

Math 255B Lecture 16 Notes

Daniel Raban

February 12, 2020

1 Closure of Quadratic Forms

1.1 Quadratic forms bounded below

Last time, we introduced the notion of quadratic forms: Let $q : D \rightarrow \mathbb{R}$ be a symmetric quadratic form. We say that q is **bounded below** if there is a c such that $q(u) \geq -c\|u\|^2$ for $u \in D$.

Example 1.1. Let $H = L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $V \in L^1(\mathbb{T}; \mathbb{R})$. Let

$$q(u) = \int (|u'|^2 + V|u|^2) dx, \quad D = D(q) = H^1(\mathbb{T}) \subseteq L^\infty(\mathbb{T}).$$

Formally, $q(u) = \langle Pu, u \rangle_{L^2}$ with $P = -\partial_x^2 + V(x)$.

We claim that q is bounded below: For every $\varepsilon > 0$, there is some $V^\sharp \in L^\infty(\mathbb{T})$ such that $\|V - V^\sharp\|_{L^1} \leq \varepsilon$. Then, keeping in mind that $\|u\|_{H^1}^2 = \|u'\|_{L^2}^2 + \|u\|_{L^2}^2$,

$$\begin{aligned} q(u) &= \int |u'|^2 + \underbrace{\int V^\sharp |u|^2}_{\geq -C_\varepsilon \|u\|_{L^2}^2} + \underbrace{\int (V - V^\sharp) |u|^2}_{\geq -O(\varepsilon) \|u\|_{H^1}^2} \\ &\geq (1 - O(\varepsilon)) \|u\|_{H^1}^2 - C_\varepsilon \|u\|_{L^2}^2 \end{aligned}$$

for all $\varepsilon > 0$. We get that there exist $c > 0, C > 0$ such that

$$q(u) \geq c\|u\|_{H^1}^2 - C\|u\|_{L^2}^2, \quad \forall u \in D(q).$$

Remark 1.1. We can always add a constant multiple of $\|u\|^2$ to q , so from now on, we will assume that q is nonnegative. This allows us to use the Cauchy-Schwarz inequality:

$$|q(u, v)| \leq q(u)^{1/2} q(v)^{1/2} \quad \forall u, v \in D(q).$$

1.2 Closed quadratic forms

Definition 1.1. Let $u_n \in D(q)$ and $u \in H$. We say that u_n is **q -convergent** to u (written $u_n \xrightarrow{q} u$) if $u_n \rightarrow u$ in H and $q(u_n - u_m) \xrightarrow{n,m \rightarrow \infty} 0$. We say that q is **closed** if whenever $u_n \xrightarrow{q} u$, $u \in D(q)$ and $q(u_n - u) \rightarrow 0$.

Remark 1.2. Let $H_q = D(q)$, equipped with the scalar product $\langle u, v \rangle_q := q(u, v) + \langle u, v \rangle$. Then q is closed if and only if H_q is a Hilbert space.

Let's return to our example.

Example 1.2. Let $q(u) = \int(|u'|^2 + V|u|^2)$ on $D(q) = H^1(\mathbb{T})$ with $V \in L^1(\mathbb{T})$. Then $q(u) \geq c\|u\|_{H^1}^2 - C\|u\|^2$, so the quadratic form $u \mapsto q(u) + C\|u\|^2$ is closed. Indeed,

$$\|u\|_q^2 := q(u) + (c+1)\|u\|^2,$$

and this inequality tells us that $\|u\|_q \sim \|u\|_{H^1}$ for $u \in D(q)$.

1.3 Closable quadratic forms

Definition 1.2. We say that q is **closable** if it has a closed extension $\tilde{q} : D(\tilde{q}) \rightarrow \mathbb{R}$: $D(q) \subseteq D(\tilde{q})$ and $\tilde{q}|_{D(q)} = q$.

Proposition 1.1. A quadratic form q is closable if and only if whenever $u_n \xrightarrow{q} 0$, then $q(u_n) \rightarrow 0$. If this condition holds, then q has a smallest closed extension \bar{q} (the **closure** of q) given by $D(\bar{q}) = \{u \in H : \exists u_n \in D(q) \text{ s.t. } u_n \xrightarrow{q} u\}$ and $\bar{q}(u) = \lim_{n \rightarrow \infty} q(u_n)$.

Proof. (\implies): Let \tilde{q} be a closed extension. If $u_n \xrightarrow{q} 0$, then $u_n \xrightarrow{\tilde{q}} 0$, so $\tilde{q}(u_n) = q(u_n) \rightarrow 0$.

(\impliedby): Assume that the condition holds, and let $u_n \xrightarrow{q} u$. We claim that $\lim_{n \rightarrow \infty} q(u_n)$ exists.

$$\begin{aligned} |q(u_n) - q(u_m)| &= |q(u_n, u_n) - q(u_m, u_m)| \\ &= |q(u_n - u_m, u_m) + q(u_m, u_n - u_m)| \\ &\stackrel{\text{C-S}}{\leq} q^{1/2}(u_n - u_m)q^{1/2}(u_n) + q^{1/2}(u_n - u_m)q^{1/2}(u_m). \end{aligned}$$

So we get

$$|q^{1/2}(u_n) - q^{1/2}(u_m)| \leq q^{1/2}(u_n - u_m) \xrightarrow{n,m \rightarrow \infty} 0.$$

The claim follows, and if $v_n \xrightarrow{q} u$, then $u_n - v_n \xrightarrow{q} 0$:

$$q^{1/2}(u_n - v_n - u_m + v_m) \leq q(u_n - u_m) + q^{1/2}(v - n - v_m) \rightarrow 0.$$

By the assumed condition, $q(u_n - v_n) \rightarrow 0$. And by the same argument as before,

$$|q^{1/2}(u_n) - q^{1/2}(v_n)| \leq q^{1/2}(u_n - v_n) \rightarrow 0.$$

We get a well-defined quadratic form \bar{q} which extends q .

We claim that \bar{q} is closed; that is, we check that $H_{\bar{q}} = D(\bar{q})$ is complete with respect to $\langle u, v \rangle_{\bar{q}} = \bar{q}(u, v) + \langle u, v \rangle$. H_q is dense in $H_{\bar{q}}$, as if $u_n \in H_q$ with $u_n \xrightarrow{q} u \in H_{\bar{q}}$, then $\bar{q}(u_n - u) \rightarrow 0$:

$$\bar{q}(u_n - u) = \lim_{m \rightarrow \infty} q(u_n - u_m) \xrightarrow{n \rightarrow \infty} 0.$$

Every Cauchy sequence in H_q has a limit in $H_{\bar{q}}$, so we have a dense subset where every Cauchy sequence has a limit. So $H_{\bar{q}}$ is complete.

Finally, one checks that if \tilde{q} is a closed extension of q , then $\bar{q} \subseteq \tilde{q}$. □

Theorem 1.1. *Let q be a nonnegative, symmetric, quadratic form. Assume that $D(q)$ is dense and that q is closed. Then there exists a unique self-adjoint operator \mathcal{A} such that $D(\mathcal{A}) \subseteq D(q)$ and $q(u, v) = \langle \mathcal{A}u, v \rangle$ for all $u \in \mathcal{A}, v \in D(q)$. Also, $D(\mathcal{A})$ is a **core** for q in the sense that $D(\mathcal{A})$ is dense in H_q .*