Math 255A Lecture 9 Notes

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1 Applications of Baire's Theorem I: The Open Mapping Theorem

1.1 The open mapping theorem

Banach used Baire's theorem to prove a number of striking results in functional analysis. Recall Baire's theorem.

Theorem 1.1 (Baire category). *Let E be a complete metric space, and let $(F_n)_{n \in \mathbb{N}}$ be closed in E containing no interior points. Then the union $\bigcup_{n=1}^\infty F_n$ has no interior points either. Moreover, $E \neq \bigcup_{n=1}^\infty F_n$.*

Definition 1.1. We say that $A \subseteq E$ is **of the first category** (or **meager**) if there exists a sequence F_n of closed sets without interior points such that $A \subseteq \bigcup_{n=1}^\infty F_n$.

Theorem 1.2 (Banach, open mapping theorem). *Let F_1, F_2 be Fréchet spaces, and let $T : F_1 \to F_2$ be linear continuous. Then either* $\operatorname{im}(T) \subseteq F_2$ *is of the first category, or else* $\operatorname{im}(T) = F_2$ *and the mapping T is open.*

Proof. Let U be an open neighborhood of 0 in F_1. We claim that $\overline{T(U)}$ contains a neighborhood of 0 in F_2, provided $\operatorname{im}(T)$ is not of the first category. Let V be a balanced neighborhood of 0 in F_1 such that $V + V \subseteq U$. Then V is absorbing (for $x \in F_1$, $\lambda x \in V$ for sufficiently small $|\lambda|$). So $F_1 = \bigcup_{n=1}^\infty nV$ means that $\operatorname{im}(T) = \bigcup_{n=1}^\infty T(nV) \subseteq \bigcup_{n=1}^\infty \overline{T(nV)}$. Since $\operatorname{im}(T)$ is not of the first category, for some n, $\overline{T(nV)} = n\overline{T(V)}$ has an interior point. Then $\overline{T(V)}$ has an interior point. So there exists $y \in F_2$ and a neighborhood W of 0 in F_2 such that $\{y\} + W \subseteq \overline{T(V)}$. Then $y \in \overline{T(V)}$. $V = -V$ since V is balanced, so $-y \in \overline{T(V)}$. So $W \subseteq \overline{T(V) + \{-y\}} \subseteq \overline{(\overline{T(V) - T(V)})} = \overline{T(V) - T(V)}$. We get $W \subseteq \overline{T(V+V)} \subseteq \overline{T(U)}$, as claimed.

Let d_{F_1} be a translation invariant metric on F_1 generating the topology on F_1, and define d_{F_2} similarly. Thus, for any $r > 0$, there exists $\rho > 0$ such that $B_{F_2}(0, \rho) \subseteq \overline{T(B_{F_1}(0, r))}$. The metrics $d_{F_1}.d_{F_2}$ are translation invariant, so for any $r > 0$, there exists a $\rho > 0$ such that for any $x \in F_1$, $B_{F_2}(Tx, \rho) \subseteq \overline{T(B_{F_1}(x, r))}$. Let $r > 0$ be arbitrary and let $r_n = r/2^n$

for $n \in N$. We get the corresponding ρ_n sequence such that $B_{F_2}(Tx, \rho_n) \subseteq \overline{T(B_{F_1}(x, r_n))}$ for all $x \in F_1$. We can arrange so that $\rho_n \downarrow 0$.

Let $y \in B_{F_2}(Tx, \rho_0)$. We shall show that there is an $x' \in F_1$ such that $d_{F_1}(x, x') \leq 2r$ and $y = Tx'$. Let $x_1 \in \overline{B_{F_1}(x, r_0)}$ be such that $d_{F_2}(y, Tx_1) < \rho_1 \iff y \in B_{F_2}(Tx_1, \rho_1) \subseteq \overline{T(B_{F_1}(x_1, r_1))}$. Let $x_2 \in B_{F_1}(x_1, r_1)$ be such that $d_{F_2}(y, Tx_2) < \rho_2$. Then $y \in B_{f_2}(Tx_2, \rho_2) \subseteq \overline{T(B_{F_1}(x_2, r_2))}$. Continuing in this fashion, we get a sequence (x_n) in F_1 such that $x_{n+1} \in \overline{T(B_{F_1}(x_n, r_n))}$. Then (x_n) is a Cauchy sequence in F_1, and $d_{F_2}(y, Tx_n) < \rho_n \to 0$. We get $x_n \to x' \in F_1$, where $d_{F_1}(x, x') \leq 2r$, and, since T is continuous, $Tx_n \to Tx'$. So $y = Tx'$.

So we get that for all $r > 0$, there exists $\rho > 0$ such that $B_{F_2}(Tx, \rho) \subseteq T(B_{F_1}(x, 2r))$. Hence, $\mathrm{im}(T) = F_2$, and T is open. $\square$

Corollary 1.1. *Let $T : F_1 \to F_2$ be an injective, linear, continuous map between Frèchet spaces. Then either the range of T is of the first category, or* $\mathrm{im}(T) = F_2$*, and T is a homeomorphism.*

1.2 Application of the open mapping theorem to partial differential equations

Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, where $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $D_{x_j} = (1/i)\partial_{x_j}$ be a partial differentiation operator (on $\mathbb{R}^n$) with constant coefficients $a_\alpha \in \mathbb{C}$. Assume that for some open set $\Omega \subseteq \mathbb{R}^n$, every solution $u \in C^m(\Omega)$ of $Pu = 0$ is in fact in $C^{m+1}(\Omega)$ (e.g. $P = \Delta$, the Laplacian). Then we have $\mathrm{Im}(\zeta) \to \infty$ if $\zeta \to \infty$ on the suface in $\mathbb{C}^n$ given by $0 = P(\zeta) = \sum_{|\alpha| \leq m} a_\alpha \zeta^\alpha$. We will do this in detail next time.