Math 250A Lecture 23 Notes

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1 Cyclic Extensions and Cyclotomic Polynomials

1.1 Cyclic extensions

Definition 1.1. A *cyclic extension* is a Galois extension with a cyclic Galois group.

Last time, we determined that a cyclic extension L/K is $K[\sqrt[n]{a}]$ if the characteristic does not divide n and $K[\alpha]$ otherwise, where $\alpha^p - \alpha - b = 0$; also note that the former element is the solution to $a^p - a = 0$. The nice thing about this is that if we know one root, α, thenwe know other roots ($\alpha\zeta^i$ and $\alpha + i$, respectively).

Which polynomials can be "solved by radicals"? What we means is that roots can be written using addition, subtraction, multiplication, and k-th roots. For example, the roots to a quadratic equation $ax^2 + bx + c$ are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.[1]

Theorem 1.1. *The Galois group is solvable iff roots can be given using radicals and Artin-Schrier equations (char > 0).*

Proof. Suppose an equation is solvable by radicals. Assume that the base field K contains all roots of 1 we need. Look at $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq L$, where L is the splitting field of the polynomial. $K_1 = K_0(\sqrt[n]{\alpha_1})$. Look at the Galois groups:

$$G \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq 1.$$

G_2 is normal in G_1, and G_1/G_2 is cyclic. G has a chain of subgroups, each normal in the next, and all quotients are cyclic. So G is solvable.

Suppose G is solvable (and K contains all roots of 1). We have

$$G \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq 1,$$

[1] Mathematicians used to duel for money and prestige, presenting each other with difficult problems to solve. Cardano came up with a general solution for finding roots of degree 4 polynomials, which became a valuable asset for him in these duels.

where G_i is normal in G_{i-1}, and G_{i-1}/G_i is cyclic of prime order. Look at the fields

$$K \subseteq \underbrace{K_1}_{=L^{G_1}} \subseteq \underbrace{K_2}_{=L^{G_2}} \subseteq \cdots \subseteq L.$$

K_{i+1}/K_i is a cyclic Galois extension, so $K_{i+1} = K_i(\sqrt[n]{\alpha_n})$ or Artin-Schrier. $\square$

Example 1.1. Consider $x^5 - 4x + 2$. The Galois group is S_5, which has order 120. The only normal subgroups are 1, A_5, and S_5. This polynomial is not solvable by radicals.

Example 1.2. $x^5 - 2$ is irreducible and of degree 5, but it can be solve by radicals. The Galois group is solvable. The field extensions look like $\mathbb{Q} \subseteq \mathbb{Q}(\zeta) \subseteq \mathbb{Q}(\zeta, \sqrt[5]{2})$. The corresponding groups of the wuotients of the Galois groups are $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$, which are cyclic.

Example 1.3. All polynomials of degree ≤ 4 can be solved by radicals (in characteristic 0), the Galois groups is a subgroup of S_4, so it is solvable. We have

$$S_4 \supseteq A_4 \supseteq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \supseteq 1.$$

1.2 Cyclotomic polynomials

Over $\mathbb{Q}$, the roots of unity are the roots of $x^n - 1 = 0$. How does this factor into irreducibles? Look at $x^{12} - 1$. This is divisible by $x^6 - 1$, $x^4 - 1$, $x^3 - 1$, etc., but these have factors in common.

Definition 1.2. The n-th *cyclotomic polynomial* $\Phi_n(x)$ is the polynomial with roots the primitive n-th roots of unity (order exactly n).

Example 1.4. Let's compute some examples:

n	$\Phi_n(x)$
1	$x - 1$
2	$x + 1$
3	$x^3 + x + 1 = \frac{x^3-1}{x-1}$
4	$x^2 + 1 = \frac{x^4-1}{x^2-1}$
5	$x^4 + x^3 + x^2 + x + 1 = \frac{x^5-1}{x-1}$
6	$x^2 - x + 1 = \frac{(x^6-1)(x-1)}{(x^3-1)(x^2-1)}$

Example 1.5. We have to make sure we're not dividing by factors multiple times, so we must put an $x - 1$ in the numerator:

$$\Phi_{12}(x) = \frac{(x^{12} - 1)(x^2 - 1)}{(x^6 - 1)(x^4 - 1)} = x^4 - x^2 + 1$$

$$x^{12} - 1 = \Phi_{12}(x)\Phi_6(x)\Phi_4(x)\Phi_3(x)\Phi_2(x)\Phi_1(x).$$

Example 1.6. Again, we make sure we don't divide by factors multiple times.

