# Math 259A Lecture 1 Notes

Daniel Raban

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## 1 Introduction to Operator Algebras

### 1.1 $*$-algebras

Let $H$ be a Hilbert space. We denote $B(H)$ to be the space of operators on $H$: $B(H)$ is the set of $T : H \to H$ such that $\sup_{\xi \in (H)_1} \|T(\xi)\| =: \|T\| < \infty$, where $(H)_1$ is the closed unit ball. $B(H)$ is an algebra.

**Definition 1.1.** An **operator algebra** is a vector subspace $B \subseteq B(H)$ closed under multiplication.

Given an operator $T$, we have an **adjoint operator** $T^*$ which satisfies $\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle$ for all $\xi, \eta \in H$. The adjoint has $\|T^*\| = \|T\|$. This defines an operation $* : B(H) \to B(H)$ sending $T \mapsto T^*$. The $*$ operation satisfies

- $(T + S)^* = T^* + S^*$
- $(\lambda T)^* = \lambda T^*$
- $(TS)^* = S^* T^*$
- $(T^*)^* = T$.

**Definition 1.2.** $B \subseteq B(H)$ is a **$*$-algebra** of operators on $B(H)$ if it is closed under the $*$ operation.

**Example 1.1.** Look at $B(\ell^2_\infty) = M_\infty(\mathbb{C})$. $\ell^2_\infty$ has an orthonormal basis $e_i$ with $(e_i)_j = \delta_{i,j}$. Elements of $B(\ell^2_\infty)$ can be multiplied like infinite matrices, and the entries can be determined by this orthonormal basis.

We always consider algebras with a unit. So $B \subseteq B(X)$ will always contain the element $1_B = \mathrm{id}_X \in B$.

## 1.2 von-Neumann algebras and group von-Neumann algebras

**Definition 1.3.** A **von-Neumann algebra** is a $*$-algebra $B \subseteq B(X)$ closed in the weak operator topology given by the seminorms $p_{\xi,\eta}(T) = |\langle T\xi, \eta\rangle|$ ($T_i \to T$ in the weak operator topology if $\langle T_i(\xi), \eta\rangle \to \langle T(\xi), \eta\rangle$ for all $\xi, \eta \in X$).

**Example 1.2.** $B(X)$ is a von-Neumann algebra.

**Definition 1.4.** An operator $U \in B(X)$ is **unitary** if $U^* = U^{-1}$.

**Example 1.3.** A representation $\pi : \Gamma \to B(X)$ of a group $\Gamma$ is called **unitary** if $\pi(g)$ is unitary for all $g \in \Gamma$. If $\pi$ is unitary, then $\operatorname{span} \pi(\Gamma)$ is a $*$-algebra on $X$. Then the closure of this space under the weak operator topology is a von-Neumann algebra.

Denote $\ell^2(I)$ as $\ell^2$ with an orthonormal basis indexed by $I$.

**Example 1.4.** Define the following representations of $\Gamma$

1. The **regular representation** is $\lambda : \Gamma \to U(\ell^2(\Gamma))$ is $\lambda(g)\xi_h = \xi_{gh}$
2. Alternatively, right group multiplication induces the unitary representation $\rho : \Gamma \to U(\ell^2(\Gamma))$ given by $\varphi(g)\xi_h = \xi_{hg^{-1}}$.

Observe that $[\lambda(g_1), \rho(g_2)] = 0$. Let $L(\Gamma)$ be the weak operator topology closure of $\operatorname{span}(\lambda(\Gamma))$, and let $R(\Gamma)$ be the weak operator topology closure of $\operatorname{span}(\rho(\Gamma))$. These are **left and right group von-Neumann algebras**. One avenue of study to study the map $\Gamma \mapsto L(\Gamma)$.

This has many applications. These operators arising from groups are related to dynamics and ergodic theory.

## 1.3 Factors and $C^*$-algebras

**Definition 1.5.** A von-Neumann algebra $M$ is a **factor** if $Z(M) = \mathbb{C}_1$, where $Z$ denotes the center of the algebra.

**Example 1.5.** $B(X)$ and $L(\Gamma)$ are factors.

Here is a question that appeared early in the theory of von-Neumann algebras: Are there any other von-Neumann factors than $B(X)$?[1] This is fundamental to understanding how much commutation there is in operator algebras. The answer is yes. In fact, $L(\mathbb{F}_2)$ and $L(S_\infty)$ are not isomorphic to $B(X)$.

[1] von-Neumann asked this question in 1935. He gave this question to a postdoc. Prior to this, he knew that any von-Neumann algebra decomposes via a measurable field of matrices as $M \cong \int_X M_t \, dt$. They solved the problem in 1936.

These two are infinite dimensional factors, and they have a trace functional on them, $\tau : M \to \mathbb{C}$ which is linear and continuous such that $\tau(x, y) = \tau(yx)$ for all $x, y \in M$. In general, if $X$ is infinite dimensional, $B(X)$ has no trace defined everywhere.

Another question: Can we axiomatize the theory of von-Neumann algebras? We have a Banach-algebra with the $*$-operation, and we have the norm with $\|T^*\| = \|T\|$. Can we construct a Hilbert space only from this information?[2]

**Definition 1.6.** A $*$-algebra $B \subseteq B(X)$ of operators on $X$ is called a **(concrete) $C^*$-algebra**.

In fact, these satisfy $\|T^*T\| = \|T\|^2$ for all $T$. (This does imply that $\|T^*\| = \|T\|$.)

**Definition 1.7.** A Banach algebra with $*$ satisfying $\|x^*x\| = \|x\|^2$ is called an **abstract $C^*$-algebra**

**Theorem 1.1** (G-N + Segal, 1943). *If $B$ is an abstract $C^*$ algebra, then it is a concrete $C^*$-algebra.*

[2] Gelfand and Naimark worked on this in 1940-1943. They did not succeed, and Grothendieck tried in the 50s.