Math 210A Lecture 8 Notes

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1 Free Groups, Normal Subgroups, and Quotient Groups

1.1 Free groups

Definition 1.1. A **word** on a set X is a symbol $x_1^{n_1} \cdots x_k^{n_k}$ where $k \geq 0$ ($k = 0$ gives e), $x_i \in X$, and $n_i \in \mathbb{Z}$ for $1 \leq i \leq k$. Write x^1 as x.

Definition 1.2. The **product** of two words is their concatenation.

$$(x_1^{n_1} \cdots x_k^{n_k}) \cdot (y_1^{n_1} \cdots y_k^{n_k}) := x_1^{n_1} \cdots x_k^{x_k} y_1^{n_1} \cdots y_k^{x_k}.$$

Definition 1.3. Two words are equivalent if they are equivalent under the equivalence relation $\sim$ generated by

1. $ww' \sim wx^0w'$
2. $wx^{m+n}w' \sim wx^m x^n w'$

for all words w, w' and $x \in X$.

Definition 1.4. A **reduced word** is a word such that $x_i^i \neq x_{i+1}$ for all $1 \leq i \leq k-1$, and $n_i \neq 0$ for all x_i.

This is a word which is the shortest in its equivalence class.

Proposition 1.1. *Every word is equivalent to a unique reduced word.*

Example 1.1. Let's reduce the word $x^4y^2z^{-1}zy^{-2}x^3$.

$$x^4y^2z^{-1}zy^{-2}x^3 \sim x^3y^2z^0y^{-2}x^2 \sim x^3y^2y^{-2}x^2 \sim x^3y^0x^2 \sim x^3x^2 \sim x^5.$$

Let F_X be the group of equivalence classes of words on X. You can check yourself that if $v \sim v'$ and $w \sim w'$, then $vw \sim v'w'$, so products on F_X are well-defined. This is a group under the product of words, where e is the identity element and the inverse is $(x_1^{n_1} \cdots x_k^{n_k})^{-1} = x_k^{-n_k} \cdots a_1^{-n_1}$.

Definition 1.5. F_X is called the **free group on** X. If $X = \{1, \ldots, n\}$, $F_n := F_X$ is called the **free group of rank** n.

Example 1.2. $F_{\{x\}} = \langle x \rangle = \{x^n : n \in \mathbb{Z}\} \cong \mathbb{Z}$.

Example 1.3. $F_{\{x,y\}} = \{x^{n_1} y^{m_1} \cdots x^{n_k} y^{m_k} : k \geq 0, n_i \neq 0 \,\forall i \geq 2, m_i \neq 0 \,\forall i \leq k-1\}$.

Proposition 1.2. *F_X is a free group on X (in the categorical sense). It is the coproduct of the functor $c_{\mathbb{Z}} : X \to \mathrm{Gp}$ which sends $i \mapsto \mathbb{Z}$ and $f \mapsto \mathrm{id}_{\mathbb{Z}}$.*

Proof. We want $\mathrm{Hom}_{\mathrm{Gp}}(F(X), G) \cong \mathrm{Maps}(X, G)$. We send $\phi \mapsto \phi|_X$. Our map $\iota : X \to F_X$ is the inclusion map. To go backwards, mapping $f \mapsto \phi$ for $f : X \to G$, we define $\phi_f(x_1^{n_1} \cdots x_k^{n_k}) = f(x_1)^{n_1} \cdots f(x_k)^{n_k}$. If we can show that ϕ_f is well defined, we will get the homomorphism we want. Observe that

$$\phi_f(wx^0w') = \phi_f(w)f(x)^0\phi_f(w') = \phi_f(w)\phi_f(w') = \phi_f(ww').$$

Check yourself that $\phi_f(wx^{m+n}w') = \phi_f(wx^n x^m w')$. Uniqueness is left as an exercise.

The coproduct property is very similar to a homework problem for this week, so we leave it as an exercise, as well. $\square$

Definition 1.6. The **free product** $*_{i \in I} G_i = \{\text{words in the groups } G_i\}/\sim$ is the coproduct in the category of groups.

1.2 Normal subgroups and quotient groups

Definition 1.7. A subgroup N of a group G is **normal**, written $N \trianglelefteq G$ if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$.

Definition 1.8. Let $H \leq G$ and $g \in G$. Then $gH = \{gh : h \in H\}$ is the **left H-coset** of g, and $Hg = \{hg : h \in H\}$ is the **right H-coset** of g.

Remark 1.1.

$$\begin{aligned} N \trianglelefteq G &\iff gNg^{-1} \leq N \,\forall g \in G \\ &\iff gNg^{-1} = N \,\forall g \in G \\ &\iff gN = Ng \,\forall g \in G. \end{aligned}$$

Remark 1.2. Let $G/H = \{gH : g \in G\}$ and $H\backslash G = \{Hg : g\, inG\}$. These are in bijection via $gH \mapsto (gH)^{-1} = Hg$.

Proposition 1.3. $N \trianglelefteq G \iff gH \cdot g'N = gg'N$ *gives a well-defined group structure on* G/N.

Definition 1.9. We call $G/N = \{gN : g \in G\}$ the **quotient group**.

Definition 1.10. The index of H in G is the number of left (or right) coests of H in G.

Example 1.4. $N\mathbb{Z} \leq \mathbb{Z}$. Since $\mathbb{Z}$ is abelian, $N\mathbb{Z} \trianglelefteq \mathbb{Z}$. Then the quotient group $\mathbb{Z}/N\mathbb{Z} = \{a + N\mathbb{Z} : 0 \leq a \leq N-1\}$.

Example 1.5. D_n is the dihedral group of symmetries of a regular n-gon. $|D_n| = 2n$, and the set of rotations is a normal subgroup.[1]

[1] Since $|D_n| = 2n$, some people call this group D_{2n}.