Representation Theory

Daniel Raban

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0.1 Note to the reader

I've always found resources on representation theory to be frustrating to read. Some omit too many details for the level of my background. Some discuss everything in terms of modules, a perspective which I don't really find intuitive at all. But representation theory is a really nice theory, and I think it should be more accessible to more people. So here's my attempt at writing the reprsentation theory introduction I've wanted to read,

These notes assume you've taken a course in abstract algebra that included group theory and that you've taken a course in linear algebra which talked about vector spaces and inner products. Representation theory needs a few fancy linear algebra concepts, so I've included an appendix which explains them. Although it's called an "appendix," I really recommend you read the whole appendix first, as if it were Chapter 0.[1]

If I've written something incorrect or unclear, please don't hesitate to send me an email letting me know. If you have ideas about how to better organize the notes or even ways I could make them look nicer, please let me know. Thanks for reading!

[1]Unless you're some linear algebra deity, skimming the appendix will at least reassure you that you know all the relevant information.

0.2 Motivation

Mathematics is often about recognizing structure that was "there all along." In the case of group theory, which can be thought of as the study of the structure of symmetries, there is a natural structure that is often brushed over: "the action of symmetries on space." If you consider a group of reflections or rotations, you are probably already imagining some sort of action on space, namely reflections and rotations. Representation theory, at the basic level, is about exploring this viewpoint of symmetry through the lens of linear algebra.

Why does this viewpoint merit its own theory? Linear algebra has a lot of structure and is very nice, compared to theories such as real analysis, which are filled with counterexamples to intuitive notions. Since group theory can be very messy, "filtering" information about a group into information in the language of linear algebra provides an easy to compute but still useful reduction of the theory. Besides, it's cool!

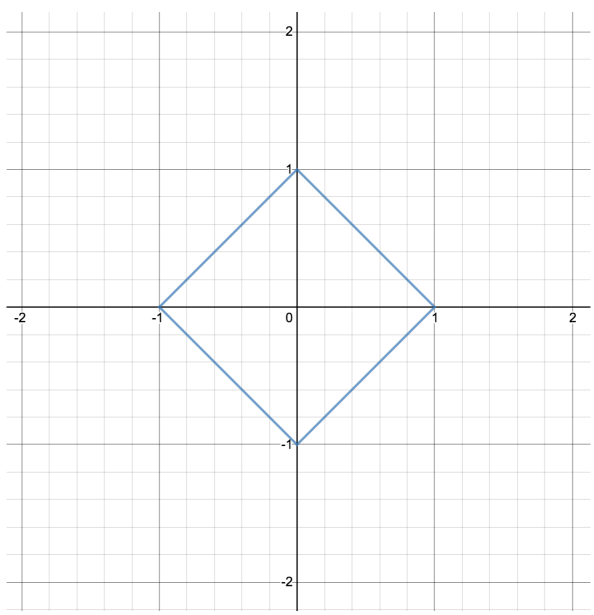
Chapter 1

Representation theory of finite groups over $\mathbb{C}$

1.1 Representations

1.1.1 Definition and examples

Example 1.1.1. Consider the dihedral group D_8, the group of rotations and reflections of a square. You may already be imagining the square as lying in the plane similar to this:



The group D_8 has 8 elements, where e is the identity rotation, s is reflection about the x-axis, and r is counterclockwise rotation by 90 degrees:

$$D_8 = \{e, s, r, r^2, r^3, sr, sr^2, sr^3\}.$$

Here, we view the element sr as meaning "first rotate, then reflect"; that is, we read the action from right to left. The group has an inherent "multiplication" by composing the reflections and rotations, and we can calculate that $r^4 = e$ (rotating 4 times by 90 degrees does nothing in total) and $s^2 = e$ (reflecting twice does nothing). We can also work out that $rs = sr^3$; these 3 relations tell us the entire multiplicative structure of the group D_8.

There is another viewpoint: these transformations can be thought of as reflections and rotations of the plane, taking the square along for the ride. We labeled these elements of the group as the letters r and s, but we can also talk about the corresponding rotations and reflections as linear transformations. That is,

$$r = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this case, D_8 is isomorphic to a group of 2×2 real matrices. Formally, we have an isomorphism $\rho : D_8 \to \mathrm{GL}_2(\mathbb{R})$, with codomain the group of invertible 2×2 real matrices.

The idea of representation theory is to do this with any group. But instead of only looking for isomorphisms to matrix groups, we look at homomorphisms to matrix groups; this will give us a more complete structure to look at.

Definition 1.1.1. Let G be a group. A **representation** of G is a pair (ρ, V), where V is a vector space and $\rho : G \to \mathrm{Aut}(V)$ is a homomorphism.

In other words, a representation is a map which takes a group G and represents every $g \in G$ as an invertible linear transformation (or a matrix).

Example 1.1.2. Let $G = \mathbb{Z}/2\mathbb{Z}$, and let $V = \mathbb{R}^2$. A natural representation of G is to think of it as a reflection about one of the axes:

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho(1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Alternatively, we could have used another reflection, such as reflection about the line $y = x$:

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As this example shows, any group, no matter how small, will have a lot of representations.

Example 1.1.3. Given any group G and a field F, the **trivial representation** (ρ, F) is defined by $\rho(g) = 1$, where $\rho(g)$ is viewed as a 1×1 matrix.

Example 1.1.4. Let S_3 be the symmetric group on 3 elements. We can represent the elements of S_3 as 3×3 permutation matrices, permuting the standard basis vectors e_1, e_2, e_3. In other words, we have a representation $(\rho, \mathbb{R}^3)$ with

$$\rho(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho\big(\begin{pmatrix} 1 & 2 \end{pmatrix}\big) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho\big(\begin{pmatrix} 1 & 3 \end{pmatrix}\big) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\rho\big(\begin{pmatrix} 2 & 3 \end{pmatrix}\big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \rho\big(\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\big) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \rho\big(\begin{pmatrix} 1 & 3 & 2 \end{pmatrix}\big) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Remark 1.1.1. Why are all the matrices $\rho(g)$ for $g \in G$ automorphisms? That is, why are they *invertible* linear transformations $V \to V$? This is because elements of a group always have inverses, so $\rho(g^{-1})$ is the inverse of $\rho(g)$:

$$\rho(g) \cdot \rho(g^{-1}) = \rho(gg^{-1}) = \rho(e) = I_V,$$

where I_V is the identity on V.

Remark 1.1.2. When the target vector space V is understood, we refer to the representation as just ρ or ρ_V. Similarly, when the representation is understood and we are referring to properties of the target vector space, it is common to refer to the representation as V itself.

Remark 1.1.3. One can consider representations over different fields, meaning the vector space V can be over any desired field. We will mostly stick with $\mathbb{C}$, the complex numbers.

Example 1.1.5. Let $\rho : G \to \mathbb{C}^\times$ be a homomorphism. Then $(\rho, \mathbb{C})$ is a representation of G, where $\rho(g)$ is viewed as a 1×1 matrix.

Example 1.1.6. We can construct a **permutation representation** of any group. Consider a group action $G \circlearrowright X$, where X is a finite set. This takes every element $g \in G$ to a permutation of X, so we get an embedding $\varphi : G \to S_n$, where $n = |X|$. We can define the representation $\rho : G \to \mathbb{C}^n$ sending g to the permutation matrix corresponding to $\varphi(g)$.

This is actually an instance of a more general construction.

Example 1.1.7. We can get representations of larger groups if we know representations of smaller ones (and vice-versa). Let $\varphi : G \to H$ be a group homomorphism. If $\rho : H \to V$ is a representation of H, then $\rho \circ \varphi$ is a representation of G.

Example 1.1.8. The group S_3 has a subgroup of order 3,

$$A_3 = \{e, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}, \}.$$

This subgroup is normal in S_3, and S_3/A_3 has order 2, so $S_3/A_3 \cong \mathbb{Z}/2\mathbb{Z}$. In particular, we have the natural quotient map $\varphi : S_3 \to \mathbb{Z}/2\mathbb{Z}$. We already saw a representation of $\mathbb{Z}/2\mathbb{Z}$, so we can "lift" this representation up to S_3; we determine $\rho(g)$ by first sending g to its image in $\mathbb{Z}/2\mathbb{Z}$ and then representing using the representation of $\mathbb{Z}/2\mathbb{Z}$:

$$\rho(e) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad \rho(\begin{pmatrix}1 & 2\end{pmatrix}) = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \qquad \rho(\begin{pmatrix}1 & 3\end{pmatrix}) = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},$$

$$\rho(\begin{pmatrix}2 & 3\end{pmatrix}) = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \quad \rho(\begin{pmatrix}1 & 2 & 3\end{pmatrix}) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \rho(\begin{pmatrix}1 & 3 & 2\end{pmatrix}) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

1.1.2 Change of basis

How does change of basis play into our representations? A representation is a homomorphism $\rho : G \to \operatorname{Aut}(V)$, but we have been writing the images $\rho(g)$ as matrices, not just linear transformations. There is an underlying choice of basis involved (the standard basis). If we choose a different basis, do we get different representations? In some sense, they should be the same.

