Math 255B Lecture 14 Notes

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February 7, 2020

1 Essential Self-Adjointness of Schrödinger Operators and Perturbations of Self-Adjoint Operators

1.1 Essential self-adjointness of Schrödinger operators

Last time, we were proving the following theorem.

Theorem 1.1 (Essential self-adjointness of the Schrödinger operator with a semibounded potential)**.** *Let $P = P(x, D) = -\Delta + q(x)$, where $q \in C(\mathbb{R}^n; \mathbb{R})$. Let P_0 be the minimal realization of P: $P_0 = \overline{P|_{C_0^\infty}}$, which is closed, symmetric and densely defined. Assume that $q \geq -C$ on $\mathbb{R}^n$. Then P_0 is self-adjoint (i.e. $P(x, D)$ is essentially self-adjoint).*

Proof. Let $P_0 = \overline{P_{C_0^\infty}}$, so $D(P_0^*) = \{u \in L^2 : P_u \in L^2\} \subseteq H^2_{\mathrm{loc}}$. We shall show that P_0^* is symmetric, which is equivalent to $\langle u, P_0^* u\rangle_{L^2} \in \mathbb{R}$ for all $u \in D(P_0^*)$.

We claim that for every $u \in D(P_0^*)$, $\nabla u \in L^2(\mathbb{R}^n)$. It suffices to show this claim when u is real (by splitting up real and imaginary parts). Consider

$$\int \psi_t^2 u P_0^* u \, dx = \int \psi_t^2 u P u \, dx,$$

where $\psi_t(x) = \psi(tx)$ for $t > 0$, $0 \leq \psi \in C_0^\infty$, and $\psi(x) = 1$ in $|x| \leq 1$. Write

$$\begin{aligned} \int \psi_t^2(x) u P u \, dx &= \int \psi_t^2 u(-\Delta + q) u \, dx \\ &= \int \psi_t^2(-\Delta u) + \int \psi_t^2 q u^2 \end{aligned}$$

Integrating by parts in the first integral (we can integrate u by parts by regularizing it, but we omit that argument),

$$= \int \nabla(\psi_t^2 u) \cdot \nabla u + \int \psi_t^2 q u^2$$

We get

$$\underbrace{\int \psi_t^2(x) uPu\,dx}_{\leq \|u\|_{L^2}\|Pu\|_{L^2}} = \int \psi_t^2 (\nabla u)^2 + \int 2\psi_t \nabla\psi_t \cdot \nabla u + \underbrace{\int q\psi_t u^2}_{-C\int \psi_t^2 u^2 \geq -C\|u\|^2}.$$

Let $I(T) = \int \psi_t (\nabla u)^2$. We get that

$$\begin{aligned} I(T) &\leq O(1) + 2\underbrace{\int |\psi_t \nabla u \cdot u \nabla \psi_t|}_{\leq CI(t)^{1/2}\|u\|_{L^2}} \\ &\leq O(1) + CI(t)^{1/2}. \end{aligned}$$

This implies that $I(t) \leq O(1)$ because $CI(t)^{1/2} \leq C^2 \frac{1}{\varepsilon} + \varepsilon I(T)$ for all $\varepsilon > 0$ by the AM-GM inequality. The claim follows by Fatou's lemma.

Let $u \in D(P_0^*)$ be complex-valued. Then

$$\psi_t^2 u\overline{Pu} = \underbrace{\int \psi_t^2 |\nabla u|^2}_{\in \mathbb{R}} + \underbrace{2\int \psi_t u \nabla \psi_t \cdot \overline{\nabla u}}_{\leq 2\|\nabla \psi_t\|_{L^\infty}\|u\|_{L^2}\|\nabla u\|_{L^2} \to 0} + \underbrace{\int q\psi_t^2 |u|^2}_{\in \mathbb{R}}$$

so the imaginary part of this goes to 0 as $t \to \infty$. Also, $\int \psi_t^2 u\overline{Pu} \to \int u\overline{Pu}$ as $t \to \infty$, so $\int u\overline{Pu} = \langle u, P_0^* u\rangle_{L^2} \in \mathbb{R}$. $\square$

Example 1.1. The quantum harmonic oscillator is the case of $q(x) = |x|^2$, so $P = -\Delta + |x|^2$ is essentially self-adjoint on C_0^∞. One can show that the domain is $D(P_0) = \{u \in L^2 : x^\alpha \partial^\beta \in L^2, |\alpha + \beta| \leq 2\}$.

Remark 1.1. If S is essentially self-adjoint, then $\overline{S} = (\overline{S})^* = S^*$. So the closure is the adjoint. In particular, there is only 1 realization.

1.2 Perturbations of self-adjoint operators

Let $A : D(A) \to H$. Then A is closed if and only if $D(A)$ is a Banach space with respect to the **graph norm**: $\|u\|_D(A) := \|u\| + \|Au\|$.

Definition 1.1. Let A, B be linear operators on H. We say that B is A-**bounded** (or **relatively bounded with respect to** A) if $D(B) \supseteq D(A)$ and if there are constants $a, b \geq 0$ such that

$$\|Bu\| \leq a\|Au\| + b\|u\|, \qquad \forall u \in D(A).$$

The infimum of all such constants a is the **relative bound** of B with respect to A.

Proposition 1.1. *Let A be closed, and let B be A-bounded with a relative bound < 1. Then $A + B$ is closed on $D(A)$.*

Proof. We have:

$$\|Bu\| \leq a\|Au\| + b\|u\|, \qquad \forall u \in D(A)$$

with $a < 1$. Check that the norms $u \mapsto \|u\| + \|Au\|$ and $u \mapsto \|u\| + \|(A + B)u\|$ are equivalent on $D(A)$. So $A + B$ is closed. □

Theorem 1.2 (Kato-Rellich[1])**.** *Let A be self-adjoint, and let B be symmetric and A-bounded with relative bound < 1. Then $A + B$ is self-adjoint on $D(A)$.*

Proof. $A + B$ is closed, symmetric, and densely defined on $D(A)$. So we only need to show that the deficiency indices are 0: that is, we want $\mathrm{Im}(A + B \pm i) = H$. In fact, we will show that there exists some $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\mathrm{Im}(A + B \pm i\lambda) = H$. □

We will prove this next time.

[1]Kato and Rellich both proved this result around the same time, independently of each other.