Math 254B Lecture 27 Notes

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1 Ergodic Decomposition of CP Distiributions and Typical Dimension Under Ergodic CP Distributions

1.1 Lemma for obtaining an adapted T-invariant distribution

Recall: We have $K_w = \phi_w[K]$, where $w \in [k]^n$. Define $\nu^w := S_*^n \nu|_{K_w}$. If $z \in \mathrm{supp}(\nu)$, then $T^t(z, \nu) = (S^t z, \nu^{\alpha_{[1,t]}(z)})$.

$\widehat{\mu}$ on $K \times P(K)$ is adapted if

$$\widehat{\mu} = \int_{P(K)} \nu \times \delta_\nu \, d\overline{\mu}(\nu) \iff \int f \, d\widehat{\mu} = \int Qf \, d\widehat{\mu}, \qquad Qf(z, \nu_: = \int f(z', \nu) \, d\nu(z').$$

We denote P_a as the collection of adapted distributions in $P(K \times P(K))$.

We need to prove the following lemma to complete our proof that we can still run the Krylov-Bogoliubov machine.

Lemma 1.1. *If $\widehat{\mu} \in P_a$, then $T_* \widehat{\mu} \in P_a$, and its second marginal is*

$$M\overline{\mu} = \int \sum_{i=1}^k \nu(K_i) \cdot \delta_{\nu^i} \, d\overline{\mu}(\nu).$$

Proof. We have

$$\begin{aligned}
\int f \circ T \, d\widehat{\mu} &= \int f(Sz, \nu^{\alpha_1(z)}) \, d\widehat{\mu}(z, \nu) \\
&= \iint f(Sz, \nu^{\alpha_1(z)}) \, \nu(z) \, d\overline{\mu}(\nu) \\
&= \int \sum_{i=1}^k \nu(K_i) \int_{K_1} f(Sz, \nu^i) \, d\nu|_{K_i}(z) \, d\overline{\mu}(\nu) \\
&= \int \sum_{i=1}^k \nu(K_i) \int_{K_1} f(Sz, \nu^i) \, d \underbrace{S_* \nu|_{K_i}(z)}_{\nu^i(z)} \, d\overline{\mu}(\nu)
\end{aligned}$$

$$= \iint f(z,\nu')\, d\nu'(z) d(M\overline{\mu})(\nu'). \qquad \square$$

1.2 Ergodic decomposition of CP distributions

Definition 1.1. A **CP distribution** is an adapted and T-invariant distribution.

Proposition 1.1. *If $\widehat{\mu}$ is a CP distribution with ergodic decomposition $\widehat{\mu} = \int_X \widehat{\mu}_x \, d\widehat{\mu}(x)$, then $\widehat{\mu}$-a.e. x is adapted (and hence CP).*

This is a bit technical, so we need the following lemma.

Lemma 1.2. *If $\widehat{\mu}$ is CP and $f \in L^1(\widehat{\mu})$, then*

$$\lim_{n \to \infty} A_n f = \lim_{n \to \infty} A_n(Qf) \qquad \widehat{\mu}\text{-a.e.}$$

Proof. By the ergodic theorem, the limits exist.

Step 1: It is enough to show that $(f - Qf) \to 0$ in $\|\cdot\|_1$ for all $f \in C(K \times P(K))$.

Step 2: It is enough to show that $A_n f \to 0$ in $\|\cdot\|_1$ when f is continuous and $Qf = 0$.

Step 3: It is enough to show that $\int g \cdot A_n f \, d\widehat{\mu} \to 0$ for all $g \in C(K \times P(K))$ if $Qf = 0$.

Step 4: By Stone-Weierstrass, it is enough to test when $g(z,\nu) = g_1(\alpha_{[1,m]}(z)) g_z(\nu)$.

