Math 255B Lecture 3 Notes

Daniel Raban

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1 The Fredholm-Riesz Theorem

1.1 The Fredholm-Riesz theorem

Theorem 1.1 (Fredholm-Riesz). *Let B be a Banach space, and let $T \in \mathcal{L}(B, B)$ be compact. Then $1 - T$ is Fredholm, and* $\operatorname{ind}(1 - T) = 0$.

Remark 1.1. If B is a Hilbert space, we can prove this more easily by using the fact that compact operators can be approximated by finite rank operators.

Proposition 1.1. *Let $T \in \mathcal{L}(B, B)$ be compact. Then*

1. $\ker(1 - T)$ *is finite dimensional.*
2. $\operatorname{im}(1 - T)$ *is closed.*

Proof.

1. Let $x_n \in \ker(1 - T)$ with $\|x_n\| \leq 1$. Then $x_n = Tx_n$ has a convergent subsequence. Then the identity map on $\ker(1-T)$ is compact, so $\dim \ker(1-T) < \infty$ (by Riesz's theorem).

2. Let $y \in \overline{\operatorname{im}(1 - T)}$, and let $x_n \in B$ be such that $y_n = (1 - T)x_n \to y$. Consider $\operatorname{dist}(x_n, \ker(1 - T)) = \inf_{z \in \ker(1-T)} \|x_n - z\|$. There exists some $z_n \in \ker(1 - T)$ realizing this infimum: $\|x_n - z_n\| = \operatorname{dist}(x_n, \ker(1 - T))$.

   We claim that the sequence $(x_n - z_n)$ is bounded: otherwise, $\|x - n - z_n\| \to \infty$ along a subsequence. Let $w_n = \frac{x_n - z_n}{\|x_n - z_n\|}$, so

   $$(1 - T)w_n = \frac{(1 - T)(x_n - z_n)}{\|x_n - z_n\|} = \frac{\eta_n}{\|x_n - z_n\|} \to 0.$$

   Passing to a subsequence, we may assume that $Tw_n \to v \in B$ and then $w_n \to v$, where $v \in \ker(1 - T)$. Now

   $$\operatorname{dist}(w_n, \ker(1 - T)) = \inf_{z \in \ker(1-T)} \frac{\|x_n - z_n - z\|}{\|x_n - z_n\|} = \frac{\operatorname{dist}(x_n, \ker(1 - T))}{\|x_n - z_n\|} = 1$$

for all n. This proves the claim.

Passing to a subsequence, we may assume that $T(x_n - z_n) \to \ell \in B$. Also, $y_n = (1-T)(x_n - z_n) \to y$, so $x_n - z_n \to y + \ell = g$. Since T is continuous, $(1-T)g = \lim_{n\to\infty}(1-T)(x_n - z_n) = y$. So $y \in \operatorname{im}(1-T)$. □

1.2 Adjoints of inclusions and quotients

To show that $\dim \operatorname{coker} < \infty$, we will use duality arguments:

Definition 1.1. If B_1, B_2 are Banach spaces with duals B_1^*, B_2^* and bilinear maps $\langle x, \xi \rangle_j : B_j \times B_j^* \to \mathbb{C}$ and if $T \in \mathcal{L}(B_1, B_2)$, then the **adjoint** $T^* \mathcal{L}(B_2^*, B_1^*)$ is defined by

$$\langle Tx, \eta \rangle_2 = \langle x, T^* \eta \rangle_1 \qquad \forall x \in B_1, \eta \in B_2^*.$$

Definition 1.2. If B is a Banach space and $W \subseteq B$ is a closed subspace, the **annihilator** $W^o \subseteq B^*$ is given by

$$W^o = \{\xi \in B^* : \langle x, \xi \rangle = 0 \, \forall x \in W\}.$$

Proposition 1.2. *Let B be a Banach space, and let $W \subseteq B$ be a closed subspace.*

1. *Let $i : W \to B$ be the inclusion map. Then $i^* : B^* \to W^*$ vanishes on W^o and induces an isometric bijection $B^*/W^o \to W^*$.*
2. *Let $q : B \to B/W$ be the quotient map. Then the adjoint $q^* : (B/W)^* \to B^*$ is an isometry with the range W^o.*

Proof.

1. We have $\langle ix, \xi \rangle = \langle x, i^* \xi \rangle$, so $i^* \xi$ is the restriction of ξ to W. So $\ker i^* = W^o$. $i^* : B^* \to W^*$ is surjective by Hahn-Banach.
2. We have $\langle qx, \eta \rangle = \langle x, q^* \eta \rangle$, so $q^* : (B/W)^* \to B^*$ sends $q^* \eta$ to $x \mapsto \langle qx, \eta \rangle$. So if $q^* \eta = 0$, then $\eta = 0$; i.e. q^* is injective. Also, $\operatorname{im} q^* \subseteq W^o$, and in fact, $\operatorname{im} q^* = W^o$: If $\xi \in W^o$, define η by $\langle qx, \eta \rangle = \langle x, \xi \rangle$ and $\xi = q^* \eta$. Check that the norms are equal. □

1.3 Proof of the Fredholm-Riesz theorem

Recall that $T \in \mathcal{L}(B, B)$ is compact. We want to show that $\operatorname{coker}(1-T)$ is finite dimensional, and we know that it is closed.

Proof. Apply $(B/W)^* \cong W^o$ with $W = \operatorname{im}(! - T)$.

$$(\operatorname{im}(1-T))^o = \{\xi \in B^* : \langle (1 - Tx, \xi \rangle = 0 \, \forall x \in B\} = \ker(1 - T^*).$$

T^* is compact, so $\dim \operatorname{im}(1-T)$. This shows that $(\operatorname{coker}(1-T))^* \cong \ker(1-T^*)$, so $\dim \operatorname{coker}(1-T) = \dim \ker(1-T^*) < \infty$. So $1-T$ is Fredholm.

Finally, for $0 \leq t \leq 1$,

$$\operatorname{ind}(1-T) = \operatorname{ind}(1-tT) = \operatorname{ind} 1 = 0.$$ □