Math 255B Lecture 4 Notes

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1 Sums of Fredholm and Compact Operators and The Toeplitz Index Theorem

1.1 Fredholm plus compact is Fredholm

Last time, we prove the Riesz-Fredholm theorem, which says that if $T \in \mathcal{L}(B, B)$ is compact, then $1 + T$ is a Fredholm operator with $\operatorname{ind}(1 + T) = 0$.

Proposition 1.1. *An operator $T \in \mathcal{L}(B_1, B_2)$ is Fredholm if and only if there exists a map $S \in \mathcal{L}(B_2, B_1)$ such that $TS - 1$ and $ST - 1$ are compact on B_2 and B_1, respectively.*

Proof. ($\Longleftarrow$): Let $S \in \mathcal{L}(B_2, B_1)$ be such that $ST = 1 + K_1$ and $TS = 1 + K_2$, where K_j is compact on B_j for $j = 1, 2$. Then $\ker T \subseteq \ker(1 + K_1)$, sso $\ker T$ is finite-dimensional. On the other hand, $\operatorname{im} T \supseteq \operatorname{im}(1 + K_2)$: Let $Y \subseteq B_2$ be such that $\dim Y = \dim \operatorname{coker}(1 + K_2)$, so $B_2 = \operatorname{im}(1 + K_2) \oplus Y$. If $Y = \operatorname{im} T \cap Y$, so $Y = Y_1 \oplus Y_2$, then $B_2 = \operatorname{im} T \oplus Y_2$. So we get $\dim \operatorname{coker} T = \dim Y_2 \leq \dim Y = \dim \operatorname{coker}(1 + K_2) < \infty$.

($\Longrightarrow$): We follow the Grushin approach: If $n_+ = \dim \ker T$ and $n_- = \dim \operatorname{coker}$, then there exist an injective $R_- : \mathbb{C}^{n_-} \to B_2$ and a surjective $R_+ : B_1 \to \mathbb{C}^{n_+}$ such that the Grushin operator

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}$$

is invertible with inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

Moreover,

$$1 = \mathcal{P}\mathcal{E} = \begin{bmatrix} R & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} TE + R_-E_- & * \\ * & * \end{bmatrix}.$$

so $TE + R_-E_- = 1$ on B_2, where R_-E_- has finite rank. Similarly, using $1 = \mathcal{E}\mathcal{P}$, we get $ET - 1 = -E_+R_+$, where E_+R_+ has finite rank on B_1. $\square$

Remark 1.1. If $S \in \mathcal{L}(B_2, B_1)$ is such that $ST - 1$ and $TS - 1$ are compact, then S is Fredholm, and $\operatorname{ind}(ST) = 0$. The logarithmic law gives $\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T$, so we get $\operatorname{ind} S = -\operatorname{ind} T$.

Theorem 1.1 (Fredholm theory)**.** *Let $T \in \mathcal{L}(B_1, B_2)$ be Fredholm, and let $S \in \mathcal{L}(B_1, B_2)$ be compact. Then $T + S$ is Fredholm, and* $\operatorname{ind}(T + S) = \operatorname{ind} T$.

Proof. Let $E \in \mathcal{L}(B_2, B_1)$ be such that $TE-1, ET-1$ are compact. Then $(T+S)E-1$ and $S(T+S)-1$ are compact, so $T+S$ is Fredholm. Moreover, $\operatorname{ind}(T+S) = \operatorname{ind}(T+tS) = \operatorname{ind} T$ for all $t \in [0, 1]$. □

1.2 The Toeplitz index theorem

Here is a nice example of a Fredholm operator.

Example 1.1. Consider $L^2((0, 2\pi)) \cong L^2(\mathbb{R}/2\pi\mathbb{Z})$. If $u \in L^2((0, 2\pi))$ and the Fourier coefficients are $\widehat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)e^{-in\theta}\, d\theta$, then $u(\theta) \sim \sum_{n\in\mathbb{Z}} \widehat{u}(n)e^{in\theta}$. Consider the **Hardy space** $H = \{u \in L^2 : \widehat{u}(n) = 0 \text{ for } n < 0\}$, which is a closed subspace of $L^2((0, 2\pi))$. The associated orthogonal projection $\pi : L^2 \to H$ sends $\sum_{n\in\mathbb{Z}} \widehat{u}(n)e^{in\theta} \mapsto \sum_{n\geq 0} \widehat{u}(n)e^{in\theta}$. Let $f \in L^\infty((0, 2\pi))$. Associated to f is the **Toeplitz operator** $\operatorname{Top}(f) : H \to H$ given by $\operatorname{Top}(f)u = \pi(fu)$. Then $\operatorname{Top}(f) \in \mathcal{L}(H, H)$, and $\|\operatorname{Top}(f)\|_{\mathcal{L}(H,H)} \leq \|f\|_\infty$.

Theorem 1.2 (Toeplitz index theorem)**.** *If $f \in C(\mathbb{R}/2\pi\mathbb{Z})$ is nonvanishing, then* $\operatorname{Top}(f)$ *is Fredholm on H, and* $\operatorname{ind}\operatorname{Top}(f) = -$*winding number*$(f)$.

To define the winding number , write $f(\theta) = r(\theta)e^{i\varphi(\theta)}$, where $r > 0$ and r, φ are continuous on $[0, 2\pi]$. Then the winding number of f is $\frac{\varphi(2\pi)-\varphi(0)}{2\pi}$.

Proof. To prove the Fredholm property of $\operatorname{Top}(f)$, we will try to invert $\operatorname{Top}(f)$ with a compact error. We claim that if $f, g \in C(\mathbb{R}/2\pi\mathbb{Z})$, then $\operatorname{Top}(f)\operatorname{Top}(f) = \operatorname{Top}(fg) + K$, where K is compact. Write $\operatorname{Top}(f) = \pi M_F$ and $\operatorname{Top}(f) = \pi M_g$, where M means a multiplication operator. Then

$$\pi M_f \pi M_g = \pi(\pi M_f + [M_f, \pi])M_g = \pi^2 M_{fg} + \pi[M_f, \pi]M_g = \operatorname{Top}(fg) + K,$$

where $[M_f, \pi] = M_f\pi - \pi M_f$ is the commutator $L^2 \to L^2$ and $K = \pi[M_f, \pi]M_g$. To show that K is compact, it suffices to show that $[M_f, \pi]$ is compact on L^2.

Case 1: If $f(\theta) = e^{in\theta}$, with $n \in \mathbb{Z}$, then

$$[M_{e^{in\theta}}\pi]e^{ik\theta} = e^{in\theta} \circ \pi - \pi \circ e^{in\theta}e^{ik\theta}$$

If $n > 0$,

$$= \begin{cases} 0 & k \geq 0 \\ -\pi(e^{i(n+k)\theta}) & k < 0, \end{cases}$$

where the latter expression $= 0$ if $k < -n$. So $[M_{e^{in\theta}}, \pi]$ is of finite rank on L^2.

We will finish the proof next time. $\square$