Definition 1.1.2. Let (ρ_V, V) and (ρ_W, W) be representations of G. A **representation homomorphism** is a linear map $\varphi : V \to W$ such that $\varphi \circ \rho_V(g) = \rho_W(g) \circ \varphi$ for all $g \in G$.

$$\begin{tikzcd} V \arrow[r, "\rho_V(g)"] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi"] \\ W \arrow[r, "\rho_W(g)"'] & W \end{tikzcd}$$

We denote the collection of representation homomorphisms from $V \to W$ as $\operatorname{Hom}_G(V, W)$.

Definition 1.1.3. Two representations (ρ_V, V) and (ρ_W, W) are **isomorphic** (denoted $V \cong W$) if there is an invertible representation homomorphism $\varphi : V \to W$; that is, $\rho_V(g) = \varphi^{-1} \circ \rho_W(g) \circ \varphi$ for all $g \in G$.

$$\begin{tikzcd} V \arrow[r, "\rho_V(g)"] \arrow[d, "\varphi"'] & V \\ W \arrow[r, "\rho_W(g)"'] & W \arrow[u, "\varphi^{-1}"'] \end{tikzcd}$$

If we express this in terms of matrices, this means that there is some change of basis matrix P such that $\rho_V(g) = P^{-1}\rho_W(g)P$ for all $g \in G$. In particular, this says that V and W are the same representation, just expressed in a different basis.

Example 1.1.9. Recall our two 2 dimensional representations of $\mathbb{Z}/2\mathbb{Z}$, corresponding to different reflections of $\mathbb{R}^2$:

$$\rho_1(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho_1(0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\rho_2(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho_2(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These representations are isomorphic: if we apply the change of basis

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

then ρ_2 in the new basis is the same as ρ_1 in the standard basis.

1.2 Vector space constructions of representations

1.2.1 Direct sum, tensor product and dual representations

Representations tend to play very nice with the vector space structure. If we think of representations as vector spaces carrying the additional structure of the action of G via the homomorphism ρ, then this section is about extending usual vector space constructions to constructions with extra structure.[1]

References for direct sums, tensor products, and dual spaces of vector spaces are in the appendix.

Definition 1.2.1. Let V, W be representations of G (with associated homomorphisms ρ_V, ρ_W). The **direct sum** of V and W, denoted $V \oplus W$, is the vector space $V \oplus W$. The associated homomorphism $\rho_{V\oplus W}$ is defined as:

$$[\rho_{V\oplus W}(g)](v, w) = ([\rho_V(g)]v, [\rho_W(g)]w).$$

In other words, if we have the actions $G \circlearrowright V$ and $G \circlearrowright W$ via two representations, then we can combine these representations by having G act on the V and W parts of $V \oplus W$ separately.

If V has ordered basis $\{v_1, \ldots, v_n\}$ and W has ordered basis $\{w_1, \ldots, w_m\}$, then $V \oplus W$ has ordered basis $\{v_1, \ldots, v_n, w_1, \ldots, w_m\}$. We then have the matrix representation of $\rho_{V\oplus W}$:

$$\rho_{V\oplus W}(g) = \begin{bmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{bmatrix}.$$

Definition 1.2.2. Let V, W be representations of G (with associated homomorphisms ρ_V, ρ_W). The **tensor product** of V and W, denoted $V \otimes W$, is the vector space $V \otimes W$. The associated homomorphism $\rho_{V\otimes W}$ is defined as:

$$[\rho_{V\oplus W}(g)](v \otimes w) = [\rho_V(g)]v \otimes [\rho_W(g)]w.$$

If V has ordered basis $\{v_1, \ldots, v_n\}$ and W has ordered basis $\{w_1, \ldots, w_m\}$, then $V \otimes W$ has the basis $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$. Suppose that in these bases, we can write

$$\rho_V(g) = A, \qquad \rho_W(g) = B.$$

[1] For the reader acquainted with the viewpoint of category theory, this section may be thought of as constructions in the category of representations of a fixed group G.

Then the matrix form of $\rho_{V\otimes W}(g)$ is (written as a block matrix):

$$\rho_{V\otimes W}(g) = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,n}B \end{bmatrix}.$$

Definition 1.2.3. Let V be a representation of G with associated homomorphism ρ_V. The **dual representation**, denoted ρ_{V^*}, is the representation V^* with associated homomorphism

$$\rho_{V^*}(g) = \rho_V(g^{-1})^*,$$

where $*$ denotes the dual map.

In terms of matrices, we have

$$\rho_{V^*}(g) = \rho_V(g^{-1})^\top.$$

Remark 1.2.1. Why do we have to take the inverse of the group element g? One way to see why is that we want ρ_{V^*} to be a homomorphism. Since $(AB)^* = B^*A^*$, we need to another map of this type (an involution) to make the homomorphism property work out right.

Here is a more satisfying line of reasoning: For $v \in V$ and $f \in V^*$, let $\langle v, f\rangle$ denote $f(v)$. If we want the dual representation to act similarly on V^* to how the original representation acts on V, it might be reasonable to want

$$\langle v, f\rangle = \langle \rho_V(g)v, \rho_{V^*}(g)f\rangle.$$

Check that our definition for the dual representation satisfies this property.[2]

1.2.2 Subrepresentations and decomposition into irreducible representations

Suppose we take representations V and W of G and construct $V \oplus W$. V can be recognized as a vector subspace of $V \oplus W$, but what about the representation structure? How do we identify when a representation contains a subspace which is actually a smaller representation?

[2] Get off your butt and do it. I mean actually get out something to write with and check that the property holds. It may be symbol pushing, but it'll help you internalize the dual representation.

Definition 1.2.4. Let V be a representation of a group G. A representation W is a **subrepresentation** of G if W is stable under the action of G (i.e. W is $\rho_V(g)$-stable for all $g \in G$.

Example 1.2.1. Let S_n act on $\mathbb{C}^n$ via its natural permutation representation (i.e. permuting the basis vectors $e_1, \ldots, e_n$). Let $W = \operatorname{span}(\{e_1 + \cdots + e_n\})$. Then W is a 1-dimensional subrepresentation of this natural representation that is isomorphic to the trivial representation.

If W is a subrepresentation of V, then $\rho_W(g) := \rho_V(g)|_W$ for each $g \in G$ defines a representation on W. But what about the remaining part of V? Can we guarantee that $V = W \oplus W'$, where W' is another representation of G?

Lemma 1.2.1. *Let W be a subrepresentation of (ρ, V) over $\mathbb{C}$. Then there is a representation W' such that $V = W \oplus W'$.*

The idea of the proof is to *symmetrize* a projection to find a complement which is also stable under the action of G. Let's say $G = \mathbb{Z}/2\mathbb{Z}$, with the representation

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho(1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the x-axis is a subrepresentation, and we want to say that the y-axis, which is also invariant under ρ, is a good choice for W'.

Start with some non-orthogonal projection onto the x-axis; you can imagine shining a light from 45 degrees and looking at the shadow. Apply $\rho(1)$ to reflect this projection across the x-axis; after reflecting, we're shining a light from 135 degrees and looking at the shadow. When taking the average of these these two (flipped and original) projections, the opposite components cancel out; in our example, the average is shining a light from 90 degrees, at high noon. This procedure gives as a suitable projection to use to find a good complement W' of W.

Proof. Let $T : V \to V$ be a projection onto W. We create an altered (symmetrized) version of T by letting

$$T' := \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ T \circ \rho(g)^{-1}.$$

We claim that this is also a projection onto W. Note that T maps V into W. Additionally, for $x \in W$, $T \circ \rho(g)^{-1} x = \rho(g)^{-1} x$, so $\rho(g) \circ p \circ \rho(g)^{-1} x = x$, which gives $T'x = x$. So T' is also a projection of V into W.

Now let $W' = \ker(T')$. We now just need to show that W' is also stable under $\rho(g)$ for all $g \in G$. Note that

$$\begin{aligned}
\rho(g) \circ T' \circ \rho(g)^{-1} &= \frac{1}{|G|} \sum_{h \in G} \rho(g)\rho(g) \circ T \circ \rho(h)^{-1}\rho(g)^{-1} \\
&= \frac{1}{|G|} \sum_{h \in G} \rho(gh) \circ T \circ \rho(gh)^{-1} \\
&= T',
\end{aligned}$$

where we have just reindexed the sum by left multiplication by g. We now have $\rho(g) \circ T' = T' \circ \rho(g)$, so for $x \in W'$,

$$T' \circ \rho(g)x = \rho(g) \circ T'x = 0.$$

Hence, $\rho(g)x \in W' = \ker(T')$, so W' is invariant under $\rho(g)$.

To summarize, we have found $V = W \oplus W'$, where W and W' are invariant under $\rho(g)$ for all $g \in G$, making W, W' subrepresentations of V. $\square$

Remark 1.2.2. This result holds for fields other than $\mathbb{C}$, as long as $\mathrm{char}(F) \nmid |G|$. This is what allows us to safely divide by $|G|$.

Remark 1.2.3. Here is a simpler proof of the lemma for inner product spaces. If we have an inner product, we can make it invariant under the action of G (i.e. $\langle v, w \rangle = \langle \rho(g)v, \rho(g)w \rangle$ for all $g \in G$) by replacing the inner product by $\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle$. Then, if we let W' be the orthogonal complement of W with respect to this G-invariant inner product, W' is also stable under the action of G.