$$\Phi_{15}(x) = \frac{(x^{15}-1)(x-1)}{(x^5-1)(x^3-1)} = x^8 - x^7 + x^5 - x^4 + x^2 - x + 1.$$

If you want to really understand cyclotomic polynomials, try out the following exercise: Find the smallest n such that $\Phi_n(x)$ has a coefficient not 0 or ± 1.[2]

Theorem 1.2. *$\Phi_n(x)$ is irreducible over $\mathbb{Q}$. It's Galois group is $(\mathbb{Z}/n\mathbb{Z})^*$.*

Proof. If b is prime, we have proved this using Eisenstein's criterion. A similar proof works for prime powers. For general n, we use a different argument. The first key idea is to reduce $\pmod p$ for primes p. The second key idea is to use the Frobenius map, $F(t) = t^p$, where the field has characteristic p; F is an automorphism.

Suppose f is an irreducible factor of $\Phi_n(x)$ (over $\mathbb{Q}$). Form $\mathbb{Z}[\zeta] = \mathbb{Z}[x]/f(x)$. This is an integral domain, and the quotient field $\mathbb{Q}(\zeta)$ is generated by a primitive n-th root ζ of 1. Use $\mathbb{Z}$, not $\mathbb{Q}$ to reduce mod p. $\mathbb{Z}[\zeta]$ contains n distinct roots of $x^n - 1$: $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$. Now choose an irreducible factor $g(x)$ of $f(x)$ in $F_p(x)$ (factor $f \pmod p$). In general, $\deg g < \deg f$. The key point is that since $x^n - 1$ has n distinct roots, $nx^{n-1} = \frac{d}{dx}(x^n - 1)$ and $x^n - 1$ are coprime.

Since ζ is a root of g (which is irreducible), ζ^p is also a root of g as $t \mapsto t^p$ is an automorphism of $F_p(\zeta)$. So in $\mathbb{Z}[\zeta]$, ζ^p is also a root of f. Then the map from roots of unity in $\mathbb{Z}[s]$ to roots of unity in $F_p[\zeta]$ is bijective. So if p does not divide n, then the roots of f are closed under the map $\zeta \mapsto \zeta^p$.

Now look at the Galois group of $\mathbb{Z}[\zeta]$. Automorphisms take $\zeta \mapsto \zeta^k$ for k, n coprime, so the Galois group is a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$. The Galois group contains $\zeta \mapsto \zeta^p$ for p, n coprime, which generate $(\mathbb{Z}/n\mathbb{Z})^*$. So the Galois group equals $(\mathbb{Z}/n\mathbb{Z})^*$, so $f = \Phi_n(x)$. $\square$

Definition 1.3. A cyclotomic[3] field is a field generated by roots of unity.

1.3 Applications of cyclotomic polynomials

1.3.1 Primes $p \equiv 1 \pmod n$

Theorem 1.3. *Suppose $n \in \mathbb{Z}$. There are infinitely many primes $p > 0$ with $p \equiv 1 \pmod n$.*[4]

Proof. The idea is to look at the primes P dividing $\Phi_n(a)$ for some a. Suppose p, n are coprime. Then all roots of $\Phi_n(x)$ are distinct mod p. So $\Phi_n(x)$ is coprime to $\Phi_m(x)$ in

[2] You may have to check $n > 100$, but do not just do this brute force. You should do small cases and notice some kind of pattern.

[3] "Cyclo" means "circle," and "tomic" means "cut."

[4] Dirichlet proved this for $p \equiv a \pmod n$ for any a coprime to n, but the proof is not as nice. There seems to be no known way to extend the nice proof to this more general case, which frustrates some people.

$F_p(x)$ for m dividing n. So if $p \mid \Phi_n(a)$, p does not divide $\Phi_m(a)$ for $m \mid n$. This says that if $\Phi_n(a) \equiv 0 \pmod p$, then $\Phi_m(a) \not\equiv 0 \pmod p$ when $m \mid n$. So if $a^n \equiv 1 \pmod p$, then $a^m \neq 1 \pmod p$ for $m \mid n$. So a has order exactly $n \pmod p$, so n divides $|(\mathbb{Z}/p\mathbb{Z})^*| = p-1$, so $p = 1 \pmod n$.

