

# Math 255B Lecture 10 Notes

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January 29, 2020

## 1 Symmetric and Self-Adjoint Operators

### 1.1 Adjoints of closable operators

If  $T : D(T) \rightarrow H$  is densely defined, we defined the adjoint  $T^*$  with  $G(T) = [\overline{V(G(T))}]^\perp$ , where  $V(u, v) = (v, -u)$ . Let's finish a proof we started last time.

**Proposition 1.1.**  *$T$  is closable if and only if  $T^*$  is densely defined.*

*Proof.* ( $\implies$ ): We did this last time.

( $\impliedby$ ): If  $D(T^*)$  is dense, then  $(T^*)^*$  is a closed operator such that

$$F(T^{**}) = [V(G(T^*))]^\perp = V(G(T^*)^\perp) = V(V(\overline{G(T)})) = \overline{G(T)},$$

where we have used  $V^2 = -1$ . So  $T$  is closable, and  $T^{**} = \overline{T}$ .  $\square$

### 1.2 Symmetric and self-adjoint operators

**Definition 1.1.** Let  $S : D(S) \rightarrow H$  be densely defined. We say that  $S$  is **symmetric** if  $\langle Sx, y \rangle = \langle x, Sy \rangle$  for all  $x, y \in D(S)$ .

**Example 1.1.**  $S = -\Delta$  on  $L^2(\mathbb{R}^n)$  with  $D(S) = C_0^\infty(\mathbb{R}^n)$  is symmetric. However, we will see that this operator is not self-adjoint.

$S$  is symmetric if and only if  $S \subseteq S^*$ .

**Proposition 1.2.** *If  $S$  is symmetric, then  $S$  is closable and  $\overline{S}$  is symmetric.*

*Proof.* If  $u_n \in D(S)$  with  $u_n \rightarrow 0$  and  $Su_n \rightarrow \ell$ . Then  $\langle u_n, S_v \rangle = \langle Su_n, v \rangle \rightarrow \langle \ell, v \rangle$ . On the other hand  $\langle u_n, Sv \rangle \rightarrow 0$ , for all  $v \in D(S)$ . So  $\ell = 0$ , and  $S$  is closable.

If  $S \subseteq S^*$ , where  $S^*$  is densely defined, then  $\overline{S} = S^{**} \subseteq S^* = \overline{S^*}$ . So  $\overline{S}$  is symmetric.  $\square$

Given a symmetric operator  $S$ , we have two natural closed extensions:  $\overline{S}$  and  $S^*$ .

**Definition 1.2.** A linear, densely defined operator  $T : D(T) \rightarrow H$  is called **self-adjoint** if  $T = T^*$ .

Note that this means that  $D(T) = D(T^*)$ . Any self-adjoint operator is closed, since adjoints are closed. We have

$$\begin{aligned} T \text{ is self-adjoint} &\iff T \text{ is symmetric and } D(T) = D(T^*) \\ &\iff \langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in D(T), \text{ and} \\ &\quad \text{if } (x, y) \in H \times H \text{ with } \langle Tz, x \rangle = \langle z, y \rangle \quad \forall z \in D(T) \implies x \in D(T). \end{aligned}$$

**Proposition 1.3.** Let  $S$  be closed and symmetric. Then  $S^*$  is symmetric, so  $S$  is self-adjoint.

*Proof.*  $S^*$  is symmetric, so  $S^* \subseteq S^{**} = S$ , as  $S$  is closed. Also,  $S \subseteq S^*$ , so  $S$  is self-adjoint.  $\square$

**Example 1.2.** Let  $H = L^2(\mathbb{R}^n)$ , and let  $m : \mathbb{R}^n \rightarrow \mathbb{R}$  be Lebesgue measurable. Let  $D(A) = \{f \in L^2 : mf \in L^2\}$  and  $Af = mf$  for all  $f \in D(A)$ . We claim that  $A$  is self-adjoint.

Check first that  $D(A)$  is dense in  $L^2$ : For any  $f \in L^2$ ,  $\frac{f}{1+|m|} \in L^2$ , as well. So if  $g \in L^2$  with  $g \perp D(A)$ ,

$$\int g \frac{f}{1+|m|} dx = 0,$$

which means that  $\frac{g}{1+|m|} = 0$ , giving  $g = 0$ .

$A$  is symmetric, as  $m$  is real. Now let  $(g, h) \in L^2 \times L^2$  be such that  $\langle Af, g \rangle = \langle f, h \rangle$  for all  $f \in D(A)$ . Then for all  $f \in L^2$ ,

$$\int \frac{mf}{1+|m|} \bar{g} = \int \frac{f}{1+|m|} h,$$

so

$$\int \left( \frac{m}{1+|m|} \bar{g} - \frac{\bar{h}}{1+|m|} \right) f = 0$$

for all  $f \in L^2$ . So  $mg = h$ , which gives  $g \in D(A)$ .

**Example 1.3.** Let  $T = -\Delta$  on  $L^2(\mathbb{R}^n)$  with  $D(T) = H^2(\mathbb{R}^n) = \{u \in L^2 : \Delta u \in L^2\} = \{u \in L^2 : \partial^\alpha u \in L^2 \forall |\alpha| \leq 2\}$ . Then  $T$  is self-adjoint.

$T$  is symmetric:  $\langle -\Delta u, v \rangle_{L^2} = \langle u, -\Delta v \rangle_{L^2}$  for all  $u, v \in H^2$ . This is true for  $u, v \in C_0^\infty(\mathbb{R}^n)$ , which is dense in  $H^2(\mathbb{R}^n)$ . Alternatively we could prove this by taking the Fourier transform, where  $T$  acts as a multiplication operator.

Let  $(g, h) \in L^2 \times L^2$  be such that  $\langle -\Delta u, g \rangle_{L^2} = \langle u, h \rangle$  for all  $g, h \in H^2$ . In particular, if  $u \in C_0^\infty$ , we get  $-\Delta g = h \in L^2$  (taken in the weak sense). Then  $g \in H^2$ , and  $Tg = -\Delta g = h$ .

### 1.3 von Neumann's extension theory for symmetric operators

Let  $S : D(S) \rightarrow H$  be a closed, symmetric (densely defined) operator. Can  $S$  be extended to a self-adjoint operator? If  $S \subseteq T = T^*$ , then  $T^* \subseteq S^*$ , so we have an operator between  $S$  and  $S^*$  in general. If  $S$  is symmetric, these are the same.

**Proposition 1.4.** *Let  $S$  be closed and symmetric. Then for any  $z \in \mathbb{C} \setminus \mathbb{R}$ ,  $S - z1 : D(S) \rightarrow H$  is injective, has closed range, and  $\|(S - z)u\| \geq |\operatorname{Im} z| \|u\|$  for  $u \in D(S)$ .*

*Proof.* Write  $z = x + iy$ . Then

$$\begin{aligned} \|(S - z)u\|^2 &= \langle (S - x)u + iyu, (S - x)u + iyu \rangle \\ &= \|(S - x)u\|^2 + y^2 \|u\|^2 \\ &\geq y^2 \|u\|^2, \end{aligned}$$

so  $S - z$  is injective.

$\operatorname{Im}(S - z)$  is closed: If  $y \in \overline{\operatorname{Im}(S - z)}$ , there exist  $x_n \in D(S)$  such that  $(S - z)x_n \rightarrow y$ . By this inequality,  $x_n \rightarrow x \in H$ . Since  $(S - z)$  is closed,  $x \in D(S)$ .  $\square$

Here is the idea due to von Neumann: Study the **Cayley transform** of  $S$ ,  $T = (S + i)(S - i)^{-1}$ .