Math 210A Lecture 5 Notes

Daniel Raban

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1 Equivalences, Cayley's Theorem, and More Limits

1.1 Equivalence of categories

Definition 1.1. An **equivalence of categories** $F : \mathcal{C} \to \mathcal{D}$ with a **quasi-inverse** $G : \mathcal{D} \to \mathcal{D}$ is a pair of functors such that there exist natural isomorphisms $\eta : F \circ g \to \mathrm{id}_{\mathcal{D}}$ and $\eta' : G \circ F \to \mathrm{id}_{\mathcal{C}}$.

Definition 1.2. A **natural isomorphism** η is a natural transformation such that η_A is an isomorphism for each A.

Example 1.1. Let $\mathcal{C}$ be the category with $\mathrm{Obj}(\mathcal{C}) = \{A\}$ and $\mathrm{Hom}_{\mathcal{C}}(A, A) = \mathrm{id}_A$, and let $\mathcal{C}$ be the category with objects B, C and morphisms $f : B \to C$, $g : C \to B$, id_B, and id_C such that $f \circ g = \mathrm{id}_C$ and $g \circ f = \mathrm{id}_B$. Let $F : \mathcal{C} \to \mathcal{D}$ be $F(A) = B$ with $F(\mathrm{id}_A) = \mathrm{id}_B$, and let $G : \mathcal{D} \to \mathcal{C}$ be $G(B) = G(C) = A$ and $G(h) = \mathrm{id}_A$ for all h. Then $G \circ F(A) = A$, $G \circ F(\mathrm{id}_A) = \mathrm{id}_A$, and you can check that $\eta : G \circ F \to \mathrm{id}_{\mathcal{C}}$ given by $\eta_A = \mathrm{id}_A$ is a natural isomorphism.

1.2 Cayley's theorem

Let $\mathcal{C}$ be a small category, and let $h^{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ be

$$h^{\mathcal{C}}(B) = h^B = \mathrm{Hom}_{\mathcal{C}}(\cdot, B)$$

and for $f : B \to C$, $h^{\mathcal{C}}(f) : h^B \to h^C$ sends $g \in \mathrm{Hom}_{\mathcal{C}}(A, B) \mapsto f \circ g$.

Lemma 1.1 (Yoneda). *$h^{\mathcal{C}}$ is fully faithful.*

Definition 1.3. The **symmetric group** on X, S_X, is the set of bijections from X to X with function composition. We call $S_n = S_{\{1,\dots,n\}}$.

Theorem 1.1 (Cayley). *Every group G is isomorphic to a subgroup of S_G.*

Proof. Let $\mathbb{G}$ be the category of the group G, where there is one object, and the group elements of G are morphisms. $h^{\mathbb{G}} : \mathbb{G} \to \mathrm{Fun}(\mathbb{G}^{op}, \mathrm{Set})$ is fully faithful. What is this functor? $h^{\mathbb{G}}(G) = h^G = \mathrm{Hom}(\cdot, G)$, and $h^{\mathbb{G}}(g) : h^G \to h^G$, where

$$h^{\mathbb{G}}(g)_G : \underbrace{h^G(G)}_{=G} \to h^G(G),$$

and

$$\rho = h^{\mathbb{G}}(\cdot)_G : G \to \mathrm{Maps}(G, G).$$

Note that

$$\rho(gh) = h^{\mathbb{G}}(gh)_G = (h^{\mathbb{G}}(g) \circ h^{\mathbb{G}}(h))_G = \rho(g)\rho(h),$$

$$\rho(e) = \mathrm{id}_G,$$

$$\mathrm{id}_G = \rho(e) = \rho(gg^{-1}) = \rho(g)\rho(g^{-1}),$$

so $\rho(g) \in S_G$. So $\rho : G \to S_G$ is a homomorphism. It is injective because if $\rho(g) = \rho(h)$, then $h^{\mathbb{G}}(g)_G = h^{\mathbb{G}}(h)_H$, so $h^{\mathbb{G}}(g) = h^{\mathbb{G}}(h)$. By Yoneda's lemma, $g = h$ because $h^{\mathbb{G}}$ is faithful. $\square$

1.3 Completeness

Definition 1.4. A category is **complete** if it admits all limits. A category is **cocomplete** if it admits all colimits.

Proposition 1.1. Set *is complete and cocomplete.*

Proof. Here is a sketch. Let $F : I \to \mathrm{Set}$. Then

$$\lim F = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} F(i) : \forall \phi : i \to j,\ F(\phi)(a_i) = a_j \right\}.$$

$$\mathrm{colim}\, F = \coprod_{i \in I} F(i) \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by the conditions $a_i \sim a_j \iff \exists \phi : i \to j$ such that $\mathbb{F}(\phi)(a_i) = a_j$ for every $a_i \in F(i)$ and $a_j \in F(j)$. $\square$

Remark 1.1. The same proof works for the category of groups.

1.4 Initial and terminal objects

Definition 1.5. An **initial object** A in a category $\mathcal{C}$ is any object such that for all $B \in \mathcal{C}$, there exists a unique morphism $f : A \to B$. A **terminal object** A in a category $\mathcal{C}$ is any object such that for all $B \in \mathcal{C}$, there exists a unique morphism $f : B \to A$.

Remark 1.2. If they exist, initial and terminal objects are unique up to unique isomorphism.

Remark 1.3. Let $\varnothing$ be the empty category, and let $F : \varnothing \to \mathcal{C}$. If $\lim F$ exists, it is a terminal object. If $\operatorname{colim} F$ exists, it is an initial object.

1.5 Sequential limits and colimits

Definition 1.6. A **sequential limit** (or **inverse limit**) $\varprojlim F$ is a limit of the diagram

$$\cdots \longrightarrow A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1$$

A **sequential colimit** (or **direct limit**) $\varinjlim F$ is a colimit of the diagram

$$A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \longleftarrow \cdots$$

Example 1.2. In CRing, $\mathbb{Z}/p^{n+1}\mathbb{Z}$ surjects onto $\mathbb{Z}/p^n\mathbb{Z}$. Then $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ is called the p-adic integers $\mathbb{Z}_p$, where

$$\mathbb{Z}_p = \left\{ a_i \in \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} : a_n = a_{n+1} \pmod{p^n} \right\}.$$