Math 255A' Lecture 11 Notes

Daniel Raban

October 21, 2019

1 Weak Closure of Convex Sets, Polars, and Alaoglu's Theorem

1.1 Weak closure of convex sets

Last time, we proved the following theorem:

Theorem 1.1. *Let X be a locally convex space.*

1. $(X, \mathrm{wk})^* = X^*$.
2. $(X^*, \mathrm{wk}^*)^* = X$.

Unlike with normed spaces, we can't just keep constructing duals and duals of duals. We can only construct $(X*, \mathrm{wk}^*)$ and its dual, (X, wk).

Theorem 1.2. *Let $A \subseteq X$. Then $\overline{\operatorname{co} A} = \overline{\operatorname{co} A}^{\mathrm{wk}}$.*

Proof. ($\subseteq$): The weak topology has fewer closed sets.

($\supseteq$): Suppose $x \notin \overline{\operatorname{co} A}$. Then there exist an $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $\operatorname{Re} f[\overline{\operatorname{co} A}] \leq \alpha < \operatorname{Re} f(x)$. So $\overline{\operatorname{co} A} \subseteq \{\operatorname{Re} f \leq \alpha\}$. ☐

Corollary 1.1. *If A is convex, $\overline{A} = \overline{A}^{\mathrm{wk}}$.*

Remark 1.1. The weak topology is the weakest topology with all closed, convex sets (in the original topology) still closed.

1.2 Polars and quotients

Definition 1.1. Let $A \subseteq X$. Its **polar** is $A^o = \{f \in X^* : |f(x)| \leq 1 \, \forall x \in A\}$.

Definition 1.2. Let $B \subseteq X^*$. Its **pre-polar** is ${}^oB := \{x \in X : |f(x)| \leq 1 \, \forall f \in B\}$.

Definition 1.3. If $A \subseteq X$, its **bipolar** is ${}^o(A^o)$.

Proposition 1.1. *Let $A \subseteq X$.*

1. *A^o is convex and balanced.*
2. *If $A_1 \subseteq A$, then $A_1^o \supseteq A^o$.*
3. *If $\alpha \in \mathbb{F} \setminus \{0\}$, then $(\alpha A)^o = \alpha^{-1} A^o$.*
4. *$A \subseteq^o A^o$.*
5. *$A^o = ({}^o A^o)^o$.*

Remark 1.2. There is an analogous version of this proposition for pre-polars if we start from $B \subseteq X^*$.

Theorem 1.3. *Let $A \subseteq X$. Then ${}^o A^o$ is the closed, convex, balanced hull of A (i.e. the intersection of all closed, convex, balanced sets containing A).*

Proof. ($\supseteq$): From the proposition, ${}^o A^o$ is closed, convex, and balanced.

($\subseteq$): Suppose there exists some convex, balanced, closed $A_1 \supseteq A$ and $x \in X \setminus A_1$; we need to show that $x \notin {}^o A^o$. Then there exist some $f \in X^*$ and $\alpha \in \mathbb{F}$ such that $\operatorname{Re} f[A_1] \leq \alpha < \operatorname{Re} f(x)$. Since $\operatorname{Re} f[A_1] \ni 0$, $\alpha \geq 0$; we can assume $\alpha > 0$. Since we have the balanced assumption, we can assume $\alpha = 1$.

If $f(x) \in \mathbb{R}$, then we are done, since $x \notin {}^o A^o$. So our only worry is that $f(x) \notin \mathbb{R}$. Then choose $w := \overline{f(x)}/|f(x)|$. Now let $g := wf$. Then $g(x) = \operatorname{Re} f(x)$, and $g[A_1] = f[wA_1] = f[A_1]$. So we can use the argument for when $f(x) \in \mathbb{R}$. $\square$

Definition 1.4. Let X be a locally convex space, and let M be a linear subspace. The **annihilator** of M is $M^\perp := \{f \in X^* : f|_M = 0\}$.

