

# Math 250A Lecture 12 Notes

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## 1 More on Projective Modules

### 1.1 Projective modules as direct sums

Recall that  $P$  is a projective module if it satisfies the following commutative diagram for exact sequences of modules  $M$  and  $N$ :

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ & \nwarrow \text{dashed} & \uparrow & & \\ & & P & & \end{array}$$

We also showed that  $P$  projective iff  $P \oplus Q$  is free for some  $Q$ . Are all submodules of free modules projective? The answer is no.

**Example 1.1.** Here is a non-projective submodule of a free module. Let  $R = K[x, y]$ , where  $K$  is a field, and let  $I = (x, y)$ , the ideal of polynomials with constant term 0. Look at  $R \oplus R \xrightarrow{g} I \rightarrow 0$  where  $(1, 0) \mapsto x$  and  $(0, 1) \mapsto y$ . If  $I$  is projective, then there exists some map  $f : I \rightarrow R \oplus R$  such that  $gf(a) = a$ . Now suppose that  $f(x) = (a, b)$  and  $f(y) = (c, d)$ . Then  $ax + by = x$  and  $cx + dy = y$ . Then  $y(a, b) = x(c, d)$ , so  $ya = xc$  and  $yb = xd$ . There are no polynomials satisfying this because  $ax + by = x$  implies that  $a = 1 + yp$  (where  $p$  is a polynomial), and  $ya = xc$  implies that  $a$  cannot be  $1 + yp$ .

### 1.2 Eilenberg-Mazur swindle

This is a technique useful for proving  $1 = 0$ . Here is a basic example.

**Example 1.2.** Start with  $1 + (-1) = 0$ . Then

$$0 = (1 + (-1)) + (1 + (-1)) + \cdots$$

$$1 = 1 + (-1 + 1) + (-1 + 1) + \cdots,$$

so we have shown that  $1 = 0$ .

We assumed two things in the above example:

1. 1 has an additive inverse  $-1$ .
2. All infinite sums make sense.

The second condition is violated in  $\mathbb{Z}$ , but we can use this technique to show that one of these two conditions does not hold.

**Example 1.3.** Knots have no inverse. Suppose we have a closed loop with a knot in it. Is there another knot we can put on the loop that will cancel out the first knot? The answer is no. Apply the swindle: add infinite numbers of knots, making each successive knot smaller so the knots all fit on the loop. Then the above contradiction would occur, so a knot must not have an additive inverse.

**Example 1.4.** Suppose  $P$  is projective. Then  $P \oplus Q = F$ , where  $F$  is free. Then  $Q$  is also projective. We can take  $Q$  to be free (in fact equal to  $F$ ). Think of free modules as 0 in some sense. So  $P \oplus Q$  is free means that  $Q$  is a sort of additive inverse of  $P$  (again, if we ignore free modules). So infinite sums are defined, and we can use the swindle to get that  $P = 0$  if we ignore free modules. What we mean here is that  $P \oplus Q$  is free for some free module  $Q$ . The catch is that this free module  $Q$  is not finitely generated.

## 2 Tensor Products

This is covered in Chapter XVI in Lang, but we will cover it here. This is something you really should know.

### 2.1 Construction and universal property

**Definition 2.1.** A *bilinear* map  $f : X \times Y \rightarrow Z$  is a map such that  $f(\cdot, y)$  is linear for fixed  $y$  and  $f(x, \cdot)$  is linear for fixed  $x$ .

**Definition 2.2.** Suppose  $R$  is a commutative<sup>1</sup> ring, and suppose that  $M$  and  $N$  are  $R$ -modules. The *tensor product*  $M \otimes N$  is the module such that if  $f : M \times N \rightarrow P$  is bilinear, then there exists a linear map  $\tilde{f} : M \otimes N \rightarrow P$  such that the following diagram commutes

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & M \otimes N \\ & \searrow f & \downarrow \tilde{f} \\ & & P \end{array}$$

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<sup>1</sup>This assumption is not necessary, but it simplifies things for now.

To construct  $M \otimes N$ , take the free module on elements  $m \otimes n$  with  $m \in M$  and  $n \in N$ . We get linear maps from this to  $P : m \otimes n \mapsto f(m, n)$ . Take the quotient by all elements of the form

$$\begin{aligned} (m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n \\ m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2 \\ (rm) \otimes n - r(m \otimes n) \\ m \otimes (rn) - r(m \otimes n). \end{aligned}$$

Taking the quotient by these elements enforces relations we want, such as

$$(rm) \otimes n = r(m \otimes n) = m \otimes (rn),$$

so the tensor product exists.

