Math 279 Lecture 14 Notes

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1 Coherence and Hairer's Reconstruction Theorem

1.1 Examples of coherence

Last time, we discussed $\mathcal{C}^\alpha_{\mathrm{loc}}$ for $\alpha \in (0,1)$ that can be characterized by (if $u \in \mathcal{C}^\alpha_{\mathrm{loc}}$ and K is compact)

$$\sup_{x \in K} \sup_{\delta \in (0,1]} \sup_{\varphi \in \mathcal{D}_0} \frac{|\langle u - u(x), \varphi_x^\delta \rangle|}{\delta^\alpha} =: [u]_{\mathcal{C}^\alpha, K} < \infty.$$

Also, if $\alpha \geq 1$, then we set

$$P_x(y) = P_x^u(y) = \sum_{|k| \leq \alpha} (\partial^k u)(x) \frac{(y-x)^k}{k!},$$

and $u \in \mathcal{C}^\alpha_{\mathrm{loc}}$ means that for K compact,

$$\sup_{x \in K} \sup_{\delta \in (0,1]} \sup_{\varphi \in \mathcal{D}_0} \frac{|\langle u - P_x, \varphi_x^\delta \rangle|}{\delta^\alpha} =: [u]_{\mathcal{C}^\alpha, K} < \infty.$$

For example, if $\alpha \in (1,2)$, then

$$\begin{aligned} u(y) - P_x(y) &= \underbrace{u(y) - u(x)}_{[u(ty+(1-t)x)]_0^1} - Du(x) \cdot (y-x) \\ &= \int_0^1 (Du(ty + (1-t)x) - Du(x)) \cdot (y-x)\, dt. \end{aligned}$$

to assert that if $u \in C^1$ and $Du \in \mathcal{C}^{\alpha-1}$, then

$$|u(y) - P_x(y)| \leq c|x-y|^\alpha$$

locally uniformly. Then we can show that the above norm is finite. However, we may use our polynomial approximation expression for our definition of $\mathcal{C}^\alpha_{\mathrm{loc}}$.

Here, we have an example of a function u that is well-approximated by a so-called germ $(P_x : x \in \mathbb{R}^d)$. Indeed, this family enjoys a regularity that we now explore. To find such a regularity, observe

$$P_a(x) = \sum_{|k|<\alpha} \partial^k u(a) \frac{(x-a)^k}{k!},$$

$$\begin{aligned} P_b(x) &= \sum_{|k|<\alpha} \partial^k u(b) \frac{(x-b)^k}{k!} \\ &= \sum_{|k|<\alpha} \left[\sum_{|r|<\alpha-|k|} \partial^{k+r} u(a) \frac{(b-a)^r}{r!} + R_k(a,b) \right] \frac{(x-b)^k}{k!}, \end{aligned}$$

where the error

$$|R_k(a,b)| \lesssim |b-a|^{\alpha-k}.$$

From now on, $f \lesssim g$ mean $f \leq cg$ for a constant c. Hence,

$$P_b(x) = \sum_{|m|<\alpha} \frac{\partial^m u(a)}{m!} \underbrace{\left(\sum_{k+r=m} \frac{(b-a)^r}{r!} \frac{(x-b)^k}{k!} m! \right)}_{(x-a)^m} + \sum_{|k|<\alpha} R_k(a,b) \frac{(x-b)^k}{k!}$$

From this, we learn that

$$P_b(x) - P_a(x) = \sum_{|k|<\alpha} R_k(a,b) \frac{(x-b)^k}{k!},$$

and hence

$$\begin{aligned} |\langle P_b - P_a, \varphi_b^\delta \rangle| &\lesssim \sum_{|k|<\alpha} |b-a|^{\alpha-|k|} \delta^{|k|} \\ &\lesssim (\delta + |b-a|)^\alpha. \end{aligned}$$

Here, we have an example of a germ, namely $(P_x : x \in \mathbb{R}^d)$ that is α-**coherent** (which will be defined later).

