{0 \neq x \in B_1} \frac{\|Tx\|_{B_2}}{\|x\|_{B_1}}.$$

Remark 2.1. To get that $\mathcal{L}(B_1, B_2)$ is complete, we only need that B_2 is complete.

Here is a special case of this construction.

Definition 2.1. Let B be a complex Banach space. The **dual space** $B^* = \mathcal{L}(B, \mathbb{C})$ is the space of linear continuous forms on B.

When $x \in B$ and $\xi \in B^*$, write $\langle x, \xi \rangle := \xi(x)$ so that the form $(x, \xi) \mapsto \langle x, \xi \rangle$ on $B \times B^*$ is bilinear.

Example 2.1. Let $B = L^1(\mathbb{R})$. Then $B^* = L^\infty(\mathbb{R})$. We claim that there exists a continuous linear form on $L^\infty(\mathbb{R})$ which is not of the form $u \mapsto \langle f, u \rangle = \int f u \, dx$. Indeed, by the Hahn-Banach theorem, there exists a linear continuous form L on $L^\infty(\mathbb{R})$ such that $L(u) = u(0)$ whenever $u \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. If we assume that for some $f \in L^1$, $L(u) = \int f u \, dx$ for all $u \in L^\infty$, then in particular, $\int f \varphi \, dx = 0$ for all continuous functions of compact support with $\varphi = 0$ near 0. This implies that $f = 0$ a.e., which is a contradiction.

Definition 2.2. The norm on B^* is given by

$$\|\xi\|_{B^*} = \sup_{0 \neq x \in B} \frac{|\langle x, \xi \rangle|}{\|x\|_B}.$$

Proposition 2.1. *For all $x \in B$,*

$$\|x\|_B = \sup_{0 \neq \xi \in B^*} \frac{|\langle x, \xi \rangle|}{\|\xi\|_{B^*}}.$$

Proof. We have $|\langle x, \xi \rangle| \leq \|x\| \|\xi\|$ by definition for all $\xi \in B^*$. So

$$\sup_{\xi \neq 0} \frac{|\langle x, \xi \rangle|}{\|\xi\|} \leq \|x\|.$$

On the other hand, let $W = \mathbb{C}x \subseteq B$, and let $\xi_0 : W \to \mathbb{C}$ be $\alpha x \mapsto \alpha\|x\|$. We have $|\xi_0(y)| = \|y\|$ for all $y \in W$, so by Hahn-Banach, ξ_0 extends to $\tilde{\xi} \in B^*$ such that $|\tilde{\xi}(y)| \leq \|y\|$ for all $y \in B$ and $\tilde{\xi}(x) = \|x\|$. So $\|\tilde{\xi}\| = 1$, which gives us

$$\|x\| = \frac{|\langle x, \tilde{\xi}\rangle|}{\|\tilde{\xi}\|} \leq \sup_{\xi \neq 0} \frac{|\langle x, \tilde{\xi}\rangle|}{\|\tilde{\xi}\|}.$$

□

Remark 2.2. This proposition implies that the natural map $\varphi : B \to B^{**}$ given by $x \mapsto (\xi \mapsto \langle x, \xi\rangle)$ is an isometry. The range is closed but may be strictly smaller than B^{**}.

2.2 Geometric version of the Hahn-Banach theorem

Definition 2.3. Let V be a normed vector space over $\mathbb{R}$. An **affine hyperplane** in V is a set of the form $H = f^{-1}(\alpha)$, where $\alpha \in \mathbb{R}$, f is linear, and $f \neq 0$.

Proposition 2.2. *The affine hyperplane $H = f^{-1}(\alpha)$ is closed if and only if f is continuous.*

Proof. It is clear that if f is continuous, then H is closed. Conversely, if H is closed, let $x_0 \in H^c$, which is open. We may assume that $f(x_0) < \alpha$. Let $r > 0$ be such that $B(x_0, r) = \{x \in V : \|x - x_0\| < r\} \cap H = \varnothing$.

We claim that $f(x) < \alpha$ for all $x \in B(x_0, r)$. If $f(x_1) > \alpha$ for some $x_1 \in B(x_0, r)$, then the line segment $\{tx_0 + (1-t)x_1 : 0 \leq t \leq 1\} \subseteq B(x_0, t)$, so $f(tx_0 + (1-t)x_1) \neq \alpha$ for all t. If $t = \frac{\alpha - f(x_0)}{f(x_1) - f(x_0)} \in (0, 1)$, we get a contradiction.

We get $f(x_0 + ry) < \alpha$ for all y with $\|y\| = 1$. So f is bounded, and hence f is continuous. □

Definition 2.4. Let V be a normed vector space over $\mathbb{R}$, and let $A, B \subseteq V$. We say that the affine hyperplane $H = f^{-1}(\alpha)$ **separates** A and B if we have $f(x) \leq \alpha$ for all $x \in A$ and $f(x) \geq \alpha$ for all $x \in B$.

Theorem 2.1 (geometric Hahn-Banach)**.** *Let V be a normed vector space over $\mathbb{R}$, and let $A, B \subseteq V$ be convex, disjoint, and nonempty. Assume also that A is open. Then there exists a closed affine hyperplane separating A and B.*

This is sometimes called the "seperation theorem." We will prove this next time. Here is the idea of the proof. Given an open convex set $C \subseteq V$, define the gauge of C as $p(x) = \inf\{t > 0 : x/t \in C\}$.

3 Proof of the Geometric Hahn-Banach Theorem

3.1 Gauges and the real geometric Hahn-Banach theorem

Theorem 3.1 (geometric Hahn-Banach)**.** *Let V be a real normed vector space with $A, B \subseteq V$ convex, nonempty and disjoint. Also assume A is open. Then there exists a closed affine hyperplane separating A and B.*

Before we prove this, we need a bit of background.

Definition 3.1. Let $C \subseteq V$ be convex and open such that $0 \in C$. Define the **gauge** of C as

$$p(x) = \inf\{t > 0 : x/t \in C\}.$$

Lemma 3.1. *The gauge of C satisfies the following properties:*

1. *$p(\lambda x) = \lambda p(x)$ for $\lambda > 0$ and $x \in V$*
2. *$p(x + y) \leq p(x) + p(y)$ for $x, y \in V$*
3. *there exists $M > 0$ such that $p(x) \leq M\|x\|$ for all $x \in V$ ($\implies$ p is continuous at 0).*
4. *$C = \{x \in V : p(x) < 1\}$*

Proof. (i) is clear.

(iii) Let $r > 0$ be such that $\{x : \|x\| \leq r\} \subseteq C$. Then for all x with $\|x\| = 1$, $rx \in C$, so $p(x) \leq 1/r$. So $p(x) \leq \|x\|/r$ for all $x \in V$.

(iv) We first show $C \subseteq \{x : p(x) < 1\}$. If $x \in C$, then $(1 + \varepsilon)x \in C$ for ε small. So $p(x) \leq 1/(1 + \varepsilon) < 1$. On the other hand, if $p(x) < 1$, then $x/t \in C$ for some $0 < t < 1$. So $x = t(x/t) + (1 - t)0 \in C$ (by convexity of C).

(ii) Let $x, y \in V$ and $\varepsilon > 0$. Then $x/(p(x) + \varepsilon), y/(p(y) + \varepsilon) \in C$, and their convex combination

$$t\frac{x}{p(x) + \varepsilon} + (1 - t)\frac{y}{p(y) + \varepsilon}$$

is also in C for $0 \leq t \leq 1$. Take $t = (p(x) + \varepsilon)/(p(x) + p(y) + 2\varepsilon)$. So

$$\frac{x + y}{p(x) + p(y) + 2\varepsilon} \in C$$

which gives us that $p(x + y) < p(x) + p(y) + 2\varepsilon$. So p is subadditive. $\square$

Lemma 3.2. *Let $C \subseteq V$ be open, convex, and nonempty, and let $x_0 \notin C$. Then there exists a continuous linear form $f : V \to \mathbb{R}$ such that $f(x) < f(x_0)$ for all $x \in C$. In particular, the closed affine hyperplane $H = f^{-1}(f(x_0))$ separates x_0 and C.*

Proof. By translation, we may assume that $0 \in C$. Let $g : \mathbb{R}x_0 \to \mathbb{R}$ send $tx_0 \mapsto t$. Then $g(tx_0) \le p(tx_0)$ for any $t \in \mathbb{R}$, where p is the gauge of C; indeed, for $t \le 0$, this is ok, and if $t > 0$, this is also ok, as $p(x_0) \ge 1$. By the analytic version of the Hahn-Banach theorem, g extends to a linear form $f : V \to \mathbb{R}$ such that $f(x_0) = 1$ and $f(x) \le p(x)$ for any $x \in V$. In particular, $f(x) < 1 = f(x_0)$ for $x \in C$. The function f is continuous as $f(x) \le p(x) \le M\|x\|$ for all $x \in V$. □

We are now ready to prove the geometric Hahn-Banach theorem.

Proof. Let $C = A - B = \{x - y : x \in A, y \in B\}$. Then C is convex because A, B are convex, $0 \notin C$, and C is open (because $C = \bigcup_{y \in B}(A - y)$, which is a union of open sets). By the previous lemma, there exists a linear continuous form f such that $f < 0$ on C. Then $f(x) < f(y)$ for $x \in A$ and $y \in B$. If $\sup_{x \in A} f(x) \le \alpha \le \inf_{y \in B} f(y)$, then $f^{-1}(\alpha)$ separates A and B. □

3.2 The complex geometric Hahn-Banach theorem

Definition 3.2. Let V be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. We say that $M \subseteq V$ is **balanced** if $\lambda x \in M$ for all $x \in M$ and $\lambda \in K$ with $|\lambda| \le 1$.

Proposition 3.1. *Let V be a normed vector space over $\mathbb{C}$, and let $C \subseteq V$ be open, convex, nonempty, and balanced. Let $x_0 \notin C$. Then there exists a complex linear continuous map $f : V \to \mathbb{C}$ such that $f(x_0) \neq f(x)$ for all $x \in C$. In particular, the closed affine hyperplane $H = f^{-1}(f(x_0))$ contains x_0 and does not meet C.*

Proof. Since C is balanced, $0 \in C$. Let p be the gauge of C. Then $C = \{x : p(x) < 1\}$, and p is a seminorm; i.e. $p(\lambda x) = |\lambda| p(x)$ and $p(x+y) \le p(x) + p(y)$. We can now conclude that there is a continuous linear form $f : V \to \mathbb{C}$ such that $f(x_0) = 1$ and $|f| \le p$ on V. Then $|f| < 1$ on C, so f is continuous. □

Remark 3.1. The gauge p of C (convex, open, balanced, contains 0) satisfies the following inequality:

$$|p(x + y) - p(y)| \le p(x) \le M\|x\|.$$

So p is Lipschitz continuous on V.

Corollary 3.1. *Let V be a normed vector space over $\mathbb{C}$, and let $A \subseteq V$ be a closed, convex, nonempty, and balanced. Let $x \notin A$. We can find a continuous linear form f on V such that* $\inf_{y \in A} |f(y) - f(x)| > 0$.

Proof. Let $\varepsilon > 0$ be so small that $(x + B(0, \varepsilon)) \cap A = \varnothing$. The set $B(0, \varepsilon) + A$ is open, convex, balanced, and does not contain x, so by the previous lemma, there is a continuous linear form f such that $f(x) \neq f(y) + f(z)$, where $y \in A$ and $z \in B(0, \varepsilon)$. Here, $f(B(0, \varepsilon)) \neq \{0\}$ is a balanced subset of $\mathbb{C}$, so it contains a neighborhood of 0. □

4 The Spanning Criterion and Runge's Theorem

4.1 The spanning criterion

For the complex geometric Hahn-Banach theorem, we don't actually need the assumption that C is balanced.

Theorem 4.1 (complex geometric Hahn-Banach)**.** *Let V be a complex normed vector space, and let $C \subseteq V$ be open, convex, and nonempty. Let $x_0 \notin C$. Then there is a continuous linear map $f : V \to \mathbb{C}$ such that $f(x) \neq f(x_0)$.*

Proof. We can regard V as a vector space over $\mathbb{R}$. Then there exists a continuous $\mathbb{R}$-linear $f_1 : V \to \mathbb{R}$ such that $f_1(x) < f_1(x_0)$ for all $x \in C$. We set $f(x) = f_1(x) - i f_1(ix)$. This is $\mathbb{C}$-linear, continuous, and $f(x) \neq f(x_0)$. $\square$

Corollary 4.1. *Let $A \subseteq V$ be closed, convex, and nonempty. Let $x \notin A$. Then there exists a linear continuous $f : V \to \mathbb{C}$ such that $\inf_{y \in A} |f(y) - f(x)| > 0$.*

We will return to the idea of a balanced set later, so our previous discussion is not a waste.

Theorem 4.2 (spanning criterion)**.** *Let V be a normed vector space over $\mathbb{C}$, and let W be a linear subspace. Then the closure $\overline{W}$ can be described as follows:*

$$\overline{W} = \{v \in V : f(v) = 0 \text{ for all } f \in V^* \text{ s.t. } f|_W = 0\}.$$

In other words,

$$\overline{W} = \bigcap_{\substack{f \in V^* \\ f|_W = 0}} \ker(f).$$

Proof. ($\subseteq$): If f is linear and continuous with $f|_W = 0$, then $f|_{\overline{W}} = 0$. So $\overline{W} \subseteq \ker(f)$.

($\supseteq$): Let $x \notin \overline{W}$. $\overline{W}$ is closed and convex, so there exists a continuous linear form $f : V \to \mathbb{C}$ such that $f(x) \neq f(y)$ for all $y \in \overline{W}$. In particular, $f(x) \neq 0$. Let $y \in \overline{W}$. Then $\lambda y \in \overline{W}$ for all $\lambda \in \mathbb{C}$. So $f(x) \neq \lambda f(y)$ for all λ. Thus, $f(y) = 0$ for all $y \in \overline{W}$. We get $f|_W = 0$ and $f(x) \neq 0$. $\square$

Remark 4.1. We can get the exact same statement in the real case, as well.

4.2 Runge's theorem

We will have two types of applications of the Hahn-Banach theorem:

1. approximation theorems
2. existence theorems.

Theorem 4.3 (Runge). *Let $K \subseteq \mathbb{C}$ be a compact set with $K^c = \mathbb{C} \setminus K$ connected. Let f be a function which is holomorphic in a neighborhood of K. Then for any $\varepsilon > 0$, there exists a holomorphic polynomial g such that $|f(z) - g(z)| \leq \varepsilon$ for all $z \in K$.*

Before we prove this, let's mention a fact from complex analysis that we will need in the proof.

Proposition 4.1. *Let $\omega \subseteq \mathbb{C}$ be a bounded, open set with C^1-boundary and let $u \in C^1(\overline{\omega})$. Then*

$$u(z) = \frac{1}{2\pi i} \int_{\partial \omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint_{\omega} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z} L(d\zeta),$$

where $L(d\zeta)$ is Lebesgue measure in $\mathbb{C}$, and

$$\frac{\partial}{\partial \overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re}(\zeta)} + i \frac{\partial}{\partial \operatorname{Im}(\zeta)} \right)$$

is the Cauchy-Riemann operator.

Proof. Here is the idea. Apply the Stokes-Green formula to the function $\zeta \mapsto u(\zeta)/(\zeta - z)$ in $\omega_\varepsilon = \{\zeta \in \omega : |\zeta - z| > \varepsilon\}$:

$$\int_{\partial \omega_\varepsilon} \frac{u(\zeta)}{\zeta - z}\, d\zeta = 2i \iint_{\omega_\varepsilon} \underbrace{\frac{\partial}{\partial \overline{\zeta}} \left(\frac{u(\zeta)}{\zeta - z} \right)}_{\frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}}} L(d\zeta)$$

and let $\varepsilon \to 0^+$. $\square$

Proof. Apply the spanning criterion with $V = C(K)$ (equipped with the sup norm) and $W = \{p|_K : p \text{ is a polynomial}\}$. Let f be holomorphic in a neighborhood of K. To show that $f|_K \in \overline{W}$, we need to show that if $L \in C(K)^*$ satisfies $L(p) = 0$ for all polynomials p, then $L(f) = 0$. By the Riesz representation theorem, the dual of $C(K)$ is the space of (Radon) measures on K. We have to show that if μ is a measure on K such that $\int_K z^n \, d\mu(z) = 0$ for all n, then $\int_K f(z)\, d\mu(z) = 0$.

Now let $f \in \operatorname{Hol}(\omega)$, where ω is a neighborhood of K. Let $\psi \in C_0^1(\omega)$ (the set of C^1 functions on ω with compact support) be such that $\psi = 1$ near K. Apply the proposition to $u = f\psi \in C_0^1(\mathbb{C})$. Then

$$f(z)\psi(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} f(\zeta) \frac{\partial \psi}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z}\, L(d\zeta)$$

for all $z \in K$.

Consider

$$\int_K f(z)\, d\mu(z) = \int_K \left(-\frac{1}{\pi} \iint_{\mathbb{C}} f(\zeta) \frac{\partial \psi}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z}\, L(d\zeta) \right) d\mu(z)$$

$$= -\frac{1}{\pi} \iint_{\mathbb{C}\setminus K} \frac{\partial \psi}{\partial \overline{\zeta}}(\zeta) f(\zeta) \left(\int_K \frac{1}{\zeta - z}\, d\mu(z) \right) L(d\zeta).$$

If suffices to show that

$$\int_K \frac{1}{\zeta - z}\, d\mu(z) = 0,$$

where $\zeta \in \mathbb{C} \setminus K$. We will finish this next time. $\square$

5 Müntz's Theorem and the Poisson Equation

5.1 Müntz's theorem

First, let's finish our proof of Runge's theorem.

Theorem 5.1 (Runge)**.** *Let $K \subseteq \mathbb{C}$ be a compact set with $K^c = \mathbb{C} \setminus K$ connected. Let f be a function which is holomorphic in a neighborhood of K. Then for any $\varepsilon > 0$, there exists a holomorphic polynomial g such that $|f(z) - g(z)| \leq \varepsilon$ for all $z \in K$.*

Proof. We had a measure μ on K such that $\int_K z^n \, d\mu(z) = 0$ for all $n \in \mathbb{N}$, and we got was

$$\int_K f(z) \, d\mu(z) = -\frac{1}{\pi} \iint_{\mathbb{C}\setminus K} \frac{\partial \psi}{\partial \overline{\zeta}} f(\zeta) M(\zeta) \, L(d\zeta),$$

where

$$M(\zeta) = \int_K \frac{1}{\zeta - z} \, d\mu(z).$$

To finish the proof, it suffices to show that $M = 0$ on $\mathbb{C}\setminus K$. Consider the Laurent expansion of M at ∞:

$$M(\zeta) = \sum_{j=0}^{\infty} \frac{1}{\zeta^{j+1}} \int_K z^j \, d\mu(z) = \sum_{j=0}^{\infty} \frac{1}{\zeta^{j+1}} 0 = 0.$$

Then $M = 0$ for large $|\zeta|$, and hence $M = 0$ in all of $\mathbb{C} \setminus K$ because $\mathbb{C} \setminus K$ is connected. □

Theorem 5.2 (Müntz)**.** *Let $(\lambda_j)_{j \in \mathbb{N}}$ be a sequence of distinct positive real numbers such that $\lambda_j \to \infty$ as $j \to \infty$. Then the closed linear span of the functions $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ in $C([0,1])$ is equal to $C([0,1])$ if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Proof. We shall only prove the sufficiency of the series condition. By the spanning criterion, we have to show the following: if μ is a finite complex Borel measure on $[0,1]$ such that $\int_{[0,1]} 1 \, d\mu(t) = \int_{[0,1]} t^{\lambda_j} \, d\mu(t) = 0$ for all j, then for all $f \in C[0,1]$, $\int f \, d\mu = 0$. We claim that if $\int_{[0,1]} t^{\lambda_j} \, d\mu(t) = 0$ for all j, then $\int_{[0,1]} t^k \, d\mu(t) = 0$ for all $k = 1, 2, \dots$. The claim implies the result by the Weierstrass approximation theorem.

We may assume that μ is concentrated on $(0,1]$ since the integrands t^k all vanish at $t = 0$. Consider the function $F(\zeta) = \int_{[0,1]} t^\zeta \, d\mu(t)$, where $\zeta \in \mathbb{C}$ with $\operatorname{Re}(\zeta) > 0$. Then F is bounded and holomorphic in $\operatorname{Re}(\zeta) > 0$. We have $F(\lambda_j) = 0$ for all j. Map the right half plane onto the disc: $G(z) = F(\zeta)$, where $\zeta = (1+z)/(1-z)$ for $|z| < 1$. Then $G \in \operatorname{Hol}(|z| < 1)$ is bounded, and $G(\alpha_j) = 0$, where $\alpha_j = (\lambda_j - 1)/(\lambda_j + 1) \to 1$.

Recall now Jensen's formula, which says that if $f \in \mathrm{Hol}(|z| < 1)$ such that $f(0) \neq 0$, and $(\alpha_k)_{j=1}^N$ are the zeros of f (counting multiplicities) such that $|\alpha_j| \leq r < 1$, then

$$\sum_{|\alpha_j| \leq r} \log \frac{r}{|\alpha_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\varphi})| \, d\varphi - \log |f(0)|.$$

So if f is bounded, the right hand side is $O(1)$ as $r \to 1$. Using that $\log(t) \geq 1 - t$ for $t \geq 0$, we get

$$\sum_{|\alpha_j| \leq r} (r - |\alpha_j|) \leq C$$

for $r < 1$. Letting $r \to 1$, we get that if $f \in \mathrm{Hol}(|z| < 1)$ is bounded and not identically 0, the zeros (α_j) of f satisfy $\sum(1 - |\alpha_j|) < \infty$.

