Math 210B Lecture 15 Notes

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1 Properties of Determinants and Change of Basis

1.1 Formulas for determinants and effect of elementary matrices

We have an isomorphism $M_n(R) \cong \mathrm{End}_R(R^{\oplus n})$ sending a matrix A to the associated linear transformation T. We say $\det(A) := \det(T)$.

Theorem 1.1. $\det(A) = \sum_{\sigma \in S_n} (\mathrm{sign}(\sigma)) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$.

Proof. Let $v_j \in R^{\oplus n}$ be the j-th column vector of A. Then $T(e_j) = v_j$ for all j. Then

$$v_1 \wedge \cdots \wedge v_n = (\det A) e_1 \wedge \cdots \wedge e_n.$$

On the other hand,

$$v_1 \wedge \cdots \wedge v_n = \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_n$$

In this sum the term will be zero unless all of the i_j are distinct. These also correspond to $\sigma \in S_n$ such that $\sigma(j) = i_j$.

$$\begin{aligned} &= \sum_{\sigma \in S_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} \\ &= \sum_{\sigma \in S_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \underbrace{\mathrm{sign}(\sigma)}_{=\mathrm{sign}(\sigma^{-1})} e_1 \wedge \cdots \wedge e_n \\ &= \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} e_1 \wedge \cdots \wedge e_n. \end{aligned}$$

$\bigwedge^n (R^{\oplus n}) \cong R$ with basis $e_1 \wedge \cdots \wedge e_n$, so we get the desired equality. $\square$

Proposition 1.1. *The determinant has the following properties:*

1. $\det(T) = \det(A^\top)$.

2. *If we switch 2 rows or columns of A to get B, then* $\det(B) = -\det(A)$.
3. *If we add an R-multiple of a row or column of A to another to get A, then* $\det(C) = \det(A)$.
4. *If we scale a row or column of A by $\alpha \in R$, to get D, then* $\det(A) = \alpha \det(A)$.

Proof. These follow from the formula for the determinant.

1. We showed this in the proof of the formula.
2. Reindex the sum by composing with a transposition.
3. If we have a repeated v_j, then the term is zero. So

$$v_1 \wedge \cdots \wedge (v_i + cv_j) \wedge \cdots \wedge v_n = v_1 \wedge \cdots \wedge v_n + c\cancelto{0}{(v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n)}.$$

4. The proof is the same as the previous part. ☐

1.2 Cofactor expansion

Definition 1.1. The (i,j) **minor** of a matrix A is the matrix $A_{i,j}$ with the i-th row and j-th column removed.

The (i,j) minor lies in $M_{n-1}(R)$.

Proposition 1.2. *For all $k \leq j \leq n$,*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

Proof. First, write

$$v_1 \wedge \cdots \wedge v_n = (-1)^{j-1} v_j \wedge (v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_n).$$

Write $v_j = \sum_{i=1}^n a_{i,j} e_i$, and write $w_k^{(i)} := v_k - a_{i,k} e_i$ for all i, k.

$$\begin{aligned}
&= (-1)^{j-1} \sum_{i=1}^{n} a_{i,j} e_i \wedge (w_1^{(i)} \wedge \cdots \wedge w_{j-1}^{(i)} \wedge w_{j+1}^{(i)} \wedge \cdots \wedge w_n^{(i)}) \\
&= (-1)^{j-1} \sum_{i=1}^{n} a_{i,j} \det(A_{i,j}) e_i \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n \\
&= \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}) e_1 \wedge \cdots \wedge e_n.
\end{aligned}$$

☐

Remark 1.1. In this formula, we could have indexed over j, instead.

1.3 Adjoint matrices

Definition 1.2. The **adjoint matrix** to A is the matrix with (i,j)-entry $(-1)^{i+j}\det(A_{j,i})$.

Proposition 1.3. $A \cdot \mathrm{ad}(A) = \det(A) \cdot I_n$.

Proof. The (i,j) entry is

$$\sum_{k=1}^{n} a_{i,k}(-1)^{k+j}\det(A_{j,K} = \begin{cases} \det(A) & i = j \\ 0 & i \neq j \end{cases}$$

because if $i \neq j$, this is the determinant of A with the j-th row replaced by the i-th row. So it is 0. ☐

Corollary 1.1. *$A \in \mathrm{GL}_n(R) \iff \det(A) \in R^{\times}$. In this case, $A^{-1} = \det(A)^{-1}\,\mathrm{ad}(A)$.*

Corollary 1.2. *If V is free of rank n, then $T : V \to V$ is invertible iff $\det(T) \in R^{\times}$.*

1.4 Change of basis

Let V, W be free R-modules of rank n, m respectively. Let $B = (v_1, \ldots, v_n)$ and $C = (w_1, \ldots, w_m)$ be ordered bases of V and W. Let $T : V \to W$ be an R-module homomorphism. Then $A = (a_{i,j})$ represents T with respect to B and C if

$$T(v_j) = \sum_{i=1}^{m} a_{i,j} w_i$$

for all $1 \leq j \leq n$.

B corresponds to $\varphi_B : R^n \to V$, where $\varphi_B(e_i) = v_i$. Given $T : V \to W$, we get $\varphi_C^{-1} \circ T \circ \varphi_B : E^n \to R^m$ is $A \in M_{m,n}(R)$ using the standard basis.

Lemma 1.1. *Let $T' : U \to V$ and $T : V \to W$ be R-module homomorphisms where the modules have bases B, C, C, and D, respectively. Let A' representa T' with respect to B and C, and lt A represent T with respect to C and D. Then AA' represents TT' with respect to B and D.*

Proof. We can see

$$\varphi_D^{-1} \circ T \circ T' \circ \varphi_B = (\varphi_D^{-1} \circ T \circ \varphi_C) \circ (\varphi_C^{-1} \circ T' \circ \varphi_B).$$

The first part is representaed by A, and the latter part is represented by A'. ☐

Definition 1.3. Let B, B' be bases of VV. The **change of basis matrix** $Q_{B,B'}$ from B to B' is the matrix representing $T_{B,B'} : V \to V$ with $T_{B,B'}(v_i) = v_i'$ with respect to B and B' is the matrix representing $\varphi_B^{-1} T_{B,B'} \varphi_B = \varphi_B^{-1} \circ \varphi_{B'}$.