

# Math 279 Lecture 5 Notes

Daniel Raban

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## 1 Integration With Respect to Rough Functions

### 1.1 Lyons' approach and Chen's relation

Today, we try to solve ODEs of the form  $\dot{y} = V(y)\dot{x}$ , where  $V$  is a  $\mathcal{C}^2$  function, and  $x$  is a Hölder continuous function, say, of exponent  $\alpha$ . If  $\alpha > 1/2$ , we can study this ODE by first making sense of integrals of the form  $\int_0^t V(y(\theta)) dx(\theta)$ . We develop a strategy to deal with such integrals when  $1/3 < \alpha \leq 1/2$ . Let's explain the idea first.

We wish to make sense of

$$y(t) - y(s) = \int_s^t V(y(\theta)) dx(\theta).$$

To make sense of the right hand side, we may try the following approximation for small  $t - s$ :

$$\int_s^t V(y(\theta)) dx(\theta) \approx V(y(s)) \underbrace{\int_s^t dx(\theta)}_{|t-s|^\alpha} + O(|t-s|^{2\alpha}).$$

If we have a very fine mesh for defining our integral, then  $\sum_i |t_i - t_{i+1}|^{2\alpha}$  is small only when  $2\alpha > 1$ . This suggests a finer Taylor expansion of the form

$$\begin{aligned} \int_s^t V(y(\theta)) dx(\theta) &= \int_s^t [V(y(s)) + DV(y(s))(y(\theta) - y(s))] dx(\theta) + O(|t-s|^{3\alpha}) \\ &= \int_s^t [V(y(s)) + DV(y(s))V(y(s))(x(\theta) - x(s)) dx(\theta)] + O(|t-s|^{3\alpha}) \\ &= V(y(s)) \int_s^t dx(\theta) + DV(y(s))V(y(s)) \int_s^t (x(\theta) - x(s)) dx(\theta) \\ &\quad + O(|t-s|^{3\alpha}) \end{aligned}$$

To make this work, we still need to make sense of

$$\int_s^t (x(\theta) - x(s)) \otimes dx(\theta) = \left[ \int_s^t (x^i(\theta) - x^i(s)) dx^j(\theta) \right]_{i,j=1}^\ell.$$

Terry Lyons' idea in 1990 was to choose a candidate for  $\mathbb{X}(s, t) = \int_s^t (x(\theta) - x(s)) \otimes dx(\theta)$ , and given  $(x(\cdot), \mathbb{X}(\cdot, \cdot))$ , we can make sense of integrals of the form  $\int_s^t V(y(\theta)) dx(\theta)$ . For example, given  $(x, \mathbb{X})$ , we can define

$$\mathcal{J}(\mathbf{x}) = \int_0^T F(\mathbf{x}) d\mathbf{x}$$

for any  $C^2$  function  $F$ , with  $\mathcal{J}(\mathbf{x})$  continuous in  $\mathbf{x}$ .

**Theorem 1.1** (Lyons-Victoire). *Given  $x \in C^\alpha$ , there exists a function  $z \in C^\alpha$  such that  $z(0) = 0$  and*

$$|z(t) - z(s) - x(s) \otimes (x(t) - x(s))| \leq x_0[x]_\alpha^2 |t - s|^{2\alpha}.$$

Here,  $[x]_\alpha = \sup_{s \neq t \in [0,1]} \frac{|x(t) - x(s)|}{|t - s|^\alpha}$ .

Here, we want to think of

$$z(t) = \int_0^t x(\theta) \otimes dx(\theta),$$

so that

$$z(t) - z(s) = \int_s^t x(\theta) \otimes dx(\theta).$$

We also want to think of

$$z(t) - z(s) - x(s) \otimes (x(t) - x(s)) = \mathbb{X}(s, t).$$

Let us write  $x(s, t) = x(t) - x(s)$ , so that we can write

$$z(s, t) := z(t) - z(s) = \mathbb{X}(s, t) + x(s) \otimes x(s, t).$$

From  $s < u < t \implies z(s, u) + z(u, t) = z(s, t)$ , we learn that  $\mathbb{X}(s, t)$  must satisfy the following formula, known as **Chen's relation**:

$$\mathbb{X}(s, u) + \mathbb{X}(u, t) = \mathbb{X}(s, t) + [x(s) \otimes x(s, t) - x(s) \otimes x(s, u) - x(u) \otimes x(u, t)]$$

Using  $x(s, t) = x(s, u) + x(s, t)$ , we get

$$= \mathbb{X}(s, t) + x(s, u) \otimes x(u, t).$$

We can now define

$$[(x(\cdot), \mathbb{X}(\cdot, \cdot))]_\alpha := [x]_\alpha + \sup_{s \neq t \in [0, T]} \frac{|\mathbb{X}(s, t)|}{|t - s|^{2\alpha}}.$$

**Remark 1.1** (Geometric Rough Path). Roughly,  $\dot{z}^{i,j} = x^i \dot{x}^j$ . Then

$$\dot{z}^{ij} + \dot{z}^{ji} = x^i \dot{x}^j + x^j \dot{x}^i = \frac{d}{dt}(x^i x^j).$$

If the product rule applies, we expect

$$z^{ij}(s, t) + z^{ji}(s, t) = x^i(t)x^j(t) - x^i(s)x^j(s).$$

In general, this may not be true. For example, Itô calculus is not geometric, while Stratonovich calculus is geometric.

