# Stat 155 Lecture 14 Notes

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# 1 Evolutionary Game Theory of Mixed Strategies and Multiple Players

## 1.1 Relationships between ESSs and Nash equilibria

We have mentioned this before, but it is worth stating explicitly.

**Theorem 1.1.** *Every ESS is a Nash equilibrium.*

*Proof.* This follows from the definition. We have that for each pure strategy $z$, $z^\top Ax \leq x^\top Ax$. Any mixed strategy is $w = \sum_{j=1}^n c_j z_j$ for $c_j \geq 0$ and $\sum_{j=1}^n c_j = 1$. Then

$$w^\top Ax = \left(\sum_{j=1}^n c_j z_j^\top\right) Ax = \sum_{j=1}^n c_j (z_j^\top Ax) \leq \sum_{j=1}^n c_j x^\top Ax = x^\top Ax. \qquad \square$$

Does this theorem have a converse?

**Definition 1.1.** A strategy profile $x^* = (x_1^*, \dots, x_k^*) \in \Delta_{S_1} \times \cdots \times \Delta_{S_k}$ is a *strict Nash equilibrium* for utility functions $u_1, \dots, u_k$ if for each $j \in \{1, \cdots, k\}$ and for each $x_k \in \Delta_{S_j}$ with $x_j \neq x_j^*$,

$$u_j(x_j, x_{-j}^*) < u_j(x_j^*, x_{-j}^*).$$

This is the same definition as for a Nash equilibrium, except that the inequality in the definition is strict. By the principle of indifference, only a pure Nash equilibrium can be a strict Nash equilibrium.

**Theorem 1.2.** *Every strict Nash equilibrium is an ESS.*

*Proof.* A strict Nash equilibirum has $z^\top Ax < x^\top Ax$ for $z \neq x$, so both conditions defining an ESS are satisfied. In particular, for the second condition, the case where $z^\top Ax = x^\top Ax$ for $z \neq x$ never occurs. $\square$

## 1.2 Evolutionary stability against mixed strategies

An ESS is a Nash equilibrium $(x^*, x^*)$ such that for all $e_i \neq x^*$, if $e_i^\top A x^* = (x^*)^\top A x^*$, then $e_i^\top A e_i < (x^*)^\top A e_i$. But what about mixed strategies?

**Definition 1.2.** A symmetric strategy $(x^*, x^*)$ is *evolutionarily stable against mixed strategies (ESMS)* if

1. $x$ is a Nash equilibrium.

2. For all mixed strategies $z \neq x^*$, if $z^\top A x^* = (x^*)^\top A x^*$, then $z^\top A z < (x^*) A z$.

Sometimes, people refer to these as ESSs.

**Theorem 1.3.** *For a two-player $2 \times 2$ symmetric game, every ESS is ESMS.*

*Proof.* Assume that $x = (q, 1-q)$ with $q \in (0,1)$ is an ESS. Let $x = (p, 1-p)$ for $p \in (0,1)$ be such that $z^\top A x = x^\top A x$. Since $e_1^\top A x \leq x^\top A x$, $e_2^\top A x \leq Ax$, and $z^\top A x = p e_1^\top A x + (1-p) e_2^\top A x$, we must have that

$$e_1^\top A x = e_2^\top A x = x^\top A x.$$

Hence, $q$ is obtained though the equalizing conditions, and

$$q = \frac{a_{1,2} - a_{2,2}}{a_{1,2} + a_{2,1} - a_{1,1} - a_{2,2}}.$$

Next, define the function $G(p) := x^\top A z = z^\top A z$. We want to show that $G$ is positive.

$$G(p) = (a_{2,1} - a_{1,1})[p^2 - pq] + (a_{1,2} - a_{2,2})[q - qp - p + p^2]$$

However, since $e_1^\top A x = x^\top A z$, by the ESS condition, we must ahve $e_1^\top A e_1 < x^\top A e_1$. The latter is equivalent to

$$a_{1,1} < q a_{1,1} + (1-q) a_{1,2},$$

which gives us that $a_{1,1} < a_{1,2}$. Similarly, $a_{2,2} < a_{2,1}$. By inspection, we see that $G(0) > 0$ and $G(1) > 0$. $G'(0) = 0$ if and only if

$$0 = (a_{2,1} - a_{1,1})[2p - q] + (a_{1,2} - a_{2,2})[-q - 1 + 2p],$$

which is equivalent to

$$2p[a_{1,2} + a_{2,1} - a_{1,1} - a_{2,2}] = q[a - 1, 2 + a_{2,1} - a_{1,1} - a_{2,2}] + a_{1,2} - a_{2,2}.$$

From this, we get that $p = q$. Moreover, $G(q) = 0$. $\square$

**Example 1.1.** Here is an example where an ESS is not an ESMS. Consider the symmetric game with matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 20 \\ 1 & 20 & 0 \end{pmatrix}.$$

$x = e_1$ is an ESS, but it is not an ESMS because for $x = (1/3, 1/3, 1/3)^\top$,

$$x^\top Ax = 5 > 1 = e_1^\top Ax.$$

## 1.3 Multiplayer evolutionarily stable strategies

Consider a symmetric multiplayer game (that is, unchanged by relabeling the players). Suppose that a symmetric mixed strategy $x$ is invaded by a small population of mutants $z$; $x$ is replaced by $(1-\varepsilon)x + \varepsilon z$. Will the mix $x$ survive? The utility for $x$ is, by linearity,

$$\begin{aligned} &u_1(x, \varepsilon z + (1-\varepsilon)x, \dots, \varepsilon z + (1-\varepsilon)x) \\ &\qquad = \varepsilon(u(x, z, x, \dots, x) + u_1(x, x, z, x, \dots, x) + \cdots + u_1(x, \dots, x, z)) \\ &\qquad\qquad + (1-(n-1)\varepsilon)u_1(x, \dots, x) + O(\varepsilon^2). \end{aligned}$$

Similarly, the utility for $z$ is

$$\begin{aligned} &u_1(z, \varepsilon z + (1-\varepsilon)x, \dots, \varepsilon z + (1-\varepsilon)x) \\ &\qquad = \varepsilon(u(z, z, x, \dots, x) + u_1(z, x, z, x, \dots, x) + \cdots + u_1(z, \dots, x, z)) \\ &\qquad\qquad + (1-(n-1)\varepsilon)u_1(z, \dots, x) + O(\varepsilon^2). \end{aligned}$$

**Definition 1.3.** Suppose, for simplicity, that the utility for player $i$ depends on $s_i$ and on the set of strategies played by the other players but is invariant to a permutation of the other players' strategies. A strategy $x \in \Delta_n$ is an *evolutionarily stable strategy (ESS)* if for any pure strategy $z \neq x$,

1. $u_1(z, x_{-1}) \leq u_1(x, x_{-1})$ ($x$ is a Nash equilibrium).
2. If $u_1(z, x_{-1}) = u_1(x, x_{-1})$, then for all $j \neq 1$, $u_1(z, z, x_{-1,-j}) < u_1(x, z, x_{-1,j})$.