Math 247A Lecture 6 Notes

Daniel Raban

January 17, 2020

1 Hunt's Interpolation Theorem

1.1 Strong type and weak type

Definition 1.1. We say that a map T on some measurable class of functions is **sublinear** if

1. $|T(cf)| \leq |c||Tf|$,
2. $|T(f,g)| \leq |T(f)| + |T(g)|$

for all constant $c \in \mathbb{C}$ and f, g in the domain of T.

Example 1.1. If T is linear, it is sublinear.

Example 1.2. If $\{T_t\}_{t \in S}$ is a family of liner maps, then

$$(Tf)(x) = \|(T_t f)(x)\|_{L_t^2}$$

is a sublinear map.

Definition 1.2. Let $1 \leq p, q \leq \infty$, and let T be a sublinear map.

1. We say that T is of **(strong) type** (p, q) if there exists a constant $C > 0$ such that
$$\|Tf\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p}, \qquad \forall f \in L^p(\mathbb{R}^d).$$
2. If $q < \infty$, we say that T is of **weak-type** (p, q) if there exists a constant $C > 0$ such that
$$\|Tf\|_{L^{q,\infty}}^* \leq \|f\|_{L^p(\mathbb{R}^d)} \qquad \forall f \in L^p(\mathbb{R}^d).$$
If $q = \infty$, we say that T is of weak-type (p, q) if it is of strong type (p, q).
3. If $p, q < \infty$, we saw that T is of **restricted weak-type** (p, q) if there exists a constant $C > 0$ such that
$$\|T\mathbb{1}_F\|_{L^{q,\infty}}^* \leq C|F|^{1/p} (\leq C\|\mathbb{1}_F\|_{L^{p,1}}^*) \qquad \forall F \subseteq \mathbb{R}^d, |F| < \infty.$$

Remark 1.1.

$$\text{Strong type } (p,q) \implies \text{weak-type } (p,q) \implies \text{restricted weak-type } (p,q).$$

For the first implication, we have $\|Tf\|^*_{L^{q,\infty}} \lesssim \|Tf\|^*_{L^{q,q}} \lesssim \|f\|_{L^p}$. For the second implication,

$$\|T\mathbb{1}_F\|^*_{L^{q,\infty}} \lesssim \|\mathbb{1}_F\|_{L^p} = \|\mathbb{1}_F\|^*_{L^{p,p}} \lesssim \|\mathbb{1}_F\|^*_{L^{p,1}} \lesssim |F|^{1/p}.$$

Exercise 1.1. For $1 < p, q < \infty$, let T be defined on functions on $(0, \infty)$ via

$$(Tf)(x) = |x|^{-1/q} \int_0^\infty |y|^{-1/p'} f(y)\, dy.$$

Then T is of restricted weak-type (p,q) but not of weak type (p,q).

Remark 1.2. Fix $1 < p, q < \infty$. If T is of restricted weak-type (p,q), then for any finite-measure sets $E, F \subseteq \mathbb{R}^d$,

$$\int |(T\mathbb{1}_F)(x)| \cdot |\mathbb{1}_E(x)|\, dx \lesssim \|T\mathbb{1}_F\|^*_{L^{q,\infty}} \|\mathbb{1}_E\|^*_{L^{q',1}} \lesssim |F|^{1/p}|E|^{1/q'}.$$

Conversely, if this condition holds for all finite measure sets $E, F \subseteq \mathbb{R}^d$, then T is of restricted weak-type (p,q). Indeed,

$$\|T\mathbb{1}_F\|^*_{L^{q,\infty}} \sim \sup_{\|g\|^*_{L^{q',1}} \leq 1} \left| \int T\mathbb{1}_F(x) g(x)\, dx \right|.$$

Take $g = \sum 2^m \mathbb{1}_{E_m}$ with E_m measurable and pairwise disjoint. Then

$$\begin{aligned}
\left| \int T\mathbb{1}_F(x) g(x)\, dx \right| &\leq \sum 2^m \int |T\mathbb{1}_F(x)| \cdot |\mathbb{1}_{E_m}(x)|\, dx \\
&\lesssim \sum 2^m |F|^{1/p} |E_m|^{1/q'} \\
&\lesssim |F|^{1/p} \|g\|^*_{L^{q',1}} \\
&\lesssim |F|^{1/p}.
\end{aligned}$$

