

# Math 247A Lecture 4 Notes

Daniel Raban

January 13, 2019

## 1 Relationships Between The Lorentz Quasinorms and $L^p$ Norms

### 1.1 Order of growth of Lorentz quasinorms in terms of $L^p$ and $\ell^q$

Last time, we had the quasinorm

$$\|f\|_{L^{p,q}(\mathbb{R}^d)}^* = p^{1/q} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^q((0,\infty), \frac{d\lambda}{\lambda})}$$

**Remark 1.1.** If  $|g| \leq |f|$ , then  $\|g\|_{L^{p,q}}^* \leq \|f\|_{L^{p,q}}^*$ .

**Proposition 1.1.** If  $f \in L^{p,q}(\mathbb{R}^d)$  for  $1 \leq p < \infty$  and  $1 \leq q \leq \infty$ , write  $f = \sum_{m \in \mathbb{Z}} f_m$ , where  $f_m(x) = f(x) \mathbb{1}_{\{x: 2^m \leq |f(x)| < 2^{m+1}\}}(x)$ . Then

$$\|f\|_{L^{p,q}}^* \sim \left\| \|f_m\|_{L^p(\mathbb{R}^d)} \right\|_{\ell_m^q(\mathbb{Z})}.$$

*Proof.* Both sides only concern  $|f|$ , so it suffices to prove this for  $f \geq 0$ . Then

$$2^m \mathbb{1}_{\{2^m \leq f(x) < 2^{m+1}\}} \leq f_m < 2^{m+1} \mathbb{1}_{\{2^m \leq f(x) < 2^{m+1}\}}.$$

Thus, by our previous remark, we may assume that  $f = \sum_{m \in \mathbb{Z}} 2^m \mathbb{1}_{F_m}$ , where  $F_m$  are measurable, pairwise disjoint sets.

$$\begin{aligned} (\|f\|_{L^{p,q}}^*)^q &= p \int_0^\infty \lambda^q |\{x : \sum_n 2^n \mathbb{1}_{F_n} > \lambda\}|^{q/p} \frac{d\lambda}{\lambda} \\ &= p \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q |\{x : \sum_n 2^n \mathbb{1}_{F_n} > \lambda\}|^{q/p} \frac{d\lambda}{\lambda} \end{aligned}$$

For  $2^{m-1} \leq \lambda < 2^m$ ,  $\{x : \sum_n 2^n \mathbb{1}_{F_n}(x) > \lambda\} = \bigcup_{n \geq m} F_n$ .

$$\sim \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q \left( \sum_{n \geq m} |F_n| \right)^{q/p} \frac{d\lambda}{\lambda}$$

$$\begin{aligned}
&\sim \sum_{m \in \mathbb{Z}} 2^{mq} \left( \sum_{n \geq m} |F_n| \right)^{q/p} \\
&\sim \left\| 2^m \left( \sum_{n \geq m} |F_n| \right)^{1/p} \right\|_{\ell_m^q}^q.
\end{aligned}$$

We wanted to show that  $\|f\|_{L^{p,q}}^* \sim \|2^m |F_m|^{1/p}\|_{\ell_m^q}$ . So we just need to show that  $\left\| 2^m \left( \sum_{n \geq m} |F_n| \right)^{1/p} \right\|_{\ell_m^q} \sim \|2^m |F_m|^{1/p}\|_{\ell_m^q}$ . We have the  $\geq$  direction, so we just need the other inequality:

$$\begin{aligned}
\left\| 2^m \left( \sum_{n \geq m} |F_n| \right)^{1/p} \right\|_{\ell_m^q} &\leq \left\| 2^m \sum_{n \geq m} |F_n|^{1/p} \right\|_{\ell_m^q} \\
&\lesssim \sum_{k \geq 0} 2^{-k} \|2^{m+k} |F_{m+k}|^{1/p}\|_{\ell_m^q}
\end{aligned}$$

Now reindex the  $\ell^q$  sum by  $n = m + k$ .

$$\begin{aligned}
&\lesssim \sum_{k \geq 0} 2^{-k} \|2^n |F_n|^{1/p}\|_{\ell_n^q} \\
&\lesssim \|2^n |F_n|^{1/p}\|_{\ell_n^q}.
\end{aligned}$$

□

## 1.2 Lorentz spaces are Banach spaces

**Lemma 1.1.** *Let  $1 \leq q < \infty$ , and let  $S \subseteq 2^{\mathbb{Z}}$ , the dyadic integers. Then*

$$\sum_{N \in S} N^q \leq \left( \sum_{N \in S} N \right)^q \leq \left( 2 \sup_{N \in S} N \right)^q \leq 2^q \sum_{N \in S} N^q.$$

In other words, if we're summing dyadic series, when we take the  $L^q$  norm, it doesn't really matter whether we have the  $q$  inside or outside the sum.

**Theorem 1.1.** *For  $1 < p < \infty$  and  $1 \leq q \leq \infty$ ,*

$$\|f\|_{L^{p,q}}^* \sim \sup \left\{ \left| \int f(x)g(x) dx \right| : \|g\|_{L^{p',q'}}^* \leq 1 \right\}.$$

*Thus,  $\|\cdot\|_{L^{p,q}}^*$  is equivalent to a norm, with respect to which  $L^{p,q}(\mathbb{R}^d)$  is a Banach space. Moreover, for  $q \neq \infty$ , the dual of  $L^{p,q}$  is  $L^{p',q'}$ , under the natural pairing.*