We can now compute

$$\int f \cdot (f \circ T^t) \, d\widehat{\mu} = \int_{P(k)} \left[\int_K g_1(\alpha_{[1,m]}(z)) f(S^t z, \nu^{\alpha_{[1,t]}(z)}) \, d\nu(z) \right] g_2(\nu) \, \overline{\mu}(z)$$

If $t > m$,

$$= \int_{P(k)} \left[\int_K g_1(\alpha_{[1,m]}(z)) \underbrace{\mathbb{E}[f(S^t(\cdot), \nu^{\alpha_{[1,t]}(z)}) \mid \alpha_{[1,t]}](z)}_{\mathbb{E}_\nu[f(S^t(\cdot),\nu^w) | K_w]} \, d\nu(z) \right] g_2(\nu) \, \overline{\mu}(z).$$

Observe that

$$\mathbb{E}_\nu[f(S^t(\cdot), \nu^w) \mid K_w] = \int f(z', \nu^w) \, dS^t_*|_{K_w}(z') = \int f(z', \nu^w) \, \nu^2(z') = Qf(z, \nu^w) = 0.$$

So for $t > m$, the quantity is 0. So

$$\int g \cdot A_n f \, d\widehat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} \int g \cdot (f \circ T^t) \, d\widehat{\mu} \to 0, \qquad \square$$

We can now prove the proposition:

Proof. Let $f \in C(L \times P(K))$. Then

$$\widehat{\mu}_x(\{(z,\nu) : \lim_n A_n f(z,\nu) = \int f \, d\widehat{\mu}_x\}) = 1,$$

$$\widehat{\mu}_x(\{(z,\nu) : \lim_n A_n Qf(z,\nu) = \int Qf \, d\widehat{\mu}_x\}) = 1.$$

So $\int f \, d\widehat{\mu}_x = \int Qf \, d\widehat{\mu}_x$ for $\widehat{\mu}$-a.e. x. □

1.3 Typical dimension under ergodic CP distributions

Convention: IF $(X, \mathcal{B}, \mu)$ is a probability space, "μ-a.e. x satisfies P" means 'there exists $A \in \mathcal{B}$ such that $\mu(A) = 1$ and x satsfies P for all $x \in A$." This allows us to not check if sets are measurable.[1]

Proposition 1.2. *If $\widehat{\mu}$ is ergodic and CP, then $\widehat{\mu}$-a.e. (z,ν) satisfies*

$$\dim(\nu) = \frac{\int H_{\nu'}(\alpha_1) \, \widehat{\mu}(z',\nu')}{\log(r^{-1})}.$$

Proof. Define $F(z,\nu) = -\log(\nu([z]_1))$. Then

$$\int F \, \widehat{\mu} = \int_{P(K)} \underbrace{\int_K - \log(\nu([z]_1)) \, d\nu(z)}_{-\sum_{i=1}^k \nu(K_i) \log(\nu(K_i))} \, d\overline{\mu}(\nu) = \int H_\nu(\alpha) \, d\widehat{\mu}(z',\nu').$$

By the pointwise ergodic theorem for $\widehat{\mu}$-a.e. (x,ν),

$$A_n F(z,\nu) = -\frac{1}{n} \sum_{t=0}^{n-1} \log(\nu|_{[z]_1^t}([z]_{t+1})) - \frac{1}{n} \log(\nu([z]_1^n) \to \int F \, d\widehat{\mu}.$$

In other words,

$$\nu([z]_1^n) = e^{-hn+o(n)}, \qquad h = \int H_{\nu'}(\alpha_1) \, d\widehat{\mu}(z',\nu').$$

as $n \to \infty$. So

$$\dim(\nu, z) = \frac{h}{\log(r^{-1})}$$

for $\widehat{\mu}$-a.e. (z,ν). Since $\widehat{\mu}$ is adapted,

$$\dim(\nu, z) = \frac{h}{\log(r^{-1})}$$

(for ν-a.e. z) for $\widehat{\mu}$-a.e. ν. □

[1] A projection of a Borel subset of $X \times Y$ need not be Borel.