Math 247A Lecture 8 Notes

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1 Interpolation and Maximal Function Estimates

1.1 Conclusion of proof of Hunt's interpolation theorem

Theorem 1.1 (Hunt's interpolation theorem). *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. Assume that T is a sublinear map satisfying $\|Tf\|_{L^{q_j,\infty}} \lesssim \|f\|^*_{L^{p_j,1}}$ for $j = 1, 2$. Then, for any $1 \le r \le \infty$ and $\theta \in (0,1)$, we have*

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}, \qquad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. We may assume $1 < p_1, p_2, q_1, q_2 < \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. We know

$$\int |T\mathbb{1}_F(x)||\mathbb{1}_E(x)|\,dx \lesssim \min\{|F_n^\ell|^{1/p_1}|E_m|^{1/q_1'}, |F_n^\ell|^{1/p_2}|E_m|^{1/q_2'}\}$$

Fix $\theta \in (0,1)$ and $1 \le e \le \infty$. We want to show that

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}$$

uniformly for $f \in L^{p_\theta,r}$. It suffices to show that

$$\left|\int Tf(x)g(x)\,dx\right| \lesssim 1,$$

where $\|f\|^*_{L^{p_\theta,r}} \sim 1$ and $g = \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{E_m}$, where E_m are measurable, pairwise disjoint, and

$$\|g\|^*_{L^{q_\theta',r'}} \sim \|2^m |E_m|^{1/q_\theta'}\|_{\ell^{r'}} \lesssim 1.$$

We write $f = \sum_{\ell \ge 1} f_\ell$, where $f_\ell = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n^\ell}$. We have

$$\left|\int Tf(x)g(x)\,dx\right| \lesssim \sum_{\ell\ge 1}\sum_{n,m} 2^n 2^m \min\{|F_n^\ell|^{1/p_1}|E_m|^{1/q_1'}, |F_n^\ell|^{1/p_2}|E_m|^{1/q_2'}\}$$

$$\begin{aligned}\lesssim \sum_{\ell\geq 1}\sum_{n,m\in\mathbb{Z}} & 2^n|F_n^\ell|^{1/p_\theta}2^m|E_m|^{1/q'_\theta}\\ &\cdot \min\{|F_n^\ell|^{(1-\theta)(1/p_1-1/p_2)}|E_m|^{(1-\theta)(1/q'_1-1/q'_2)}\\ &\qquad |F_n^\ell|^{-\theta(1/p_1-1/p_2)}|E_m|^{-\theta(1/q'_1-1/q'_2)}\}\end{aligned}$$

Using the same trick we have used before, we write this as a geometric series.

$$\lesssim \sum_{\ell}\sum_{N,M\in 2^{\mathbb{Z}}}\sum_{n:|F_n^\ell|\sim N} 2^n N^{1/p_\theta}\sum_{m:|E_m^\ell|\sim M} 2^m M^{1/q'_\theta}A(N,M)$$

where

$A(N,M)=\min\{|F_n^\ell|^{(1-\theta)(1/p_1-1/p_2)}|E_m|^{(1-\theta)(1/q'_1-1/q'_2)},|F_n^\ell|^{-\theta(1/p_1-1/p_2)}|E_m|^{-\theta(1/q'_1-1/q'_2)}\}$.

$$\begin{aligned}\lesssim \sum_{\ell\geq 1}&\left[\sum_{N,M\in 2^{\mathbb{Z}}}A(N,M)\left[\sum_{n:|F_n^\ell|\sim N}2^nN^{1/p_\theta}\right]^r\right]^{1/r}\\ &\cdot\left[\sum_{N,M\in 2^{\mathbb{Z}}}A(N,M)\left[\sum_{m:|E_m|\sim M}2^mM^{1/q'_\theta}\right]^r\right]^{1/r}.\end{aligned}$$

Note that $\sup_N\sum_{M\in 2^{\mathbb{Z}}}A(N,M)\lesssim 1$ and $\sup_{M\in 2^{\mathbb{Z}}}\sum_{N\in 2^{\mathbb{Z}}}A(N,M)\lesssim 1$. Fix $M\in 2^{\mathbb{Z}}$. Let $n_0^{1/p_1-1/p_2}\sim M^{-(1/q'_1-1/q'_2)}$. Then

$$\sum_{N\in 2^{\mathbb{Z}}}A(N,M)=\sum_{N\leq N_0}N^{(1-\theta)(1/p_1-1/_2)}M^{(1-\theta)(1/q'_1-1/q'_2)}+\sum_{N>N_0}N^{-\theta(1/p_1-1/p_2)}M^{-\theta(1/q'_1-1/q'_2)}.$$

