

# Math 206A Lecture 2 Notes

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October 1, 2018

## 1 Helly's Theorem

### 1.1 Proof of Helly's theorem

**Theorem 1.1** (Helly). *Suppose  $X_1, \dots, X_n \subseteq \mathbb{R}^d$  are convex sets such that  $X_I \neq \emptyset$  for all  $|I| = d + 1$ , where  $X_I = \bigcup_{i \in I} X_i$ . Then  $X_1 \cap \dots \cap X_n \neq \emptyset$ .*

When  $d = 1$ , we have a collection of intervals where every pair of intervals intersect; then all intervals intersect. In this case the proof is elementary. Take the largest left endpoint  $a^*$  and the smallest right endpoint  $b^*$  of one of the intervals. Then  $a^* < b^*$ , so a point between  $a^*$  and  $b^*$  is contained in all the intervals.

However, when  $d = 2$ , the result is a little less obvious.

*Proof.* Let's prove the theorem for  $d = 2$ ,  $n = 4$ . Let  $J = \{1, \dots, n\}$ . Let  $y_i = X_{J \setminus \{i\}}$ . Either one of the  $y_i$  lies in the triangle formed by the three others or the  $y_i$  form a convex shape. In the first case, without loss of generality,  $y_4 \in X_1 \cap X_2 \cap X_3$ . But if  $y_1, y_2, y_3 \in X_4$ , then  $y_4 \in X_4$ . In the second case, find the point  $z$  at the intersection of the line segments connecting  $y_1$  to  $y_3$  and  $y_2$  to  $y_4$ . Then  $z \in X_2 \cap X_4$ , and  $z \in X_1 \cap X_3$ . So  $z \in X_J$ .

Now proceed by induction on  $n$ . Why does  $n$  imply  $n + 1$ ? The proof is the same, except we just include the points in  $y_i \in X_5, X_6, \dots$ . So in the first case, we just ignore the extra points, we get

$$z \in (X_2 \cap X_4 \cap X_5 \cap \dots \cap X_{n+1}) \cap (X_1 \cap X_3 \cap X_5 \cap \dots \cap X_{n+1}) = X_J$$

for the second case.

Before we prove the general case, we will state a lemma. □

**Lemma 1.1** (Radon). *Let  $y_1, \dots, y_m \in \mathbb{R}^d$ , where  $m \geq d + 2$ . Then there exist  $I, I' \neq \emptyset$  such that  $I \cap I' = \emptyset$  and the convex hull of  $\{y_i : i \in I\}$  intersects the convex hull of  $\{y_j : j \in I'\}$ .*

*Proof.* Let  $y_i = (y_i, \dots, y_d) \in \mathbb{R}^d$  with  $i = 1, \dots, m$ ,  $m \geq d + 1$ . Consider the system of equations  $\sum_{i=1}^m \tau_i = 0$  and  $\sum_{i=1}^m \tau_i y_{i,j} = 0$  for  $j \in \{1, \dots, d\}$ . These are  $d + 1$  equations. So there exist  $(\tau_1, \dots, \tau_m) \neq 0$  which satisfies the system. Let  $I = \{i : \tau_i > 0\}$  and  $I' = \{i : \tau_i < 0\}$ . Then

$$\frac{\sum_{i \in I} \tau_i y_i}{c} = \frac{\sum_{j \in I'} (-\tau_j) y_j}{c},$$

where  $c = \sum_{i \in I} \tau_i = \sum_{j \in I'} -\tau_j$ . □

Now we can prove the general case of Helly's theorem.

*Proof.* For general  $d$ , we induct on  $n$ . The base case is  $n = d + 1$ . By the lemma, we get  $z \in X_r$ , where  $r \notin I$ , and  $z \in X_s$ , where  $s \notin I'$ . So  $z \in X_J$ . □

## 1.2 Applications of Helly's theorem

**Corollary 1.1.** *Let  $R_1, \dots, R_n \subseteq \mathbb{R}^2$  be axis-parallel rectangles. Suppose  $R_1 \cap R_j \neq \emptyset$  for all  $i, j$ . Then  $R_1 \cap \dots \cap R_n \neq \emptyset$ .*

We could have proved this like we proved the case of  $d = 1$  because the intersection of rectangles is the pair of intersections of the corresponding intervals.

**Corollary 1.2.** *Let  $A \subseteq \mathbb{R}^2$  be a fixed convex set, and let  $X_1, \dots, X_n \subseteq \mathbb{R}^2$  be convex sets such that for  $|I| = 3$ , there exists some  $c \in \mathbb{R}^2$  such that  $X_i \cap (A + c) \neq \emptyset$  for all  $i \in I$ . Then there exists some  $c \in \mathbb{R}^2$  such that  $X_i \cap (A + c) \neq \emptyset$  for all  $i \in \{1, \dots, n\}$ .*

This says that if there is some translation where  $A$  intersects some of the  $X_i$  there is some translation where  $A$  intersects all of them.

*Proof.* Pick some point in  $a \in A$ , and look at all  $A$  translated by the extreme points of  $X_i$ . Let  $\hat{X}_i$  be the convex hull of the translated copies of  $a$ . Then  $\hat{X}_i$  is convex, so  $\hat{X}_I \neq \emptyset$  for all  $|I| = 3$ . By Helly's theorem,  $\hat{X}_J \neq \emptyset$ , which completes the proof. □

**Remark 1.1.** If we take  $A$  to be a point, we get the original statement of Helly's theorem for  $d = 2$ .