Math 259A Lecture 12 Notes

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1 Spectral Scales

1.1 Spectral scales

Last time, we stated the following lemma.

Lemma 1.1. *Let $x = x^* \in \mathcal{B}(H)_h$ and let $f_n, g_n \geq 0$ be increasing sequences of continuous functions on $\operatorname{Spec}(x)$ that are both uniformly bounded. If $\sup_n f_n(t) \leq \sup_n g_n(t)$ for all $t \in \operatorname{Spec}(x)$, then $\sup_n f_n(x) \leq \sup_n g_n(x)$.*

Proof. We will prove that for any fixed n and $\varepsilon > 0$, there exists an m_n such that $f_n - \varepsilon \leq g_{m_n}$; this will complete the proof because then $\sup_m g_m(x) \geq f_n(x) - \varepsilon$ for all n and $\varepsilon > 0$.

If $t \in \operatorname{Spec}(x)$, then $f_n(t) - \varepsilon < f_n(t) < \sup_n f_n(t) \leq \sup_m g_m(t)$. So there exists m_n such that $g_{m_n}(t) \geq f_n(t) - \varepsilon$. So there is a neighorhoof V_t of t such that $f_n(s) - \varepsilon < g_{m_n}(s)$ for all $s \in V_t$. By the compactness of $\operatorname{Spec}(x)$, there exist $V_{t_1}, \dots, V_{t_k}$ covering $\operatorname{Spec}(x)$ and corresponding $m_{n_1}, \dots, m_{n_k}$. If we let $m_n := \max\{m_{n_j} : 1 \leq j \leq k\}$, then $f_n(s) - \varepsilon < g_{n_m}(s)$ for all $s \in \operatorname{Spec}(x)$. $\square$

Corollary 1.1. *The* ***spectral scales*** $e_{(-\infty,t]}(x) := \sup\{f(x) : f \in C(\operatorname{Spec}(x)), f \leq \mathbb{1}_{(-\infty,t)}\}$ *are well-defined.*

We have that $\mathbb{1}_{(-\infty,t]}(x) = \bigwedge_{s>t} e_s(x)$. If $Y \subseteq \operatorname{Spec}(x)$ is Borel, then $e_Y = \mathbb{1}_Y(x)$ exists and be called **spectral projections**. These are all contained in the von Neumann algebra generated by x.

Proposition 1.1. *Let $e_{[t,\infty)} = 1 - e_t$. Then*

1. $e_{[\alpha,\beta)} = e_{(-\infty,\alpha)} e_{[\beta,\infty)} = e_\alpha(1 - e_\beta)$.
2. $e_{Y_1} e_{Y_2} = e_{Y_1 \cap Y_2}$.
3. $e_{Y_1} \vee e_{Y_2} = e_{Y_1 \cup Y_2}$.
4. $x e_t \leq t e_t$, *and* $x(1 - e_t) \geq t(1 - e_t)$.

Corollary 1.2. *If $t \leq s$, then*

$$t(e_s - e_t) \leq x(e_s - e_t) \leq s(e_s - e_t).$$

Let $m = \inf \operatorname{Spec}(x)$ and $M = \sup \operatorname{Spec}(x)$. Given a partition $m = t_0 < t_1 < \cdots < t_n = M$, we can construct **Riemann-Darboux sums**

$$s(\Delta) = \sum_{i=1}^{n} t - i - 1(e_{t_i} - e_{t_{i-1}}), \qquad S(\Delta) = \sum_{i=1}^{n} t_i(e_{t_i} - e_{t_{i-1}})$$

If the mesh size is $< \varepsilon$, then $\|S(\Delta) - s(\Delta)\| < \varepsilon$. Then we can define the **vector valued Stieltjes integral** which satisfies

$$x = \int_{-\infty}^{\infty} \lambda \, de_\lambda.$$

Remark 1.1. This integral takes values in $\mathcal{B}(H)$. Since $\operatorname{Spec}(x) \subseteq [m, M]$, this is really an integral over a compact set. The convergence is convergence in norm.

Corollary 1.3. *If $x \in (\mathcal{B}(H)_+)_1$, then there exist projections $\{p_n\}_n$ in $\mathcal{A}$, the von Neumann algebra generated by x, such that $x = \sum_{n \geq 1} 2^{-n} p_n$.*

This is called the **dyadic decomposition** of x.

Proof. Define the projections recursively: Start with $p_1 = e_{[1/2,1)}(x)$, so $\|x - \frac{1}{2}p_1\| \leq 1/2$. Then take $p_2 = e_{[1/4,1/2]}(1 - \frac{1}{2}p_1)$ to be this projection applied to the previous result. Continuing like this, we get all the projections. $\square$

So the von Neumann algebra is generated by x is generated by these projections, as well.

1.2 Cyclic and separating vectors

Definition 1.1. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **cyclic vector** of M if $\overline{M\xi} = H$.

Definition 1.2. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **separating vector** of M if when $x \in M$ satisfies $x\xi = 0$, $x = 0$.

Proposition 1.2. *Let $M = \mathcal{A}$ be an abelian von Neumann algebra. Then ξ is separating if and only if it is cyclic.*