Proposition 1.2. *Let X be a vector space over $\mathbb{F}$, and let M be a linear subspace. Let p be a seminorm on X, and define*

$$\overline{p}(x + M) := \inf\{p(x + y) : y \in M\}.$$

Then the function $\overline{p}$ is a seminorm on X/M. If X is an LCS and $\mathcal{P}$ is the collection of continuous seminorms on X, then $\{\overline{p} : p \in \mathcal{P}\}$ generates the quotient topology on X/M. This is an LCS if M is closed.

Remark 1.3. This doesn't work unless we take $\mathcal{P}$ to be the collection of all continuous seminorms on X. What we need is $\overline{p_1 + p_2} \geq \overline{p}_1 + \overline{p}_2$, so we want a generating family of seminorms that is closed under addition (max is okay, too).

Theorem 1.4. *Let $Q : X \to X/M$ be the quotient map. Define $(X/m)^* \to M^\perp$ sending $f \mapsto f \circ Q$. This is an isomorphism of LCSs.*

Proof. Onto: Let $g \in M^\perp$. Then $g = f \circ Q$ for some linear $f : X/M \to \mathbb{F}$; we need to show that f is continuous. We have that $|g|$ is a continuous seminorm on X. Then $\overline{|g|}(x + M) = |f|(x)|$, so $|f|$ is a continuous seminorm. So f is continuous.

To check that the topologies are the same, he have that $\{f \in (X/M)^* : |f(x+M)| < \varepsilon\}$ corresponds to $\{g \in M^\perp : |g(x)| < \varepsilon\}$. These generate the respective topologies for the domain and codomain. $\square$

Theorem 1.5. *The map $X^* \to M^*$ sending $f \mapsto f|_M$ quotients to $X^*/M^\perp \to M^*$. This is an isomorphism of LCSs.*

Remark 1.4. We want $X^*/M^\perp$ to be Hausdorff, so we want $M^\perp$ to be closed. But $M^\perp$ is always closed, so this is okay.

Proof. Onto: If $g \in M^*$, then there is a continuous seminorm p on X such that $g \leq p$. Now apply Hahn-Banach to extend g to a continuous seminorm bounded by p. $\square$

1.3 Alaoglu's theorem

Theorem 1.6. *Let X be a normed space, and let $B^* = \{f \in X^*; \|f\| \leq 1\}$ be the closed unit ball in the dual space of X. Then B^* is weak*-compact.*

Proof. Consider the map $\varphi : B^* \to \prod_{x \in X, \|x\| \leq 1} \overline{\mathbb{D}}$, where $\mathrm{d} = \{z \in \mathbb{C} : |z| \leq 1\}$, given by $f \mapsto \langle f(x) \rangle_{\|x\| \leq 1}$. We claim that φ is a homeomorphism to a closed subset of $\prod_{\|x\| \leq 1} \overline{\mathbb{D}}$.

We claim that we have

$$\varphi[B^*] = \{\langle \alpha(x) \rangle_{\|x\| \leq 1} \in \prod \overline{\mathbb{D}} : \alpha(x+cy) = \alpha(x) + c\alpha(y) \text{ if } \|x\|, \|y\| \leq 1, c \in \mathbb{F}, \|x + cy\| \leq 1\}.$$

For ($\supseteq$): If α is in the right hand side, define $f(x) := \varepsilon^{-1}\alpha(\varepsilon x)$ for all $x \in X$ with $\varepsilon < 1/\|x\|$. Then $f \in B^*$.

Closed: Suppose $\alpha \notin \varphi[B^*]$. Then there are x, y, c such that

$$|\alpha(x + cy) - \alpha(x) - c\alpha(y)| > \varepsilon > 0.$$

If $|\alpha'(x) - \alpha(x)|$, $|\alpha'(y) - \alpha(y)|$, $|\alpha'(x+cy) - \alpha(x+cy)| < \varepsilon/3$, then $\alpha'(x+cy) \neq \alpha'(x) + c\alpha'(y)$. So $\varphi[B^*]$ is closed.

Check that the topologies agree. $\square$

Theorem 1.7. *For any normed space $\mathcal{X}$, there exists a compact Hausdorff space Z such that $\mathcal{X}$ embeds isometrically as a subspace of $C(Z)$.*

Proof. Let $Z = B^*$. For the mapping, take $x \mapsto \hat{x}|_{B^*}$. $\square$