Now that we have constructed the tensor product, what does it look like? We have the identity

$$(M_1 \oplus M_2) \otimes N \cong (M_1 \otimes N) \oplus (M_2 \otimes N),$$

which says that a bilinear map  $(M_1 \oplus M_2) \otimes N \rightarrow P$  is the same as a pair of bilinear maps from  $(M_1 \otimes N) \rightarrow P$  and  $(M_2 \otimes N) \rightarrow P$ . Similarly, we have the identity

$$R \otimes M \cong M,$$

which says that bilinear maps  $R \times M \rightarrow P$  are the same as linear maps from  $M \rightarrow P$ .

**Example 2.1.**

$$R^m \otimes R^n \cong R^{m+n}$$

If  $V, W$  are vector spaces with bases  $\{v_i\}$  and  $\{w_j\}$ , then  $V \otimes W$  has basis  $v_i \otimes w_j$ .

## 2.2 Exact sequences and the tensor product

**Proposition 2.1.** *Suppose  $A \rightarrow B \rightarrow C \rightarrow 0$  is exact. Then so is*

$$A \otimes M \rightarrow B \otimes M \rightarrow C \otimes M \rightarrow 0.$$

**Remark 2.1.** This does not hold if we put a  $0 \rightarrow$  before both of these sequences. We say that  $\otimes M$  is *right exact*.

*Proof.* To prove things about the tensor product, forget the construction of the tensor product using relations and instead use the universal property.

Homomorphisms  $A \otimes B \rightarrow C$  are bilinear maps  $A \times B \rightarrow C$ , which are linear maps  $A \rightarrow \text{Hom}_R(B, C)$ . Think of this as an analogue of the fact that functions  $R \times S \rightarrow T$  are the same as functions from  $R$  to the set of functions from  $S$  to  $T$ .

The key point of this proof is that  $A \rightarrow B \rightarrow C \rightarrow 0$  is exact if and only if

$$\text{Hom}(A, M) \leftarrow \text{Hom}(B, M) \leftarrow \text{Hom}(C, M) \leftarrow 0$$

is exact. We leave this as an exercise.<sup>2</sup>

We want to show that  $A \otimes N \rightarrow B \otimes N \rightarrow C \otimes N \rightarrow 0$  is exact. Then this is equivalent to the following sequence being exact:

$$\text{Hom}(A \otimes N, M) \leftarrow \text{Hom}(B \otimes N, M) \leftarrow \text{Hom}(C \otimes N, M) \leftarrow 0.$$

Then, using our identification of homomorphisms  $A \otimes N \rightarrow M$  with linear maps  $A \rightarrow \text{Hom}_R(N, M)$ , this is equivalent to the following sequence being exact:

$$\text{Hom}(A, \text{Hom}(N, M)) \leftarrow \text{Hom}(B, \text{Hom}(N, M)) \leftarrow \text{Hom}(C, \text{Hom}(N, M)) \leftarrow 0.$$

And this is exact by applying the key point again.  $\square$

We can now calculate  $M \otimes N$ . Pick  $R^a \rightarrow R^b \rightarrow M \rightarrow 0$ , where  $R^a, R^b$  are free. Pick relations generating  $\ker(R^b \rightarrow M)$  and pick a set of  $b$  generators of  $M$ . Tensoring with  $N$  gives us that

$$R^a \otimes N \rightarrow R^b \otimes N \rightarrow M \otimes N \rightarrow 0$$

is exact. So we get

$$N^a \rightarrow N^b \rightarrow M \otimes N \rightarrow 0,$$

which makes  $M \otimes N = N^b / \text{im}(N^a \rightarrow N^b)$ .

**Example 2.2.** We can find  $M \otimes N$  for finitely generated abelian groups  $M, N$ . Recall that finitely generated abelian groups are direct sums of copies of  $\mathbb{Z}$  and  $\mathbb{Z}/n\mathbb{Z}$ . Since  $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$ , it is enough to work out a few cases:

1.  $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$
2.  $\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}.m\mathbb{Z}$
3.  $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}/m\mathbb{Z}$
4.  $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/(\text{gcd}(m, n)\mathbb{Z})$ .

To obtain this last result, take the exact sequence

$$\mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0.$$

Then the sequence

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}/n\mathbb{Z} \rightarrow (\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}) \rightarrow 0$$

is exact, so

$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/n\mathbb{Z})/m(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(\text{gcd}(m, n)\mathbb{Z}).$$

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<sup>2</sup>This was an exercise from last lecture, but Professor Borchers suspects that no one actually does them.

**Example 2.3.**

$$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = 0$$

$$\mathbb{Z}/9\mathbb{Z} \otimes \mathbb{Z}/12\mathbb{Z} = \mathbb{Z}/3\mathbb{Z}$$

The tensor product is not left exact. Look at  $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ . The following sequence is not exact:

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

## 2.3 More examples and properties

**Definition 2.3.** An *algebra*  $S$  over a ring  $R$  is a commutative ring with a homomorphism  $R \rightarrow S$  that makes  $S$  an  $R$ -module.

