Math 255A Lecture 11 Notes

Daniel Raban

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1 Applications of Baire's Theorem III: The Uniform Boundedness Principle

1.1 Equicontinuity

Definition 1.1. A subset M of a locally convex space V is **bounded** if every continuous seminorm p is bounded on M: $\sup_{x \in M} p(z) \leq C < \infty$.

When V_1, V_2 are locally convex, we let $\mathcal{L}(V_1, V_2)$ be the space of all linear continuous maps $V_1 \to V_2$.

Definition 1.2. We say that $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is **equicontinuous** if for every neighborhood U_2 of 0 in V_2, there is a neighborhood U_1 of 0 in V_1 such that $x \in U_1 \implies T_x \in U_2$ for every $T \in \Phi$.

If p_j is a continuous seminorm on V_j $(j = 1, 2)$ that $U_j = \{x \in V_j : p_j(x) < 1\}$, then the equicontinuity of Φ means that $p_1(x) < 1 \implies p_2(Tx) < 1$ for all $T \in \Phi$. This implies that $p_2(Tx) \leq p_1(x)$ for all $x \in V_1$ and $T \in \Phi$. We get that $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is equicontinuous if and only if there exist a continuous seminorm p_1, p_2 on V_1, V_2 such that

$$p_2(Tx) \leq p_1(x)$$

for all $x \in V_1$ and $T \in \Phi$.

Remark 1.1. If V_1, V_2 are normed spaces, then $\Phi \subseteq \mathcal{L}(V_1, V_2)$ is equicontinuous means that there exists $C > 0$ such that $\|Tx\|_{V_1} \leq C\|x\|_{V_1}$ for all $x \, in V_1$ and $T \in \Phi$. That is, $\|T\|_{\mathcal{L}(V_1, V_2)} \leq C$ for every $T \in \Phi$.

1.2 Proof of the uniform boundedness principle

Theorem 1.1 (Banach-Steinhaus, uniform boundedness principle)**.** *Let F be a Fréchet space, and let V be a locally convex space. If $\Phi \subseteq \mathcal{L}(F, V)$ is such that for each $x \in F$ the set $\{Tx : T \in \Phi\} \subseteq V$ is bounded, then Φ is equicontinuous. On the other hand, if Φ is not equicontinuous, then the set of all $x \in F$ such that $\{Tx : T \in \Phi\}$ is bounded is a set of the first category.*

Proof. Let U be an open, convex, balanced neighborhood of 0 in V, and consider the set $A = \{x \in F : Tx \in \overline{U}\ \ \forall T \in \Phi\} = \bigcap_{T \in \Phi} T^{-1}(U)$. A is an intersection of closed sets, so it is closed. A is convex as the intersection of convex sets. Also, A is symmetric. Distinguish between two different cases:

1. A has an interior point for any choice of U: Then there exists $x_0 \in F$ and a convex, symmetric neighborhood of 0 in F (call it V) such that $\{x_0\} + V \subseteq A$. Since V is balanced, $\{-x_0\} + V \subseteq A$, and the convexity of V gives
$$V = \frac{1}{2}(\{x_0\} + V) + \frac{1}{2}(\{-x_0\} + V) \subseteq A.$$
We get that $V \subseteq \bigcap_{T \in \Phi} T^{-1}(\overline{U})$, so $T(V) \subseteq \overline{U}$ for all $T \in \phi$. So Φ is equicontinuous.

2. There exists a neighborhood U such that $A = \bigcap_{T \in \Phi} T^{-1}(\overline{U})$ has empty interior. Then $\bigcup_{n=1}^{\infty} nA \subseteq F$ is of the first category, and we claim that it contains the set $\{x \in F : \{Tx : T \in \Phi\} \text{ is bounded}\}$. Take a continuous seminorm p on V such that $\{y : p(y) < 1\} \subseteq U$. Then, since $p(Tx) \leq C$ for all $T \in \Phi$, there exists some $n \in \mathbb{N}$ such that $p(Tx/n) < 1$ for all $T \in \Phi$. So $T(x/n) \in U$ for all $T \in \Phi$, and so $x/n \in A$, which makes $x \in nA$.

To summarize, if $\{Tx : T \in \Phi\}$ is bounded for all $x \in F$, then we are necessarily in case 1 by the open mapping (aka Baire's) theorem. If Φ is not equicontinuous, we are in case 2, and the set $\{x \in F : \{Tx : T \in \Phi\} \text{ is bounded}\}$ is of the first category in F. $\square$

1.3 Applications of the uniform boundedness principle

Corollary 1.1. *Let F be a Fréchet space, and let V be locally convex and metrizable.*[1] *Let $T_j \in \mathcal{L}(F, V)$ be such that for all $x \in F$, the sequence $(T_j x)$ converges in V. Let $Tx = \lim_{j \to \infty} T_j x$. Then $T \in \mathcal{L}(F, V)$.*

Proof. Linearity is preserved under limits, so T is linear. For any continuous seminorm p on V and for all $x \in F$, $p(T_j x) \leq C(x)$ for all j. By the Banach-Steinhaus theorem, (T_j) is equicontinuous. That is, for every continuous seminorm p_2 on V, there exists a continuous seminorm p_1 on F such that $p_2(T_j x) \leq p_1(x)$. for all $x \in F$ and for all j. If we let $j \to \infty$, we get $p_2(Tx) \leq p_1(x)$, so $T \in \mathcal{L}(F, V)$. $\square$

Let $f \in C(\mathbb{R})$ be 2π-periodic. Associated to f is its Fourier series $\sum_{-\infty}^{\infty} c_n(f) e^{inx}$, where
$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$
are the Fourier coefficients. Let $S_N(f, x) = \sum_{-N}^{N} c_n(f) e^{inx}$. Next time, we will show that or all 2π-periodic $f \in C(\mathbb{R})$ outside of a set of the first category, $(S_N(f, x))_{N=1}^{\infty}$ is unbounded for all $x \in \mathbb{Q}$.

[1] The metrizability of V is not actually necessary in this result.