In our case, $\alpha_j = (\lambda_j - 1)/(\lambda_j + 1)$, and we may assume that $\alpha_j > 0$. Then

$$\sum(1 - |\alpha_j|) = \sum(1 - \frac{\lambda_j - 1}{\lambda_j + 1}) = \sum \frac{2}{\lambda_{j+1}} = \infty.$$

Thus, $G = 0$, so $F(\zeta) = \int_{[0,1]} t^\zeta \, d\mu(t) = 0$ for $\mathrm{Re}(\zeta) > 0$. $\square$

5.2 Solving the Poisson equation using Hahn-Banach

We will try to solve the Poisson equation. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $f \in L^2(\Omega)$ be real-valued. Let $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ be the Laplacian. We would like to solve the equation $\Delta u = f$ in some sense. The existence of solutions to this equation can be reduced to the proof of an inequality.

Proposition 5.1. *There exists a constant $A > 0$ such that for any $\varphi \in C_0^2(\Omega)$ (C^2 functions on Ω with compact support), we have*

$$\|\varphi\|_{L^2(\Omega)} \leq A \|\Delta \varphi\|_{L^2(\Omega)}.$$

We will prove this next time.

Remark 5.1. An inequality of this form holds for all differential operator with constant coefficients, in place of Δ.

6 Weak Solutions of the Poisson Equation and Strengthened Hahn-Banach

6.1 Weak solutions of the Poisson equation

Last time, we were trying to solve the equation $\Delta u = f$ for $f \in L^2(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open and bounded.

Proposition 6.1. *There exists a constant $A > 0$ such that for any $\varphi \in C_0^2(\Omega)$ (C^2 functions on Ω with compact support), we have*

$$\|\varphi\|_{L^2(\Omega)} \le A\|\Delta\varphi\|_{L^2(\Omega)}.$$

Proof. For simplicity of notation, we assume φ is real. Then, using integration by parts,

$$\int_\Omega \varphi\Delta\varphi\,dx = \sum_j \int_\Omega \varphi\frac{\partial^2\varphi}{\partial x_j^2}\,dx = -\sum_j \int_\Omega \left(\frac{\partial\varphi}{\partial x_j}\right)^2\,dx = -\int_\Omega |\nabla\varphi|^2\,dx.$$

Also,

$$\int_\Omega x_1\,\underbrace{2\varphi\frac{\partial\varphi}{\partial x_1}}_{=\partial_{x_1}(\varphi^2)}\,dx = -\int_\Omega \varphi^2\,dx$$

implies that, using Cauchy-Schwarz,

$$\|\varphi\|_{L^2}^2 \le 2C\int_\Omega |\varphi||\partial_{x_1}\varphi|\,dx \le 2C\|\varphi\|_{L^2}\|\nabla\varphi\|_{L^2}.$$

Thus,

$$\|\varphi\|_{L^2} \le 2C^2\|\Delta\varphi\|_{L^2}.$$

□

Now let $\Delta C_0^2(\Omega) = \{\Delta\varphi : \varphi \in C_0^2(\Omega)\} \subseteq L^2(\Omega)$. Consider the linear form $L : \Delta C_0^2(\Omega) \to \mathbb{C}$ that sends $\Delta\varphi \mapsto \int_\Omega f\varphi\,dx$, where $f = \Delta u$. This form L is well-defined (thanks to the proposition), and we get

$$|L(\Delta\varphi)| \le \|f\|_{L^2}\|\varphi\|_{L^2} \le 4C^2\|f\|_{L^2}\|\Delta\varphi\|_{L^2}.$$

By Hahn-Banach, L extends to a continuous linear form $\tilde{L}$ on all of $L^2(\Omega)$ such that

$$|\tilde{L}(v)| \le 4C^2\|f\|_{L^2}\|v\|_{L^2}$$

for any $v \in L^2(\Omega)$. By the Riesz representation theorem, there exists $u \in L^2(\Omega)$ such that $\tilde{L}(v) = \int_\Omega vu\,dx$ for any $v \in L^2$ and $\|u\|_{L^2} \le 4C^2\|f\|_{L^2}$.

When $v = \Delta\varphi$ with $\varphi \in C_0^2(\Omega)$, we get

$$\int_\Omega u\Delta\varphi\,dx = \int_\Omega f\varphi\,dx.$$

If u were of class $C^2(\Omega)$, we would get $\int \Delta u\varphi = \int f\varphi\ \forall\varphi$, so $\Delta u = f$ a.e. In general, $u \in L^2(\Omega)$ satisfying the above equation is called a **weak solution** of $\Delta u = f$.

6.2 Strengthened Hahn-Banach theorem

Theorem 6.1. *Let V be a normed vector space over $K = \mathbb{R}$ or $\mathbb{C}$, and let $T : V \to V$ be a continuous linear map such that $\|T\| \le 1$ ($\|Tx\| \le \|x\|$ for all $x \in V$). Assume that T has a fixed point $x_0 \neq 0$ such that $Tx_0 = x_0$. Then there is a linear continuous form $f : V \to K$ such that $\|f\| = 1$, $f(x_0) = \|x_0\|$, and $f(Tx) = f(x)$ for all $x \in V$.*

Proof. Let us define $\|x\|_T = \inf \|\sum_{n=0}^{\infty} \lambda_n T^n x\|$, where the inf is taken over all $\lambda_n \ge 0$ such that $\sum \lambda_n = 1$, where only finitely many are nonzero in this sum.[1] We claim that $x \mapsto \|x\|_T$ is a seminorm on V. We only need to check the triangle inequality. Let $x, y \in V$ and $\varepsilon > 0$. Then there exist $\lambda_n \ge 0$ and $\mu_n \ge 0$ with $\sum \lambda_n = \sum \mu_n = 1$ such that

$$\left\|\sum \lambda_n T^n x\right\| < \|x\|_T + \varepsilon, \qquad \left\|\sum \mu_n T^n y\right\| < \|y\|_T + \varepsilon.$$

By the triangle inequality,

$$\begin{aligned}\left\|\left(\sum \lambda_n T^n\right)\left(\sum \mu_n T^n\right)(x+y)\right\| &\le \left\|\left(\sum \lambda_n T^n\right)\underbrace{\left(\sum \mu_n T^n\right)}_{\text{norm}\le 1} x\right\| \\ &\quad + \left\|\underbrace{\left(\sum \lambda_n T^n\right)}_{\text{norm}\le 1}\left(\sum \mu_n T^n\right) y\right\| \\ &\le \|x\|_T + \|y\|_T + 2\varepsilon.\end{aligned}$$

Now observe that

$$\|x+y\|_T \le \left\|\sum \left(\sum_{n+m=j} \lambda_n \mu_m\right) T^j (x+y)\right\| = \left\|\left(\sum \lambda_n T^n\right)\left(\sum \mu_n T^n\right)(x+y)\right\|.$$

Apply Hahn-Banach with respect to this seminorm. Let $g : Kx_0 \to \mathbb{C}$ send $\alpha x_0 \mapsto \alpha\|x_0\|$. Then $|g(y)| = \|y\| = \|y\|_T$ for all $y \in Kx_0$. Then g extends to a linear form f such that $f(x_0) = \|x_0\|$ and $|f(x)| \le \|x\|_T$ for $x \in V$. Finally, check that $f(Tx) = f(x)$:

$$|f(Tx) - f(x)| = |f(Tx - x)| \le \|Tx - x\|_T = 0$$

for all x, where the last equality comes from

$$\left\|\frac{1}{N}(1 + T + \cdots + T^{N-1})(Tx - x)\right\| = \left\|\frac{1}{N}(T^N x - x)\right\| \le \frac{2\|x\|}{N} \xrightarrow{N\to\infty} 0. \qquad \square$$

Remark 6.1. When T is the identity, this is the usual Hahn-Banach theorem.

[1] This is a really clever choice for a seminorm.

6.3 Generalized Banach limits

Here is an application due to Banach himself. Let $V = \ell^\infty(\mathbb{N})$ with elements $x = (x_1, x_2, \ldots,)$ with $x_j \in \mathbb{C}$. Let the shift operator be $(Tx)_j = x_{j+1}$. By the theorem, there is a continuous linear form $f : \ell^\infty \to \mathbb{C}$ such that $\|f\| = 1$, $f(1, 1, \ldots) = 1$, $f(Tx) = f(x)$ for all $x \in \ell^\infty$. Note that

$$|f(x)| = |f(T^n x)| \leq \sup_{j>n} |x_j|,$$

so

$$|f(x)| \leq \limsup_{n\to\infty} |x_n|.$$

For all $c \in \mathbb{C}$ plugging in $x + (c, c, \ldots)$ gives

$$|f(x) - c| \leq \limsup_{n\to\infty} |x_n - c|.$$

So if (x_n) converges, then $f(x) = \lim_{n\to\infty} x_n$.

Remark 6.2. If x is a real sequence, $\liminf x_n \leq f(x) \leq \limsup x_n$.

7 Locally Convex Spaces

7.1 Topologies induced by seminorms

Let V be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, and let $(p_\alpha)_{\alpha \in A}$ be a family of seminorms on V. We may introduce a topology on V as follows:

$O \subseteq V$ is open if for any $x \in O$, there exists $\varepsilon > 0$ and finitely many seminorms $p_{\alpha_1}, \dots, p_{\alpha_J}$ such that $N_{\alpha_1,\dots,\alpha_J,\varepsilon} = \bigcap_{j=1}^J \{y \in V : p_{\alpha_j}(y - x) < \varepsilon\} \subseteq O$. This defines a topology on V, and the sets $N_{\alpha_1,\dots,\alpha_J,\varepsilon}$ are open. We shall assume that $(p_\alpha)_{\alpha \in A}$ separates points: $p_\alpha(x) = 0$ for all α iff $x = 0$.

Remark 7.1. An open neighborhood of 0 of the form $\bigcap_{j=1}^J \{x \in V : p_{\alpha_j}(x) < \varepsilon\}$ is balanced and convex.

Definition 7.1. A vector space with a topology defined by a family of seminorms is called a **locally convex space**.

Proposition 7.1. *A locally convex space is Hausdorff.*

Proof. Let $x, y \in V$ be distinct, and let $\alpha \in A$ be such that $p_\alpha(x - y) \neq 0$. Then the open sets $O_x = \{z \in V : p_\alpha(x - z) < p_\alpha(x - y)/4\}$ and $O_x = \{z \in V : p_\alpha(y - z) < p_\alpha(x - y)/4\}$ are disjoint. □

Remark 7.2. In a locally convex space V, the vector operations $+ : V \times V \to V$ and $\cdot : K \times V \to V$, given by $(x, y) \mapsto x + y$ and $(a, x) \mapsto ax$ respectively, are continuous. In particular, translations $x \mapsto x + y$ are homeomorphisms.

Remark 7.3. In a locally convex space, the convex, balanced, open sets of the form $\bigcap_{j=1}^J \{x \in V : p_{\alpha_j}(x) < \varepsilon\}$ form a fundamental system of neighborhoods of 0. Conversely, assume that V is a vector space with a Hausdorff topology in which the vector operations are continuous. Assume that 0 has a fundamental system of neighborhoods which are convex and balanced. Then V is a locally convex space. Let N be such a neighborhood of 0, and let p be gauge of N, $p(x) = \inf\{t > 0 : x/t \in N\}$. Then we know that p is a seminorm on V, and $N = \{x \in V : p(x) < 1\}$.

7.2 Continuity of seminorms

Proposition 7.2. *Let V be a locally convex space with the topology defined by $(p_\alpha)_{\alpha \in A}$. A seminorm p on V is continuous if and only if there is some constant $C > 0$ and $\alpha_1, \dots, \alpha_J$ such that $p(x) \leq C \sum_{j=1}^J p_{\alpha_j}(x)$ for all $x \in V$.*

Proof. We have $|p(x + y) - p(y)| \leq p(x)$, so p is continuous if and only if p is continuous at 0. To show that the condition for continuity is sufficient, we need that for all $\varepsilon > 0$, there exists a neighborhood of $0 \in V$ such that $x \in U \implies p(x) < \varepsilon$. We can take $p_{\alpha_j}(x) < \delta$ for $1 \leq j \leq J$ and $\delta > 0$.

On the other hand, if p is continuous at 0, then there is a neighborhood U of 0 such that $x \in U \implies p(x) < 1$. Thus, there exist $\varepsilon > 0$ and seminorms $p_{\alpha_1}, \ldots, p_{\alpha_J}$ such that $p_{\alpha_j}(x) < \varepsilon \, \forall j \in \{1, \ldots, J\} \implies p(x) < 1$. Equivalently, if $t > 0$ and x is replaced by tx, $tp_{\alpha_j}(x) < \varepsilon \implies tp(x) < 1$. Take

$$t = \frac{\varepsilon}{\sum_{j=1}^{J} p_{\alpha_j}(x) + \mu},$$

where $\mu > 0$. We get

$$p(x) < \frac{1}{\varepsilon} \left(\sum_{j=1}^{J} p_{\alpha_j}(x) + \mu \right)$$

for all $\mu > 0$. □

Remark 7.4. Assume that we have 2 systems of seminorms on V, $(p_\alpha)_{\alpha \in A}$ and $(q_\beta)_{\beta \in B}$. The locally complex topology defined by (p_α) is stronger (has more open sets) the locally convex topology generated by (q_β) if and only if for any $\beta \in B$, we have $q_\beta(x) \le C \sum_{j=1}^{J} p_{\alpha_j}(x)$ for all $x \in V$.

Example 7.1. The space $V = C(\mathbb{R})$ becomes a locally convex space with the topology defined by the seminorms $p_n(f) = \sup_{|x| \le n} |f(x)|$. This topology cannot be defined by a single seminorm p. Otherwise, we would have that for every n, there is a constant $C_n > 0$ such that $p_n(f) \le C_n p(f)$ for every $f \in C(\mathbb{R})$. We can choose $f \in C(\mathbb{R})$ such that $f(n) = nC_n$ for all n, contradicting this inequality when n is large.

Next time, we will show that a locally convex topology is metrizable if and only if it can be defined by countable many seminorms.

8 Metrizability and Fréchet Spaces

8.1 Metrizability of locally convex spaces

Last time, we introduced the idea of a locally convex vector space V where the topology is defined by a family of seminorms $(p_\alpha)_{\alpha \in A}$. Here, $O \subseteq V$ is open if for all $x \in I$, there exists an $\varepsilon > 0$ and $p_{\alpha_1}, \dots, p_{\alpha_J}$ such that $p_{\alpha_j}(y - x) < \varepsilon \,\forall j \implies y \in O$.

Theorem 8.1. *A locally convex space V is metrizable if and only if the topology can be defined by a countable family of seminorms.*[2] *The metric can be chosen to be translation invariant : $d(x, y) = d(x - y)$.*

Proof. ($\implies$): Each neighborhood of 0 contains a set of the form $\{x \in V : d(x, 0) < 1/n\}$ for $n \in \mathbb{N}$. If the locally convex topology on V is defined by the seminorms (p_α), them for all n, there exists $p^{(n)}$, a positive linear combination of finitely many p_α such that if $p^{(n)}(x) < 1$, then $d(x, 0) < 1/n$. So every neighborhood of 0 contains a set of the form $\{x \in V : p^{(n)}(x) < 1\}$, and thus the seminorms $(p^{(n)})_{n \in \mathbb{N}}$ define the topology.

($\impliedby$): Let us assume that the locally convex topology on V is generated by the seminorms $(p_n)_{n \in \mathbb{N}}$ such that $p_n(x) = 0 \,\forall n \iff x = 0$. Set

$$d(x) = \sum_{n=1}^\infty 2^{-n} \frac{p_n(x)}{1 + p_n(x)}$$

for each $x \in V$. We have

1. $d(x) > 0$ for $x \neq 0$
2. $d(-x) = d(x)$
3. $d(x + y) \leq d(x) + d(y)$: We need to check that $f(t) = t/(1 + t)$ for $t \geq 0$ is increasing and subadditive. It is increasing because $f(t) = 1 - \frac{1}{1+t}$. $f(t)/t$ is decreasing, so $f(t)/t \geq f(t + s)/(t + s)$ when $t, s > 0$. So $f(t) + f(s) \geq f(t + s)$.

We get that $d(x, y) = d(x - y)$ is a metric on V.

We check now that the topology defined by d is the same as the topology defined by the p_n. If $d(x) < \varepsilon 2^{-N}$ for some $\varepsilon \in (0, 1)$, then $2^{-n} p_n(x)/(1 + p_n(x)) < \varepsilon 2^{-N}$ for $n \leq N$. Then $p_n(x) < \varepsilon/(1 - \varepsilon)$ for $n \leq N$. So any set of the form "a finite intersection of $\{x \in V : p_n(x) < \varepsilon\}$" contains an open d-ball around 0.

Conversely, if $p_n(x) < \varepsilon/2$ for all $n \leq N$, then

$$d(x) = \sum_{n=0}^N 2^{-n} \underbrace{\frac{p_n(x)}{1 + p_n(x)}}_{\leq \varepsilon/(2+\varepsilon)} + \sum_{n=N+1}^\infty 2^{-n} \underbrace{\frac{p_n(x)}{1 + p_n(x)}}_{<1} < 2 \frac{\varepsilon}{2 + \varepsilon} + 2^{-N} < \varepsilon$$

[2] We should also include the condition here that V is Hausdorff, but we assume this is always true in our definition of locally convex spaces because we assume that the seminorms separate points.

for N large enough such that $2^{-N} < \varepsilon/2$. Thus, any open d-ball around 0 contains all finite intersections of sets of the form $\{x \in V : p_n(x) < \varepsilon\}$. $\square$

Remark 8.1. If $(x_j)_{j\in\mathbb{N}}$ is in V, then $x_j \to x \iff d(x_j, x) \to 0 \iff p_n(x_j - x) \to 0$ for each n.

8.2 Fréchet spaces

Definition 8.1. A locally convex, metrizable, and complete space is called a **Fréchet space**.

Example 8.1. Let $\Omega \subseteq \mathbb{R}^n$ be open. The space $C(\Omega)$ is a Frèchet space with the topology defined by the seminorms $u \mapsto \sup_{x\in K} |u(x)|$ with compact $K \subseteq \Omega$. The topology is metrizable as it suffices to use $u \mapsto \sup_{K_j} |u|$, where $K_j = \{x \in \Omega : |x| \leq j, d(x, \Omega^c) \geq 1/j\}$. If (u_j) is a Cauchy sequence in $C(\Omega)$ (for compact $K \subseteq \Omega$, if $\sup_K |u_j - u_k| \xrightarrow{j,k\to\infty} 0$), then there exists $u \in C(\Omega)$ such that $u_j \to u$ in $C(\Omega)$. If $\Omega \subseteq \mathbb{C}$ is open, then the space $\mathrm{Hol}(\Omega)$ is a Fréchet space viewed as a subspace of $C(\Omega)$ because a uniform limit of holomorphic functions is holomorphic.

Example 8.2. Let $\Omega \subseteq \mathbb{R}^n$ be open and $j \in \mathbb{N} \cup \{\infty\}$. Then the space $C^j(\Omega)$ is a Fréchet space with the topology given by the seminorms $u \mapsto \sup_{x\in K} |\partial^\alpha u(x)|$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, and $|\alpha| := \sum_{k=1}^n \alpha_k \leq j$.

Let $(V_1, (p_n)), (V_2, (q_n))$ be Fréchet spaces. A linear map $T : V_1 \to V_2$ is continuous if and only if for any n, there exists $\varepsilon > 0$ and $p_{i_1}, \dots, p_{i_m}$ such that $p_{i_j}(x) < \varepsilon \, \forall j \implies q_n(Tx) < 1$. This condition is equivalent to $q_n(Tx) \leq C_n \sum_{j=1}^m p_{i_j}(x)$ for all n.

Example 8.3. A linear form $u : C^\infty(\Omega) \to \mathbb{C}$ is continuous if and only if there exist $C > 0$, $m \in \mathbb{N}$, and a compact $K \subseteq \Omega$ such that

$$|u(f)| \leq C \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha f|$$

for $f \in C^\infty(\Omega)$.

9 Applications of Baire's Theorem I: The Open Mapping Theorem

9.1 The open mapping theorem

Banach used Baire's theorem to prove a number of striking results in functional analysis. Recall Baire's theorem.

Theorem 9.1 (Baire category). *Let E be a complete metric space, and let $(F_n)_{n\in\mathbb{N}}$ be closed in E containing no interior points. Then the union $\bigcup_{n=1}^{\infty} F_n$ has no interior points either. Moreover, $E \neq \bigcup_{n=1}^{\infty} F_n$.*

Definition 9.1. We say that $A \subseteq E$ is **of the first category** (or **meager**) if there exists a sequence F_n of closed sets without interior points such that $A \subseteq \bigcup_{n=1}^{\infty} F_n$.