If W is a subrepresentation of V, then there is some basis in which the matrices $\rho_V(g)$ all look like

$$\rho_V(g) = \begin{bmatrix} \rho_W(g) & 0 \\ 0 & \rho_{W'}(g) \end{bmatrix}.$$

We can then think of the notion of breaking down representations into smaller subrepresentations. Since the dimension decreases each time we take subrepresentations, this procedure must terminate after finitely many steps. So there have to be subrepresentations which cannot be broken down any further. What do these look like?

Definition 1.2.5. A representation V of G is **irreducible** if the only subrepresentations of V are $\{0\}$ and V itself.

Example 1.2.2. Any 1 dimensional representation is irreducible.

Example 1.2.3. S_3 has the following two dimensional irreducible representation (which we will not prove the irreducibility of at the moment). If $S_3 \circlearrowleft \mathbb{C}^3$ via the natural permutation representation, then let $W = \operatorname{span}(\{e_1 - e_2, e_2 - e_3\})$. W is irreducible, and if we let $W' = \operatorname{span}(\{e_1 + e_2 + e_3\})$, $\mathbb{C}^3 \cong W \oplus W'$.

Theorem 1.2.1 (Maschke). *Let V be a finite dimensional representation over $\mathbb{C}$. Then V admits a decomposition into irreducible representations: $V \cong \bigoplus_{i=1}^r V_i$, where the V_i are irreducible.*

Proof. Proceed by induction on $n = \dim(V)$. If $n = 1$, then V is irreducible, Now suppose that the theorem is true for representations of dimension $\leq n$. If $\dim(V) = n+1$, and V is not irreducible, then let W be a nontrivial, proper subrepresentation. By the previous lemma, $V \cong W \oplus W'$ for some other subrepresentation W'. Applying the inductive hypothesis to W and W' (as $\dim(W), \dim(W') \leq n$) gives a decomposition of V into irreducible representations. $\square$

1.2.3 $\operatorname{Hom}(V, W)$, $\operatorname{Hom}_G(V, W)$, and Schur's lemma

We can define a representation on $\operatorname{Hom}(V, W)$, the collection of linear transformations $T : V \to W$, as follows.

Definition 1.2.6. Let V, W be representations of G (with associated homomorphisms ρ_V, ρ_W). Then $\operatorname{Hom}(V, W)$ has the representation defined as follows: if $T : V \to W$ is a linear transformation,

$$[\rho_{\operatorname{Hom}(V,W)}(g)]T := \rho_W(g) \circ T \circ \rho_V(g^{-1}).$$

It turns out that we can express this representation in terms of our previous constructions:

Proposition 1.2.1. *Let V, W be finite dimensional representations of G. Then $\operatorname{Hom}(V, W) \cong V^* \otimes W$ via the isomorphism $\varphi : V^* \otimes W \to \operatorname{Hom}(V, W)$ defined by*

$$\varphi(f \otimes w) = (v \mapsto f(v)w)$$

and extended to all of $V^* \otimes W$ bilinearly.

In other words, we can view a simple tensor $f \otimes w$ as a linear transformation from $V \to W$ as follows: given $v \in V$, f "eats" v and returns the result as the coefficient in front of w.[3]

[3] The proof of this proposition is notationally hard to follow symbol-pushing. I recommend you try to prove it yourself and consult the proof here whenever you get stuck.

Proof. We first need to show that φ is invertible. We will find an inverse $\psi : \mathrm{Hom}(V,W) \to V^* \otimes W$. Let $\{v_1, \dots, v_n\}$ be a basis of V, and let $\{v_1^*, \dots, v_n^*\}$ be the corresponding dual basis. Then define

$$\psi(T) = \sum_{i=1}^n v_i^* \otimes T v_i.$$

To show that ψ is the inverse of ϕ, we check that for $T \in \mathrm{Hom}(V,W)$ and $f \otimes w \in V^* \otimes W$,

$$\varphi(\psi(T)) = \varphi\left(\sum_{i=1}^n v_i^* \otimes T v_i\right) = \sum_{i=1}^n (v \mapsto v_i^*(v) T v_i) = \left(v \mapsto T\left(\sum_{i=1}^n v_i^*(v) v_i\right)\right) = T,$$

$$\psi(\varphi(f \otimes w)) = \psi((v \mapsto f(v)w)) = \sum_{i=1}^n v_i^* \otimes f(v_i) w = \left(\sum_{i=1}^n f(v_i) v_i^*\right) \otimes w = f \otimes w.$$

So φ is an invertible linear transformation.

To show that φ is an isomorphism of representations, we need to check that $[\rho_{\mathrm{Hom}(V,W)}(g)]T = [\psi^{-1} \circ \rho_{V^* \otimes W}(g) \circ \psi]T$ for all $T \in \mathrm{Hom}(V,W)$. If $g \in G$, $T \in \mathrm{Hom}(V,W)$, and $v \in V$, then

$$\begin{aligned}
([\psi^{-1} \circ \rho_{V^* \otimes W}(g) \circ \psi]T)v &= \left[\psi^{-1}\left(\rho_{V^* \otimes W}(g)\left(\sum_{i=1}^n v_i^* \otimes T v_i\right)\right)\right] v \\
&= \left[\psi^{-1}\left(\sum_{i=1}^n \rho_{V^*}(g) v_i^* \otimes \rho_W(g) T v_i\right)\right] v
\end{aligned}$$

Using the fact that ψ^{-1} is φ,

$$\begin{aligned}
&= \sum_{i=1}^n (\rho_{V^*}(g) v_i^*)(v) \cdot \rho_W(g) T v_i \\
&= \rho_W(g) T \left(\sum_{i=1}^n (\rho_{V^*}(g) v_i^*)(v) v_i\right) \\
&= \rho_W(g) T \left(\sum_{i=1}^n v_i^*(\rho_V(g^{-1})v) v_i\right)
\end{aligned}$$

For any $u \in V$, $u = \sum_{i=1}^n v_i^*(u) v_i$; this is because $v_i^*(u)$ is just the coefficient of v_i in the decomposition of u with respect to this basis.

$$\begin{aligned}
&= \rho_W(g) \circ T \circ \rho_V(g^{-1}) v \\
&= [\rho_{\mathrm{Hom}(V,W)}(g)] T v.
\end{aligned}$$

$\square$

Let $\mathrm{Hom}_G(V, W)$ be the collection of representation homomorphisms $\varphi : V \to W$. Just like $\mathrm{Hom}(V, W)$ has structure as a vector space, $\mathrm{Hom}_G(V, W)$ has structure related to the representation structure of $\mathrm{Hom}(V, W)$.

Proposition 1.2.2. $\mathrm{Hom}_G(V, W)$ *is the subspace of* $\mathrm{Hom}(V, W)$ *of linear maps fixed by the action of* G*. That is,*

$$A \in \mathrm{Hom}_G(V, W) \iff A = \rho_V(g^{-1})A\rho_W(g) \quad \forall g \in G.$$

Proof. Recall that ρ_V, ρ_W are homomorphisms.

$$\begin{aligned} A \in \mathrm{Hom}_G(V, W) &\iff \rho_V(g)A = A\rho_W(g) \quad \forall g \in G \\ &\iff A = (\rho_V(g))^{-1}A\rho_W(g) \quad \forall g \in G \\ &\iff A = \rho_V(g^{-1})A\rho_W(g) \quad \forall g \in G. \end{aligned}$$

□

Moreover, there exists a natural projection of $\mathrm{Hom}(V, W)$ onto $\mathrm{Hom}_G(V, W)$.

Proposition 1.2.3. *Let* V, W *be representations of* G*. Then the linear map* $\varphi :$ $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ *defined by*

$$\varphi T := \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V,W)}(g)T$$

is a projection from $\mathrm{Hom}(V, W)$ *onto* $\mathrm{Hom}_G(V, W)$*.*

Proof. First, we show that $\mathrm{im}(\varphi) \subseteq \mathrm{Hom}_G(V, W)$. Let $S = \varphi(T)$, where $T \in \mathrm{Hom}(V, W)$. Then, for any $h \in G$,

$$\begin{aligned} \rho_{\mathrm{Hom}(V,W)}(h)S &= \rho_{\mathrm{Hom}(V,W)}(h)\varphi T \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V,W)}(h)\rho_{\mathrm{Hom}(V,W)}(g)T \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V,W)}(hg)T \end{aligned}$$

Left multiplication of all the elements of G by h just reindexes the sum.

$$\begin{aligned} &= \varphi T \\ &= S. \end{aligned}$$

So S is fixed by the action of G, meaning $S \in \mathrm{Hom}_G(V, W)$. So $\mathrm{im}(\varphi) \subseteq \mathrm{Hom}_G(V, W)$, as claimed.

On the other hand, suppose that $T \in \mathrm{Hom}_G(V, W)$ (i.e. T is fixed by the action of G on $\mathrm{Hom}(V, W)$). Then

$$\varphi T = \frac{1}{|G|} \sum_{g \in G} \rho_{\mathrm{Hom}(V,W)}(g) T = \frac{1}{|G|} \sum_{g \in G} T = T.$$

This shows that $\mathrm{Hom}(V, W) \subseteq \mathrm{im}(\varphi)$ and that $\varphi^2 = \varphi$. $\square$

The following lemma tells us even more about the structure of $\mathrm{Hom}_G(V, W)$, in the case where V and W are irreducible representations. It says that there are barely any representation homomorphisms between irreducible representations.

