Statistics 210A Lecture 2 Notes

Daniel Raban

August 31, 2021

1 Estimation and Introduction to Exponential Families

1.1 Review of measure theory

Last time, we introduced some ideas from measure theory. Let's review:

A **measure** μ assigns a "weight" to subsets $A \subseteq \mathcal{X}$ (for $A \in \mathcal{F}$).

Example 1.1. The **counting measure** is $\#(A) = \mathrm{card}(A)$.

Example 1.2. **Lebesgue measure** gives $\lambda(A) = \mathrm{vol}(A)$ (in $\mathbb{R}^n$).

Example 1.3. The **Gaussian distribution** gives $P(A) = \int_A \phi(x)\,dx$.

Measures give rise to integrals:

$$\int f(x)\,d\mu(x) = \begin{cases} \mu(A) & f(x) = \mathbb{1}_{\{x \in A\}} \\ \sum_i c_i \mu(A_i) & f(x) = \sum_i c_i \mathbb{1}_{\{x \in A_i\}} \\ \text{limit} & f(x) \text{ nice enough.} \end{cases}$$

If $P \ll \mu$ (meaning $\mu(A) = 0 \implies P(A) = 0$), there is a **density** $p(x)$ with $p : \mathcal{X} \to [0, \infty)$ such that $\int f\,dP = \int fp\,d\mu$ for all (nice) f.

The **outcome space** Ω containing outcomes ω is equipped with a measure $\mathbb{P}$. Random variables are functions with $X(\omega) \in \mathcal{X}$ (e.g. $\mathcal{X} = \mathbb{R}$). You can think of X "decoding" the randomness ω to tell you what the value in our nicer space $\mathcal{X}$ is. We write $X \sim Q$ if $\mathbb{P}(X \in B) = Q(B)$.

1.2 Estimation

In statistics, there are multiple possible distributions that could have generated the data.

Definition 1.1. A **statistical model** is a family of candidate probability distributions $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ for a random variable $X \sim P_\theta$. X is called the **data**, and θ is called the **parameter**.

The data X is observed by the statistical analyst, whereas θ is unobserved by the analyst. For now, θ is fixed and unknown.[1] The goal of estimation is to observe $X \sim P_\theta$ and guess the value of some estimand $g(\theta)$.

Example 1.4. Flip a biased coin n times. The parameter $\theta \in [0,1]$ is the probability of heads, and $X \sim \text{Binom}(n, \theta)$ is the number of heads after n flips. X has a density $p_\theta(x) = \theta^x(1-\theta)^{n-x}\binom{n}{x}$ for $x = 0, 1, \dots, n$ (this is a density with respect to counting measure on $\{0, 1, \dots, n\}$).

Definition 1.2. A **statistic** is any function $T(X)$ of X.

In particular, a statistic is not a function of θ. It is something the statistical analyst can calculate.

Definition 1.3. An **estimator** $\delta(X)$ of $g(\theta)$ is a statistic intended to guess $g(\theta)$.

Example 1.5. In our coin flipping example, the natural estimator for θ is $\delta_0(X) = X/n$.

1.3 Loss and risk

How can we tell if an estimator is good?

Definition 1.4. The **loss function** $L(\theta, d)$ measures how badly an estimate is.

Example 1.6. One important loss function is the **squared error loss** $L(\theta, d) = (d - g(\theta))^2$.

Usually, $L(\theta, d) \geq 0$ for all θ, d with $L(\theta, g(\theta)) = 0$.

Definition 1.5. The **risk function** $R(\theta; \delta(\cdot)) = \mathbb{E}_\theta[L(\theta, \delta(X))]$ is the expected loss as a function of θ.

Remark 1.1. The $\mathbb{E}_\theta$ notation refers to the expectation with respect to X, where θ is the true parameter. This is in contrast to other disciplines which use the notation $\mathbb{E}_X$ to denote what variables we are conditioning on in the expectation. We will use the notation $\mathbb{E}[f(X, X') \mid X']$ when we want to only integrate over certain random variables.

Example 1.7. The **mean squared error** is the risk function $\text{MSE}(\theta, \delta_0(\cdot)) = \mathbb{E}_\theta[(\delta(x) - \theta)^2]$.

Example 1.8. In our coin flipping example, we have the estimator $\delta_0(X) = X/n$ with $\mathbb{E}_\theta[X/n] = \theta$ (this is an **unbiased estimator**). The loss is

$$\text{MSE}(\theta, \delta_0(\cdot)) = \mathbb{E}_\theta[(\delta(x) - \theta)^2]$$