Theorem 9.2 (Banach, open mapping theorem). *Let F_1, F_2 be Fréchet spaces, and let $T : F_1 \to F_2$ be linear continuous. Then either* $\operatorname{im}(T) \subseteq F_2$ *is of the first category, or else* $\operatorname{im}(T) = F_2$ *and the mapping T is open.*

Proof. Let U be an open neighborhood of 0 in F_1. We claim that $\overline{T(U)}$ contains a neighborhood of 0 in F_2, provided $\operatorname{im}(T)$ is not of the first category. Let V be a balanced neighborhood of 0 in F_1 such that $V + V \subseteq U$. Then V is absorbing (for $x \in F_1$, $\lambda x \in V$ for sufficiently small $|\lambda|$). So $F_1 = \bigcup_{n=1}^{\infty} nV$ means that $\operatorname{im}(T) = \bigcup_{n=1}^{\infty} T(nV) \subseteq \bigcup_{n=1}^{\infty} \overline{T(nV)}$. Since $\operatorname{im}(T)$ is not of the first category, for some n, $\overline{T(nV)} = n\overline{T(V)}$ has an interior point. Then $\overline{T(V)}$ has an interior point. So there exists $y \in F_2$ and a neighborhood W of 0 in F_2 such that $\{y\} + W \subseteq \overline{T(V)}$. Then $y \in \overline{T(V)}$. $V = -V$ since V is balanced, so $-y \in \overline{T(V)}$. So $W \subseteq \overline{T(V) + \{-y\}} \subseteq \overline{(\overline{T(V) - T(V)})} = \overline{T(V) - T(V)}$. We get $W \subseteq \overline{T(V + V)} \subseteq \overline{T(U)}$, as claimed.

Let d_{F_1} be a translation invariant metric on F_1 generating the topology on F_1, and define d_{F_2} similarly. Thus, for any $r > 0$, there exists $\rho > 0$ such that $B_{F_2}(0, \rho) \subseteq T(B_{F_1}(0, r))$. The metrics d_{F_1}, d_{F_2} are translation invariant, so for any $r > 0$, there exists a $\rho > 0$ such that for any $x \in F_1$, $B_{F_2}(Tx, \rho) \subseteq \overline{T(B_{F_1}(x, r))}$. Let $r > 0$ be arbitrary and let $r_n = r/2^n$ for $n \in N$. We get the corresponding ρ_n sequence such that $B_{F_2}(Tx, \rho_n) \subseteq \overline{T(B_{F_1}(x, r_n))}$ for all $x \in F_1$. We can arrange so that $\rho_n \downarrow 0$.

Let $y \in B_{F_2}(Tx, \rho_0)$. We shall show that there is an $x' \in F_1$ such that $d_{F_1}(x, x') \leq 2r$ and $y = Tx'$. Let $x_1 \in \overline{B_{F_1}(x, r_0)}$ be such that $d_{F_2}(y, Tx_1) < \rho_1 \iff y \in B_{F_2}(Tx_1, \rho_1) \subseteq \overline{T(B_{F_1}(x_1, r_1))}$. Let $x_2 \in B_{F_1}(x_1, r_1)$ be such that $d_{F_2}(y, Tx_2) < \rho_2$. Then $y \in B_{F_2}(Tx_2, \rho_2) \subseteq \overline{T(B_{F_1}(x_2, r_2))}$. Continuing in this fashion, we get a sequence (x_n) in F_1 such that $x_{n+1} \in \overline{T(B_{F_1}(x_n, r_n))}$. Then (x_n) is a Cauchy sequence in F_1, and $d_{F_2}(y, Tx_n) < \rho_n \to 0$. We get $x_n \to x' \in F_1$, where $d_{F_1}(x, x') \leq 2r$, and, since T is continuous, $Tx_n \to Tx'$. So $y = Tx'$.

So we get that for all $r > 0$, there exists $\rho > 0$ such that $B_{F_2}(Tx, \rho) \subseteq T(B_{F_1}(x, 2r))$. Hence, $\operatorname{im}(T) = F_2$, and T is open. $\square$

Corollary 9.1. *Let $T : F_1 \to F_2$ be an injective, linear, continuous map between Frèchet spaces. Then either the range of T is of the first category, or $\mathrm{im}(T) = F_2$, and T is a homeomorphism.*

9.2 Application of the open mapping theorem to partial differential equations

Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, where $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $D_{x_j} = (1/i)\partial_{x_j}$ be a partial differentiation operator (on $\mathbb{R}^n$) with constant coefficients $a_\alpha \in \mathbb{C}$. Assume that for some open set $\Omega \subseteq \mathbb{R}^n$, every solution $u \in C^m(\Omega)$ of $Pu = 0$ is in fact in $C^{m+1}(\Omega)$ (e.g. $P = \Delta$, the Laplacian). Then we have $\mathrm{Im}(\zeta) \to \infty$ if $\zeta \to \infty$ on the suface in $\mathbb{C}^n$ given by $0 = P(\zeta) = \sum_{|\alpha| \leq m} a_\alpha \zeta^\alpha$. We will do this in detail next time.

10 Applications of Baire's Theorem II: The Closed Graph Theorem

10.1 Differential operators and the open mapping theorem

Last time, we had a differential operator $P(D)$ on $\mathbb{R}^n$ with constant coefficients and of order m such that if $u \in C^m(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open, then $Pu = 0 \implies u \in C^{m+1}(\Omega)$. Write $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$.

Proposition 10.1. *if $|\operatorname{Im}(\zeta)| \to \infty$ as $|\zeta| \to \infty$, then $\zeta \in P^{-1}(0) \subseteq \mathbb{C}^n$, where $P(\zeta) = \sum_{|\alpha| \le m} a_\alpha \zeta^\alpha$.*

Example 10.1. If $P(D) = -\Delta = \sum_{j=1}^n D_{x_j}^2$, then $P(\zeta) = \sum_{j=1}^n \zeta_j^2 = \zeta \cdot \zeta$ for $\zeta \in \mathbb{C}^n$. We get $P^{-1}(0) = \{\zeta \in \mathbb{C}^n : |\operatorname{Re}(\zeta)| = |\operatorname{Im}(\zeta)|, \operatorname{Re}(\zeta) \cdot \operatorname{Im}(\zeta) = 0\}$. So $|\zeta| \to \infty$ along $P^{-1}(0) \iff |\operatorname{Im}(\zeta)| \to \infty$ along $P^{-1}(0)$.

Example 10.2. Consider also the Schrödinger equation: $i\partial_t u = -\Delta_x u$, where $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Then $P(D_x, D_t) = \sum D_{x_j}^2 + D_t$ gives us the polynomial $P(\xi, \tau) = \xi \cdot \xi + \tau$, where $\xi \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. If $|\xi| + |\tau| \to \infty$ along $P^{-1}(0)$, the Schrödinger equation has a solution in $C^2 \setminus C^3$.

Proof. Let $F_1 = \{x \in C^{m+1}(\Omega) : Pu = 0\}$ and $F_2 = \{x \in C^m(\Omega) : Pu = 0\}$. Then F_1 and F_2 are Fréchet spaces. Our assumption is that the inclusion map $F_1 \to F_2$ is surjective. By the open mapping theorem, the inverse $F_2 \to F_1$ is continuous. So for any compact set $K \subseteq \Omega$, there exists a compact set $K' \subseteq \Omega$ and $C > 0$ such that

$$\sum_{|\alpha| \le m+1} \sup_K |\partial^\alpha u| \le C \sum_{|\alpha| \le m} \sup_K |\partial^\alpha u|$$

for any $u \in F_1 = F_2$. If $\zeta \in \mathbb{C}^n$ is such that $P(\zeta) = 0$, then apply this inequality, where $u(x) = e^{ix \cdot \zeta}$. Then $P(e^{ix\cdot\zeta}) = P(\zeta) e^{ix \cdot \zeta} = 0$. So we get

$$\sum_{|\alpha| \le m+1} \sup_K |\zeta^\alpha| e^{-x \cdot \operatorname{Im}(\zeta)} \le C \sum_{|\alpha| \le m} |\zeta^\alpha| \sup_K e^{-x \cdot \operatorname{Im}(\zeta)}.$$

So there exists $C > 0$ such that

$$\sum_{|\alpha| \le m+1} |\zeta^\alpha| \le C e^{C|\operatorname{Im}(\zeta)|} \sum_{|\alpha| \le m} |\zeta^\alpha| = O((1 + |\zeta|)^m).$$

It follows that $|\operatorname{Im}(\zeta)| \to \infty$ when $|\zeta| \to \infty$ and $P(\zeta) = 0$. $\square$

10.2 The closed graph theorem

Definition 10.1. Let $T : D(T) \to F_2$, where $D(T) \subseteq F_1$ and F_1, F_2 are Fréchet spaces. We say that T is **closed** if when $x_n \in D(T)$ with $x_n \to x \in F_1$ and $Tx_n \to y \in F_2$, then $x \in D(T)$ and $y = Tx$.

Note that T is closed iff the graph of T, $G(T) = \{(x, Tx) : x \in D(T)\}$ is closed in $F_1 \oplus F_2$. If T is linear and closed, then the graph of T is a Fréchet space (as a closed linear subspace of a Fréchet space).

Theorem 10.1 (closed graph theorem)**.** *Let $T : D(T) \to F_2$ be a closed linear map, where $D(T) \subseteq F_1$. Then either $D(T)$ is of the first category in F_1, or $D(T) = F_1$ and T is continuous. The range of T is either of the first category, or it is all of F_2.*

Proof. For the first statement, apply the open mapping theorem to the linear, continuous, injective map $G(T) \to F_1$ given by $(x, Tx) \mapsto x$. For the second statement, apply the open mapping theorem to the map $G(T) \to F_2$ given by $(x, Tx) \mapsto Tx$. □

Corollary 10.1. *Let H be a Hilbert space, and let $T : H \to H$ be linear such that $F(T) = H$ and T is symmetric ($\langle Tx, y\rangle = \langle x, Ty\rangle$). Then T is continuous.*

Proof. Check that T is closed. If $x_n \to x \in H$ and $Tx_n \to y \in H$, then $\langle Tx_{,n}, z\rangle = \langle x_n, Tz\rangle$ for all $x \in H$. Then $\langle y, z\rangle = \langle x, Tz\rangle = \langle Tx, z\rangle$ for all z, so $y = Tx$. □

Corollary 10.2. *Let B_0, B_1, B_2 be Banach spaces, and let T_j be closed linear maps $D(T_j) \to B_j$ with $D(T_j) \subseteq B_0$ for $j = 1, 2$. If $D(T_1) \subseteq D(T_2)$, then there exists some $C > 9$ such that $\|T_2 x\| \leq C(\|T_1 x\|_{B_1} + \|x\|_{B_0})$ for any $x \in D(T_1)$.*

Proof. Consider the map $\hat{T} : G(T_1) \to B_2$ sending $(x, T_1 x) \mapsto T_2 x$. It suffices to show that $\hat{T}$ is closed. Suppose that (x_n, Tx_n) converges in $G(T_1)$ and $(T_2 x_n)$ converges in B_2. T_1 is closed, so $x_n \to x \in D(T_1)$, and $T_1 x_n \to T_1 x$. T_2 is closed, so $x \in D(T_2)$, and $T_2 x_n \to T_2 x$. □

11 Applications of Baire's Theorem III: The Uniform Boundedness Principle

11.1 Equicontinuity

Definition 11.1. A subset M of a locally convex space V is **bounded** if every continuous seminorm p is bounded on M: $\sup_{x \in M} p(z) \leq C < \infty$.

When V_1, V_2 are locally convex, we let $\mathcal{L}(V_1, V_2)$ be the space of all linear continuous maps $V_1 \to V_2$.

Definition 11.2. We say that $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is **equicontinuous** if for every neighborhood U_2 of 0 in V_2, there is a neighborhood U_1 of 0 in V_1 such that $x \in U_1 \implies T_x \in U_2$ for every $T \in \Phi$.

If p_j is a continuous seminorm on V_j $(j = 1, 2)$ that $U_j = \{x \in V_j : p_j(x) < 1\}$, then the equicontinuity of Φ means that $p_1(x) < 1 \implies p_2(Tx) < 1$ for all $T \in \Phi$. This implies that $p_2(Tx) \leq p_1(x)$ for all $x \in V_1$ and $T \in \Phi$. We get that $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is equicontinuous if and only if there exist a continuous seminorm p_1, p_2 on V_1, V_2 such that

$$p_2(Tx) \leq p_1(x)$$

for all $x \in V_1$ and $T \in \Phi$.

Remark 11.1. If V_1, V_2 are normed spaces, then $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is equicontinuous means that there exists $C > 0$ such that $\|Tx\|_{V_1} \leq C\|x\|_{V_1}$ for all $x\ in V_1$ and $T \in \Phi$. That is, $\|T\|_{\mathcal{L}(V_1,V_2)} \leq C$ for every $T \in \Phi$.

11.2 Proof of the uniform boundedness principle

Theorem 11.1 (Banach-Steinhaus, uniform boundedness principle)**.** *Let F be a Fréchet space, and let V be a locally convex space. If $\Phi \subseteq \mathcal{L}(F, V)$ is such that for each $x \in F$ the set $\{Tx : T \in \Phi\} \subseteq V$ is bounded, then Φ is equicontinuous. On the other hand, if Φ is not equicontinuous, then the set of all $x \in F$ such that $\{Tx : T \in \Phi\}$ is bounded is a set of the first category.*

Proof. Let U be an open, convex, balanced neighborhood of 0 in V, and consider the set $A = \{x \in F : Tx \in \overline{U}\ \forall T \in \Phi\} = \bigcap_{T \in \Phi} T^{-1}(U)$. A is an intersection of closed sets, so it is closed. A is convex as the intersection of convex sets. Also, A is symmetric. Distinguish between two different cases:

1. A has an interior point for any choice of U: Then there exists $x_0 \in F$ and a convex, symmetric neighborhood of 0 in F (call it V) such that $\{x_0\} + V \subseteq A$. Since V is balanced, $\{-x_0\} + V \subseteq A$, and the convexity of V gives

$$V = \frac{1}{2}(\{x_0\} + V) + \frac{1}{2}(\{-x_0\} + V) \subseteq A.$$

We get that $V \subseteq \bigcap_{T\in\Phi} T^{-1}(\overline{U})$, so $T(V) \subseteq \overline{U}$ for all $T \in \phi$. So Φ is equicontinuous.

2. There exists a neighborhood U such that $A = \bigcap_{T\in\Phi} T^{-1}(\overline{U})$ has empty interior. Then $\bigcup_{n=1}^{\infty} nA \subseteq F$ is of the first category, and we claim that it contains the set $\{x \in F : \{Tx : T \in \Phi\} \text{ is bounded}\}$. Take a continuous seminorm p on V such that $\{y : p(y) < 1\} \subseteq U$. Then, since $p(Tx) \leq C$ for all $T \in \Phi$, there exists some $n \in \mathbb{N}$ such that $p(Tx/n) < 1$ for all $T \in \Phi$. So $T(x/n) \in U$ for all $T \in \Phi$, and so $x/n \in A$, which makes $x \in nA$.

To summarize, if $\{Tx : T \in \Phi\}$ is bounded for all $x \in F$, then we are necessarily in case 1 by the open mapping (aka Baire's) theorem. If Φ is not equicontinuous, we are in case 2, and the set $\{x \in F : \{Tx : T \in \Phi\} \text{ is bounded}\}$ is of the first category in F. $\square$

11.3 Applications of the uniform boundedness principle

Corollary 11.1. *Let F be a Fréchet space, and let V be locally convex and metrizable.*[3] *Let $T_j \in \mathcal{L}(F, V)$ be such that for all $x \in F$, the sequence $(T_j x)$ converges in V. Let $Tx = \lim_{j\to\infty} T_j x$. Then $T \in \mathcal{L}(F, V)$.*

Proof. Linearity is preserved under limits, so T is linear. For any continuous seminorm p on V and for all $x \in F$, $p(T_j x) \leq C(x)$ for all j. By the Banach-Steinhaus theorem, (T_j) is equicontinuous. That is, for every continuous seminorm p_2 on V, there exists a continuous seminorm p_1 on F such that $p_2(T_j x) \leq p_1(x)$. for all $x \in F$ and for all j. If we let $j \to \infty$, we get $p_2(Tx) \leq p_1(x)$, so $T \in \mathcal{L}(F, V)$. $\square$

Let $f \in C(\mathbb{R})$ be 2π-periodic. Associated to f is its Fourier series $\sum_{-\infty}^{\infty} c_n(f) e^{inx}$, where

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

are the Fourier coefficients. Let $S_N(f, x) = \sum_{-N}^{N} c_n(f) e^{inx}$. Next time, we will show that or all 2π-periodic $f \in C(\mathbb{R})$ outside of a set of the first category, $(S_N(f, x))_{N=1}^{\infty}$ is unbounded for all $x \in \mathbb{Q}$.

[3] The metrizability of V is not actually necessary in this result.

12 Unbounded Fourier Coefficients and Bilinear Maps

12.1 Unbounded partial sums of Fourier coefficients

Last time we introduced an application of the Banach-Steinhaus theorem. Let $S_n(f,0) = \sum_{-N}^{N} c_n(F)$, where $c_n(f)$ is the n-th Fourier coefficient of f.

Proposition 12.1. *There exists a 2π-periodic $f \in C(\mathbb{R})$ such that the sequence $(S_N(f,0))_{N=1}^{\infty}$ is unbounded.*

Proof.

$$S_n(f,0) = \sum_{n=-N}^{N} c_n(f) = \int_{-\pi}^{\pi} D_N(x) f(x)\, dx,$$

where $D(x) = \sum_{-N}^{N} e^{inx}$ is the DIrichlet kernel. We have

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

If the claim does not hold, we have that $(S_n(f,0))$ is bounded for all $f \in B$, the Banach space of continuous 2π-periodic functions with $\|f\|_B = \sup_{[-\pi,\pi]} |f|$. By the Banach-Steinhaus theorem, there exists $C > 0$ such that $|S_N(f,0)| \le C\|f\|_B$ for all $f \in B$ and $N \in \mathbb{N}^+$. So

$$\left| \int_{-\pi}^{\pi} D_N(x) d(x)\, dx \right| \le \|f\|_B \implies \|D_N\|_{L^1(-\pi,\pi)}.$$

On the other hand,

$$\begin{aligned}
\|D_N\|_{L^1(-\pi,\pi)} &= \frac{2}{2\pi} \int_0^{\pi} \frac{|\sin((N+1/2)x)|}{\sin(x/2)} \\
&\ge \frac{4}{2\pi} \int_0^{\pi} \frac{|\sin((N+1/2)x)|}{x}\, dx \\
&= \frac{2}{\pi} \int_0^{(N+1/2)\pi} \frac{|sin(x)|}{x}\, dx \\
&\ge \frac{2}{\pi} \sum_{n=1}^{N-1} \int_{n\pi}^{(n+1)\pi} \frac{|\sin(x)|}{x}\, dx \\
&\ge \frac{4}{\pi^2} \sum_{n=2}^{N} \frac{1}{n} \\
&= \frac{4}{\pi^2} \log(N) + O(1)
\end{aligned}$$

as $N \to \infty$. If follows that the set of all $f \in B$ such that $(S_N(f,0))_{N=1}^{\infty}$ is bounded is of the first category. By translation invariance, we get the same statement for $(S_N(f,x))_{N=1}^{\infty}$

for each fixed $x \in \mathbb{R}$. Taking the union over all $x \in \mathbb{Q}$, we get a set of the first category such that if f is in the complement, then $(S_n(f,x))_{N=1}^{\infty}$ is unbounded for all $x \in \mathbb{Q}$. $\square$

Remark 12.1. Notice that for all $f \in B$, we have $S_N(f,x) = o(\log(N))$ uniformly in x, as $N \to \infty$. This follows as $\|D\|_{L^1} = O(\log(N))$ and $S_N(f,x) = O(1)$ for 2π-periodic $f \in C^1(\mathbb{R})$ (dense in B).

12.2 Bilinear maps

Let E, F, G be locally convex spaces, and let $B : E \times F \to G$ be bilinear.

Proposition 12.2. *Assume that B is continuous at $0 \in E \times F$. Then B is continuous.*

Proof. Let U_G be a neighborhood of $0 \in G$, and let U_E, U_F be neighborhoods of 0 in E, F such that if $x \in U_E$ and $y \in U_F$, then $B(x,y) \in U_G$. Write $B(x+x_0, y+y_0) = B(x,y) + B(x,y_0) + b(x_0,y) + B(x_0,y_0)$. As U_E, U_F are absorbing, let $\varepsilon > 0$ be such that $\varepsilon x_0 \in U_E$ and $\varepsilon y \in U_G$. Then $B(x,y_0) = B(x/\varepsilon, \varepsilon y_0) \in U_G$ if $x/\varepsilon \in U_E$. Similarly, $B(x_0,y) \in U_G$ if $y/\varepsilon in U_F$. When $x \in U_E \cap \varepsilon U_E$ and $y \in U_F \cap \varepsilon E_F$, $B(x+x_0, y+y_0) - B(x_0,y_0) \in U_G + U_G + U_G$. $\square$

We have that $B : E \times F \to G$ is continuous iff for every continuous seminorm p_G on G, there exist continuous seminorms p_E on E and p_F on F such that

$$p_G(B(x,y)) \leq p_E(x) p_F(y)$$

for all $x \in E$ and $y \in F$.

Definition 12.1. We say that a bilinear form B is **separately continuous** if the linear forms $x \mapsto B(x,y)$ for fixed y and $y \mapsto B(x,y)$ for fixed x are continuous.

