Math 206B Lecture Notes

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1 Teaser of Course Topics

This lecture will be an advertisement for the topics in the course.[1]

1.1 Combinatorics of S_n and applications

We denote $[n] := \{1, \dots, n\}$.

Definition 1.1. The **symmetric group** S_n is the group of bijections $\sigma : [n] \to [n]$.

We can write permutations as products of cycles.

Example 1.1. The permutation $\sigma = \begin{pmatrix} 5 & 1 & 2 & 4 & 3 \end{pmatrix}$ represents the bijection sending $1 \mapsto 5$, $2 \mapsto 1$, etc.

The conjugacy classes of S_n are the different cycle types. These correspond to partitions of n. Let $p(n)$ be the number of conjugacy classes of S_n. Euler showed that

$$1 + \sum_{n=1}^{\infty} p(n) t^n = \prod_{i=1}^{\infty} \frac{1}{1 - t^i}.$$

What does this all have to do with the symmetric group itself? Here is Percy MacMahon's version of the story. If we have a partition, we can think of it as a sequence of numbers, padded with zeros at the end to make it infinite. We can write

$$\sum_{\lambda \in \mathcal{P}} t^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1}{1 - t^i}.$$

If we write the partition over and over in a grid (chopping off an element from the front each time, we can get what is called a **plane partition**. This gives us

$$\sum_{A \in \mathcal{PP}} t^{|A|} = \prod_{i=1}^{\infty} \frac{1}{(1 - t^i)^i},$$

where $|A| = \sum_{i,j} a_{i,j}$ is the sum of the numbers in the plane partition. Why should this be true? This is actually related to the representation theory of S_n.

MacMahon made a conjecture for higher dimensions:

Theorem 1.1.

$$\sum_{A \in \mathcal{P}^{(d)}} t^{|A|} = \prod \frac{1}{(1 - t^i)^{(i_{d-1})}}.$$

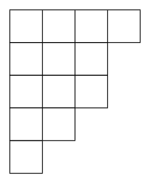
[1] This is an advertisement of an advertisement, much like the ad before watching a trailer online.

This does not work. It actually fails for $d = 3$ and a low coefficient like the coefficient of t^7. MacMahon did not understand why the formula was true, even though he proved it. Irreducible representations of S_n will correspond to partitions of n $\mathcal{P}_n := \{\lambda \in \mathcal{P} : |\lambda| = n\}$ because these correspond to conjugacy classes of S_n.

Theorem 1.2 (A. Young, 1897)**.** *Let f^λ be the dimension of S^λ. Then f^λ is the number of standard Young tableau with shape λ.*

Definition 1.2. Given a partition λ, the *young diagram* of λ is the partition expressed as stacked rows of boxes.

Example 1.2. Take $\lambda = (4, 3, 3, 2, 1)$. The Young diagram of λ is



Definition 1.3. A **Young tableau** is a Young diagram where we fill in the boxes with the numbers 1 to n, according to the rule that the numbers have to be increasing going to the right and going down.

Example 1.3. Here is a Young tableau:

1	2	3	7
4	5	10	
6	8		
9	12		
11			

Theorem 1.3 (FRT, c.1960)**.** $f^\lambda = \frac{n!}{\prod_{i,j} h_{i,j}}$, *where $h_{i,j}$ is the length of the hook starting from position i, j and going to the right and downwards.*

From representation theory, we can get that $f^\lambda \mid |S_n|$, so we know that f^λ is $n!$ divided by something. The magic is in what that something is.

1.2 Representation theory of $\mathrm{GL}_n(\mathbb{C})$

A basic representation of $\mathrm{GL}_n(\mathbb{C})$ is ρ_M is the matrix M acting on $\mathbb{C}^n$. Another representation is the determinant map. We will find that representations of $\mathrm{GL}_n(\mathbb{C})$ correspond to sequences $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Theorem 1.4 (Weyl)**.** $\dim(\rho_\lambda)$ *equals the number of semistandard Young tableau with shape λ.*

This will be another remarkable product formula. This, paired with the hook-length formula, will pave the way for a nice proof of MacMahon's formula.

1.3 Young graph

Definition 1.4. The **Young graph** is the graph with vertices $\lambda \in \Gamma$, and edges $(\lambda, \mu$ where $\mu \setminus \lambda$ is a single difference.

Essentially, we have taken all young diagrams and made an undirected graph, partially ordering them by containment.

Theorem 1.5. *The number of loops of length* $2n$ *(that do not zigzag up and down) in the Young graph is* $n!$.

Proof. We can prove this using basic representation theory.

$$\#\text{ loops} = \sum_{\lambda \in \mathcal{P}_n} (\#\text{ paths } \phi \to \lambda)^2 = \sum_{\lambda \in \mathcal{P}_n} \mathrm{SYT}(\lambda)^2 = \sum_{\lambda \in \mathcal{P}_n} (f^\lambda)^2 = |S_n| = n! \qquad \square$$

What about general loops?

Theorem 1.6. *The number of general loops is* $(2n-1)!!$.

2 Representation Theory of Finite Groups

This is meant to remind you of basic results in representation theory. Good references are Chapter 1 of Sagan's book and the book by Fulton and Harris.[2]

2.1 Conjugacy classes and characters

Let G be a finite group. There is an action of G on G in the following way: $a \cdot g = aga^{-1}$. This is the action of conjugation, where conjugacy classes are the orbits of the action. In other words, $g \sim h$ if $g = aha^{-1}$ for some $a \in G$; conjugacy classes are equivalence classes under this relation. We will denote $c(G)$ as the number of conjugacy classes of G.

Example 2.1. Let $G = \mathbb{Z}_n$ be the cyclic group of order n. Then $c(\mathbb{Z}_n) = n$.

Example 2.2. Let $G = S_n$. Then $c(S_n) = p(n)$, the number of integer partitions of n.

Definition 2.1. A **character** $\chi_1 : G \to \mathbb{C}$ is a function such that $\chi(g) = \chi(h)$ whenever $g \sim h$.

Example 2.3. The trivial character is $\chi(g) = 1$ for all g.

Example 2.4. Let $G = S_n$. The sign character is $\chi(\sigma) = \mathrm{sign}(\sigma)$.

Theorem 2.1. $\dim(\mathrm{span}(\textit{characters } \chi) = c(G)$.

Definition 2.2. The **inner product** on characters is defined as

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

Example 2.5. If $n \geq 2$, then

$$\langle \chi, \mathrm{sign} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \cdot 1 = \frac{1}{n!} \left(\frac{n!}{2} - \frac{n!}{2} \right) = 0.$$

Example 2.6. Suppose $G = \mathbb{Z}_n$. Let $\omega = e^{2\pi i/n}$. For each j, we have a character $\chi_j(k) = \omega^{jk}$.

2.2 Linear representaitons

Let $V = \mathbb{C}^d$. The group $\mathrm{GL}(V) = \mathrm{GL}_d(\mathbb{C})$ is the group of automorphisms of V.

Definition 2.3. A **representation** is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

[2]This is Professor Pak's opinion. I hate this book.

Here are operations we can do on representations:

1. If we have $\rho : G \to \mathrm{GL}(V), \pi : G \to \mathrm{GL}(W)$, then we can form $\rho \oplus \pi : G \to \mathrm{GL}(V \oplus W)$ by acting on the individual parts of the vector space by the respective actions.

2. We can form $\rho \otimes \pi : G \to \mathrm{GL}(V \otimes W)$. The dimension of $\rho \otimes \pi$ is $\dim(V)\dim(W)$.

3. Reduced representations: If we have $\rho : G \to \mathrm{GL}(V)$ and $H \leq G$, we can define $\rho \downarrow_H^G$ as the restriction of ρ to H.

4. Induced representations: If we have $\pi : H \to GL(W)$, there is an induced representation $\pi \uparrow_H^G : G \to \mathrm{GL}(W^{\otimes h})$, where $h := [G : H] = |G|/|H|$.

Example 2.7. The **trivial representation** maps $g \mapsto \mathrm{id}_V$ for all $g \in G$.

Example 2.8. The **regular representation** $\pi : G \to \mathrm{GL}(\mathbb{C}^{|G|})$ acts on a basis indexed by all $g \in G$ by $a \cdot v_g = v_{ag}$.

Example 2.9. The **natural representation** $\rho : S_n \to \mathrm{GL}_n(\mathbb{C})$ sends σ to its permutation matrix (applying the permutation to the basis vectors $\{e_1, \dots, e_n\}$).

If we have a representation $\rho : G\, to\, \mathrm{GL}(V)$, we can define its character $\chi_r ho$ by $\chi_\rho(g) = \mathrm{tr}([\rho(g)])$. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$. So χ_ρ is in fact a character in the previous sense.

Remark 2.1. Even though $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, then $\mathrm{tr}(ABC) \neq \mathrm{tr}(CBA)$.

We also have $\mathrm{tr}(A+B) = \mathrm{tr}(A)+\mathrm{tr}(B)$. However, we do not have $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

Example 2.10. Let π be the regular representation of G. Then

$$\chi_\pi(\sigma) = \begin{cases} n! & \sigma = 1 \\ 0 & \sigma \neq 1. \end{cases}$$

Example 2.11. Let ρ be the natural representation of S_n. Then $\chi_\rho(\sigma)$ is the number of fixed points of σ.

Example 2.12. Here is an example of a reduced representation. Let $G = S_n$, and $H = \mathbb{Z}_n$. Let ρ be the natural representation of S_n. Then $\rho \downarrow_{\mathbb{Z}_n}^{S_n}$ is the regular representation of $\mathbb{Z}_n$.

3 Irreducible Representations and Characters

3.1 Irreducible representations

Recall that if π, ρ are representations of G, then $\pi \oplus \rho$ is well-defined.

Definition 3.1. A representation ρ is **irreducible** if $\rho \neq \pi \oplus \pi'$.

Theorem 3.1. *Let G be a finite group. There exists a finite number of irreducible representations $\rho_0, \rho_1, \dots$ such that every $\pi = \bigoplus m_i \rho_i$ for some coefficients m_i. Moreover, $m_i = \langle \chi_\pi, \chi_{\rho_i} \rangle$.*

In other words, $\{\chi_{\rho_i}\}$ is an orthonormal basis in the space of functions $f : G \to \mathbb{C}$ that are constant on conjugacy classes.

Remark 3.1. The decomposition $\pi = \bigoplus m_i \rho_i$ is non-unique. G can act on $V = \mathbb{C}^d$ trivially. This representation is $\pi = \bigoplus d\rho_0$. Any basis will work. When all m_i are 0 or 1, this decomposition is in fact unique.

Example 3.1. Let π be the regular representation of G. Let χ_i be the character of an irreducible representation. Then $m_i = \langle \chi_\pi, \chi_i \rangle = |G|^{-1} \sum_{g \in G} \chi_\pi(g) \overline{\chi_i(g)} = |G|^{-1} \chi_\pi(1) \chi_i(1) = |G|^{-1} |G| \dim(\rho_i)$.

3.2 Irreducible characters and character tables

Let $\chi_0, \dots, \chi_{c-1}$ be the characters of the irreducible representations of G, where $c = c(G)$. Then $\langle \chi_i, \chi_j \rangle = \delta_{i,j}$. Let $d_i = \chi_i(1) = \dim(\rho_i)$.

Theorem 3.2. $\sum_i d_i^2 = |G|$.

Theorem 3.3. *$d_i \mid |G|$ for all i.*

Theorem 3.4. *All χ_i are real iff $C^{-1} = C$ for all conjugacy classes C of G.*

Let's calculate character tables of irreducible representations of S_n.

Example 3.2.

S_2	id	$(1\,2)$
χ_0	1	1
χ_1	1	-1

For every n, there is the character $\chi_{\text{sign}}(\sigma) = (-1)^{\text{inv}(\sigma)}$, where $\text{inv}(\sigma)$ is the number of the inversions.

Proposition 3.1. $\text{inv}(\sigma\tau) \cong \text{inv}(\sigma) + \text{inv}(\tau) \pmod 2$..

Here is the character table for S_3:

Example 3.3.

S_3	id	(1 2)	(1 2 3)
χ_0	1	1	1
χ_{sign}	1	-1	1
χ_1	2	0	-1

How do we find the values for χ_1? The first value is the dimension of the representation. Since $\chi_0 + \chi_{\text{sign}} + 2\chi_1 = \chi_\pi$, we can figure out the rest of the values.

What is the representation corresponding to the character χ_1? We can calculate $\langle \chi_0, \chi_{\text{nat}} \rangle = \dim(\rho^G)$ or, combinatorially, $= |G|^{-1} \sum_{\sigma \in S_n}$ fixed pts$(\sigma) \cdot 1 = \frac{1}{n!} \sum_{i=1}^{n} \sum_{\sigma(i)=i} 1 = \frac{1}{n!} n(n-1)! = 1$. Then $\chi_1 = \chi_{\text{nat}} - \chi_0$.

Example 3.4.

S_4	id	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
χ_0	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_1	3	1	0	-1	-1
χ_1'	3	-1	0	-1	1
χ_2	2	0	-1	2	0

We can figure out χ_1 as the number of fixed points minus 1. Here, $\chi_1' = \chi_1 \cdot \chi_{\text{sign}}$. We can figure out χ_2 using the regular representation, as before.

4 Characters of S_n

4.1 Induced M^μ representations and μ-flags

Last time, we found a character of S_4, but we didn't quite know what representation it corresponded to. Let's try to understand this a little better.

Definition 4.1. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of n. Define

$$M^\mu = \operatorname{ind}_{S_{\mu_1} \times \cdots \times S_{\mu_k}}^{S_n} 1.$$

Example 4.1. Let $\mu = (n-1, 1)$. then $M^{n-1,1} = \operatorname{ind}_{S_{n-1}}^{S_n} 1$. $\dim(M^{n-1,1}) = n$. This is the natural representation.

Definition 4.2. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of n. A **μ-flag** on $[n]$ is $\varnothing \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq [n]$ such that $|A_1| + \mu_1$, $|A_2| = \mu_1 + \mu_2$, and so on.

Example 4.2. Let $\mu = (n-k, k)$, where $1 \leq k \leq n/2$. Then μ-flags are in correspondence with $(n-k)$ subsets of $[n]$: $\varnothing \subseteq A_1 \subseteq [n]$.

Example 4.3. Let $\mu = (1^n)$. μ-flags are in correspondence with S_n, where $\sigma \mapsto A_1 \subseteq \cdots \subseteq A_n$, and $A_i = \{\sigma(1), \sigma(2), \ldots \sigma(i)\}$.

4.2 Structure of the M^μ representations of S_n

Definition 4.3. Let G be a finite group, and let X be a finite set. Let $G \circlearrowright X$. Then there is a **permutation representation** $\varphi : G \to S_X$.

Equivalently, there is a permutation representation $\rho_\varphi : G \to \mathrm{GL}(V)$ over $\mathbb{C}$, where $V = \mathbb{C}\langle x \rangle$.

