Electrical Engineering 229A Lecture 3 Notes

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September 2, 2021

1 Entropy Over Countable Alphabets and Features of Conditional Entropy

1.1 Entropy of distributions over countable sets

Let's adjust our definitions to allow for distributions over countable sets. Let X be a random variable taking values in $\mathscr{X}$, a finite or countably infinite set, and let $(p(x), x \in \mathscr{X})$ be its probability distribution. Its **entropy** is

$$H(X) = H((p(x), x \in \mathscr{X})) = -\sum_x p(x) \log p(x).$$

This is well-defined, even if $\mathscr{X}$ is countably infinite, because all the terms have the same sign.

Remark 1.1. In general, to define $\sum_{x \in \mathscr{X}} a(x)$, where $\mathscr{X}$ is countably infinite, define it to be $(\sum_{x \in \mathscr{X}} a^+(x)) - (\sum_{x \in \mathscr{X}} a^-(x))$, where $a^+(x) := \max(a(x), 0)$ and $a^-(x) := \max(-a(x), 0)$. This definition makes sense when at least one of $\sum_{x \in \mathscr{X}} a^+(x), \sum_{x \in \mathscr{X}} a^-(x)$ is finite.

To avoid subtracting infinities when dealing with entropies over countable sets, proceed as follows: Given a pair of random variables X, Y taking values taking values in (finite or countably infinite) $\mathscr{X}, \mathscr{Y}$, respectively, for each $y \in \mathscr{Y}$, define $H(X \mid Y = y)$ to be the entropy of the conditional distribution of X given $Y = y$:

$$H(X \mid Y = y) = -\sum_{x \in \mathscr{X}} p(x \mid y) \log p(x \mid y).$$

We can alternatively express

$$H(X) = \mathbb{E}\left[\log \frac{1}{p(X)}\right], \qquad \mathbb{E}\left[\log \frac{1}{p(X \mid Y)} \mid Y = y\right],$$

as before.

Define the **conditional entropy** of X given Y to be $\sum_y p(y)H(X \mid Y = y)$, denoted $H(X \mid Y)$. So

$$H(X \mid Y) = \mathbb{E}\left[\log \frac{1}{p(X \mid Y)}\right].$$

Now $H(X,Y) = H(Y)+H(X \mid Y)$ becomes a theorem, called the chain rule for entropy.

Theorem 1.1 (Chain rule)**.**

$$H(X,Y) = H(Y) + H(X \mid Y).$$

Proof.

$$\mathbb{E}[\log \frac{1}{p(X,Y)}] = \mathbb{E}\left[\log \frac{1}{p(Y)}\right] + \mathbb{E}\left[\log \frac{1}{p(X \mid Y)}\right]. \qquad \square$$

We define $D(p \mid\mid q)$ for $(p(x), x \in \mathscr{X})$, $(q(x), x \in \mathscr{X})$ as

$$D(p \mid\mid q) = \sum_x p(x) \log \frac{p(x)}{q(x)}$$

To see that this is well-defined, observe that

$$= \sum_x q(x) \frac{p(x)}{q(x)} \log \frac{p(x)}{q(x)}.$$

Then this is well-defined because the function $u \mapsto u \log u$ defined on $\mathbb{R}^+$ is bounded below.

Then, we can define $I(X;Y) := D(p(x,y) \mid\mid p(x)p(y))$, and our previous definition for mutual information becomes a theorem:

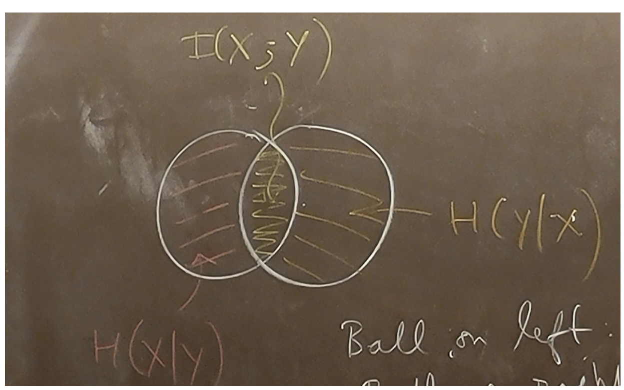
Theorem 1.2.

$$H(X) = I(X,Y) + H(X \mid Y).$$

Proof.

$$\mathbb{E}\left[\log \frac{1}{p(X)}\right] = \mathbb{E}\left[\log \frac{p(X,Y)}{p(X)p(Y)}\right] + \mathbb{E}\left[\log \frac{1}{p(X \mid Y)}\right]. \qquad \square$$

These “theorems” or (X,Y) can be schematically visualized via a Venn diagram.



