Math 255B Lecture 2 Notes

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1 Perturbation of Fredholm Operators and The Logarithmic Law

1.1 Perturbation of Fredholm operators

Last time, said that $T \in \mathcal{L}(B_1, B_2)$ is **Fredholm** if $\dim \ker T < \infty$ and $\dim B_2 / \operatorname{im} T < \infty$. We were proving that the Fredholm property is preserved under small perturbations.

Theorem 1.1. *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator. If $S \in \mathcal{L}(B_1, B_2)$ is such that $\|S\|_{\mathcal{L}(B_1,B_2)}$ is sufficiently small, then $T + S$ is Fredholm, and $\operatorname{ind}(T+S) = \operatorname{ind} T$.*

Proof. Produce

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+},$$

where $n_+ = \dim \ker T$ and $n_- = \dim \operatorname{coker} T$. We get that

$$\widetilde{\mathcal{P}} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix}$$

is invertible as well, with inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus \mathbb{C}^{n_-},$$

where

$$E : B_2 \to B_1, \qquad E_+ : \mathbb{C}^{n_+} \to B_1, \qquad E_- : B_2 \to \mathbb{C}^{n_-}, \qquad E_{-+} : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}.$$

Since $\mathcal{E}$ is a right inverse for $\widetilde{\mathcal{P}}$,

$$\begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_+E_+ \end{bmatrix}.$$

This is the identity map, so $R_+E_+ = 1$ on $\mathbb{C}^{n_+}$. So E_+ is injective. Similarly, we get $E_-R_- = 1$ on $\mathbb{C}^{n_-}$, so E_- is surjective.

We claim that $T+S$ is Fredholm. For the kernel, we have $x \in \ker(T+S) \iff (T+S)x = 0$. We can write this as

$$\widetilde{\mathcal{P}} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_+ \end{bmatrix},$$

where $a_= = R_+x \in \mathbb{C}^{n_+}$. Using the inverse $\mathcal{E}$, this is

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} 0 \\ a_+ \end{bmatrix} = \begin{bmatrix} E_+a_+ \\ E_{-+}a_+ \end{bmatrix}.$$

So we get that $x \in \ker(T+S) \iff x = E_+a_+$ for some $a_+ \in \ker E_+$. So $E_+ : \ker E_{-+} \to \ker(T+S)$ is surjective. Since we already know E_+ is injective, we get that $\ker(T+S)$ is finite dimensional with $\dim\ker(T+S) = \dim\ker(E_{-+}) \leq n_+$.

Next consider $B_2/\operatorname{im}(T+S)$: Given y,

$$\begin{aligned} (T+S)x = y \iff \widetilde{\mathcal{P}} \begin{bmatrix} x \\ 0 \end{bmatrix} &= \begin{bmatrix} y \\ a_+ \end{bmatrix} \\ \iff \begin{bmatrix} x \\ 0 \end{bmatrix} &= \mathcal{E} \begin{bmatrix} y \\ a_+ \end{bmatrix}. \end{aligned}$$

So we get that $x = Ey + E_+a_+$ and $0 = E_-y + E_{-+}a_+$. We get that $y \in \operatorname{im}(T+S) \iff E_-y \in \operatorname{im} E_{-+}$. Now consider $B_2/\operatorname{im}(T+S) \to \mathbb{C}^{n_-}/E_{-+}$ sending $y + \operatorname{im}(T+S) \mapsto E_-y + \operatorname{im} E_{-+}$. This map is surjective, as E_- is surjective, and it is also injective. So $\dim\operatorname{coker}(T+S) = \dim\operatorname{coker} E_{-+} < \infty$.

So $T+S$ is Fredholm, and

$$\operatorname{ind}(T+S) = \dim\ker E_{-+} - \dim\operatorname{coker} E_{-+} = \operatorname{ind} E_{-+} = n_+ - n_- = \operatorname{ind}(T). \qquad \square$$

Corollary 1.1. *The set of Fredholm operators is open in $\mathcal{L}(B_1, B_2)$, and $T \mapsto \operatorname{ind}(T)$ is locally constant.*

The proof also gives the following:

Corollary 1.2. *$T \mapsto \dim\ker T$ is upper-semicontinuous on the set of Fredholm operators.*

1.2 The logarithmic law

Proposition 1.1. *Let $T_1 \in \mathcal{L}(B_1, B_2)$ and $T_2 \in \mathcal{L}(B_2, B_3)$ be Fredholm. Then T_2T_1 is Fredholm, and we have the* ***logarithmic law****:*

$$\operatorname{ind} T_2T_1 = \operatorname{ind} T_2 + \operatorname{ind} T_1.$$

Proof. Consider $T_1' : \ker T_2T_1 \to \ker T_2$ sending $x \mapsto T_1 x$. Then $\ker T_1' = \ker T_1$, so $\dim(\ker(T_2T_1)/\ker T_1) \le \dim\ker T_2$. So $\dim\ker T_2T_1 < \infty$.

Now consider

$$0 \longrightarrow B_2/\operatorname{im} T_1 \xrightarrow{T_2'} B_3/\operatorname{im} T_2T_1 \xrightarrow{q} B_3/\operatorname{im} T_2 \longrightarrow 0,$$

where

$$T_2'(x + \operatorname{im} T_1) = T_2 x + \operatorname{im} T_2T_1, \qquad q(x + \operatorname{im} T_2T_1) = x + \operatorname{im} T_2.$$

The sequence is exact at $B_3/\operatorname{im} T_2T_1$: $\operatorname{im} T_2' \subseteq \ker q$ by definition, and if $x + \operatorname{im} T_2T_1 \in \ker q$, then $x \in \operatorname{im} T_2$, so $x + \operatorname{im} T_2T_1 \in \operatorname{im} T_2'$. We have

$$\dim(\operatorname{coker}(T_2T_1)/\ker q) \le \dim\operatorname{coker} T_2, \qquad \dim\ker q = \dim\operatorname{im} T_2' \le \dim\operatorname{coker} T_1,$$

so

$$\dim\operatorname{coker}(T_2T_1) \le \dim\operatorname{coker} T_1 + \dim\operatorname{coker} T_2.$$

To compute the index, consider

$$L(t) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} I_2\cos t & I_2 \sin t \\ -I_2 \sin t & I_2\cos t \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} : B_1 \oplus B_2 \to B_2 \oplus B_3, \qquad t \in \mathbb{R}, I_2 = \mathrm{id}_{B_2}.$$

This is a product of three Fredholm operators, so $L(t)$ is Fredholm for all t and $t \mapsto \mathcal{L}(t)$ is continuous. So $\operatorname{ind} L(t)$ is independent of t! When $t = 0$,

$$L(0) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

so $\operatorname{ind} L(0) = \operatorname{ind} T_1 + \operatorname{ind} T_2$. When $t = -\pi/2$, we get

$$L(-\pi/2) = \begin{bmatrix} 0 & -I_2 \\ T_2T_1 & 0 \end{bmatrix}.$$

So

$$L(-\pi/2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ T_2T_1x \end{bmatrix},$$

which gives $\ker L(-\pi/2) = \ker T_2T_1 \oplus \{0\}$. We get $\operatorname{ind} L(-\pi/2) = \operatorname{ind}(T_2T_1)$. Since the index is locally constant, we get the logarithmic law. $\square$