Remark 1.3. If $1 < p, q < \infty$, then T is of restricted weak-type (p,q) if and only if there is a constant $C > 0$ such that

$$\|Tf\|^*_{L^{q,\infty}} \leq C \|f\|^*_{L^{p,1}} \qquad \forall f \in L^{p,1}(\mathbb{R}^d).$$

1.2 Hunt's interpolation theorem

Theorem 1.1 (Hunt's interpolation theorem)**.** *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$ with $p_1 < p_2$ and $q_1 \ne q_2$. Assume that T is a sublinear map satisfying $\|Tf\|_{L^{q_j,\infty}} \lesssim \|f\|^*_{L^{p_j,1}}$ for $j = 1, 2$. Then, for any $1 \le r \le \infty$ and $\theta \in (0,1)$, we have*

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}, \qquad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Remark 1.4.

1. If $p_\theta \le q_\theta$, then T is of strong type (p_θ, q_θ). Indeed, taking $r = q_\theta$, we get
$$\|Tf\|_{L^{q_\theta}} \lesssim \|f\|^*_{L^{p_\theta,q_\theta}} \lesssim \|f\|_{L^{p_\theta}}.$$

2. The condition $p_\theta \le q_\theta$ is needed to obtain the strong-type conclusion. For example, let $(Tf)(x) = f(x)|x|^{-1/2}$. Then $T : L^p(0,\infty) \to L^{2p/(p+2)}(0,\infty)$ boundedly for any $2 \le p < \infty$. But T is not bounded from L^p to $L^{2p/(p+2)}$ for all $2 < p < \infty$. To see that $T : L^p \to L^{2p/(p+2),\infty}$ is bounded, we use the Hölder inequality in Lorentz spaces (which we will prove later): If $1 \le p_1, P_2, p < \infty$ and $1 \le q_1, q_2, q \le \infty$, then
$$\|f_1 f_2\|^*_{L^{p,q}} \lesssim \|f_1\|^*_{L^{p_1,q_1}} \|f_1\|^*_{L^{q_1,q_1}}, \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$
Then
$$\|Tf\|^*_{L^{2p/(p_2),\infty}} \lesssim \||x|^{-1/2}\|^*_{L^{2,\infty}} \|f\|^*_{L^{p,\infty}} \lesssim \|f\|_{L^p}.$$
Take
$$f(x) = |x|^{-1/p} |\log(x + 1/x)|^{-(p+2)/(2p)}.$$
We get
$$\begin{aligned} \|f\|^p_{L^p} &= \int_0^\infty |\log(x+1/x)|^{-(p+2)/p} \frac{dx}{x} \\ &= \sum_{n\in\mathbb{Z}} \int_{2^n} 2^{n+1} |\log(x+1/x)|^{-1-p/2} \frac{dx}{x} \\ &\sim \sum_{n\in\mathbb{Z}} |n|^{-1-p/2} \\ &< \infty \end{aligned}$$
On the other hand,
$$\begin{aligned} \|Tf\|^{2p/(p+2)}_{L^{2p/(p+2)}} &= \int_0^\infty |\log(x+1/x)|^{-1} \frac{dx}{x} \\ &\sim \sum_{n\in\mathbb{Z}} |n|^{-1} \end{aligned}$$

$$= \infty$$

We know that $T : L^{p_1,p_1} \to L^{2p_1/(p_1+2),\infty}$ and $T : L^{p_2,p_2} \to L^{2p_2/(p_2+2),\infty}$ for $2 \leq p_1 < p_2 < \infty$. Hunt's interpolation theorem gives that for all $1 \leq r \leq \infty$, $T : L^{p_\theta,r} \to L^{2p_\theta/(p_\theta+2),r}$. Note that $\frac{2p_\theta}{p_\theta+2} < p_\theta$.

Next time, we will prove the following as a consequence of Hunt's interpolation theorem.

Corollary 1.1 (Marcinkiewicz interpolation theorem)**.** *Let* $1 \leq p_1 \leq q_1 \leq \infty$ *and* $1 \leq p_2 \leq q_2 \leq \infty$ *with* $p_1 \leq p_2$ *and* $q_1 \neq q_2$*. Let* T *be a sublinear map that satisfies*

$$\|Tf\|^*_{L^{q_j,\infty}} \lesssim \|f\|_{L^{p_j}}, \qquad j = 1, 2.$$

Then for any $\theta \in (0,1)$*,* T *is of strong type* (p_θ, q_θ)*, where*

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

We will also prove Hunt's theorem next time.