Math 210A Lecture 6 Notes

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1 Inverse Limits, Direct Limits, and Adjoint Functors

1.1 Inverse and direct limits

Example 1.1. Consider the colimit of this diagram in Ab:

$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot p} \cdots \xrightarrow{\cdot p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\cdot p} \cdots$$

Then $\varinjlim_n \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Q}_p/\mathbb{Z}_p \subseteq \mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}_p$ is the free field of $\mathbb{Z}_p$. We can also show that $\mathbb{Q}_p/\mathbb{Z}_p : \{a \in \mathbb{Q}/\mathbb{Z} : p^n a = 0 \text{ for some } n \geq 0\}$.

Definition 1.1. A **directed set** I is a set with a partial ordering such that for all $i, j \in I$, there is a $k \in I$ such that $i \leq k$, $j \leq k$.

Definition 1.2. A **directed category** is a category where the objects are elements of a directed set I, and there are morphisms $i \to j$ iff $i \leq j$. A **codirected category** $\mathcal{I}$ is a category where $\mathcal{C}^{op}$ is directed.

Definition 1.3. Suppose $\mathcal{I}$ is codirected with $\mathrm{Obj}(\mathcal{I}) = I$ and $F : \mathcal{I} \to \mathbb{C}$. A limit of F is called the **inverse limit** of the $F(i)$ for all $i \in I$. We write $\lim F = \varprojlim_{i \in I} F(i)$.

$$\begin{tikzcd}
& & F(i) \\
\varprojlim F(i) \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & F(k) \arrow[ur] \arrow[dr] & \\
& & F(j)
\end{tikzcd}$$

If $\mathcal{I}$ is directed with $\mathrm{Obj}(\mathcal{I}) = I$ and $F : \mathcal{I} \to \mathcal{C}$. A colimit of F is called the **direct limit** of the $F(i)$ for all $i \in I$. We write $\operatorname{colim} F = \varinjlim_{i \in I} \operatorname{colim} F$.

$$
\begin{tikzcd}
 & & F(i) \arrow[dl] \arrow[dll, bend right] \\
\varprojlim F(i) & F(k) \arrow[l] & \\
 & & F(j) \arrow[ul] \arrow[ull, bend left]
\end{tikzcd}
$$

Definition 1.4. A small category $\mathcal{I}$ is **filtered** if

1. for all $i, j \in I$, there exists $k \in I$ such that there exist morphisms $i \to k, j \to k$,
2. for all $\kappa, \kappa' : i \to j$ in I there exists a morphism $\lambda : j \to k$ such that $\lambda \circ \kappa = \lambda \circ \kappa'$

A category it **cofiltered** if the opposite category is filtered.

Cofiltered limits and diltered limits generalize inverse and direct limits, respectively.

Example 1.2. Let I be cofiltered with an initial object c. Then if $F : I \to \mathcal{C}$, $\lim F = F(e)$.

1.2 Adjoint functors

Definition 1.5. A functor $F : \mathcal{C} \to \mathcal{D}$ is **left adjoint** to a functor $G : \mathcal{D} \to \mathcal{C}$ if for each $C \in \mathcal{C}$, $D \in \mathcal{D}$, there exist bijections $\eta - C, D : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \; to \, \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ such that η is a natural transformation between functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Sets}$. That is,

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(F(C), D) & \xrightarrow{\eta_{C,D}} & \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \\
\Big\downarrow{\scriptstyle h \mapsto g \circ h \circ F(f)} & & \Big\downarrow{\scriptstyle h \mapsto G(g) \circ h \circ f} \\
\mathrm{Hom}_{\mathcal{D}}(F(C'), D') & \xrightarrow{\eta_{C',D'}} & \mathrm{Hom}_{\mathcal{C}}(C', G(D'))
\end{array}
$$

G is **right adjoint** to F if F is left adjoint to G.

Remark 1.1. If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are quasi-inverses and $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ is a natural isomorphism, then we can define $\phi_{C,D} : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ given by $h \mapsto G(h) \circ \eta_C$. Check that $\phi_{C,D}$ is a bijection. So F is left-adjoint to G. Similarly, G is left-adjoint to F.

Proposition 1.1. *Suppose S is a set, and consider $h_S : \mathrm{Set} \to \mathrm{Set}$ given by $h_S(T) = \mathrm{Maps}(S, T)$ and $h_S(f : T \to T') = g \mapsto f \circ g$. Then h_S is right adjoint to $t_S : \mathrm{Set} \to \mathrm{Set}$ given by $t_S(T) = T \times S$ and $t_S(f) = (f, \mathrm{id}_S) : T \times S \to T' \times S$.*

Proof. We need to find a bijection $\tau_{T,U} : \mathrm{Maps}(T \times S, U) \to \mathrm{Maps}(T, \mathrm{Maps}(S, U))$. We can send $f \mapsto (t \mapsto (s \mapsto f(s,t)))$. To show that this is a bijection, we can go backward by sending $\varphi \mapsto ((t,s) \mapsto \varphi(t)(s))$. Check that these maps are inverses of each other and that this is a natural transformation. $\square$

Proposition 1.2. *Suppose all limits $F : I \to \mathcal{C}$ exist. Then the functor* $\lim : \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ *given by $F \mapsto \lim F$ and $(\eta : F \to F') \mapsto (\lim F \mapsto \lim F')$ has a left adjoint $\Delta : \mathcal{C} \to$* $\mathrm{Fun}(I, \mathcal{C})$ *such that $\Delta(A) = c_A$ is the constant functor $I \to \mathcal{C}$ with value A.*

Proof. We want a bijection $\eta : \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(c_A, F) \to \mathrm{Hom}_{\mathcal{C}}(A, \lim F)$. Let $\eta : c_A \to F$ be $\eta_i : \underbrace{c_A(i)}_{=A} \to F(i)$ such that

$$\begin{tikzcd} A \arrow[r, "\eta_i"] \arrow[d, "\mathrm{id}_A = c_A(f)"'] & F(i) \arrow[d, "F(f)"] \\ A \arrow[r, "\eta_j"] & F(j) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_i"] \arrow[dr, "\eta_j"] & F(i) \arrow[d, "F(f)"] \\ & F(j) \end{tikzcd}$$

for all $f : i \to j$. So $\eta_j = F(f) \circ \eta_i$ for all $f : i \to j$. There exists a unique morphism $g : A \to \lim F$ such that

$$\begin{tikzcd} & A \arrow[d, dashed, "g"] \arrow[ddl, "\eta_j"'] \arrow[ddr, "\eta_i"] & \\ & \lim F \arrow[dl, "p_j"] \arrow[dr, "p_i"'] & \\ F(j) & & F(i) \arrow[ll, "F(f)"] \end{tikzcd}$$

Send η to g. Conversely if we have $g : A \to \lim F$, $\eta_i = p_i \circ g$ is a morphism from $A \to F(i)$. So we get $\eta \in \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(c_A, F)$. $\square$

Definition 1.6. A contravariant functor $F : \mathcal{C} \to \mathrm{Set}$ is **representable** if there exists an object $B \in \mathcal{C}$ and a natural isomorphism $h^B \to F$, where $h^B = \mathrm{Hom}_{\mathcal{C}}(\cdot, B)$. We say that B **represents** F.

Example 1.3. The functor $P : \mathrm{Set} \to \mathrm{Set}$ given by $S \mapsto \mathcal{P}(S)$ and $(f : S \to T) \mapsto (V \mapsto f^{-1}(V))$ is representable by $\{0, 1\}$.