Math 142 Lecture 15 Notes

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1 Covering Spaces and Induced Maps of Homotopic Maps

1.1 Covering spaces

Recall from last time that if G is a group acting "nicely" on a space X, then we get an identification space X/G and a projection map $\pi : X \to X/G$. We saw that if X is path-connected and simply connected, then $\pi_1(X/G) \cong G$. Here is a new point of view:

Definition 1.1. Given a space X, a continuous function $\pi : \tilde{X} \to X$ is a *covering (space) map* and say that $\tilde{X}$ is a *covering space* (or *cover*) of X if for all $x \in X$, there exists an open neighborhood U_x of x such that $\pi^{-1}(U_x) = \bigcup_\alpha \tilde{U}_\alpha$, each $\tilde{U}_\alpha$ is open, $\tilde{U}_\alpha \cap \tilde{U}_{\alpha'} = \varnothing$, and $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U_\alpha$ is a homeomorphism.

Example 1.1. If G is a group acting nicely on X, then $\pi : X \to X/G$ is a covering space map.

Assume X and $\tilde{X}$ are path-connected.[1] Then the same proofs as before give the following lifting lemmas.

Theorem 1.1 (path lifting). *If $p \in X$ and $q \in \pi^{-1}(p)$, then every path σ in X such that $\sigma(0) = p$ has a unique lift $\tilde{\sigma}$ in $\tilde{X}$ such that $\tilde{\sigma}(0) = q$.*

Theorem 1.2 (path lifting). *If σ, σ' are two paths in X from p to p, and $\sigma \simeq_F \sigma'$ rel $\{0,1\}$, thene there exists a unique lift $\tilde{F}$ of F to $\tilde{X}$ such that $\tilde{\sigma} \simeq_{\tilde{F}} \tilde{\sigma}'$ rel $\{0,1\}$.*

Definition 1.2. If $\pi : \tilde{X} \to X$ is a covering space map, and $\pi^{-1}(x)$ is finite for all $x \in X$ ($|\pi^{-1}(x)| = n \in \mathbb{N}$), then we say that $\tilde{X}$ is an *n-sheeted* (or *n-fold*) covering space.

Check that if X and $\tilde{X}$ are path-connected, then this is well-defined.

[1] If $X, \tilde{X}$ are not path connected, then each component of X will have a path-connected component of $\tilde{X}$ as its covering space, so we might as well just talk about path-connected spaces.

Example 1.2. Let $f_n L S^1 \to S^1$ send $e^{2\pi i x} \mapsto e^{2\pi i n x}$ (where $n > 0$ is an integer). Then $f_n^{-1}(\{1\}) = \{1, e^{2\pi i/n}, e^{2\pi i(2/n)}, \ldots, e^{2\pi i(n-1)/n}\}$, so $|f_n^{-1}(1)| = n$. Check that f_n is a covering map. Then S^1 is an n-fold cover of S^1 for any $n \geq 1$.

Here, our theorem about orbit spaces doesn't apply, but $(f_n)_* : \mathbb{Z} \to \mathbb{Z}$ sending $1 \mapsto n$ is an induced homomorphism between the fundamental groups. Note that the quotient $\pi_1(S^1, 1)/(f_n)_*(\pi_1(S^1, 1)) \cong \mathbb{Z}/n\mathbb{Z}$, which has order n.

1.2 Induced maps of homotopic maps

Theorem 1.3. *If $f, g : X \to Y$ and $f \simeq_F g$, then $g_* : \pi_1(X, p) \to \pi_1(Y, g(p))$ is equal to*

