Mathematics 222B Lecture 7 Notes

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1 Compactness of Sobolev Embeddings and Poincaré-Type Inequalities

1.1 Compactness of embeddings of Hölder spaces into Hölder spaces

Last time we defined the notion of compact operators.

Definition 1.1. Let X, Y be normed spaces, and let $T : X \to Y$ be linear. We say that T is a **compact operator** if $T(B_X)$, the image of the unit ball in X, is compact in Y. Equivalently, we may require that for all bounded $\{x_n\} \subseteq X$, $\{Tx_n\}$ has a convergent subsequence.

The proof will resemble the proof of the Arzelà-Ascoli theorem.

Theorem 1.1 (Arzelà-Ascoli)**.** *Let K be a compact set and $\mathcal{A} \subseteq C(K)$. Suppose that*

1. *$\mathcal{A}$ is **locally bounded**, i.e. for any $x \in K$, there is an $M(x)$ such that for all $f \in \mathcal{A}$, $|f(x)| \leq M(x)$.*
2. *$\mathcal{A}$ is **equicontinuous**, i.e. for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{A}$,*

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon, \qquad \forall x, y \in K.$$

Then $\mathcal{A}$ is compact.

There is a weaker notion of convergence in $C(K)$, pointwise convergence. The link between pointwise and uniform convergence is given by the equicontinuity assumption. In short, we use extra regularity to help us prove compactness.

Theorem 1.2 (Compactness of $C^{0,\alpha}(U) \subseteq C^{0,\alpha'}(U)$)**.** *Let U be a bounded open subset of $\mathbb{R}^d$, and assume $0 < \alpha' < \alpha < 1$ (so that $C^{0,\alpha}(U) \subseteq C^{0,\alpha'}(U)$). The embedding $C^{0,\alpha}(U) \to C^{0,\alpha'}(U)$ is compact.*

Here is a sketch of the proof.

Proof.

(i) The first observation is to note that the embedding $C^{0,\alpha}(U) \to C(U)$ is compact (this is by Arzelà-Ascoli).

(ii) By (i), if $\{u_n\} \subseteq C^{0,\alpha}(U)$ is bounded: $\|u_n\}_{C^{0,\alpha}} \le M$, then there is a subsequence u_{n_j} such that $\{u_{n_j}\}$ is convergent in $C(U)$ (to u_∞). We claim that in fact,

$$\|u_{n_j} \to u_\infty\|_{C^{0,\alpha'}(U)} \to 0.$$

The key idea here is **interpolation**. Because

$$\|v\|_{C^{0,\alpha'}} = \|v\|_{L^\infty} + [v]_{C^{0,\alpha'}},$$

we need to show that

$$[v]_{C^{0,\alpha'}} \le \|v\|_{L^\infty} [v]_{C^{0,\alpha}}^{\alpha'/\alpha},$$

where the α'/α exponent comes from dimensional analysis concerns. If we have this, then

$$[u_{n_j} - u_\infty]_{C^{0,\alpha'}} \le \underbrace{\|u_{n_j} - u_\infty\|^{1-\alpha'/\alpha}}_{\to 0 \text{ by (i)}} \underbrace{[u_n - u_\infty]_{C^{0,\alpha}}^{\alpha'/\alpha}}_{\text{bdd}}.$$

To prove this inequality, write

$$\frac{|v(x) - v(y)|}{|x-y|^{\alpha'}} \le (|v(x)| + |v(y)|)^{1-\alpha'/\alpha} \left(\frac{|v(x) - v(y)|}{|x-y|^{\alpha'}} \right)^{\alpha'/\alpha}.$$

Then take the sup over $x, y \in U$ with $x \ne y$ on both sides. □

1.2 Rellich-Kondrachov compactness of embedding Sobolev spaces into L^p spaces

Theorem 1.3 (Rellich-Kondrachov)**.** *Let $d \ge 2$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U. (Recall that if $1 \le p < d$, we have the embedding $W^{1,p}(U) \to L^{p^*}(U)$, where $\frac{d}{p^*} = \frac{d}{p} - 1$.) Let $1 \le p < d$, and let $1 \le q < p^*$. Then the embedding $W^{1,p}(U) \to L^q(U)$ is compact.*

As in the discussion of Arzelà-Ascoli, we will approximate a bounded sequence by a part which is compact and leverage some sort of uniform control. Here is a property of mollifiers that will be useful for us: Recall that if $v \in L^p(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\int \varphi = 1$, $\varphi_\varepsilon * v \to v$ in $L^p(\mathbb{R}^d)$. This is a qualitative statement that doesn't tell us how fast this converges with respect to ε. However, if we have more information, we can rectify this.