Theorem 12.1. *Let E be locally convex and metrizable, F a Fréchet space, and G a locally convex space. If the bilinear form $B : E \times F \to G$ is separately continuous, then B is continuous.*

Proof. Let U be a an open, convex, symmetric neighborhood of $0 \in G$. Let $V_1 \supseteq V_2 \supseteq \cdots$ be a fundamental system of neighborhoods of 0 in E. Let

$$A_j = \{y \in F : B(x,y) \in \overline{U} \ \forall c \in V_j\} = \bigcap_{x \in V_j} B^{-1}(x, \cdot)(\overline{U})$$

As $y \mapsto B(x,y)$ is continuous, A_j is closed. It is also convex and symmetric. For any $y \in F$, $x \mapsto B(x,y)$ is continuous, so there exists j such that $x \in V_j \implies B(x,y) \in \overline{U}$. In other words, $\bigcup_{j=1}^{\infty} A_j = F$. By the open mapping theorem, there exists some j such that A_j has an interior point. Arguing as in the proof of the Banach-Steinhaus theorem, we get that 0 is an interior point of A_j; i.e. there exists a neighborhood N of $0 \in F$ such that if $y \in N$ and $x \in V_j$, $B(x,y) \in U$. Thus, B is continuous at 0 and hence continuous. $\square$

Remark 12.2. It suffices to have a locally convex topology on E defined by countably many seminorms (no Hausdorff property is needed).

13 Non-Solvability of Lewy's Pperator

13.1 Continuity of bilinear forms

Here is a slight reformulation of a theorem we proved last lecture.

Theorem 13.1. *Let E be a locally convex space with the topology defined by countably many seminorms (not necessarily Hausdorff), F be a Fréchet space, and let G be locally convex space. Let $B : E \times F \to G$ be bilinear such that for all $x \in E$, $y \mapsto B(x,y)$ is continuous. If B is not continuous, then the set of all $y \in F$ such that $x \mapsto B(x,y)$ is a set of the first category.*

The proof is roughly the same, as well. We sketch it briefly.

Proof. Let $A_j = \{y \in F : B(x,y) \in \overline{U}\ \forall x \in V_j\}$, where U is a neighborhood of 0 in G and V_j form a fundamental system of neighborhoods of 0 in E. Then A_j is closed, convex, and symmetric. We claim that if $y \in F$ is such that $x \mapsto B(x,y)$ is continuous, then $y \in A_j$ for some j. If A_j has a nonempty interior for some j, then B is continuous. Thus if B is not continuous, the set $\{y \in F : x \mapsto B(x,y) \text{ continuous}\} \subseteq \bigcup_j A_j$ is of the first category. $\square$

13.2 Non-solvability of Lewy's operator

Theorem 13.2 (H. Lewy, 1957)**.** *There exists $f \in C^\infty(\mathbb{R}^3)$ such that the differential equation $Pu = (D_{X_1} + iD_{x_2} + 2i(x_1 + ix_2)D_{x_3})u = f$ does not have a distributional solution u in any neighborhood of 0. Here, $D_{x_j} = \partial_{x_j}/i$.*

Remark 13.1. One can show that this differential equation cannot be solved in any open set in $\mathbb{R}^3$.

Proof. This argument is due to Hörmander. Let $\Omega \subseteq \mathbb{R}^3$ be an open neighborhood of 0. What it means for $u \in D^1(\Omega)$ to solve this equation is that for all test functions $\varphi \in C_0^\infty(\Omega)$,

$$\underbrace{P_u(\varphi)}_{u(-P\varphi)} = f(\varphi) = \int f\varphi\, dx.$$

Therefore, for any compact set $K \subseteq \Omega$, there exist C, m such that

$$|f(\varphi)| \le C \sum_{|\alpha| \le m} \sup_K |\partial^\alpha (P\varphi)|$$

when $\varphi \in C_0^\infty(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq K$.

Let $W = \{\varphi \in C_0^\infty(\Omega) : \operatorname{supp}(\varphi) \subseteq L\}$ with the locally convex topology given by the seminorms $\varphi \mapsto \sum_{|\alpha| \le m} \sup |\partial^\alpha P\varphi|$ (only countable many seminorms occur). $F = C^\infty(\mathbb{R}^3)$, which is Fréchet. Now consider the bilinear map $B : E \times F \to \mathbb{C}$ given by

$(\varphi, f) \mapsto \int f\varphi\, dx$. B is continuous in f for any fixed φ. B is also continuous in φ if the equation $Pu = f$ has a solution $u \in D^1(\Omega)$, in view of the above inequality.

We claim that the map B is not continuous provided that $0 \in \mathrm{int}(K)$. Assume that B is continuous. Then there exist a compact $L \subseteq \mathbb{R}^3$, C, and m such that

$$|f(\varphi)| \leq C \left(\sum_{|\alpha| \leq m} \sup |\partial^\alpha P\varphi| \right) \left(\sum_{|\alpha| \leq m} \sup_L |\partial^\alpha f| \right)$$

for all $\varphi \in C_0^\infty$ with $\mathrm{supp}(\varphi \subseteq K$ and $f \in C^\infty(\mathbb{R}^3)$.

The idea is to show that the estimate is not valid by constructing a quasimode of P; we want to have φ such that $P\varphi \approx 0$ and $\varphi \approx 1$.[4] The form of P gives us that $P(x_1^2 + x_2^2 + ix_3) = 0$. Consider

$$w(x) = \frac{1}{i} \left[-x_1^2 - x_2^2 - ix_3 + (x_1^2 + x_2^2 + ix_3)^2 \right].$$

This satisfies $Pw = 0$. Note that $w = \frac{1}{i}\left[-|x|^2 - ix_3 + O(|x|^3)\right]$, so $\mathrm{Im}(w) = |x|^2 + O(|x|^3) \sim |x|^2$ near 0. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be such that $\chi = 1$ near 0 and such that $\mathrm{Im}(w) \geq |x^2|/2$ on $\mathrm{supp}(\chi)$. Let $V_\lambda(x) = \chi(x)e^{i\lambda w(x)} \in C_0^\infty$ with $\lambda \gg 1$. Then $\mathrm{supp}(v_\lambda) \subseteq K$, and $|v_\lambda| \sim e^{-\lambda|x|^2}$. Take $v_\lambda = \varphi$ in the inequality. Then $Pv_\lambda = (P\chi)e^{i\lambda w} = O(e_\lambda^{-c\lambda})$ with $c > 0$. We get

$$\sum_{|\alpha| \leq m} \sup |\partial^\alpha P\varphi| = O(\lambda^m e^{-c\lambda}) \xrightarrow{\lambda \to \infty} 0.$$

Take $f(x) = f_\lambda(x) = e^{i\lambda x_3}\lambda^3 h(\lambda x)$ for $0 < h \in C_0^\infty$ with $\int h = 1$. The right hand side in the inequality is $O(\lambda^m e^{-c\lambda}\lambda^{3+M}$, which goes to 0 as $\lambda \to \infty$. The left hand side is

$$\int e^{i\lambda x_3}\lambda^3 h(\lambda x)\chi(x)e^{i\lambda w(x)}\, dx = \int e^{ix_3} h(x)\chi(x/\lambda)e^{i\lambda w(x/\lambda)}\, dx \xrightarrow{\lambda \to \infty} \int h = 1.$$

We get that the set of $f \in C^\infty$ such that the equation $pu = f$ has a solution $u \in D^1(\Omega)$ is of the first category. □

[4]Up to this point in the proof, we have not used the form of the operator P at all. This argument shows that if we can find a quasimode for any operator P with this property, then we can show that P has no solutions in this sense.

14 Fredholm Operators

14.1 Fredholm operators

Definition 14.1. Let B_1, b_2 be complex Banach spaces. An operator $T \in \mathcal{L}(B_1, B_2)$ is called a **Fredholm operator** if $\ker(T)$ and $\operatorname{coker}(T) = B_2/\operatorname{im}(T)$ are finite dimensional.

This is an operator that may fail to be injective and surjective by only finitely many dimensions.

Definition 14.2. The **index** of a Fredholm operator T is defined as $\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\operatorname{coker}(T))$.

Remark 14.1. If $T \in \mathcal{L}(B_1, B_2)$, then $\ker(T) \subseteq B_1$ is closed. However, $\operatorname{im}(T)$ need not be closed. For example, take $B_1 = B_2 = C([0,1])$, and $Tf(x) = \int_0^x f(y)\,dy$.

Theorem 14.1. *Let* $T \in \mathcal{L}(B_1, B_2)$ *be such that* $\dim(\operatorname{coker}(T)) = \operatorname{codim}(\operatorname{im}(T)) < \infty$. *Then* $\operatorname{im}(T) \subseteq B_2$ *is closed.*

Proof. We can assume that T is injective; otherwise, consider $\tilde{T} : B_1/\ker(T) \to B_2$ given by $x + \ker(T) \mapsto Tx$. Then $\tilde{T}$ is injective, and $\operatorname{im}(\tilde{T}) = \operatorname{im}(T)$. Let $\dim(B_2/\operatorname{im}(T)) = n < \infty$, and ket $x_1, \dots, x_n$ be such that $x_1 + \operatorname{im}(T), \dots, x_n + \operatorname{im}(T)$ form a basis for $B_2/\operatorname{im}(T)$. Then for an $y \in B_2$, we can write

$$y = Tz + \sum_{j=1}^{n} a_j x_j$$

for $z \in B_1$.

The linear continuous map $S : \mathbb{C}^n \to B_2$ given by $(a_1, \dots, a_n) \mapsto \sum_{j=1}^n a_j x_j$ is injective, and $B_2 = \operatorname{im}(T) \oplus \operatorname{im}(S)$. It follows that the map $T_1 : B_1 \oplus \mathbb{C}^n \to B_2$ sending $(x, a) \mapsto Tx + Sa$ is a linear, continuous bijection, and by the open mapping theorem, T_1 is a homeomorphism. We get $\operatorname{im}(T) = T_1(B_1 \oplus \{0\})$, which the image of a closed set. So $\operatorname{im}(T) \subseteq B_2$ is closed. $\square$

In particular, any Fredholm operator has closed image.

14.2 Perturbing Fredholm operators

Lemma 14.1. *Let B be a Banach space, and let $S \in \mathcal{L}(B, B)$ be such that $\|S\| < 1$. Then the operator $I - S$ has an inverse in $\mathcal{L}(B, B)$.*

Proof. Consider the Neumann series $R = \sum_{k=0}^{\infty} S^K$. This converges in $\mathcal{L}(B, B)$ since $\sum_{k=1}^{\infty} \|S^k\| \le \sum_{k=0}^{\infty} \|S\|^k = 1/(1 - \|S\|) < \infty$. We have $R(I - S) = (i - S)R = I$. $\square$

Remark 14.2. Let $T \in \mathcal{L}(B_1, B_2)$ be bijective. Then T^{-1} is continuous by the open mapping theorem, and $T + S = T(I + T^{-1}S)$ is invertible, provided that $\|T^{-1}\|\|S\| < 1$.

Theorem 14.2. *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator. If $S \in \mathcal{L}(B_1, B_2)$ is such that $\|S\|$ is sufficiently small, then $T + S$ is Fredholm and* $\operatorname{ind}(T+S) = \operatorname{ind}(T)$*.*

Proof. Let $T : B_1 \to B_2$ be Fredholm, and let $n_+ = \dim(\ker(T))$ and $n_- = \dim(\operatorname{coker}(T))$. Let $R_- : \mathbb{C}^{n_-} \to B_2$ be linear, continuous, and injective such that $B_2 = \operatorname{im}(T) \oplus R_-(\mathbb{C}^{n_-})$. Let $e_1, \ldots, e_{n+}$ be a basis for $\ker(T)$, and let $\varphi_1, \ldots, \varphi_{n+} \in B_1^*$ such that $\varphi_j(e_k) = \delta_{j,k}$; these exist by Hahn-Banach. Let $R_+ : B_1 \to \mathbb{C}^{n_+}$ send $x \mapsto (\varphi_1(x), \ldots, \varphi_{n_+}(x))$. Then R_+ is linear, continuous, and surjective, and $R_+|_{\ker(T)}$ is bijective.

Let us introduce the operator[5]

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}.$$

We claim that $\mathcal{P}$ is bijective. If $x \in B_1$ and $a \in C^{n_-}$, then

$$\mathcal{P} \begin{bmatrix} x \\ a \end{bmatrix} = \begin{bmatrix} Tx + R_- a \\ R_+ x \end{bmatrix}.$$

$\mathcal{P}$ is injective since $R_+|_{\ker(T)}$ was given to be bijective. By construction, $\mathcal{P}$ is surjective. It follows that

$$\tilde{\mathcal{P}} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix}$$

is also invertible, provided that $\|S\|$ is small enough. Let $\mathcal{E} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus \mathbb{C}^{n_-}$ be the inverse of $\tilde{\mathcal{P}}$:

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

We have $E : B_2 \to B_1$, $E_+ : \mathbb{C}^{n_+} \to B_1$, $E_- : B_2 \to \mathbb{C}^{n_-}$, and $E_{-+} : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}$. Observe that

$$\tilde{P}\mathcal{E} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_+E_+ \end{bmatrix},$$

so $R_+E_+ = I$. So E_+ has a left inverse, which means it is injective. Similarly, E_-R_- is the identity on $\mathbb{C}^{n_-}$, so E_- is surjective. We will finish the proof next time. ☐

[5] This operator is sometimes called the Grushin operator.

15 Perturbation of Fredholm Operators and the Logarithmic Law

15.1 Perturbation of Fredholm Operators

Last time, we were showing that Fredholm operators are stable under small perturbations. Let's finish the proof.

Theorem 15.1. *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator. If $S \in \mathcal{L}(B_1, B_2)$ is such that $\|S\|$ is sufficiently small, then $T + S$ is Fredholm and* $\operatorname{ind}(T+S) = \operatorname{ind}(T)$.

Proof. We take a "Grushin approach." Let $\mathcal{P} : B_1 \otimes \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}$ be

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix},$$

where $n_+ = \dim(\ker(T))$, $n_- = \dim(\operatorname{coker}(T))$, $R_- : \mathbb{C}^{n_-} \to B_2$ is injective, and $R_+ : B_1 \to \mathbb{C}^{n_+}$ is surjective. Then $\mathcal{P}$ is invertible, so

$$\tilde{\mathcal{P}} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix}$$

is also invertible with the inverse $\mathcal{E} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus C^{n_-}$ given by

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

We have

$$\tilde{\mathcal{P}}\mathcal{E} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_+E_+ \end{bmatrix},$$

so R_+E_+ is the identity on $\mathbb{C}^{n_+}$. So E_+ is injective. Similarly,

$$\mathcal{E}\tilde{\mathcal{P}} = \begin{bmatrix} * & * \\ * & E_-R_- \end{bmatrix},$$

so E_- is surjective.

We now show that $T+S$ is Fredholm.

$$\begin{aligned} x \in \ker(T+S) &\iff (T+S)x = 0 \\ &\iff \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_+ \end{bmatrix} \\ &\iff \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} 0 \\ a_+ \end{bmatrix} = \begin{bmatrix} E_+a_+ \\ E_{-+}a_+ \end{bmatrix}, \end{aligned}$$

where $a_+ = R_+x \in \mathbb{C}^{n_+}$. We get that $x \in \ker(T+S)$ if and only if $x = E_+a_+$, were $a_+ \in \ker(E_{+-})$. Thus, $E_+ : \ker(E_{-+}) \to \ker(T+S)$ is surjective. So it is injective, since $\ker(E_{-+})$ is finite dimensional. So $\dim(\ker(T+S)) = \dim(\ker(E_{-+})) \leq n_+$. In particular, we get that $\dim(\ker(T+S)) \leq \dim(\ker(T))$.

Also,

$$\begin{aligned}
(T+S)x = y &\iff \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ a_+ \end{bmatrix} \\
&\iff \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} y \\ a_+ \end{bmatrix} \\
&\iff x = Ey + e_+a_+, 0 = E_-y + E_{-+}a_+.
\end{aligned}$$

Thus, $\operatorname{im}(T+S) = \{y \in B_2 : \exists a_+ \in \mathbb{C}^{n_+} \text{ s.t.} E_-y = -E_{-+}a_+\}$. We get a map from $B_2/\operatorname{im}(T+S) \to \mathbb{C}^{n_-}/\operatorname{im}(E_{-+})$ given by $y + \operatorname{im}(T+S) \mapsto E_-y + \operatorname{im}(E_{-+})$. The map is injective and surjective since E_- is surjective. We get $\dim(\operatorname{coker}(T+S)) = \dim(\operatorname{coker}(E_{-+})) < \infty$. Thus, $T+S$ is Fredholm and

$$\begin{aligned}
\operatorname{ind}(T+S) = \operatorname{ind}(E_{-+}) &= \dim(\ker(E_{-+})) - \dim(\mathbb{C}^n/\operatorname{im}(E_{-+})) \\
&= n_+ - n_- = \dim(\ker(T)) - \dim(\operatorname{im}(T)) = \operatorname{ind}(T).
\end{aligned}$$

□

Corollary 15.1. *The set $\{T \in \mathcal{L}(B_1, B_2) : T$ is Fredholm$\}$ is open in $\mathcal{L}(B_1, B_2)$, and the index is constant on each component of this set. Moreover, $\dim(\ker(T))$ is upper semicontinuous.*

15.2 The logarithmic law

Proposition 15.1. *Let $T_1 \in \mathcal{L}(B_1, B_2)$ and $T_2 \in \mathcal{L}(B_2, B_3)$ be Fredholm. Then $T_2T_1 \in \mathcal{L}(B_1, B_3)$ is also Fredholm, and we have "the logarithmic law"*

$$\operatorname{ind}(T_2T_1) = \operatorname{ind} T_2) + \operatorname{ind}(T_1).$$

Proof. Consider $T_1 : \ker(T_2T_1) \to \ker(T_2)$ sending $x \mapsto T_1x$. From linear algebra, we have $\dim(\ker(T_2T_2)/\ker(T_1)) \leq \dim(\ker(T_2)$. So

$$\dim(\ker(T_2T_1)) \leq \dim(\ker(T_1)) + \dim(\ker(T_1)) + \dim(\ker(T_2)).$$

Also, we have the exact sequence

$$B_2/\operatorname{im}(T_1) \xrightarrow{T_2'} B_3/\operatorname{im}(T_2T_1) \xrightarrow{q} B_3/\operatorname{im}(T_2)$$

where T_2' sends $x + \operatorname{im}(T_2) \mapsto T_2x + \operatorname{im}(T_2T_1)$, and q sends $x + \operatorname{im}(T_2T_1) \mapsto x + \operatorname{im}(T_2)$. So we have $\operatorname{im}(T_2') = \ker(q)$. It follows that $\dim(B_3/\operatorname{im}(T_2T_1)) < \infty$. So T_2T_1 is Fredholm.

To prove the logarithmic law, consider the family of operators $B_1 \oplus B_2 \to B_2 \oplus B_3$ given by

$$L(t) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \cos(t) I_2 & \sin(t) I_2 \\ -\sin(t) I_2 & \cos(t) I_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix},$$

where I_2 is the identity on B_2, and $t \in \mathbb{R}$. Then $L(t)$ is a product of 3 Fredholm operators and is Fredholm for each t.

The map $t \mapsto L(t)$ is continuous (w.r.t. the operator norm on $\mathcal{L}(B_1 \oplus B_2, B_2 \oplus B_3)$). Then $\operatorname{ind}(L(t))$ is locally constant, so it is constant. If $t = 0$, we get

$$L(0) = \begin{bmatrix} T_1 & 0 \\ 0 & 2 \end{bmatrix},$$

so $\operatorname{ind}(L(0)) = \operatorname{ind}(T_1) + \operatorname{ind}(T_2)$. If $t = -\pi/2$,

$$L(-\pi/2) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0 & -I_2 \\ T_2 T_1 & 0 \end{bmatrix}.$$

That is,

$$L(-\pi/2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ T_2 T_1 x \end{bmatrix}.$$

Since $\operatorname{ind}(L(-\pi/2)) = \operatorname{ind}(T_2 T_1)$, we get the logarithmic law. $\square$

15.3 Introduction to compact operators

Definition 15.1. A linear operator $T : B_1 \to B_2$ between Banach spaces is called **compact** if the closure of the image of the unit ball in B_1 is compact in B_2: $\overline{T(\{\|x\| \le 1\})}$ is compact in B_2.

In other words, T is compact if and only if for $\|x_n\| \le 1$, $(Tx_n)_{n \in \mathbb{N}}$ has a convergent subsequence. Also, compact operators are continuous.

16 Compact Operators and Riesz's Theorem

16.1 Compact operators

Last time, we said that a map $T : B_1 \to B_2$ is compact if given $\|x_n\| \leq 1$, then $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Example 16.1. Let $B_1 = C^1([0,1])$ with $\|f\|_{B_1} = \|f\|_{L^\infty} + \|f'\|_{L^\infty}$ and $B_2 = C([0,1])$ eith $\|f\|_{B_2} = \|f\|_{L^\infty}$. Then the inclusion map $B_1 \to B_2$ is compact by Ascoli's theorem.

Example 16.2. Let $k \in C([0,1]\times[0,1])$, and consider $Kf(x) = \int_0^1 k(x,y)f(y)\,dy$. Then $K : L^2((0,1)) \to L^2((0,1))$ is compact by Ascoli's theorem.

