Math 245B Lecture 15 Notes

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1 Locally Convex Topological Vector Spaces

1.1 A note on the uniform boundedness principle

Here is another perspective on the uniform boundedness principle.

Theorem 1.1 (uniform boundedness principle, weaker version)**.** *Suppose that (X, ρ) is a complete metric space, $(\mathcal{Y}, \|\cdot\|)$ is a normed space, and $\mathcal{F} \subseteq C(X, \mathcal{Y})$ is such that for all $x \in X$, $\sup_{f \in \mathcal{F}} \|f(x)\| < \infty$. Then there exists a nonempty, open $U \subseteq X$ such that $\sup\{\|f(x)\| : x \in U, f \in \mathcal{F}\} < \infty$.*

Proof. Let

$$\begin{aligned} E_n &= \{x \in X : \|f(x)\| \leq n \,\forall f \in \mathcal{F}\} \\ &= \bigcap_{f \in \mathcal{F}} \{\|f(\cdot)\| \leq n\}. \end{aligned}$$

Then each E_n is closed, and $X = \bigcup_n E_n$, so by the Baire category theorem. There exists an n such that $E_n^o \neq \varnothing$. □

If X is Banach and $\mathcal{F} \subseteq L(\mathcal{X}, \mathcal{Y})$, then we actually get $\sup_{T \in \mathcal{F}} \|T\|_{\text{op}} < \infty$.

1.2 Topological vector spaces and convexity

Proposition 1.1. *Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. Then*

1. *The addition map $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ sending $(x, y) \mapsto x + y$ is continuous.*
2. *The scalar multiplication map $K \times \mathcal{X} \to \mathcal{X}$ given by $(\lambda, x) \mapsto \lambda x$ is continuous.*

Proof. Use the fact that these maps are continuous over the scalar field. □

Definition 1.1. A **topological vector space** is a pair $(\mathcal{X}, \mathcal{T})$ such that $\mathcal{X}$ is a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{T}$ is a topology on $\mathcal{X}$, and addition and scalar multiplication are continuous.

Definition 1.2. Let $\mathcal{X}$ be a vector space over K. A subset $A \subseteq \mathcal{X}$ is **convex** if $x, y \in A \implies tx + (1-t)y \in A$ for all $t \in [0,1]$.

Definition 1.3. A topological vector space is **locally convex** if the origin in $\mathcal{X}$ has a neighborhood base consisting of convex open sets.

1.3 Topologies induced by seminorms

Theorem 1.2. *Let $(p_\alpha)_\alpha$ be a family of seminorms on $\mathcal{X}$. If $x \in \mathcal{X}$, $\alpha \in A$, and $\varnothing > 0$, define $U_{x,\alpha,\varepsilon} = \{y : p_\alpha(x-y) < \varepsilon\}$. Let $\mathcal{T}$ be the topology generated by the $U_{x,\alpha,\varepsilon}$.*

1. *For $x \in \mathcal{X}$, the set $\{\bigcap_{i=1}^n U_{x,\alpha_i,\varepsilon} : \alpha_i \in A, \varepsilon > 0\}$ is a neighborhood base at x.*
2. *If (x_n) is a sequence in $\mathcal{X}$, then $x_n \to x$ in $\mathcal{T}$ iff $p_\alpha(x_n - x) \to 0$ for all α.*
3. *$(\mathcal{X}, \mathcal{T})$ is a locally convex topological vector space.*

Proof. Here are the idea.

1. Suppose $x \in \bigcap_{i=1}^n U_{x_i,\alpha_i,\delta}$. Then $p_{\alpha_i}(x - x_i) < \delta_i$ for each i. Pick $\varepsilon_i < \delta_i - p_{\alpha_i}(x - x_i)$. Now $x \in U_{x_i,\alpha_i,\delta} \subseteq U_{x_i,\alpha_i,\delta_i}$. Let $\varepsilon = \min(\varepsilon_1, \dots, \varepsilon_n)$.
2. Try it yourself!
3. We must show that addition and multiplication are continuous. Pick $\bigcap_{i=1}^n U_{x+y,\alpha_i,\varepsilon} \ni x + y$. Let $x' \in \bigcap_i U_{x,\alpha,\varepsilon/2}$ and same for y. Multiplication is the same.