1.2 Relationship between mutual information and independence

It is important to recognize that the condition for $I(X;Y) = 0$ is $p(x,y) = p(x)p(y)$ for all x, y, i.e. X, Y are independent (denoted $X \amalg Y$). Since $I(X;Y) = H(X) + H(Y) - H(X,Y)$ (inclusion-exclusion),

$$X \amalg Y \iff H(X,Y) = H(X) + H(Y).$$

1.3 General form of the chain rule

If we apply the chain rule twice, we get

$$\begin{aligned} H(X_1, X_2, X_3) &= H(X_1 \mid X_2, X_3) + H(X_2, X_3) \\ &= H(X_1, X_2, X_3) + H(X_2 \mid X_3) + H(X_3). \end{aligned}$$

Similarly, using the notation (X_1^n) to denote $(X_1, \dots, X_n)$, we get the general chain rule:

Theorem 1.3 (Chain rule, general form)**.**

$$H(X_1, \dots, X_n) = H(X_1) + H(X_2 \mid X_1) + H(X_3 \mid X_1, X_2) + \cdots + (X_n \mid X_1^{n-1}).$$

Example 1.1. Consider an urn[1] with 3 balls, two white and 1 red. Pull out all 3 balls in a random order. Let X_1 be the color of the first ball, let X_2 be the color of the second ball, and let X_3 be the color of the third ball. Then

$$H(X_1) = H(X_2) = H(X_3) = \frac{1}{3}\log 3 + \frac{2}{3}\log\frac{3}{2} = \log 3 - \frac{2}{3}.$$

We can also calculate the conditional entropies:

$$\begin{aligned} H(X_2 \mid X_1) &= \mathbb{P}(X_1 = \text{red})H(X_2 \mid X_1 = \text{red}) + \mathbb{P}(X_1 = \text{white})H(X_2 \mid X_1 = \text{white}) \\ &= \frac{2}{3}\log 2 \\ &= \frac{2}{3}. \end{aligned}$$

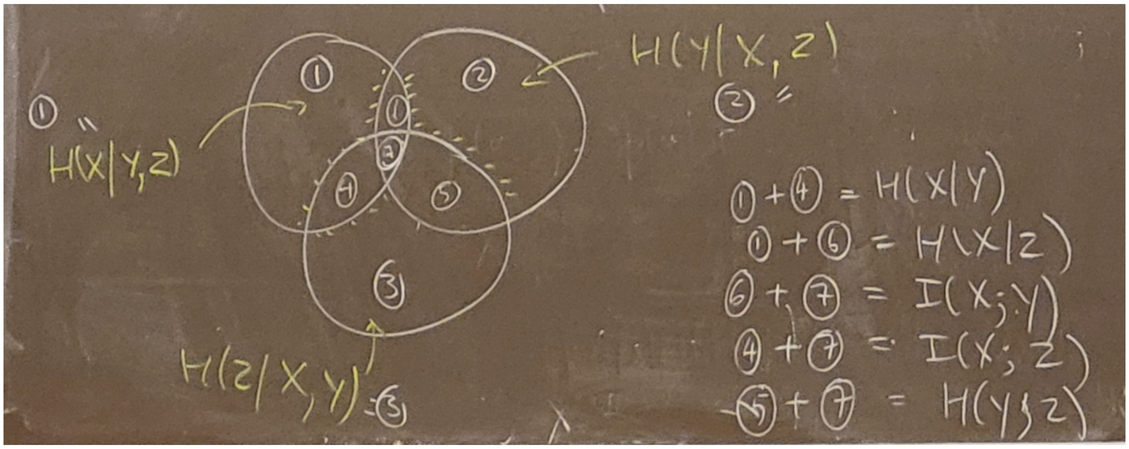
On the other hand, $H(X_3 \mid X_1, X_2) = 0$ because X_3 is determined by X_1, X_2. So the chain rule gives

$$\begin{aligned} H(X_1, X_2, X_3) &= H(X_1) + H(X_2 \mid X_1) + H(X_3 \mid X_1, X_2) \\ &= \log 3 - \frac{2}{3} + \frac{2}{3} + 0 \\ &= \log 3. \end{aligned}$$

[1]No one in the 21st century has ever seen an urn.

