Math 245C Lecture 27 Notes

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1 Applying Distributions to Convolutions

1.1 Uniform estimates of functions on bounded sets

Last time, we proved the first half this theorem:

Theorem 1.1. *Let $\phi \in C_c^\infty(\Omega)$, and let $T \in \mathcal{D}'(\Omega)$. Set $f(y) = T(\phi_y)$ for $y \in O_\phi$.*

1. *$f \in C^\infty(O_\phi)$, and*
$$D^\alpha f(y) = (-1)^{|\alpha|} T((D^\alpha \phi)_y).$$
2. *If $\psi \in L^1(O_\phi)$ has compact support, then*
$$T(\psi * \phi) = \int_{O_\phi} \psi(y) f(y)\, dy.$$

To prove the second half, we first make some remarks.

Remark 1.1. Fix $R > 0$, and set $Q = [-R, R]^d$. There are $a : (0, \infty) \to (0, \infty)$ and $m : (0, \infty) \to \mathbb{N}$ such that for all $\varepsilon > 0$,

$$\lim_{\varepsilon \downarrow 0} a(\varepsilon) = 0$$

and such that for every $\varepsilon > 0$, there is a partition $\{Q\}_{i=1}^{m(\varepsilon)}$ of squares of diameters less than $a(\varepsilon)$.

These conclusions extend to any set $\Omega \subseteq [-R, R]^d$ with $\Omega_i = Q_i \cap \Omega$.

Definition 1.1. Let $A \subseteq \mathbb{R}^d$, and let $f : A \to \mathbb{R}$. We define the **oscillation** of f as

$$\operatorname{osc}(f, A, \delta) = \sup_{x,y \in A} \{|f(x) - f(y)| : |x - y| \leq \delta\}.$$

Remark 1.2. Assume $A = \Omega$ and $f : \Omega \to \mathbb{R}$ is uniformly continuous. Then

$$\int_\Omega f(x)\,dx = \sum_{i=1}^{m(\varepsilon)} \int (f(x) - f(x_i))\,dx + |\Omega_i^\varepsilon| f(x_i^\varepsilon),$$

where $x_i^\varepsilon \in \Omega_i^\varepsilon$. As a consequence,

$$\left| \int_\Omega f(x)\,dx - \sum_{i=1}^{m(\varepsilon)} |\Omega_i^\varepsilon| f(x_i^\varepsilon) \right| \leq i|\Omega| \operatorname{osc}(f, \Omega, a(\varepsilon)).$$

Remark 1.3. If $\phi \in C_c^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, we set

$$\phi_y(x) = \phi(x - y), \qquad x \in y + \operatorname{supp}(\phi),$$

and $y \mapsto T(\phi_y)$ is continuous on $O_\phi = \{y \in \mathbb{R}^d : y + \operatorname{supp}(\phi) \subseteq \Omega\}$.

1.2 Proof of the theorem

Now we can prove the theorem.

Proof. Let $\psi \in L^1(O_\phi)$ be such that $\operatorname{supp}(\psi) \subseteq O_\phi$, We are to show that

$$\int_{O_\phi} \psi(y) T(\phi_y)\,dy = T(\psi * \psi).$$

Case 1: $\psi \in C_c^\infty(O_\phi)$. Since $y \mapsto f(y) := \psi(y) T(\phi(y))$ is uniformly continuous on O_ϕ,

$$\left| \int_{O_\phi} \psi(y) T(\phi_y)\,dy - \sum_{i=1}^{m(\varepsilon)} \psi(y_i^\varepsilon) T(\phi_{y_i^\varepsilon}) |\Omega_i^\varepsilon| \right| \leq \operatorname{osc}(f, O_\phi, a(\varepsilon)) |O_\phi|$$

for some $y_i^\varepsilon \in \Omega_i^\varepsilon$ independent of T, ϕ, ψ. Set $\eta^\varepsilon(x) = \sum_{i=1}^{m(\varepsilon)} \psi(y_i^\varepsilon) \phi(x - y_i^\varepsilon)$. Let K_1 be the closure of the set $\bigcup_{y \in O_\phi} (y + \operatorname{supp}(\phi)) \subseteq \Omega$. Then K_1 is compact.

For any multi-index $\alpha \in \mathbb{N}^d$,

$$\partial^\alpha \eta^\varepsilon(x) = \sum_{i=1}^{m(\varepsilon)} \psi(y_i^\varepsilon) \partial^\alpha \phi(x - y_i^\varepsilon) |\Omega_i^\varepsilon|.$$

This converges to $\int_{O_\phi} \psi(y) \partial^\alpha \phi(x - y)\,dy = \psi * \partial^\alpha$ uniformly:

$$\left| \int_{O_\phi} \psi(y) \partial^\alpha \phi(x - y)\,dy - \partial^\alpha \eta^\varepsilon(x) \right| \leq |\Omega| \operatorname{osc}(g_\varepsilon^x, \Omega, a(\varepsilon)) \xrightarrow{\varepsilon \to 0} 0,$$

where $g_\varepsilon^x(y) = \psi(y)\partial^\alpha\phi(x-y)$. This means $(\eta_\varepsilon)_\varepsilon$ converges to $\psi * \phi$ in $C_c^\infty(O_\phi)$. Consequently,

$$T(\phi * \psi) = \lim_{\varepsilon\to 0} T(\eta_\varepsilon) = \lim_{\varepsilon\to 0}\sum_{i=1}^{m(\varepsilon)} |\Omega_i^\varepsilon| T(\psi_{y_i^\varepsilon}) = \int_{O_\phi} \psi(y)T(\phi_y)\,dy.$$

Case 2: $\psi \in L^1(\emptyset_\phi)$ and $\operatorname{supp}(\psi) \subseteq O_\phi$: For each $\delta > 0$, let $\psi_\delta \in C_c^\infty(\emptyset_\phi)$ be such that $\int_{O_\phi} |\psi - \psi_\delta|\,dx \le \delta$, and assume there exists a compact K_2 such that $\operatorname{supp}(\psi_\delta) \subseteq K_2 \subseteq O_\phi$. Note that for a multi-index $\alpha \in \mathbb{N}^d$,

$$\partial_\alpha(\psi_\delta * \phi) = \partial^\alpha\phi * \psi_\delta \to \partial^\alpha\phi * \psi$$

uniformly on K_2. Hence, $\psi_\delta * \phi \to \psi * \phi$ uniformly as $\delta \to 0$. We conclude that

$$T(\psi * \phi) = \lim_{\delta\to 0} T(\psi_\delta * \phi) = \lim_{\delta\to 0}\int_{O_\phi} \psi_\delta(y)T(\phi_y)\,dy = \int_{O_\phi} \phi(y)T(\phi_y),$$

using the dominated convergence theorem. □

Let $\phi \in C^1(\Omega$, and assume that $\int_\Omega |\phi|^p\,dx + \int_\Omega |\nabla\phi|^p\,dx < \infty$. Then $\nabla\phi$ as a distribution is equal to the usual $\nabla\phi$.

A consequence of our result will be that for every y and a.e. x,

$$\phi(x+y) - \phi(y) = \int_0^1 \nabla\phi(x+ty)\cdot y\,dt.$$