Math 254A Lecture 8 Notes

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1 Integral Formula for the Fenchel-Legendre Transform

1.1 The Fenchel-Legendre transform and the integral formula

Last time, we defined the **Fenchel-Legendre transform** $s^* = \sup_x s(x) + \langle y, x \rangle$, which is convex, lower semicontinuous, is $s^* : X^* \to (-\infty, \infty]$, and is not always $+\infty$.[1] We also saw that $s = (s^*)_*$, so we can recover s from its Fenchel-Legendre transform.

Let's focus on the $X = Y^*$ case, since this also subsumes the $X = \mathbb{R}^k$ case. Also assume $\lambda \neq 0$.

Theorem 1.1. *In this generalized type counting problem for $X = Y^*$,*

$$s^*(y) = \log \int e^{\langle y, \varphi \rangle} \, d\lambda$$

for $y \in Y$.

Before proving this, observe:

$$\begin{aligned} s^*(ty + (1-t)w) &= \log \int e^{t \langle \varphi \rangle + (1-t) \langle w, \varphi \rangle} \, d\lambda \\ &= \int e^{t \langle y, a \rangle} \cdot e^{(1-t) \langle w, \varphi \rangle} \, d\lambda \end{aligned}$$

Using Hölder's inequality,

$$\leq \left(\int e^{\langle y, \varphi \rangle} \, d\lambda \right)^t + \left(\int e^{\langle w, \varphi \rangle} \, d\lambda \right)^{1-t},$$

so taking logs gives that this expression is convex. We can also check that this expression is lower semicontinuous.

[1] Many authors study $\widetilde{s} = -s$ throughout and then get $s^*(y) = \sup_x \langle y, x \rangle - \widetilde{s}(x)$ and $\widetilde{s}(z) = \sup_y \langle y, z \rangle - s^*(y)$. We use a different convention.

1.2 Proofs of the upper bound and the lower bound

Proof. ($\leq$): Since $s^*(y) = \sup_x s(x) + \langle y, x\rangle$, we need to show that

$$s(x) + \langle y, x\rangle \leq \log \int e^{\langle y,\varphi\rangle}\, d\lambda$$

for all x. Let $\varepsilon > 0$, and consider $U = \{x' : \langle y, x'\rangle > \langle y, x\rangle - \varepsilon\}$. We know that

$$\begin{aligned}
s^{n\cdot(s(x)+\langle y,x\rangle)} &\leq e^{n(s(U)+\langle y,x\rangle)} \\
&= e^{o(n)} e^{n\langle y,x\rangle} \lambda^{\times n}\left(\left\{p : \frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\right\}\right) \\
&= e^{o(n)} e^{n\langle y,x\rangle} \lambda^{\times n}\left(\left\{p : \sum_{i=1}^n \langle y, \varphi(p_i)\rangle > n\langle y, n\rangle - n\varepsilon\right\}\right)
\end{aligned}$$

Exponentiate both sides in the inequality and apply Markov's inequality:[2]

$$\begin{aligned}
&\leq e^{o(n)} e^{n\langle y,x\rangle} e^{n\varepsilon - n\langle y,x\rangle} \int e^{\sum_{i=1}^n \langle y,\varphi(p_i)\rangle}\, d\lambda^{\times n} \\
&= e^{o(n)+n\varepsilon} \int_{M^n} \prod_{i=1}^n e^{\langle y,\varphi(p_i)\rangle}\, d\lambda^n \\
&= e^{o(n)} e^{\varepsilon n}\left(\int e^{\langle y,\varphi\rangle}\, d\lambda\right)^n,
\end{aligned}$$

so

$$n(s(U) + \langle y, x\rangle) \leq o(n) + \varepsilon n + n\log\int e^{\langle y,\varphi\rangle}\, d\lambda.$$

Divide by n and send $n \to \infty$ to get

$$s(x) + \langle y, x\rangle \leq\ s(U) + \langle y, x\rangle \leq \varepsilon + \log\int e^{\langle y,\varphi\rangle}\, d\lambda.$$

Since ε is arbitrary, we get ($\leq$).