Proposition 16.1. *Compact operators have the following properties:*

1. *The space $\mathcal{L}_C(B_1, B_2)$ of compact linear maps $B_1 \to B_2$ is a closed subspace of $\mathcal{L}(B_1, B_2)$.*

2. *Compact operators form an ideal: if $T_1 \in \mathcal{L}(B_1, B_2)$, $T_2 \in \mathcal{L}(B_2, B_3)$, and either T_1 or T_2 is compact, then $T_2T_2 \in \mathcal{L}_C(B_1, B_3)$.*

3. *If $T \in \mathcal{L}(B_1, B_2)$ has finite rank ($\dim(\mathrm{im}(T)) < \infty$), then T is compact.*

Proof. We prove the properties one at a time:

1. T is compact $\iff$ $T(\{\|x\| \leq 1\})$ compact $\iff$ $\overline{T(\{\|x\| \leq 1\})}$ is complete and totally bounded $\iff$ $T(\{\|x\| \leq 1\})$ is totally bounded. Let $T_n \in \mathcal{L}_C(B_1, B_2)$ be such that $T_n \to T$ in $\mathcal{L}(B_1, B_2)$. Let $\varepsilon > 0$ be given, and let N be such that $\|T - T_N\| < \varepsilon/2$. Then, since $T_N(\{\|x\| \leq 1\})$ is totally bounded, $T_N(\{\|x\| \leq 1\}) \subseteq \bigcup_{j\in I \text{ finite}} B(x_j, \varepsilon/2)$. We get that $T(\{\|x\| \leq 1\}) \subseteq \bigcup_{j\in I} B(x_j, \varepsilon/2)$, so T is compact.

2. This property is clear.

3. We have a factorization $T : B_1 \to B_1/\ker(T) \to B_2$ given by $x \mapsto x + \ker(T) \mapsto Tx$. The space $B_1/\ker(T) \cong \mathrm{im}(T)$ is finite dimensional, and since the identity operator ina finite dimensional space is compact, we get that T is compact. $\square$

16.2 Riesz's Theorem

Theorem 16.1 (F. Riesz)**.** *If the identity map on the Banach space B is compact, then B is finite dimensional.*

Remark 16.1. This is clear if B is a Hilbert space; consider an orthonormal basis.

Lemma 16.1. *Let $B_1 \subsetneq B$ be a proper, closed subspace. Then for every $\varepsilon > 0$, there exists $x \in B$ such that $\|x\| = 1$, and $\operatorname{dist}(x, B_1) = \inf_{y\in B_1} \|x - y\| \geq 1 - \varepsilon$.*

Proof. Let $x \in B \backslash B_1$, and let $d = \text{dist}(z, B_1) > 0$. Let $x_1 \in B_1$ be such that $d \leq \|z - x_1\| < d/(1-\varepsilon)$. We can take $x = (z -_1)/\|z - x_1\|$. For any $y \in B_1$, we have

$$\frac{\|x - y\| = z - x_1 - y\|z - x_1\|\|}{\|z - x_1\|} \geq \frac{d}{\|z - x_1\|} > 1 - \varepsilon. \qquad \square$$

Now we can prove Riesz's theorem.

Proof. If B is infinite-dimensional, there exists a strictly increasing sequence $B_1 \subsetneq B_2 \subsetneq \cdots$ of finite dimensional subspaces of B. Using the lemma, we find $x_j \in B_j$ such that $\text{dist}(x_j, B_{j-1}) \geq 1/2$. In particular, $\|x_j - x_k\| \geq 1/2$ for $k < j$, so (x_j) has no convergent subsequence. $\square$

Theorem 16.2 (Fredholm-Riesz)**.** *Let B be a Banach space, and let $T \in \mathcal{L}_C(B, B)$. Then $I - T$ is Fredholm, and* $\text{ind}(I - T) = 0$.

Before we prove this, let's prove a proposition.

Proposition 16.2. *Let $T \in \mathcal{L}_C(B, B)$. Then*

1. $\dim(\ker(I - T)) < \infty$.

2. $\text{im}(T - T)$ *is closed.*

Proof. This is a crucial observation to any proof of the Fredholm-Riesz theorem.

1. Let $x_n \in \ker(I - T)$ with $\|x_n\| \leq 1$. Then $x_n = Tx_n$ has a convergent subsequence. By Riesz's theorem, $\dim(\ker(I - T)) < \infty$.

2. Let $y \in \overline{\text{im}(I - T)}$ and let $x_n \in B$ be such that $y_n = (1 - T)x_n \to y$. Consider $\text{dist}(x_n, \ker(I - T))$. This equals $\|x_n - z_n\|$ for some $z_n \in \ker(I - T)$ because $y \mapsto \|x_n - y\|$ is continuous and goes to ∞ as $y \to \infty$. We have that $y_n = (I - T)(x_n - z_n) = x_n - z_n - T(x_n - z_n)$.

   We claim that $(x_n - z_n)$ is a bounded sequence. Otherwise, we can assume that $\|x_n \to z_n\| \to \infty$. Let $w_n = (x_n - z_n)/\|x_n - z_n\|$. Then $\|w_n\| = 1$, and $(I - T)w_n = y_n/\|x_n - z_n\| \to 0$ as (y_n) converges. Passing to a subsequence, we may assume that $Tw_n \to v \in B$, so $w_n \to V$. So $(I - T)v = 0$. On the other hand,

   $$\text{dist}(w_n, \ker(I - T)) = \frac{\text{dist}(x_n, \ker(I - T))}{\|x_n - z_n\|},$$

   so $\text{dist}(v, \ker(I - T)) \leq 1$, and we get the claim. We will finish the proof next time. $\square$

17 Adjoint Operators and Annihilators

17.1 Translates of compact operators

Last time, we had that if $T : B \to B$ is compact, $\dim(\ker(I - T)) < \infty$.

Proposition 17.1. $\operatorname{im}(I - T)$ *is closed.*

Proof. Last time we showed that there exists a bounded sequence $x_n \in B$ such that $(I - T)x_n \to y$. We can assume that $Tx_n \to \ell \in B$, so x_n converges. In particular, $x_n \to y + \ell$. If $g = y + \ell$, then $(I - T)g = \lim_{n\to\infty}(I - T)x_n = y$. So $y \in \operatorname{im}(I - T)$. □

To show that $\dim(\operatorname{coker}(I - T)) < \infty$, we use duality arguments.

17.2 Adjoint operators

Let B_1, B_2 be Banach spaces with dual spaces B_1^*, B_2^* and the bilinear maps $B_j \times B_j^* \to \mathbb{C}$ given by $(x, \xi) \mapsto \langle x, \xi\rangle$.

Theorem 17.1. *For every $T \in \mathcal{L}(B_1, B_2)$, there exists a unique operator $T^* \in \mathcal{L}(B_2^*, B_1^*)$ such that $\langle Tx, \eta\rangle_2 = \langle x, T^*\eta\rangle_1$ for all $x \in B_1$ and $\eta \in B_2^*$. Moreover, the map $\mathcal{L}(B_1, B_2) \to \mathcal{L}(B_2^*, B_1^*)$ given by $T \mapsto T^*$ is a linear isometry.*

Proof. Let $\eta \in B_2^*$ be fixed. The map $x \mapsto \langle Tx, \eta\rangle_2$ for $x \in B_1$ is a linear continuous form on B_1 with norm $\sup_{x\neq 0} |\langle Tx, \eta\rangle_2|/\|x\| \le \|T\|\|\eta\|$. Thus there exists a unique element $\xi \in B_1^*$ such that $\langle Tx, \eta\rangle_2 = \langle x, \xi\rangle_1$ and $\|\xi\| \le \|T\|\|\eta\|$. The map $B_2^* \to B_1^*$ given by $\eta \mapsto \xi$ is linear and continuous of norm $\le \|T\|$. Thus, there exists a unique operator $T^* \in \mathcal{L}(B_2^*, B_1^*)$ such that $\langle Tx, \eta\rangle_2 = \langle x, T^*\eta\rangle_1$ and $\|T^*\| \le \|T\|$.

Now, from an earlier consequence of Hahn-Banach,

$$\|Tx\| = \sup_{\eta\neq 0} \frac{|\langle Tx, \eta\rangle_2|}{\|\eta\|} = \sup_{\eta\neq 0} \frac{|\langle x, T^*\eta\rangle_1|}{\|\eta\|} \le \|x\|\|T^*\|.$$

So $\|T\| \le \|T^*\|$, and the result follows. □

Definition 17.1. The operator T^* is called the **adjoint** operator of T.

17.3 Annihilators

Definition 17.2. Let B be a Banach space, and let $W \subseteq B$ be a closed subspace. The **annihilator** of W is defined as $W^o = \{\xi \in B^* : \langle x, \xi\rangle = 0 \,\forall x \in W\}$.

The annihilator is a closed subspace.

Theorem 17.2. *Let W be a closed subspace of a Banach space B.*

1. *Let $i : W \to B$ be the inclusion map. Then $i^* : B^* \to W^*$ vanishes on W^o and induces an isometric bijection $B^*/W^o \to W^*$.*

2. *Let $q : B \to B/W$ be the quotient map. Then $q^* : (B/W)^* \to B^*$ is an isometry with the range W^o.*

We have the natural isomorphisms $B^*/W^o \cong W^*$ and $(B/W)^* \cong W^o$.

Proof. The proof mainly consists of checking the definitions:

1. We have $\langle ix, \xi \rangle = \langle x, i^*\xi \rangle$ for $x \in W$ and $\xi \in B^*$. Thus, $i^*\xi$ is the restriction of ξ to W. So $\ker(i^*) = W^*$. By the Hahn-Banach theorem, every continuous linear form on W can be extended to an element of B^*. So $i^* : B^* \to W^*$ is surjective. One can check that for all $\xi \in B^*$, $\|i^*\xi\|_{W^*} = \inf_{\eta \in W^o} \|\xi + \eta\|_{B^*}$.

2. Let $q : B \to B/W$. Then $\langle qx, \eta \rangle = \langle x, q^*\eta \rangle$, where $x \in B$ and $\eta \in (B/W)^*$. Then q^* is injective, as its kernel is trivial. If $x \in W$, $0 = \langle qx, \eta \rangle = \langle x, q^*\eta \rangle$, so $\operatorname{im}(q^*) \subseteq W^o$. On the other hand, if $\xi \in W^o$, we can factor

$$B \xrightarrow{w} B/W \xrightarrow{q(x) \mapsto \langle x, \xi \rangle} \mathbb{C}.$$

So if η is the second map, then $\xi = q^*\eta$. So $\operatorname{im}(q^*) = W^o$. We can check that $\|\xi\|_{B^*} = \|\eta\|_{(B/W)^*}$. $\square$

Theorem 17.3. *Let $T \in \mathcal{L}(B_1, B_2)$ and assume that $\operatorname{im}(T)$ is closed. Then $\operatorname{im}(T^*)$ is also closed, $(\ker(T))^o = \operatorname{im}(T^*)$, $(\operatorname{im}(T))^o = \ker(T)^*$, $\dim(\ker(T)) = \dim(\operatorname{coker}(T^*))$, and $\dim(\ker(T^*)) = \dim(\operatorname{coker}(T))$.*

Proof. Factorize $T = T_3T_2T_1$, where $T_1 : B_1 \to B_1/\ker(T)$ is the quotient map, $T_2 : B_2/\ker(T) \to \operatorname{im}(T)$ is an isomorphism, and $T_3 : \operatorname{im}(T) \to B_2$ is the inclusion map. Then $T^* = T_1^*T_2^*T_3^*$. $T_3^* : B_2^* \to (\operatorname{im}(T)) \cong B_2^*/(\operatorname{im}(T))^o$ is surjective. $T_2^* : (\operatorname{im}(T))^* \to (B_1/\ker(T)) \cong (\ker(T))^o$ is an isomorphism. $T_1^* : (\ker(T))^o \to B_1^*$ is the inclusion map. We get that $\operatorname{im}(T^*) = (\ker(T))^o$ is closed.

If $T : B \to B$, we get $(B/\operatorname{im}(T))^* \cong (\operatorname{im}(T))^o = \ker(T)$. So $\dim(\operatorname{coker}(T)) = \dim(\ker(T^*))$. The other identities can be derived similarly. $\square$

18 The Riesz-Fredholm Theorem

18.1 Conclusion of the proof of the Riesz-Fredholm theorem

Last time, we showed that if $T \in \mathcal{L}(B_1, B_2)$ with $\operatorname{im}(T)$ closed, then $\operatorname{im}(T^*) = (\ker(T))^o$ is closed, and $(B_2/\operatorname{im}(T))^* \cong \ker(T^*)$. In particular, $\dim(\operatorname{coker}(T)) = \dim(\ker(T^*))$. Apply this when $T = I + T'$, where $T' : B \to B$ is compact.

Proposition 18.1. *Let $T : B_1 \to B_2$ be compact. Then $T^* : B_2^* \to B_1^*$ is compact.*

Proof. Let $\xi_n \in B_2^*$ be bounded, $\|\xi_n\| \le 1$ for $n = 1, 2, \ldots$. Set $K = \overline{T(\{\|x\| \le 1\}} \subseteq B_2$ compact. Consider the sequence of continuous functions $\varphi_n(x) = \langle x, \xi_n \rangle$ for $x \in K$. We have:

1. $|\varphi_n(x)| \le \|x\|\|\xi_n\| \le C$ for all $n = 1, 2, \ldots$ and $x \in K$
2. $|\varphi_n(x) - \varphi_n(y)| = |\langle x - y, \xi_n \rangle_2| \le \|x - y\|_1$ for $x, y \in K$.

By Ascoli's theorem, there exists a uniformly convergent subsequence (φ_{n_k}). In particular, $\sup_{\|x\| \le 1} |\varphi_{n_k}(Tx) - \varphi_{n_\ell}(Tx)| \to 0$ as $k, \ell \to \infty$. So

$$\sup_{\|x\| \le 1} |\langle Tx, \xi_{n_k} \rangle_2 - \langle Tx, \xi_{n_\ell} \rangle| = \sup_{\|x\| \le 1} |\langle x, T^*\xi_{n_k} \rangle_2 - \langle x, T^*\xi_{n_\ell} \rangle| = \|T^*\xi_{n_k} - T^*\xi_{n_\ell}\|_{B_1^*} \to 0$$

as $k, \ell \to \infty$. So $(T^*\xi_{n_k})$ converges, which makes T^* compact. □

This completes our proof of the Riesz-Fredholm theorem.

Theorem 18.1 (Riesz-Fredholm). *Let B be a Banach space, and let $T \in \mathcal{L}_C(B, B)$. Then $I - T$ is Fredholm, and* $\operatorname{ind}(I - T) = 0$.

Proof. Let $K : B \to B$ be compact. Then $\dim(\ker(I + K)) < \infty$, $\operatorname{im}(I + K)$ is closed, $\dim(\operatorname{coker}(I + K)) = \dim(\ker(I + K^*)) < \infty$. Thus, $I + K$ is Fredholm and $\operatorname{ind}(I + K) = \operatorname{ind}(I + \lambda K) = \operatorname{ind}(I) = 0$. □

18.2 Atkinson's theorem and stronger Riesz-Fredholm

We can actually upgrade the statement of the Riesz-Fredholm theorem to get a stronger theorem.

Proposition 18.2 (Atkinson's theorem). *An operator $T \in \mathcal{L}(B_1, B_2)$ is Fredholm if and only if there exists $S \in \mathcal{L}(B_2, B_1)$ such that $TS - I$ and $ST - I$ are compact in B_2 and B_1, respectively.*

Proof. Sufficiency: Let $S \in \mathcal{L}(B_2, B_1)$ be such that $ST = I + K_1$ and $TS = I + K_2$, where K_1, K_2 are compact. Then $\ker(T) \subseteq \ker(I + K_1)$, so $|dim(\ker(T)) < \infty$. Similarly, $\operatorname{im}(T) \supseteq \operatorname{im}(I + K_2)$, so $\dim(\operatorname{coker}(T)) \leq \dim(\operatorname{coker}(I + K_2)) < \infty$. So T is Fredholm.

Necessity: Take the Grushin approach: if T is Fredholm, write $n_+ = \dim(\ker(T))$, and $n_- = \dim(\operatorname{coker}(T))$. Then there exist an injective $R_- : \mathbb{C}^{n_-} \to B_2$ with $B_2 = \operatorname{im}(T) \oplus R_-(\mathbb{C}^{n_-})$ and a surjective $R_+ : B_1 \to \mathbb{C}^{n_+}$ such that $R_+|_{\ker(T)}$ is bijective. Then the operator $\mathcal{P} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}$ given by

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix}$$

is invertible. It has the inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

We get that

$$\mathcal{P}\mathcal{E} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} TE + R_-E_- & * \\ * & * \end{bmatrix},$$

$$\mathcal{E}\mathcal{P} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} = \begin{bmatrix} ET + E_+R_+ & * \\ * & * \end{bmatrix},$$

so $TE - I = -R_0 E$ and $ET - I = -E_+R_+$. The maps R_-E_- and E_+R_+ have finite rank, so they are compact. □

Remark 18.1. If $ET - I$ and $TE - I$ are compact, then $E \in \mathcal{L}(B_2, B_1)$ is Fredholm, and $\operatorname{ind}(ET) = \operatorname{ind}(I + K) = 0$. By the logarithmic law, this equals $\inf(E) + \operatorname{ind}(T)$. So $\operatorname{ind}(E) = -\operatorname{ind}(T)$.

Theorem 18.2. *Let $T \in \mathcal{L}(B_1, B_2)$ be Fredholm and $S \in \mathcal{L}_C(B_1, B_2)$. Then $T + S$ is Fredholm, and* $\operatorname{ind}(T + S) = \operatorname{ind}(T)$.

Proof. Let $E \in \mathcal{L}(B_2, B_1)$ be such that $TE - I$ and $ET - I$ are compact. Then $(T + S)E - I$ and $E(T + S) - I$ are compact. So $T + S$ is Fredholm. Also, $\operatorname{ind}(T + S) = \operatorname{ind}(T + \lambda S)$ for $\lambda \in \mathbb{C}$. Letting $\lambda \to 0$, we get $\operatorname{ind}(T + S) = \operatorname{ind}(T)$. □

18.3 Applications to differential equations

Proposition 18.3. *Let $a, b \in C([0, 1])$, and consider the boundary value problem $u'' + au + bu = f$ with boundary conditions $u(0) = u(1) = 0$. Here, $f \in C([0, 1])$, and $u \in C^2([0, 1])$. The boundary value problem has a unique solution for any $f \in C([0, 1])$ if and only if the homogeneous problem when $f = 0$ only has the trivial solution.*

Proof. Let $B_1 = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}$ be a Banach space. Let $B_2 = C([0,1])$ with

$$\|u\|_{B_2} = \sum_{j=0}^{2} \|u^{(j)}\|_{L^\infty}.$$

Then $T : B - 1 \to B_2$ sending $u \mapsto u''$ is bijective, so $\mathrm{ind}(T) = 0$. The map $S : B_1 \to B_2$ sending $u + au' + bu$ is compact, so $T + S$ is Fredholm with $\mathrm{ind}(T + S) = 0$. So $T + S$ is bijective if and only if $T + S$ is injective. $\square$

Our next application will be the Toeplitz index theorem. Here is the idea. Let H be the closed subspace of $L^2(\mathbb{R}/2\pi\mathbb{Z})$ with Fourier coefficients $\hat{u}(n) = 0$ for $n < 0$. Let $\Pi : L^2 \to H$ be an orthogonal projection. Let $f \in C(\mathbb{C}/2\pi\mathbb{Z}) \mapsto \mathrm{Top}(f)$, which sends $u \mapsto \Pi(fu)$. This is called the **Toeplitz operator**.

Theorem 18.3. $\mathrm{Top}(f)$ *is Fredholm if and only if* $f \neq 0$. *Moreover,* $\mathrm{ind}(\mathrm{Top}(f)) =$ *−winding number of* f.

19 The Toeplitz Index Theorem

19.1 Hardy space

Let $H = \{u \in L^2((0,2\pi)) : \hat{u}(n) = 0 \, \forall n < 0\} \subseteq L^2((0,2\pi))$, where the Fourier coefficients are $\hat{u}(n) = (1/2\pi)\int_0^{2\pi} e(\theta)e^{-in\theta}\,d\theta$. If $u \in H$, then $u(\theta) \sim \sum_{n=0}^{\infty} \hat{u}(n)e^{in\theta}$ can be viewed as the boundary values of the holomorphic function $\sum_{n=0}^{\infty} \hat{u}(n)z^n$ with $|z| < 1$. The space H is called the **Hardy space**.

Let $\Pi : L^2((0,2\pi)) \to H$ be the orthogonal projection sending $u \sim \sum_{n=0}^{\infty} \hat{u}(n)e^{in\theta} \mapsto \sum_{n=0}^{\infty} \hat{u}(n)e^{in\theta}$. Given $f \in L^\infty((0,2\pi))$, associated to f is the **Toeplitz operator** $\operatorname{Top}(f) : H \to H$ sending $u \mapsto \Pi(fu)$. We have $\|\operatorname{Top}(f)\|_{\mathcal{L}(H,H)} \le \|f\|_{L^\infty}$.

19.2 The Toeplitz index theorem

Theorem 19.1 (Toeplitz index theorem)**.** *Let f be continuous 2π-periodic, and assume that f has no zeros. Then* $\operatorname{Top}(f)$ *is Fredholm, and* $\operatorname{ind}(\operatorname{Top}(f)) = -$*winding number*$(f)$*.*

To define the winding number, write $f(\theta) = r(\theta)e^{i\varphi(\theta)}$ with $r > 0$ and $0 \le \theta \le 2\pi$. The winding number of f is $(\varphi(2\pi) - \varphi(0))/2\pi$. If $f \in C^1$, then the winding number of f is

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\theta)}{f(\theta)}\,d\theta.$$

Proof. To establish the Fredholm property, we try to invert $\operatorname{Top}(f)$ modulo a compact error. Here is a claim: Let f, g be continuous 2π-periodic. Then $\operatorname{Top}\operatorname{Top}(g) = \operatorname{Top}(fg) +$ compact operator.

