Math 255B Lecture 5 Notes

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1 The Toeplitz Index Theorem and Analytic Fredholm Theory

1.1 The Toeplitz index theorem

Last time, we had the Hardy space $H \subseteq L^2(\mathbb{R}/2\pi\mathbb{Z})$ of functions u with $\widehat{u}(n) = 0$ for $n < 0$. Given $f \in C(\mathbb{R}/2\pi\mathbb{Z})$, we defined $\mathrm{Top}(f) = \pi M_f$.

Theorem 1.1 (Toeplitz index theorem)**.** *If $f \in C(\mathbb{R}/2\pi\mathbb{Z})$ is nonvanishing, then* $\mathrm{Top}(f)$ *is Fredholm on H, and* $\operatorname{ind}\mathrm{Top}(f) = -$*winding number*$(f)$.

Proof. We had the claim that for all $f, g \in C(\mathbb{R}/2\pi\mathbb{Z})$, then $\mathrm{Top}(f)\,\mathrm{Top}(g) - \mathrm{Top}(fg)$ is compact. We saw that this is $\pi[M_f, \pi]M_g$, so we only need to show that $[M_f, \pi]$ is compact from $L^2 \to L^2$. If $f(\theta) = e^{in\theta}$ (or more generally, a trigonometric polynomial), then $[M_f, \pi]$ is of finite rank; we showed this last time.

In general, given $f \in C(\mathbb{R}/2\pi\mathbb{Z})$, let f_n b trigonometric polynomials such that $f_n \to f$ uniformly on $\mathbb{R}/2\pi\mathbb{Z}$. Then

$$\|[M_f, \pi] - [M_{f_n}, \pi]\| = \|[M_{f-f_n}, \pi\| \leq 2\|f - f_n\|_u \to 0.$$

So $[M_f, \pi]$ is compact, and we get the claim.

If $f \neq 0$, we take $g = 1/f$, so $\mathrm{Top}(f)\,\mathrm{Top}(g) - I$ is compact. So $\mathrm{Top}(f)$ is Fredholm. To compute $\operatorname{ind}\mathrm{Top}(f)$, observe that if g, h are continuous (and nonvanishing), then

$$\operatorname{ind}\mathrm{Top}(gh) = \operatorname{ind}(\mathrm{Top}(g)\,\mathrm{Top}(h)) = \operatorname{ind}\mathrm{Top}(g) + \operatorname{ind}\mathrm{Top}(f).$$

Write $f(\theta) = r(\theta)e^{-\varphi(\theta)}$ with r, φ continuous on $[0, 2\pi]$ and $r > 0$. Then

$$\operatorname{ind}\mathrm{Top}(f) = \operatorname{ind}\mathrm{Top}(r) + \operatorname{ind}\mathrm{Top}(e^{i\varphi})$$

We have $\operatorname{ind}\mathrm{Top}(r) = \operatorname{ind}\mathrm{Top}(r_t))$ for $0 \leq t \leq 1$, where $r_t(\theta) = (1-t)r(\theta) + t1 > 0$. So $\operatorname{ind}\mathrm{Top}(r) = 0$.

$$= \operatorname{ind}\mathrm{Top}(e^{i\varphi}).$$

To compute $\operatorname{ind}\operatorname{Top}(e^{i\varphi})$, consider $f_t(\theta) = e^{(1-t)i\varphi(\theta)+iNt\theta}$ for $0 \leq t \leq 1$, where $N = \frac{\varphi(2\pi)-\varphi(0)}{2\pi}$ is the winding number. Then f_t is 2π-periodic and continuous in t. We get

$$\begin{aligned}\operatorname{ind}\operatorname{Top}(e^{i\varphi}) &= \operatorname{ind}\operatorname{Top}(f_t)\\ &= \operatorname{ind}\operatorname{Top}(e^{iN\theta})\end{aligned}$$

In general, if T is Fredholm, $\operatorname{ind} T = \dim\ker T - \dim\ker T^*$.

$$= \dim\ker\operatorname{Top}(e^{iN\theta}) - \dim\ker\operatorname{Top}(e^{iN\theta})^*$$

To find the adjoint, we have $\langle \operatorname{Top}(f)u, v\rangle_{L^2} = \langle \pi(fu), v\rangle_{L^2} = \langle fu, v\rangle_{L^2} = \left\langle u, \overline{f}v\right\rangle_{L^2} = \left\langle \pi u, \overline{f}v\right\rangle_{L^2} = \left\langle i, \operatorname{Top}(\overline{f})v\right\rangle$. So $\operatorname{Top}(f)^* = \operatorname{Top}(\overline{f})$.

$$= \dim\ker\operatorname{Top}(e^{iN\theta}) - \dim\ker\operatorname{Top}(e^{-iN\theta}).$$

Here, we have

$$\dim\ker\operatorname{Top}(e^{iN\theta}) = \begin{cases} 0 & N \geq 0 \\ -N, N < 0 & \end{cases}.$$

Altogether, we get

$$\operatorname{ind}\operatorname{Top}(f) = -N. \qquad \square$$

1.2 Analytic Fredholm Theory

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ a **holomorphic family** $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ is a family such that $\Omega \to \mathcal{L}(B_1, B_2)$ sending $z \mapsto T(z)$ is holomorphic (as an operator-valued function).

Remark 1.1. We can define holomorphic operator-valued functions in two ways: $z \mapsto T(z)$ is **holomorphic** if

1. For all $z \in \Omega$, $\|\frac{T(z+h)-T(z)}{h} - T'(z)\| \to 0$ as $h \to 0$ for some $T'(z) \in \mathcal{L}(B_1, B_2)$.
2. For every $x \in B_1$ and $\xi \in B_2^*$, $z \mapsto \langle T(z)x, \xi\rangle$ is holomorphic.

Theorem 1.2 (analytic Fredholm theory)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ be a holomorphic family of Fredholm operators. Assume that there exists a $z_0 \in \Omega$ such that $T(z_0) : B_1 \to B_2$ is bijective. Then the set*

$$\Sigma = \{z \in \Omega : T(z) \text{ is not bijective}\}$$

is discrete.

Proof. Notice first that $\operatorname{ind} T(z) = \operatorname{ind} T(z_0) = 0$ for all z. Let $z_1 \in \Omega$, and write $n_0(z_1) = \dim \ker T(z_1) = \dim \operatorname{coker} T(z_1)$. Introduce the Grushin operator

$$\mathcal{P}_{z_1}(z) = \begin{bmatrix} T(z) & R_-(z_1) \\ R_+(z_1) & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_0(z_1)} \to B_2 \oplus \mathbb{C}_{n_0(z_1)}.$$

We know that $\mathcal{P}_{z_1}(z_1)$ is invertible. So there is a connected neighborhood $N(z_1) \subseteq \Omega$ of z_1 such that $\mathcal{P}_{z_1}(z)$ is bijective for $z \in N(z_1)$, depending holomorphically on z. Let

$$\mathcal{E}_{z_1}(z) = \mathcal{P}_{z_1}(z)^{-1} : B_2 \oplus \mathbb{C}^{n_0(z_1)} \to B_1 \oplus \mathbb{C}^{n_0(z_1)}$$

$$\mathcal{E}_{z_1}(z) = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix},$$

depending holomorphically on z.

We claim that for $z \in N(z_1)$, $T(z) : B_1 \to B_2$ is bijective $\iff$ $E_{-+}(z) : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}$ is bijective. This will allow us to analyze invertibility of $T(z)$ via a holomorphic function, $\det E_{-+}(z)$. $\square$