You can think of algebras as modules with multiplication.

**Example 2.4.** Let  $S, T$  be algebras over  $R$ . Then  $S \otimes_R T$  is a push-out of  $S, T$  over  $R$ .

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \otimes_R T \end{array}$$

Check that  $S \otimes_R T$  is a commutative ring. We need a bilinear map  $(S \otimes T) \times (S \otimes T) \rightarrow (S \otimes T)$ . This is a linear map from  $S \otimes T \otimes S \otimes T \rightarrow S \otimes T$ . This relies on associativity of the tensor product;  $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$  because maps from each to  $M$  are trilinear maps  $A \times B \times C \rightarrow M$ . We have a map  $S \otimes S \rightarrow S$  given by the product on  $S$ . Same for  $T \otimes T \rightarrow T$ . So we get a map  $S \otimes T \otimes S \otimes T \rightarrow S \otimes S \otimes T \otimes T \rightarrow S \otimes T$  by sending  $(s_1 \otimes t_1) \times (s_2 \otimes t_2) \rightarrow s_1 s_2 \otimes t_1 t_2$ . We leave verification of the pushout property as an exercise.

**Example 2.5.**  $S = K[x]$  and  $T = K[y]$  with bases  $\{x^m\}$  and  $\{y^n\}$ , respectively.  $S \otimes_R T$  has a basis  $x^m \otimes y^n$ . This can be identified as the polynomial ring  $K[x, y]$  via the map  $x^m \otimes y^n \rightarrow x^m y^n$ .

**Example 2.6.**  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$  is a ring.  $\mathbb{C}$  has basis  $\{1, i\}$ , so  $\mathbb{C} \otimes \mathbb{C}$  has basis  $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\}$ . Calculating a few products, we get

$$(i \otimes i)(i \otimes i) = i^2 \otimes i^2 = -1 \otimes -1 = 1 \otimes 1$$

$$(1 \otimes 1)(a \otimes b) = (a \otimes b)$$

$$(1 \otimes 1 + i \otimes i)^2 = 2(1 \otimes 1 + i \otimes i).$$

Call  $e = (1 \otimes 1 + i \otimes i)/2$ . Then  $e^2 = e$ , so  $e$  is idempotent. Then this ring splits as a product, so  $\mathbb{C} \otimes \mathbb{C} = e(\mathbb{C} \otimes \mathbb{C}) \times (1 - e)(\mathbb{C} \otimes \mathbb{C}) \cong \mathbb{C} \times \mathbb{C}$ .

**Example 2.7.** The tensor product satisfies all the axioms of a commutative *semiring* (a ring but without subtraction)

1.  $(A \otimes B) \otimes C$
2.  $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$
3.  $A \otimes B \cong B \otimes A$
4.  $A \oplus B \cong B \oplus A$
5.  $(A \oplus B) \oplus C \cong A \oplus (B \oplus C)$
6.  $R \otimes A \cong A$ .

If we want to construct a ring out of this structure, we have a few problems:

1. The set of all modules is not a set.
2. There is no subtraction.

This can be circumvented by constructing the set of all pairs  $M - N$  for  $M, N$  modules under some equivalence relation.

3. By the swindle,  $M = 0$  for any  $M$ .

We circumvent problems 1 and 3 by only considering finitely generated modules.<sup>3</sup>

**Example 2.8.** Take  $R = \mathbb{Z}$ , the integers. The finitely generated modules are all of the form  $\mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{n_2} \oplus (\mathbb{Z}/4\mathbb{Z})^{n_4} \oplus (\mathbb{Z}/8\mathbb{Z})^{n_8} \oplus \cdots \oplus (\mathbb{Z}/3\mathbb{Z})^{n_3} \oplus \cdots$ . So we get a basis  $\{n_i b_i\}$ , where we allow the  $n_i$  to be positive or negative. The product is  $b_0 \times b_n = b_n$  and  $b_{p^a} \times b_{p^b} = b_{p^{\min(a,b)}}$ .

## 2.4 Tensor products of noncommutative rings

When  $R$  is a noncommutative ring,  $M \otimes_R N$  is only defined for  $M$  a right module and  $N$  a left module. This is because we need

$$(mr) \otimes n = m \otimes (rn).$$

Secondly,  $M \otimes_R N$  is only an abelian group, not an  $R$ -module. We have that

$$mr \otimes n = m \otimes rn,$$

but multiplying by  $s$  gives us

$$mrs \otimes n = m \otimes srn,$$

even though we want  $m(rs) \otimes n = m \otimes (rs)n$ .

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<sup>3</sup>This leads into  $K$ -theory, where you consider the ring of finitely generated modules over a ring  $R$ .