# Math 210A Lecture 16 Notes

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## 1 Sylow Theorems

### 1.1 Sylow $p$-subgroups

For this lecture, we will assume that a $p$-group is finite and of order $p^k$. Let $G$ be a finite group. Take $p \mid |G|$ and say that $p \parallel |G|$ if $p^n \mid |G|$ but $p^1 \nmid |G|$.

**Definition 1.1.** A **$p$-subgroup** of $G$ is a subgroup of order $p^K$ for some $k \leq n$.

**Definition 1.2.** A **Sylow $p$-subgroup** of $G$ is a $p$-subgroup of $G$ which is not properly contained in any other $p$-subgroup.

**Example 1.1.** The symmetric group $S_5$ has order $120 = 2^3 \cdot 3 \cdot 5$. For $p = 5$, a Sylow 5-subgroup will look like $\langle (a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5) \rangle$. There are $6 = 4!/4$ of these, For $p = 3$, a Sylow 3-subgroup will look like $\langle (a_1 \quad a_2 \quad a_3) \rangle$. There are 10 of these. For $p = 2$, a Sylow 2-subgroup will look like $\langle (a_1 \quad a_2 \quad a_3 \quad a_4), (a_1 \quad a_3) \rangle$. There are 15 of these.

Observe that the number of each type of Sylow $p$-subgroup divides the order of the group. In general, this is unusual.

### 1.2 Sylow theorems

Let $n_p(G)$ be the number of $p$-Sylow subgroups of $G$, and let $\mathrm{Syl}_p(G)$ be the set of Sylow $p$-subgroups of $G$. Our goal will be to prove the following.

**Theorem 1.1** (Sylow theorems)**.** *Let $G$ be a finite group.*

1. *Every Sylow $p$-subgroup of $G$ has order $p^n$, where $p^n \parallel |G|$.*
2. *Any two Sylow $p$-subgroups are conjugate.*
3. $n_p(G) \mid |G|$, *and* $n_p(G) \equiv 1 \pmod{p}$.

Recall that if $P$ is a $P$-group, $X$ is a finite set, and $P \circlearrowright X$, then $|X| \equiv |X^p| \pmod{p}$.

**Lemma 1.1.** *Let $G$ be finite, and let $H$ be a $p$-subgroup of $G$. Then*

$$[G : H] \equiv [N_G(H) : H] \pmod{p}.$$

*Proof.* Let $L = G/H$ be the set of right cosets of $H$. Then $|L| = [G : H]$. $H \circlearrowright L$ by $h \cdot (aH) = (ha)H$. If $aH \in L^H$, then for all $h \in H$, $haH = aH$, which means that $a^{-1}haH = H$, which is the same thing as $a^{-1}ha \in H$ for all $h \in H$. $\square$

**Theorem 1.2.** *If $H \leq G$, and $|H| = p^k$ for $k < n$, then there is some $P \leq G$ with $H \trianglelefteq P$ and $|P| = p^{k+1}$.*

*Proof.* If $|H| \neq P^n$, then $p \mid [G : H]$, so $p \mid [N_G(H) : H] = |N_G(H)/H|$. So $N_G(H)/H$ has a subgroup $P/H$ of order $p$. Then $P \leq N_G(H)$, and $|P| = P^{k+1} = |P/H||H|$. So $H \trianglelefteq P$. $\square$

This proves the first Sylow theorem. Let's prove the second theorem.

*Proof.* Take $P, Q \in \mathrm{Sly}_p(G)$. We know that $|P| = |Q| = p^n$. Let $Q \circlearrowright G/P$. Since $p \nmid |G/P|$, $p \nmid |(G/P)^Q|$. So $(G/P)^Q \neq \varnothing$, and we get some $xP$ such that $qxP = xP$ for all $q \in Q$. This means that $(x^{-1}qx)P = P$, so $x^{-1}qx \in P$ for all $q \in Q$. So $x^{-1}Qx \subseteq P$. Since $P$ and $x^{-1}Qx$ have the same order, $x^{-1}Qx = P$. $\square$

Now let's prove the third Sylow theorem.

*Proof.* Let $G \circlearrowright \mathrm{Syl}_p(G)$ by conjugation. By the second Sylow theorem, this action is transitive. Let $P$ be a Sylow $p$-subgroup of $G$. By orbit-stabilizer,

$$n_p(G) = |\,\mathrm{Syl}_p(G)| = [G : \mathrm{Stab}(P)] = [G : N_G(P)].$$

We have that

$$[G : P] = [G : N_G(P)][N_P(G) : P]$$

and

$$[G : P] \equiv [N_G(P) : P] \not\equiv 0 \pmod{p},$$

so

$$[G : N_G(P)] \equiv 1 \pmod{p}.$$ $\square$

**Example 1.2.** Let $|G| = 42$. We will show that $G$ has a nontrivial normal subgroup. $n_7(G) \mid 42$ and $7 \nmid n_7(G)$, so $n_7(G) \mid 6$. So $n_7(G) = 1$. So if $|H| = 7$, then $H \trianglelefteq G$.

**Example 1.3.** Let $|G| = 30$. We show that $G$ has a nontrivial normal subgroup. Then $G$ has 9 nontrivial normal subgroups. $n_5(G) \mid 30$, so $n_5(G) \mid 6$. Then $n_5(G) = 1$ or 6. Similarly, $n_3(G) \mid 10$, so $n_3(G) = 1$ or 10. Assume that $n_5(G), n_3(G) > 1$. Then we have 6 5-subgroups. Each one has 4 elements of order 5. So there are 24 elements of order 5. If $n_3(G) = 10$, there are 20 different elements of order 3. This is impossible because $24 + 20 > 30$.