**Remark 1.2.** For  $p = 1, q \neq 1$ , there cannot be a norm equivalent to  $\|\cdot\|_{L^{1,q}}^*$ . Let's see this for  $q = \infty$  and  $d = 1$ . Assume, towards a contradiction, that  $\|\cdot\|_{L^{1,\infty}}^* \sim \|\cdot\|$ . Let  $f(x) = \sum_{n=1}^N \frac{1}{|x-n|}$  for  $N \gg 1$ . Then

$$\left\| \frac{1}{|x-n|} \right\|_{L^{1,\infty}}^* = \sup_{\lambda > 0} \lambda \left| \left\{ x : \frac{1}{|x-n|} > \lambda \right\} \right| = 2,$$

so

$$\sum_{n=1}^N \left\| \frac{1}{|x-n|} \right\| \sim \sum_{n=1}^N \left\| \frac{1}{|x-n|} \right\|_{L^{1,\infty}}^* = 2N.$$

Then we have

$$\|f\|_{L^{1,\infty}}^* = \sup_{\lambda > 0} \lambda \left| \left\{ x : \sum_{n=1}^N \frac{1}{|x-n|} > \lambda \right\} \right|.$$

We claim that  $\{x : \sum_{n=1}^N \frac{1}{|x-n|} > \frac{1}{10} \log N\} \supseteq [0, N]$ . If  $x = 0$ , then  $\sum 1/n > \log(N+1) \geq \frac{1}{10} \log N$ . Now do the same with  $x = 1, x = 2, \dots$ . The worst case scenario is when  $x \approx N/2$ , but the inequality holds in this case, too. So we have

$$\|f\|_{L^{1,\infty}}^* \geq \frac{1}{10} \log N \left| \left\{ x : \sum_{n=1}^N \frac{1}{|x-n|} > \frac{1}{10} \log N \right\} \right| \geq \frac{N \log N}{10}.$$

So we have shown that

$$\|f\| \sim \|f\|_{L^{1,\infty}}^* \geq \frac{N \log N}{10}.$$

This gives

$$N \log N \lesssim \|f\| \leq \sum_{n=1}^N \left\| \frac{1}{|x-n|} \right\| \sim N.$$

Let  $N \rightarrow \infty$  to get a contradiction.

Now let's prove the theorem.

*Proof.* We may assume  $f \geq 0, g \geq 0$ . As both sides are positive homogeneous, we may assume that  $\|f\|_{L^{p,q}}^* = 1$ . We may assume  $f = \sum 2^n \mathbb{1}_{F_n}$  and  $g = \sum 2^m \mathbb{1}_{E_m}$  with  $F_n$  measurable, pairwise disjoint and  $E_n$  measurable, pairwise disjoint. Then

$$\begin{aligned} 1 &= (\|f\|_{L^{p,q}}^*)^q \\ &\sim \|2^n |F_n|^{1/p}\|_{\ell_q}^q \\ &\sim \sum_{n \in \mathbb{Z}} 2^{nq} |F_n|^{q/p} \end{aligned}$$

$$\begin{aligned}
&\sim \sum_{N \in 2^{\mathbb{Z}}} \sum_{n: N \leq |F_n| < 2N} 2^{nq} |F_n|^{q/p} \\
&\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q/p} \sum_{n: |F_n| \sim N} 2^{nq}
\end{aligned}$$

By the lemma,

$$\begin{aligned}
&\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q/p} \left( \sum_{n: |F_n| \sim N} 2^n \right)^q \\
&\sim \sum_{N \in 2^{\mathbb{Z}}} \left( \sum_{n: |F_n| \sim N} 2^n |F_n|^{1/p} \right)^q.
\end{aligned}$$

Similarly,

$$1 \geq (\|g\|_{L^{p',q'}})^{q'} \sim \sum_{M \in 2^{\mathbb{Z}}} \left( \sum_{m: |E_m| \sim M} 2^m |E_m|^{1/p'} \right)^{q'}.$$

Now

$$\begin{aligned}
\int f(x)g(x) dx &= \sum_{n,m} 2^n 2^m |F_n \cap E_m| \\
&\lesssim \sum_{N,M \in 2^{\mathbb{Z}}} \sum_{n: |F_n| \sim N} \sum_{m: |E_m| \sim M} 2^n |F_n|^{1/p} 2^m |E_m|^{1/p'} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \\
&\lesssim \sum_{N,M \in 2^{\mathbb{Z}}} \left( \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \right)^{1/q+1/q'} \sum_{n: |F_n| \sim N} 2^n |F_n|^{1/p} \sum_{m: |E_m| \sim M} 2^m |E_m|^{1/p'}
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\lesssim \left[ \sum_{N,M \in 2^{\mathbb{Z}}} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \left( \sum_{n: |F_n| \sim N} 2^n |F_n|^{1/p} \right)^q \right]^{1/q} \\
&\quad \cdot \left[ \sum_{N,M \in 2^{\mathbb{Z}}} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \left( \sum_{m: |E_m| \sim M} 2^m |E_m|^{1/p'} \right)^{q'} \right]^{1/q'}.
\end{aligned}$$

Now we just need  $\sum_{M \in 2^{\mathbb{Z}}} \frac{\min\{N, M\}}{N^{1/p} M^{1/p'}} \lesssim 1$ . This comes from

$$\sum_M \min \left\{ \left( \frac{N}{M} \right)^{1/p'}, \left( \frac{M}{N} \right)^{1/p} \right\} \lesssim \sum_{M \leq N} \left( \frac{M}{N} \right)^{1/p} + \sum_{M > N} \left( \frac{N}{M} \right)^{1/p'} \lesssim 1,$$

as we get a geometric series.<sup>1</sup>

□

---

<sup>1</sup>Instead of using Hölder's inequality and the subsequent steps, we could alternatively use Schur's test for convergence of series. This kind of argument will be common in this course.