Lemma 1.2.2 (Schur)**.** *Let V, W be irreducible representations of a group G over $\mathbb{C}$, and let $\varphi : V \to W$ be a representation homomorphism between them. Then, if $V \cong W$, $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$. If V, W are not isomorphic, then $\varphi = 0$. In other words,*

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathrm{span}(\{I\}) & V \cong W \\ \{0\} & V \ncong W. \end{cases}$$

Proof. Consider $\ker(\varphi)$ and $\mathrm{im}(\varphi)$. These are vector subspaces of V and W, respectively, so they are subrepresentations of V and W. Since V and W are irreducible, each of these is either $\{0\}$ or the whole space. If $\ker(\varphi) = V$, then $\varphi = 0$.

Otherwise, $\ker(\varphi) = \{0\}$, which makes $\mathrm{im}(\varphi) \neq \{0\}$. Then $\mathrm{im}(\varphi) = W$, so φ is a linear bijection between V and W, making $V \cong W$.

In this case, note that φ must have an eigenvalue λ since $\mathbb{C}$ is algebraically closed. This means that $\ker(\varphi - \lambda I) \neq \{0\}$ because φ has an eigenvector; invoking irreducibility again, $\ker(\varphi - \lambda I) = V$, so $\varphi - \lambda I = 0$. So $\varphi = \lambda I$. $\square$

Remark 1.2.4. This result holds for fields other than $\mathbb{C}$, as long as they are algebraically closed.

1.3 Characters of Representations

1.3.1 Characters and class functions

Representations can be a lot of information to deal with. Here is an extremely useful reduction which does not give away too much information. We take the trace of the representation.[4]

Definition 1.3.1. Given a representation (ρ_V, V) of a group G, the **character** $\chi_V : G \to \mathbb{C}$ is the function

$$\chi_V(g) := \operatorname{tr}(\rho_V(g)).$$

Example 1.3.1. Let $S_n \circlearrowright \mathbb{C}^n$ by its natural permutation representation. What is the character of this representation? The trace of a permutation matrix is the number of fixed points of the permutation. So for all $\sigma \in S_n$,

$$\chi(\sigma) = |\{1 \leq i \leq n : \sigma(i) = i\}|.$$

Example 1.3.2. Let W be the 2 dimensional irreducible representation of S_3 introduced in the previous section. Then if $\tau = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$,

$$\rho_W(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \rho_W(\tau) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So we get $\chi_W(\sigma) = -1$, $\chi_W(\tau) = 0$, and $\chi_W(e) = 2$.

Example 1.3.3. The character of the trivial representation, $\chi(g) = 1$ for all $g \in G$, is called the **trivial character**.

Here is a case where no information is lost in reducing a representation to its character.

Example 1.3.4. If $\varphi : G \to \mathbb{C}^\times$ is a homomorphism, we can view φ as a 1 dimensional representation, where $\varphi(g)$ is thought of as a 1×1 matrix. In this case, the character $\chi = \varphi$.

In the same vein, if ρ is a 1-dimensional representation of G, $\chi(g) = \rho(g)$ for all $g \in G$, since the trace of a 1×1 matrix is the entry contained within. In this case, χ is called an **abelian character**.

[4]The trace of a matrix is a kind of divine magic, the workings of which have been lost to the ages. The immense power of this arcane magic is what makes this part of the theory so nice.

The following proposition should convince you that not so much information is lost in general when passing from a representation to its character.

Proposition 1.3.1. *Let V be a representation of dimension n. Then*

$$\chi_V(e) = \dim(V).$$

Proof. The trace of the identity matrix is the number of columns in the matrix. This is the dimension of the vector space V. $\square$

Actually, characters are a simpler reduction than you would expect at first glance. You only need to know the values of χ on a representative of each conjugacy class of G.

Definition 1.3.2. A **class function** is a function on a group G that is constant on conjugacy classes of G.

Proposition 1.3.2. *Let (ρ_V, V) be a representation and χ_V be its character. Then χ_V is a class function.*

Proof. Let $a, b \in G$ share the same conjugacy class; then there exists some $g \in G$ such that $b = gag^{-1}$. Since the trace is invariant under conjugation,

$$\chi_V(b) = \mathrm{tr}(\rho_V(gag^{-1})) = \mathrm{tr}(\rho_V(g)\rho_V(a)(\rho_V(g))^{-1}) = \mathrm{tr}(\rho_V(a)) = \chi_V(a). \quad \square$$

Here is how characters play with our vector space constructions of representations.

Proposition 1.3.3. *Given representations $(\rho_V, V), (\rho_W, W)$ of a group G, $\forall g \in G$, the following identities hold:*

$$\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g),$$

$$\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g),$$

$$\chi_{V^*}(g) = \chi_V(g^{-1}).$$

If V is a vector space over $\mathbb{C}$, we have

$$\chi_{V^*}(g) = \overline{\chi_V(g)},$$

where the bar denotes complex conjugation.

Proof. For the first two identities, consider the block matrix forms of $\rho_{V\oplus W}$ and $\rho_{V\otimes W}$:

$$\rho_{V\oplus W}(g) = \begin{bmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{bmatrix},$$

$$\rho_{V\otimes W}(g) = \begin{bmatrix} (\rho_V(g))_{1,1}\, \rho_W(g) & (\rho_V(g))_{1,2}\, \rho_W(g) & \cdots & (\rho_V(g))_{1,n}\, \rho_W(g) \\ (\rho_V(g))_{2,1}\, \rho_W(g) & (\rho_V(g))_{2,2}\, \rho_W(g) & \cdots & (\rho_V(g))_{2,n}\, \rho_W(g) \\ \vdots & \vdots & \ddots & \vdots \\ (\rho_V(g))_{n,1}\, \rho_W(g) & (\rho_V(g))_{n,2}\, \rho_W(g) & \cdots & (\rho_V(g))_{n,n}\, \rho_W(g) \end{bmatrix}.$$

For the third identity, recall that $\rho_{V^*}(g) = (\rho_V(g^{-1}))^\top$. Since the trace is invariant under transposition, the identity follows.

Now suppose V is a vector space over $\mathbb{C}$. G is a finite group, so for each $g \in G$, $\rho_V(g)$ has finite order. Then for some $n \in \mathbb{N}$, $(\rho_V(g))^n = I$, so the eigenvalues of $\rho_V(g)$ are n-th roots of unity. Since the sum of the eigenvalues of a linear map is its trace, $\chi_V(g)$ is the sum of roots of unity, $\chi_V(g) = \lambda_1 + \cdots + \lambda_n$. The trace of $\rho_V(g^{-1})$ is then $\lambda_1^{-1} + \cdots + \lambda_n^{-1}$. And since the complex conjugate of a root of unity is the same as the multiplicative inverse ($\zeta^{-1} = 1/\zeta = |\zeta|^2/\zeta = \overline{\zeta}$), we have

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \lambda_1^{-1} + \cdots + \lambda_n^{-1} = \overline{\lambda}_1 + \cdots + \overline{\lambda}_n = \overline{\chi_V(g)}. \qquad \square$$

1.3.2 Orthogonality of characters

We can introduce a Hermitian inner product on the vector space of class functions.

Definition 1.3.3. Let $\varphi, \psi : G \to \mathbb{C}$ be class functions. Their **inner product** is

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g\in G} \varphi(g)\psi(g^{-1}) = \frac{1}{|G|} \sum_{g\in G} \varphi(g)\overline{\psi(g)}.$$

The following theorem is the lynchpin upon which all the nice results of representation theory rely. In some sense, all of the topics to this point were picked so we could prove this. After the proof, results will start falling into our laps.

Theorem 1.3.1 (Orthogonality of characters)**.** *Let $(\rho_V, V), (\rho_W, W)$ be irreducible representations. Then*

$$(\chi_V, \chi_W) = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W. \end{cases}$$

Proof.

$$\begin{aligned}(\chi_V, \chi_W) &= \frac{1}{|G|}\sum_{g\in G}\chi_V(g^{-1})\chi_W(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{V^*}(g)\chi_W(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{V^*\otimes W}(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Hom}(V,W)}(g)\end{aligned}$$

Recalling the definition of character and commuting the trace with sums,

$$= \mathrm{tr}\left(\frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}(V,W)}(g)\right).$$

Recall that $\frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}(V,W)}(g)$ is the projection map $\mathrm{Hom}_(V, W) \to \mathrm{Hom}_G(V, W)$. Applying Schur's lemma, we have that $\frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}(V,W)}(g)$ is a projection map onto a one dimensional subspace of $\mathrm{Hom}(V, W)$ or it is the zero map; i.e.

$$\frac{1}{|G|}\sum_{g\in G}\rho_{\mathrm{Hom}(V,W)}(g) = \begin{cases}\begin{bmatrix}1 & 0 & \cdots\\ 0 & 0 & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix} & V \cong W\\ \\ \begin{bmatrix}0 & 0 & \cdots\\ 0 & 0 & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix} & V \ncong W.\end{cases}$$

Taking the trace completes the proof. ∎

Corollary 1.3.1. *Let $V \cong V_1 \oplus \cdots \oplus V_r$ be a decomposition into irreducible representations, and let W be an irreducible representation. Then $\langle \chi_V, \chi_W\rangle$ is the number of V_i isomorphic to W.*

Proof. Since $\chi_V = \chi_{V_1} + \cdots + \chi_{V_r}$,

$$\langle \chi_V, \chi_W\rangle = \langle \chi_{V_1}, \chi_W\rangle + \cdots + \langle \chi_{V_r}, \chi_W\rangle.$$

∎

In this situation, we say that $n_i := \langle \chi_V, \chi_W \rangle$ is the "number of copies of W contained in V."

