Math 247A Lecture 11 Notes

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1 A_p Weights and The Vector-Valued Maximal Function

1.1 Use of reverse Hölder in the characterization of A_p weights

Last time, we proved the following theorem:

Theorem 1.1. *Fix $1 < p < \infty$. Then $\omega \in A_p$ if and only if $M : L^p(\omega\, dx) \to L^p(\omega\, dx)$ boundedly.*

We showed that ($\Longleftarrow$) holds if $L : L^p(\omega\, dx) \to L^{p,\infty}(\omega\, dx)$. For the ($\Longrightarrow$) direction, we had 3 ingredients:

1. $M : L^1(\omega\, dx) \to L^{q,\infty}(\omega, dx)$ for all $1 \leq q < \infty$.
2. A reverse Hölder inequality yields if $\omega \in A_p$, then $\omega \in A_q$ for some $q < p$.
3. $M : L^\infty(\omega,\, dx) L^\infty(\omega\, dx)$ boundedly.

The reverse holds inequality says

Lemma 1.1. *If $\omega \in A_p$, then there exist an $r > 1$ and $c > 0$ such that*

$$\left(\frac{1}{|B|} \int_B \omega(y)^r\, dy \right)^{1/r} \leq \frac{c}{|B|} \int_B \omega(y)\, dy.$$

We will not prove this. Here's how we use it:

Proof. Apply this to $\sigma(y) = \omega(y)$. Recall that $\omega \in A_p \iff \sigma \in A_p$. Then there exist $r > 1$ and $c > 0$ depending on σ (and hence on ω) so that

$$\left[\frac{1}{|B|} \int_B \omega(y)^{-rp'/p}\, dy \right]^{1/r} \leq \frac{C}{|B|} \int_B \omega(y)^{-p'/p}\, dy.$$

So we get

$$\omega \in A_p \iff \sup_B \frac{1}{|B|} \omega(B) \left(\frac{1}{|B|} \int_B \omega(y)^{-p'/p}\, dy \right)^{p/p'} \lesssim 1$$

$$\implies \frac{1}{|B|}\int_B \omega(y)^{-p'/p}\,dy \lesssim \left(\frac{|B|}{\omega(B)}\right)^{p'/p} \lesssim \left(\frac{|B|}{\omega(B)}\right)^{1/(p-1)}.$$

We get

$$|B|^{-1/r}\left(\int_B \omega(y)^{-rp'/p}\,dy\right)^{1/r} \lesssim \left(\frac{|B|}{\omega(B)}\right)^{1/(p-1)}.$$

Write

$$\begin{aligned} rp'/p = q'/q &\iff r\frac{1}{p-1} = \frac{1}{q-1} \\ &\iff a-1 = \frac{p-1}{p} < p-1 \\ &\iff q = 1+\frac{p-1}{p} \in (1,p). \end{aligned}$$

We get

$$\begin{aligned} |B|^{-q/q'\cdot 1/(p-1)}\left(\int_B \omega(y)^{-q'/q}\,dy\right)^{q/q'\cdot 1/(p-1)} &\lesssim \left(\frac{|B|}{\omega(B)}\right)^{1/(p-1)} \\ |B|^{-1-(q-1)}\omega(B)\left(\int_B \omega(y)^{-q'/q}\,dy\right)^{q/q'} &\lesssim 1. \end{aligned}$$

So $\omega \in A_q$. $\square$

Theorem 1.2. *Fix $1 \le p < \infty$. If $d\mu$ is a nonnegative Borel measure such that $M : L^p(d\mu) \to L^{p,\infty}(d\mu)$ boundedly, then $d\mu = \omega\,dx$ and $\omega \in A_p$.*

Proof. It suffices to show that $d\mu$ is absolutely continuous with respect to Lebesgue measure. Write $d\mu = \omega\,dx + d\nu$ with $d\nu$ singular with respect to Lebsegue measure. Let K be a compact set such that $|K| = 0$ and $\nu(K) > 0$. For $n \ge 1$, let $U_n = \{: d(x,k) < 1/n\}$. Note that $U_n \setminus K \supseteq U_{n+1} \setminus K$ and $\bigcap(U_n \setminus K) = \varnothing$. Let $f_n = \mathbb{1}_{U_n\setminus K}$, so $f_{n+1} \le f_n$ and $f_n \to 0$.

We claim that $d\mu$ is finite on compact sets. Assuming the claim, by the monotone convergence theorem,

$$\int |f_n|^p\,d\mu \xrightarrow{n\to\infty} 0.$$

For $x \in K$,

$$\begin{aligned} Mf_n(x) &= \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} \mathbb{1}_{U_n\setminus K}(y)\,dy \\ &\ge \frac{1}{|B(x,1/n)|}\int_{B(x,1/n)} \mathbb{1}_{U_n\setminus K}(y)\,dy \\ &= \frac{1}{|B(x,1/n)|}\int_{K^c} \mathbb{1}_{B(x,1/n)}(y)\,dy \end{aligned}$$

As $|K| = 0$,

$$
\begin{aligned}
&= \frac{1}{|B(x,1/n)|} \int_{\mathbb{R}^d} \mathbb{1}_{B(x,1/n)}(y)\,dy \\
&= 1
\end{aligned}
$$

Then

$$
\mu(K) \le \mu(\{x : Mf_n(x) > 1/2\}) \lesssim \int |f_n|^p \, d\mu \xrightarrow{n\to\infty} 0,
$$

so we get a contradiction.