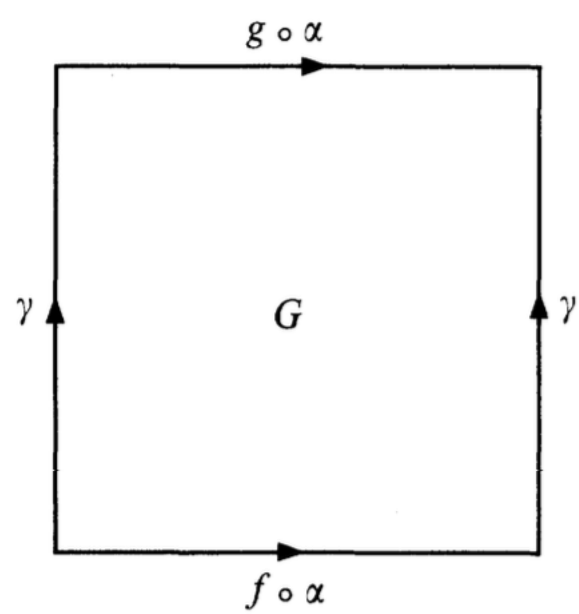
$$\pi_1(X, p) \xrightarrow{f_*} \pi_1(Y, f(p)) \xrightarrow{\gamma_*} \pi_1(Y, g(p)),$$

where $\gamma : [0, 1] \to Y$ is the path $\gamma(x) = F(p, x)$.

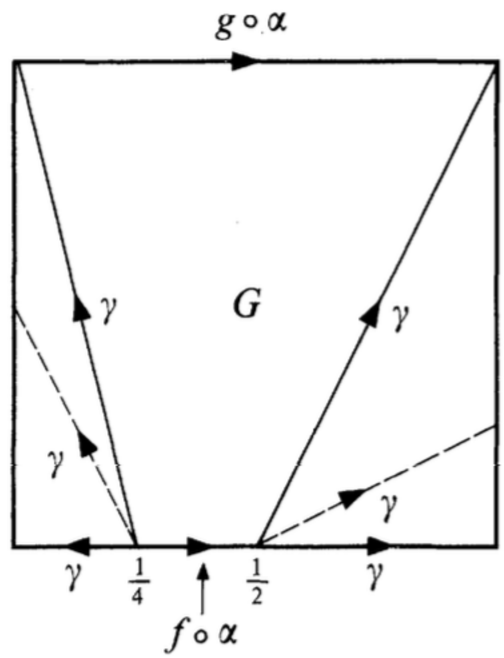
Proof. Let $\alpha : [0, 1] \to X$ with $\alpha(0) = \alpha(1) = p$ be a path. Then $g_*([\alpha]) = [g \circ \alpha]$, and

$$\gamma_*(f_*([\alpha])) = \gamma_*([f \circ \alpha]) = [(\gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma)].$$

We want to show that these two are equal. Let $G : [0, 1] \times [0, 1] \to Y$ send $(x, t) \mapsto F(\alpha(x), t)$. Drawing x on the horizontal axis and t on the vertical axis, we have the following picture for G:



Now define $H : [0,1] \times [0,1] \to Y$ according to the following picture:[2]



Then $H(x,0) = \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma$, $H(0,1) = g \circ \alpha$, $H(0,t) = \gamma(1) = g(p)$, and $H(1,t) = \gamma(1) = g(p)$. $\square$

Corollary 1.1. *If X and Y are path-connected and $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. If $f : X \to Y$ and $g : Y \to X$ are maps such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, then the previous theorem tells us that $(g \circ f)_* = g_* \circ f_* = \gamma_* \circ (\mathrm{id}_X)_*$ for some path γ. Then γ_* and $(\mathrm{id}_X)_*$ are isomorphisms, so $g_* \circ f_*$ is an isomorphism, as well. Since $g_* \circ f_*$ is injective, f_* is injective. Additionally, since $g_* \circ f_*$ is surjective, g_* is surjective. Similarly, $f_* \circ g_*$ is an isomorphism, so f_* is surjective, and g_* is injective. So f_* and g_* are isomorphisms. $\square$

Example 1.3. $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$, the cyclinder, and the Möbius strip. So

$$\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(\text{cylinder}) \cong \pi_1(\text{Möbius strip}) \cong \mathbb{Z}.$$

Also, the cylinder is isomorphic to $S^1 \times [0,1]$, so

$$\pi_1(\text{cylinder}) \cong \pi_1(S^1) \times \underbrace{\pi_1([0,1])}_{\cong 1} \cong \pi_1(S^1) \cong \mathbb{Z},$$

which gives us a consistent answer.

[2] An explicit formula for H is given in the proof of theorem 5.17 in the Armstrong textbook. These pictures are also taken from the Armstrong textbook.