Math 246A Lecture 12 Notes

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1 Existence of Power Series and the Maximum Principle

1.1 Existence of power series of holomorphic functions in a maximal ball

Theorem 1.1. *Let Ω be a domain, $f \in H(\Omega)$, and $B(z_0, r) \subseteq \Omega$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and the series converges on $B(z_0, R)$.

Proof. Fix $z \in B(z_0, R)$ and send $\zeta \in B(z_0, R) \setminus \{z\}$ to

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}.$$

Then $\lim_{\zeta \to z} (\zeta - z) g(\zeta) = 0$. So g extends to be analytic on $B(z_0, R)$, and there exists G analytic on $B(z_0, R)$ such that $G' = g$ there. If $|z - z_0| < r < R$, then

$$\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} g(\zeta) \, d\zeta = 0.$$

That is,

$$\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = 0.$$

If we separate the integral and solve to isolate $f(z)$, we get

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} \, d\zeta = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right] (z - z_0)^n.$$

So the series has radius of convergence $\geq R$. $\square$

Corollary 1.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ with positive radius of convergence R. Then $R = \sup\{r > 0 : f$ has an analytic extension to $\{z : |z - z_0| < r\}\}$.*

Corollary 1.2 (Cauchy's estimates)**.** *Assume* $f \in H(B(z_0, R))$ *and* $|f| \le M < \infty$ *there. Then* $|f^{(n)}(z_0)| \le Mn!/R^n$.[1]

Proof. Let $0 < r < R$. Then

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz,$$

$$|a_n| \le \frac{1}{2\pi} \frac{M}{r^{n+1}} 2\pi r.$$

Then note that $n! a_n = f^{(n)}(z_0)$. □

Definition 1.1. An **entire** function is a function in $H(\mathbb{C})$.

Corollary 1.3 (Liouville)**.** *If f is bounded and entire, it is constant.*

The fundamental theorem of algebra is a corollary of Liouville's theorem. We will discuss this later.

Corollary 1.4. *Let* $f \in H(\Omega)$ *be such that* $|f| \le M$ *on* Ω*. Then* $|f'(z_0)| \le M/\operatorname{dist}(z, \partial\Omega)$ *for all* $z_0 \in \Omega$.

1.2 The maximum principle

Lemma 1.1. *Let* Ω *be a domain,* $f \in H(\Omega)$*, and* $\overline{B(z_0, R)} \subseteq \Omega$*. Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

Proof.

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) ire^{i\theta}}{re^{i\theta}}\, d\theta.$$

□

Theorem 1.2 (maximum principle)**.** *Let* Ω *be a domain,* $f \in H(\Omega)$*, and* $\overline{B(z_0, R)} \subseteq \Omega$*. Then*

1. $|f(z_0)| = \sup_{\overline{B(z_0,R)}} |f(z)| \implies f = f(z_0)$ *for all* $z \in \Omega$
2. $\operatorname{Re}(e^{i\alpha} f(z_0)) = \sup_{\overline{B(z_0,R)}} \operatorname{Re}(e^{i\alpha} f(z_0)) \implies f = f(z_0)$ *in* Ω.

[1] Professor Garnett says that inequalities like this should be treasured. After all, inequalities are more general than equalities!

Proof. To prove the first statement, we may assume that $f(z_0) = |f(z_0)|$ because we can replace f with $(\overline{f(z_0)}/|f(z_0)|)f$ for $f(z_0) \neq 0$. Then

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\mathrm{Re}(f(z_0) - f(z_0 + re^{i\theta}))}_{\geq 0}\, dt,$$

so $f(z_0 + re^{it}) = f(z_0)$ for all $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq R$.

Now use connectedness. Let $U = \{z \in \Omega : f(z) = f(z_0) \text{ in a neighborhood of } z\}$. U is open by the previous part of the proof, and U is closed in Ω. Ω is connected, so $U = \Omega$.

For part 2, observe that

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\mathrm{Re}(e^{i\theta} f(z_0) - e^{i\alpha} f(z_0 + re^{it}))}_{\geq 0}\, dt.$$

Therefore, $e^{i\alpha} f(z) = e^{i\alpha} f(z_0)$ for all $z \in B(z_0, R)$. □

Theorem 1.3. *Every meromorphic function on $\mathbb{C}^*$ is rational.*

Proof. Assume that f is meromorphic and nonconstant. Let $a_1, \ldots, a_N$ be the zeros of f in $\mathbb{C}$, counted with multiplicity, and let $b_1, \ldots, b_M$ be the poles of $f \in \mathbb{C}$, counted with multiplicity. Without loss of generality, $N \geq M$. Let

$$g = \frac{\prod_{j=1}^{N}(z - a_j)}{\prod_{k=1}^{M}(z - b_k)}.$$

Then $\lim_{z\to\infty} f/g$ is finite. Then f/g has no zeros nor poles. The maximum principle implies that f/g is constant, so $f = cg$ for some $c \in \mathbb{C}$. □