Proposition 4.1. *M^μ is a permutation representation of S_n on μ-flags of $[n] = \{1, \ldots, n\}$.*

Example 4.4. Let $\mu = (2^2)$. Then M^μ is the permutation representation of S_n on 2-subsets of [4]. $\dim(M^{2^2}) = \binom{4}{2} = 6$. We claim that $M^{2^2} = S^{(4)} \oplus S^{(3,1)} \oplus S^{(2^2)}$, where these refer to the irreducible representations of S_4. Let's calculate $\chi_{M^{2^2}}$:

λ	1^4	$2\,1^2$	2^2	$3\,1$	4
$\chi_{M^{2^2}}$	6	2	2	0	0
z_λ	1	6	3	8	6

Proposition 4.2. *Let $\nu = (\nu_1, \nu_2, \ldots) = 1^{m_1(\nu)} 2^{m_2(\nu)} \cdots$ be a partition of n, where m_i is the number of is in ν. Then*

$$z_\nu = \frac{n!}{(m_1! 1^{m_1})(m_2! 2^{m_2}) \cdots}.$$

Theorem 4.1. $M^\mu = \bigoplus_\lambda m_{\mu,\lambda} S^\lambda$, *where*

$$m_{\mu,\lambda} := \langle M^\mu, S^\lambda \rangle = \frac{1}{n!} \sum_\nu z_\nu \chi_{M^\mu}[\nu] \chi_{S^\lambda}[\nu].$$

Theorem 4.2. *The matrix* $[m_{\mu,\lambda}]$ *has nonnegative integer entries and is upper triangular with 1s on the diagonal, where* $\lambda \leq \mu$ *is the lexicographic order.*

5 Tableau

5.1 Young diagrams and tableau

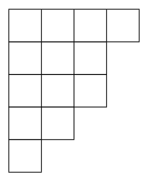
We want to work toward the following theorem:

Theorem 5.1. *$M^\mu = \bigoplus_{\lambda \geq \mu} m_{\mu,\lambda} S^\lambda$, where*

$$m_{\mu,\lambda} := \langle M^\mu, S^\lambda \rangle = \frac{1}{n!} \sum_{\nu} z_\nu \chi_{M^\mu}[\nu] \chi_{S^\lambda}[\nu].$$

Definition 5.1. Given a partition λ, the **Young diagram** of λ is the partition expressed as stacked rows of boxes.

Example 5.1. Take $\lambda = (4, 3, 3, 2, 1)$. The Young diagram of λ is



Definition 5.2. A **standard Young tableau**[3] is a Young diagram where we fill in the boxes with the numbers 1 to n, according to the rule that the numbers have to be increasing going to the right and going down.

Example 5.2. Here is a Young tableau:

1	2	3	7
4	5	10	
6	8		
9	12		
11			

5.2 Poly-tabloids and their relation to irreducible representations of S_n

Definition 5.3. A **tabloid** is a Young diagram filled with the numbers $\{1, \ldots, n\}$. A **standard tabloid** is the same thing, except you need that numbers are increasing going to the right.

Example 5.3. Here is a tabloid:

3	5	6	8	10	11
1	2	7	13		
4	12				

[3]The plural of tableau is tableaux.

Observe that λ-tabloids are in bijection with λ-flags.

Definition 5.4. A **poly-tabloid** is an equivalence class of tabloids under the action of $S_{\lambda_1} \times S_{\lambda_2} \times \cdots$.

Let $R^\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$ be the group of row permutations on λ-tabloids, and let $C^\lambda = S_{\lambda'_1} \times S_{\lambda'_2} \times \cdots$ be the group of column permutations on λ-tabloids.

Let

$$\mathcal{X}_\lambda = \sum_{\sigma \in C^\lambda} \operatorname{sign}(\sigma)\sigma$$

be an element of the group algebra $\mathbb{C}[S_n]$.

Example 5.4. Let $\lambda = (3\,2)$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

Then $\mathcal{X}_\lambda = (1 - (1\,4))(1 - (2\,5))$. where 1 is the identity in the group algebra.

We can also think of M^λ as S_n acting on $\mathbb{C}\langle \lambda - \text{poly-tabloids}\rangle$. Define $e_t := \mathcal{X}_t\{t\}$, where $\mathcal{X}_t$ is the projection of $\mathcal{X}$ onto $\mathbb{C}\langle\{t\}\rangle$.

Example 5.5. Suppose

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

Then

$$e_t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 4 & 5 & 2 \\ \hline 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 3 & 5 & 2 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

Theorem 5.2. *The S_n action on e_t for t a λ-tabloid is an irreducible representation S^λ.*

What is the idea here? Start with λ, and construct a λ-tabloid t. Now $\mathcal{X}_t$ is the element of $\mathbb{C}[S_n]$ corresponding to t. Then $e_t = \mathcal{X}_t\{t\}$ is a linear combination of poly-tabloids. If we let $W_\lambda = \mathbb{C}\langle\{t\}\rangle$ be the linear span of poly-tabloids (M^λ acts on W_λ), then $\sigma \cdot e_t \in E_\lambda$. Now we can think of $\mathbb{C}\langle\{\sigma e_t : \sigma \in S_n\}\rangle$. The claim is that this is isomorphic to S^λ. That is, if we define S^λ like this, then the claim is that these are all irreducible and distinct.

6 Examples, Dimension, and Irreducibility of S^λ

6.1 Examples of S^λ

Let t be a tabloid, and consider $\mathbb{C}\langle \sigma \cdot e_t : \sigma \in S_n \rangle = S^\lambda$.

Example 6.1. Let $\lambda = (n)$. There is a single poly-tabloid,

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n \\ \hline \end{array}$$

S_n acts trivially on this, so $S^{(n)} = 1$.

Example 6.2. Let $\lambda = (1^n)$. The poly-tabloids correspond to permutations:

$$\sigma \mapsto \begin{array}{|c|}\hline \sigma(1) \\ \hline \sigma(2) \\ \hline \vdots \\ \hline \sigma(n) \\ \hline \end{array}$$

Let $\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\sigma = \mathcal{X} \in \mathbb{C}[S_n]$. Then $\pi \cdot \mathcal{X} = \operatorname{sign}(\pi)\mathcal{X}$.

Example 6.3. Let $\nu_k = (n-k, 1^k)$. Then $M^{\nu_k} = \operatorname{ind}_{S_{n-k} \times 1 \times \cdots \times 1}^{S_n} 1$. The dimension is $\dim(M^{\nu_k}) = n!/(n-k)!$. What is S^{ν_k}? Permutations do not act on the columns, but they permute the elements in the first column. Let $v = \sum_{\sigma \in S_{k+1}} \operatorname{sign}(\sigma)\sigma$. What is $\pi \cdot v = w$? In general $\mathbb{C}\langle \pi e_t \rangle$ will be a representation of S_n.

Example 6.4. Let $\lambda = (2,2)$. Let

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Then

$$v = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 4 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

Then $(1\,2)v = v$ and $(3\,4)v = v$. We can also calculate

$$(3\,4)v = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} := w.$$

Then M^μ is a 2-dimensional vector space spanned by v, w.

6.2 Dimension and irreducibility of S^λ

Lemma 6.1. *Let S^λ be as above.*

1. $\dim(S^\lambda) = f^\lambda = \#\,\mathrm{SYT}(\lambda)$.
2. $n! \leq \sum_\lambda (f^\lambda)^2$, *where equality holds iff S^λ is irreducible for each λ.*

Definition 6.1. The **dominance order** on partitions is the partial order $\lambda \trianglelefteq \mu$ if $\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$ for all k.

We had that $M^\lambda = \bigoplus_{\mu \trianglelefteq \lambda} m_{\lambda,\mu} S^\mu$. The proof of the inequality in the 2nd statement comes from this fact.

How should we prove the equality in statement 2 of the lemma? Suppose G is a finite group. We have both a left and a right action of $G \circlearrowright \mathbb{C}[G]$. This gives us a $G \times G$ representation, $\bigoplus_\pi \pi \otimes \pi$, where the sum is over all irreducible representations. We want to view this equation as $n!$ being the order of the group, and the $(f^\lambda)^2$ terms being the tensor product of two irreducible representations. It is hard to do this in general, but it will work in the symmetric group. This is called the RSK correspondence.

7 The Robinson-Schensted Algorithm

7.1 Description of the algorithm

Recall Burnside's identity:

Proposition 7.1.

$$\sum_{\lambda}(f^{\lambda})^2 = n!,$$

where

$$f^{\lambda} = \dim(S^{\lambda}) = \chi^{\lambda}(1) = \#\,\mathrm{SYT}(\lambda).$$

We want a bijection between the symmetric group and the set of pairs (A, B), where A, B are $\mathrm{SYT}(\lambda)$ for some partition λ of n. We usually write the bijection Φ as $\sigma \mapsto (P, Q)$, where P is called the **insertion tableau** and Q is called the **recording tableau**.

Example 7.1. Let $\sigma = 4\ 2\ 7\ 3\ 6\ 1\ 5$ be a permutation. For each partial reading of the string representing σ, the algorithm will produce an outcome.

First, we start with the 4. Here are our two tableaux.

$$\begin{array}{|c|}\hline 4\\\hline\end{array}\qquad\qquad \begin{array}{|c|}\hline 1\\\hline\end{array}$$

Now let's add the 2. It can't go to the right of the 4, so it pushes the 4 down. We record the move into our right (recording) tableau:

$$\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\qquad\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$$

Now we add the 7. It is bigger than the 2, so it goes to the right.

$$\begin{array}{|c|c|}\hline 2 & 7\\\hline 4 \\\cline{1-1}\end{array}\qquad\qquad \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array}$$

Now we add the 3. It bumps down the smallest number in the first row that is bigger than it, the 7. When the 7 moves down, it does not need to bump the 4:

$$\begin{array}{|c|c|}\hline 2 & 3\\\hline 4 & 7\\\hline\end{array}\qquad\qquad \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline\end{array}$$

Now let's add in the 6.

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 6\\\hline 4 & 7 \\\cline{1-2}\end{array}\qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5\\\hline 2 & 4 \\\cline{1-2}\end{array}$$

What happens when we put in the 1? It is smaller than 2, so it bumps the 2 down. Doing the same process, the 2 is going to bump down the 4:

$$\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&7\\\cline{1-2} 4\\\cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

Finally, we can add in the 5:

$$P = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4&7\\\cline{1-2}\end{array} \qquad\qquad Q = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6&7\\\cline{1-2}\end{array}$$

Theorem 7.1 (R-S, 1961)**.** *The map* $\Phi : S_n \to \{(A, B) : A, B \in \mathrm{SYT}(\lambda), |\lambda| = n\}$ *is a bijection.*

Proof. Here are the steps:

1. The map Φ is well defined.
2. $P, Q \in \mathrm{SYT}(\lambda)$ for some partition λ of n.
3. Φ^{-1} is well-defined.

The first two are easy to convince yourself of. For the third, run the algorithm in reverse and see that it outputs the original permutation.[4] $\square$

7.2 Properties of the RS algorithm

This bijection is natural in some sense. It actually exhibits the following remarkable property!

Theorem 7.2. *Suppose* $\Phi(\sigma) = (P, Q)$*. Then* $\Phi(\sigma^{-1}) = (Q, P)$*.*

We will not prove this now, but we will prove it later in the course.

Theorem 7.3 (Schensted)**.** *Let* $\Phi(\sigma) = (P, Q)$*, where* $P, Q \in \mathrm{SYT}(\lambda)$*. Then* λ_1 *is the length of the longest increasing subsequence in* σ*.*

Proof. Proceed by induction. We claim that λ_1 of P_i is the longest increasing subsequence of $\sigma = \sigma_1 \ \cdots \ \sigma_i$, where P_i is the insertion tableau at the i-th step. Suppose we have $[a_1 \ \cdots \ a_r]\sigma_{i+1}$, where $a_1 \cdots a_r$ is the longest increasing subsequence in the permutation. If $\sigma_{i+1} > a_r$, then we must add a number to the first row because nothing gets bumped down. If $a_r > \sigma_{i+1}$, then something in the first row gets bumped down. $\square$

[4] Back in the 60s, people used to listen to songs backwards to find hidden messages. This is same same, except there is actually a message if you go backwards.

How does one come up with an algorithm like this? Schensted was a graduate student at either Berkeley or Stanford.[5] Schensted had a roommate, Floyd, who later became a famous computer scientist. They were interested in sorting things like solitaire (place the newest card on the heap with the smallest number). Schensted saw the above property and found it interesting. The rest of the story will have to wait until next time.

[5] Professor Pak doesn't remember which.

8 The RSK Algorithm

8.1 Semi-standard Young tablaeux

We will extend the RS-algorithm, which gave a bijection $\Phi : S_n \to \coprod_{|\lambda|=n} \mathrm{SYT}(\lambda)^2$. RSK will give a bijection $\Phi : M(\overline{a}, \overline{b}) \to \coprod_{|\lambda|=N} \mathrm{SSYT}(\lambda, \overline{a}) \times \mathrm{SSYT}(\lambda, \overline{b})$. Here, $\overline{a} = (a_1, \dots, a_n)$, $\overline{b} = (b_1, \dots, b_n)$, and $N = |\overline{a}| = |\overline{b}| = a_1 + \cdots + a_n = b_1, \dots, b_n$. $M(\overline{a}, \overline{b})$ is the set of $\mathbb{N}^+$ $n \times n$ matrices with row sums $a_1, \dots, a_n$ and column sums $b_1, \dots, b_n$. $\mathrm{SSYT}(\lambda, \overline{a})$ is the set of semi-standard Young tableaux of shape λ and weight $\overline{a}$.

Definition 8.1. A **semi-standard Young tableau** of shape λ is a Young tableau where we are allowed to have numbers reused, and we have numbers are weakly increasing as we go to the right. The **weight** of a semi-standard Young tableau is $(m_1, m_2, \dots)$, where m_i is the number of is in A.

Example 8.1. If $\overline{a} = \overline{e} = (1, \dots, 1)$, then $\mathrm{SSYT}(\lambda, \overline{a}) = \mathrm{SYT}(\lambda)$. Then $M(\overline{a}, \mathbf{a})$ is the number of 0-1 matrices with ros sums equal to 1 and column sums equal to 1. So the number of such matrices is $|S_n| = n!$. This special case is the case of R-S.

Example 8.2. Let $n = 2$, and $\overline{a} = (m, m) = \overline{b}$. Then $\# M(\overline{a}, \overline{b}) = m + 1$ because the entire matrix is determined by the upper left entry:

$$\begin{bmatrix} * & \\ & \end{bmatrix}$$

The right hand side is $\coprod_{|\lambda|=2m} \mathrm{SSYT}(\lambda, \overline{a})^2$. Note that $\lambda = (\lambda_1, \lambda_2)$. Otherwise, $\mathrm{SSYT}(\lambda, \overline{a}) = 0$. The number of such λ is $m + 1$:

1	1	$\cdots$	1	1	1
2	$\cdots$	2			

This case is very different from the R-S case.

