Math 246A Lecture 10 Notes

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1 Exact 1-Forms, Harmonic Functions, and Local Behavior

1.1 Exact 1-forms

A 1-form on Ω is an expression $\omega = P(x,y)dx + Q(x,y)dy$ where $P, Q : \Omega \to \mathbb{C}$ are continuous.

$$\int_\gamma \omega = \int_\gamma P\,dx + \int_\gamma Q\,dy = \int_0^1 P(z(t))x'(t)\,dt + \int_0^1 Q(z(t))y'(t)\,dt,$$

where $\gamma\{z(t) : 0 \le t \le 1\}$.

ω is independent of γ if for all γ,

$$\int_\gamma \omega = \int_{\gamma'} \omega$$

if $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$.

Theorem 1.1. *The following are equivalent.*

1. $\int_\gamma \omega$ *is independent of path*
2. *there exists* $U : \Omega \to \mathbb{C}$, $U \in C^1$ *such that* $\frac{\partial U}{\partial x} = P$ *and* $\frac{\partial U}{\partial y} = Q$.
3. $\int_\gamma \omega = 0$ *for all closed paths* γ *(*$\gamma(0) = \gamma(1)$*) consisting of horizontal and vertical segments.*

Proof. (1) $\implies$ (2): Fix $z_0 \in \Omega$. Let $\gamma_{z_0,z}$ be a path with $\gamma_{z_0,z}(0) = z_0$ and $\gamma_{z_0,z}(1) = z$. Set

$$U(z) = \int_{\gamma_{z_0,z}} \omega.$$

By the hypothesis, U is independent of the path $\gamma_{z_0,z}$. Let $h \in \mathbb{R}$. Then

$$U(z+h) - U(z) = \int_z^{z+h} P(x,y)\,dx.$$

Taking $h \to 0$, we get $\frac{\partial u}{\partial x} = P(x, y)$. Likewise, $\frac{\partial U}{\partial y} = Q$.

(2) $\implies$ (1): We omit this.

(1) $\implies$ (3): (3) is a special case of (1).

(3) $\implies$ (2): Since Ω is connected, we can connect any two points by a polygonal path with horizontal and vertical segments. Proceed with the same argument as before. $\square$

Corollary 1.1. *Let $f \in H(\Omega)$. Then*

$$\int_\gamma f(z)\ dz = 0$$

for all closed paths γ if and only if there exists $F \in H(\Omega)$ such that $F' = f$.

Example 1.1. Let $\Omega = \{z : 0 < |z - a| < R\}$. Let $f(z) = 1/(z - a)$, and let γ be a circle around a. Then

$$\int_\gamma f(z)\, dz = 2\pi i.$$

That is,

$$\int_0^{2\pi} \frac{1}{e^{it} - a} i e^{it}\, dt = 2\pi i.$$

1.2 Harmonic functions

Definition 1.1. A function $u(x, y) : \Omega \to \mathbb{C}$ is **harmonic** if $u \in C^2$, $\Delta u = u_{xx} + u_{yy} = 0$.

Example 1.2. Let $f \in H(\Omega)$ and $u = \mathrm{Re}(f)$, so $f = u+iv$.The Cauchy-Riemann equations say that $u_x = v_y$ and $u_y = -v_x$. So u is harmonic.

Theorem 1.2. *Let $u : \Omega \to \mathbb{R}$ be harmonic. Define the conjugate differential $*du := \frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$.*[1] *There exists $f \in H(\Omega)$ such that $u = \mathrm{Re}(f)$ if and only if $\int_\gamma *du = 0$ for all closed curves γ. This is also equivalent to the same condition but with closed curves consisting of vertical and horizontal segments.*

Proof. These are all equivalent to the existence of a function v such that $v_x = -u_y$ and $v_y = u_x$. $\square$

Example 1.3. Let $\Omega = \{z : r < |z| < R\}$, and let $u = \log |z| = \frac{1}{2} \log(x^2 + y^2)$. Then

$$u_x = \frac{x}{x^2 + y^2}, \qquad u_y = \frac{y}{x^2 + y^2}.$$

Check yourself that $u_{xx} + u_{yy} = 0$. Then $*du = (ydx - xdy)/(x^2 + y^2)$. Let γ be the circle $|z| = (r + R)/2$. Then

$$\int_\gamma *du = i \int_\gamma \frac{1}{z}\, dz.$$

This is not the real part of an analytic function.

[1] This is the Hodge star operator.

1.3 Local behavior of analytic functions

Theorem 1.3. *Let $f, g \in H(\Omega)$ for some domain Ω. Suppose $z_0 \in \Omega$, and $(z_j)_{j\in\mathbb{N}}$ is in $\Omega \setminus \{z_0\}$ with $z_j \to z_0$, and suppose $f(z_j) = g(z_j)$. Then $f = g$ in Ω.*

Proof. Let $h = f - g$. Then h has a power series expansion

$$h(z) = \sum_{n=0}^{\infty} z_n (z - z_0)^n$$

in $\{z : |z - z_0| < R\} \subseteq \Omega$. Let a_{N_0} be the first a_n which is nonzero; if this doesn't exist, $a_n = 0$ for all n, and we are done. Then

$$H(z) = \frac{h(z)}{(z - z_0)^{N_0}} = a_{N_0} + \sum_{k=1}^{\infty} a_{N_0+k} (z - z_0)^k$$

converges in $\{z : |z - z_0| < R\}$. But $H(z_j) = 0$ for all j, and $a_{N_0} = H(z) = 0$. This is a contradiction.

Therefore, the set $U = \{z \in \Omega : h = 0 \text{ on a disc containing } z_0\} \neq \varnothing$. U is open. U is closed relative to Ω because if $(z_j) \in U$ converges to z_0, then the previous argument gives us that $z \in U$. So U is a nonempty open and closed subset of a connected set, Ω. Hence, $U = \Omega$. $\square$