Math 255B Lecture 6 Notes

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January 17, 2020

1 Consequences of Analytic Fredholm Theory

1.1 Analytic Fredholm theory

Last time, we were proving the analytic Fredholm theory.

Theorem 1.1 (analytic Fredholm theory)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ be a holomorphic family of Fredholm operators. Assume that there exists a $z_0 \in \Omega$ such that $T(z_0) : B_1 \to B_2$ is bijective. Then the set*

$$\Sigma = \{z \in \Omega : T(z) \text{ is not bijective}\}$$

is discrete.

Proof. Let $z_1 \in \Omega$. Then there is a neighborhood $N(z_1)$ of z_1 such that for every $z \in N(z_1)$, the Grushin operator

$$\mathcal{P}_{z_1}(z) = \begin{bmatrix} T(z) & R_-(z) \\ R_+(z) & 0 \end{bmatrix}$$

is bijective with the inverse

$$\mathcal{E}_{z_1}(z) = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_0} \to B_1 \oplus \mathbb{C}^{n_0}.$$

We claim that for $z \in N(z_1)$, $T(z) : B_1 \to B_2$ is bijective $\iff$ $E_{-+}(z) : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}$ is bijective.[1] Check:

$$\begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies TE + R_-E_- = 1, TE_+ + R_-E_{-+} = 0.$$

[1] What we lose from this reduction is that if $T(z)$ has some simple dependence of z (e.g. polynomial), $E_{-+}(z)$ may not have a simple dependence. In some contexts, the operator E_{-+} is called the effective Hamiltonian.

If E_{-+}^{-1} exists, then $R_- = -TE_+E_{-+}^{-1}$, so

$$T(E - E_-E_{-+}^{-1}E_-) = 1.$$

So T^{-1} exists and

$$T^{-1}(z) = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z).$$

Using that $\mathcal{E}\mathcal{P} = 1$, so $E_-R_- = 1$ and $E_-T + E_{-+}R_+ = 0$, we get T^{-1} exists $\implies E_{-+}$ exists.

We get for $z \in N(z_1)$ that $T(z)$ is invertible if and only if $\det E_{-+}(z) \neq 0$. The function $\det E_{-+}(z)$ is holomorphic on $N(z_1)$. So either $\det E_{-+}(z) \equiv 0$, or $\det E_{-+}(z) \neq 0$ in a punctured neighborhood of z_1. Let $\Omega_1 = \{z \in \Omega : T(z') \text{ is invertible } \forall z' \neq z \text{ near } z\}$ and $\Omega_2 = \{z \in \Omega : T(z') \text{ is not invertible } \forall z' \neq z \text{ near } z\}$. Then each Ω_j is open, $\Omega_1 \cup \Omega_2 = \Omega$, and $\Omega_1 \neq \varnothing$ (as $z_0 \in \Omega_1$). Since Ω is connected, $\Omega_1 = \Omega$ and thus, $\Sigma = \{z \in \Omega : T(z) \text{ is not invertible}\}$ is discrete. $\square$

Remark 1.1. The map $\Omega \setminus \Sigma \to \mathcal{L}(B_2, B_1)$ sending $z \mapsto T(z)^{-1}$ is holomorphic. Consider $T(z)^{-1}$ for z in a punctured neighborhood of $w \in \Sigma$: We have

$$T^{-1}(z) = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z),$$

where E, E_+, E_- are all holomorphic in a neighborhood of w. We have that

$$E_{-+}(z)^{-1} = \frac{\text{holomorphic near } w}{\det E_{-+}(z)},$$

so we have a Laurent expansion

$$E_{-+}(z)^{-1} = \frac{R_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{R_{-1}}{z-w} + \mathrm{Hol}(z),$$

where $1 \leq N_0 < \infty$ and the R_j are of finite rank. Combining these formulas, we get that $z \mapsto T(z)^{-1}$ has a pole of order N_0 at $z = w$:

$$T(z)^{-1} = \frac{A_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{A_{-1}}{z-w} + Q(z), \qquad Q(z) \text{ holomorphic near } w,$$

where for $1 \leq j \leq N_0$, the $A_{-j} \in \mathcal{L}(B_2, B_1)$ n be expressed in terms of $R_{-N_0}, \ldots, R_{-1}$ and are therefore of finite rank.

