# Math 222A Lecture 13 Notes

Daniel Raban

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## 1 Homogeneous Distributions of Order $-1$, Convolution, and Fundamental Solutions

### 1.1 Special homogeneous distributions of order $-1$

#### 1.1.1 The principal value of $1/x$ as a complex limit

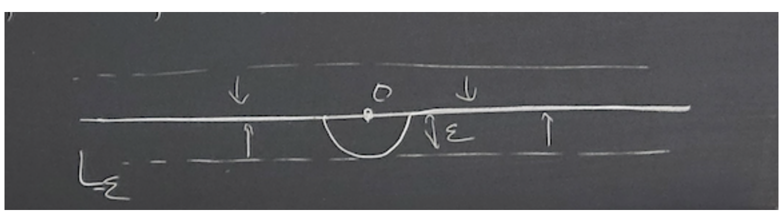
Last time, we were discussing homogeneous distributions. When classifying homogeneous distributions of order -1 in 1 dimension, we saw two interesting distributions:

$$\delta_0, \qquad \mathrm{PV}\,\frac{1}{x}.$$

If you like complex analysis, you can consider the function

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}.$$

Then $f(z) = \frac{1}{x-i\varepsilon}$ on the line $L_{-\varepsilon}$ below the real line:




What is $\lim_{\varepsilon \to 0} \frac{1}{x-i\varepsilon}$? Apply this to a test function:

$$\begin{aligned} \frac{1}{x-i\varepsilon}(\varphi) &= \int \frac{\varphi(x)}{x-i\varepsilon}\,dx \\ &\approx \int_{\mathbb{R}\setminus[\varepsilon,\varepsilon)} \frac{\varphi(x)}{x-i\varepsilon} + \int_{\frac{1}{2}C_\varepsilon} \frac{\varphi(z)}{z}\,dz \end{aligned}$$

$$\approx \operatorname{PV}\frac{1}{x}(\varphi) + \varphi(0)\cdot\int_{\frac{1}{2}C_\varepsilon}\frac{1}{z}\,dz$$

Write $\ln z = \ln|z| + i\arg z$. Then $z = \varepsilon e^{i\theta}$ for $\theta\in[\pi,2\pi]$

$$\begin{aligned}&= \operatorname{PV}\frac{1}{x}(\varphi) + \varphi(0)\cdot\int_{\pi}^{2\pi}\frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}}\,d\theta\\ &= \operatorname{PV}\frac{1}{x}(\varphi) + \varphi(0)\pi i.\end{aligned}$$

So

$$\lim_{\varepsilon\to0}\frac{1}{x-i\varepsilon} = \operatorname{PV}\frac{1}{x} + \pi i\delta_0.$$

If we do the same approximation from the line $L_\varepsilon$ above the real line, we get

$$\lim_{\varepsilon\to0}\frac{1}{x+i\varepsilon} = \operatorname{PV}\frac{1}{x} - \pi i\delta_0.$$

What is $\partial_x \operatorname{PV}\frac{1}{x}$? We can calculate that

$$-\lim_{\varepsilon\to0}\frac{1}{(x-i\varepsilon)} = \left(\operatorname{PV}\frac{1}{x}\right)' + \pi i\delta_0',$$

and repeat this idea to find the derivatives of $\operatorname{PV}\frac{1}{x}$.

### 1.1.2 $1/|x|$ as a distribution

What is $\frac{1}{|x|}$ as a distribution?

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{[-1,1]\setminus[-\varepsilon,\varepsilon)}\frac{1}{|x|}\varphi(x)\,dx &= \int\frac{1}{|x|}(\varphi(x)-\varphi(0))\,dx + \varphi(0)\int\frac{1}{|x|}\,dx\\ &\to \int_{-1}^{1}\frac{1}{|x|}(\varphi(x)-\varphi(0))\,dx + 2\varphi|\log\varepsilon|.\end{aligned}$$

But this does not converge as $\varepsilon\to0$. So we can try to **renormalize**, calculating the integral when we subtract out the divergent term:

$$\frac{1}{|x|}(\varphi) := \lim_{\varepsilon\to0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon}\frac{1}{|x|}(\varphi(x)-\varphi(0))\,dx - 2\varphi(0)|\log\varepsilon|$$

However, this breaks the homogeneity.