Write $\operatorname{Top}(f) = \Pi M_f$ and ΠM_g, where M_f, M_g are multiplication operators by f and g. Then $\operatorname{Top}(f)\operatorname{Top}(g) = \Pi M_f \Pi M_g = \Pi(\Pi M_f + [M_f\Pi])M_g$, where $[M_f, \Pi] = M_f\Pi - \Pi M_f$ is the commutator. So we get

$$\Pi(\Pi M_f + [M_f\Pi])M_g = \Pi M_f M_g + \Pi[M_f,\Pi]M_g = \operatorname{Top}(fg) + \Pi[M_f,\Pi]M_g.$$

It suffices to show that $[M_f, \Pi] : L^2 \to L^2$ is compact. We split into cases.

If $f(\theta) = e^{in\theta}$ with $n \in \mathbb{Z}$, $n \ne 0$, then if $n > 0$,

$$[M_f,\Pi]e^{ik\theta} = (M_f\Pi - \Pi M_f)e^{ik\theta} = \begin{cases} 0 & k \ge 0 \\ -\Pi(e^{i(k+n)\theta}) & k < 0. \end{cases}$$

Now observe that $-\Pi(e^{i(k+n)\theta}) = 0$ if $k < -n$, so the operator is of finite rank and is therefore compact. The computation is similar for $n < 0$.

If f is a trigonometric polynomial $f(\theta) = \sum_{-N}^{N} a_n e^{in\theta}$, then $[M_f, \Pi]$ is also of inite rank and is hence compact. If f is an arbitrary continuous, 2π-periodic function, let f_n be a sequence of trigonometric polynomials such that $f_n \to f$ uniformly. Then

$$\|[M_{f_n},\Pi] - [M_f,\Pi]\| = \|[M_{f_n} - M_f,\Pi]\|$$

$$\leq \|M_{f_n - f}\Pi\| + \|\Pi M_{f_n - f}\|$$
$$\leq 2\|f_n - f\| \to 0.$$

Thus, $[M_f, \Pi]$ is compact.

So the claim holds. Now if $f \neq 0$, write $\mathrm{Top}(f)\,\mathrm{Top}(1/f) = I + \text{compact}$, and same for $\mathrm{Top}(1/f)\,\mathrm{Top}(f)$. So we get that $\mathrm{Top}(f)$ is Fredholm. Notice also that if f, g are continuous and nonvanishing, then $\mathrm{ind}(\mathrm{Top}(fg)) = \mathrm{ind}(\mathrm{Top}(f) + \mathrm{Top}(g) + \text{compact}) = \mathrm{ind}(\mathrm{Top}(f)) + \mathrm{ind}(\mathrm{Top}(g))$.

Now write $f(\theta) = r(\theta)e^{i\varphi(\theta)}$. Then we get $\mathrm{ind}(\mathrm{Top}(f)) = \mathrm{Top}(r) + \mathrm{Top}(e^{i\varphi})$. Take $t_t(\theta) = (1-t)r(\theta) = (1-t)r(\theta) + t > 0$ with $0 \leq t \leq 1$. To compute $\mathrm{ind}(\mathrm{Top}(e^{i\varphi}))$, write N for the winding number of f, and let $g_t(\theta) = e^{i(1-t)\varphi(\theta) + iNt\theta}$. Then g_t is periodic in θ and continuous in t. So $\mathrm{ind}(\mathrm{Top}(e^{i\varphi})) = \mathrm{ind}(\mathrm{Top}(e^{iN\theta})) = -N$. $\square$

20 Analytic Fredholm Theory

20.1 Analytic Fredholm theory

Theorem 20.1 (analytic Fredholm theory)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ be a family for Fredholm operators depending holomorphically on z; that is $T : z \mapsto T(z)$ is holomorphic with respect to the operator norm on $L(B_1, B_2)$. Assume that there exists $z_0 \in \Omega$ such that $T(z_0) : B_1 \to B_2$ is invertible. Then there exists a set $\Sigma \subseteq \Omega$ having no limit point in Ω such that for all $z \in \Omega \setminus \Sigma$, the operator $T(z) : B_1 \to B_2$ is is bijective.*

Proof. Notice that $z \mapsto \operatorname{ind}(T(z))$ is constant, so $\operatorname{ind}(T(z)) = \operatorname{ind}(T(z_0)) = 0$ for all $z \in \Omega$. Let $z_1 \in \Omega$, and write $n_0(z_1) = \dim(\ker(T(z_1))) = \dim(\operatorname{coker}(T(z_1)))$. Consider the Grushin operator for $T(z_1)$:

$$\mathcal{P}^{z_1} = \begin{bmatrix} T(z_1) & R_-(z_1) \\ R_+(z_1) & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_0(z_1)} \to B_2 \oplus \mathbb{C}^{n_0(z_1)},$$

which is invertible. There exists a connected open neighborhood $N(z_1) \subseteq \Omega$ of z_1 such that for $z \in N(z_1)$, the operator

$$\mathcal{P}^{z_1}(z) = \begin{bmatrix} T(z) & R_-(z_1) \\ R_+(z_1) & 0 \end{bmatrix}$$

is bijective and depends holomorphically on $z \in N(z_1)$.

Let

$$\mathcal{E}^{z_1}(z) = (\mathcal{P}^{z_1}(z))^{-1} = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix}$$

be the inverse of $\mathcal{P}^{z_1}(z)$, depending holomorphically on $z \in N(z_1)$. We claim that for $z \in N(z_1$, we have $T(z) : B_1 \to B_2$ is bijective if and only if $E_{-+}(z) : \mathbb{C}^{n_0(z_1)} \to \mathbb{C}^{n_0(z_1)}$ is bijective.

$$\begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

so we get $TE + R_-E_- = I$ and $TE_+ + E_-E_{-+}$. If E_{-+}^{-1} exists, then $TE_+E_{-+}^{-1} = R_-$, so $T(E - E_+E_{-+}^{-1}E_-) = I$. Thus, T is surjective, so because T is Fredholm of index 0, T is bijective, and $T^{-1} = E - E_+E_{-+}^{-1}E_-$. The converse is checked similarly.

E_{-+} is a holomorphic function with values in $n_0(z_1) \times n_0(z_1)$ matrices. So it is bijective iff $\det(E_{-+}) \neq 0$. We have that either $\det(E_{-+}(z)) = 0$ on $N(z_1)$ or $\det(E_{-+}) \neq 0$ in a deleted neighborhood of z_1. Let $\Omega_1 = \{z \in \Omega : T(z') \text{ is invertible } \forall z' \neq z \text{ near } z\}$, and let $\Omega_2 = \{z \in \Omega : T(z') \text{ is not invertible } \forall z' \neq z \text{ near } z\}$. Then $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 are open. $\Omega_1 \neq \varnothing$, so $\Omega_2 = \varnothing$, and thus the set $\Sigma = \{z \in \Omega : T(z) \text{ is not invertible}\}$ is a closed set with only isolated points. $\square$

20.2 Behavior of inverses near singularities

Remark 20.1. We have $z \mapsto T(z)^{-1}$ is holomorphic on $\Omega \setminus \Sigma$. Consider the behavior of $T(z)^{-1}$ near $w \in \Sigma$. Write $T(z)^{-1} = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z)$. Then $E_{-+}(z)^{-1}$ has a pole at $z = w$ (because we are dividing by the determinant, which may has zeros of at most finite multiplicity), so

$$E_{-+}(z)^{-1} = \frac{R_{N_0}}{(z-w)^{N_0}} + \cdots + \frac{R_{-1}}{z-w} + \mathrm{Hol}(z).$$

Here, $\mathrm{rank}(R_j) \leq n_0$. It follows that $z \mapsto T(z)^{-1}$ has a poleof order N_0 at $z = w$:

$$T(z)^{-1}(z) = \frac{A_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{A_{-1}}{(z-w)} + Q(z),$$

where $Q(z)$ is holomorphic in a neighborhood of w and takes values in $\mathcal{L}(B_2, B_1)$. The operators $A_{-N_0}, \ldots, A_{-1} \in \mathcal{L}(B_2, B_1)$ can be expressed in terms of $R_{-N_0}, \ldots, R_{-1}$ and $E_{\pm}^{(j)}(w)$ and are of finite rank.

Definition 20.1. The **spectrum** of $T : B_1 \to B_2$ is

$$\mathrm{Spec}(T) = \{z \in \mathbb{C} : T - zI \text{ is not invertible}\}.$$

Analytic Fredholm theory shows that if T is Fredholm, then $\mathrm{Spec}(T)$ consists of isolated points.

21 Spectral Theory for Compact Operators

21.1 Applications of analytic Fredholm theory

Last time, we proved analytic Fredholm theory, which said that if $T(z)$ is a family of operators in $\mathcal{L}(B_1, B_2)$ that is holomorphic in z (in some domain $\Omega \subseteq \mathbb{C}$) and if $T^{-1}(z_0)$ exists for some $z_0 \in \Omega$, then $\Sigma = \{z \in \Omega : T(z) \text{ is not invertible}\}$ is a discrete subset of Ω.

Example 21.1. Let M be a compact C^∞ Riemannian manifold (e.g. torus, sphere,etc.). Let $V \in L^\infty(M, \mathbb{C})$, and consider the **Schrödinger operator** $P = -\Delta + V : H^2(M) \to L^2(M)$. where $H^2(M) = \{u \in L^2(M) : \partial^\alpha u \in M^2(M), |\alpha| \leq 2\}$ is a Sobolev space. What is $\mathrm{Spec}(P)$? We need 2 basic facts (that we will accept without proof).

Proposition 21.1. *The inclusion map $H^2(M) \to L^2(M)$ is compact.*

Proposition 21.2. *For all $x \in \mathbb{C} \setminus \mathbb{R}$, $-\Delta - zI : H^2 \to L^2$ is bijective and*

$$\|(-\Delta - zI)^{-1}\|_{\mathcal{L}(L^2, L^2)} \leq \frac{1}{\mathrm{Im}(z)}.$$

This second fact follows form the fact that $-\Delta$ is self adjoint. Now observe that

$$P - z = \underbrace{(-\Delta + iI)}_{\text{bijective}} + \underbrace{B - zI - iI}_{\text{compact}},$$

so $P - z$ is Fredholm of index 0 and is holomorphic in z. We claim that there exists some $z_0 = it$ such that $P - z_0 I : H^2 \to L^2$ is bijective. Write

$$P - z_0 I = -\Delta + V - z_0 I = (I + V(-\Delta - z_0)^{-1})(-\Delta - z_0).$$

To show that $(I + V(-\Delta - z_0)^{-1})$ is invertible, we can make $\|V(-\Delta - z_0)^{-1})\| < 1$. So

$$\|V(-\Delta - z_0)^{-1}\| \leq \frac{\|V\|_{L^\infty}}{|\mathrm{Im}(z_0)|},$$

so we can take z_0 with large enough imaginary part to make this small. By the analytic Fredholm theory, we get $\mathrm{Spec}(P) \subseteq \mathbb{C}$ is discrete and $\mathrm{Spec}(P) \subseteq \{z : |\mathrm{Im}(z)| \leq C\}$. Moreover, the spectrum consists entirely of eigenvalues.

Example 21.2. When operators are not Fredholm, the spectrum may not have eigenvalues. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be $u(x) \mapsto \sin(x)u(x)$. Then $T \in \mathcal{L}(P^2, L^2)$, and $\mathrm{Spec}(T) = [-1, 1]$, while T has no eigenvalues. If $Tu = \lambda u$, for $\lambda\ in \mathbb{C}$, then $u = 0$ a.e. Take $\lambda \in [-1, 1]$, and show that an estimate of the form $\|u\|_{L^2} \leq C\|(T - \lambda I)u\|_{L^2}$ cannot hold: $m(\underbrace{\{x : |\sin(x) - \lambda| < \varepsilon\}}_{E_{\lambda,\varepsilon}}) > 0$ for all $\varepsilon > 0$. Letting $u = \mathbb{1}_{E_{\lambda,\varepsilon}}/m(E_{\lambda,\varepsilon})^{1/2}$, sending $\varepsilon \to 0$ gives us the conclusion.

21.2 Spectral theory for compact operators

Theorem 21.1 (spectral theory for compact operators)**.** *Let B be an infinite dimensional Banach space, and let $T : B \to B$ be compact.*

1. $0 \in \operatorname{Spec}(T)$.
2. *If* $0 \neq \lambda \in \operatorname{Spec}(T)$*, then* $\ker(T - \lambda I) \neq 0$.
3. *One of the following occurs:*
   (a) $\operatorname{Spec}(T) = \{0\}$.
   (b) $\operatorname{Spec}(T) \setminus \{0\}$ *is a finite set.*
   (c) $\operatorname{Spec}(T) \setminus \{0\}$ *is a countable set* $= \{\lambda_1, \lambda_2, \dots\}$*, and* $\lambda_n \to 0$.

Proof. These statements are consequences of the results we have already proven.

1. This follows from Riesz's theorem.
2. If $\lambda \neq 0$¡ then $T - \lambda I = (-\lambda)(I - (1/\lambda)T)$ is Fredholm of index 0. If $\lambda \in \operatorname{Spec}(T)$, then $\ker(T - \lambda I) \neq \{0\}$.
3. Apply the analytic Fredholm theory to $F(\lambda) = (-\lambda)(I - (1/\lambda)T)$. $F(\lambda)$ is invertible for large λ. So $\operatorname{Spec}(T) \setminus \{0\}$ consists of at most countably many isolated points. □

Example 21.3. Let $B = L^2(0, 1)$, and let $Tf(x) = \int_0^x f(y)\,dy$ be the **Volterra operator**. T is compact. We claim that $\operatorname{Spec}(T) = \{0\}$. If $\lambda \in \operatorname{Spec}(T) \setminus \{0\}$, then there exists some $f \in L^2$ such that $\int_0^x f(y)\,dy = \lambda f(x)$. This implies $f(x) = \lambda f'(x)$ with $f(0) = 0$. So $f = 0$.

Let $\lambda \in \operatorname{Spec}(T) \setminus \{0\}$, and consider the resolvent of T in a neighborhood of λ. We have the Laurent expansion $(T - (\lambda + z)I)^{-1} = \sum_{j=-N}^{\infty} A_j z^j$ for $0 < |z|$ small, $1 \leq N < \infty$, and $A_j \in \mathcal{L}(B, B)$, where $A_{-N}, \dots, A_{-1}$ are of finite rank. In complex analysis, the coefficient of z^{-1} is the residue at λ. What is the significance here?

Proposition 21.3. *Let $N_\lambda = \bigcup_{k=1}^{\infty} \ker(T - \lambda I)^k$ be the generalized eigenspace of T associated to λ. Then $-A_{-1}$ is a projection onto N_λ. So N_λ is finite dimensional.*

We will prove this next time.

22 Riesz Projection and Spectra of Self-Adjoint Operators

22.1 Riesz projection

Let $T : B \to B$ be compact and $\lambda \in \operatorname{Spec}(T) \setminus \{0\}$. Then $(T - (\lambda + z)I)^{-1} = \sum_{j=-N}^{\infty} A_j z^j$ for some $1 \leq N < \infty$.

Proposition 22.1 (Riesz)**.** *The operator $-A_{-1}$ is a projection onto the finite dimensional generalized eigenpace $N_\lambda = \bigcup_{k=1}^{\infty} \ker(T - \lambda I)^k$.*

Proof. Multiply the Laurent expansion by z^{-j-1} for $-N \leq -1 \leq -1$, and integrate over $\partial D(0, r)$ with $0 < r \ll 1$. We get

$$A_j = \frac{1}{2\pi i} \int_{\partial D(0,r)} (T - (\lambda + z)I)^{-1} z^{-j-1} \, dz,$$

so we get

$$\Pi = -A_{-1} = \frac{1}{2\pi i} \int_{\partial D(0,r)} ((\lambda + z)I - T)^{-1} \, dz.$$

We now claim that Π is a projection. Let $0 < r_1 < r_2 \ll 1$, and write

$$\begin{aligned}
\Pi^2 &= \int_{\partial D(0,r_2)} \int_{\partial D(0,r_1)} ((\lambda + w)I - T)^{-1} ((\lambda + z)I - T)^{-1} \frac{1}{2\pi i} \, dz \frac{1}{2\pi i} \, dw \\
&= \int_{\partial D(0,r_2)} \int_{\partial D(0,r_1)} \frac{1}{w - z} ((\lambda + z)I - T)^{-1} \frac{1}{2\pi i} \, dz \frac{1}{2\pi i} \, dw \\
&\quad - \int_{\partial D(0,r_2)} \int_{\partial D(0,r_1)} \frac{1}{w - z} ((\lambda + w)I - T)^{-1} \frac{1}{2\pi i} \, dz \frac{1}{2\pi i} \, dw
\end{aligned}$$

Apply Cauchy's integral formula to both terms. The second term equals 0.

$$= \Pi.$$

Now in the Laurent expansion, multiply by $T - (\lambda + z)I$ on the left to get

$$I = (T - \lambda I) A_{-N} z^{-N} + \sum_{j=-N+1}^{\infty} ((T - \lambda I) A_j - A_{j-1}) z^j,$$

which gives

$$(T - \lambda I) A_{-N} = A_{-N} (T - \lambda I) = 0$$

$$(T - \lambda I) A_j - A_{j-1} = \begin{cases} 0 & j \neq 0, j \geq N + 1 \\ 1 & j = 0. \end{cases}$$

So $[T, A_j] = 0$ for all j, and

$$A_{-N} = (T - \lambda I)A_{-N+1} = (T - \lambda I)^2 A_{-N+2} = \cdots = (T - \lambda I)^{N-I} A_{-1}.$$

We get that $(T - \lambda I)^N A_{-1} = 0$. Also, $I + A_{-1} = (T - \lambda I)A_0$, so applying $(T - \lambda I)^N$ gives us

$$(T - \lambda I)^N = A_0 (T - \lambda I)^{N+1}.$$

Thus, if $(T - \lambda I)^{N+1} x = 0$, then $(T - \lambda I)^N x = 0$. It follows that $N_\lambda = \ker(T - \lambda I)^N$, so $\dim(\ker(T_\lambda)) < \infty$ because $T - \lambda I$ is Fredholm of index 0.

It remains to show that $\operatorname{im}(A_{-1}) = N_\lambda = \ker((T - \lambda I)^N)$. If $x \in N_\lambda$, then $x + A_{-1}x = (T - \lambda I)A_0 x = (T_\lambda I)^2 A_1 x = \cdots = (T - \lambda I)^N A_{N-1} x = 0$. So $\operatorname{im}(A_{-1}) = N_\lambda$. $\square$

We can write $B = N_\lambda \oplus \ker(\Pi)$. This is a T-invariant decomposition. Moreover, $(T - \lambda I)|_{N_\lambda}$ is nilpotent, and $(T - \lambda I)_{\ker(\Pi)}$ is bijective.

22.2 Spectra of self-adjoint operators

Assume now that $B = H$ is a complex Hilbert space.

Definition 22.1. An operator is **self-adjoint** if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$.

Example 22.1. Let $H = L^2((0,1))$, and let $Tu(x) = \int_0^1 K(x,y)u(y)\, dy$, where $K \in C([0,1] \times [0,1])$ is such that $\overline{K(x,y)} = K(y,x)$.

Proposition 22.2. *Let $T \in \mathcal{L}(H,H)$ be self-adjoint. Then $\operatorname{Spec}(T) \subseteq \mathbb{R}$, and the resolvent $R(z) = (T - zI)^{-1} \in \mathcal{L}(H,H)$ satisfies $\|R(z)\|_{\mathcal{L}(H,H)} \leq 1/|\operatorname{Im}(z)|$ for $z \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. Let $z = i + iy$ with $y \neq 0$, and compute

$$\begin{aligned}
\|(T - zI)u\|^2 &= \langle (T - xI)u - iyu, (T - xI)u - iyu \rangle \\
&= \|(T - x)u\|^2 + \underbrace{i \langle ((T - x)u, yu \rangle - i \langle yu, (T - x)u \rangle}_{=0} + y^2 \|u\|^2.
\end{aligned}$$

We get

$$\|(T - z)u\|^2 = \|(T - x)u\|^2 + y^2\|u\|^2 \geq y^2 \|u\|^2,$$

so $\|(T - zI)u\| \geq |\operatorname{Im}(z)| \|u\|$, so $T - zI$ is injective and $\operatorname{im}(T - zI)$ is closed. So $H = \operatorname{im}(T - z) \oplus \operatorname{im}(T - z)^\perp$, where $\operatorname{im}(T - z)^\perp = \{x : \langle (T - z)y, x \rangle = 0 \, \forall y \in H\} = \ker(T - \overline{z}I) = \{0\}$. So we get that $T - zI : H \to H$ is bijective, and $\|(T - z)^{-1}\|_{\mathcal{L}(H,H)} \leq 1/|\operatorname{Im}(z)|$. $\square$

Remark 22.1. Let $T \in \mathcal{L}(H,H)$. Then T is uniquely determined by the function $x \mapsto \langle Tx, x \rangle$. If $\langle Tx, x \rangle = 0$ for all x, then we polarize:

$$\langle T(y + z), y + z \rangle = 0, \qquad \langle T(y + iz), y + iz \rangle = 0,$$

for all $y, z \in H$, so

$$\langle Ty, z\rangle + \langle Tz, y\rangle = 0, \qquad \langle Ty, z\rangle - \langle Tz, y\rangle = 0,$$

which give us $\langle Ty, z\rangle = 0$. So $T = 0$. So T is self adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

Now let T be compact and self adjoint. Let $\lambda \in \mathrm{Spec}(T) \setminus \{0\}$. Then $z \mapsto (T - zI)^{-1}$ has a pole at $z = \lambda$, and the pole is simple. We get

$$(T - zI)^{-1} = \frac{\Pi\lambda}{\lambda - z} + \mathrm{Hol}(z)$$

for $0 < |z - \lambda| \ll 1$. $\Pi\lambda$ is projection onto $\ker(T - \lambda I)$, and Π_λ is self-adjoint. Indeed, $\Pi_\lambda = \lim_{z\to\lambda}(\lambda - z)(T - zI)^{-1}$, and if z approaches λ along the real axis, then this is self-adjoint.