To get the lower bound, let's look at the proof of the upper bound and try to make it as tight as possible. The first inequality is close if U is a small enough neighborhood of x. In the Chernoff bound, we want to see when this is close to equality. To prove ($\geq$), we will look at the Chernoff bound step; here's the idea: Consider

$$e^{n\langle y,n\rangle} \lambda^{\times n}\left(\left\{p : \frac{1}{n}\sum_{i=1}^n \langle y, \varphi(p_i)\rangle \in U\right\}\right),$$

where we want to make U small enough around x to force this to be $\approx \langle, y, x\rangle$. We then get

$$e^{n\langle y,x\rangle} \lambda^{\times n}\left(\left\{p : \exp\sum_{i=1}^n \langle y, \varphi(p_i)\rangle \approx e^{\langle y,x\rangle}, \frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\right\}\right).$$

This is

$$\approx e^{\pm\varepsilon n} \int_{\{\frac{1}{n}\sum_{i=1}^n \varphi(p_i)\in U\}} e^{\sum_{i=1}^n \langle y, \varphi(p_i)\rangle} \, d\lambda^{\times n} \le e^{\varepsilon n} \int_{M^n} e^{\sum_{i=1}^n \langle y, \varphi(p_i)\rangle} \, d\lambda^{\times n}.$$

So the question becomes: Can we find an x where most of the mass lies in the set $\{\frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\}$?

Now let's prove $(\ge)$ carefully. First assume two conditions:

1. $Z = \int e^{\langle y, \varphi\rangle} \, d\lambda < \infty$.
2. p takes values in a compact subset K of X.

In this case, we can define a new probability measure on M by

$$d\theta(p) = \frac{1}{Z} e^{\langle y, \varphi(p)\rangle} \, d\lambda(p)$$

(using assumption 1). Now, for any $A \subseteq M^n$,

$$\int_A e^{\sum_{i=1}^n \langle y, \varphi(p_i)\rangle} \, d\lambda^{\times n} = Z^n \theta^{\times n}(A).$$

With $A = \{\frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\}$, we get

$$Z^n \theta^{\times n}\left(\left\{\frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\right\}\right)$$

This suggests we can use the Weak Law of Large Numbers for θ and φ.[3] To do this carefully, we need assumption 2: p takes values in $K \subseteq X$, so it has a barycenter with respect to θ: a unique $x \in K$ such that

$$\int \langle y, \varphi\rangle \, d\theta = \langle y, x\rangle \qquad \forall y \in Y.$$

And now a vector-valued Weak Law of Large Numbers holds: for this x and any weak* neighborhood $U \ni x$, we get

$$\theta^{\times n}\left(\left\{\frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\right\}\right) = 1 - o(1)$$

as $n \to \infty$. As a result, for any weak* neighborhood of this x, we now get

$$\int_{\{\frac{1}{n}\sum_{i=1}^n \varphi(p_i)\in U\}} = Z^n \theta^{\times n}\left(\left\{\frac{1}{n}\sum_{i=1}^n \varphi(p_i) \in U\right\}\right) \ge Z^n e^{o(n)}.$$

[3] This is the key idea of the lower bound proof. It is called the **change of measure** idea.

Insert this to reverse the previous upper bound proof to get an x such that $(x) + \langle y, x\rangle \geq \log Z - \varepsilon$. This gives $s^*(y) = \log Z$.

To remove assumptions 1 and 2, recall that (M, λ) is σ-finite and $X = \bigcup_n K_n$, so for any $a < \int e^{\langle y,\varphi\rangle}\, d\lambda$, there exists a measurable $A \subseteq M$ such that $\infty > \int_A e^{\langle y,\varphi\rangle}\, d\lambda > a$, and $\varphi(A)$ takes values in some K_n. Now run the previous argument with $d\lambda'(p) = \mathbb{1}_A(p)\, d\lambda(p)$ to get that for every ε, there is an x such that $s(x)+\langle y, x\rangle \geq \log a - \varepsilon$. Since $a < \int e^{\langle y,\varphi\rangle}\, d\lambda$ was arbitrary, we get

$$s^*(y) = \log \int e^{\langle y,\varphi\rangle}\, d\lambda,$$

even if this is $+\infty$. $\square$