## 1.2 Properties of convolution

**Definition 1.1.** Let $\varphi, \psi \in \mathcal{D}$. The **convolution** is the function

$$(\varphi * \psi)(x) = \int \varphi(y)\psi(x-y)\,dy.$$

Observe that this is smooth in $x$. What about the support?

**Proposition 1.1.**

$$\operatorname{supp} \varphi * \psi \subseteq \operatorname{supp} \varphi + \operatorname{supp} \psi$$

*Proof.* If we want to know the support, call $K = \operatorname{supp} \varphi$ and $K_1 = \operatorname{supp} \psi$. If $(\varphi * \psi)(x) \neq 0$, then we must have $x \in K + K_1$. $\square$

So we can think about convolution as a function

$$* : \mathcal{D} \times \mathcal{D} \to \mathcal{D}.$$

**Proposition 1.2** (commutativity of convolution)**.**

$$\varphi * \psi = \psi * \varphi.$$

*Proof.* Make the change of variables $z = x - y$ in the integral. $\square$

**Proposition 1.3** (associativity of convolution)**.**

$$\varphi * (\psi * \zeta) = (\varphi * \psi) * \zeta.$$

So $(\mathcal{D}, +, *)$ is a commutative algebra. We have another commutative algebra structure on $\mathcal{D}$, $(\mathcal{D}, +, \cdot)$. We will later see that these structures are not unrelated; they are mirror images of each other.

With multiplication, we have the Leibniz rule:

$$\partial(\psi\varphi) = \partial\psi \cdot \varphi + \psi \cdot \partial\psi.$$

We don't exactly have a Leibniz rule for convolution:

**Proposition 1.4.**

$$\varphi(\psi * \varphi) = \psi * \partial\varphi = \varphi * \partial\psi.$$

**Proposition 1.5.** *If $\varphi \in L^1$ and $\psi \in L^\infty$, then*

$$\|\varphi * \psi\|_{L^\infty} \leq \|\varphi\|_{L^1} \|\psi\|_{L^\infty}.$$

*Proof.*

$$\begin{aligned}|(\varphi * \psi)(x)| &\leq \int |\varphi| \cdot \sup |\psi| \\ &= \|\varphi\|_{L^1} \|\psi\|_{L^\infty}.\end{aligned}$$

□

When you think of convolution, you want to think of two things: regularity and support. If $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{E}$, then we lose information about the support, so $\varphi * \psi \in \mathcal{E}$. So $\mathcal{D} * \mathcal{E} \to \mathcal{E}$. On the other hand, if we take a derivative of the convolution, we just need to be able to take a derivative of one of the factors. Here is the takeaway:

- For the support of the convolution, we need the support of both factors.
- For regularity, we need the regularity of just one factor!

We can think of convolutions as distributions: If $\varphi \in \mathcal{E}$ and $\psi \in \mathcal{D}$,

$$\varphi * \psi(x) = \varphi(\psi(x - \cdot)).$$

This right hand side is well-defined even if $\varphi \in \mathcal{D}'$. So we see that

$$\mathcal{D}' * \mathcal{D} \to \mathcal{E}.$$

Similarly, we have

$$\mathcal{E}' * \mathcal{D} \to \mathcal{D}.$$

What about $\mathcal{E}' * \mathcal{E}'$? If $u, v, \varphi \in \mathcal{D}$, then

$$(u * v)(\varphi) = \iint u(y)v(x-y)\,dy\varphi(x)\,dx$$

Change variables using $z = x - y$ so $\varphi(x) = z + y$.

$$\begin{aligned}&= \iint u(y)v(z)\varphi(z+y)\,dy\,dz \\ &= \int u(y) \underbrace{\int v(z)\varphi(z+y)\,dz}_{v(\varphi(y+\cdot))}\,dy \\ &= u(v(\varphi(y+\cdot))).\end{aligned}$$

This conclusion makes sense even if $u, v \in \mathcal{E}'$. We can make this precise if we can approximate elements of $\mathcal{E}'$ by elements in $\mathcal{E}$. So we get

$$\mathcal{E}' * \mathcal{E}' \to \mathcal{E}'.$$

However, $\mathcal{D}' * \mathcal{D}'$ is undefined.

