Math 255A Lecture 8 Notes

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1 Metrizability and Fréchet Spaces

1.1 Metrizability of locally convex spaces

Last time, we introduced the idea of a locally convex vector space V where the topology is defined by a family of seminorms $(p_\alpha)_{\alpha \in A}$. Here, $O \subseteq V$ is open if for all $x \in I$, there exists an $\varepsilon > 0$ and $p_{\alpha_1}, \dots, p_{\alpha_J}$ such that $p_{\alpha_j}(y - x) < \varepsilon \, \forall j \implies y \in O$.

Theorem 1.1. *A locally convex space V is metrizable if and only if the topology can be defined by a countable family of seminorms.*[1] *The metric can be chosen to be translation invariant : $d(x, y) = d(x - y)$.*

Proof. ($\implies$): Each neighborhood of 0 contains a set of the form $\{x \in V : d(x, 0) < 1/n\}$ for $n \in \mathbb{N}$. If the locally convex topology on V is defined by the seminorms (p_α), them for all n, there exists $p^{(n)}$, a positive linear combination of finitely many p_α such that if $p^{(n)}(x) < 1$, then $d(x, 0) < 1/n$. So every neighborhood of 0 contains a set of the form $\{x \in V : p^{(n)}(x) < 1\}$, and thus the seminorms $(p^{(n)})_{n \in \mathbb{N}}$ define the topology.

($\impliedby$): Let us assume that the locally convex topology on V is generated by the seminorms $(p_n)_{n \in \mathbb{N}}$ such that $p_n(x) = 0 \, \forall n \iff x = 0$. Set

$$d(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{1 + p_n(x)}$$

for each $x \in V$. We have

1. $d(x) > 0$ for $x \neq 0$
2. $d(-x) = d(x)$
3. $d(x + y) \leq d(x) + d(y)$: We need to check that $f(t) = t/(1 + t)$ for $t \geq 0$ is increasing and subadditive. It is increasing because $f(t) = 1 - \frac{1}{1+t}$. $f(t)/t$ is decreasing, so $f(t)/t \geq f(t + s)/(t + s)$ when $t, s > 0$. So $f(t) + f(s) \geq f(t + s)$.

[1] We should also include the condition here that V is Hausdorff, but we assume this is always true in our definition of locally convex spaces because we assume that the seminorms separate points.

We get that $d(x,y) = d(x-y)$ is a metric on V.

We check now that the topology defined by d is the same as the topology defined by the p_n. If $d(x) < \varepsilon 2^{-N}$ for some $\varepsilon \in (0,1)$, then $2^{-n} p_n(x)/(1+p_n(x)) < \varepsilon 2^{-N}$ for $n \leq N$. Then $p_n(x) < \varepsilon/(1-\varepsilon)$ for $n \leq N$. So any set of the form "a finite intersection of $\{x \in V : p_n(x) < \varepsilon\}$" contains an open d-ball around 0.

Conversely, if $p_n(x) < \varepsilon/2$ for all $n \leq N$, then

$$d(x) = \sum_{n=0}^{N} 2^{-n} \underbrace{\frac{p_n(x)}{1+p_n(x)}}_{\leq \varepsilon/(2+\varepsilon)} + \sum_{n=N+1}^{\infty} 2^{-n} \underbrace{\frac{p_n(x)}{1+p_n(x)}}_{<1} < 2\frac{\varepsilon}{2+\varepsilon} + 2^{-N} < \varepsilon$$

for N large enough such that $2^{-N} < \varepsilon/2$. Thus, any open d-ball around 0 contains all finite intersections of sets of the form $\{x \in V : p_n(x) < \varepsilon\}$. $\square$

Remark 1.1. If $(x_j)_{j \in \mathbb{N}}$ is in V, then $x_j \to x \iff d(x_j, x) \to 0 \iff p_n(x_j - x) \to 0$ for each n.

1.2 Fréchet spaces

Definition 1.1. A locally convex, metrizable, and complete space is called a **Fréchet space**.

Example 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be open. The space $C(\Omega)$ is a Frèchet space with the topology defined by the seminorms $u \mapsto \sup_{x \in K} |u(x)|$ with compact $K \subseteq \Omega$. The topology is metrizable as it suffices to use $u \mapsto \sup_{K_j} |u|$, where $K_j = \{x \in \Omega : |x| \leq j, d(x, \Omega^c) \geq 1/j\}$. If (u_j) is a Cauchy sequence in $C(\Omega)$ (for compact $K \subseteq \Omega$, if $\sup_K |u_j - u_k| \xrightarrow{j,k\to\infty} 0$), then there exists $u \in C(\Omega)$ such that $u_j \to u$ in $C(\Omega)$. If $\Omega \subseteq \mathbb{C}$ is open, then the space $\mathrm{Hol}(\Omega)$ is a Fréchet space viewed as a subspace of $C(\Omega)$ because a uniform limit of holomorphic functions is holomorphic.

Example 1.2. Let $\Omega \subseteq \mathbb{R}^n$ be open and $j \in \mathbb{N} \cup \{\infty\}$. Then the space $C^j(\Omega)$ is a Fréchet space with the topology given by the seminorms $u \mapsto \sup_{x \in K} |\partial^\alpha u(x)|$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, and $|\alpha| := \sum_{k=1}^n \alpha_k \leq j$.

Let $(V_1, (p_n)), (V_2, (q_n))$ be Fréchet spaces. A linear map $T : V_1 \to V_2$ is continuous of and only if for ant n, there exists $\varepsilon > 0$ and $p_{i_1}, \dots, p_{i_m}$ such that $p_{i_j}(x) < \varepsilon \, \forall j \implies q_n(Tx) < 1$. This condition is equivalent to $q_n(Tx) \leq C_n \sum_{j=1}^m p_{i_j}(x)$ for all n.

Example 1.3. A linear form $u : C^\infty(\Omega) \to \mathbb{C}$ is continuous if and only if there exist $C > 0$, $m \in \mathbb{N}$, and a compact $K \subseteq \Omega$ such that

$$|u(f)| \leq C \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha f|$$

for $f \in C^\infty(\Omega)$.