8.2 Description of the algorithm

Given $M \in M(\overline{a}, \overline{b})$, we first need to turn M into a word.

Example 8.3. Here is how we turn a matrix into a word.

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 1 \end{bmatrix} \to 1\ 1\ 3 3\ 3\ 1\ 2\ 2\ 2\ 2\ 3\ 1\ 1\ 1\ 2\ 3$$

We have 2 1s, then 0 2s, then 3 3s. Then we have 1 1, 4 2s, and 1 3. Continue like this.

Now we will proceed by applying the R-S bumping procedure to this word.

Example 8.4. Start with the previous word. Our partial outputs are

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \cline{1-3} \end{array}$$

and so on.

This gives us an $A \in \mathrm{SSYT}(\lambda, \bar{b})$. How do we make our corresponding recording tableau? The numbers that come from row i on the matrix get recorded in our recording tableau as i. We fill in the shape of the insertion tableau as the shape of the insertion tableau evolves.

Example 8.5. Our partial outputs for the recording tableau are

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} \end{array}$$

and so on.

8.3 Properties and relationship to representation theory

Theorem 8.1 (Knuth, c. 1980)**.** *RSK is a bijection.*

Proof. Here is what we need to show:

1. Φ is well-defined.

2. Φ^{-1} is well-defined.

Like before, we can prove these step by step and induct on the number of steps. $\square$

Theorem 8.2. *If $\Phi(M) = (A, B)$, then $\Phi(M^\top) = (B, A)$.*

From representation theory, we had the decomposition $M^\mu = \bigoplus_{|\lambda|=n} m_{\lambda,\mu} S^\lambda$. Here, $M^\mu = \mathrm{ind}_{S_{\mu_1}\times S_{\mu_2}\times\cdots}^{S_n} 1$. Take $\overline{a} = (a_1, \ldots, a_k)$. We can define $M^{\overline{a}} = \mathrm{ind}_{S_{a_1}\times S_{a_2}\times\cdots}^{S_N} 1$, where $N = a_1 + \cdots + a_k$.

Theorem 8.3.

$$M^{\overline{a}} = \bigoplus_{|\lambda|=N} m_{\lambda,\overline{a}} S^\lambda,$$

where $m_{\lambda,\overline{a}} = \#\,\mathrm{SSYT}(\lambda, \overline{a})$.

Why should this be true?

$$\#M(\overline{a}, 1^N) = \frac{N!}{a_1!a_2!\cdots} = \dim(M^{\overline{a}}).$$

$$M(\overline{a}, 1^N) = \sum_{|\lambda|=N} \# \in SSYT(\lambda, \mu) \cdot \#\,\mathrm{SYT}(\lambda).$$

Now

$$M(\overline{a}, \overline{b}) = \langle \chi_{M^{\overline{a}}}, \chi_{M^{\overline{b}}} \rangle = \dim \mathrm{Hom}(M^{\overline{a}}, M^{\overline{b}}).$$

9 Geometric RSK

9.1 Longest value of a path in RSK

Let Φ denote the R-S algorithm, and let $\hat{\Phi}$ denote RSK. For Φ, we had a property about the longest increasing subsequence of a permutation. What about for $\hat{\Phi}$?

Example 9.1. When we run RSK on

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 1 \\ 5 & 3 & 1 \end{bmatrix}$$

we get the pair of tableau

1	1	1	1	1	1	1	1	2	2	2	3
2	2	2	2	3							
3	3	3									

1	1	1	1	1	2	2	2	3	3	3	3
2	2	2	3	3							
3	3	3									

Then $\lambda_1 = 12$, which happens to be the sum of the numbers in the longest path from top left to bottom right : $2+1+5+3+1 = 12$.

Theorem 9.1. *Let $\hat{\Phi}(M) = (A, B)$ have shape λ. Then $\lambda = \gamma(M)$, where γ is the maximum total value of a path from $(1,1)$ to (n,n).*

9.2 Geometric RSK

How would we feed a semistandard Young tableau to a computer? We want to think of $A = (\lambda \supseteq \cdots \supseteq \mu^{(2)} \supseteq \mu^{(1)})$, where μ_i is the shape of the tableau, only looking at the numbers $\leq i$.

Example 9.2. For the tableaux

1	1	1	1	1	1	1	1	2	2	2	3
2	2	2	2	3							
3	3	3									

1	1	1	1	1	2	2	2	3	3	3	3
2	2	2	3	3							
3	3	3									

we get

$$\begin{array}{ccc} 12 & 11 & 8 \\ & 5 & 4 \\ & & 3 \end{array} \qquad \begin{array}{ccc} 12 & & \\ 8 & 5 & \\ 5 & 3 & 3 \end{array}$$

We can put these together into a matrix

$$\begin{bmatrix} 12 & 11 & 8 \\ 8 & 5 & 4 \\ 5 & 3 & 3 \end{bmatrix}.$$

The output of RSK can then be sent to a computer as $x = (x_{i,j})$ such that

- $x_{i,j} \geq 0$
- $x_{i,j} \leq x_{i,j+1}, x_{i+1,j}$
- $\sum_{i-j=n-c} x_{i,j} = a_1 + \cdots + a_c$
- $\sum_{j-i=n-c} x_{i,j} = b_1 + \cdots + b_c$, where $0 \leq c \leq n$.

This defines a polytope. So we can think of $\tilde{\Phi} : \tilde{M}(\overline{a}, \overline{b}) \to \tilde{X}(a, b)$, where the left hand side takes a polytope defined by a matrix, and the the right side outputs a polytope defined by a matrix.

Theorem 9.2. *$\tilde{\Phi}$ is piecewise linear, volume preserving, and continuous.*

These polytopes were invented by Gelfand and Tseitlin. This was further developed in a paper by Gelfand and Zelevinsky.

9.3 Further generalization of RSK

Let $|\nu| = k$ be a Young diagram. Let

$$P_\nu(\overline{a}, \overline{b}) = \left\{ f : \nu \to \mathbb{R}_+ \text{ s.t. } f(i,j) \leq f(i,j+1), f(i+1,j), \sum_j f(i,j) = a_i, \sum_i f(i,j) = b_j \right\}$$

$$Q_\nu(\overline{a}, \overline{b}) = \left\{ g : \nu \to \mathbb{R}_+ \text{ s.t. } \sum_{i-j=c} g(i,j) = d_c \; \forall c \right\},$$

where d_c is something.

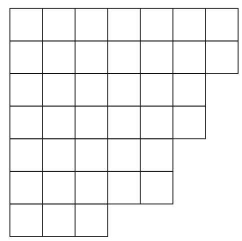
Theorem 9.3. *There exists Φ^* sending $P_\nu(\overline{a}, \overline{b}) \to Q_\nu(\overline{a}, \overline{b})$ which is piecewise linear, continuous, volume preserving and a bijection $\Phi^* : P_\nu \cap \mathbb{Z}^k \to Q_\nu \cap \mathbb{Z}^k$.*

10 RSK: The Final Chapter

10.1 Diagonals in Young diagrams and generalizing geometric RSK

Let ν be a partition of k, and let $f : \nu \to \mathbb{R}_+$ be a function on squares of the Young diagram. Define the diagonal sums $\alpha_c := \sum_{i-j=c} f(i,j)$, where $c \in \mathbb{Z}$. Similarly, define $\beta_c := \sum_{i=1}^{r_c} \sum_{j=1}^{s_c} f(i,j)$, where $c \in \mathbb{Z}$.

Example 10.1. Consider the following partition.



Then

$$\alpha_{-1} = f(1,2) + f(2,3) + f(3,4) + f(4,5)$$

and $(r_c, s_c) = (5,4)$ is the position of the lowermost square on this diagonal.

Theorem 10.1. *Fix $\overline{d}$, and let*

$$P_\nu(\overline{d}) = \{f : \nu \to \mathbb{R}_+ \mid f(i,j) \le f(i+1,j) \le f(i,j+1), \alpha_c(f) = dc \,\forall c\},$$

$$Q_\nu(\overline{d}) = \{g : \nu \to \mathbb{R}_+ \mid \beta_c(f) = d_c \,\forall c\}.$$

Then there exists some $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$ such that Φ is

1. *piecewise linear,*
2. *volume-preserving,*
3. *continuous,*
4. $\Phi : P_\nu \cap \mathbb{Z}^L \to Q_\nu \cap \mathbb{Z}^K$.

Moreover, Φ commutes with transposition.

Corollary 10.1. *The number of integer points in $P_\nu(\overline{d})$ is the same as the number of integer points in $Q_\nu(\overline{d})$.*

Corollary 10.2 (reduction to RSK)**.**

$$\#M(\overline{a}, \overline{b}) = \sum_\lambda \#\,\mathrm{SSYT}(\lambda, \overline{a}) \times \#\,\mathrm{SSYT}(\lambda, \overline{b})$$

Example 10.2. Let $\nu(\ell^\ell)$ be an $\ell \times \ell$ square. If we split it up along the diagonal, we get two tableaux, a $\mathrm{SSYT}(\lambda, (d_0 - d_{-1}, d_{-1} - d_{-2}, \dots))$ and a $\mathrm{SSYT}(\lambda, (d_0 - d_1, d_1 - d_2, \dots))$. So in the case of a square, we get RSK.

10.2 Description and proof of generalized geometric RSK

Let's prove the theorem.

Proof. Proceed by induction. If $\lambda = \varnothing$, we are done, and if λ is a square, we are also done because P and Q are the same. Let $r - s = c$, so $(r_c, s_c) = (r, s)$. And let $\overline{\nu} = \nu - (r, s)$; we are removing a box from the diagram at position (r, s) on the boundary of the diagram. We have $\Phi_{\overline{\nu}} : P_{\overline{nu}} \to Q_{\overline{\nu}}$, and we want to get Φ_ν.

Draw a diagonal from the square (r, s) up and left. We want to alter boxes on the diagonal. Take ξ sending $f(i, j) \mapsto \max\{f(i-1, j), f(i, j-1)\} + \min\{f(i, +1, j), f(i, j+1)\} - f(i, j)$. Call this $\overline{f}(i, j)$. Then we get $f \mapsto \overline{f} \in P_{\overline{\nu}} \mapsto \overline{g} \in Q_{\overline{\nu}}$. How do we get $g \in Q_\nu$ from $\overline{g}$? Just add the last square by setting $g(r, s) := f(r, s) - \max\{f(r-1, s), f(r, s-1)\}$.

Why is Φ is well-defined? Note that no two adjacent diagonals can contain corners. Now think about the order of the squares we chop off and replace. If we write this order in reverse, we get a Young tableau. We claim that if Γ is a graph on $\mathrm{SYT}(\nu)$ with $(i, i+1)$ swaps allowed in distinct diagonals, then Γ_ν is connected. If we keep switching to put every number in our tableau in lexicographic order, we will eventually get the full lexicographic ordering. So the graph Γ_ν is connected, which makes this process well-defined. ☐

Example 10.3. Take the element of P_ν

1	1	4
2	3	4
4	4	5

Chop off the 5 in the bottom right hand corner, and alter the diagonal of that 5. After replacing the space of the 5, we get

0	1	4
2	3	4
4	4	1

Now chop off the 4 on the right in the bottom row and alter its diagonal. After replacing that 4, we get

0	1	4
1	3	4
4	0	1

Continue like this, replacing one square at a time in the corner (of the diagram, only counting squares we haven't chopped off and replaced) and altering its diagonal until we've altered everything.

0	2	4
1	3	0
4	0	1

1	2	4
1	1	0
4	0	1

1	2	2
1	1	0
4	0	1

Continuing like this, we eventually get

1	1	2
0	1	0
3	0	1

11 RSK: The (final) Final Chapter

11.1 Properties of generalized RSK

Theorem 11.1. $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$ *is*

1. *piecewise linear,*
2. *volume-preserving,*
3. *continuous,*
4. $\Phi : P_\nu \cap \mathbb{Z}^K \to Q_\nu \cap \mathbb{Z}^K$.

We had $\Phi = \xi_{(r_n,s_n)} \circ \xi_{(r_{n-1},s_{n-1})} \circ \cdots$, where $\xi_{(r_i,s_i)}$ is the PL map corresponding to removal of the $(n+1-i)$-th square in $A \in \mathrm{SYT}(\nu)$. Last time, we showed that Φ is well-defined.

Proof. Invertibility of Φ follows from construction, since we can reverse the process.

To prove that Φ is piecewise linear, let ξ send $f(i,j) \mapsto \max\{f(i-1,j), f((i,j-1)\} + \min\{f(i+1,j), f(i,j+1)\} - f(i,j)$. Then ξ sends $f \mapsto \overline{f}$, so this is also volume preserving, as it has determinant ± 1.

We now have to show that $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$; that is, if $M \in P_\nu(\overline{d})$ and $\alpha_c(M) = d_c$, then $\Phi(M) \in Q_\nu(\overline{d})$, and $\beta_c(\Phi(M)) = d_c$ for all c. We also need that if $M = [m_{i,j}]$ and $\Phi(M) = [m'_{i,j}]$, then if $m_{i,j} \geq 0$ and $m_{i,j} \leq m_{i,j+1}, m_{i+1,j}$, then $m'_{i,j} \geq 0$. Let's show this 2nd part. We only need to understand this for when we change a corner:

$$\overline{f}(r,s) = f(r,s) - \max\{f(r-1,s), f(r,s-1)\} \geq 0.$$

To show that $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$, proceed by induction on each step. Suppose $\xi_{r,s}$ sends $M \mapsto \overline{M}$, where $\overline{M}$ has share $\nu - (r,s)$. We only alter boxes on the diagonal, so $\alpha_c(M) = \alpha_c(\overline{M})$ for all $c \neq r-s$. Now let $u = r-s$. Then

$$\alpha_u(\overline{M}) = \alpha_{u+1}(M) + \alpha_{u-1}(M) - \alpha_u(M) + m'_{r,s}.$$

We also get, using inclusion-exclusion, that

$$\begin{aligned}
\beta_c(M') &= \beta_{c+1}(\Phi(M)) + \beta_{c-1}(\Phi(M)) - \beta_c(\Phi(M) - (r,s)) + m'_{r,s} \\
&= \alpha_{c+1}(M) + \alpha_{c-1}(M) - \alpha_{c-1}(M) + m'_{r,s} \\
&= \alpha_u(\Phi(M)).
\end{aligned}$$

Comparing these two equalities proves the property by induction. $\square$

11.2 Maximal sum over a path

Proposition 11.1. *Let (r, s) be a corner of ν. Then $m_{r,s}$ is equal to the maximal sum over a path from $(1, 1)$ to (r, s) in M'.*

Proof. Look at the formula

$$m'_{r,s} = \overline{f}(r, s) = f(r, s) - \max\{f(r-1, s), f(r, s-1)\}.$$

Then we can prove this by induction. $\square$

When ν is a square, this is the corresponding property for RSK.