## 1.3 Fundamental solutions to PDEs

Now suppose we have the PDE

$$P(\partial)u = f,$$

where $P$ is linear with constant coefficients and $f$ is a distribution. The simplest $f$ we can consider is $\delta_0$, which gives us the equation

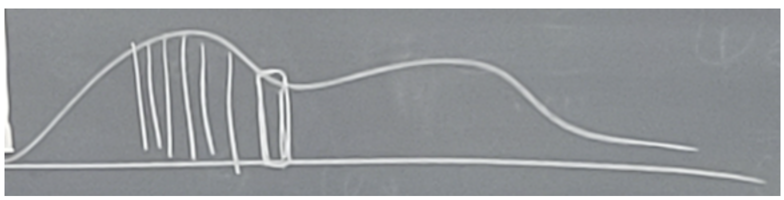
$$P(\partial)K = \delta_0$$

The next simplest $f$ we can consider is $\delta_{x_0}$. So we get

$$P(\partial)K(\cdot - x_0) = \delta_{x_0}$$

by invariance with respect to translations.

Can we write a general function as a superposition of $\delta$ functions? If we have a Riemann integral, we can approximate it by a sum of pieces which look like Dirac masses.



So can we make sense of something that looks like

$$f = \int f(x_0)\delta_{x_0}\,dx_0?$$

We can define this by applying $f$ to a test function:

$$\varphi(\varphi) = \int f(x_0)\underbrace{\delta_{x_0}(\varphi)}_{=\varphi(x_0)}\,dx_0.$$

So if we can deal with a Dirac masses, we can deal with a lienar combination of Dirac masses and hence any function as a superposition of Dirac masses. So the solution should looks like

$$u(x) = \int f(x_0)K(x - x_0)\,dx_0.$$

This was some intuition[1], but here are some definitions.

**Definition 1.2.** $K$ is a **fundamental solution** of $P(\partial)$ if

$$P(\partial)K = \delta_0.$$

[1] Or maybe confusion!

**Proposition 1.6.** *The function $u = K * f$ solves the equation*

$$P(\partial)u = f.$$

*Proof.*

$$\begin{aligned} P(\partial)u &= P(\partial)(K * f) \\ &= P(\partial K) * f \\ &= \delta_0 * f. \end{aligned}$$

We are done if $f * \delta_0 = f$. If $f \in \mathcal{D}$, then

$$f * \delta_0(x) = \delta_0(f(x - \cdot)) = f(x).$$

The same works for $f \in \mathcal{D}'$. $\square$

In this proof, we saw that $\delta_0$ is the identity with respect to $*$. For multiplication, **1** is the identity. The constant **1** function has support on all of $\mathbb{R}^n$, but it has regularity; conversely, $\delta_0$ has 1 point as it support but no regularity. You can think of these as opposites.

**Example 1.1.** With our notation, the fundamental theorem of calculus looks like this:

**Theorem 1.1.** *If $\partial_x u = f$ in $\mathbb{R}$, then*

$$u = \int f(x)\,dx + C.$$

*If we specify that $u(-\infty) = 0$, then*

$$u(x) = \int_{-\infty}^{x} f(y)\,dy.$$

We want to interpret this as a convolution. First, let's compute the fundamental solution:

$$\partial_x K = \delta_0, \qquad K(-\infty) = 0.$$

This tells us that

$$K = H(x)$$

is the Heaviside function. By our proposition, $u = K * f$. We can write this as

$$u(x) = \int H(x - y) f(y)\,dy$$

For $H(x - y)$ to give 1 and not 0, we need $x - y > 0$.

$$= \int_{-\infty}^{x} f(y)\,dy.$$

Is the fundamental solution $K$ unique? In general, if $K$ is a constant solution, then $K + C$ is a fundamental solution for any constant $C$. If we ask for $K = 0$ at $-\infty$, we get $K = H$. But if we ask for $K = 0$ at $+\infty$, we get $K = H - 1$. If we ask for $K$ to be odd, we get $K = H - 1/2$.