Thus,

$$\begin{aligned}\left|\int Tf(x)g(x)\,dx\right| &\lesssim \sum_{\ell\geq 1}\left\{\sum_N\left(\sum_{n:|F_n^\ell|\sim N}2^nN^{1/p_\theta}\right)^r\right\}^{1/r}\left\{\sum_M\left(\sum_{m:|E_m|\sim M}2^mM^{1/q'_\theta}\right)^r\right\}^{1/r}\\ &\lesssim \sum_{\ell\geq 1}\underbrace{\left(\sum_n 2^{nr}|F_n^\ell|^{r/p_\theta}\right)^{1/r}}_{\|f_\ell\|^*_{L^{p_\theta,r}}}\underbrace{\left(\sum_m 2^{mr'}|E_m|^{r'/q'_\theta}\right)^{1/r'}}_{\|g\|^*_{L^{q'_\theta,r'}}}\\ &\lesssim \sum_{\ell\geq 1}\|f_\ell\|^*_{L^{p_\theta,r}}\end{aligned}$$

Since $|f_\ell|\leq\frac{1}{2^{\ell-1}}|f|$,

$$\begin{aligned}&\lesssim \|f\|^*_{L^{p_\theta,r}}\\ &\sim 1.\end{aligned}$$ $\square$

Remark 1.1. We did not use anything specific about Lebesgue measure in our proof. So these theorems hold for arbitrary measures μ.

1.2 Maximal and vector maximal functions

Recall the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

Theorem 1.2.

1. *If $f \in L^p(\mathbb{R}^d)$ for some $1 \le p \le \infty$, then Mf is finite almost everywhere.*
2. *M is of weak-type $(1,1)$ and strong-type (p,p) for $1 < p \le \infty$.*

Remark 1.2.

1. M is not of strong-type $(1,1)$. Let $\varphi \in C_c^\infty(B(0,1/2))$. For $|x| \le 1$, $M\varphi(x) \sim 1$. If $|x| > 1$, then $M\varphi(x) \sim \frac{1}{|x|^d}$. So $M\varphi(x) \sim \langle x \rangle^{-d}$, where this notation means $\langle x \rangle := (1+|x|^2)^{1/2}$.[1] So $M\varphi \notin L^1$.
2. M is of weak-type $(1,1)$ means

   $$|\{x : Mf(x) > \lambda\}| \lesssim \frac{\|f\|_1}{\lambda}$$

   uniformly in $\lambda > 0$ and $f \in L^1$. The decay in λ on the right hand side cannot be improved. To see this, consider φ as above. Then $M\varphi \in L^\infty$, so only the small λ are relevant. For small λ,

   $$\begin{aligned} |\{x : Mf(x) > \lambda\}| &= |\{x : \langle x \rangle^{-d} \gtrsim \lambda\}| \\ &= |\{x : \langle x \rangle \lesssim \lambda^{-1/d}\}| \\ &\lesssim \lambda^{-1}. \end{aligned}$$

   Also, $M\varphi \notin L^{1,q}(\mathbb{R}^d)$ for any $q < \infty$ because

   $$\|M\varphi\|_{L^{1,q}}^* \sim \int_0^\infty \lambda^q \underbrace{|\{x : M\varphi(x) > \lambda\}|^q}_{\lesssim \lambda^{-q}} \frac{d\lambda}{\lambda} = \infty.$$

[1] This is known as “Japanese bracket notation” everywhere except Japan, where they just call it “bracket notation.”

Theorem 1.3. *Let $\omega : \mathbb{R}^d \to [0, \infty)$ be a locally integrable function (a weight), to which we associate a measure via*

$$\omega(E) = \int_E \omega(x)\, dx.$$

Then

1. *$M : L^1(M\omega\, dx) \to L^{1,\infty}(\omega\, dx)$ maps boundedly; that is,*

$$\omega(\{x : Mf(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int |f(y)|(M\omega)(y)\, dy.$$

2. *$M : L^p(M\omega\, dx) \to L^p(\omega\, dx)$ boundedly for all $1 < p \leq \infty$; that is,*

$$\int |Mf(x)|^p \omega(x)\, dx \lesssim \int |f(y)|^p (M\omega)(y)\, dy.$$

Remark 1.3.

1. If $\omega \equiv 1$, then $M\omega \equiv 1$, so we recover the previous theorem.

2. In order for the statement to be non-vacuous, we need $M\omega$ is finite somewhere. This happens precisely when $\frac{1}{r^d} \int_{|x| \leq r} \omega(x)\, dx \lesssim 1$ uniformly for sufficiently large r.

   ($\implies$): If $x = 0$, we are done, so assume $x \neq 0$. For $r > 2d(x, 0)$,

$$M\omega(x) \geq \frac{1}{|B(x, r)|} \int_{B(x,r)} \omega(y)\, dy \gtrsim \frac{1}{r^d} \int_{|x| \leq r/2} \omega(y)\, dy.$$

   ($\impliedby$): Choose x to be a Lebesgue point. The Lebesgue differentiation theorem controls the maximal function at small scales, and the same argument controls the maximal function at large scales.