Let us have another example, namely what we had before in Gubinelli's version (the sewing lemma) of Lyons and Victoire's result: Imagine that we have $A(s,t)$ with

$$|A(s,t) + A(u,t) - A(s,t)| \lesssim |t-s|^{\alpha+\beta}, \qquad s < u < t, \alpha + \beta > 1.$$

Then by the sewing lemma, we can find h such that

$$|h(t) - h(s) - A(s,t)| \lesssim |t-s|^{\alpha+\beta}.$$

For example, we may have $A(s,t) = f(s)(g(t)-g(s))$ with $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$. As we stated before, we may consider the germ $(F_s : s \in \mathbb{R})$, where $F_s = f(s)g'$; what the condition A means is this: Observe that $A(s,t) = \langle F_s, \mathbb{1}_{[s,t]}\rangle$. Hence

$$\langle F_u - F_s, \mathbb{1}_{[s,t]}\rangle \lesssim |t-s|^{\alpha+\beta}.$$

If $\varphi = \mathbb{1}_{[0,1]}$,

$$|\langle F_u - F_s, \varphi_s^\delta\rangle| \lesssim \delta^{-1}(|u-s|+\delta)^{\alpha+\beta} = \delta^{-1}(|u-s|+\delta)^{\gamma+1},$$

where $\gamma = \alpha+\beta-1 > 0$. In summary, we have an example of a germ that is γ-coherent, or more specifically $(-1,\gamma)$-coherent.

Motivated by these two examples, we formulate some definitions.

Definition 1.1. By a **germ**, we mean a measurable map $F : \mathbb{R}^d \to \mathcal{D}'$ sending $x \mapsto F_x$.

Definition 1.2. We call a germ $(-\tau,\gamma)$**-coherent** with $\tau = \tau_K$ only depending on a compact set K and with respect to a test function $\phi \in \mathcal{D}$, $\int \varphi \neq 0$, if the following condition is true:

$$\langle F_x - F_y, \varphi_y^\delta\rangle \lesssim \delta^{-\tau_K}(|x-y|+\delta)^{\gamma+\tau_K}$$

uniformly for $x, y \in K$. Here, we assume that $\tau_K \geq 0$ and $\gamma + \tau_K \geq 0$.

We say γ**-coherent** when we mean $(-\tau,\gamma)$-coherent for some τ which does not matter.

1.2 Martin Hairer's reconstruction theorem

Theorem 1.1 (Martin Hairer's reconstruction theorem)**.** *Assume that F is a γ-coherent germ F with respect to some $\varphi \in \mathcal{D}$. Then there exists $u \in \mathcal{D}'$ such that*

$$|\langle i - F_x, \psi_x^\delta\rangle| \lesssim \begin{cases} \delta^\gamma & \gamma \neq 0 \\ 1 + |\log\delta| & \gamma = 0. \end{cases}$$

uniformly for $x \in K$ and ψ such that $\operatorname{supp}\psi \subseteq B_1(0)$ *and $\|\psi\|_{C^r} \leq 1$ with $r = r_K$.*

Remark 1.1. If $\gamma > 0$ is positive, then the u in the theorem is unique.

Proof. If u and u' satisfy the same inequality, and $T = u - u'$, then $|T(\psi_x^\delta)| \lesssim \delta^\gamma$. Let us take $f \in L^1_{\text{loc}}$ and consider $\zeta \in \mathcal{D}$ and consider $f * \zeta$. Here,

$$(f*\zeta)(x) = \int \zeta(x-y)f(y)\,dy = \int (\tau_y\zeta)(x)\,f(y)\,dy.$$

We claim that

$$T(f*\zeta) = T\left(\int T_y\zeta f(y)\,dy\right) = \int T(\tau_y\zeta)f(y)\,dy.$$

This can be done by Riemann approximation of the integral. Now

$$T(\zeta) = \lim_{\delta \to 0} T(\zeta * \psi^{\delta}) = \lim_{\delta \to 0} \int T(\tau_y \psi^{\delta}) \zeta(y) \, dy = 0,$$

as $|T(\psi^{\delta})| \lesssim \delta^{\gamma}$. $\square$