   To get local convexity, if $y, z \in U_{x,\alpha,\varepsilon}$ and $t \in [0,1]$, then $p_\alpha(x - ty - (1-tz) \leq p_\alpha(tz - ty) = p_\alpha((1-t)x - (1-tz)) = tp_\alpha(x-y) + (1-t)p_\alpha(x-z) < \varepsilon$. Any intersection of convex sets is convex. □

Example 1.1. Let $\mathbb{R}^{\mathbb{N}}$ have the product topology. Let $p_i(x) = |x_i|$ for each i. These generate the product topology. Alternatively, we could define $\tilde{p}_m(x) = \max_{i \leq m} |x_i|$. Actually, we could also take $r_u(x) = |x_1| + \cdots + |x_i|$. This is a locally convex vector space. However, there is no norm that gives the product topology on $\mathbb{R}^{\mathbb{N}}$.

Example 1.2. There is a locally convex topology on $C(\mathbb{R}^n)$ that captures the notion of locally uniform convergence. Define the seminorms $p_m(f) = \|f|_{\overline{B_m(0)}}\|_u$ for each $m \in \mathbb{N}^+$. Now $f_n \to f$ in $\mathcal{T}$ iff $f_n \to f$ locally uniformly.

Example 1.3. Look at $L^1_{\text{loc}}(\mathbb{R}^n)$, Define the seminorms $P_m(f) = \int_{[-m,m]^n} |f| \, dx$. Then $f_n \to f$ in this topology iff $f_n \mathbb{1}_B \to f \mathbb{1}_B$ in L^1 for all bounded, measurable $B \, subseteq \mathbb{R}^n$.

Here is a non-example.

Example 1.4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and define $L^0(\mu)$ to be the set of equivalence classes of measurable functions $X \to \mathbb{R}$ that agree μ-a.e. Let $\mathcal{T}$ be the topology generated by all sets of the form $V(f, \varepsilon) := \{g \in L^0(\mu) : \mu(\{|f - g| > \varepsilon\}) < \varepsilon\}$, where $f \in L^0(\mu)$ and $\varepsilon > 0$. Then $f_n \to f$ iff $f_n \to f$ in measure, but $\mathcal{T}$ is not locally convex.

In normed spaces, we saw that continuity was equivalent to boundedness. How does this play out in locally convex spaces?

1.4 Continuity in locally convex spaces

Proposition 1.2. *Let $\mathcal{X}, \mathcal{Y}$ be locally convex spaces generated by $(p_\alpha)_\alpha$ and $(q_\beta)_\beta$, respectively. Let $T : \mathcal{X} \to \mathcal{Y}$ be linear. The following are equivalent:*

1. *T is continuous.*
2. *For all $\beta \in B$, there exist $\{\alpha_1, \dots, \alpha_n\} \subseteq A$ and $C > 0$ such that $q_\beta(Tx) \leq C \sum_{i=1}^n p_{\alpha_i}(x)$.*

Proof. (1) $\implies$ (2): Pick $\beta \in B$. If T is continuous, then $\{x : q_\beta(Tx) < 1\}$ is open in $\mathcal{X}$ and contains 0. So there exist $\alpha_1, \dots, \alpha_n \in A$ and $\varepsilon > 0$ such that $\bigcap_{i=1}^n U_{0,\alpha_i,\varepsilon} \subseteq \{q_\beta \circ T < 1\}$. In particular, if $x \in \mathcal{X}$ and $\sum_{i=1}^n p_{\alpha_i}(x) < \varepsilon$, then $x \in U$, so $q_\beta(Tx) < 1$. That is, if $(1/\varepsilon) \sum_{i=1}^n p_{\alpha_i}(x) < 1$, then $q_\beta(Tx) < 1$. By homogeneity of order 1, we get $q_B \circ T \leq (1/\varepsilon) \sum_{i=1}^n p_{\alpha_i}$. $\square$

Example 1.5. Take $\mathbb{R}^{\mathbb{N}}$ with the 3 families of seminorms $p_i(x) = |x_i|$, $q_i(x) = \max_{j \leq i} |x_j|$, and $r_i(x) = |x_1 + \cdots + |x_i|$. If we had that $\mathbb{R}^{\mathbb{N}}$ had a topology given by a norm, then $\|x\| \leq C \sum_{i=1}^n |x_i|$ for some C and n. But then, if we pick x to be nonzero but 0 in the first n coordinates, it has to have norm 0. This is impossible.