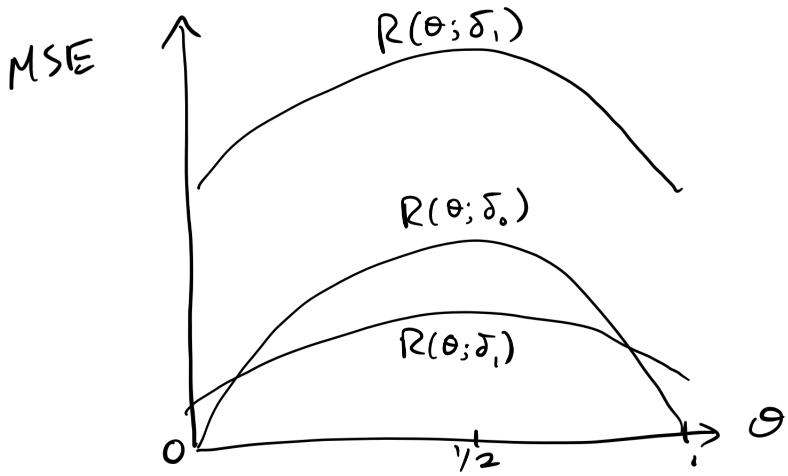
[1] This is a frequentist perspective. With a Bayesian perspective, we may assume that θ follows some distribution.

$$
\begin{aligned}
&= \mathrm{Var}_\theta(X/n) \\
&= \frac{\theta(1-\theta)}{n}.
\end{aligned}
$$

Here are other choices of estimators. We could take

$$\delta_1(X) = \frac{X+3}{n}.$$

$$\delta_2(X) = \frac{X+3}{n+6}$$



There is no estimator which is always the best; if $\theta = 3/4$, then the constant estimator $\delta(X) = 3/4$ would be better than any estimator which has a chance of suggesting anything other than $3/4$.

1.4 Comparing estimators

Definition 1.6. An estimator $\delta(X)$ is **inadmissible** if there exists another estimator $\delta^*(X)$ such that

(a) $R(\theta; \delta^*) \le R(\theta; \delta)$ for all θ,

(b) $R(\theta; \delta^*) < R(\theta; \delta)$ for some θ.

In our previous example, δ_0 rendered δ_1 inadmissible.
Here are some strategies to resolve the ambiguity:

1. Summarize $R(\theta)$ by a scalar:

(a) **Average-case risk**: Minimize $\int_\Theta R(\theta; \delta)\, d\Lambda(\theta)$. The minimizer δ is called the **Bayes estimator**.

(b) **Worse-case risk**: Minimize $\sup_{\theta \in \Theta} R(\theta; \delta)$. The minimizer δ is called the **minimax estimator**.

2. Constrain the choice of estimator:

   (a) Only consider **unbiased** $\delta(X)$ ($\mathbb{E}_\theta[\delta(X)] = g(\theta)$).

1.5 Exponential families

Definition 1.7. An s-**parameter exponential family** is a family $\mathcal{P} = \{P_\eta : \eta \in \Xi\}$ with densities $p_\eta(x)$ with respect to a common dominating measure μ on $\mathcal{X}$ of the form

$$p_\eta(x) = e^{\eta^\top T(x) - A(\eta)} h(x),$$

where

- $T : \mathcal{X} \to \mathbb{R}^s$ is called the **sufficient statistic**,
- $h : \mathcal{X} \to [0, \infty)$ is called the **carrier/base density**,
- $\eta \in \Xi \subseteq \mathbb{R}^s$ is called the **natural parameter**,
- $A : \mathbb{R}^s \to \mathbb{R}$ is called the **cumulant generating function** (or the **normalizing constant**).

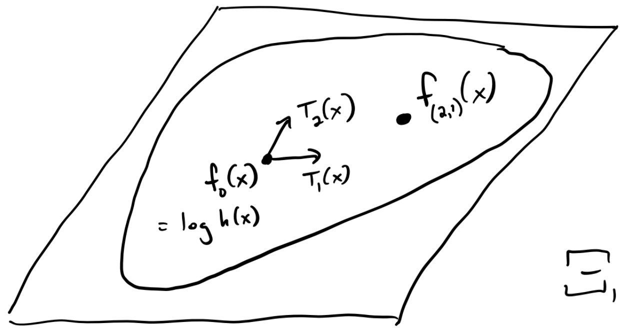
Remark 1.2. $A(\eta)$ is totally determined by h, T, since we always must have $\int_\mathcal{X} p_\eta \, d\mu = 1$ for all η. So we can solve

$$A(\eta) = \log \left[\int_\mathcal{X} e^{\eta^\top T(x)} h(x)\, d\mu(x) \right] \leq \infty.$$

Definition 1.8. p_η is **normalizable** if $A(\eta) < \infty$. The **natural parameter space** is $\Xi_1 = \{\eta : A(\eta) < \infty\}$. We say $\mathcal{P}$ is in **canonical form** if $\Xi = \Xi_1$.

Remark 1.3. $A(\eta)$ is a convex function, so Ξ_1 is a convex set.

In general, you can think of an s-parameter exponential family as describing an s-dimensional hyperplane in the space of log-densities.


$f_{(2,1)}(x)$
$T_2(x)$
$T_1(x)$
$f_0(x)$
$= \log h(x)$
Ξ_1