Math 222A Lecture 7 Notes

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1 Existence of Solutions to Nonlinear First Order Scalar PDEs

1.1 Proving existence and uniqueness given initial data

Last time, we were looking at fully nonlinear equations

$$\begin{cases} F(x,u,\partial u) = 0 \\ u = u_0 \text{ on } \Sigma. \end{cases}$$

If u solves this equation, then $(x,u,\partial_j u)$ solves the characteristic system

$$\begin{cases} \dot{x} = F_p(x,z,p) \\ \dot{z} = F_p(x,z,p) \cdot p \\ \dot{p} = -F_x(x,z,p) - F_z(x,z,p) \cdot p. \end{cases}$$

The initial data for the characteristic system on Σ is

$$\begin{cases} x(0) = x_0 \\ z(0) = u_0(x_0) \\ p(0) = p_0, \end{cases}$$

where p_0 has a tangential component $\partial_\tau u_0$ and a normal component given by solving $F(x_0,u_0,p_0)$. In this last part, we had a local solvability condition $F_p \cdot N \neq 0$, where N is the normal to Σ. This is the same as the noncharacteristic condition.

Our objective is to turn this into an existence proof.

Theorem 1.1. *Assume that F is of class C^2, Σ is C^2, $u_0 \in C^2$, and the problem is noncharacteristic, i.e. there exists p_0 on Σ such that $F_{p_0} \cdot N \neq 0$, $F(x_0,u_0,p_0) = 0$, and $(p_0)_\tau = \partial_\tau u_0$. Then there exists a unique local solution $u \in C^2$ near Σ such that $u|_\Sigma = u_0$ and $\partial u|_\Sigma = p_0$.*

Proof. First, an outline:

Step 1: Solve the characteristic system with initial data (x_0, u_0, p_0) on Σ. This gives us

$$(x(s, x_0), u(s, x_0), p(s, x_0)),$$

which we can solve by using ODE theory.

Step 2: Show that the map

$$\Sigma \times [-\varepsilon, \varepsilon] \ni (x_0, s) \mapsto x(x_0, s) \in \mathbb{R}^n$$

is a local diffeomorphism with inverse

$$x \mapsto (x_0, s).$$

Step 3: Define

$$u(x(s, x_0)) = z(s, x_0).$$

This is true if a solution u exists.

The main difficulty is that at the end of our construction, we get the functions

$$z(s, x_0) = u(x), \qquad x = x(s, x_0), \qquad p_j(s, x_0) \stackrel{?}{=} \partial_j z(x).$$

Our final goal is to prove that $p_j(s, x_0) = \partial_j z(s, x_0)$. By construction of our initial data, we know this is true at $s = 0$. Ideally, we might want to show that $\frac{\partial}{\partial s}(p_j - \partial_j z) = 0$. Instead, we will have a weaker version that works:

$$\frac{\partial}{\partial s}(p_j - \partial_j z) = \text{coeff}(p_j - \partial_j z),$$

which is a linear ODE for $p_j - \partial_j z$.

Our preliminary step is to show that $F(x, z, p) = 0$. This is true on Σ, i.e. when $s = 0$.[1] Compute

$$\frac{d}{ds} F(x, z, p) = F_x \cdot \dot{x} + F_z \cdot \dot{z} + F_p \cdot \dot{p} = 0.$$

Next, compute $\frac{\partial}{\partial s}(p_j - \partial_j z)$. We have

$$\frac{\partial}{\partial s} = (-F_{x_j} - F_z \cdot p_j),$$

but to calculate $\frac{\partial}{\partial s} \partial_j z$, we need to use $\dot{z} = F_p \cdot p$. We have $\frac{\partial}{\partial s} = F_{p_k} \cdot \frac{\partial}{\partial x_k}$, where F_{p_k} has variable coefficients. So the derivatives do not commute. We can explicitly compute

$$\frac{\partial}{\partial s} \partial_j z = F_{p_k} \partial_k \partial_j z,$$

[1] This is the same thing we wanted to do with $p_j - \partial_j z$, but that is more difficult to work with because that is a vector equation, rather than just a scalar equation.

$$\partial_j \dot{z} = \partial_j (F_{p_k})\partial_k z) = F_{p_k} \partial_j \partial_k z + \partial_j F_{p_k} \cdot \partial_k z,$$

which gives

$$\frac{\partial}{\partial s} \partial_j z = \partial_j \dot{z} - \partial_j F_{p_k} \cdot \partial_k z.$$

