Math 249 Lecture 10 Notes

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1 Homogeneous and Power Sum Symmetric Functions

Recall from last lecture that we have 2 bases for the symmetric functions: the monomials $\{m_\lambda\}$ and the elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_n}$, where $e_k = m_{1^k}$. We also have that

$$(z - x_1) \cdots (z - x_n) = z^n - e_1 z^{n-1} + e_2 z^{n-2} - \cdots \pm e_n.$$

Example 1.1. The function

$$\prod_{i<j} (x_i - x_j)^2 = \Delta(e_1, \dots, e_n)$$

is a symmetric function. This is the discriminant of a polynomial. So we can calculate discriminants using the e_k terms.

1.1 Homogeneous symmetric functions

We introduce another basis for the symmetric functions: the homogeneous symmetric functions.

Definition 1.1. The *homogeneous symmetric functions* are the functions h_λ for partitions λ such that

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}, \qquad h_k = \sum_{|\lambda|=k} m_\lambda.$$

Proposition 1.1.

$$h_\lambda = \sum_\mu b_{\lambda,\mu} m_\mu,$$

where $b_{\lambda,\mu}$ is the number of matrices with entries in $\mathbb{N}$, row-sums λ, and column-sums μ.

Proof. We use a similar matrix argument to the proof for expressing the e_λ in terms of the m_μ. Since symmetric functions must be invariant under permuting the variables, to find the coefficient $b_{\lambda,\mu}$, we need only find the coefficient in front of a single monomial in each m_μ; we pick the monomial $x_1^{\mu_1}x_2^{\mu_2}\cdots x_\ell^{\mu_\ell}$. Each monomial in each h_{λ_i} will be a product of variables with combined degree λ_i; then we can find the coefficient of a monomial in h_λ by finding the number of ways to trace back where its x_i terms could have come from in the product of the h_{λ_i}.

Now consider the following matrix (represented as a table):

	μ_1	μ_2	μ_3	$\cdots$
λ_1	2	0	1	$\cdots$
λ_2	0	3	0	$\cdots$
λ_3	1	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We fill in this matrix as follows: Think of the column j as picking an x_j from some of $h_{\lambda_1}, h_{\lambda_2}, \ldots, h_{\lambda_\ell}$ (this is "where the x_j term came from" when you multiply the h_{λ_i}), and place a 1 in the space i, j if an x_j term comes from h_{λ_i} and a 0 otherwise. The product of all the x_j terms in a column should give $x_j^{\mu_j}$ (since this is the monomial in e_λ that we are looking at), so this is a matrix with column-sums μ_j. Similarly, each monomial in h_{λ_i} has variables with combined multiplicity λ_i, so the 1s in row i should add to be λ_i. We have produced a bijection between these matrices and the number of ways to get $x_1^{\mu_1}x_2^{\mu_2}\cdots x_\ell^{\mu_\ell}$ from the product of the h_{λ_i}, so we are done. □

1.1.1 Generating functions for the e_k and h_k

Define the generating function for the e_n terms: $E(t) = \sum_n e_n t^n$.

$$E(t) = \sum_n e_n t^n = \prod_i (1 + tx_i)$$

because e_n is the sum of monomials where we choose n different x_i terms to multiply together. Also define the generating function for the h_n terms: $H(t) = \sum_n h_n t^n$.

$$H(t) = \sum_n h_n t^n = \prod_i (1 + x_i t + x_i^2 t^2 + \cdots) = \prod_i \frac{1}{1 - x_i t},$$

where the second equality follows from the observation that multiplying the $(1 + x_i t + \cdots)$ terms keeps track of the number of powers of each x_i, where the powers of the x_i terms correspond to a partition of n; then the coefficient of t^n is the sum of all polynomials corresponding to partitions of n.

This gives us that

$$E(t) = 1/H(-t), \qquad E(t)H(-t) = 1,$$

and looking at the t^k term in the latter series expansion gives us

$$e_k - e_{k-1}h_1 + e_{k-2} - \cdots + (-1)^k h_k = 0.$$

This gives us e_k in terms of $h_1, \ldots, h_k$ and h_k in terms of $e_1, \ldots, e_k$, and the expressions for each in terms of the other are the same! We also get that $\{h_\lambda : \lambda_1 \leq n\}$ is a basis for $\lambda(x_1, \ldots, x_n)$

Also, if we define for $\Lambda(x_1, x_2, \ldots)$ the map $\omega : \Lambda \to \Lambda$ sending $e_k \mapsto h_k$, we have $\omega_k^{-1} : \Lambda \to \Lambda$ given by $h_k \mapsto e_k$. This is because the expression for h_k in terms of the e_k is the exact same as the expression for e_k in terms of the h_k. So $\omega^2 = \mathrm{id}_\Lambda$. This makes ω an involution.

1.2 Power sum symmetric functions

Definition 1.2. The *power sum symmetric functions* are the functions

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}, \text{ where } p_k = x_1^k + x_2^k + \cdots = m_{(k)}.$$

Example 1.2. We can calculate a few p_λ in terms of our other bases.

$$p_2 = m_{(2)},$$
$$p_{(1,1)} = p_1^2 = m_{(2)} + 2m_{(1,1)},$$
$$e_2 = m_{(1,1)} = \frac{p_1^2 - p_2}{2},$$
$$h_2 = m_{(1,1)} + m_{(2,2)} = \frac{p_1^2 + p_2}{2}.$$

The power-sum symmetric functions are related to the homogeneous symmetric functions and the sizes of conjugacy classes of the symmetric group.

Proposition 1.2.

$$h_n = \sum_\lambda \frac{p_\lambda}{z_\lambda}, \ \textit{where} \ \frac{n!}{z_\lambda} = |C_\lambda|.$$

Proof. Use the generating function $H(t) = \sum_n h_n t^n = \prod_i \frac{1}{1-x_i t}$. Then

$$\log(H(t)) = \sum_i \log\left(\frac{1}{1-x_i t}\right) = \sum_i -\log(1-x_i t) = \sum_i \sum_{k=1}^\infty \frac{x_i^k t^k}{k} = \sum_{k=1}^\infty \frac{p_k t^k}{k}.$$

So, using the generating function for the sizes of conjugacy classes of the symmetric group (proved in lecture 3), we have

$$H(t) = \exp\left(\sum_k \frac{p_k t^k}{k}\right) = \sum_n \left(\sum_{|\lambda|=n} \frac{p_\lambda}{z_\lambda}\right) t^n.$$

□