## 1.2 Convergence of the integral

**Theorem 1.2** (Lyons). *Let  $(x, \mathbb{X})$  be as above (Chen's relation +  $[(x, \mathbb{X})]_\alpha < \infty$ ), and let  $F \in \mathcal{C}^2$ . Then we can define*

$$\mathcal{J}(F) = \int_0^t F(\mathbf{x}) \cdot d\mathbf{x} = \lim_{|\pi| \rightarrow 0} \underbrace{\sum_i [F(x(t_i)) \cdot x(t_i, x_{i+1}) + DF(x(t_i))^* \mathbb{X}(t, t_{i+1})]}_{\mathcal{R}(\pi)},$$

where  $\pi = \{0 < t_1 < \dots < t_n < t\}$  and  $|\pi| = \max_i |t_{i+1} - t_i|$ . Moreover,

$$\left| \int_s^t F(\mathbf{x}) \cdot d\mathbf{x} - (F(x(s)) \cdot x(s, t) + \underbrace{DF(x(s))^* \mathbb{X}(s, t)}_{A(s)}) \right| \leq c(\alpha) \|F\|_{\mathcal{C}^2} [(x, \mathbb{X})]_\alpha^3 |t - s|^{3\alpha}.$$

*Proof.* Take a partition  $\pi = \{s < t_0 < \dots < t_{n-1} < t_n < t = t_{n+1}\}$ . Pick some  $i$ , and compare  $\mathcal{R}(\pi)$  with  $\mathcal{R}(\pi - \{t_i\})$ :

$$\begin{aligned} \mathcal{R}(\pi) - \mathcal{R}(\pi - \{t_i\}) &= F(x(t_{i-1}))x(t_{i-1}, t_i) + F(x(t_i))x(t_i, t_{i+1}) \\ &\quad + A(t_{i-1})\mathbb{X}(t_{i-1}, t_i) + A(t_i)\mathbb{X}(t_i, t_{i+1}) \\ &\quad - F(x(t_{i-1}))x(t_{i-1}, t_{i+1}) + A(t_{i-1})\mathbb{X}(t_{i-1}, t_{i+1}) \\ &= y(t_{i-1}, t_i)x(t_i, t_{i+1}) + A(t_{i-1}, t_i)\mathbb{X}(t_i, t_{i+1}) \\ &\quad - A(t_{i-1})x(t_{i-1}, t_i) \otimes x(t_i, t_{i+1}) \\ &= [y(t_{i-1}, t_i)x(t_i, t_{i+1}) - A(t_{i-1})x(t_{i-1}, t_i) \otimes x(t_i, t_{i+1})] \\ &\quad + A(t_{i-1}, t_i)\mathbb{X}(t_i, t - i + 1). \end{aligned}$$

So we may estimate the error as

$$\begin{aligned} |\mathcal{R}(\pi)| &= |[y(t_{i-1}, t_i)x(t_i, t_{i+1}) - A(t_{i-1})x(t_{i-1}, t_i) \otimes x(t_i, t_{i+1})] + A(t_{i-1}, t_i)\mathbb{X}(t_i, t - i + 1)| \\ &\leq \|F\|_{\mathcal{C}^2} |t_{i+1} - t_i|^{3\alpha} [x]_\alpha^3 + \|F\|_{\mathcal{C}^2} |t_{i+1} - t_{i-1}|^{3\alpha} \|F\|_{\mathcal{C}^2} [\mathbb{X}]_{2\alpha}. \end{aligned}$$

Choose  $i$  so that  $|t_{i+1} - t_i| \leq 2(t - s)/n$ ,

$$\mathcal{R}(\pi) - \mathcal{R}(\pi - \{t_i\}) \leq c_0 \frac{|t - s|^{3\alpha}}{n^{3\alpha}} 2^{3\alpha}.$$

Do this inductively to obtain

$$|\mathcal{R}(\pi) - \mathcal{R}(\emptyset)| \leq c_0 |t - s|^{3\alpha}.$$

From our proof, we can also deduce that  $\mathcal{R}(\pi)$  converges as  $|\pi| \rightarrow 0$ . □