1.2 Application: the residue of the resolvent

Here is an example/special case of the analytic Fredholm theory.

Assume that $B_1 \subseteq B_2$ with continuous inclusion, and let $T(z) = T - z$ for $z \in \Omega$, where T is some operator. Assume that $T(z)$ is Fredholm for each z and that $T(z_0)^{-1}$ exists for some $z_0 \in \Omega$. We get a Laurent expansion for the resolvent $(T-z)^{-1}$ at $w \in \Sigma$:

$$(z-T)^{-1} = \frac{A_{-N_0}}{(z-z_0)^{N_0}} + \cdots + \frac{A_{-1}}{z-w} + Q(z), \qquad Q(z) \text{ holomorphic near } w.$$

for $0 < |z-w| \ll 1$.

Proposition 1.1. *The operator $\Pi := A_{-1}$ is a projection*[2] *on B_2 which commutes with T (on B_1).*

Proof. Integrate the Laurent expansion along $\gamma_r = \partial D(w,r)$ for $0 < r \ll 1$. Then

$$\Pi = \frac{1}{2\pi i}\int_{\gamma_r} (z-T)^{-1}\, dz.$$

We claim that $\Pi^2 = \Pi$: Let $0 < r_1 < r_2 \ll 1$, and write

$$\Pi^2 = \int_{\gamma_{r_2}}\int_{\gamma_{r_1}} (z-T)^{-1}(\widetilde{z}-T)^{-1}\,\frac{dz}{2\pi i}\,\frac{d\widetilde{z}}{2\pi i}$$

Using $(\widetilde{z}-T)^{-1} - (z-T)^{-1} = (\widetilde{z}-T)^{-1}(z-\widetilde{z})(z-T)^{-1}$, we have

$$= \int_{\gamma_{r_2}}\int_{\gamma_{r_1}} \frac{1}{\widetilde{z}-z}(z-T)^{-1}\,\frac{dz}{2\pi i}\,\frac{d\widetilde{z}}{2\pi i} - \int_{\gamma_{r_2}}\int_{\gamma_{r_1}} \frac{1}{\widetilde{z}-z}(\widetilde{z}-T)^{-1}\,\frac{dz}{2\pi i}\,\frac{d\widetilde{z}}{2\pi i}$$

The second term is 0 by applying the Cauchy integral formula on the inner integral.

So we get

$$\Pi^2 = \int_{\gamma_{r_1}} \underbrace{\frac{1}{2\pi i}\int_{\gamma_{r_2}} \frac{1}{\widetilde{z}-z}\, d\widetilde{z}}_{=1}(z-T)^{-1}\,\frac{dz}{2\pi i} = \Pi. \qquad \square$$

Remark 1.2. We know that $T(\operatorname{Ran}\Pi) \subseteq \operatorname{Ran}\Pi \subseteq B_1$, where $\operatorname{Ran}\Pi$ is finite dimensional, and let us check that $(T-z_0)|_{\operatorname{Ran}\Pi}$ is nilpotent:

$$\begin{aligned}(T - z_0\Pi) &= \frac{1}{2\pi i}\int_{\gamma_r} (T-z_0)(z-T)^{-1}\, dz \\ &= \underbrace{\frac{1}{2\pi i}\int_{\gamma_r} (T-z)(z-T)^{-1}\, dz}_{=0} + \frac{1}{2\pi i}\int_{\gamma_r} (z-z_0)(z-T)^{-1}\, dz\end{aligned}$$

[2]This is sometimes called the Riesz projection.

$$= \frac{1}{2\pi i} \int_{\gamma_r} (z - z_0)(z - T)^{-1}\, dz.$$

It follows that

$$(T - z_0)^j \Pi = \frac{1}{2\pi i} \int_{\gamma_r} (z - z_0)^j (z - T)^{-1}\, dz.$$

And if $j = N_0$, we get $(T - z_0)^{N_0} \Pi = 0$, as $(z - z_0)^{N_0}(z - T)^{-1}$ is holomorphic.