Math 254A Lecture 5 Notes

Daniel Raban

April 7, 2021

1 Eventual Finiteness of $s_n(U)$ and Point Function Conditions

1.1 Recap

From last time, we have a σ-finite measure space (M, λ), a locally convex topological vector space X, and a measurable map $\varphi : M \to X$. We also let $\mathcal{U}$ be the convex open subsets of X. In this case, the equivalent of type classes is $T_n(U) = \{p \in M^n : \frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\}$, and we may let $s_n(U) := \log \lambda^{\times n}(T_n(U))$. We have shown that $T_{n+m}(U) \supseteq T_n(U) \times T_m(U)$, which implies that $s_{n+m}(U) \geq s_n(U) + s_m(U)$ (taking values in $[-\infty, \infty]$), and so, by Fekete,

$$s(U) = \lim_n \frac{s_n(U)}{n} = \sup_n \frac{s_n(U)}{n},$$

provided we show that either $s_n(U) = -\infty$ or $s_n(U) > -\infty$ for all sufficiently large n.

1.2 Eventual finiteness of $s_n(U)$

Lemma 1.1. *Either $s_n(U) = -\infty$ or $s_n(U) > -\infty$ for all sufficiently large n.*

Proof. Suppose $s_m(U) > -\infty$, i.e. $\lambda^{\times m}(T_m(U)) > 0$. Then $T_{km}(U) \supseteq T_m(U)^k$, so $s_{km}(U) > -\infty$. We need to control the indices in between.

Step 1: Reduce to the case where $U \ni 0$.[1] To do this, let $x \in U$ and now consider $\varphi'(m) = \varphi(m) - x$. Then $U' = U - x$ is a neighborhood of 0, and $\{p : \frac{1}{n}\sum_{i=1}^n \varphi'(p_i) \in U'\} = \{p : \frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\}$.

Step 2: Since U is convex and $U \ni 0$, $tU \subseteq U$ for all $t \in [0,1]$. Also, since U is open, $U = \bigcup_{0 \leq t < 1} t \cdot U = \bigcup_{r \in \mathbb{N}} \frac{r}{r+1} U$; this is because $x \in U$ implies there is some $r \in \mathbb{N}$ such that $\frac{r+1}{r} x \in U$, i.e. $x \in \frac{r}{r+1} U$. The countable union is for measure theory purposes. So $T_n(U) = \bigcup_r T_n(\frac{r}{r+1} U)$, and so

$$\lambda^{\times n}(U) = \lim_{r \to \infty} \lambda^{\times n}\left(T_n\left(\frac{r}{r+1} U\right)\right).$$

[1] This step is not strictly necessary, but it makes our notation easier.

So there exists some $r \in \mathbb{N}$ such that

$$\lambda^{\times m}\left(T_m\left(\frac{r}{r+1}U\right)\right) > 0.$$

Step 3: On the other hand, $X = \bigcup_{q \in \mathbb{N}} q \cdot U$, so for all n, we have $\lambda^{\times n}(T_n(q \cdot U)) > 0$ for some q.

Step 4: Let $n \gg m$ with $n = km + \ell$ with $\ell \in \{0, \dots, m-1\}$. Suppose $p \in M^n$. Then

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\varphi(p_i) &= \frac{1}{n}\left(\sum_{i=1}^{n}\varphi(p_i) + \sum_{i=m+1}^{2m}\varphi(p_i) + \cdots + \sum_{i=(k-1)m+1}^{km}\varphi(p_i) + \sum_{i=km+1}^{n}\varphi(p_i)\right) \\
&= \frac{m}{n}\left(\frac{1}{m}\sum_{i=1}^{n}\varphi(p_i) + \sum_{i=m+1}^{2m}\varphi(p_i) + \cdots + \frac{1}{m}\sum_{i=(k-1)m+1}^{km}\varphi(p_i)\right) \\
&\qquad + \frac{\ell}{n}\cdot\underbrace{\frac{1}{\ell}\sum_{i=km+1}^{n}\varphi(p_i)}_{*}.
\end{aligned}$$

