Math 206A Lecture 12 Notes

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1 The Blind-Mani Theorem

1.1 Acyclic orientations

Let's prove the Blind-Mani theorem.

Theorem 1.1 (Blind-Mani)**.** *Let $P \subseteq \mathbb{R}^d$ be a simple, convex polytope. Then the face lattice $\alpha(P)$ is determined by the graph $\Gamma(P)$ of the polytope.*

Example 1.1. Here are non-simple convex polytopes with don't satisfy this theorem. Let $\Gamma = K_6$ be the complete graph on 6 vertices. Then the simplex Δ^5 has graph Γ. But there also exists a polytope $Q \subseteq \mathbb{R}^4$ such that $f_0 = 6$ and $\Gamma(Q) = K_6$. To construct Q, think of $\mathbb{R}^4$ as $\mathbb{R}^2 \times \mathbb{R}^2$. Take two triangles, one in each copy of $\mathbb{R}^2$, and connect them together. So $Q = \Delta^2 \times \Delta^2$. Note that $\alpha(Q) \not\cong \alpha(\Delta^5)$. This is an example in a large family of polytopes called **neighborly polytopes**, which have $\Gamma(P) \cong K^n$.

Proof. (Kalai[1]) Let $\Gamma = \Gamma(P)$. This is connected. Let $d = \deg(\Gamma)$. Γ is d-regular. Let O be the acyclic orientation of the edges E (so the edges all receive an orientation such that no cycles form). Now define h_i^O be the number of vertices $v \in V$ with out degree equal to i. This is to take the place of Morse functions in our proof. $\square$

Define O to be good if $T \in \alpha(P)$ has a unique source. How do we know if an orientation is good?

Lemma 1.1. *Let $\alpha(P) := h_0^O + 2h_1^O + 4h_2^O + \cdots + 2^d h_d^O$. Then $\alpha(O) \geq f_0 + f_1 + \cdots + f_d =: \beta(P)$. Moreover, $\alpha(O) = \beta(P)$ if and only if O is good.*

This is Theorem 8.6 in Professor Pak's textbook. Let's prove the lemma.

[1] The original proof was "plain boring," according to Professor Pak. But this proof is more interesting than the theorem itself.

Proof. Suppose O is an acyclic orientation coming from a Morse function φ on $P \subseteq \mathbb{R}^d$. Then $h_i^O = h_i^\varphi$. Then from the Dehn-Sommerville equations, $f_k = \sum_{i=k}^d \binom{i}{k} h_i^O$. Then $\beta(P) = \sum_{k=0}^d f_k = \mathcal{F}_P(1) = \mathcal{G}_P(2) = \sum_{i=0}^d h_i^O 2^i$. If O is good, then, we have the same equality ($\alpha(O) = \beta(P)$) because our proof of the Dehn-Sommerville equations only relied on the fact that each face had a unique source.

If O is any orientation, we write the same thing, except $f_k \leq \sum_{k=0}^d h_i^O \binom{i}{k}$. So $\alpha(O) \geq \beta(P)$. Then the only way to get an exact equality is if we never count a face twice. This is only if every face has a unique source. $\square$

Now we need to use this characterization to find out when a subgraph of $\Gamma(P)$ is the graph of a face.

1.2 The face criterion

Let $\Gamma = \Gamma(P)$ be the graph of a simple d-dimensional polytope, and let O be a good acyclic orientation of Γ. Think of a face as $\Gamma(F) \subseteq \Gamma$, where $V(F) \subseteq V(P)$. Suppose $\deg(\Gamma(F)) = k$.

Proposition 1.1. *$H \subseteq \Gamma(F)$ is a graph of a face if and only if the following two conditions are satisfied:*

1. *$\Gamma(F)$ is k-regular.*
2. *There exists a good orientation O such that $V(F)$ is final (no edges from outside $V(F)$ are oriented into $V(F)$).*

Proof. Suppose $F \subseteq \alpha(P)$ is a k-dimensional face. Then $H = \Gamma(F)$ is k-regular. There also exists a final O on H; take a hyperplane containing that face, perturb it a little, and take a Morse function that defines O.

For the opposite direction, take the minimum point (since O is final). Create 2 graphs, one spanned by $\Gamma(F)$ and one containing everything you can reach from the minimum vertex. They are both k-regular and one contains the other, so they are equal. $\square$