What is the moral here? We generalized RSK so far that we can prove all these properties from just two equations involving α and β. Later, we will approach RSK from a different angle, involving f^λ.

12 Applications of The Hilman-Grassl Bijection

12.1 MacMahon's theorem

We showed the following last time:

Theorem 12.1 (Stanley)**.**

$$\sum_{A \in \mathrm{RPP}(\lambda)} t^{|A|} = \prod_{(i,j)\in\lambda} \frac{1}{1-t^{h_{i,j}}},$$

where $h_{i,j}$ is the hook length of the square i,j in the Young diagram of λ.

Definition 12.1. A **plane partition** of shape λ is a tableau which is weakly decreasing going down or to the right.

Example 12.1. Here is a plane partition of shape $(4,4,3,2)$.

7	7	6	4
4	3	1	1
3	3	1	
2			

Theorem 12.2 (MacMahon, c. 1900)**.**

$$\sum_{A\in \mathrm{PP}} t^{|A|} = \prod_{k=1}^{\infty} \frac{1}{(1-t^k)^k}$$

Proof. Let $\lambda = m^m$; this is a square diagram. Then Stanley's theorem says

$$\sum_{A\in\mathrm{RPP}(m^m)} t^{|A|} = \prod_{k=1}^{m} \frac{1}{(1-t^k)^k} \cdot \prod_{k=m+1}^{2m-1} \frac{1}{(1-t^k)^{2m-k}}.$$

Now

$$\sum_{A\in PP} t^{|A|} = \lim_{m\to\infty} \sum_{A\in PP\cap(m^m)} t^{|A|}$$

because this limit stabilizes for each coefficient of the power series.

$$\begin{aligned} &= \lim_{m\to\infty} \sum_{A\in RPP(m^m)} t^{|A|} \\ &= \lim_{m\to\infty} \prod_{k=1}^{m} \frac{1}{(1-t^k)^k} \cdot \prod_{k=m+1}^{2m-1} \frac{1}{(1-t^k)^{2m-k}} \\ &= \prod_{k=1}^{m} \frac{1}{(1-t^k)^k}. \end{aligned}$$

□

MacMahon's theorem is analogous to the following result.

Theorem 12.3 (Euler, 1738)**.**

$$\sum_{\lambda \in P} t^{|\lambda|} = \prod_{k=1}^{\infty} \frac{1}{1-t^k}.$$

12.2 The hook length formula

We can also prove the hook length formula using Stanley's theorem.

Theorem 12.4 (Frame-Robinson-Thrall, 1954)**.**

$$f^\lambda = \#\,\mathrm{SYT}(\lambda) = n! \prod_{(i,j)\in\lambda} \frac{1}{h_{i,j}}.$$

Example 12.2. Suppose $\lambda = (m, m)$. Here is a standard Young tableau of this shape:

1	2	4	5	9
3	6	7	8	10

Then the hook lengths of each square look like

$m+1$	m	$\cdots$	3	2
m	$m-1$	$\cdots$	2	1

In general,

$$f^{(m,m)} = \frac{(2m)!}{m!(m-1)!} = \frac{1}{m+1}\binom{2m}{m} = \mathrm{Cat}(m),$$

the m-th Catalan number.

Proof. Write

$$\mathrm{RPP}(\lambda) = \bigcup_{A \in \mathrm{SYT}(\lambda)} C_A,$$

where C_A is where we pick the numbers in order of sums of indices in the diagram. That is $0 \le x_{1,1} \le x_{1,2} \le x_{2,1} \le x_{1,3} \le x_{2,2} \le x_{1,4} \le x_{2,3} \le x_{3,1} \le \cdots$; this is a cone in $\mathbb{R}^n$. Now look at the number of $A \in \mathrm{RPP}(\lambda)$ such that $|A| \le N$. Asymptotically, this is about

$$\sum_{T \in \mathrm{SYT}(\lambda)} \#\{A \in C_T : |A| \le N\} = \#\,\mathrm{SYT}(\lambda) \cdot \#\{0 \le z_1 \le \cdots \le z_n : z_1 + \cdots + z_n \le N\}$$
$$\sim \#SYT(\lambda) \cdot N^n \operatorname{vol}(\Delta),$$

where $\Delta = \{0 \le z_1 \le z_2 \le \cdots \le z_n : z_1 + \cdots + z_n \le 1\}$. The vertices of Δ are when we have equalities. So we have

$$\begin{gathered} v_0 = (0, 0, \ldots, 0) \\ v_1 = (0, \ldots, 0, 1) \\ v_2 = (0, \ldots, 0, 1/2, 1/2) \\ v_3 = (0, \ldots, 0, 1/3, 1/3, 1/3) \\ \vdots \\ v_n = (1/n, \ldots, 1/n). \end{gathered}$$

This is a triangular matrix, so the volume ($1/n!$ times the determinant), is $1/n!$ times the product of the diagonal entries. That is,

$$\mathrm{Vol}(\Delta) = \frac{1}{n!} \cdot \frac{1}{n!}.$$

So we get

$$|\{A \in \mathrm{RPP}(\lambda) : |A| \le N\}| \sim \frac{N^n}{(n!)^2} \# \mathrm{SYT}(\lambda).$$

The right hand side of Stanley's theorem is

$$\sum t^{\sum b_{i,j} h_{i,j}},$$

where the sum is over matrices $B = (b_{i,j})$ such that $b_{i,j} \ge 0, \sum b_{i,j} \le N$. This is asymptotically $N^n \operatorname{vol}(\Delta')$, where $\Delta' = \{0 \le y_{i,j}, \sum y_{i,j} h_{i,j} \le 1\}$. The vertices of Δ' are

$$\begin{gathered} (0, 0, \ldots, 0) \\ (1/h_{1,1}, 0, \ldots, 0) \\ (0, 1/h_{1,2}, 0, \ldots, 0) \\ \vdots \end{gathered}$$

So we get that

$$\mathrm{Vol}(\Delta') = \frac{1}{n!} \cdot \prod_{(i,j) \in \Delta} \frac{1}{h_{i,j}}.$$

By Stanley's theorem, we have

$$|\{A \in \mathrm{RPP}(\lambda) : |A| \le N\| = |\{B : b_{i,j} \ge 0, \sum b_{i,j} \le N\}|$$

So the asymptotics have to be the same. Then we get that

$$\# \mathrm{SYT}(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}}. \qquad \square$$

We will prove the hook length formula in different ways, as well.

13 The NPS Algorithm

13.1 Motivation

Here is the hook length formula.

Theorem 13.1 (FRT, 1958)**.**

$$f^\lambda = \#\,\mathrm{SYT}(\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

The following is a general result from representation theory.

Theorem 13.2. *Let χ be a character of a finite group G, and let $d = \chi(1)$ be the dimension of the representation. Then $d \mid |G|$.*

In this case, we get the following.

Corollary 13.1. $\#\,\mathrm{SYT}(\lambda) \mid n!$.

Why should this be true?

Example 13.1. Let $\lambda = (n-k, 1^k)$. Then

$$f^\lambda = \binom{n-1}{k} = \frac{(n-1)!}{k!(n-k)!} \mid (n-1)!,$$

so it divides $n!$.

Example 13.2. Let $\lambda = (m, m)$, where $n = 2m$. Then

$$f^\lambda = \frac{1}{m+1}\binom{2m}{m} = \frac{(2m)!}{m!(m+1)!} = \frac{n!}{m!(m+1)!}.$$

Why should this be an integer?

Today, we will construct $\Phi_\lambda : S_n \to \mathrm{SYT}(\lambda)$, such that $\Phi_\lambda^{-1}(A)$ is a constant depending on λ. In reality, this constant will be $\prod h_{i,j}$.

Example 13.3. Let $\lambda = (n)$. Then $f^\lambda = 1$. In this case, Φ_λ will be some sorting algorithm.

Example 13.4. Let $\lambda = (1^n)$. Then $f^\lambda = 1$. In this case, Φ_λ will sort vertically.

13.2 Construction of Φ_λ

This will sort of be like 2-dimensional bubble sort.

Take $\lambda = (5, 5, 3, 2)$, and choose a permutation:

9	8	11	12	4
15	3	14	2	5
10	7	1		
13	6			

Look at the last column. The column is sorted, so we are ok. Now include the next column. 12 is bigger than 2, so we must switch them. Same with 12 and 5.

9	8	11	2	4
15	3	14	5	12
10	7	1		
13	6			

Now look at the 1 in the next column. This is sorted. Now look at the 14. This is bigger than the 1, so we need to switch it.

9	8	11	2	4
15	3	1	5	12
10	7	14		
13	6			

Going up the columns, we need to switch the 11 with something. It must be switched with the 11. Then we need to switch the 11 with the 5.

9	8	1	2	4
15	3	5	11	12
10	7	14		
13	6			

Move on to the next column. The 6 is fine, but when we move up to the 7, we see that we have to switch it with the 6.

9	8	1	2	4
15	3	5	11	12
10	6	14		
13	7			

The 3 is fine where it is. But the 8 above needs to be switched. We switch it with the 1, then the 2, and then the 4 until this part is sorted.

9	1	2	4	8
15	3	5	11	12
10	6	14		
13	7			

The 13 is okay, but the 10 needs to be switched. Switch it with the 6.

9	1	2	4	8
15	3	5	11	12
6	10	14		
13	7			

The 15 has a long way to go. See if you can figure out where it needs to go:

9	1	2	4	8
3	5	11	12	15
6	10	14		
13	7			

Finally, we move the 9 where it needs to go:

1	2	4	8	9
3	5	11	12	15
6	10	14		
13	7			

We now have a standard Young tableau. We had no choice at each step, so we can see that this algorithm is well-defined. Notice that this is similar to jeu-de-taquin.

13.3 Construction of Ψ_λ

Now we will construct $\Psi_\lambda : S_n \to \prod_{(i,j)\in\lambda}[-\lambda'_j + 1, \cdots, \lambda_i - i]$. This range has size $h_{i,j}$.

Lemma 13.1. $(\Phi_\lambda, \Psi_\lambda)$ *gives a bijection between* S_n *and the Cartesian product of* $\mathrm{SYT}(\lambda)$ *with the above product.*

Example 13.5. When we are just doing bubblesort, Ψ_λ will give us $(a_1, \ldots, a_2, a_1)$, where a_i is the number of steps the i-th number made.

Take our example from earlier:

9	8	11	12	4
15	3	14	2	5
10	7	1		
13	6			

Let's record the sorting procedure and keep track of what numbers move.

9	8	11	12	4
15	3	14	2	5
10	7	1		
13	6			

0	0	0	0	0
0	0	0	0	0
0	0	0		
0	0			

12 is moved down and then to the right. Our rule is that when we move to the right, we add 1. When we move a number down, we switch the two numbers and subtract 1 from the top.

9	8	11	2	4
15	3	14	12	5
10	7	1		
13	6			

0	0	0	-1	0
0	0	0	0	0
0	0	0		
0	0			

9	8	11	2	4
15	3	14	5	12
10	7	1		
13	6			

0	0	0	-1	0
0	0	0	1	0
0	0	0		
0	0			

When we move the 14 down, we get:

9	8	11	2	4
15	3	1	5	12
10	7	14		
13	6			

0	0	0	-1	0
0	0	-1	1	0
0	0	0		
0	0			

When 11 goes down, we switch the two blocks and then subtract 1 from the new top:

9	8	1	2	4		0	0	−2	−1	0
15	3	11	5	12		0	0	0	1	0
10	7	14				0	0	0		
13	6					0	0			

Now the 11 has to move to the right:

9	8	1	2	4		0	0	−2	−1	0
15	3	5	11	12		0	0	1	1	0
10	7	14				0	0	0		
13	6					0	0			

The 7 in the next column has to be moved down:

9	8	1	2	4		0	0	−2	−1	0
15	3	5	11	12		0	0	1	1	0
10	6	14				0	−1	0		
13	7					0	0			

Now we have to move the 8. We only change the numbers in the "column and row of action."

9	1	2	4	8		0	3	−2	−1	0
15	3	5	11	12		0	0	1	1	0
10	6	14				0	−1	0		
13	7					0	0			

Now move the 13:

9	1	2	4	8		0	3	−2	−1	0
15	3	5	11	12		0	0	1	1	0
10	6	14				0	−1	0		
13	7					1	0			

Continuing like this, we get

1	2	4	8	9
3	5	11	12	15
6	10	14		
13	7			

4	3	−2	−1	0
4	0	1	1	0
1	−1	0		
1	0			

In summary, right moves change the numbers on the right as

$$\begin{array}{|c|}\hline x \\ \hline y \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline x+1 \\ \hline y \\ \hline \end{array}$$

and down moves change the numbers on the right as

$$\begin{array}{|c|}\hline x \\ \hline y \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline y-1 \\ \hline x \\ \hline \end{array}$$

Check that we can invert this algorithm. Next time, we will hear the story of this algorithm and learn about the GNW algorithm.

14 Hook Walks and Representation Theory of Stanley's Formula

14.1 Hook walks

We want to prove the hook length formula in a more elegant way. This is a construction due to Greene, Nijenhuis, and Wilf in 1979.[6]

We will probabilistically construct a standard Young tableau. First, take the empty diagram. We need to put n into one of the corner spaces. We get

$$\mathbb{P}(n \text{ is at } (r,s)) = \frac{f^{\lambda^*}}{f^\lambda} \overset{\text{WTS}}{=} \frac{(n-1)!}{n!} \prod \frac{h_{i,j}}{h^*_{i,j}},$$

where λ^* is $\lambda - (r,s)$. Using this, if we had a guess for the hook length formula, we could try to prove the hook length formula by induction. However, this is actually not that easy. So GNW constructed a random process to do this.

The idea is a **hook walk**.

1. Start at a random $(i,j) \in \lambda$.
2. Mover to a random square in $\mathrm{Hook}_\lambda(i,j)$.
3. Repeat step 2 until you get to the corner.

Lemma 14.1. *$\mathbb{P}($corner is at $(r,s)) = f^{\lambda^*}/f^\lambda$.*

Given this lemma, using the fact that $\sum \mathbb{P}(\text{corner is at } (r,s)) = 1$, we get $f^\lambda = \sum_{\lambda^*} f^{\lambda^*}$. This lets us use the guess for the hook length formula to use our induction. Here is a stronger result, which is easier to prove.

Lemma 14.2. *Let α_i be the column number of the i-th square in this algorithm, and let β_i be the row number of the i-th square. Then*

$$\mathbb{P}((i,j) \to (r,s) \text{ via } \overline{\alpha}, \overline{\beta}) = \frac{1}{n} \prod \left(1 + \frac{1}{h_{i,\alpha}1}\right) \prod \left(1 + \frac{1}{h_{\beta,j} - 1}\right).$$

For the precise statement of the lemma, look at Exercise 3.17 in the textbook (Sagan's *The Symmetric Group*).