Next time, we will show that

$$(T - zI)^{-1} = \sum_{\lambda_j \in \mathrm{Spec}(T)\setminus\{0\}} \frac{\Pi\lambda_j}{\lambda_j - z}$$

for $\mathrm{Im}(z) \neq 0$.

23 The Spectral Theorem for Compact, Self-Adjoint Operators

23.1 Orthogonal projections and the resolvent

Let H be a Hilbert space, and let $T : H \to H$ be compact and self-adjoint. Then $\mathrm{Spec}(T) \subseteq \mathbb{R}$. If $\lambda \in \mathrm{Spec}(T) \setminus \{0\}$, then $R(z) = (T - zI)^{-1} = \frac{\Pi_\lambda}{\lambda - z} + \mathrm{Hol}(z)$ for $0 < |z - \lambda| < \varepsilon \ll 1$, where Π_λ is an orthogonal projection with $\mathrm{im}(\Pi_\lambda) = \ker(T - \lambda I)$.

Let λ_1, λ_2 be distinct nonzero eigenvalues of T, and notice that

$$\Pi_{\lambda_j}\Pi_{\lambda_k} = 0$$

if $j \neq k$. This follows from the fact that $\ker(-\lambda_j I) \perp \ker(T - \lambda_k I)$. It follows that the series $\sum_{j \geq 1} \pi_{\lambda_j} x$ converges in H for all x. Indeed,

$$\sum_{j=1}^{N} \|\Pi_{\lambda_j} x\|^2 \leq \|x\|^2$$

for each N by Bessel's inequality, so the same bound holds for the infinite sequence. Then

$$\left\| \sum_{j=1}^{N} \Pi_{\lambda_j} x - \sum_{j=1}^{M} \Pi_{\lambda_j} x \right\|^2 = \left\| \sum_{j=M+1}^{N} \Pi_{\lambda_j} x \right\|^2 = \sum_{j=M+1}^{N} \|\Pi_{\lambda_j} x\|^2 \xrightarrow{N,M \to \infty} 0$$

If we let $\Pi x = \sum_{j \geq 1} \Pi_{\lambda_j} x$, then $\Pi \in \mathcal{L}(H, H)$ is an orthogonal projection.

Proposition 23.1. *For all $x \in H$ and $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$R(z)x = (T - zI)^{-1}x = \sum_{j=1}^{\infty} \frac{\Pi_{\lambda_j} x}{\lambda_j - z}.$$

The series in the right hand side converges with $\| \sum_{j=1}^{\infty} \frac{\Pi_{\lambda_j} x}{\lambda_j - z} \| \leq \|x\| / |\operatorname{Im}(z)|$.

Proof. Consider

$$f(z) = \langle R(z)x, y \rangle - \sum_{j=0}^{\infty} \frac{\langle \Pi_{\lambda_j} x, y \rangle}{\lambda_j - z}$$

for all $x, y \in H$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then f is holomorphic on $\mathbb{C} \setminus \{0\}$, $\|f(z)| \leq 2\|x\|\|y\| / |\operatorname{Im}(z)$, and $|f(z)| \leq O(1/|z|^2)$ as $|z| \to \infty$. Indeed,

$$R(z) = (T - zI)^{-1} = ((-z)(I - T/z))^{-1} = -\frac{1}{z} I + O(1/|z|^2)$$

$$\sum_{j=1}^{\infty} \frac{\Pi_{\lambda_j}}{\lambda_j - z} = -\frac{1}{z} \underbrace{\sum \Pi_{\lambda_j}}_{=I} + I(1/|z|^2),$$

and we get the decay of f. We can write the Laurent expaansion at $z = 0$,

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \qquad \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{j+1}} \, dz$$

for $0 < |z| < \infty$.

We claim that $a_j = 0$ for all j. If $j + 1 \geq 0$, let $R \gg 1$. Then $|a_j| \leq O(1/R^2)R \to 0$. So $f(z) = \sum_{j=-\infty}^{-2} a_j z^j$. If $j + 1 < 0$, then let $k = -j - 1 > 0$. Then, assuming that $\int f(z) z^{k-2} \, dz = 0$.

$$\int_{|z|=R} f(z) z^k \, dz = \int_{|z|=R} f(z) z^{k-2} (z^2 - R^2) \, dz,$$

so

$$\begin{aligned} \left\| \int_{|z|=R} f(z) z^k \, dz \right\| &\leq 2\|x\| \|y\| \int_{|z|=R} \frac{R^{k-2}}{|\operatorname{Im}(z)|} |z^2 - R^2| \, |dz| \\ &\overset{z=Rw}{=} R \frac{2\|x\|\|y\|}{R} R^{k-2} R^2 \int_{|w|=1} \frac{|w^2 - 1|}{|\operatorname{Im}(w)|} \, |dw| \\ &= 2\|x\|\|y\| R^k \underbrace{\int_{|w|=1} \frac{|w^2 - 1|}{|\operatorname{Im}(w)|} \, |dw|}_{<\infty}. \end{aligned}$$

Letting $R \to 0$, we get $a_j = 0$ for all j. $\square$

Remark 23.1. Observe that

$$\|R(z)x\|^2 = \sum_{j=0}^{\infty} \frac{1}{|\lambda - z|^2} \|\Pi_{\lambda_j} x\|^2.$$

We also get that

$$\|(T - zI)^{-1}\|_{\mathcal{L}(H,H)} = \frac{1}{\operatorname{dist}(z, \operatorname{Spec}(T))}.$$

This estimate remains valid for all self-adjoint $T \in \mathcal{L}(H, H)$, but we will not prove that in this course.

23.2 The missing projection

Write $\lambda_0 = 0$ and $\Pi_{\lambda_0} = I - \Pi$. Then Π_{λ_0} is an orthogonal projection.

Proposition 23.2. Π_{λ_0} *is the orthogonal projection onto* $\ker(T)$*.*

Proof. Write

$$x = (T - zI)R(z)x = \sum_{j=1}^{\infty} \frac{(T - zI)\Pi_{\lambda_j} x}{\lambda_j - z} + \frac{(T - z)\Pi_{\lambda_0} x}{-z} = \underbrace{\sum_{j=1}^{\infty} \Pi_{\lambda_j} x + \Pi_{\lambda_0} x}_{=x} - \frac{T\Pi_{\lambda_0} x}{z}$$

So $T\Pi_{\lambda_0} = 0$, and $\operatorname{im}(\Pi_{\lambda_0} \subseteq \ker(T)$. IF $x \in \ker(T)$, then

$$-x/z = R(z)x = \underbrace{\sum_{j=1}^{\infty} \frac{\Pi_{\lambda_j} x}{\lambda_j - z}}_{=0} - \frac{\Pi_{\lambda_0} x}{z}.$$

So $x = \Pi_{\lambda_0} x$, making $x \in \operatorname{im}(\Pi_{\lambda_0})$. $\square$

We can write $x = \sum_{j=0}^{\infty} \Pi_{\lambda_j} x$ for all x, which is equivalent to $H = \bigoplus_{j \geq 0} H_j$, where $H_j = \Pi_{\lambda_j} H = \ker(T - \lambda_j I)$.

Theorem 23.1 (spectral theorem for compact, self-adjoint operators)**.** *Let* $T \in \mathcal{L}(H, H)$ *be compact and self-adjoint. Then* H *has an orthonormal basis consisting of eigenvectors of* T*.*

Proof. Choose an orthonormal basis in $H_j, j \geq 0$. $\square$

24 Duality and Weak Topologies

24.1 The weak topology

Definition 24.1. Let F and G be two vector spaces over $K = \mathbb{R}$ or $\mathbb{C}$, and suppose $\langle \cdot, \cdot \rangle : F \times G \to K$ sending $(x, y) \mapsto \langle x, y \rangle$ is a bilinear form. The form is said to define a **duality** between F and G if

1. If $\langle x, y \rangle = 0$ for all $y \in G$, then $x = 0$.

2. If $\langle x, y \rangle = 0$ for all $x \in F$, then $y = 0$.

Example 24.1. Let F be a Banach space B, and let $G = B^*$. Then F and G are in duality by Hahn-Banach.

Definition 24.2. The locally convex topology in F defined by the seminorms $x \mapsto |\langle x, y \rangle|$ for $y \in G$ is called the **weak topology** in F and is denoted by $\sigma(F, G)$.

We also have a weak topology $\sigma(G, F)$ in G. What are open sets in $\sigma(F, G)$? A set $O \subseteq F$ is open in $\sigma(F, G)$ iff for all $x_0 \in O$, there exists $\varepsilon > 0$ and $y_1, \ldots, y_N \in G$ such that $\{x \in F : |\langle x - x_0, y_j \rangle < \varepsilon \,\forall 1 \leq j \leq N\} \subseteq O$.

$\sigma(F, G)$ and $\sigma(G, F)$ are Hausdorff topologies.

24.2 Continuity and convergence in the weak topology

Lemma 24.1. *A linear form $L : F \to K$ is continuous for $\sigma(F, G)$ if and only if there exists a unique $y \in G$ such that $L(x) = \langle x, y \rangle$ for all $x \in F$.*

Proof. ($\Longleftarrow$): This follows immediately from the definition of the topology.

($\Longrightarrow$): L is continuous for $\sigma(F, G)$ iff there exist $y_1, \ldots, y_N$ and $C > 0$ such that $|L(x)| \leq C \sum_{j=1}^{N} |\langle x, y_j \rangle|$ for $x \in F$. We get that $\langle x, y_1 \rangle = \cdots = \langle x, y_N \rangle = 0 \implies L(x) = 0$ so that $L(x) = \sum_{j=1}^{N} \alpha_j \langle x, y_j \rangle = \left\langle x, \sum_{j=1}^{N} \alpha_j y_j \right\rangle$.[6] $\square$

Definition 24.3. Let (x_n) be a seuqence in F, and let $x \in F$. We say that $x_n \to x$ in $\sigma(F, G)$ (or **converges weakly**) if for all $y \in G$, $\langle x_n, y \rangle \to \langle x, y \rangle$.

Proposition 24.1. *Let $F = B$ be a Banach space, and let $x_n \to x$ in $\sigma(B, B^*)$. Then (x_n) is bounded: $\|x_n\| \leq C$ for $n = 1, 2, \ldots$ and $\|x\| \leq \liminf_{n \to \infty} \|x_n\|$.*

Proof. For all $\xi \in B^*$, $\langle x_n, \xi \rangle$ is bounded, so by Banach Steinhaus, there exists some $C > 0$ such that $|\langle x_n, \xi \rangle| \leq C\|\xi\|$ for $n = 1, 2, \ldots$. So $\|x_n\| \leq C$.

Since $|\langle x_n, \xi \rangle| \leq \|\xi\| \|x_n\|$, $|\langle x, \xi \rangle| \leq \|\xi\| \liminf_{n \to \infty} \|x_n\|$. So $\|x\| \leq \liminf_{n \to \infty} \|x_n\|$. $\square$

[6] The proof that L is a linear combination of these forms follows from a problem on Homework 1.

Example 24.2. Let $\xi_j \in \mathbb{R}^n$ be such that $|\xi_j| \to \infty$. Then, for $\varphi \in L^2$, $\varphi_n(x) := e^{ix\cdot\xi_j}\varphi(x)$ satisfies $\|\varphi_j\|_{L^2} = \|\varphi\|$, and $\varphi_j \to 0$ in $\sigma(L^2, L^2)$. Indeed, for $f \in L^2$,

$$\langle \varphi_j, f\rangle = \int e^{ix\cdot\xi_j} \underbrace{\varphi(x)f(x)}_{\in L^1}\,dx \xrightarrow{j\to\infty} 0$$

by Riemann-Lebesgue.

24.3 Closed and convex sets in the weak topology

Proposition 24.2. *Let B be a Banach space, and let $C \subseteq B$ be convex and nonempty. Then C is closed in $\sigma(B, B^*)$ if and only if C is closed in the usual (strong) sense.*

Proof. ($\implies$): If C is closed in $\sigma(B, B^*)$, then C^c is open in $\sigma(B, B^*)$, so C^c is open in the strong sense. So C is closed in the strong sense.

($\impliedby$): Let $C \subseteq B$ be convex and strongly closed. We claim that C^c is open in $\sigma(B, B^*)$. Le $x_0 \notin C$. By he geometric Hahn-Banach theorem, there exists a continuous linear form f on B such that $\inf_{x\in C}(\mathrm{Re}(f(x)) - \mathrm{Re}(f(x_0)) > 0$. Thus, there exists an $\alpha \in \mathbb{R}$ such that $\mathrm{Re}(f(x_0)) < \alpha < \mathrm{Re}(f(x))$ for $x \in X$. The set $N = f^{-1}(\{z : \mathrm{Re}(z) < a\})$ is open in $\sigma(B, B^*)$ (as f is continuous on $\sigma(B, B^*)$), $x_0 \in N$, and $N \cap C = \varnothing$. It follows that C^c is weakly open. So C is weakly closed. $\square$

Here is a fact to help with intuition for what the weak topology is like.

Proposition 24.3. *Let B be an infinite dimensional Banach space, and let $S = \{x \in B : \|x\| = 1\}$ be the unit sphere. The closure of S in $\sigma(B, B^*)$ is $\{x \in B : \|x\| \leq 1\}$.*

Proof. The closed ball $\{x : \|x\| \leq 1\}$ is convex, so it is weakly closed, and $\overline{S}^{\sigma(B,B^*)} \subseteq \{x : \|x\| \leq 1\}$. On the other hand, let $\|x_0\| < 1$. We check that any neighborhood U of x_0 in $\sigma(B, B^*)$ meets S. We can assume that $U = \{x : |\langle x - x_0, \xi_j\rangle| < \varepsilon\ \forall\}$ with $\xi_j \in B^*$. Notice that $\bigcap_{j=1}^N \ker(\xi_j) \neq \{0\}$. Let $y_0 \neq 0 \in \bigcap_{j=1}^\infty \ker(\xi_j)$. Then $x_0 + \lambda y_0 \in U$ for all λ, so the function $g(\lambda) = \|x_0 + \lambda y_0\|$ for $\lambda \geq 0$ is continuous and goes to ∞ at ∞. Since $g(0) < 1$, we get a λ such that $x_0 + \lambda y_0 \in S \cap U$. $\square$

24.4 The weak* topology

Definition 24.4. Let B be a Banach space. The weak topology $\sigma(B^*, B)$ is called the **weak* topology**.

Remark 24.1. The weak * topology on B^* is weaker than the weak topology $\sigma(B^*, B^{**})$.

Next time, we will prove the following theorem.

Theorem 24.1 (Banach-Alaoglu)**.** *The closed unit ball $U = \{\xi \in B^* : \|\xi\|_{B^*} \leq 1\}$ is compact in $\sigma(B^*, B)$.*

25 The Weak* Topology and the Banach-Alaoglu Theorem

25.1 Completeness of the weak* topology

Proposition 25.1. *Let $\xi_n \in B^*$ be such that $\xi_n - \xi_m \to 0$ in $\sigma(B^*, B)$ as $n, m \to \infty$. Then there exists $\xi \in B^*$ such that $\xi_n \to \xi$ in $\sigma(B^*, B)$.*

Proof. We have $\langle x, \xi_m \rangle - \langle x, \xi_m \rangle \to 0$ for each $x \in B$, so the limit $\lim_{n\to\infty} \langle x, \xi_n \rangle$ exists pointwise, and we can let $\langle x, \xi \rangle = \lim_{n\to\infty}$ for $x \in B$. Then $\xi \in B^*$ by the Banach-Steinhaus theorem. $\square$

25.2 Tychonov's theorem and the Banach-Alaoglu theorem

Theorem 25.1 (Banach-Alaoglu)**.** *Let B be a Banach space. Then the closed unit ball $U = \{\xi \in B^* : \|\xi\| \leq 1\}$ is compact in $\sigma(B^*, B)$.*

The main point in the proof is Tychonov's theorem from point set topology. Let's review this.

Let $(X_\alpha)_{\alpha \in J}$ be a collection of topological spaces. Then the product space $X = \prod_{\alpha \in J} X_\alpha = \{f : J \to \bigcup_{\alpha \in J} X_\alpha \mid f(\alpha) \in X_\alpha \ \forall \alpha \in J\}$. is equipped with the product topology, the weakest topology such that the projection maps $p_\alpha : X \to X_\alpha$ sending $x \mapsto x_\alpha$ (where $x = \{x_\alpha\}_{\alpha \in J}$) are continuous for all α. A base for the product topology is given by the finite intersection $\bigcap_{\text{finite}} p_\alpha^{-1}(O_\alpha)$, where $O_\alpha \subseteq X_\alpha$ is open.

Theorem 25.2 (Tychonov)**.** *If X_α is compact for all $\alpha \in J$, then the space $X = \prod_{\alpha \in J} X_\alpha$ is compact in the product topology.*

We will not prove this, but we will use this in our proof of the Banach-Alaoglu theorem.

Proof. When $x \in B$, let $D_x = \{z \in K : |z| \leq \|x\|\}$. If $\xi \in U = \{\xi \in B^* : \|\xi\| \leq 1\}$, then $\langle x, \xi \rangle \in D_x$ for all x. Consider the injective map $\gamma : U \to D = \prod_{x \in B} D_x$ sending $\xi \mapsto \{\langle x, \xi \rangle\}_{x \in B}$. Equip U with the weak* topology and D with the product topology.

We claim that γ is continuous. Let O be an open set in D. We can assume that $O = \{f = (f_x)_{x \in B} : |f_{x_j} - cx_j| < \varepsilon_{x_j}, \varepsilon_{x_j} > 0, c_{x_j} \in D_{x_j}, 1 \leq j \leq N\}$. Then the inverse image $\gamma^{-1}(O) = \{\xi \in U : |\langle x_j, \xi \rangle - c_{x_j}| < \varepsilon_{x_j}, 1 \leq j \leq N\}$ is open in $\sigma(B^*, B)$. Similarly, $\gamma^{-1} : \mathrm{im}(\gamma) \to U$ is continuous. So $\gamma : U \to \mathrm{im}(\gamma)$ is a homeomorphism.

It suffices to check that $\mathrm{im}(\gamma) \subseteq D$ is compact in the product topology. By Tychonov's theorem, D is compact, so we only need that $\mathrm{im}(\gamma)$ is closed. We have that

$$\begin{aligned}
\mathrm{im}(\gamma) &= \{f = (f_x)_{x \in B} \in D : f_{x+y} = f_x + f_y, f_{\lambda x} = \lambda f_x \ \forall x, y \in B, \forall \lambda \in \mathbb{C}\} \\
&= \bigcap_{x, y \in B} \{f : f_{x+y} = f_x + f_y\} \cap \bigcap_{\substack{\lambda \in \mathbb{C} \\ x \in B}} \{f : f_{\lambda_x} = \lambda f_x\}.
\end{aligned}$$

We now claim that $E_{x,\lambda} := \{f = (f_y)_{y \in B} : f_{\lambda x} = \lambda f_x\}$ is closed in D. Let $f_0 \in \overline{E_{x,\lambda}}$. An open neighborhood of f_0 is a set of the form $V_{x,\varepsilon} := \{f \in D : |f_x - f_{0,x}| < \varepsilon\}$. Let $f \in E_{x,\lambda} \cap V_{\lambda x,\varepsilon} \cap V_{x,\varepsilon}$; $V_{\lambda x,\varepsilon} \cap V_{x,\varepsilon}$ is an open neighborhood of f_0. Then

$$\begin{aligned} |f_{0,\lambda x} - \lambda f_{0,x}| &= |f_{0,\lambda x} - f_{\lambda x} + \lambda f_x - \lambda f_{0,x}| \\ &\leq |f_{0,\lambda x} - f_{\lambda x}| + |\lambda||f_x - f_{0,x}| \\ &\leq \varepsilon + |\lambda|\varepsilon, \end{aligned}$$

so $f_0 \in E_{x,\lambda}$. The result follows. □

Now that we have proved the theorem in full generality, it is worth noting that for separable Banach spaces, there is an elementary proof.

Proposition 25.2. *Let B be a separable Banach space, and let $x_1, x_2, \ldots$ be a dense subset. Then the seminorms $\xi \mapsto |\langle x_k, \xi \rangle|$ for $j = 1, 2, \ldots$ define the same ropology as $\sigma(B^*, B)$.*

Proof. □

Corollary 25.1. *Let B be a separable Banach space. Then $U = \{\xi \in B^* : \|\xi\| \leq 1\}$ is a compact metrizable space in the weak* topology $\sigma(B^*, B)$.*

Proof. If $\|\xi_n\| \leq 1$ for $n = 1, 2, \ldots$, then there exists a subsequence (ξ_{n_k}) converging in $\sigma(B^*, B)$. □

26 Applications of Banach-Alaoglu

26.1 Banach-Alaoglu for separable Banach spaces

Last time, we stated the following proposition.

