Math 247A Lecture 4 Notes

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1 Relationships Between The Lorentz Quasinorms and L^p Norms

1.1 Order of growth of Lorentz quasinorms in terms of L^p and ℓ^q

Last time, we had the quasinorm

$$\|f\|_{L^{p,q}(\mathbb{R}^d)}^* = p^{1/q} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^q((0,\infty), \frac{d\lambda}{\lambda})}$$

Remark 1.1. If $|g| \leq |f|$, then $\|g\|_{L^{p,q}}^* \leq \|f\|_{L^{p,q}}^*$.

Proposition 1.1. *If $f \in L^{p,q}(\mathbb{R}^d)$ for $1 \leq p < \infty$ and $1 \leq q \leq \infty$, write $f = \sum_{m \in \mathbb{Z}} f_m$, where $f_m(x) = f(x) \mathbb{1}_{\{x : 2^m \leq |f(x)| < 2^{m+1}\}}(x)$. Then*

$$\|f\|_{L^{p,q}}^* \sim \left\| \|f_m\|_{L^p(\mathbb{R}^d)} \right\|_{\ell_m^q(\mathbb{Z})}.$$

Proof. Both sides only concern $|f|$, so it suffices to prove this for $f \geq 0$. Then

$$2^m \mathbb{1}_{\{2^m \leq f(x) < 2^{m+1}\}} \leq f_m < 2^{m+1} \mathbb{1}_{\{2^m \leq f(x) < 2^{m+1}\}}.$$

Thus, by our previous remark, we may assume that $f = \sum_{m \in \mathbb{Z}} 2^m \mathbb{1}_{F_m}$, where F_m are measurable, pairwise disjoint sets.

$$\begin{aligned}
(\|f\|_{L^{p,q}}^*)^q &= p \int_0^\infty \lambda^q |\{x : \sum_n 2^n \mathbb{1}_{F_n} > \lambda\}|^{q/p} \, \frac{d\lambda}{\lambda} \\
&= p \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q |\{x : \sum_n 2^n \mathbb{1}_{F_n} > \lambda\}|^{q/p} \, \frac{d\lambda}{\lambda}
\end{aligned}$$

For $2^{m-1} \leq \lambda < 2^m$, $\{x : \sum 2^n \mathbb{1}_{F_n}(x) > \lambda\} = \bigcup_{n \geq m} F_n$.

$$\sim \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q \left(\sum_{n \geq m} |F_n| \right)^{q/p} \frac{d\lambda}{\lambda}$$

$$\sim \sum_{m\in\mathbb{Z}} 2^{mq} \left(\sum_{n\geq m} |F_n| \right)^{q/p}$$
$$\sim \left\| 2^m \left(\sum_{n\geq m} |F_n| \right)^{1/p} \right\|_{\ell^q_m}^q .$$

We wanted to show that $\|f\|^*_{L^{p,q}} \sim \|2^m |F_m|^{1/p}\|_{\ell^q_m}$. So we just need to show that $\left\| 2^m \left(\sum_{n\geq m} |F_n| \right)^{1/p} \right\|_{\ell^q_m} \sim \|2^m |F_m|^{1/p}\|_{\ell^q_m}$. We have the $\geq$ direction, so we just need the other inequality:

$$\begin{aligned}
\left\| 2^m \left(\sum_{n\geq m} |F_n| \right)^{1/p} \right\|_{\ell^q_m} &\leq \left\| 2^m \sum_{n\geq m} |F_n|^{1/p} \right\|_{\ell^q_m} \\
&\lesssim \sum_{k\geq 0} 2^{-k} \|2^{m+k} |F_{m+k}|^{1/p}\|_{\ell^q_m}
\end{aligned}$$

Now reindex the ℓ^q sum by $n = m + k$.

$$\begin{aligned}
&\lesssim \sum_{k\geq 0} 2^{-k} \|2^n |F_n|^{1/p}\|_{\ell^q_n} \\
&\lesssim \|2^n |F_n|^{1/p}\|_{\ell^q_n}. \qquad \square
\end{aligned}$$

1.2 Lorentz spaces are Banach spaces

Lemma 1.1. *Let $q \leq q < \infty$, and let $S \subseteq 2^{\mathbb{Z}}$, the dyadic integers. Then*

$$\sum_{N\in S} N^q \leq \left(\sum_{N\in S} N \right)^q \leq \left(2 \sup_{N\in S} N \right)^q \leq 2^q \sum_{N\in S} N^q.$$

In other words, if we're summing dyadic series, when we take the L^q norm, it doesn't really matter whether we have the q inside or outside the sum.

Theorem 1.1. *For $1 < p < \infty$ and $1 \leq q \leq \infty$,*

$$\|f\|^*_{L^{p,q}} \sim \sup \left\{ \left| \int f(x)g(x)\, dx \right| : \|g\|^*_{L^{p',q'}} \leq 1 \right\}.$$

Thus, $\|\cdot\|^*_{L^{p,q}}$ is equivalent to a norm, with respect to which $L^{p,q}(\mathbb{R}^d)$ is a Banach space. Moreover, for $q \neq \infty$, the dual of $L^{p,q}$ is $L^{p',q'}$, under the natural pairing.