[6] This is actually before the NPS algorithm, which is from 1992.

14.2 Stanley's formula, explained

Let $W_n = \mathbb{C}[x_1, \dots, x_n]$, thought of as an infinite dimensional representation of S_n. Let $H_n \subseteq W_n$ be the set of **harmonic polynomials**, nonconstant polynomials $f \in W_n$ such that $h \cdot f = 0$ for all $h \in \mathbb{C}[\partial/\partial x_1, \dots, \partial/\partial x_n]^{S_n}$.

Example 14.1. Let $n = 2$. Then what survives when we apply $\partial/\partial x_1, \partial/\partial x_2$ or $\partial/\partial x_1 \partial x_2$? We get that $H_2 = \mathbb{C}\langle 1, x_1 - x_2\rangle$.

Theorem 14.1 (Chevalley)**.** *Let W_n, H_n be as above.*

1. *$W_n = H_n \otimes I_n$, where $I_n = W^{S_n}$ is the symmetric polynomials.*
2. *H_n is the regular S_n-representation.*

Write $W_n = \bigoplus_{k=0}^{\infty} W_n^k$, $H_n = \bigoplus_{k=0}^{\binom{n}{2}} H_n^k$, and $I_n = \bigoplus_{k=0}^{\infty} I_n^k$. This is a grading by the degree of the polynomials. Write

$$P_n(t) = \sum_{k=0}^{\infty} (\dim(W_n^k) t^k = 1/(1-t)^n$$

$$I_n(t) = \sum_{k=0}^{\infty} (\dim(T_n^k)) t^n = \frac{(1-t)(1-t^2)\cdots(1-t^n)}{.}$$

$$H_n(t) = \frac{P_n(t)}{I_n(t)} = \prod_{i=1}^{n} \frac{1-t^i}{1-t} = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1}).$$

Now

$$P_\lambda(t) = \sum_{k=0}^{\infty} \dim(\mathrm{Hom}(W_n^k, S^\lambda)) t^k = \prod_{(i,j)\ in \lambda} \frac{t^{i-1}}{1-t^{h_{i,j}}}.$$

Then

$$H_\lambda(t) = \sum \dim(\mathrm{Hom}(B_n^k, S^\lambda)) t^k = H_n(t) \prod_{(i,j)\in\lambda} \frac{1}{(h_{i,j})_t},$$

the t-analogue of $h_{i,j}$. Then the hook length formula is

$$H_\lambda(1) = f^\lambda = \frac{n!}{\prod h_{i,j}}.$$

This is an algebraic interpretation of how Stanley's formula implies the hook length formula.

15 Odds, Ends, and Things to Come

15.1 Recap episode

We have learned a few things so far:

1. Representation theory of S_n
2. Combinatorics of Young tableau and combinatorial algorithms
   (a) RSK algorithm
   (b) Hook length formula
   (c) Stanley's formula and the Hilman-Grassl algorithm.

We will learn about two more things:

1. $\mathrm{GL}_N(\mathbb{C})$ representation theory
2. Symmetric functions.

These are related to further topics we won't cover, namely Schubert calculus and enumerative algebraic geometry.

15.2 Combinatorial questions about partitions

Theorem 15.1 (MacMahon, c.1900)**.** *Let* $\mathrm{PP}(\lambda, c)$ *be the set of plane partitions of shape* λ *numbers* $\leq c$*. Then* $PP(a^b, c)$ *(rectangles with numbers at most* c*) satisfies*

$$|\,\mathrm{PP}(\lambda, c)| = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

Here is a q-version.

Theorem 15.2.

$$\sum_{A\in\mathrm{PP}(a^b,c)} q^{|A|} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{(i+j+k-1)_q}{(i+j+k-2)_q},$$

where $(m)_q = (1-q^m)/(1-q)$*.*

So when $q \to 1$, we get the original theorem.

The number of Lozenge tilings of an $a\times b\times c$ hexagon is the number of $\mathrm{PP}(a^b, c)$. View the tilings as stacked 3-dimensional cubes, a 3-dimensional partition of an $a\times b\times c$ box. This is a bit strange. For any graph G, the number of perfect matching in G is the determinant of some matrix. We can view a tiling as a matching in terms of the dual graph. How can this determinant have a product formula?

15.3 Related ideas in representation theory

Theorem 15.3 (Frobenius, c.1902)**.** *Let ξ^α be the character corresponding to M^α, let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $\rho = (\ell, \ldots, 1)$ be a staircase partition, and for a permutation ω, let $\omega\alpha = (\alpha_{\omega(1)}, \alpha_{\omega(2)}, \ldots)$. Then*

$$\chi^\lambda = \sum_{\omega \in S_\ell} \operatorname{sign}(\omega) \xi^{\lambda - \omega\rho + \rho}.$$

Here, the convention is $\xi^\alpha = 0$ for all $\alpha \notin \mathbb{N}^\ell$ (if we get negative entries, discard the term).

This formula looks a little like a determinant. There is a relationship between these formulas.

Theorem 15.4 (Kostka)**.** *Let $K_{\lambda,\mu} = \#\,\mathrm{SSYT}(\lambda, \mu)$. Then*

$$\xi^\mu = \sum_{|\lambda|=n} K_{\lambda,\mu} \chi^\lambda.$$

This explains the triangular nature of everything because you cannot have a semistandard Young tableau with shape λ and weight μ if $\lambda \leq \mu$. Forbenius's theorem gives a form for the inverse of the matrix

$$(K_{\lambda,\mu})^{-1}_{|\lambda|=|\mu|=n}.$$

These values are called the **Kostka numbers**. Here is what we will learn:

Definition 15.1. The **Schur functions** are $s_\lambda = \sum_{A \in \mathrm{SSYT}(\lambda)} x_a^{m_1(A)} \cdots x_N^{m_N(A)}$, where $m_i(A)$ is the number of i in A.

Theorem 15.5. *The s_λ are symmetric functions ($\mathbb{C}[x_1, \ldots, x_N]^{S_N}$).*

Definition 15.2. The **elementary symmetric polynomials** are $e_\alpha = e_{\alpha_1} e_{\alpha_2} \cdots$.

Theorem 15.6 (Jacobi-Trudy)**.** *Using the convention that $e_r = 0$ for all $r < 0$,*

$$s_\lambda = \det[\tilde{e}_{\lambda_i + j - i}]_{i,j=1,\ldots,\ell}.$$

It turns out that the Jacobi-Trudy theorem is saying the same thing as Frobenius's theorem. Also, the definition of Schur functions is equivalent to Kostka's theorem.[7] The idea is then that

$$|\,\mathrm{PP}(a^b, c)| = s_{(a,b)}(\underbrace{1, 1, \ldots, 1}_{c}).$$

Since the Schur functions have a determinant formula, we can see why there is a determinant formula for Lozenge tilings. To get the q-analogue, we can look at $s_{(a,b)}(1, q, q^2, \ldots, q^{c-1})$.

This is the general picture we will be going through in the next few weeks.

[7] This is in the same sense that whoever came up with the definition of variance understood a lot about scalar products. The definition is designed to be compatible with the nice structure.

16 Relationships of Schur Functions to Characters of $\mathrm{GL}_N(\mathbb{C})$ and to Young Tableau

16.1 Schur functions and characters of $\mathrm{GL}_N(\mathbb{C})$

Definition 16.1. The ring of **symmetric functions of degree** n is $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$. The ring of **symmetric functions** is $\Lambda = \varprojlim_n \Lambda_n$.

Example 16.1. Λ_3 is spanned by $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$ and $x_1x_2x_3$.

A representation is a homomorphism $\rho : V \mathit{to} \operatorname{Aut}(V)$, where $V = \mathbb{C}^d$. Essentially, what we do with representations of finite groups can be done with compact groups, as well.

Theorem 16.1. *All representations of* $\mathrm{GL}_N(\mathbb{C})$ *are rational. That is, if* $M = (x_{i,j})$*, then* $\rho(M) = (f_{pq}(x_{i,}))$*, where the* $f_{p,q}$ *are rational polynomials. Moreover,* $\rho(M) = (\det)^k$ *times a polynomial representation.*

Example 16.2. Let ρ be the determinant map sending $M \mapsto \det(M)$. This is a 1-dimensional representation. Then $\rho_k(M) = (\det(M))^k$ is also a representation for $k \in \mathbb{Z}$.

Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_N$ with $\lambda \in \mathbb{N}$. If ρ is an irreducible representation of $\mathrm{GL}_N(\mathbb{C})$, then consider diagonal matrices M and "characters of ρ" given by $\operatorname{tr}(\rho[M])$. Then these is in Λ_N.

Theorem 16.2 (Weyl[8]). *If* π *is an irreducible representation of* $\mathrm{GL}_N(\mathbb{C})$ *corresponding to* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$*, then* $\operatorname{tr}(\pi) = s_\lambda$*, where* $s_\lambda = a_{\lambda+\rho}/a_\rho$ *is a Schur function.*

Here, if $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $a_\alpha = \det(x_i^{\alpha_j})_{i,j=1,\ldots,N}$ is like a Vandemonde determinant, and $\rho = (N-1, N-2, \ldots, 0)$ (so a_ρ is a Vandermonde determinant).

Example 16.3. Let $N = 2$ and $\lambda = (4, 3)$. Then

$$s_\lambda = \frac{\det \begin{bmatrix} x^4 & y^4 \\ x^3 & y^3 \end{bmatrix}}{x - y} = \frac{x^4y^3 - y^4x^3}{x - y} = x^3y^3.$$

Remark 16.1. The situation in the theorem is actually something that happens for all compact groups.

16.2 Formula for Schur functions in terms of Young tableau

Theorem 16.3. *Let* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$*. Then*

$$s_\lambda = \sum_{A \in \mathrm{SSYT}(\lambda, \leq N)} x_1^{m_1(A)} \cdots x_N^{m_N(A)},$$

where $m_i(A)$ *is the number of is in* A*.*

[8] Igor does not remember whose theorem this is, so Weyl is a guess.

Lemma 16.1. *This sum is a symmetric polynomial of degree N.*

Proof. We show that the sum is invariant under the transposition $(i\, i+1) \in S_N$ for all $i = 1, \ldots, N-1$. Given a semistandard Young tableau, look at each row. If we look at rows with i and $i+1$, look at parts where we do not have $i+1$ squares directly below i squares. Then we can switch the number of is and $(i+1)$s in this part of each row and still get a semistandard Young tableau. $\square$

Proof. We can prove the theorem by showing that a_ρ times the sum is $a_{\lambda+\rho}$.

$$\prod_{1 \le i < j \le N} (x_i - x_j) = \sum_{\sigma \in S_N} \operatorname{sign}(\sigma) s_{\sigma(1)}^0 x - \sigma(2)^1 \cdots x_{\sigma(N)}^{N-1}$$

Here is Gessel's proof of this fact. It is easier to consider $\prod_{i<j}(1 - x_j/x_i)$. $\square$

17 Bases, Involution, and Scalar Product of Symmetric Functions

17.1 Five bases of symmetric functions

Let $\Lambda = \varprojlim_n \lambda_n$ be the ring of symmetric functions. Then $\Lambda \subseteq \mathbb{C}[x_1, x_2, \dots]$; that is $f \in \Lambda = \sum c_\alpha x^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \dots)$, $\sum_\alpha \alpha_i < \infty$, and $\alpha_i \in \mathbb{N}$.

Example 17.1. $e_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 + x_1x_5 + \cdots$.

Definition 17.1. The **monomial symmetric functions** are

$$m_\lambda = \left[\sum_{\substack{\sigma \in S_\ell \\ i_1 < \cdots < i_\ell}} x_{i_1}^{\lambda_{\sigma(1)}} x_{i_2}^{\lambda_{\sigma(2)}} \cdots x_{i_\ell}^{\lambda_{\sigma(\ell)}} \right] / \prod_{i=1}^{\ell} m_i(\lambda)!,$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 17.1. *The monomial symmetric functions form a basis for Λ.*

Definition 17.2. The **elementary symmetric functions** are

$$e_k = m_{(1^k)} = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},$$

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 17.2. *The elementary symmetric functions form a basis for Λ.*

Proof. $f \in \Lambda$ is $c_\lambda x^\lambda + \cdots$, when written in lexicographic order. □

Theorem 17.1. *The elementary symmetric functions are free generators of Λ as a ring; i.e. they do not satisfy any algebraic equations.*

Definition 17.3. The **power symmetric functions** are

$$p_k = m_{(k)} = x_1^k + x_2^k + \cdots ,$$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 17.3. *The power symmetric functions form a basis for Λ.*

Definition 17.4. The **complete symmetric functions** are

$$h_k = \sum_{|\lambda|=k} m_\lambda = \sum_{i_1 \le i_2 \le \cdots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.

Proposition 17.4. *The complete symmetric functions form a basis for Λ.*

Definition 17.5. The **Schur functions** are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} = \sum_{A \in \mathrm{SSYT}(\lambda)} x^A,$$

$$x^A = x_1^{m_1(A)} x_2^{m_2(A)} \cdots .$$

Theorem 17.2. *The Schur functions form a basis for Λ.*

17.2 Involution and scalar product on symmetric functions

Here is a dictionary relating symmetric functions and representation theory of S_n

Symmetric functions	Representations of S_n
s_λ	S^λ
h_λ	$M^\lambda = \mathrm{ind}_{S_{\lambda_1}\times\cdots\times a_{\lambda_\ell}}^{S_n} 1$
e_λ	$M^\lambda \otimes \mathrm{sgn}$

This correspondence tells that we should have an involution $\omega : \Lambda \to \Lambda$ sending $e_\lambda \mapsto h_\lambda$ corresponding to $\otimes\,\mathrm{sgn}$.

Theorem 17.3. *The involution $\omega : \Lambda \to \Lambda$ sends $s_\lambda \mapsto s_{\lambda'}$.*

Theorem 17.4. *The involution $\omega : \Lambda \to \Lambda$ sends $p_\lambda \mapsto \varepsilon_\lambda p_\lambda$.*

There is a scalar product on Λ that relates to the scalar product on characters of representations of S_n.

Definition 17.6. Define a **scalar product** on Λ by its value on the basis of Schur functions:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

If $f = \sum c_\lambda s_\lambda$ and $g = \sum r_\lambda s_\lambda$, then

$$\langle f, g \rangle = \sum_\lambda c_\lambda r_\lambda.$$

Proposition 17.5. *$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$ for all λ, μ.*

Proof. Write $M^\mu = \bigoplus K_{\lambda,\mu} S^\lambda$. Then $h_\mu = \sum K_{\lambda,\mu} s_{\lambda,\mu}$. We also have $s_\lambda = \sum K_{\lambda,\mu} m_\mu$. $\square$

Proposition 17.6. *$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$, where*

$$z_\lambda = \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots}$$

is the size of the conjugacy class corresponding to λ in S_n.