Proposition 26.1. *If G is separable with the dense subset $\{x_1, x_2, \dots\}$, then the seminorms $\xi \mapsto |\langle x_j, \xi\rangle|$ for $\xi \in U = \{\xi : \|x\| \leq 1\}$ and $j = 1, 2, \dots$ define the same topology as $\sigma(B^*, B)$ on U.*

Proof. It sufficies to check that if $O \subseteq U$ is open for $\sigma(B^*, B)$, then O is open for the topology determined by these seminorms. Let $\xi_0 \in U$ and N be an open neighborhood of ξ_0 in $\sigma(B^*, B)$. We can assume that $N = \{\xi \in U : |\langle y_j, \xi - \xi_0\rangle| < \varepsilon \, \forall 1 \leq j \leq M\}$. We claim that N contained a neighborhood of ξ_0 in the topology defined by the seminorms. For each $j \in \{1, \dots, M\}$, pick k_j such that $\|y_j - x_{k_j}\| < \varepsilon/4$. If $|\left\langle x_{k_j}, \xi - \xi_0\right\rangle| < \varepsilon/2$ for $1 \leq j \leq M$, then

$$|\langle y_j, \xi - \xi_0\rangle| \leq \underbrace{|\left\langle y_j - x_{k_j}, \xi - \xi_0\right\rangle|}_{\leq \|y_j - x_{k_j}\| \|\xi - \xi_0\| < \varepsilon/2} + \underbrace{|\left\langle x_{k_j}, \xi - \xi_0\right\rangle|}_{< \varepsilon/2} < \varepsilon$$

for all $\xi \in N$ and $1 \leq j \leq M$. Then $N \supseteq \{\xi \in U : |\left\langle x_{k_j}, \xi - \xi_0\right\rangle| < \varepsilon/2, 1 \leq j \leq M\}$, and the result follows. $\square$

Corollary 26.1. *If B is separable, then $U = \{\xi \in B^* : \|\xi\| \leq 1\}$ is a complete metrizable space for $\sigma(B^*, B)$.*

Example 26.1. Let $f_n \in L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$ be bounded: $\|f_n\|_{L^p} \leq C$. Then there exists a subsequence f_{n_k} and $f' \in L^p$ such that

$$\int f_{n_k} g \, dx \to \int f g \, dx$$

for $g \in L^q$ with $1/p + 1/q = 1$.

When $p = 1$, $L^1(\mathbb{R}^d) \subseteq M(\mathbb{R}^d)$, the space of bounded measures on $\mathbb{R}^d$. If $\|f_n\|_{L^1} \leq C$, then there exists a subsequence f_{n_k} and a $\mu \in M(\mathbb{R}^d)$ such that

$$\int f_{n_k} g \, dx \to \int g d\mu$$

for all $g \in C_0(\mathbb{R}^d)$, the space of functions which vanish at ∞.

26.2 Applications of Banach-Alaoglu to minimizing functionals

Definition 26.1. A Banach space B is **reflexive** if the natural linear isometry $J : B \to B^{**}$ sending $x \mapsto (\xi \mapsto \langle x, \xi \rangle)$ is a bijection.

Proposition 26.2 (Minimization of functionals[7])**.** *Let B be a reflexive Banach space with B^* separable, and let $J : B \to \mathbb{R}$ be a function such that*

1. *J is convex: $J(\lambda u + (1-\lambda)v) \leq \lambda J(u) + (1-\lambda)J(v)$ fir $0 \leq \lambda \leq 1$ and $u, v \in B$.*
2. *J is norm lower semicontinuous: For all $x \in \mathbb{R}$, the set $\{u \in B : J(u) > a\}$ is open iff if $u_n \to u_0$ in B, then $J(u_0) \leq \liminf_{n\to\infty} J(u_n)$.*
3. *J is coercive: There exists $C > 1$ such that $J(u) \geq \|u\|^q/C - C$ for all u, for some $q \geq 1$.*

In particular, $\mu = \inf_{u \in B} J(u) > -\infty$. Then there exists some $u_0 \in B$ such that $J(u_0) = \mu$.

Proof. Let $u_n \in B$ be such that $J(u_n) \to \mu$. Property 3 implies that (u_n) is bounded: $\|u_n\| \leq C$. By Banach-Alaoglu, there exists a subsequence (u_{n_k}) and $u_0 \in B$ such that $u_{n_k} \to u_0$ in $\sigma(B, B^*)$. Now J is convex norm lower semicontinuous, so $\{u \in B : J(u) \leq a\}$ is closed and convex. By convexity, it is weakly closed, so J is lower semicontinuous with respect to $\sigma(B, B^*)$. If $u_{n_k} \to u_0$ in $\sigma(B, B^*)$, then $H(u_0) \leq \liminf J(u_{n_k}) = \mu$, and we get the claim. $\square$

Here is a concrete application.

Example 26.2. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and let $H^1(\Omega) = \{u \in L^2(\Omega) : \partial_{x_j} u \in L^2(\Omega)\ \forall 1 \leq j \leq n\}$ be a Hilbert space with the inner product $\langle u, v \rangle_{H^1} = \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2}$. Define $H_0^1(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. This is not all of H^1; roughly, $u \in H_0^1(\Omega)$ iff $u \in H^1(\Omega)$ and "$u|_{\partial\Omega} = 0$."

Apply the abstract discussion when $B = H_0^1(\Omega)$ and $J(u) = (1/2)\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx$ where $f \in L^2(\Omega)$.[8] We claim that there exists some $u_0 \in H_0$ which is a minimizer of J. Observe that:

1. J is convex.
2. J is continuous.
3. J is coercive: $J(u) \geq \|\nabla u\|_{L^2}^2 - \|f\|_{L^2}\|u\|_{L^2} \geq \|\nabla u\|_{L^2}^2 - \varepsilon\|u\|_{L^2} - (1/\varepsilon)\|f\|_{L^2}^2$. Since $\|u\||_{L^2} \leq C\|\nabla u\|_{L^2}$ for $u \in C_0^\infty$, we get $J(u) \geq (1/C)\|u\|_H^2 - C$.

[7] This application comes from calculus of variations.

[8] This is sometimes called the Dirichlet functional.

So we have a minimizer $J(u_0) \leq J(u_0 + \varepsilon v)$ for all ε. Then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J(u_0 + \varepsilon v) = 0,$$

where $-\Delta u_0 = f$ and $u_0 \in H_0(\Omega)$.

27 Reflexive Spaces and Kakutani's theorem

27.1 Helly's lemma

Let B be a Banach space. Recall that B is called reflexive if the natural map $J : B \to B^{**}$ given by $x \mapsto (\xi \mapsto \langle x, \xi \rangle)$ is a bijection. We want to characterize reflexive spaces. First we need a lemma.

Lemma 27.1 (Helly)**.** *Let B be a Banach space, and let $\xi_1, \dots \xi_m \in B^*$ and $\alpha_1, \dots, \alpha_n \in K$ $(= \mathbb{R}$ or $\mathbb{C})$. The following conditions are equivalent:*

1. *For each $\varepsilon > 0$, there exists $x_\varepsilon \in B$ such that $\|x_\varepsilon\| \leq 1$ and $|\langle x_\varepsilon, \xi_j \rangle - \alpha_j| < \varepsilon$ for $1 \leq j \leq n$.*

2. *For all $\beta_1, \dots, \beta_n \in K$, $|\sum_{j=1}^n \beta_j \alpha_j| \leq \| \sum_{j=1}^n \beta_j \xi_j \|$.*

Proof. (1) $\implies$ (2): Let $\beta_1, \dots, \beta_n \in K$ be given, and let $S = \sum_{j=1}^n |\beta_j|$. For $\varepsilon > 0$, we have

$$\left| \sum_{j-1}^n \beta_j \langle x_\varepsilon, \xi_j \rangle - \sum_{j=1}^n \beta_j \alpha_j \right| < \varepsilon S \implies \left| \sum_{j=1}^n \beta_j \alpha_j \right| \leq \left\| \sum_{j=1}^n \beta_j \xi_j \right\| \underbrace{\|x_\varepsilon\|}_{\leq 1} + \varepsilon S.$$

Letting $\varepsilon \to 0$ gives us $|\sum_{j=1}^n \beta_j \alpha_j| \leq \| \sum_{j=1}^n \beta_j \xi_j \|$.

(2) $\implies$ (1): Consider the linear, continuous map $F : B \to K^n$ sending $x \mapsto (\langle x, \xi_1 \rangle, \dots, \langle x, \xi_n \rangle)$. Then condition 1 holds if and only if $(\alpha_1, \dots, \alpha_n) \in \overline{F(\{x \in B : \|x\| \leq 1\})}$. Assume that $(\alpha_1, \dots, \alpha_n) \notin \overline{F(\{x \in B : \|x\| \leq 1\})}$ (which is closed and convex in K^n). By the geometric Hahn-Banach theorem, there exists a continuous, linear form f on K^n and $\gamma \in \mathbb{R}$ such that $\mathrm{Re}(f(y)) < \gamma < \mathrm{Re}(f(\alpha_1, \dots, \alpha_n))$ for all $y \in F(\{x \in B : \|x\| \leq 1\})$. Writing $f(y) = \beta \cdot y = \sum_{j=1}^n \beta_j y_j$, we get

$$\mathrm{Re} \left(\sum_{j=1}^n \beta_j \langle x, \xi_j \rangle \right) < \gamma < \mathrm{Re} \left(\sum_{j=1}^n \alpha_j \beta_j \right) \leq \left| \sum_{j=1}^n \alpha_j \beta_j \right|$$

for all $x \in B$ with $\|x\| \leq 1$. So

$$\mathrm{Re} \left\langle x, \sum_{j=1}^n \beta_j \xi_j \right\rangle < \gamma < \left| \sum_{j=1}^n \alpha_j \beta_j \right|$$

for all $x \in B$ with $\|x\| \leq 1$. Replacing x by $e^{i\theta} x$ (with $\theta \in \mathbb{R}$), we get by varying θ that

$$\left\langle x, \sum_{j=1}^n \beta_j \xi_j \right\rangle < \gamma < \left| \sum_{j=1}^n \alpha_j \beta_j \right|$$

if $\|x\| \leq 1$. That is,

$$\left\|\sum_{j=1}^{n} \beta_j \xi_j\right\| \leq \gamma < \left|\sum_{j=1}^{n} \alpha_j \beta_j\right|,$$

which contradicts condition 2. $\square$

Lemma 27.2. *Let B be a Banach space. Then the set $J(\{x \in B : \|x\| \leq 1\})$ is dense in $\{z \in B^{**} : \|z\| \leq 1\}$ for the weak topology $\sigma(B^{**}, B^*)$.*

Proof. Let $z \in B^{**}$ with $\|z\| \leq 1$, and let V be an open neighborhood of z in the topology $\sigma(B^{**}, B^*)$. We claim that $V \cap J(\{x \in B : \|x\| \leq 1\}) \neq \varnothing$. We may assume that V has the form $V = \{y \in B^{**} : |\langle \xi_j, y - z\rangle| < \varepsilon \, \forall 1 \leq j \leq n\}$ with $\varepsilon > 0$ and $\xi_j \in B^*$. We must show that there exists $x \in B$ with $\|x\| \leq 1$ such that $|\langle x, \xi_j\rangle - \langle \xi_j, z\rangle| < \varepsilon$ for $1 \leq j \leq n$. Letting $\alpha_j = \langle \xi_j, z\rangle$, we notice that for all $\beta_1, \ldots, \beta_n \in K$,

$$\left|\sum_{j=1}^{n} \beta_j \alpha_j\right| = \left|\left\langle \sum_{j=1}^{n} \beta_j \xi_j, z\right\rangle\right| \leq \left\|\sum_{j=1}^{n} \beta_j \xi_j\right\|_{B^*} \underbrace{\|z\|_{B^{**}}}_{\leq 1} \leq \left\|\sum_{j=1}^{n} \beta_j \xi_j\right\|_{B^*}.$$

By the previous lemma, there exists an $x_\varepsilon \in B$ with $\|x_\varepsilon\| \leq 1$ such that $|\langle x_\varepsilon, \xi_j\rangle - \alpha_j| < \varepsilon$. Thus, $J(x_\varepsilon) \in J(\{x \in B : \|x\| \leq 1\}) \cap V$. $\square$

Remark 27.1. Notice that $J(\{x \in B : \|x\| \leq 1\}) \subseteq \{z \in B^{**} : \|z\| \leq 1\}$ is closed in the strong sense.

27.2 Kakutani's theorem

Proposition 27.1. *Let B_1, B_2 be Banach spaces, and let $T \in \mathcal{L}(B_1, B_2)$. Then $T : (B_1, \sigma(B_1, B_1^*)) \to (B_2, \sigma(B_2, B_2^*))$ is continuous.*

Proof. Let $O \subseteq B_2$ be open for $\sigma(B_2, B_2^*)$. We may assume that $O = \{y \in B_2 : |\langle y - x, \eta_j\rangle| < \varepsilon \, \forall 1 \leq j \leq n\}$, where $x \in B_2$, $\eta_j \in B_2^*$, and $\varepsilon > 0$. Then

$$\begin{aligned} T^{-1}(O) &= \{z \in B_1 : |\langle Tz - x, \eta_j\rangle| < \varepsilon \, \forall 1 \leq j \leq n\} \\ &= \{z \in B_1 : |\langle Tz, \eta_j\rangle - \langle x, \eta_j\rangle| < \varepsilon \, \forall 1 \leq j \leq n\} \\ &= \{x \in B_1 : |\langle z, T^* \eta_j\rangle - \langle x, \eta_j\rangle| < \varepsilon \, \forall 1 \leq j \leq n\}, \end{aligned}$$

which is open in B_1 for $\sigma(B_1, B_1^*)$ since $T^* \eta_j \in B_1^*$. $\square$

Theorem 27.1 (Kakutani)**.** *A Banach space B is reflexive if and only if the closed unit ball $\{x \in B : \|x\| \leq 1\}$ is compact for the weak topology $\sigma(B, B^*)$.*

Proof. Assume first that B is reflexive. Then $J(\{x \in B : \|x\|_B \leq 1\}) = \{y \in B^{**} : \|y\|_{B^{**}} \leq 1\}$ is compact in the weak* topology $\sigma(B^{**}, B^*)$ by Banach Alaoglu. We only have to check that $J^{-1} : (B^{**}, \sigma(B^{**}, B^*)) \to (B, \sigma(B, B^*))$ is continuous (as a continuous image of a compact set is compact). When $O = \{y \in B : |\langle y - x, \xi \rangle| < \varepsilon\}$ with $x \in B$, $\xi \in B^*$ and $\varepsilon > 0$ is open in $\sigma(B, B^*)$, it suffices to check that $(J^{-1})^{-1}(O) = J(O)$ is open in B^{**} with respect to $\sigma(B^{**}, B^*)$. This follows from $J(O) = \{z \in B^{**} : |\langle \xi, z \rangle - \langle x, \xi \rangle| < \varepsilon\}$.

Assume that $\{x \in B : \|x\| \leq 1\}$ is compact for the weak topology $\sigma(B, B^*)$. We claim that the map $J : (B, \sigma(B, B^*)) \to (B^{**}, \sigma(B^{**}, B^{***}))$ is continuous. Indeed, $J : B \to B$ is strongly continuous (as an isometry), and so the claim follows. Now $B^* \subseteq B^{***}$, so the topology $\sigma(B^{**}, B^*)$ on B^{**} is weaker than $\sigma(B^{**}, B^{***})$, and it follows that $J : (B, \sigma(B, B^*)) \to (B^{**}, \sigma(B^{**}, B^*))$ is continuous. We get that $J(\{x \in B : \|x\| \leq 1\})$ is compact for $\sigma(B^{**}, B^*)$ as the continuous image of a compact set is compact. It is also dense in $\{x \in B^{**} : \|z\| \leq 1\}$ for the topology $\sigma(B^{**}, B^*)$. Therefore, $J(\{x \in B : \|x\| \leq 1\}) = \{z \in B^{**} : \|z\| \leq 1\}$ and hence, $J(B) = B^{**}$. So B is reflexive. $\square$

28 Properties of Reflexive Spaces

28.1 Reflexivity of subspaces and the dual space

Last time we proved Kakutani's theorem, that a Banach space B is reflexive if and only if $\{x \in B : \|x\| \leq 1\}$ is compact for $\sigma(B, B^*)$.

Proposition 28.1. *Let B be a reflexive Banach space, and let $M \subseteq B$ be a closed subspace. Then M is reflexive.*

Proof. We have to show that $\{x \in M : \|x\| \leq 1\}$ is compact for $\sigma(M, M^*)$. Now $\sigma(M, M^*)$ agrees with the topology induced on M by $\sigma(B, B^*)$. We can write $\{x \in M : \|x\| \leq 1\} = M \cap \{x \in B : \|x\| \leq 1\}$, where $\{x \in B : \|x\| \leq 1\}$ is compact for $\sigma(B, B^*)$. M is closed and convex, so it is closed for $\sigma(B, B^*)$. Therefore, $M \cap \{x \in B : \|x\| \leq 1\}$ is compact for $\sigma(B, B^*)$, so it is compact for $\sigma(M, M^*)$. By Kakutani's theorem, M is reflexive. $\square$

Corollary 28.1. *A Banach space B is reflexive if and only if B^* is reflexive.*

Proof. ($\Longrightarrow$): By Banach-Alaoglu, $\{\xi \in B^* : \|\xi\| \leq 1\}$ is compact for $\sigma(B^*, B)$. B is reflexive, so this topology agrees with $\sigma(B^*, B^{**})$, as B is reflexive. This is the weak topology on B^*. By Kakutani's theorem, B^* is reflexive.

($\Longleftarrow$): If B^* is reflexive, by the first part of the proof, B^{**} is reflexive. Now $J : B \to B^{**}$ is isometric, so $J(B) \subseteq B^{**}$ is closed. So $J(B)$ is reflexive. We claim that B is reflexive. In general, if B_1 and B_2 are Banach spaces with B_2 reflexive and there exists $T \in L(B_1, B_2)$ is bijective, then B_1 is reflexive. The adjoint $T^* : B_2^* \to B_1^*$ is bijective; for all $\xi \in B_1^*$, there exists a unique $\eta \in B_2^*$ sicj tjat $\xi = T^*\eta$. Let $y \in B_1^{**}$, and consider (for $\xi \in B_1^*$),

$$\langle \xi, y \rangle = \langle T^*\eta, y \rangle = \langle \eta, T^{**}y \rangle = \langle \eta, x \rangle \,,$$

where $x \in B_2$, since B is reflexive (so we can view $T^{**} : B_1^{**} \to B_2$). We get

$$\langle \xi, y \rangle = \langle x, \underbrace{(T^*)^{-1}}_{=(T^{-1})^*} \xi \rangle = \langle \underbrace{T^{-1}x}_{\in B_1}, \xi \rangle.$$

This shows that B_1 is reflexive, and we get that B is reflexive. $\square$

We record the general statement we have proved here for completeness.

Proposition 28.2. *Let B_1 and B_2 be Banach spaces with B_2 reflexive, and let $T \in L(B_1, B_2)$ is bijective. Then B_1 is reflexive.*

Example 28.1. $L^1(\mathbb{R}^n)$ is not reflexive, so $L^\infty(\mathbb{R}^n)$ is not reflexive. This differs from the spaces L^p for $1 < p < \infty$, which are reflexive.

28.2 Compactness properties of the weak topology

Corollary 28.2. *Let B be a reflexive Banach space, and let $K \subseteq B$ be closed, bounded, and convex. Then K is compact for $\sigma(B, B^*)$.*

Proof. K is closed and convex, so K is closed for $\sigma(B, B^*)$. Moreover, $K \subseteq \{x \in B : \|x\| \leq C\}$, which is compact for $\sigma(B, B^*)$. So K is compact. $\square$

Recall: Let B be a separable Banach space, and let $\xi_n \in B^*$ be a such that $\|\xi_n\| \leq C$. Then there exists a subsequence (ξ_{n_k}) which converges in $\sigma(B^*, B)$. We have a similar statement for reflexive Banach spaces which need not be separable.

First, we state a basic fact that we will use.

Proposition 28.3. *Let B be a Banach space. If B^* is separable, then so is B.*

We do not have time to prove this statement, but you can either do the proof yourself or see the proof in Folland's textbook (exercise 25 in chapter 5).

Theorem 28.1. *Let B be a reflexive Banach space, and let (x_n) be a bounded sequence. There exists a subsequence (x_{n_k}) which converges in $\sigma(B, B^*)$.*

Proof. Let $M_0 \subseteq B$ be the space of finite linear combinations of the x_ns. M_0 is separable (using rational coefficients), and so is $M = \overline{M}_0$. Then $x_n \in M$ for all n, and M^* is separable and reflexive. Then $J(M)$ is separable, and $J(M) = M^{**}$. Since M^{**} is separable, we get that M^* is separable. It follows that the weak topology $\sigma(M, M^*)$ on $\{x \in M : \|x\| \leq 1\}$ is metrizable. Thus, $\{x \in M : \|x\| \leq 1\}$ is a compact metric space for $\sigma(M, M^*)$, and there exists a subsequence (x_{n_k}) which converges in $\sigma(M, M^*)$. In other words, $\langle x_{n_k}, \eta \rangle \to \langle x_0, \eta \rangle$ for all $\eta \in M^*$. If $\xi \in B^*$, then $\xi|_M \in M^*$ and so $x_{n_k} \to x_0$ in $\sigma(B, B^*)$. $\square$

Remark 28.1. If B is a Banach space, then B is separable and reflexive if and only if B^* is separable and reflexive.