Math 255A Lecture 1 Notes

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1 The Hahn-Banach Theorem

1.1 The real Hahn-Banach theorem

Theorem 1.1 (Hahn-Banach, analytic form)**.** *Let V be a vector space over $\mathbb{R}$, and let $p : V \to \mathbb{R}$ be a map which satisfies*

1. *positive homogeneity: $p(\lambda x) = \lambda p(x)$ for all $x \in V$, $\lambda > 0$,*
2. *subadditivity: $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.*

Let $W \subseteq V$ be a linear subspace and let $g : W \to \mathbb{R}$ be a linear form such that $g(x) \leq p(x)$ for all $x \in W$. Then there exists a linear form $f : V \to \mathbb{R}$ which agrees with g on W such that $f(x) \leq p(x)$ for all $x \in V$.

Proof. We will use Zorn's lemma to obtain f. For notation, we write $D(f)$ as the domain of f. Let us consider the set

$$\begin{aligned} P = \{h \mid h &: D(h) \to \mathbb{R}, D(h) \subseteq V \text{ is a linear subspsace s.t. } W \subseteq D(h), \\ &h|_W = g, h(x) \leq p(x), x \in D(h)\}. \end{aligned}$$

$P \neq \varnothing$ because $g \in P$. P is equipped with the partial order relation $\leq$:

$$h_1 \leq h_2 \iff D(h_1) \subseteq D(h_2) \text{ and } h_2 \text{ extends } h_1.$$

Claim: The set P is inductive, in the set that any totally ordered subset $Q \subseteq P$ has an upper bound; i.e. there exists $x \in P$ such that $a \leq x$ for all $a \in Q$. Write $Q = (h_j)_{j \in I}$. Let $D(h) = \bigcup_{i \in I} D(h_j)$, and define h by saying $x \in D(h_j) \implies h(x) = h_j(x)$. The function h is well defined, $h \in P$, and $h_j \leq h$ for all $j \in I$.

By Zorn's lemma, we conclude that P has a maximal element f, in the sense that if $f \leq h \in P$, then $h = f$. We have to check that $D(f) = V$; proceed by contradiction. If $D(f) \neq V$, let $x_0 \in V \setminus D(f)$, and define h by $D(h) = D(f) + \mathbb{R}x_0$ and for $x \in D(f)$, $h(x + tx_0) = f(x) + t\alpha$, where $\alpha \in \mathbb{R}$ is to be chosen such that $h \in P$ ($h(x) \leq p(x)$ for $x \in D(h)$).

We have to arrange: $f(x)+t\alpha \le p(x+tx_0)$ for all $t \in \mathbb{R}$ and $x \in D(f)$. By the positive homogeneity of p, we need only check when $t = \pm 1$. So we need to satisfy:

$$f(x)+\alpha \le p(x+x_0) \qquad f(x)-\alpha \le p(x-x_0).$$

In other words, we have to choose α so that

$$\sup_{y \in D(f)} f(y)-p(y-x_0) \le \alpha \le \inf_{x \in D(f)} p(x+x_0)-f(x).$$

This is possible as $f(y)-p(y-x_0) \le p(x+x_0)-f(x)$ for all $x, y \in D(f)$, which follows from $f(x+y) \le p(y-x_0)+p(x+x_0)$ (by $p(x+y) \ge f(x+y)$). We conclude that $f \le h$, $h \ne f$, which contradicts the maximality of f. $\square$

1.2 The complex Hahn-Banach theorem

Definition 1.1. Let V be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. A function $p : V \to [0, \infty)$ is a **seminorm** if

1. $p(\lambda x) = |\lambda| p(x)$ for all $x \in V$, $|lambda \in K$
2. $p(x+y) \le p(x)+p(t)$ for all $x, y \in V$.

Theorem 1.2 (Hahn-Banach, complex version)**.** *Let V be a vector space $\mathbb{C}$, $W \subseteq V$ a $\mathbb{C}$-linear subspace, and $p : V \to [0, \infty)$ a seminorm. Let $g : W \to \mathbb{C}$ be $\mathbb{C}$-linear such that $|g(x)| \le p(x)$ for all $x \in W$. Then g can be extended to a $\mathbb{C}$-linear form $f : V \to \mathbb{C}$ such that $|f(x)| \le p(x)$ for all $x \in V$.*

Proof. Let $g = g_1 + ig_2$, where $g_1(x) = \mathrm{Re}(g(x))$ and $g_2(x) = \mathrm{Im}(g(x))$; g_1, g_2 are $\mathbb{R}$-linear. and defined on W. Note that $g_1(iy) = \mathrm{Re}(g(iy)) = \mathrm{Re}(ig(y)) = -g_2(y)$, so we can recover g_2 from g_1. Now $g_1(y) \le p(y)$ for all $y \in W$, so by the real version of the Hahn-Banach theorem, there exists an $\mathbb{R}$-linear $f_1 : V \to \mathbb{R}$ such that $f_1|_W = g_1$ and $f_1(x) \le p(x)$ for all $x \in V$. Let $f(x) = f_1(x) - i(f(ix))$. Then, by our previous observation, $f|_W = g$. Note that f is $\mathbb{R}$-linear and $f(ix) = f_1(ix) - if_1(-x) = i(f_1(x) - if_1(ix)) = if(x)$, so f is $\mathbb{C}$-linear. Finally, we check that $|f(x)| \le p(x)$ for all $x \in V$. If $f(x) \ne 0$, write $f(x) = |f(x)|e^{i\varphi}$ with $\varphi \in \mathbb{R}$. Then

$$|f(x)| = e^{-i\varphi} f(x) = f(e^{-i\varphi}x) = f_1(e^{-i\varphi}x) \le p(e^{-i\varphi}x) = p(x). \qquad \square$$

1.3 Introduction to dual spaces

Definition 1.2. Let B be a complex Banach space. The **dual space** B^* is the space of linear continuous maps $\xi : B \to \mathbb{C}$.

The form on $B \times B^*$ given by $(x, \xi) \mapsto \xi(x) = \langle x, \xi \rangle$ is bilinear. There may exist linear forms B^* which are not of the form $\xi \mapsto \langle x, \xi \rangle$.