So if $p \mid \Phi_n(a)$, then either $p \mid n$ or $p \equiv 1 \pmod n$. Suppose $p_1, \ldots, p_k$ are $1 \pmod n$. Choose p dividing $\Phi_n(np_1 \cdots p_k)$. $\Phi_n(x) = 1 + x + \cdots$, so this is $1 \pmod n p_1 \cdots p_k$, so p does not divide $p_1 \cdots p_k$. Then p does not divide n. So we have found p, a new prime $\equiv 1 \pmod n$. $\square$

Example 1.7. Let $n = 8$. Then $\Phi_8(a) = a^4 + 1$. if $a = 1$, we get 2, which divides 8. If $a = 2$, we get 9, which is $1 \pmod 8$. If $a = 3$, we get $82 = 41 \times 2$; $41 \equiv 1 \pmod 8$, and $2|8$.

1.3.2 Galois extensions over $\mathbb{Q}$

Recall the hard problem: given finite G, is G a Galois group of $K/\mathbb{Q}$ for some K?

Theorem 1.4. *If G is abelian, there exists some $K/\mathbb{Q}$, such that G is the Galois group of $K/\mathbb{Q}$.*

Proof. Write G as a product of cyclic groups:

$$G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots .$$

Choose distinct primes $p_1 \equiv 1 \pmod n_1$, $p_2 \equiv 1 \pmod n_2, \cdots$. $(\mathbb{Z}/n_1\mathbb{Z})$ is a quotient of $(\mathbb{Z}/p+1\mathbb{Z})^*$. So G is a quotient of $\mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \times \cdots)^* = (\mathbb{Z}/p_1p_2\cdots\mathbb{Z})^*$, which is the Galois group of $x^{p_1,\cdots p_n} - 1$. So any quotient G/H is the Galois group of some extension $K/\mathbb{Q}$. $\square$

Here is a type of converse, which we will not prove.

Theorem 1.5 (Kronecker-Weber-Hilbert)**.** *If K is a Galois extension of $\mathbb{Q}$ with abelian Galois group, then $K \subseteq \mathbb{Q}(\zeta)$ for some root of unity ζ.*

1.3.3 Finite division algebras

Can we find finite analogues of the quaternions $\mathbb{H}$? This is a division algebra that is a "non-commutative field."

Theorem 1.6 (Wedderburn)**.** *Any finite division algebra is a field (commutative).*

Proof. Recall that any group G is the union of its conjugacy classes, which have sizes $|G| / |H|$, where H is a subgroup centralizing a representative element of a conjugacy class.

Let L be a finite division algebra, and let K be its center, a field F_q of order q for some prime power q. Look at the group $G = L^*$, which has order $q - 1$. Suppose $a \in G$. The centralizer of a in L is a subfield of order q^k for some k, so the centralizer of a in G is a subfield of order $q^k - 1$ ($0 \notin G$). So

$$q^{n-1} = q - 1 + \sum_i \frac{q^n - 1}{q^{k_i - 1}},$$

where the conjugacy classes of orders > 1. Note that $k_1 < n$.

Now note that q^{n-1} is divisible by $\Phi_n(q)$. Also note that so is $(q^n - 1)/(q^{k_i - 1})$, as $k_1 < n$. So $q - 1$ is divisible by $\Phi_n(x) = \prod_{i \in (\mathbb{Z}/n\mathbb{Z})^*}(q - \zeta^i)$. But observe that $|q - \zeta_i| > q - 1$ unless $\zeta^i = 1$. So $n = 1$. So $L = K$, which makes L commutative. $\square$

Definition 1.4. The *Brauer group* is the group of isomorphism classes of a finite dimensional division algebras over a field K with center K.

Example 1.8. The Brauer group of $\mathbb{R}$ has 2 elements: $\mathbb{R}$, and $\mathbb{H}$.

If D_1, D_2 are division algebras, $D_1 \otimes_K D_2 \cong M_n(D_3)$ for some n, D_3, where D_3 is the product of D_1, D_2 in the Brauer group.

Remark 1.1. Wedderburn's theorem shows that the Brauer group of a finite field is trivial.

1.4 Norm and trace in finite extensions

Let L/K be a finite extension, and choose $a \in L$. Multiplication by a is a linear transformation from $L \to L$, where L is viewed as a vector space over K.

Definition 1.5. The *trace* of a is defined as the trace of a as a linear transformation. The norm of a is the determinant of a as a linear transformation.

Definition 1.6. The *norm* of a is the determinant of a as a linear transformation.[5]

Example 1.9. Take $\mathbb{C}/\mathbb{R}$ and $a = x + iy \in \mathbb{C}$. A basis for $\mathbb{C}/\mathbb{R}$ is $\{1, i\}$. $a \cdot 1 = x + iy$, and $a \cdot i = -y + ix$. So a is given by the matrix

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}.$$

So the trace of a is $2x$, and the norm is $x^2 + y^2$.

[5] Ignore Lang's definition. Professor Borcherds thinks it is "silly."