Math 245C Lecture 12 Notes

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1 More Properties of Convolutions and Generalized Young's Inequality

1.1 Uniform continuity and vanishing of convolutions

Let's continue the proof of this statement from last time.

Theorem 1.1. *Let $1 \leq p, q, \leq \infty$ be conjugate exponents. Let $f \in L^p$ and $g \in L^q$. Then*

1. *$f * g(x)$ exists for each $x \in \mathbb{R}^n$, and*

$$|f * g| \leq \|f\|_p \|g\|_q.$$

2. *$f * g$ is uniformly continuous.*

3. *If $1 < p < \infty$, then $f * g \in C_0(\mathbb{R}^n)$.*

Proof. We have already proven the first statement. To prove the second, it suffices to show that

$$\lim_{y \to 0} \|(f * g) - f * g\|_u = 0.$$

Note that if $1 \leq p < \infty$,

$$\tau_y(f * g) - f * g = ((\tau_y f) - f) * g.$$

So

$$\|\tau_y(f * g) - f * g\|_u \leq \|\tau_y f - f\|_p \|g\|_q \xrightarrow{y \to 0} 0,$$

When $p = \infty$, $q = 1$, and we interchange the role of f and g.

Assume $1 < p < \infty$ so that $1 < q < \infty$. Choose $(f_k)_k, (g_k)_k \in C_c(\mathbb{R}^n)$ such that

$$\lim_{k \to \infty} \|f - f_k\|_p = 0 = \lim_{k \to \infty} \|g - g_k\|_q.$$

By the first proposition stated last time, $f_k * g_k \in C_c(\mathbb{R}^n)$. We have

$$f * g - f_k * g_k = f * (g - g_k) + (f - f_k) * g_k,$$

so

$$\|f * g - f_k * g_k\|_u \leq \|f\|_p \|f - f_k\|_q + \|f - f_k\|_p \|g_k\|_q \xrightarrow{k \to \infty} 0.$$

Since $C_0(\mathbb{R}^n)$ is the closure of $C_c(\mathbb{R}^n)$ in the uniform norm, we get the result. $\square$

1.2 Generalized Young's inequality

Theorem 1.2. *Let $1 \leq p, q, r \leq \infty$ be such that $1 + r^{-1} = p^{-1} + q^{-1}$. Let $f \in L^p$.*

1. *(Generalized Young's inequality) If $g \in L^q$, then*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

2. *Further assume $1 < p, q, r < \infty$ and $g \in$ weak L^q, Then there is a constant $C_{p,q}$ independent of f, g such that*

$$\|f * g\|_r \leq_C p, q \|f\|_p [g]_q.$$

3. *If $p = 1$ (so $q = r < \infty$), there exists a constant C_q independent of f such that for any $g \in$ weak L^q,*

$$[f * g]_r \leq C_p \|f\|_1 [g]_q.$$

Proof. For now, we only prove the first statement. Split into cases:

1. $r = \infty$: This is part 1 of the previous theorem (Young's inequality).

2. $p = 1, q = r$: We have already proven this.

3. $1 < p, q, r < \infty$. Since $r^{-1} = p^{-1} + q^{-1} - 1 < q^{-1}$, $q/r \in (0, 1)$. Set $t = 1 - q/r$. Define the operator T as

$$(Tf)(x) = f * g(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy, \qquad K(x, y) = g(x - y).$$

We want to use an interpolation theorem. Let $\frac{1}{p_1} = \frac{1-t}{1} + \frac{t}{p}$ so that $p_1 > p$. By definition,

$$\frac{1}{r} = \frac{1-t}{q} = \frac{1-t}{q} + \frac{t}{\infty}.$$

Note that

$$\|Tf\|_{p_1} = \|f * g\|_{p_1} \leq \|f\|_1 \underbrace{\|g\|_{p_1}}_{:=M_0}, \qquad \|Tf\|_\infty = \|f * g\| \leq \|f\|_p \underbrace{\|g\|_q}_{:=M_1}$$

Applying the theorem, we get

$$\|Tf\|_r \leq M_0^{1-t} M_1^t \|f\|_p = \|g\|_q^{1-t} \|g\|_q^t \|f\|_p = \|f\|_p \|g_q\|.$$ $\square$