Remark 1.2. For $p = 1, q \neq 1$, there cannot be a norm equivalent to $\|\cdot\|^*_{L^{1,q}}$. Let's see this for $q = \infty$ and $d = 1$. Assume, towards a contradiction, that $\|\cdot\|^*_{L^{1,\infty}} \sim |||\cdot|||$. Let $f(x) = \sum_{n=1}^N \frac{1}{|x-n|}$ for $N \gg 1$. Then

$$\left\|\frac{1}{|x-n|}\right\|^*_{L^{1,\infty}} = \sup_{\lambda>0} \left|\left\{x : \frac{1}{|x-n|} > \lambda\right\}\right| = 2,$$

so

$$\sum_{n=1}^N \left|\left|\left|\frac{1}{|x-n|}\right|\right|\right| \sim \sum_{n=1}^N \left\|\frac{1}{|x-n|}\right\|^*_{L^{1,\infty}} = 2N.$$

Then we have

$$\|f\|^*_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \left|\left\{x : \sum_{n=1}^N \frac{1}{|x-n|} > \lambda\right\}\right|.$$

We claim that $\{x : \sum_{n=1}^N \frac{1}{|x-n|} > \frac{1}{10}\log N\} \supseteq [0, N]$. If $x = 0$, then $\sum 1/n > \log(N+1) \geq \frac{1}{10}\log N$. Now do the same with $x = 1, x = 2, \ldots$. The worst case scenario is when $x \approx N/2$, but the inequality holds in this case, too. So we have

$$\|f\|^*_{L^{1,\infty}} \geq \frac{1}{10}\log N \left|\left\{x : \sum_{n=1}^N \frac{1}{|x-n|} > \frac{1}{10}\log N\right\}\right| \geq \frac{N\log N}{10}.$$

So we have shown that

$$|||f||| \sim \|f\|^*_{L^{1,\infty}} \geq \frac{N\log N}{10}.$$

This gives

$$N\log N \lesssim |||f||| \leq \sum_{n=1}^N \left|\left|\left|\frac{1}{|x-n|}\right|\right|\right| \sim N.$$

Let $N \to \infty$ to get a contradiction.

Now let's prove the theorem.

Proof. We may assume $f \geq 0$, $g \geq 0$. AS both sides are positive homogeneous, we may assume that $\|f\|^*_{L^{p\cdot q}} = 1$. We may assume $f = \sum 2^n \mathbb{1}_{F_n}$ and $g = \sum 2^m \mathbb{1}_{E_m}$ with F_n measurable, pairwise disjoint and E_n measurable, pairwise disjoint. Then

$$\begin{aligned} 1 &= (\|f\|^*_{L^{p,q}})^q \\ &\sim \|2^n |F_n|^{1/p}\|^q_{\ell^q} \\ &\sim \sum_{n\in\mathbb{Z}} 2^{nq}|F_n|^{q/p} \end{aligned}$$

$$\sim \sum_{N \in 2^{\mathbb{Z}}} \sum_{n : N \leq |F_n| < 2N} 2^{nq} |F_n|^{q/p}$$
$$\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q/p} \sum_{n : |F_n| \sim N} 2^{nq}$$

By the lemma,

$$\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q/p} \left(\sum_{n : |F_n| \sim N} 2^n \right)^q$$
$$\sim \sum_{N \in 2^{\mathbb{Z}}} \left(\sum_{n : |F_n| \sim N} 2^n |F_n|^{1/p} \right)^q .$$

Similarly,

$$1 \geq \left(\|g\|_{L^{p',q'}} \right)^{q'} \sim \sum_{M \in 2^{\mathbb{Z}}} \left(\sum_{m : |E_m| \sim M} 2^m |E_m|^{1/p'} \right)^{q'} .$$

Now

$$\int f(x) g(x)\, dx = \sum_{n,m} 2^n 2^m |F_n \cap E_m|$$
$$\lesssim \sum_{N,M \in 2^{\mathbb{Z}}} \sum_{n : |F_n| \sim N} \sum_{m : |E_m| \sim M} 2^n |F_n|^{1/p} 2^m |E_m|^{1/p'} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}}$$
$$\lesssim \sum_{N,M \in 2^{\mathbb{Z}}} \left(\frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \right)^{1/q + 1/q'} \sum_{n : |F_n| \sim N} 2^n |F_n|^{1/p} \sum_{m : |E_m| \sim M} 2^m |E_m|^{1/p'}$$

By Hölder's inequality,

$$\lesssim \left[\sum_{N,M \in 2^{\mathbb{Z}}} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \left(\sum_{n : |F_n| \sim N} 2^n |F_n|^{1/p} \right)^q \right]^{1/q}$$
$$\cdot \left[\sum_{N,M \in 2^{\mathbb{Z}}} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \left(\sum_{m : |F_m| \sim M} 2^m |E_n|^{1/p'} \right)^{q'} \right]^{1/q'} .$$

Now we just need $\sum_{M \in 2^{\mathbb{Z}}} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \lesssim 1$. This comes from

$$\sum_M \min \left\{ \left(\frac{N}{M} \right)^{1/p'}, \left(\frac{M}{N} \right)^{1/p} \right\} \lesssim \sum_{M \leq N} \left(\frac{M}{N} \right)^{1/p} + \sum_{M > N} \left(\frac{N}{M} \right)^{1/p'} \lesssim 1,$$

as we get a geometric series.[1] $\square$

[1] Instead of using Hölder's inequality and the subsequent steps, we could alternatively use Schur's test for convergence of series. This kind of argument will be common in this course.