Math 142 Lecture 8 Notes

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1 Connectivity, Path-Connectivity, Separation, and Metrization

1.1 Connectivity involving continuous functions and product spaces

Theorem 1.1. *If $f : X \to Y$ is continuous, and X is connected, then $f(X)$ is connected.*

Proof. For simplicity, assume $Y = f(X)$. If $Y = A \cup B$ with A, B open and $A \cap B = \varnothing$, then $X = f^{-1}(Y) = f^{-1}(A) \cup f^{-1}(B)$. We know that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint (because $A \cap B = \varnothing$) and open (because f is continuous). X is connected, so $f^{-1}(A)$ or $f^{-1}(B)$ is $\varnothing$. Note that $f(\varnothing) = \varnothing$ and $f(f^{-1}(A)) = A$ by the surjectivity of f, so A or $B = \varnothing$. $\square$

Lemma 1.1. *If $\{A_i\}$ is a collection of connected subspaces of X, and $\cap_i A_i \neq \varnothing$, then $\bigcup_i A_i$ is connected.*

Proof. Let $p \in \bigcap_i A_i$. Suppose that $\bigcup_i A_i = B \cup C$ for open, disjoint B, C. Then $p \in B$ without loss of generality. For each A_i, $A_i = (B \cap A_i) \cup (C \cap A_i)$. These are disjoint and open (in the subspace topology). For each A_i, A_i is connected, so $B \cap A_i = \varnothing$ or $C \cap A_i = \varnothing$. But $B \cap A_i \neq \varnothing$, as it contains p. So $A_i \cap C = \varnothing$, meaning $A_i \subseteq B$. So $\bigcup_i A_i \subseteq B$, which implies that $C = \varnothing$. So $\bigcup_i A_i$ is connected. $\square$

Theorem 1.2. *X and Y are connected iff $X \times Y$ is connected.*

Proof. ($\Longleftarrow$) The projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous and surjective, so by our previous theorem, X and Y are connected.

($\Longrightarrow$) If $x \in X$, then $\{x\} \times Y \cong Y$ (check this yourself). So $\{x\} \times Y$ is connected; similarly, $X \times \{y\} \cong X$ is connected for any $y \in Y$. Let $A_{x,y} = (X \times \{y\}) \cup (\{y\} \times Y)$. This is connected by our lemma, since $(X \times \{y\}) \cup (\{y\} \times Y) = \{(x, y)\} \neq \varnothing$. Fix $y_0 \in Y$. Then $X \times Y = \bigcup_{x \in X} A_{a,y_0}$, and $\bigcap_{x \in X} A_{a,y_0} = X \times \{y_0\} \neq \varnothing$. So the lemma implies that $X \times Y$ is connected. $\square$

Corollary 1.1. *$\mathbb{R}^n$ is connected.*

Proof. Use the fact that $\mathbb{R}$ is connected, and induct on n. $\square$

Corollary 1.2. *$S^n \setminus \{point\}$ is connected.*

Proof. We already showed that $\mathbb{R}^n \cong S^n \setminus \{\text{north pole}\}$. It doesn't matter which point we remove. $\square$

Corollary 1.3. *S^n is connected.*

Proof. $S^n = (S^n \setminus \{\text{north pole}\}) \cup (S^n \setminus \{\text{south pole}\})$, which are both connected and have nonempty intersection. Our lemma from before shows that S^n is connected. $\square$

1.2 Connected components

What if a space is not connected? We can try to find "maximal" connected pieces.

Definition 1.1. A *(connected) component* of a space X is a subspace $A \subseteq X$ such that A is connected, and if $A \subsetneq B$, then B is not connected.

Example 1.1. If X is connected, it has one component: the set X itself.

Example 1.2. The set $[0, 1] \cup [2, 3]$ has two components: $[0, 1]$ and $[2, 3]$.

Example 1.3. The set $[0, 1) \cup (1, 2]$ has two components: $[0, 1)$ and $(1, 2]$.

1.3 Path connectivity

Definition 1.2. A *path* in a space X is a continuous function $\gamma : [0, 1] \to X$. A path is said to be a path from $\gamma(0)$ to $\gamma(1)$; here, $\gamma(0)$ is the beginning of the path, and $\gamma(1)$ is the end of the path.

Intuitively, people like to think of a "path" as the image of γ. But in fact, if we parametrize γ differently, the path may be different, even though the image will be the same (e.g. if the image is traversed more slowly with respect to t in one area).

Definition 1.3. A space X is *path-connected* if $\forall x, y \in X$ with $x \neq y$, there exists a path from x to y.

Theorem 1.3. *If X is path-connected, X is connected.*

Proof. If $X = A \cup B$ with A, B nonempty, disjoint, and open, let $x \in A$ and $y \in B$. Then let $\gamma : [0, 1] \to X$ be a path from x to y. So $[0, 1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$, and these are open because γ is continuous. These are also disjoint and nonempty ($0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$), contradicting the fact that $[0, 1]$ is connected. So X is connected. $\square$

1.4 Separation and Metrization

This section won't be tested, but it is included for interest. A good reference is Munkres sections 31 to 34.

We have already discussed Hausdorff spaces. There are other types of separation axioms for topological spaces.

Definition 1.4. A topological space X is *regular* if

1. $\{x\}$ is closed for all $x \in X$.

2. For all $x \in X$ and $A \subseteq X$ closed with $x \notin A$, there exist open sets U_x and U_A with $x \in U_x$, $A \subseteq U_A$, and $U_x \cap U_A = \varnothing$.

Definition 1.5. A topological space X is *normal* if for all pairs $A, B \subseteq X$ that are closed and disjoint, there exist open sets U_A, U_B such that $A \subseteq U_A$, $B \subseteq U_B$, and $U_A \cap U_B = \varnothing$.

Theorem 1.4 (Urysohn's metrization lemma)**.** *If X is regular, and there exists a countable base or its topology, then X is metrizable; i.e. we can put a metric on X such that the topology induced from (X, d) is the same as the original topology.*

Remark 1.1. All metric spaces are regular, but not all metric spaces have a countable base. This second part is harder to prove.