Corollary 1.3.2. *The number of copies of each irreducible representation in V is independent of the decomposition into irreducible representations.*

Proof. This is dependent only on $\langle \chi_V, \chi_W \rangle$ for each irreducible W, which is invariant of the decomposition. □

In other words, there is only 1 way to decompose a representation into irreducible representations.

Corollary 1.3.3. *The character χ_V uniquely determines the representation (ρ_V, V). That is, if $(\rho_{V'}, V')$ has the character χ_V, then $V' \cong V$.*

Proof. Since $(\rho_{V'}, V')$ has the character χ_V, V' contains the same number of copies of each irreducible representation as V. So V' and V have the same decomposition into irreducible representations. □

Corollary 1.3.4. *$\langle \chi_V, \chi_V \rangle = 1$ if and only if V is irreducible.*

Proof. If V is irreducible, then $\langle \chi_V, \chi_V \rangle = 1$ by the orthogonality of characters. Conversely, suppose $\langle \chi_V, \chi_V \rangle = 1$, and decompose $V \cong V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$ into irreducible representations, where n_i is the number of copies of V_i in V. Then

$$1 = \langle \chi_V, \chi_V \rangle = \sum_{i=1}^{r} \langle n_i \chi_{V_i}, n_i \chi_{V_i} \rangle = \sum_{i=1}^{r} n_i^2.$$

Since all the n_i are positive integers, we must have all n_i are 0 except one, which is 1. So $V \cong V_i$ for some irreducible V_i. □

Corollary 1.3.5. *V is irreducible if and only if V^* is irreducible.*

Proof. Since $V^{**} \cong V$, it suffices to show that if V is irreducible, so is V^*. If V is irreducible, then $\langle \chi_V, \chi_V \rangle = 1$. So

$$\langle \chi_{V^*}, \chi_{V^*} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{V^*}(g) \overline{\chi_{V^*}(g)} = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_V(g) = \langle \chi_V, \chi_V \rangle = 1,$$

making V^* irreducible, as well. □

To conclude the section, here is another interpretation of the inner product of characters.

Proposition 1.3.4. *Let V, W be representations of G. Then*

$$\langle \chi_V, \chi_W \rangle = \dim(\mathrm{Hom}_G(V, W)).$$

Proof. The dimension of the subspace of $\mathrm{Hom}(V, W)$ fixed by the action of G, $\dim(\mathrm{Hom}_G(V, W))$, is the number of copies of the trivial representation in $\mathrm{Hom}(V, W)$ (since in $\mathrm{Hom}_G(V, W)$, every $\rho_{\mathrm{Hom}(V,W)}(g)$ acts as the identity). So we get

$$\begin{aligned}
\dim(\mathrm{Hom}_G(V, W)) &= \left\langle \chi_{\mathrm{triv}}, \chi_{\mathrm{Hom}(V,W)} \right\rangle \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{triv}}(g) \overline{\chi_{\mathrm{Hom}(V,W)}(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V^*}(g) \chi_W(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} \\
&= \langle \chi_V, \chi_W \rangle .
\end{aligned}$$

□

Remark 1.3.1. In particular, the inner product of characters will always be a natural number. This is pretty remarkable; you would not expect that from the definition of the inner product![5]

1.3.3 The regular representation

Every group can be viewed as a group of symmetries of some set via a group action $G \circlearrowright X$. Some groups have a "natural" choice of X that they can act on, such as dihedral groups, each of which acts on the vertices of a polygon. But there is a universal choice for any group: action of G on itself by left multiplication. If we extend this action to an representation on a vector space, we can learn a lot about the representations of G.

Definition 1.3.4. The **group algebra**, $\mathbb{C}G$, is the vector space with basis elements labeled by the elements of G. Multiplication is inherited from multiplication of elements of g and extended linearly:

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) := \sum_{g,h \in G} a_g a_h gh.$$

[5]The trace is magic.

Remark 1.3.2. This is the same object as the group ring $\mathbb{C}[G]$.

Definition 1.3.5. The **regular representation** is the vector space $\mathbb{C}G$, with the action of left multiplication:

$$\rho_{\mathbb{C}G}(g)\left(\sum_{h\in G} a_h h\right) = \sum_{h\in G} a_h gh.$$

Since left multiplication by a group element permutes the elements of the group, matrices in the regular representation look like permutation matrices.

Despite seeming like such a large object, the regular representation has a very simple character.

Proposition 1.3.5. *The character of the regular representation is*

$$\chi_{\mathbb{C}G}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e. \end{cases}$$

Proof. The trace of $\rho_{\mathbb{C}G}(g)$ is the number of diagonal entries of the matrix of $\rho_{\mathbb{C}G}(g)$, i.e. the number of $h \in G$ such that $gh = h$. This is 0 if $g \neq e$ and is $|G|$ if $g = e$. $\square$

Corollary 1.3.6. *The regular representation contains n_i copies of each irreducible representation V_i, where $n_i = \dim(V_i)$.*

Proof. Using the expression for the character $\chi_{\mathbb{C}G}$,

$$\langle \chi_{\mathbb{C}G}, \chi_{V_i}\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_{\mathbb{C}G}(g)\overline{\chi_{V_i}(g)} = \frac{1}{|G|}|G|\overline{\chi_{V_i}(e)} = \overline{\dim(V_i)} = n_i. \qquad \square$$

Corollary 1.3.7. *Let n_i be the dimensions of the irreducible representations of G.*

$$|G| = \sum_{i=1}^{r} n_i^2.$$

Proof. On one hand,

$$\langle \chi_{\mathbb{C}G}, \chi_{\mathbb{C}G}\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_{\mathbb{C}G}(g)\overline{\chi_{\mathbb{C}G}(g)} = \frac{1}{|G|}|G|^2 = |G|.$$

On the other hand, using the decomposition $\mathbb{C}G \cong V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$ and the orthogonality of characters,

$$\langle \chi_{\mathbb{C}G}, \chi_{\mathbb{C}G}\rangle = \sum_{i=1}^{r}\sum_{j=1}^{r} \langle n_i\chi_{V_i}, n_j\chi_{V_j}\rangle = \sum_{i=1}^{r} \langle n_i\chi_{V_i}, n_i\chi_{V_i}\rangle = \sum_{i=1}^{r} n_i^2.$$

Setting these expressions equal gives the result. $\square$

Remark 1.3.3. This result is very useful for figuring out the irreducible representations of a group G. It gives you information about the dimensions of the irreducible representations of G, as well as how many there are.

The orthogonality of characters shows that characters of irreducible representations form an orthonormal set in the space of class functions on G. We can actually show more, now; they form a basis.

Theorem 1.3.2. *Characters of irreducible representations form an orthonormal basis in the space of class functions on G.*

Proof. Linear independence: Orthogonal sets are linearly independent.

Spanning: To show that characters of irreducible representations span the space, it suffices to show that if $\psi : G \to \mathbb{C}$ is a class function with $\langle \psi, \chi_{V_i} \rangle = 0$ for all irreducible V_i, then $\psi = 0$. Consider the map $T_i : V_i \to V_i$ given by

$$T_i v := \sum_{g \in G} \psi(g) \rho_{V_i}(g) v.$$

Observe that $T \in \mathrm{Hom}_G(V_i, V_i)$, as

$$\rho_{\mathrm{Hom}(V_i,V_i)}(g) T_i = \sum_{h \in G} \psi(h) \rho_{V_i}(g) \rho_{V_i}(h) \rho_{V_i}(g^{-1}) = \sum_{h \in G} \psi(ghg^{-1}) \rho_{V_i}(ghg^{-1})) = T_i,$$

where we have used the fact that ψ is a class function and the fact that conjugation by h just reindexes the sum.

By Schur's lemma, we get that $T_i = \lambda_i I_{V_i}$. To determine the value of λ_i, we can compute:

$$\lambda_i = \frac{1}{n_i} \operatorname{tr}(T_i) = \frac{1}{n_i} \sum_{g \in G} \psi(g) \chi_{V_i}(g) = \frac{1}{n_i} \sum_{g \in G} \psi(g) \overline{\chi_{V_i^*}(g)} = \frac{|G|}{n_i} \left\langle \psi, \chi_{V_i^*} \right\rangle = 0,$$

as V_i^* is also irreducible. So we get that $T_i = 0$ for all i.