Lemma 1.1 (Accelerated convergence of modifiers)**.** *Let* $1 \leq p < \infty$*, and suppose* $v \in W^{k,p}$*. Choose* $\varphi \in C_c^\infty(\mathbb{R}^d)$ *such that* $\int \varphi \, dx = 1$ *and* $\int x^\alpha \varphi \, dx = 0$ *for all* $1 \leq |\alpha| < k$*.*[1] *Then*

$$\|\varphi_\varepsilon * v - v\|_{L^p} \leq C\varepsilon^k \|\partial^{(k)} v\|_{L^p}.$$

Here is the proof of this lemma when $k = 2$. The argument is the same for other values of k.

Proof. First, write

$$\int \varphi_\varepsilon(y) v(x-y) \, dy - \underbrace{v(x)}_{=\int \varphi_\varepsilon(y) v(x) \, dy} = \int \varphi_\varepsilon(y)(v(x-y) - v(x)) \, dy.$$

Here, we should think of $|y| \lesssim \varepsilon$. To quantify the convergence of the v part, we Taylor expand in y. We will be using the integral form of the Taylor expansion with remainder.[2] Here is the $k = 2$ case: Write $\int_0^1 \frac{d}{ds} v(x - sy) \, ds = -\int \frac{d}{ds}(1-s) \frac{d}{ds} v(x - sy) \, dx = \frac{d}{dsv(x-sy)|_{s=0} + \int_0^1 (1-s) \frac{[}{d}^2} ds^2 v(x-sy) \, ds$. The first term gves $y \cdot \nabla v(x)$, and the second term gives $y^i y^j \int_0^1 (1-s) \partial_i \partial_j v(x - sy) \, ds$. The contribution of the first term is 0 by the moment condition, and we are left with the remainder, which we can control. In all, we get

$$\left| \int \varphi_\varepsilon(y) v(x-y) \, dy - v(x) \right| \leq \int |\varphi_\varepsilon(y)| |y|^2 \int_0^1 |\partial^2 \varphi(x - sy)| \, ds \, dy.$$

This tells us that

$$\begin{aligned} \|\cdot\|_{L^p} &\leq \|\partial^2 v\|_{L^p} \int |\varphi_\varepsilon(y)| \underbrace{|y|^2}_{\lesssim \varepsilon^2} \, dy \\ &\lesssim \varepsilon^2 \|\partial^2 v\|_{L^p}. \end{aligned}$$

□

Now let's prove the theorem.

Proof.

Step 1: Reduce to the compactness of $W^{1,p}(U) \to L^p(U)$. This is sufficient because of the following two cases:

Case 1: $W^{1,p} \to L^q(U)$ with $1 \leq q \leq p$. In this case, if U is bounded, then Hölder gives $\|v\|_{L^q(U)} \leq |U|^{1/q - 1/p} \|v\|_{L^p}$, and we already have control in L^p.

[1] The conditions $\int x^\alpha \varphi \, dx = 0$ are called **moment conditions**.

[2] Sung-Jin Oh says that this is the only version of Taylor's theorem you should ever use; this is a lesson he learned later than he would have liked.

Case 2: $W^{1,p} \to L^q(U)$ with $p < q < p^*$. Again by Hölder, we have

$$\|v\|_{L^q} \leq \|v\|_{L^p}^{\theta} \|v\|_{L^{p^*}}^{1-\theta},$$

where $\frac{d}{q} = \frac{d}{p}\theta + \frac{d}{p^*}(1-\theta)$. The condition that $p < q < p^*$ tells us that $0 < \theta < 1$. The L^p term goes ti 0 by compactness of $W^{1,p} \to C^p$, and the L^{p^*} term goes to 0 by the Sobolev inequality.

Step 2: Prove compactness of $W^{1,p}(U) \to L^p(U)$: Given $\{u_n\} \subseteq W^{1,p}(U)$ with $\|u_n\|_{W^{1,p}(U)} \leq M < \infty$, by extension, we can find a sequence of extensions $\widetilde{u}_n$ of u_n defined on $\mathbb{R}^d$ such that

$$\|\widetilde{u}_n\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u_n\|_{W^1,p(U)} \leq CM$$

and $\operatorname{supp} \widetilde{u}_n \subseteq V$, where V is a bounded open set containing $\overline{U}$. It suffices to find a subsequence of $\widetilde{u}_n$ that converges in L^p. Introduce $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\int \varphi \, dx = 1$, and write

$$\widetilde{u}_n = \underbrace{\varphi * \widetilde{u}_n}_{v_{n\varepsilon}} + \underbrace{(\widetilde{u}_n - \varphi * \widetilde{u}_n)}_{e_{n,\varepsilon}}.$$

