Math 255A Lecture 14 Notes

Daniel Raban

October 29, 2018

1 Fredholm Operators

1.1 Fredholm operators

Definition 1.1. Let B_1, b_2 be complex Banach spaces. An operator $T \in \mathcal{L}(B_1, B_2)$ is called a **Fredholm operator** if $\ker(T)$ and $\operatorname{coker}(T) = B_2/\operatorname{im}(T)$ are finite dimensional.

This is an operator that may fail to be injective and surjective by only finitely many dimensions.

Definition 1.2. The **index** of a Fredholm operator T is defined as $\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\operatorname{coker}(T))$.

Remark 1.1. If $T \in \mathcal{L}(B_1, B_2)$, then $\ker(T) \subseteq B_1$ is closed. However, $\operatorname{im}(T)$ need not be closed. For example, take $B_1 = B_2 = C([0,1])$, and $Tf(x) = \int_0^x f(y)\,dy$.

Theorem 1.1. *Let $T \in \mathcal{L}(B_1, B_2)$ be such that $\dim(\operatorname{coker}(T)) = \operatorname{codim}(\operatorname{im}(T)) < \infty$. Then $\operatorname{im}(T) \subseteq B_2$ is closed.*

Proof. We can assume that T is injective; otherwise, consider $\tilde{T} : B_1/\ker(T) \to B_2$ given by $x + \ker(T) \mapsto Tx$. Then $\tilde{T}$ is injective, and $\operatorname{im}(\tilde{T}) = \operatorname{im}(T)$. Let $\dim(B_2/\operatorname{im}(T)) = n < \infty$, and ket $x_1, \dots, x_n$ be such that $x_1 + \operatorname{im}(T), \dots, x_n + \operatorname{im}(T)$ form a basis for $B_2/\operatorname{im}(T)$. Then for an $y \in B_2$, we can write

$$y = Tz + \sum_{j=1}^n a_j x_j$$

for $z \in B_1$.

The linear continuous map $S : \mathbb{C}^n \to B_2$ given by $(a_1, \dots, a_n) \mapsto \sum_{j=1}^n a_j x_j$ is injective, and $B_2 = \operatorname{im}(T) \oplus \operatorname{im}(S)$. It follows that the map $T_1 : B_1 \oplus \mathbb{C}^n \to B_2$ sending $(x, a) \mapsto Tx + Sa$ is a linear, continuous bijection, and by the open mapping theorem, T_1 is a homeomorphism. We get $\operatorname{im}(T) = T_1(B_1 \oplus \{0\})$, which the image of a closed set. So $\operatorname{im}(T) \subseteq B_2$ is closed. $\square$

In particular, any Fredholm operator has closed image.

1.2 Perturbing Fredholm operators

Lemma 1.1. *Let B be a Banach space, and let $S \in \mathcal{L}(B,B)$ be such that $\|S\| < 1$. Then the operator $I - S$ has an inverse in $\mathcal{L}(B,B)$.*

Proof. Consider the Neumann series $R = \sum_{k=0}^{\infty} S^K$. This converges in $\mathcal{L}(B,B)$ since $\sum_{k=1}^{\infty} \|S^k\| \leq \sum_{k=0}^{\infty} \|S\|^k = 1/(1-\|S\|) < \infty$. We have $R(I-S) = (i-S)R = I$. $\square$

Remark 1.2. Let $T \in \mathcal{L}(B_1, B_2)$ be bijective. Then T^{-1} is continuous by the open mapping theorem, and $T + S = T(I + T^{-1}S)$ is invertible, provided that $\|T^{-1}\|\|S\| < 1$.

Theorem 1.2. *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator. If $S \in \mathcal{L}(B_1, B_2)$ is such that $\|S\|$ is sufficiently small, then $T + S$ is Fredholm and* $\operatorname{ind}(T+S) = \operatorname{ind}(T)$.

Proof. Let $T : B_1 \to B_2$ be Fredholm, and let $n_+ = \dim(\ker(T))$ and $n_- = \dim(\operatorname{coker}(T))$. Let $R_- : \mathbb{C}^{n_-} \to B_2$ be linear, continuous, and injective such that $B_2 = \operatorname{im}(T) \oplus R_-(\mathbb{C}^{n_-})$. Let $e_1, \dots, e_{n+}$ be a basis for $\ker(T)$, and let $\varphi_1, \dots, \varphi_{n+} \in B_1^*$ such that $\varphi_j(e_k) = \delta_{j,k}$; these exist by Hahn-Banach. Let $R_+ : B_1 \to \mathbb{C}^{n_+}$ send $x \mapsto (\varphi_1(x), \dots, \varphi_{n_+}(x))$. Then R_+ is linear, continuous, and surjective, and $R_+|_{\ker(T)}$ is bijective.

Let us introduce the operator[1]

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}.$$

We claim that $\mathcal{P}$ is bijective. If $x \in B_1$ and $a \in C^{n_-}$, then

$$\mathcal{P} \begin{bmatrix} x \\ a \end{bmatrix} = \begin{bmatrix} Tx + R_- a \\ R_+ x \end{bmatrix}.$$

$\mathcal{P}$ is injective since $R_+|_{\ker(T)}$ was given to be bijective. By construction, $\mathcal{P}$ is surjective. It follows that

$$\tilde{\mathcal{P}} = \begin{bmatrix} T + S & R_- \\ R_+ & 0 \end{bmatrix}$$

is also invertible, provided that $\|S\|$ is small enough. Let $\mathcal{E} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus \mathbb{C}^{n_-}$ be the inverse of $\tilde{\mathcal{P}}$:

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

We have $E : B_2 \to B_1$, $E_+ : \mathbb{C}^{n_+} \to B_1$, $E_- : B_2 \to \mathbb{C}^{n_-}$, and $E_{-+} : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}$. Observe that

$$\tilde{P}\mathcal{E} = \begin{bmatrix} T + S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_+ E_+ \end{bmatrix},$$

so $R_+ E_+ = I$. So E_+ has a left inverse, which means it is injective. Similarly, $E_- R_-$ is the identity on $\mathbb{C}^{n_-}$, so E_- is surjective. We will finish the proof next time. $\square$

[1] This operator is sometimes called the Grushin operator.