Now if we define $T : \mathbb{C}G \to \mathbb{C}G$ by

$$Tv := \sum_{g \in G} \psi(g) \rho_{\mathbb{C}G}(g) v,$$

then this gives us that $T = 0$, as well. In other words,

$$\sum_{g \in G} \psi(g) \rho_{\mathbb{C}G}(g) = 0.$$

But the linear maps $\rho_{\mathbb{C}G}(g)$ are linearly independent, as $\rho_{\mathbb{C}G}(g)e = g$, which are independent for different choices of g. So the coefficients $\psi(g)$ must all be 0. This completes the proof. $\square$

Corollary 1.3.8. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. Let C be the collection of conjugacy classes of G. The space of class functions has the basis $\{\mathbb{1}_{C_\alpha} : C_\alpha \in C\}$, where

$$1_{C_\alpha}(g) = \begin{cases} 1 & g \in C_\alpha \\ 0 & g \notin C_\alpha. \end{cases}$$

So the space of conjugacy classes has dimension $|C|$. Since characters of irreducible representations span this space as well, $|C|$ equals the number of irreducible representations. □

As another corollary, we also get another orthogonality relation. We will discuss the interpretation of this in the next section.

Theorem 1.3.3 (2nd orthogonality relation)**.** *Let $V_1, \ldots, V_r$ be the irreducible representations of G, let $g \in G$, and let C_g be the conjugacy classe of g. Then for any $h \in G$,*

$$\sum_{i=1}^{r} \chi_{V_i}(h)\overline{\chi_{V_i}(g)} = \begin{cases} |G|/|C_g| & h \in C_g \\ 0 & h \notin C_g. \end{cases}$$

Proof. Consider the class function $\mathbb{1}_{C_g}$. Since $\chi_{V_1}, \ldots, \chi_{V_r}$ form an orthonormal basis for class functions on G, we have

$$\mathbb{1}_{C_g} = a_1\chi_{V_1} + \cdots + a_r\chi_{V_r}, \qquad \text{where } a_i = \left\langle \mathbb{1}_{C_g}, \chi_{V_i} \right\rangle.$$

Writing out the definition of the inner product, we get

$$a_i = \frac{1}{|G|}\sum_{x \in G} \mathbb{1}_{C_g}(x)\overline{\chi_{V_i}(x)} = \frac{1}{|G|}\sum_{x \in C_g} \overline{\chi_{V_i}(x)} = \frac{|C_g|}{|G|}\overline{\chi_{V_i}(g)}.$$

Plugging these values back into the expression for $\mathbb{1}_{C_g}$, we get

$$\sum_{i=1}^{r} \overline{\chi_{V_i}(g)}\chi_{V_i} = \frac{|G|}{|C_g|}\mathbb{1}_{C_g}.$$

Now evaluate both sides at h. □

1.3.4 Character tables

Now that we've built up the basic theory of representations and characters, we can work though examples.[6] In particular, we can figure out all the characters of irreducible representations (and hence all the characters) of G. We will keep track of the values of these characters in a character table.

Definition 1.3.6. Let $V_1, \dots, V_r$ be the irreducible representations of G, and let $g_1, \dots, g_r$ be representatives of the conjugacy classes of G. A **character table** of G is a matrix A with $a_{i,j} = \chi_{V_i}(g_j)$.

Although the character table can be formally thought of as a matrix, we will write it out as a table for clarity.

Example 1.3.5. Let's compute the character table of S_3. The conjugacy classes of S_3 correspond to the different cycle types of permutations, so the conjugacy classes are $[e]$, $[\begin{pmatrix}1 & 2\end{pmatrix}]$, and $[\begin{pmatrix}1 & 2 & 3\end{pmatrix}]$. This also means that we have 3 irreducible representations to look for. We have $6 = |S_3| = \sum_{i=1}^{3} n_i^2$, where n_i is the dimension of the i-th irreducible representation, so we must have two 1 dimensional irreducible representations and one 2 dimensional irreducible representation.

As with all representations, we have the trivial representation. The other 1 dimensional representation comes from the sign homomorphism $\mathrm{sgn} : S_3 \to \{-1, 1\} \subseteq \mathbb{C}$. So we have two characters already:

$$\chi_{\mathrm{triv}}(e) = 1, \qquad \chi_{\mathrm{triv}}(\begin{pmatrix}1 & 2\end{pmatrix}) = 1, \qquad \chi_{\mathrm{triv}}(\begin{pmatrix}1 & 2 & 3\end{pmatrix}) = 1,$$

$$\chi_{\mathrm{sgn}}(e) = 1, \qquad \chi_{\mathrm{sgn}}(\begin{pmatrix}1 & 2\end{pmatrix}) = -1, \qquad \chi_{\mathrm{sgn}}(\begin{pmatrix}1 & 2 & 3\end{pmatrix}) = 1.$$

For the last character, χ_{V_3}, we have the choice of a few techniques:

- Solve for the values by using $\chi_{\mathbb{C}G} = \chi_{\mathrm{triv}} + \chi_{\mathrm{triv}} + 2\chi_{V_3}$.
- Recall that the natural permutation representation V_{nat} contains a copy of V_{triv}: $V_{\mathrm{nat}} \cong V_{\mathrm{triv}} \oplus W$ for some representation W. Then $\chi_W = \chi_{\mathrm{nat}} - \chi_{\mathrm{triv}}$. Since we know all the 1-dimensional representations, we can check that $\langle \chi_W, \chi_{\mathrm{triv}} \rangle = \langle \chi_W, \chi_{\mathrm{sgn}} \rangle = 0$, which tells us that W does not contain any 1-dimensional representations in its decomposition into irreducibles. Since $\dim(W) = 2$, W must be the irreducible representation we're looking for.

[6] I use the word "basic." but don't assume I know so much more than this. I wish I did.

- Observe that $S_3 \cong D_6$, the dihedral group. D_6 has a natural 2 dimensional representation W, by viewing D_6 as the symmetries of the plane that fix a triangle. We can then check that $\langle \chi_W, \chi_W \rangle = 1$, so W is irreducible.

No matter which technique we use, we get the following character table:

S_3	$[e]$	$[(1\ \ 2)]$	$[(1\ \ 2\ \ 3)]$
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{V_3}	2	0	-1

Remark 1.3.4. The orthogonality of characters says that the rows of a character table are orthogonal. The second orthogonality relation says that the columns are orthogonal.

As the following example shows, computing the character table of an abelian group is easier than for a nonabelian group, since we just need to look for 1 dimensional representations.

Example 1.3.6. Let's compute the character table of $\mathbb{Z}/n\mathbb{Z}$. Since $\mathbb{Z}/n\mathbb{Z}$ is abelian, every element is alone in its conjugacy class. So there are n irreducible representations. And since $n = |\mathbb{Z}/n\mathbb{Z}| = \sum_{i=1}^{n} n_i^2$, we must have that all the irreducible representations are 1-dimensional. This means that they all arise from homomorphisms $\varphi_j : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$. Since $\varphi_j(k)^n = \varphi_j(nk) = \varphi_j(0) = 1$ for all $k \in \mathbb{Z}/n\mathbb{Z}$, all elements must be sent to n-th roots of unity. So we get the n homomorphisms, which are determined by what n-th root of unity 1 is sent to:

$$\varphi_j(k) = \zeta^{kj}, \qquad \text{where } \zeta = e^{2\pi i/n}.$$

This gives us $\chi_j = \varphi_j = \varphi_1^j$, so we get the character table:

$\mathbb{Z}/n\mathbb{Z}$	0	1	2	$\cdots$	$n-1$
$\chi_{\text{triv}} = \chi_0$	1	1	1	$\cdots$	1
χ_1	1	ζ	ζ^2	$\cdots$	ζ^{n-1}
χ_2	1	ζ^2	ζ^4	$\cdots$	$\zeta^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_{n-1}	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\cdots$	$\zeta^{(n-1)^2}$

As groups get larger, the character table generally becomes more difficult to determine, but the theory we have developed gives enough tools for us to work out some larger character tables without too much work.

Example 1.3.7. Let's compute the character table of S_4. The conjugacy classes of S_4 correspond to the different cycle types of permutations, so the conjugacy classes are $[e]$, $[\begin{pmatrix}1 & 2\end{pmatrix}]$, $[\begin{pmatrix}1 & 2 & 3\end{pmatrix}]$, $[\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}]$, and $[\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}]$. So we need to look for 5 irreducible representations. As before, we have the characters χ_{triv} and χ_{sgn} of 1-dimensional representations. And since $24 = |S_4| = \sum_{i=1}^5 n_i^2$, we must have $n_3 = 2$ and $n_4 = n_5 = 3$ (without loss of generality).

There is a normal subgroup $N = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \trianglelefteq S_4$, and $S_4/N \cong S_3$. So we get representations of S_4 by factoring through representations of S_3. This gives χ_3, which corresponds to the 2-dimensional irreducible representation of S_3.

To figure out χ_4, recall that we have $V_{\text{nat}} \cong V_{\text{triv}} \oplus W$ for some representation W, since the subspace $\operatorname{span}(\{e_1 + e_2 + e_3 + e_4\})$ is acted upon trivially in the natural permutation representation. We then get $\chi_W = \chi_{\text{nat}} - \chi_{\text{triv}}$:

$$\chi_W(e) = 3, \qquad \chi_W(\begin{pmatrix}1 & 2\end{pmatrix}) = 1, \qquad \chi_W(\begin{pmatrix}1 & 2 & 3\end{pmatrix}) = 0,$$

$$\chi_W(\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}) = -1, \qquad \chi_W(\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}) = -1.$$

Checking that $\langle \chi_W, \chi_W \rangle = 1$, we get that χ_W is the fourth irreducible character.