Theorem 17.5. $s_\lambda = \sum_\mu \chi_\lambda[\mu] p_\mu$.[9]

So we can talk about characters of the symmetric group by only talking about symmetric functions.[10]

[9] Maybe there is a factor of z_λ in here. Professor Pak doesn't remember.
[10] Newton studied p_μ. That's how old the idea of symmetric functions is.

18 Generating Functions of Symmetric Functions and Cauchy-Type Identities

18.1 Relationship between e_k and h_k

We want to prove a correspondence between e_n and h_n. Define $E(t) = \sum_{n=0}^{\infty} e_n t^n$ and $H(t) = \sum_{n=0}^{\infty} h_n t^n$. So we just need to prove a relation between E and H.

Proposition 18.1. *$E(t) = \prod_{i=1}^{\infty}(1 + x_i t)$.*

Proof. This follows from $e_n = \sum_{i_1<\cdots<i_n} x_{i_1} \cdots x_{i_n}$.[11] ☐

Proposition 18.2. *$H(t) = \prod_{i=1}^{\infty} \frac{1}{1-x_i t}$.*

Proof. This follows from $h_n = \sum_{i_1\leq\cdots\leq i_n} x_{i_1} \cdots x_{i_n}$. ☐

Corollary 18.1. *$E(t)H(-t) = 1$.*

Definition 18.1. The ω-**involution** $\omega : \Lambda \to \Lambda$ is $\omega(e_k) = h_k$ for all k.

Proposition 18.3. *$\omega^2 = \mathrm{id}_\Lambda$.*

Proof. This is equivalent to $\omega(h_k) = e_k$ for all k. The relation for E and H shows that $\sum_{i=0}^{n}(-1)^{n-i} e_i h_{n-i} = 0$ for all n. The map ω is an algebra homomorphism, so applying ω to the sum should still give 0. We can then recursively determine that $\omega(h_k) = e_k$. ☐

Example 18.1. Say we are looking at n variable symmetric functions. Then

$$e_k(1, \ldots, 1) = \binom{n}{k},$$

$$h_k(1, \ldots, 1) = \binom{n+k-1}{k}.$$

Some people call this latter quantity $\binom{-n}{k}$. The generating function of $\binom{n}{k}$ is

$$\sum_{k=0}^{\infty} \binom{n}{k} t^n = (1 + t)^n,$$

while the generating function of $\binom{-n}{k}$ is

$$\sum_{k=0}^{\infty} \binom{-n}{k} t^k = (1 - t)^{-n}.$$

So this is the symmetric function analogue of these classic generating function identities.

[11] Igor says that this is the kind of thing he thinks is so simple, he doesn't actually know how to really prove it. He's not being facetious.

18.2 Cauchy-type identities

Consider the multivariate generating function

$$Q(\overline{x},\overline{y}) = \prod_{i,j=1} \frac{1}{1-x_i y_j}.$$

Proposition 18.4.

$$Q(\overline{x},\overline{y}) = \sum_{\alpha,\beta} |\operatorname{Mat}(\alpha,\beta)| m_\alpha(x) m_\beta(y) = \sum_\lambda m_\lambda(x) h_\lambda(y),$$

where $\operatorname{Mat}(\alpha,\beta)$ *are matrices with row-sums* α *and column-sums* β*.*

Proof. For the first part, the $x^\alpha y^\beta$ term in the product is a sum of terms of products of $(x_i y_j)^{a_{i,j}}$. If we put the $a_{i,j}$ into a matrix, the matrix has row-sums α and column-sums β.

For the second part, use $h_\lambda = \sum_\alpha |\operatorname{Mat}(\alpha,\lambda)| m_\alpha(x)$. □

Proposition 18.5.

$$Q(\overline{x},\overline{y}) = \sum_\lambda z_\lambda p_\lambda(\overline{x}) p_\lambda(\overline{y}).$$

Theorem 18.1.

$$Q(\overline{x},\overline{y}) = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Proof. This follows from RSK. The coefficient of $x^\alpha y^\beta$ in $s_\lambda(x)s_\lambda(y)$ is $|\operatorname{SSYT}(\lambda,\alpha)| \cdot |\operatorname{SSYT}(\lambda,\beta)|$. □

When Richard Stanley wrote his textbook, he didn't believe that Cauchy proved this identity. After looking through hundreds of pages of Cauchy's works, he determined that Cauchy did not actually prove this identity. After more research, he found that Cascoux was the first person to cite the identities as theorems of Cauchy. So in fact, none of these Cauchy-type identities are due to Cauchy.

Proposition 18.6. $\{f_\alpha\}$ *is an orthonormal basis of* Λ *if*

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda f_\lambda(x) f_\lambda(y).$$

Corollary 18.2. $\{p_\lambda/\sqrt{z_\lambda}\}$ *is an orthonormal basis of* λ*.*

19 The Jacobi-Trudi Identity

19.1 Connection to Frobenius' theorem

Theorem 19.1 (Jacobi-Trudy[12])**.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$*. Then*

$$s_\lambda = \det(h_{\lambda_{1-i+j}})_{i,j=1}^{\ell}.$$

This should remind you of the following theorem:

Theorem 19.2 (Frobenius)**.**

$$\chi^\lambda = \sum_{\omega \in S_\ell} \operatorname{sgn}(\omega) \zeta^{\lambda + \omega\rho - \rho},$$

where ζ^μ *is the character of* M^μ *and* $\rho(\ell-1, \ell-2, \ldots, 1, 0)$*.*

In fact, these are the same.

Example 19.1. Let $\lambda = (4,2,1)$, so $\ell = 3$. Then

$$s_{(4,3,2)} = \begin{vmatrix} h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = h_{(4,2,1)} - h_{(5,1,1)} - h_{(4,3)} + h_{(6,1)}$$

The Frobenius formula in this case says

$$\begin{aligned} \chi^{(4,2,1)} &= \zeta^{(4,2,1)-(2,1,0)+(2,1,0)} - \zeta^{(4,2,1)-(1,2,0)+(2,1,0)} \\ &\quad - \zeta^{(4,2,1)-(2,0,1)+(2,1,0)} + \zeta^{(4,2,1)-(0,2,1)+(2,1,0)} + 0 + 0 \\ &= \zeta^{(4,2,1)} - \zeta^{(5,1,1)} - \zeta^{(4,3)} + \zeta^{(6,1)}. \end{aligned}$$

19.2 Proof of the identity

The idea of the proof is term cancellation. We expand the determinant and show that a lot of terms cancel. Here is an analogy.

Proposition 19.1. *Let* $A = (a_{i,j})$*,* $B = (b_{i,j})$ *be* $n \times n$ *matrices. Then* $\det(AB) = \det(A)\det(B)$*.*

You prove this by representing both sides as counting something and showing that the extra terms in $\det(AB)$ compared to $\det(A)\det(B)$ cancel out in pairs.

[12] You can find this in section 7.16 of Richard Stanley's Enumerative Combinatorics volume 2.

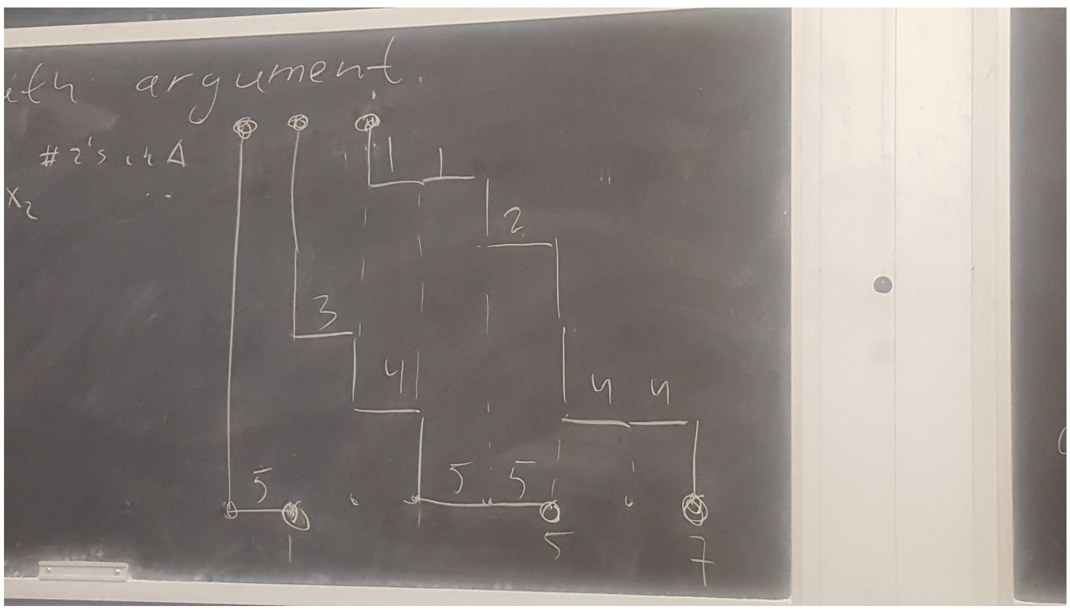
Proof. The idea is a non-crossing path argument, like you can use to prove the statement about determinants.

$$S_\lambda = \sum_{A\,\mathrm{SSYT}(\lambda)} x^A = \sum_A x_1^{\#1\mathrm{s}} x_2^{\#2\mathrm{s}} \cdots$$

Take A and construct a system of paths. For example, if

$$A = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 \\ \hline 3 & 4 & 5 & 5 & \\ \cline{1-4} 5 & & & & \\ \cline{1-1} \end{array}$$

construct the system of paths



Fix the starting points and ending points. Now consider all systems of paths from the starting points to the ending points. A path Q will produce the monomial $x^Q = x_1^{\#1\mathrm{s}} x_2^{\#2\mathrm{s}} \cdots$, and if we switch two paths that intersect, we get a $\mathrm{sgn}(\sigma)$ coefficient in front. Then each system with a pair of paths that intersect somewhere will not be counted because we can just switch the paths after the intersection point; in this case, the contribution of the paths to the sum will cancel. Since the paths define polynomials h_λ with this definition, we are done. $\square$

20 Littlewood-Richardson Coefficients

20.1 Multiplying symmetric functions

Recall

$$s_\lambda = \sum_{A \in \mathrm{SSYT}(\lambda)} x^A, \qquad x^A = x_1^{\#1\text{s in } A} x_2^{\#2\text{s in } A} \cdots$$

We can multiply many of the different bases of Λ:

$$e_\lambda e_\mu = e_{\lambda \cup \mu},$$

$$h_\lambda h_\mu = h_{\lambda \cup \mu},$$

$$p_\lambda p_\mu = p_{\lambda \cup \mu}.$$

And multiplying $m_\lambda m_\mu$ is straightforward. What about multiplying Schur functions?

Let $|\mu| + |\nu| = 1$. Then

$$s_\mu s_\nu = \sum_{|\lambda|=n} c^\lambda_{\mu,\nu} s_\lambda$$

What are the coefficients $c^\lambda_{\mu,\nu}$?

Proposition 20.1. $c^\lambda_{\mu,\nu} \in \mathbb{N}$.

Proof. Let S^ν, S^λ be irreducible representations. Then $s_\mu s_\nu$ corresponds to $\mathrm{ind}_{S_k \times S_{n-K}}^{S_n} S^\mu \otimes S^\nu$. So $c^\lambda_{\mu,\nu}$ is the inner product of S^λ with this induced character. This is the dimension of the irreducible representation S^λ in this representation. $\square$

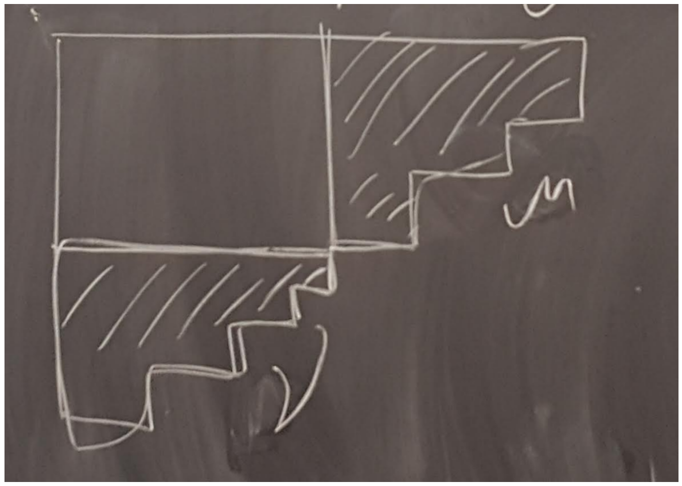
Theorem 20.1. $c^\lambda_{\mu,\nu} = \#\,\mathrm{LR}(\lambda/\mu, \nu)$, *the number of a certain type of semistandard Young tableaux.*

This is difficult to prove.[13]

[13] It is so difficult that Stanley did not actually prove it in his textbook.

20.2 Multiplying Schur functions

Let $\mu \circ \nu$ be the skew shape



Then

$$s_\mu s_\nu = s_{\mu\circ\nu} = \sum_{A \in \mathrm{SSYT}} x^A = \sum_{|\lambda|=n} c^\lambda_{\mu,\nu} s_\lambda$$

How do we determine a tableau with shape $\mu \circ \nu$? Take the skew-shape and reduce it using Jeu-de-taquin.

Example 20.1. We reduce the skew tableau

		1	1
		3	
1	2		
3	3		

to the tableau

1	1	1
2	3	3
3		

So $c^\lambda_{\mu,\nu}$ is the multiplicity of any $P \in \mathrm{SSYT}(\lambda)$ as a jeu-de-taquin of $B \circ C$, where $B \in \mathrm{SSYT}(\mu)$ and $C \in \mathrm{SSYT}(\nu)$.

Corollary 20.1. *$c^\lambda_{\mu,\nu} \in \#P$.*

There is a polynomial algorithm, jeu-de-taquin, for determining if B and C produce the correct tableau. But this is a very messy combinatorial interpretation. There is a better interpretation.

20.3 Ballot sequences

Definition 20.1. $(a_1, \ldots, a_n)$ is a **ballot sequence** if for all $k \in [n]$, the number of is among $a_1, \ldots, a_k$ is greater than the number of $(i+1)$s among $a_1, \ldots, a_k$ for all i.

Example 20.2. The sequence $(1, 1, 2, 1, 1, 2, 3, 3, 1, 2, 3)$ is a ballot sequence.

$\mathrm{Cat}(n)$ is the number of ballot sequences with n 1s and n 2s. Young tableau are basically the same as ballot sequences; if the number i in our tableau is in row j, we can make the i-th term in the sequence j.

When we have a pair of tablueux that we arrange into a skew shape, form a sequence by listing the numbers in each row from left to right, going down in rows.

Example 20.3.

$$\begin{array}{cccc} & & 1 & 1 \\ & & 3 & \\ 1 & 2 & & \\ 3 & 3 & & \end{array}$$

gives us the sequence $(1, 1, 3, 3, 1, 3, 3)$.