By the lemma,

$$\|e_{n\varepsilon}\|_{L^p} \leq C\varepsilon M,$$

independent of n. Also, note that using Hölder's inequality (specifically using that $\int |\widetilde{u}_n(x-y)\varphi_\varepsilon(x-y)| \, dy \leq \|\widetilde{u}_n\|_{L^p} \|\varphi_\varepsilon\|_{L^{p'}}$),

$$\|v_{n,\varepsilon}\|_{L^\infty} + \|\nabla v_{n,\varepsilon}\|_{L^\infty} \leq C_\varepsilon.$$

For each ℓ, there exists a subsequence $\widetilde{u}_{n_\ell}$ such that

$$\|e_{n_\ell,\varepsilon}\| < 2^{-\ell}$$

and such that

$$\|v_{n_{\ell'},\varepsilon} - v_{n_{\ell'},\varepsilon}\|_{L^p} < 2^{-\ell} \qquad \forall \ell', \ell'' > \ell.$$

(The second statement is by Arzelà-Ascoli. Now use a diagonal argument to extract a convergent subsubsequence; i.e. apply this recursively to subsequences and then extract a diagonal subsequence that converges. $\square$

1.3 Poicaré-type inequalities

A **Poincaré-type inequality** refers to any inequality that controls u in terms of information on Du, along with some additional condition to fix the ambiguity.

Theorem 1.4 (Poincaré inequality)**.** *Let $1 \le p < \infty$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U. For $u \in W^{1,p}(U)$ with $\int_U u\,dx = 0$,*

$$\|u\|_{L^p} \le C_U \|Du\|_{L^p}.$$

Remark 1.1. For $p = 1$, the proof requires a bit more effort than what we will say.

Here is a proof from Evans' book. This is a typical application of Rellich-Kondrachov compactness.

Proof. We argue by contradiction. For contradiction, assume that for each $n \ge 1$, there exists $u_n \in W^{1,p}(U)$ such that $\int u_n = 0$ and

$$\|u_n\|_{L^p} \ge n \|\nabla u_n\|_{L^p}.$$

By normalization, we may assume that $\|u_n\|_{L^p} = 1$. Then it follows that

$$\|\nabla u_n\|_{L^p} \le \frac{1}{n}.$$

In particular, this means that $\|u_n\|_{W^{1,p}(U)} \le 2$, and by Rellich-Kondrachov compactness, there is a subsequence such that $u_n \to u_\infty$ in L^p. Moreover, $1 = \|u_n\|_{L^p} \to \|u_\infty\|_{L^p}$. Since $Du_n \to Du$ weakly in L^p, we must have $Du = 0$. That is, u is constant on U. But $0 = \int u_n \to \int u$, which tells us that $u = 0$ on U. However, this contradicts $\|u\|_{L^p} = 1$. $\square$

In most applications of this compactness arguments, u will satisfy linear relations that imply that it equals 0. Then you can show that it's not 0.

Remark 1.2. Another popular form of the Poincaré inequality is

$$\left\| u - \frac{1}{|U|} u \right\|_{L^p} \le C_U \|Du\|_{L^p}.$$

Here are some other examples of Poincaré-type inequalities:

Theorem 1.5 (Friedrich inequality)**.** *Let $1 \le p < \infty$, and let U be a bounded domain in $\mathbb{R}^d$ with C^1 boundary ∂U. For $u \in W^{1,p}(U)$ with $u|_{\partial U} = 0$,*

$$\|u\|_{L^p} \le C_U \|Du\|_{L^p}.$$

We can prove this in the same way using compactness. On the other hand, we can also prove this just from the Sobolev inequality for $W_0^{1,p}(U)$.

Theorem 1.6 (Hardy's inequality)**.**

(i) If $u \in W^{1,p}(U)$ *and* $u|_{\partial U} = 0$*, then*

$$\left\| \frac{1}{\operatorname{dist}(\cdot, \partial U)} u \right\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}.$$

(ii) If $u \in W^{1,p}(\mathbb{R}^d)$ *with* $p < d$*, then*

$$\left\| \frac{1}{|x|} u \right\|_{L^p} \leq C \|Du\|_{L^p}.$$

We can view Hardy's inequality as a refinement of Friedrich's inequality. We will discuss the proof next time.