We can now find χ_5 by the decomposition of the regular representation:

$$\chi_{\mathbb{C}G} = \chi_{\text{triv}} + \chi_{\text{sgn}} + 2\chi_3 + 3\chi_4 + 3\chi_5.$$

So we get the character table:

S_4	e	$\begin{pmatrix}1 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}$	$\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}$
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

Remark 1.3.5. Instead of using the natural representation to find χ_4, we could have used both orthogonality relations to deduce χ_4 and χ_5 simultaneously.

Remark 1.3.6. In some cases, such as with the fifth irreducible representation of S_4, it is much easier to figure out the character than it is to find the representation. Nevertheless, there is actually a general theory which gives all the irreducible representations and charachers of the symmetric groups S_n.

The following proposition will be useful in our next example.

Proposition 1.3.6. *The number of 1-dimensional (irreducible) representations of* G *is the the order of the abelianization* $|G^{\text{ab}}|$.

Proof. Recall that a 1-dimensional representation is a homomorphism $\varphi : G \to \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is abelian, this homomorphism factors through the abelianization of G:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & \mathbb{C}^\times \\ G^{\text{ab}} \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

So the number of 1-dimensional representations of G is the number of homomorphisms $\tilde{\varphi} : G^{\text{ab}} \to \mathbb{C}^\times$. That is, the number of 1-dimensional representations of G is the number of 1-dimensional representations of G^{ab}. G^{ab} is abelian, so every one of its elements is alone in its own conjugacy class. So there are $|G^{\text{ab}}|$ 1-dimensional, irreducible representations of G^{ab}. $\square$

Example 1.3.8. Let's compute the character table of A_4. The conjugacy classes of S_4 correspond to the different cycle types of even permutations, but one of the classes gets split in half when passing to A_4, since the only permutations which conjugated elements between these two classes were odd permutations (which are not included in A_4. So the conjugacy classes are $[e]$, $[\begin{pmatrix}1 & 2 & 3\end{pmatrix}]$, $[\begin{pmatrix}1 & 3 & 2\end{pmatrix}]$, and $[\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}]$. So we need to look for 4 irreducible representations.

To find out the number of 1 dimensional representations, observe that $A_4^{\text{ab}} \cong \mathbb{Z}/3\mathbb{Z}$. So the proposition says that there are 3 1 dimensional representations. Moreover, since $A_4^{\text{ab}} = A_4/[G, G]$ (where $[G, G]$ is the commutator subgroup), we have a surjective homomorphism $A_4 \to \mathbb{Z}/3\mathbb{Z}$. So these 1 dimensional representations are the representations of $\mathbb{Z}/3\mathbb{Z}$, lifted up to A_4. So we get χ_{triv},

$$\chi_2(e) = 1, \quad \chi_2(\begin{pmatrix}1 & 2 & 3\end{pmatrix}) = \zeta, \quad \chi_2(\begin{pmatrix}1 & 3 & 2\end{pmatrix}) = \zeta^2, \quad \chi_2(\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}) = 1,$$

$$\chi_3(e) = 1, \quad \chi_3(\begin{pmatrix}1 & 2 & 3\end{pmatrix}) = \zeta^2, \quad \chi_3(\begin{pmatrix}1 & 3 & 2\end{pmatrix}) = \zeta, \quad \chi_3(\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}) = 1,$$

where $\zeta = e^{2\pi i/3}$.

To find the remaining character, we check $12 = |A_4| = \sum_{i=1}^4 n_i^2$, which gives us that $n_4 = 3$. We can now use the decomposition of the regular representation to solve for χ_4:

$$\chi_4 = \frac{\chi_{\mathbb{C}G} - \chi_{\text{triv}} - \chi_2 - \chi_3}{3}.$$

So we get the character table:

A_4	e	$\begin{pmatrix}1 & 2 & 3\end{pmatrix}$	$\begin{pmatrix}1 & 3 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}$
χ_{triv}	1	1	1	1
χ_2	1	ζ	ζ^2	1
χ_3	1	ζ^2	ζ	1
χ_4	3	0	0	-1

Example 1.3.9. Let's compute the character table of

$$D_8 = \left\langle r, s \mid s^2 = r^4 = 1, rs = sr^3 \right\rangle .$$

The conjugacy classes are $[e]$, $[s]$, $[r]$, $[rs]$, and $[r^2]$, so we need to find 5 irreducible representations. $D_8^{\text{ab}} = D_8/\{e, r^2\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so the four 1 dimensional representations come from homomorphisms $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ (with $r^2 \mapsto 1$). The images of elements in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ must have multiplicative order 2, so they must be 1 or -1. This gives us the four 1 dimensional representations, determined by whether r and s get sent to 1 or -1.

We now have $8 = |D_8| = \sum_{i=1}^{5} n_i^2$, so $n_5 = 2$. The natural 2 dimensional representation is the remaining irreducible representation; we can check that it is irreducible by checking the decomposition of the regular representation. So we get the character table:

D_8	e	s	r	rs	r^2
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
χ_4	1	-1	-1	1	1
χ_5	2	0	0	0	-2

Remark 1.3.7. You can check that the quaternion group Q_8 has the same character table as D_8. So a character table does not uniquely determine a group.

Appendix A

Linear Algebra Background

A.1 The trace

In linear algebra, the determinant of matrix gives a lot of useful information about the associated linear transformation. The determinant is a homomorphism $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^\times$, so it tells us about the multiplicative structure of matrices (composition of the linear transformation). This section is about the trace, a homomorphism $\mathrm{tr} : M_{n\times n}(\mathbb{C}) \to \mathbb{C}$, which tells us about the additive structure of matrices.

Definition A.1.1. Let A be a matrix with entries $a_{i,j}$. The **trace** of A is

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{i,i}.$$

Adding together the diagonal entries of a matrix seems arbitrary. But it turns out that this is a very good quantity to study. From the defnition, we can see $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. But hte trace has other nice properties. For instance, it is invariant of the choice of basis.

Lemma A.1.1. *Let A, B be $n \times n$ matrices. Then*

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Proof. Write out the definitions of the trace and of matrix multiplication:

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} [AB]_{i,i} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,k}b_{k,i} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{k,i}a_{i,k} = \sum_{k=1}^{n} [BA]_{k,k} = \mathrm{tr}(BA). \quad \square$$

Proposition A.1.1. *The trace is invariant under change of basis. In particular,*

$$\operatorname{tr}(A) = \operatorname{tr}(BAB^{-1}).$$

Proof. By the previous lemma,

$$\operatorname{tr}(B(AB^{-1})) = \operatorname{tr}((AB^{-1})B) = \operatorname{tr}(A). \qquad \square$$

Since the trace is invariant under change of basis, as you might suspect, it has an interpretation dependent only on the linear transformation, no matrix shenanigans needed.

Proposition A.1.2. *Let A be an $n \times n$ matrix over $\mathbb{C}$. Then*

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i,$$

where the λ_i are the eigenvalues of A, counted with multiplicity.

Here is a proof, assuming you know about the Jordan canonical form of a matrix.

Proof. Write A in its Jordan canonical form, so $A = PJP^{-1}$, where J is upper triangular with the eigenvalues of A long the diagonal. Then

$$\operatorname{tr}(A) = \operatorname{tr}(PJP^{-1}) = \operatorname{tr}(J) = \sum_{i=1}^{n} \lambda_i. \qquad \square$$

Here is another proof, which does not assume knowledge of the Jordan canonical form.

Proof. By the fundamental theorem of algebra, the characteristic polynomial of A factors as $c_A(t) = (-1)^n(t - \lambda_1) \cdots (t - \lambda_n)$. When we multiply this out, each term corresponds to a choice of, for each i, multiplying either a t or a λ_i; for example, the t^n term comes from choosing all the ts, and the constant term comes from choosing all the λ_i. The coefficient of t^{n-1} in this polynomial is the sum of all the terms where we only pick one of the λ_i to multiply. So it is

$$(-1)^n(-\lambda_1 - \lambda_2 - \cdots - \lambda_n) = (-1)^{n-1} \sum_{i=1}^{n} \lambda_i.$$

On the other hand, the characteristic polynomial is $\det(A - tI)$. If we evaluate this, we will only get terms with a t^{n-1} from the product of diagonal entries of $A - tI$;

indeed, any term containing $a_{i,j}$ off the diagonal of $A-tI$ will not contain $a_{i,i}-t$ or $a_{j,j}-t$, so we can have at most t^{n-2} in this term. This means that the coefficient of t^{n-1} in $c_A(t)=\det(A-tI)$ is the coefficient of t^{n-1} in $(a_{1,1}-t)(a_{2,2}-t)\cdots(a_{n,n}-t)$. By the same argument as above, this coefficient is

$$(-1)^{n-1}(a_{1,1}+a_{2,2}+\cdots+a_{n,n})=(-1)^{n-1}\operatorname{tr}(A).$$

These two expressions are both the coefficient of t^{n-1} in the characteristic polynomial of A, so they are equal. We get

$$\operatorname{tr}(A)=\sum_{i=1}^{n}\lambda_i. \qquad \square$$

Here are some vague thoughts to help you interpret the trace: What does a diagonal element of a matrix refer to? If we fix a basis $\{v_1,\ldots,v_n\}$ of V, $a_{i,i}$ is how much in the direction of v_i Av_i is. If we assume this to be an orthonormal basis, $a_{i,i}=\langle Av_i,v_i\rangle$. The trace is invariant under change of basis, and we can think of this in the sense of "if we sum the amount that A keeps vectors in the same direction, but in every direction, we should get an quantity invariant of which vectors we picked for our directions."