Now we prove the claim. Let E be a compact set such that $0 < \mu(E) < \infty$.

$$
\begin{aligned}
M\mathbb{1}_E(x) &= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \mathbb{1}_E(y)\,dy \\
&\gtrsim \frac{|E|}{[d(x,E) + \operatorname{diam}(E)]^d}
\end{aligned}
$$

So $M\mathbb{1}_E$ is bounded from below uniformly on compact sets: if F is a compact set, then for $x \in F$,

$$
M\mathbb{1}_E(x) \lesssim \frac{|E|}{[\operatorname{dist}(F,E) + \operatorname{diam} F + \operatorname{diam} E]^d} =: C(F,E).
$$

Then

$$
\mu(F) \le \mu\left(\left\{x : M\mathbb{1}_E(x) > \frac{1}{2}C(F,E)\right\}\right) \lesssim_{F,E} \int |\mathbb{1}_E(y)|^p \, d\mu(y) \lesssim_{F,E} \mu(E) < \infty. \qquad \square
$$

1.2 The vector-valued maximal function

Definition 1.1. Let $F : \mathbb{R}^d \to \ell^2$, $f(x) = \{f_n(x)\}_{n\ge 1}$. We write

$$
|f(x)| = \|\{f_n(x)\}_{n\ge1}\|_{\ell^2}, \qquad \|f\|_{L^p} = \left(\int |f(x)|^p \, dx\right)^{1/p}.
$$

The **vector-valued maximal function** is

$$
\overline{M}f(x) = \|\{Mf_n(x)\}_{n\ge1}\|_{\ell^2}.
$$

Theorem 1.3.

1. *$\overline{M}$ is of weak-type $(1,1)$.*
2. *For $1 < p < \infty$, $\overline{M}$ is of strong type (p,p).*

Remark 1.1. We no longer have a trivial L^∞ bound. In fact, it fails. Take $d = 1$. For $n \geq 1$, take $f_n = \mathbb{1}_{[2^{n-1}, 2^n)}$.

$$|f(x)| = \sqrt{\sum_{n \geq 1} |f_n|^2(x)} = \mathbb{1}_{[1,\infty)}(x) \in L^\infty$$

For $|x| \leq 2^n$,

$$\begin{aligned} Mf_n(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \mathbb{1}_{[2^{n-1},2^n)}(y)\, dy \\ &\geq \frac{1}{2 \cdot 2^{n+1}} \int_{x-2^{n+1}}^{x+2^{n+1}} \mathbb{1}_{[2^{n-1},2^n)}(y)\, dy \\ &= \frac{1}{2 \cdot 2^{n+1}} 2^{n-1} \\ &= \frac{1}{8}. \end{aligned}$$

Now

$$\begin{aligned} \overline{M} f(x) &= \sqrt{\sum_{n \geq 1} |Mf_n(x)|^2} \\ &\geq \sqrt{\sum_{n : 2^n \geq |x|} \left(\frac{1}{8}\right)^2} \\ &= \infty \end{aligned}$$

So $\overline{M} f \notin L^\infty$.

Remark 1.2. Boundedness of $\overline{M}$ on L^2 follows from the scalar case:

$$\begin{aligned} \|\overline{M} f\|_{L^2}^2 &= \int \sum_{n \geq 1} |Mf_n(x)|^2 = \sum_{n \geq 1} \|Mf_n\|_{L^2}^2 \lesssim \sum_{n \geq 1} \|f_n\|_{L^2}^2 \\ &= \sum_{n \geq 1} \int |f_n(x)|^2\, dx \leq \int |f(x)|^2\, dx = \|f\|_{L^2}^2. \end{aligned}$$

Let's prove boundedness of $\overline{M}$ on L^p for $2 < p < \infty$.

Proof. If $\omega \geq 0$ with $\omega \in L^1_{\text{loc}}$, then

$$\int |Mf_n|^2 \omega\, dx \lesssim \int |f_n|^2 (M\omega)\, dx$$

uniformly in n. Summing in n, we get

$$\int |\overline{M}f(x)|^2 \omega(x)\, dx \lesssim \int |f(x)|^2 (M\omega)(x)\, dx.$$

Then

$$\begin{aligned}
\|\overline{M}f\|_{L^p}^2 &= \||\overline{M}f|^2\|_{L^{p/2}} \\
&= \sup_{\|\omega\|_{L^{(p/2)'}} \le 1} \int |\overline{M}f|^2(x)\omega(x)\, dx \\
&\lesssim \sup_{\|\omega\|_{L^{(p/2)'}} \le 1} \int \underbrace{|f(x)|^2}_{\in L^{p/2}} \underbrace{(M\omega)(x)}_{\in L^{(p/2)'}}\, dx \\
&\lesssim \||f|^2\|_{L^{p/2}} \sup_{\|\omega\|_{L^{(p/2)'}} \le 1} \underbrace{\|M\omega\|_{L^{(p/2)'}}}_{\lesssim \|\omega\|_{L^{(p/2)'}}} \\
&\lesssim \|f\|_{L^p}^2.
\end{aligned}$$

To prove M is of strong-type (p,p) for $1 < p < \infty$, it suffices (by Marcinkiewicz) to show that $\overline{M}$ is of weak-type $(1,1)$. □

We will use the following.

Lemma 1.2 (A Calderón-Zygmund decomposition). *If $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$, then we can decompose $f = g + b$ such that*

1. $|g(x)| \le \lambda$ *for almost every* $x \in \mathbb{R}^d$.
2. $\operatorname{supp} b$ *is a union of cubes whose interiors are pairwise disjoint and*

$$\lambda < \frac{1}{|Q_k|} \int_{Q_k} |b(x)|\, dx \le 2^d \lambda.$$

3. $g = f[1 - \mathbb{1}_{\bigcup Q_k}]$.

Next time, we will prove this decomposition and use it to prove the weak $(1,1)$ bound.