For each of these k terms, we have positive measure for the event that $\frac{1}{m}\sum_{i=*}^{*+m}\varphi(p_i) \in \frac{r}{r+1}U$. Hence, we have positive measure that $\frac{1}{k}(\frac{1}{m}\sum_{i=1}^{m}\varphi(p_i)+\cdots+\frac{1}{m}\sum_{i=(k-1)m+1}^{km}\varphi(p_i)) \in \frac{r}{r+1}U$ (and we can even replace this by $\frac{mk}{n}$ times this). By step 3, we have positive measure that $* \in q \cdot U$ for some q independent of n and hence $\frac{\ell}{n} \cdot * \in \frac{q^{\ell}}{n}U$. If all of these positive measure events occur, then

$$\frac{1}{n}\sum_{i=1}^{n}\varphi(p_i) \in \frac{r}{r+1}\cdot U + \frac{q^{\ell}}{n}U.$$

Provided $n \geq q \cdot \ell \cdot (r+1)$, this implies

$$\frac{1}{n}\sum_{i=1}^{n}\varphi(p_i) \in \frac{r}{r+1}U + \frac{1}{r+1}U = U.$$

Hence, $s_n(U) > -\infty$ for this n. $\square$

Remark 1.1. It is also possible that $\lambda^{\times n}(T_n(U) = +\infty$, so $s_n(U) = +\infty$, and we may get $s(U) = +\infty$. Fekete's lemma still works, but the result is not meaningful. You usually want to look for additional reasons of why s is locally finite. The simplest condition is that if $\lambda(M) < \infty$, then $\lambda^{\times n}(T_n(U)) \leq \lambda(M)^n$ for all n.

1.3 Checking conditions to extend s to a point function

Next, we want to switch to point functions $s(x) = \inf\{s(U) : U \in \mathcal{U}, U \ni x\}$.

Proposition 1.1. *Under the same conditions as before, s is concave.*

Proof. $T_{n+m}(U) \supseteq T_n(U) \times T_m(U)$. Similarly, let $x \in T_n(U)$ and $y \in T_m(V)$ (where $U, V \in \mathcal{U}$). Then the concatenation $z = xy$ satisfies

$$\frac{1}{2n}\sum_{i=1}^{2n} \varphi(z_i) = \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i) + \frac{1}{n}\sum_{i=1}^{n}\varphi(y_i)\right) \in \frac{1}{2}U + \frac{1}{2}V.$$

So $T_{2n}(\frac{1}{2}U + \frac{1}{2}V) \supseteq T_n(U) \times T_n(V)$, which tells us that

$$\frac{s_{2n}(\frac{1}{2}U + \frac{1}{2}V)}{2n} \geq \frac{1}{2}\left(\frac{s_n(U)}{n} + \frac{s_n(V)}{n}\right).$$

After letting $n \to \infty$, we get

$$s\left(\frac{1}{2}U + \frac{1}{2}V\right) \geq \frac{1}{2}(s(U) + s(V)).$$

By a previous lemma (the argument with dyadic rationals and applying upper semicontinuity), this gives that the point function $s(x)$ is concave. $\square$

Next, we quickly check that condition (S1) holds: If $U \subseteq U_1 \cup \dots \cup U_k$, then $T_n(U) \subseteq T_n(U_1) \cup \dots \cup T_n(U_k)$.. Using subadditivity and taking logs, we get

$$\frac{s_n(U)}{n} \leq \frac{\log K}{n} + \max_i \frac{s_n(U_i)}{n},$$

which gives

$$s(U) \leq \max_i s(U_i).$$

We also need conditions under which we can check (S2): $s(U) = \sup\{s(K) : K \subseteq U, K \text{ compact}\}$, where $s(K) = \inf\{\max_i s(U_i) : K = U_1 \cup \dots \cup U_k, U_i \in \mathcal{U}\} = \sup_{x \in K} s(x)$ (by a previous lemma). To deduce (S2) in the setting of generalized type-counting, we need to assume:

Every open convex set U can be written as a countable union of compact, convex sets.

Example 1.1. In $\mathbb{R}^d$, by intersecting with balls, we can write every U as a countable union of bounded, open, convex sets, and then we can express each of these as a countable union of compact convex sets by looking at the set of points under a certain distance from the boundary.

Example 1.2. If $X = Y^*$ with the weak*-topology, this property also holds, but we will show this later.