So we get

$$\begin{aligned}
\frac{\partial}{\partial s}(p_j - \partial_j z) &= -F_{x_j} - F_z \cdot p_j - \partial_j \dot{z} + \partial_j (F_{p_k}) \cdot \partial_k z \\
&= -F_{x_j} - F_z \cdot p_j - \partial_j (F_{p_k} \cdot p_k) + \partial \cdot (F_{p_k}) \partial_k z \\
&= -F_{x_j} - F_z \cdot p_j - F_{p_k} \partial_j p_k \underbrace{-p_k (F_{x_j p_k} + F_{z p_k} \partial_j z + F_{p_\ell p_k} \partial_j p_\ell) + \partial_k z (\text{same})}_{-(p_k - \partial_k z) \cdot \partial_j F_{p_k}} \\
&= -F_{x_j} - F_z \cdot p_j - F_{p_k} \partial_j p_k + \text{good}.
\end{aligned}$$

We also have

$$F_{x_j} + F_z \cdot \partial_j z + F_{p_k} \partial_j p_k = 0$$

by taking $\frac{\partial}{\partial x_j}$ of our earlier computation. This last term $F_{p_k} \cdot \partial_j p_k$ is the same worst term in the above expression. If we substitute, we get

$$\frac{\partial}{\partial_s}(p_j - \partial_j z) = -F_z (p_j - \partial_j z) - \partial_j F_{p_k} (p_k - \partial_k z),$$

which is a linear system.

Therefore, z is the solution to our equation, and we are done. □

1.2 Problems in standard form

Example 1.1. Begin with the equation

$$u_t + F(t, x, u, \partial u) = 0$$

We will label u_t as τ, u as z, and ∂u as p. So we get the equation

$$\widetilde{F}(t, x, z, \tau, p) = \tau + F(t, x, z, p) = 0,$$

and the system

$$\begin{cases} \dot{t} = 1 \text{ (so } s = t) \\ \dot{x} = F_p \\ \dot{z} = \tau + F_p \cdot p = F_p \cdot p - F \\ \dot{p} = -F_x - F_z \cdot p \\ \dot{\tau} = -F_t - F_z \cdot \tau \end{cases}$$

In the middle 3 equations, we have no τ terms, so we can discard the last equation. Another way to think of this is that $\widetilde{F} = 0$, so τ is already given as $-F$. So we get a smaller system

$$\begin{cases} \dot{x} = F_p \\ \dot{z} = F_p \cdot p - F \\ \dot{p} = -F_x - F_z \cdot p. \end{cases}$$

The price we pay is the extra F term in the second equation, compared to before.

Remark 1.1. Solutions are local, near Σ, until characteristics may intersect. There is no way to continue solutions in general past this intersection of characteristics. For specific classes of problems, however, there is hope.

Example 1.2. Suppose we have an equation $H(x, \partial u) = 0$ which does not depend directly on u. Then we get

$$\begin{cases} \dot{x} = H_p \\ \dot{p} = -H_x \\ \dot{z} = H_p \cdot p - H. \end{cases}$$

The first two equations do not depend on z, so we can discard the last equation, solve the first two equations first, and integrate the last equation at the end.

This type of system is called a **Hamilton flow**.[2] Many PDEs can be interpreted as Hamiltonian flows. The **Hamilton-Jacobi** equations are of the form

$$u_t + H(x, \partial u) = 0.$$

Next time, we will do a bit of variational calculus to not only solve Hamilton-Jacobi equations but to also see how we may extend a solution past a point where characteristics intersect. In a Hamilton flow, the characteristics only depend on (x, p). When characteristics intersect, they may have the same x but different $p = \partial u$. We will try to continue the solution in a way such that ∂u has a jump discontinuity.

Example 1.3. Consider the equation

$$\begin{cases} u_t + \frac{1}{2}|\partial_x u|^2 = 0 \\ u(0) = u_0. \end{cases}$$

Here, $H(p) = \frac{1}{2}p^2$, and we get the system

$$\begin{cases} \dot{x} = p \\ \dot{p} = 0. \end{cases}$$

Here, the characteristics are straight lines, with $p(0) = \partial_x u_0$.

[2] Hamilton flows play a role in symplectic geometry.

Example 1.4 (Eikonal equation)**.** The equation

$$|u_t|^2 - |\partial_x u|^2 = 0.$$

is not in the form we have talked about already. This gives

$$u_t = \pm|\partial_x u,$$

so we will get 2 solutions.