Math 255B Lecture 10 Notes

Daniel Raban

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1 Symmetric and Self-Adjoint Operators

1.1 Adjoints of closable operators

If $T : D(T) \to H$ is densely defined, we defined the adjoint T^* with $G(T^*) = [\overline{V(G(T))}]^\perp$, where $V(u, v) = (v, -u)$. Let's finish a proof we started last time.

Proposition 1.1. *T is closable if and only if T^* is densely defined.*

Proof. ($\implies$): We did this last time.

($\impliedby$): If $D(T^*)$ is dense, then $(T^*)^*$ is a closed operator such that

$$F(T^{**}) = [V(G(T^*))]^\perp = V(G(T^*)^\perp) = V(V(\overline{G(T)})) = \overline{G(T)},$$

where we have used $V^2 = -1$. So T is closable, and $T^{**} = \overline{T}$. $\square$

1.2 Symmetric and self-adjoint operators

Definition 1.1. Let $S : D(S) \to H$ be densely defined. We say that S is **symmetric** if $\langle Sx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in D(S)$.

Example 1.1. $S = -\Delta$ on $L^2(\mathbb{R}^n)$ with $D(S) = C_0^\infty(\mathbb{R}^n)$ is symmetric. However, we will see that this operator is not self-adjoint.

S is symmetric if and only if $S \subseteq S^*$.

Proposition 1.2. *If S is symmetric, then S is closable and $\overline{S}$ is symmetric.*

Proof. If $u_n \in D(S)$ with $u_n \to 0$ and $Su_n \to \ell$, then $\langle u_n, Sv\rangle = \langle Su_n, v\rangle \to \langle \ell, v\rangle$. On the other hand $\langle u_n, Sv\rangle \to 0$, for all $v \in D(S)$. So $\ell = 0$, and S is closable.

If $S \subseteq S^*$, where S^* is densely defined, then $\overline{S} = S^{**} \subseteq S^* = \overline{S}^*$. So $\overline{S}$ is symmetric. $\square$

Given a symmetric operator S, we have two natural closed extensions: $\overline{S}$ and S^*.

Definition 1.2. A linear, densely defined operator $T : D(T) \to H$ is called **self-adjoint** if $T = T^*$.

Note that this means that $D(T) = D(T^*)$. Any self-adjoint operator is closed, since adjoints are closed. We have

$$\begin{aligned} T \text{ is self-adjoint } &\iff T \text{ is symmetric and } D(T) = D(T^*) \\ &\iff \langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in D(T), \text{ and} \\ &\quad\ \text{if } (x,y) \in H \times H \text{ with } \langle Tz, x\rangle = \langle z, y\rangle\ \forall z \in D(T) \implies x \in D(T). \end{aligned}$$

Proposition 1.3. *Let S be closed and symmetric. Then S^* is symmetric, so S is self-adjoint.*

Proof. S^* is symmetric, so $S^* \subseteq S^{**} = S$, as S is closed. Also, $S \subseteq S^*$, so S is self-adjoint. $\square$

Example 1.2. Let $H = L^2(\mathbb{R}^n)$, and let $m : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Let $D(A) = \{f \in L^2 : mf \in L^2\}$ and $Af = mf$ for all $f \in D(A)$. We claim that A is self-adjoint.

Check first that $D(A)$ is dense in L^2: For any $f \in L^2$, $\frac{f}{1+|m|} \in L^2$, as well. So if $g \in L^2$ with $g \perp D(A)$,

$$\int g \frac{f}{1+|m|}\, dx = 0,$$

which means that $\frac{g}{1+|m|} = 0$, giving $g = 0$.

A is symmetric, as m is real. Now let $(g, h) \in L^2 \times L^2$ be such that $\langle Af, g\rangle = \langle f, h\rangle$ for all $f \in D(H)$. Then for all $f \in L^2$,

$$\int \frac{mf}{1+|m|}\overline{g} = \int \frac{f}{1+|m|}\overline{h},$$

so

$$\int \left(\frac{m}{1+|m|}\overline{g} - \frac{\overline{h}}{1+|m|} \right) f = 0$$

for all $f \in L^2$. So $mg = h$, which gives $g \in D(A)$.

Example 1.3. Let $T = -\Delta$ on $L^2(\mathbb{R}^n)$ with $D(T) = H^2(\mathbb{R}^n) = \{u \in L^2 : \Delta u \in L^2\} = \{u \in L^2 : \partial^\alpha u \in L^2\ \forall |\alpha| \leq 2\}$. Then T is self-adjoint.

T is symmetric: $\langle -\Delta u, v\rangle_{L^2} = \langle u, -\Delta v\rangle_{L^2}$ for all $u, v \in H^2$. This is true for $u, v \in C_0^\infty(\mathbb{R}^n)$, which is dense in $H^2(\mathbb{R}^n)$. Alternatively we could prove this by taking the Fourier transform, where T acts as a multiplication operator.

Let $(g, h) \in L^2 \times L^2$ be such that $\langle -\Delta u, g\rangle_{L^2} = \langle u, h\rangle$ for all $g, h \in H^2$. In particular, if $u \in C_0^\infty$, we get $-\Delta g = h \in L^2$ (taken in the weak sense). Then $g \in H^2$, and $Tg = -\Delta g = h$.

1.3 von Neumann's extension theory for symmetric operators

Let $S : D(S) \to H$ be a closed, symmetric (densely defined) operator. Can S be extended to a self-adjoint operator? If $S \subseteq T = T^*$, then $T^* \subseteq S^*$, so we have an operator between S and S^* in general. If S is symmetric, these are the same.

Proposition 1.4. *Let S be closed and symmetric. Then for any $z \in \mathbb{C} \setminus \mathbb{R}$, $S - z1 : D(S) \to H$ is injective, has closed range, and $\|(S-z)u\| \geq |\operatorname{Im} z|\|u\|$ for $u \in D(S)$.*

Proof. Write $z = x + iy$. Then

$$\begin{aligned}\|(S-z)u\|^2 &= \langle (S-x)u + iyu, (S-x)u + iyu\rangle \\ &= \|(S-x)u\|^2 + y^2\|u\|^2 \\ &\geq y^2\|u\|^2,\end{aligned}$$

so $S - z$ is injective.

$\operatorname{Im}(S-z)$ is closed: If $y \in \overline{\operatorname{Im}(S-z)}$, there exist $x_n \in D(S)$ such that $(S-z)x_n \to y$. By this inequality, $x_n \to x \in H$. Since $(S-z)$ is closed, $x \in D(S)$. □

Here is the idea due to von Neumann: Study the **Cayley transform** of S, $T = (S+i)(S-i)^{-1}$.