Math 255B Lecture 8 Notes

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1 Realizations of Partial Differential Operators

1.1 Maximal and minimal realizations

Last time, we considered the unbounded operator $T = -\Delta$ on $L^2(\mathbb{R}^n)$ with $D(T) = C_0^\infty(\mathbb{R}^n)$. We saw that $\overline{T} = -\Delta$ with domain $D(\overline{T}) = H^2(\mathbb{R}^n) = \{u \in L^2 : \Delta u \in L^2\}$.

Example 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $P = P(x, D_x)$ be a linear partial differential operator with C^∞ coefficients:

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \qquad a_\alpha \in C^\infty, D_{x_j} = \frac{1}{i}\partial_{x_j}.$$

The operator P_Ω on $L^2(\Omega)$ with $D(P_\Omega) = C_0^\infty(\Omega)$ is densely defined and closable: if $u_n \in C_0^\infty(\Omega)$ with $u_n \xrightarrow{L^2} u_n$ and $Pu_n \xrightarrow{L^2} v$, then $v = 0$. The closure of P_Ω, denoted by $P_{\min}$, is called the **minimal realization** of P_Ω with domain $D(P_{\min}) = \{u \in L^2 : \exists u_n \in C_0^\infty \text{ s.t. } u_n \to u, Pu_n \text{ conv. in } L^2\}$.

If $u \in D(P_{\min})$, then $Pu \in L^2(\Omega)$, where P_u is defined in the sense of distributions. So $D(P_{\min}) \subseteq \{u \in L^2 : Pu \in L^2\}$. We also introduce the **maximal realization** $P_{\max}$ of P_Ω, given by $D(P_{\max}) = \{u \in L^2 : P_u \in L^2\}$ with $P_{\max}u = Pu$ for all $u \in D(P_{\max})$. We get $P_\Omega \subseteq P_{\min} \subseteq P_{\max}$, meaning $D(P_\Omega) \subseteq D(P_{\min}) \subseteq D(P_{\max})$ and $P_{\max} = P_{\min}$ on $D(P_{\min})$. Both $P_{\min}$ and $P_{\max}$ are closed.

1.2 Realizations of order 1 partial differential operators with smooth coefficients

Proposition 1.1. *Let $P = \sum_{k=1}^n a_k(x) D_{x_k} + b(x)$ be an operator of order 1 on $\mathbb{R}^n$ with $a_k, b \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\nabla a_k \in L^\infty$. Then the minimal and the maximal realizations of P agree: $D(P_{\min}) = D(P_{\max})$.*

Proof. Let $u \in D(P_{\max})$. We have to show that $u \in D(P_{\min})$; that is, we show there exists a sequence $u_n \in C_0^\infty(\mathbb{R}^n)$ such that $u_n \xrightarrow{L^2} u$ and $Pu_n \xrightarrow{L^2} Pu$. Notice first that if

$\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ near 0 and $\chi_j(x) = \chi(x/j)$ for $j = 1, 2, \ldots$, then $\chi_j u \xrightarrow{L^2} u$. We may write $P(\chi_j u) = \chi_j P u + [P, \chi_j]u$. The first term goes to P_u in L^2, and

$$[P, \chi_j] = (P \circ \chi_j - \chi_j P)u = \sum_{k=1}^{n} a_k(x) \frac{1}{j} (D_{x_k} \chi)(x/j) \xrightarrow{L^2} 0.$$

Thus, when proving that $u \in D(P_{\min})$ can be approximated by C_0^∞ functions, we may assume that u has compact support.

Regularize u: Let $0 \le \varphi \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi = 1$, let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x/\varepsilon)$, and let $J_\varepsilon u = (u * \varphi_\varepsilon)(x) \in C_0^\infty(\mathbb{R}^n)$. Then $I_\varepsilon \xrightarrow{L^2} u$. Compute:

$$P(I_\varepsilon u) = J_\varepsilon P u + [P, J_\varepsilon] u$$

Since P_u is compactly supported and in L^2, the first term goes to P_u in L^2. Let's get rid of the b term:

$$[b, J_\varepsilon]u = \underbrace{b(J_\varepsilon u)}_{\to bu} - \underbrace{J_\varepsilon(bu)}_{bu} \xrightarrow{L^2} 0.$$

Since $[P, J_\varepsilon] = \sum [a_k D_{x_k}, J_\varepsilon] + [b, J_\varepsilon]$ it now suffices to show that $[a_k D_{x_k}, J_\varepsilon]u \to 0$ in L^2 for all $u \in L^2$. This is Friedrich's lemma. $\square$

Lemma 1.1 (Friedrich's lemma)**.** *Let $u \in L^2(\mathbb{R}^n)$ and $a_k \in C_0^\infty(\mathbb{R}^n)$. Then*

$$[a_k D_{x_k}, J_\varepsilon] u \xrightarrow{L^2} 0.$$

Proof. Observe first that if $u \in C_0^\infty(\mathbb{R}^n)$, then

$$[a_k D_{x_k}, J_\varepsilon] u = a_k D_{x_k}(J_\varepsilon u) - J_\varepsilon(a_k D_{x_k} u)$$

Since $a_k D_{x_k} u \in C_0^\infty$, the second term goes to $a_j D_{x_k} u$ in L^2. The first term also goes to $a_k D_{x_k} u$ in L^2. So this goes to 0.

It only remains to show that $\|[a_k D_{x_k}, J_\varepsilon]u\|_{L^2} \le C\|u\|_{L^2}$ for $1 < \varepsilon \le 1$ and $u \in C_0^\infty$. Compute

$$\begin{aligned}
W_\varepsilon(x) &= [a_k D_{x_k}, J_\varepsilon]u(x) \\
&= a_k D_{x_k} \int u(y) \frac{1}{\varepsilon^n} \varphi\left(\frac{x-y}{\varepsilon}\right) - \int a_k(x-y) D_{x_k} u(x-y) \frac{1}{\varepsilon^n} \varphi\left(\frac{y}{\varepsilon}\right)\, dy \\
&= \int a_k(x) u(x - \varepsilon y) \frac{1}{\varepsilon} (D_k \varphi)(y) - \int a_k(x - \varepsilon y) \underbrace{(D_{x_k} u)(x - \varepsilon y)}_{= -1/\varepsilon D_{y_k}(u(x-\varepsilon y))} \varphi(y)\, dy
\end{aligned}$$

Integrate by parts in the second integral.

$$= \int a_k(x) u(x - \varepsilon y) \frac{1}{\varepsilon} (D_{x_k} \varphi)(y) - \int a_k(x - \varepsilon y) u(x - \varepsilon y) D_k \varphi.$$

So we get

$$|W_\varepsilon(x)| \leq C_a |u| * \psi_\varepsilon. \qquad \square$$