Theorem 20.2. *$c^{\lambda}_{\mu,\nu} = \#\,\mathrm{SSYT}(\nu, \lambda \setminus \mu)$ such that the sequence obtained from $B \circ C$ is a ballot sequence, where $B \in \mathrm{SSYT}(\mu)$ and $C \in \mathrm{SSYT}(\nu)$.*

Next time we will discuss the following.

Corollary 20.2. *$c^{\lambda}_{\mu,\nu}$ is the number of integer points in a polytope defined by the vectors λ, μ, ν.*

Theorem 20.3. *It can be determined in polynomial time whether $c^{\lambda}_{\mu,\nu} = 0$.*

21 Complexity Aspects of Young Tableau

21.1 Generating random Young tableau

Given λ, we want to generated a random $A \in \mathrm{SYT}(\lambda)$. We have two algorithms for doing this:

1. NPS algorithm: the sort of 2-dimensional bubblesort on tableaux
2. GNW algorithm: Place n according to a hook walk, update λ to be $\lambda \setminus \{n\}$, and repeat.

Which one is faster?

Example 21.1. Let $\lambda = (n)$, which will produce a 1-row tableau. Then NPS will take $\mathbb{E}[\mathrm{inv}(\sigma)] = \Theta(n^2)$ steps. Alternatively, GNW will take $\Theta(n \log(n))$ steps.

Example 21.2. Let λ be the shape of a $k \times k$ square tableau, $\lambda = (k^k)$. Here, $n = k^2$. Then GNW will run in $\Theta(n \log(n))$ time. What about NPS? It goes column by column, and each column takes $\Theta(k^2)$ time. So does NPS run in $\Theta(k^3) = \Theta(n^{3/2})$ time? We really want to look at the average complexity. It turns out that the average complexity is $\Theta(k^3) = \Theta(n^{3/2})$ anyway.

21.2 Generating random partitions

Given n, we want to uniformly generate a random partition $|\lambda| = n$. Here is a (not too) slow algorithm:

Compute $p_k(n)$, the number of partitions λ of n such that $\lambda_1 = k$ parts. Then $p_k(n) = p_1(n-k) + p_n(n-k) + \cdots + p_k(n-k)$ and $p_1(n) = 1$, so we can solve the recurrence relation. This takes $\Theta(n^{3.5})$ steps.

What is a smarter algorithm? We know

$$\sum_{n=0}^{\infty} p(n) t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i}$$

Write this as the following:

$$= \sum_{\lambda \in \mathcal{P}} t^{|\lambda|}.$$

Now think of this probabilistically. Let $Z_i + 1 \sim \mathrm{Geom}(1-t^i)$. Then let $\lambda = 1^{Z_1} 2^{Z_2} \cdots$. Now

$$\mathbb{P}(\lambda) = \mathbb{P}(Z_1 = m_1(\lambda), Z_2 = m_2(\lambda), \dots) = \prod_{i=1}^{\infty} t^{i m_i}(1-t^i) = t^{\sum i m_i} \prod_{i=1}^{n} (1-t^i).$$

If we generate λ like this, then $|\lambda| = n \iff Z_1 + \cdots + 2Z_2 + \cdots = n$. So we can generate all these and output λ. So we get

$$\mathbb{P}(|\lambda| = n) = p(n)t^n \prod_{i=1}^{n}(1 - t^i) \sim e^{c\sqrt{n}} t^n \prod_{i=1}^{n}(1 - t^i).$$

Optimize over all possible values of t. Then the number of steps becomes $\Theta(n^{3/4})$ or so. This idea is called **Boltzmann sampling**.

21.3 Generating random 3-dimensional partitions

Given n, we want a random 3-dimensional (called solid/plane) partition A. We proved[14] the generating function

$$\sum_{A \in \mathcal{PP}} t^{|A|} = \sum_{n=0}^{\infty} \mathcal{PP}(n) t^n = \prod_{i=1}^{\infty} \frac{1}{(1 - t^i)^i}.$$

We can do the same type of sampling as before. Let $Z_{i,j} \sim \text{Geom}(1 - t^i)$, where $1 \le j \le i$. Place the $Z_{i,j}$ in a matrix:

$$\begin{matrix} Z_{1,1} & Z_{2,1} & Z_{3,1} & \cdots \\ Z_{2,2} & Z_{3,2} & \cdots & \\ Z_{3,3} & \cdots & & \end{matrix}$$

Flip the matrix across the (non-main) diagonal and apply the Hilman-Grassl algorithm.

21.4 Generating random skew shapes

Suppose we have the skew shape $\lambda \setminus \mu$. How do we generate a random $A \in \text{SYT}(\lambda \setminus \mu)$? The Jacobi-Trudy identity gives us

$$s_\lambda = \det([h_{\lambda_i - i + j}]).$$

Then we get

$$f^\lambda = n! \det\left(\left[\frac{1}{(\lambda_i - i + j)!}\right]\right)$$

We can calculate the probability of putting n into a corner using

$$\mathbb{P}(n \text{ in corner}) = \frac{f^{\lambda \setminus \{n\}}}{f^\lambda}$$

Here, we use a generalized version of the Jacobi-Trudy identity:

[14] Maybe the word 'proved' should be taken lightly.

Theorem 21.1 (Jacobi-Trudy)**.** *Let*

$$s_{\lambda\backslash\mu} := \sum_{A \in \mathrm{SSYT}(\lambda\backslash\mu)} t^{|A|}.$$

Then

$$s_{\lambda\backslash\mu} = \det([h_{\lambda_i-\mu_i-i+j}])$$

22 Topics on Skew Tableau and Skew Schur Functions

22.1 Complexity of the number of skew tableau

Recall that **skew Schur functions** satisfy

$$s_{\lambda\setminus\mu} = \sum_{A \in \mathrm{SSYT}(\lambda\setminus\mu)} x^A.$$

Let $f^{\lambda\setminus\mu}$ be the number of $\mathrm{SSYT}(\lambda \setminus \mu)$.

Theorem 22.1 (Jacoi-Trudy)**.**

$$f^{\lambda\setminus\mu} = n! \det\left[\left(\frac{1}{(\lambda_i - \mu_i + j - i)!}\right)\right]$$

Corollary 22.1. *$f^{\lambda\setminus\mu}$ can be computed in polynomial time.*

Theorem 22.2 (Pittmer-Pak)**.** *Let D be an arbitrary shape of a diagram (not necessarily a skew shape). Then f^D is $\#P$ complete.*

This is equivalent to the following theorem.

Definition 22.1. $\mathrm{Bruhat}(\sigma) = \{\omega \in S_n : \omega \leq \sigma\}$.

Theorem 22.3. $|\mathrm{Bruhat}(\sigma)|$ *is $\#P$ complete.*

22.2 Littlewood-riichardson coefficients for skew Schur functions

Recall that $s_\mu s_\nu = \sum_{|\lambda|=n} c^\lambda_{\mu,\nu} s_\lambda$, where $c^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients.

Theorem 22.4.

$$s_{\lambda\setminus\mu} = \sum_{|\nu|=n-k} c^\lambda_{\mu,\nu} s_\nu.$$

Proof. The idea is to perform jeu-de-taquin on the skew tableau A of shape $\lambda \setminus \nu$. $\square$

One interpretation of this is that we can use Schur functions to construct $s_{\lambda\setminus\nu}$, which are symmetric functions that end up being nice.

Recall that

$$\begin{aligned} c^\lambda_{\mu,\nu} &= \left\langle S^\lambda, S^\mu \otimes S^\nu \uparrow^{S_n}_{S_k\times S_{n-k}} \right\rangle \\ &= \left\langle S^\lambda \downarrow^{S_n}_{S_k\times S_{n-k}}, S^\mu \otimes S^\nu \right\rangle. \end{aligned}$$

by Frobenius reciprocity. Then if π_λ is an irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ corresponding to λ and s_λ is a character of π_λ, then

$$c^\lambda_{\mu,\nu} = \langle \pi_\lambda, \pi_\mu \otimes \pi_\nu \rangle.$$

Theorem 22.5 (Knutsen-Tao, c. 2000). *For all k, $c^{\lambda}_{\mu,\nu} > 0$ iff $c^{k\lambda}_{k\mu,k\nu} > 0$.*

Corollary 22.2. *It can be decided if $c^{\lambda}_{\mu,\nu} = 0$ in polynomial time.*

Proof. $c^{\lambda}_{\mu,\nu} \neq 0$ if there exists a rational point in some certain polytope $P(\lambda, \mu, \nu)$ containing $c^{\lambda}_{\mu,\nu}$ integer points. □

Theorem 22.6. *Computing $c^{\lambda}_{\mu,\nu}$ is $\#P$ complete.*

This is related to the following problem. When does $A + B = C$, when A, B, C are Hermitian matrices?

Theorem 22.7 (Klyacko). *Let A, B, C be hermition matrices with (vector of) eigenvalues μ, ν, λ, respectively. Then there exist matrices A, B, C solving $A + B = C$ iff $c^{\lambda}_{\mu,\nu} \neq 0$.*

23 Inequalities in Algebraic Combinatorics

23.1 Largest number of tableau for a partition of n

Proposition 23.1. $(f^\lambda)^2 \leq n!$

Proof. This is because $\sum_{|\lambda|=n}(f^\lambda)^2 = n!$. □

Here is a restatement of this fact:

Corollary 23.1. $|\operatorname{SYT}(\lambda)|^2 \leq n!$.

This seems much less obvious, and requires RSK to prove it directly.

Corollary 23.2. *Denote $D(n) = \max_{|\lambda|=n} f^\lambda$. Then $D(n) \geq \sqrt{n!/p(n)}$, where $p(n)$ is the number of partitions of n.*

Here is a conjecture:

Theorem 23.1. *The number of $|\lambda| = n$ such that $f^\lambda = D(n)$ is $O(1)$.*

This is open, even though it seems like it should be obvious. In fact, we don't know if it is $e^{O(\sqrt{n})}$. The following, however, is known.

Theorem 23.2 (V-K)**.** *$D(n) < \sqrt{n!}\alpha^{\sqrt{n}}$ for some $\alpha > 1$.*

So we have this upper bound and the lower bound $\sqrt{n!}/\beta^{\sqrt{n}}$. Here is a conjecture that Professor Pak wants to prove:

Theorem 23.3. *The following limit exists:*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\left(\frac{D(n)}{\sqrt{n!}}\right).$$

In 1954, someone at Los Alamos, used extra computing power to compute character tables of S_n for $n \leq 15$. They became interested in $D(n)$ and conjectured that $D)n_{\leq}\sqrt{n!}/n$. This was proven false about 15 years later.

Theorem 23.4 (Bufetov)**.** *Let $H(n) = 1/p(n)\sum_{|\lambda|=n} f^\lambda$. Then the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\left(\frac{H(n)}{\sqrt{n!}}\right).$$

Theorem 23.5 (V-K)**.** *Let $f^\lambda = D(n)$. Then the shape of λ (scaled by $\sqrt{n}$) looks like a rotated version of the graph*

$$\Phi(x) = \frac{2}{\pi}(x \arcsin(x/\sqrt{2}) + \sqrt{2-x}).$$

Corollary 23.3. *The average longest increasing subsequence of a permutation is $2\sqrt{n}$.*

23.2 Bounds for Littlewood-Richardson coefficients

Theorem 23.6 (PPY)**.** $(c^{\lambda}_{\mu,\nu})^2 \leq \binom{n}{k}$.

The upper bound is actually somewhat tight: the idea is to show that

$$\sum_{|\lambda|=n} \sum_{|\mu|=k, |\nu|=n-k} (c^{\lambda}_{\mu,\nu})^2 = \sum_{\alpha \in \mathrm{conj}(H=S_k \times S_{n-k})} \frac{z_\alpha(S_n)}{z_\alpha(H)} \alpha \overset{\geq}{=} 1 \binom{n}{k}.$$

Then

$$\max_{\lambda,\mu,\nu} c^{\lambda}_{\mu,\nu} \geq \frac{\sqrt{\binom{n}{k}}}{\sqrt{p(k)}p(n-k)p(n)}.$$

Proof. The idea of the proof of the theorem is to show that $\binom{n}{k} f^\mu f^\nu$ is the dimension of $\mathrm{ind}^{S_n}_{S_k \times S_{n-k}} S^\mu \otimes S^\nu$ and decompose the representation into irreducible repesentations. Then

$$\sum_{|\mu|=k} \sum_{|\nu|=n-k} c^{\lambda}_{\mu,\nu} f^\mu f^\nu = f^\lambda,$$

and

$$\sum_{|\lambda|=n} (c^{\lambda}_{\mu,\nu})^2 \leq \sum_{|\lambda|=n} c^{\lambda}_{\mu,\nu} \frac{f^\lambda}{f^\mu f^\nu} = \frac{1}{f^\mu f^\nu} f^\mu f^\nu \binom{n}{k} = \binom{n}{k}.$$

So $(c^{\lambda}_{\mu,\nu})^2 \leq \binom{n}{k}$. □

Theorem 23.7 (PPY, 2018)**.** *There exist $\lambda, \mu\, \nu$ such that $c^{\lambda}_{\mu,\nu} = 2^n / e^{O(-\sqrt{n})}$.*

23.3 Bounds on the number of skew tableau of size n

Let $f^{\lambda \setminus \mu} = |\operatorname{SYT}(\lambda \setminus \mu)|$. We know that this number is the determinant of a matrix we get from λ, μ. Can we understand this number better? Our previous considerations give us the following:

Proposition 23.2. *Let $|\lambda| = n$ and $|\mu| = k$. Then*

$$f^{\lambda \setminus \mu} \leq \sqrt{\binom{n}{k}} p(n-k) \sqrt{(n-k)!}.$$

Proof. This follows from the previous inequalities applied to the identity:

$$f^{\lambda \setminus \mu} = \sum_{|\nu|=n-k} c^{\lambda}_{\mu,\nu} f^\nu.$$ □

What about lower bounds?

Theorem 23.8 (Naruse,MPP)**.**

$$f^{\lambda\backslash\mu} = n! \sum_{D\in\mathcal{E}(\lambda\backslash\mu)} \prod_{(i,j)\notin D} \frac{1}{h_{i,j}},$$

where $\mathcal{E}(\lambda \setminus \mu)$ is the set of "excited diagrams" (start with chips in the removed shape μ, and move them to the right or down to get a configuration in $\lambda \setminus \mu$).

Example 23.1. Suppose $\lambda = (3, 3)$ and $\mu = (2)$. Then we start with

$$\begin{array}{|c|c|c|}\hline \star & \star & \\ \hline & & \\ \hline \end{array}$$

We get 3 excited diagrams:

$$\begin{array}{|c|c|c|}\hline \star & \star & \\ \hline & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline \star & & \\ \hline & & \star \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & \star & \star \\ \hline \end{array}$$

If we take only the first term of the sum, we get the following lower bound:

Corollary 23.4.

$$f^{\lambda\backslash\mu} \geq n! \prod_{(i,j)\in\lambda\backslash\mu} \frac{1}{h_{i,j}}$$

24 D-Finite and Polynomially Recursive Series

Note: Today's lecture is a guest lecture. The lecture material is from section 6.4 of Stanley's Enumerative Combinatorics (Volume 2).