On the other hand, the trace is also the sum of the eigenvalues (with multiplicities). In this sense, the picture is even cleaner. The trace is the sum of how much A keeps vectors in the same direction, for every direction. In the case where you have a basis of eigenvectors, Av_i is now entirely in the direction of v_i. So the trace is really just a measure of how much A moves or does not move vectors into a different direction from where they started.

A.2 Direct sums and products

The Cartesian product is a way of combining sets to get ordered pairs of elements between sets:

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

For infinite products (indexed by some infinite set I), we get something that looks like this:

$$\prod_{\alpha \in I} A_\alpha := \{(a_\alpha)_{\alpha \in I} : a_\alpha \in A_\alpha \ \forall \alpha \in I\}.$$

In a similar way, we can construct a product of vector spaces.

Definition A.2.1. If V_α are vector spaces, the **product** $\prod_{\alpha \in I} V_\alpha$ is a vector space with componentwise addition and scalar multiplication:

$$c(a_\alpha)_{\alpha \in I} + (b_\alpha)_{\alpha \in I} = (ca_\alpha + b_\alpha)_{\alpha \in I}.$$

As you can imagine, products can be very large. Even if $V_\alpha = \mathbb{Q}$, viewed as a vector space over $\mathbb{Q}$, we can get uncountable products by taking a product of uncountably many such V_α. However, there is another way to combine spaces which produces spaces which are not as large.

Definition A.2.2. If V_α are vector spaces, the **direct sum** $\bigoplus_{\alpha \in I} V_i$ is the following subspace of $\prod_{\alpha \in I} V_\alpha$:

$$\bigoplus_{\alpha \in I} V_\alpha := \{(a_\alpha)_{\alpha \in I} : a_\alpha = 0 \text{ for all but finitely many } \alpha\}.$$

Remark A.2.1. For finite direct sums, the definition is the same as the product:

$$V_1 \oplus \cdots \oplus V_n = V_1 \times \cdots \times V_n.$$

The direct sum is not such an unfamiliar concept. Here is an example you are already familiar with.

Example A.2.1. Let $\mathbb{P}$ be the collection of polynomials with coefficients in $\mathbb{C}$. This is a vector space over $\mathbb{C}$. By looking at the coefficients of a polynomial, polynomials in $\mathbb{P}$ look like ordered tuples of coefficients (which can be 0); for example, the polynomial $5x^3 - \sqrt{2}x + i$ can be thought of as the sequence $(i, -\sqrt{2}, 0, 5, 0, \dots)$. Every polynomial only has finitely many nonzero terms, so

$$\mathbb{P} \cong \bigoplus_{j=1}^{\infty} \mathbb{C}.$$

This is not the same as $\prod_{j=1}^{\infty} \mathbb{C}$. In the product, we have elements such as $(1, 1, 1, \dots)$, which correspond to power series such as $1 + x + x^2 + \cdots$. These are not polynomials, though. In this case, the product, $\prod_{j=1}^{\infty} \mathbb{C}$ is the vector space of **formal power series**[1] with coefficients in $\mathbb{C}$.

[1] The word formal here indicates that we only care about the symbols, without any regard to whether these series actually converge.

A.3 Tensor products

I got a little lazy. Bug me to actually write this part.

A.4 Dual spaces

Dual spaces are a common construction in advanced linear algebra.

Definition A.4.1. Let V be a vector space over a field F. Then the **dual vector space** V^* is the vector space of linear functions $T : V \to F$.

Remark A.4.1. At first glance, this definition seems fairly arbitrary and unmotivated. Here's why we care about dual spaces. One of the central ideas of functional analysis is the study of (infinite dimensional) vector spaces of functions, such as $C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$. In this setting, many familiar functions are elements of the dual space, such as the integration map: $T(f) := \int_0^1 f(x)\,dx$.

Here is a way for us to talk about vectors in the dual space.

Proposition A.4.1. *Let V be a finite dimensional vector space with basis $\{v_1, \ldots, v_n\}$. Then $\{v_1^*, \ldots, v_n^*\}$ is a basis for V^*, where*

$$v_i^*(v_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases}$$

Proof. Let v_i^* be defined as above; that is, for each i,

$$v_i^*(a_1 v_1 + \cdots + a_n v_n) = a_i.$$

The v_i^* are linearly independent: if $a_1 v_1^* + \cdots + a_n v_n^* = 0$, then for each i,

$$0 = a_1 v_1^*(v_i) + \cdots + a_n v_n^*(v_i) = a_i.$$

So all the a_i equal 0.

To see that the v_i^* span V^*, let $f \in V^*$. Since f is linear and V is finite dimensional, f is uniquely determined by $f(v_i)$ for $1 \leq i \leq n$. Letting $a_i = f(v_i)$ for each i, we get that

$$f = a_1 v_1^* + \cdots + a_n v_n^*,$$

as these agree when applied to each of the v_i. $\square$

Linear maps between vector spaces give rise to linear maps between dual spaces.

Definition A.4.2. Given a linear map $T : V \to W$, the induced **dual map** $T^* : W^* \to V^*$ is given by $f \mapsto f \circ T$.

$$V \xrightarrow{T} W$$

$$V^* \xleftarrow{T^*} W^*$$

Here's why this construction is called the "dual" space.

Proposition A.4.2. *If $T : V \to W$ is a linear transformation with matrix A, then the dual map $T^* : W^* \to V^*$ has matrix representation $A^\top$.*

Proof. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be ordered bases of V and W, respectively. Let $\{v_1^*, \ldots, v_n^*\}$ and $\{w_1^*, \ldots, w_m^*\}$ be the corresponding dual bases. We have that

$$\begin{aligned}
(T^* w_i^*)\left(\sum_{j=1}^n b_j v_j\right) &= w_i^*\left(\sum_{j=1}^n b_j T v_j\right) \\
&= w_i^*\left(\sum_{j=1}^n b_j \sum_{k=1}^m a_{k,j} w_k\right) \\
&= \sum_{j=1}^n \sum_{k=1}^m b_j a_{k,j} \underbrace{w_i^*(w_k)}_{\delta_{i,k}} \\
&= \sum_{j=1}^n b_j a_{i,j} \\
&= \sum_{j=1}^n a_{i,j} v_j^*\left(\sum_{\ell=1}^n b_\ell v_\ell\right),
\end{aligned}$$

so $T^* w_i^* = \sum_{j=1}^n a_{i,j} v_j^*$. The definition of matrix multiplication gives us $[T^*]_{j,i} = a_{i,j}$, and we are done. □

Here's the picture: if vectors in V are column vectors, then vectors in the dual space are row vectors. In the vector space V, matrices act on the left as Av. In the dual space, matrices act on the right, and the action of the corresponding matrix for the dual transformation looks like $v^* A^\top$, where v^* is a row vector.

In this picture, what does $v_i^*(v_j)$ look like? If these are the basis vectors for their respective spaces, we get $\delta_{i,j}$. This agrees with multiplication of the vectors $v_i^\top v_j$:

$$v_i^*(v_j) = \begin{bmatrix} \quad v_i \quad \end{bmatrix} \begin{bmatrix} \\ v_j \\ \\ \end{bmatrix}.$$

Now notice that if we fix v_i^*, this becomes linear in v if we replace v_j by any $v \in V$. Correspondingly. if we fix v_j, this is linear in f if we replace v_i^* by any $f \in V^*$. So, in general, $f(v)$ gives the same value as multiplication of the vectors $f^\top v$:

$$f(v) = \begin{bmatrix} & f & \end{bmatrix} \begin{bmatrix} \\ v \\ \\ \end{bmatrix}.$$

For this reason, if $v \in V$ and $f \in V^*$, we often denote $f(v)$ as the **pairing** $\langle v, f \rangle$. With this notation, the action of matrices looks like

$$\langle Av, f \rangle = f(Av) = (A^* f)(v) = \langle v, A^* f \rangle .$$

This gives an analogy between the relationship between a vector and its dual space and the relationship between an inner product space and itself.

Warning A.4.1. This should not be confused with the concept of a Hermitian inner product; with a Hermitian inner product on a complex vector space, the adjoint $A^\dagger$ of A satisfies $\langle Av, w \rangle = \langle v, A^\dagger w \rangle$, but the matrix of the adjoint is the *conjugate* transpose of A, not just the transpose.

Bibliography

[FH13] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 2013.

[SS12] L.L. Scott and J.P. Serre. *Linear Representations of Finite Groups*. Graduate Texts in Mathematics. Springer New York, 2012.