

Math 206A Lecture 16 Notes

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1 Cauchy's Arm Lemma

1.1 The arm lemma

Lemma 1.1. *Let $Q = [x_1, \dots, x_n]$, $Q' = [x'_1, \dots, x'_n]$ be two noncongruent convex polygons with equal corresponding lengths $|x_i - x_{i+1}| = |x'_i - x'_{i+1}|$ for $i = 1, \dots, n \pmod n$. Then there exist at least 4 sign changes in $\delta_i = \angle x_{i-1}x_i x_{i+1} - \angle x'_{i-1}x'_i x'_{i+1}$.*

Example 1.1. Suppose $n = 4$, let Q be a square, and let Q' be a rhombus with angles α and $\pi - \alpha$. Then $\bar{\delta} = \{+, -, +, -\}$. The idea is that the diagonal length increases, which is impossible. Let's make this more rigorous.

Proof. Proceed by contradiction. Then $\bar{\delta} = \{+, +, \dots, +, -, -, \dots, -\}$. Then on one side of the polygon, the angles are increasing. \square

Lemma 1.2 (arm lemma). *Let $P = [y_1, \dots, y_k]$, $P' = [y'_1, \dots, y'_k]$ be convex polygons with $|y_i - y_{i+1}| = |y'_i - y'_{i+1}|$ for $i = 1, \dots, k - 1$. If for $i = 1, \dots, k - 2$*

$$\angle y_i y_{i+1} y_{i+2} \leq \angle y'_i y'_{i+1} y'_{i+2},$$

then $|y_1 - y_k| \leq |y'_1 - y'_k|$.

1.2 Cauchy and Zaremba's proofs

Cauchy proved this in 1813 but incorrectly.¹ Let's go through Cauchy's proof.

Proof. Proceed by induction. When $n = 3$, we use the law of cosines. For the inductive step, increase all the angles except 1. Then, applying the law of cosines to the triangle formed by the triangle $x_1 x_n x_{n+1}$, we get that the length $x_1 - x_{n+1}$ increases. \square

Where does this proof fail? It does not use convexity, and this theorem is not true for nonconvex polygons. There are cases where the inductive step does not work.

¹He was about 19 at the time. Legendre gave him this as a project.

Proof. This is a proof by Zaremba.² In a case where Cauchy's proof doesn't work, first, increase the angle $\angle x_3x_2x_1$ until x_n lies on the segment connecting x_1 and x_{n-1} . This is as far as we can expand the angle without losing convexity. By the inductive hypothesis, the polygon $[x'_1, \dots, x_{n-1}]$ has the desired property, that is the line x'_1 to x_{n-1} has gotten bigger (compared to x_1x_n). So if you append a triangle onto this side to get a polygon with 1 more vertex, x_n , the length of the segment connecting x_1 and x_n is smaller than the length of the segment connecting x'_1 and x'_n . \square

²Zaremba and Schonberg corresponded, coming up with iterative constructions for this proof. Eventually, Zaremba came up with this proof. They published all three of their proofs. Basically, they just published their correspondence. But everyone only cares about the last proof.