24.1 D-finite series

Let K be a field. We have the ring of formal power series $K[\![x]\!]$.

Definition 24.1. If $u = \sum_n f(n)x^n$, then the **formal derivative** is $\frac{d}{dx}u = u' = \sum_n nf(n)x^{n-1}$.

Proposition 24.1. *Let $u \in K[\![x]\!]$. The following are equivalent.*

1. $\dim_{K(x)}(K(x)u + K(x)u' + K(x)u'' + \cdots) < \infty$.
2. *There are $p_0, \ldots, p_\ell \in K[x]$ with $p_\ell \neq 0$ such that*
$$p_d u^{(d)} + p_{d-1}u^{(d-1)} + \cdots + p_1 u' + p_0 u = 0.$$
3. *There are $q_0, \ldots, q_m, q \in K[x]$ with $q_m \neq 0$ such that*
$$q_m u^{(m)} + q_{m-1}u^{(m-1)} + \cdots + q_1 u' + q_0 u = q.$$

Proof. (1) $\implies$ (2): Suppose $\dim = d$. Then $u, u', \ldots, u^{(d)}$ are linearly dependent over $K(x)$. Write down the dependence relation, and clear the denominators to get the p_i.

(2) $\implies$ (3): This is a special case.

(3) $\implies$ (2): Suppose that $\deg_x(q(x)) = t \geq 0$. Differentiate the polynomial relation $t+1$ times to get a homogeneous relation involving the derivatives of u. We get $p_d = q_m \neq 0$. Solve for $u^{(d)}$ to get
$$u^{(d)} \in K(x)u + K(x)u' + \cdots + K(x)u^{(d-1)}.$$
Writing $u^{(d)}$ as a linear combination of $u, \ldots, u^{(d-1)}$ and differentiating gives $u^{(d+1)} \in K(x)u + \cdots + K(x)u^{(d)} = K(x)u + \cdots + K(x)u^{(d-1)}$. By induction, for all $k \geq 0$, $u^{(d+k)} \in K(x)u + \cdots + K(x)u^{(d-1)}$. $\square$

This allows us to make the following definition.

Definition 24.2. $u \in K[\![x]\!]$ is ***D*-finite** if any of these three conditions hold for u.

Example 24.1. $u = e^x = \sum_n x^n/n!$ is D-finite. This satisfies $u' = u$, so $u' - u = 0$. In general, $u = x^m e^{ax}$ is D-finite, as $u' = mx^{m-1}e^{ax} + ax^m e^{ax} = (m/x + a)u$.

Example 24.2. Suppose $u = \sum_{n\geq 0} n!x^n$. Then $(xu)' = \sum_{n\geq 0}(n+1)!x^n$, so we see that $1 + x(xu)' = u$. That is, $x^2u' + (x-1)u = -1$, so u is D-finite.

24.2 Polynomially recursive series

Definition 24.3. Say $f : \mathbb{N} \to K$ is **P-recursive** (or **polynomially recursive**) if there are $P_0, \dots, P_e \in K[x]$ with $P_e \neq 0$ such that

$$P_e(n)f(n+e) + P_{e-1}(n)f(n+e-1) + \cdots + P_0(n)f(n) = 0$$

for all $n \in \mathbb{N}$.

Proposition 24.2. *Let $u = \sum_{n \geq 0} f(n)x^n \in K[\![x]\!]$. Then u is D-finite if and only if f is P-recursive.*

Proof. First, suppose u is D-finite. Then we have polynomials p_I such that

$$p_e u^{(d)} + \cdots + p_1 u' + p_0 u = 0.$$

Then $x^j u^{(i)} = \sum_{n \geq 0} (u+i-j)_i f(u+i-j)x^n$, where $(u+i-j)_i$ is the falling factorial. For $k \gg 0$, equate coefficients of x^{n+k} in the polynomial relation to get the recurrence. Note that if $[x^j]p_d(x) \neq 0$, then $[n^d]P_{d-j+k}(n) \neq 0$ (where brackets denote the coefficient of the term inside).

Now suppose f is P-recursive. Then f satisfies a relation

$$P_e(n)f(n+e) + P_{e-1}(n)f(n+e-1) + \cdots + P_0(n)f(n) = 0$$

with $P_e \neq 0$. For each i, $\{(n+i)_j : j \geq 0\}$ is a basis for the K-vector space $K[u]$. So $P_i(n)$ is a K-linear combination of $(n+i)_j$s. So $\sum_n P_i(n)f(n+i)x^n$ is a K-linear combination of series of the form $\sum_{n \geq 0} (n+i)_j f(n+i)x^n$. Now $\sum_{n \geq 0} (n+i)_j f(n+i)x^n = R_i(x) + x^{j-i}u^{(j)}$, where $R_i \in x^{-1}K[x^{-1}]$. For example, $x^{-1}u' = \sum_{n \geq -1} (n+2)f(n+2)x^n = f(1)x^{-1} + \sum_{n \geq 0} (n+2)f(n+2)x^n$. Now multiply the relation by x^n and sum over $n \geq 0$ to get

$$0 = \sum a_{i,j} x^{j-i} u^{(j)} + R(x).$$

This sum is finite, $a_{i,j} \in K$ are not all 0, and $R(x) \in x^{-1}K[x^{-1}]$. Multiply by x^q with $q \gg 0$. We get that u is D-finite. □

Example 24.3. Let $u = e^x$. Another way to show that u is D-finite is to show that $f(n) = n!$ is P-recursive: $f(n+1) - (n+1)f(n) = 0$.

Example 24.4. Let $f(n) = \binom{2n}{n}$. This is P-recursive: $(n+1)f(n+1) - 2(2n+1)f(n) = 0$. So $u = \sum_{n \geq 0} \binom{2n}{n} x^n$ is D-finite. This is the series for $1/\sqrt{1-4x}$.

Proposition 24.3. *Suppose $f, g : \mathbb{N} \to K$ is P-recursive and $f(n) = g(n)$ for all sufficiently large n. Then g is P-recursive.*

Proof. Suppose $f(n) = g(n)$ for $n \geq n_0$. Then

$$\left(\prod_{j=0}^{n_0-1}(n-j)\right)[P_e(n)g(n+e)+\cdots+P_0(n)g(n)] = 0,$$

so g is P-recursive. $\square$

Theorem 24.1. *Suppose $u \in K[\![x]\!]$ is algebraic over $K(x)$ of degree d. Then u is D-finite and satisfies a polynomial equation of order d.*

Proof. Let $P(Y) \in K(x)[Y]$ be the minimal polynomial of u over $K(x)$. Suppose $P(Y) = \sum_{i=0}^{d} p_i Y^i$. Then $P(u) = 0$. Differentiate to get

$$0 = (P(u))' = \left(\sum_{i=0}^{d} p_i u^i\right)' = \underbrace{\sum_{i=0}^{d} p_i' u^i}_{=Q(u)} + \underbrace{\sum_{i=0}^{d} i p_i u^{i-1} u'}_{=(\frac{\partial P}{\partial Y}(u))u'}$$

The derivarive $\frac{\partial P}{\partial Y}(u) \neq 0$. So $u' = Q(u)/(\frac{\partial P}{\partial Y}(u)) \in K(x)(u)$. Similarly, $u^{(n)} \in K(x)(u)$ for all N. Then we get linear dependence among $u, u', \ldots, u^{(d)}$. $\square$

25 Operations on D-Finite Series

Note: Today's lecture is a guest lecture.

25.1 Addition, multiplication, and composition of D-finite series

Last time, we showed that $u \in K[\![x]\!]$ is D-finite iff $\dim_{K(x)}(\text{span}(\{u, u', u'', \dots\})) < \infty$.

Theorem 25.1. *The set D of D-finite $u \in K[\![x]\!]$ is a subalgebra of $K[\![x]\!]$. If $u, v \in D$ and $\alpha, \beta \in K$, then $\alpha u + \beta v \in D$, and $uv \in D$.*

Proof. Given $w \in K[\![x]\!]$, let $V_w = \text{span}_{K(x)}(\{w, w', w'', \dots\}) \subseteq K((x))$. Suppose $u, v \in D$, $\alpha, \beta \in K$, and let $y = \alpha u + \beta v$. Then $y, y', y'', \dots \in V_u + V_v$. Thus, $\dim(V_y) \leq \dim(V_u + V_v) \leq \dim(V_u) + \dim(V_v) < \infty$.

Next, let $u, v \in D$. Consider $\phi : V_u \otimes_{K(x)} V_r \to K((x))$ defined by $\phi(y \otimes z) = yz$ for all $y \in V_u$ and $z \in V_v$. The product rule implies $V_{uv} \subseteq \phi(V_u \otimes_{K(x)} V_v)$; indeed, $(uv)^{(i)} = \sum_{j=0}^{i} \binom{i}{j} u^{(i)} v^{(i-j)}$. Thus, $\dim(V_{uv}) \leq \dim(V_u \otimes_{K(x)} V_v) = \dim(V_u)\dim(V_v) < \infty$, so $uv \in D$. $\square$

Theorem 25.2. *Let $u \in D$ and $v \in K_{\text{alg}}[\![x]\!]$ (i.e. $v \in K[\![x]\!]$ and v is algebraic over $K(x)$) with $v(0) = 0$. Then $u(v(x)) \in D$.*

Proof. Let $y = u(v(x))$. Then $y' = u'(v(x))v'(x)$, $(u'(v(x)))' = u''(v(x))v'(x)$, etc. In general, $y^{(i)}$ is a linear combination of $u(v(x)), u'(v(x)), u''(v(x)), \dots$ with coefficients in $K[v, v', v'', \dots]$. Since v is algebraic over $K(x)$, $v^{(i)} \in K(x, v)$ for all i (proved last time). Thus, $K[v, v', \dots] \subseteq K(x, v)$.

Let $V = \text{span}_{K(x)}(\{u(v(x)), u'(v(x)), \dots\}) \ni y^{(i)}$ for all i. We want to show that $\dim_{K(x)}(V) < \infty$. Since u is D-finite, $\dim_{K(x)}(\text{span}_{K(x)}(\{u(x), u'(x), \dots\})) < \infty$. By "specializing x at v," $\dim_{K(x)}(\text{span}_{K(x)}(\{u(v(x)), u'(v(x)), \dots\})) < \infty$. So we get that $\dim_{K(x,v)}(\text{span}_{K(x,v)}(\{u(v(x)), u'(v(x)), \dots\})) < \infty$. Then $\dim_{K(x,v)}(V) < \infty$, and (since v is algebraic over $K(x)$) $[K(x, v) : K(x)] < \infty$, so

$$\dim_{K(x)}(V) = (\dim_{K(x,v)}) \cdot [K(x, v) : K(x)] < \infty. \qquad \square$$

Example 25.1. We know that $\sum_{n\geq 0} n!x^n$, e^x, and $\frac{x}{\sqrt{1-4x}}$ are in D. So we can get that $u = (\sum_{n\geq 0} n!x^n)e^{x/\sqrt{1-4x}} \in D$. This would be difficult to do by hand without the results we have proved.

25.2 Hadamard products of P-recursive series

Given $h : \mathbb{N} \to K$ and $R(n) \in K(n)$, we want to define $Rh : \mathbb{N} \to K$ by $Rh(n) = R(n)h(n)$. But this could be undefined when $R(n) = \infty$. Here is the solution: given $h_1, h_2 : \mathbb{N} \to K$, say $h_1 \sim h + 2$ if $h_1(u) = h_2(u)$ for all $n \gg 0$. Call $[h]$, the equivalence class of h, the **germ** of h. Define $\mathcal{G} = \{[h] \mid h : \mathbb{N} \to K\}$; this is a $\mathbb{K}(n)$ vector space. Note: if $g \sim h$, the nh is P-recursive iff g is P-recursive . For each $h : \mathbb{N} \to K$, set $\mathcal{G}_h = \text{span}_{K(n)}(\{[h(n)], [h(n+1)], [h(n+2)], \dots\})$.

Fact: h is P-recursive iff $\dim_{K(n)} \mathcal{G}_h < \infty$.

Definition 25.1. Given $u = \sum_n f(n)x^n$ and $v = \sum_n g(n)x^n$, define the **Hadamard product** $u * v = \sum_{n \geq 0} f(n)g(n)x^n$.

Theorem 25.3. *If $f, g : \mathbb{N} \to K$ are P-recursive, then so is fg. That is, $u, v \in D \implies u * v \in D$.*

Proof. It is sufficient to show that if $[f], [g]$ are P-recursive (element member of the germ is recursive), then so is $[fg]$. Define $\phi : \mathcal{G}_f \otimes_{K(n)} \mathcal{G}_g \to \mathcal{G}$ such that for each i, j, on simple tensors, $\phi([f(n+i)] \otimes [g(n+j)]) = [f(n+i)][g(n+j)] = [f(n+i)g(n+j)]$. So the image of ϕ contains $\mathcal{G}_{fg} = \text{span}_{K(n)}(\{[f(n)g(n)], [f(n+1)g(n+1)], \dots\})$. So

$$\dim_{K(n)}(\mathcal{G}_{fg}) \leq \dim_{K(n)}(\mathcal{G}_f \otimes \mathcal{G}_g) = (\dim(\mathcal{G}_f))(\dim(\mathcal{G}_g)) < \infty.$$

So fg is P-recursive. $\square$

25.3 Fun facts

Here are some fun facts from Professor Pak's 2016 206A notes:

Theorem 25.4. *Let $S \subseteq \mathbb{Z}^d$ with $|S| < \infty$. Let a_n be the number of walks $0 \to 0$ of length n on $\mathbb{Z}^d$ with steps in S. Then (a_n) is P-recursive.*

Theorem 25.5. *If $a_n = |\{\sigma \in S_n : \sigma^2 = 1\}|$, then $a_n = a_{n-1} + (n-1)a_{n-2}$, so (a_n) is P-recursive.*

Definition 25.2. Given $F = \sum f(n_1, n_2, \dots, n_r)x_1^{n_1} \cdots x_r^{n_r} \in F[\![a_1, \dots, x_r]\!]$, define the **diagonal**, $\text{diag}(F) \in K[\![t]\!]$, by

$$(\text{diag}(F))(t) = \sum_n f(n, n, \dots, n)t^n.$$

Theorem 25.6 (Furstenberg)**.** *Suppose $F(s,t) \in K[\![s,t]\!] \cap K(s,t)$. Then $\text{diag}(F)$ is algebraic. If $P, Q \in \mathbb{Z}[x_1, \dots, x_r]$, then $\text{diag}(P/Q)$ is D-finite.*

The proof of this theorem involves Puiseaux series.

Remark 25.1. The converse is also true. An algebraic single variable power series is the diagonal of such a multivariable series.