1.4 Problems with intuiting mutual information

Here is the Venn diagram for (X_1, X_2, X_3):



What does region 6 represent? This could be $I(X; Y \mid Z)$, the conditional relative entropy between the joint distribution (X, Y), conditioned on Z and the product distribution with the corresponding marginals, conditioned on Z. That is, region 6 is

$$H(X \mid Z) - H(X \mid Y, Z).$$

What does region 7 represent? This region is

$$I(X; Y) - I(X; Y \mid Z).$$

Here is a big problem, not for the math but for any hope of intuition: This can be *negative*. In particular, this says that in the presence of Z, Y can tell you more about X than it does alone.

Example 1.2. Let $X \amalg Y$, with $X \in \{1, -1\}$, $Y \in \{1, -1\}$, $\mathbb{P}(X = 1) = 1/2$, and $\mathbb{P}(Y = 1) = 1/2$. Let $Z = X, Y$ do $Z \in \{1, -1\}$ with $\mathbb{P}(Z = 1) = 1/2$. Then $Y \amalg Z$ and $X \amalg Z$, but X, Y, Z are not mutually independent. Since $X \amalg Y$, we have $I(X; Y) = 0$. However,

$$\begin{aligned} I(X; Y \mid Z) &= \mathbb{P}(Z = 1) I(X; Y \mid Z = 1) + \mathbb{P}(Z = -1) I(X; Y \mid Z = -1) \\ &= \mathbb{P}(Z = 1)(H(X \mid Z = 1) - H(X \mid Y, Z = 1)) \\ &\quad + \mathbb{P}(Z = -1)(H(X \mid Z = -1) - H(X \mid Y, Z = -1)) \end{aligned}$$

Since $X \amalg Z$, $H(X \mid Z = 1) = H(X \mid Z = -1) = H(X) = \log 2 = 1$. Also, $H(X \mid Y, Z = 1) = 0$ because $X = Y$ when $Z = 1$ and $H(X \mid Y, Z = 1) = 0$ because $X = -Y$ when $Z = -1$. So

$$= \frac{1}{2}(1 - 0) + \frac{1}{2}(1 - 0)$$

$$= 1.$$

This is strictly bigger than $I(X;Y)$.

Let's define $I(X;Y \mid Z)$ in a way that works for a countably infinite alphabet. We first define, given $p(x,y,z)$,

$$\sum_z p(z) D(p(x \mid z) \mid\mid p(y \mid z)),$$

denoted $D(p(x \mid z) \mid\mid p(y \mid z) \mid p(z))$ to be the conditional relative entropy of $p(x,z)$ with respect to $p(y,z)$ given z. Then $D(p(x,y \mid z) \mid\mid p(x \mid z)p(y \mid z) \mid p(z))$ would then be $I(X;Y \mid Z)$. That is,

$$\begin{aligned} I(X;Y \mid Z) &:= \sum_z p(z) \sum_{x,y} p(x,y \mid z) \log \frac{p(x,y \mid z)}{p(x \mid z)p(y \mid z)} \\ &= \mathbb{E}\left[\log \frac{p(X,Y \mid Z)}{p(X,Z)p(Y,Z)}\right] \\ &= H(X \mid Z) + H(Y \mid Z) - H(X,Y \mid Z). \end{aligned}$$

Then the chain rule gives

$$I(X;Y \mid Z) = H(X \mid Z) - H(X \mid Y,Z).$$

1.5 The chain rule for relative entropy

Theorem 1.4 (Chain rule for relative entropy)**.**

$$D(p(x,y) \mid\mid q(x,y)) = D(p(x) \mid\mid q(x)) + D(p(y \mid x) \mid\mid q(y \mid x) \mid p(x)).$$

Proof.

$$\begin{aligned} D(p(x,y) \mid\mid q(x,y)) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{q(x,y)} \\ &= \mathbb{E}_p\left[\log \frac{p(X,Y)}{q(X,Y)}\right] \\ &= \mathbb{E}_p\left[\log \frac{p(X)}{q(X)}\right] + \mathbb{E}_p\left[\log \frac{p(Y \mid X)}{q(Y \mid X)}\right] \\ &= D(p(x) \mid\mid q(x)) + D(p(y \mid x) \mid\mid q(y \mid x) \mid p(x)). \end{aligned}$$

□

Similarly, there is a chain rule for mutual information

Theorem 1.5 (Chain rule for mutual information)**.**

$$I(X;Y_1,\dots,Y_n) = I(X;Y_1) + I(X;Y_2 \mid Y_1) + \cdots + I(X;Y_n \mid Y_1^{n-1}).$$