Math 259A Extra Note

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1 Student Presentations

In this class, every enrolled student gave a presentation on a topic. Here are notes I took for each presentation.

1.1 Kadison's transitivity theorem

Definition 1.1. If M is a C^*-algebra acting on a Hilbert space H, M is said to act **topologically irreducibly** if H has no proper, closed, invariant subspaces under M. M is said to act **algebraically irreducibly** if H has no proper, invariant subspaces under M.

From the definitions, we have that algebraically irreducible C^*-algebras are topologically irreducible.

Theorem 1.1 (Kadison's transitivity theorem)**.** *If M is topologically irreudicble, it is algebraically irreducible.*

Why is this called the transitivity theorem? We will show that M acts n-transitively on H; i.e. for all linearly independent $x_1, \dots, x_n \in H$ and any $y_1, \dots, y_n \in H$, there is an $A \in M$ such that $Ax_i = y_i$ for all $1 \leq i \leq n$.

Lemma 1.1. *Let $x_1, \dots, x_n \in H$ be orthonormal, and let $z_1, \dots, z_n \in H$ with $\|z_i\| \leq r$. Then there exists an operator $B \in \mathcal{B}(H)$ such that $Bx_i = z_i$ for all i and $\|B\| \leq \sqrt{2n}r$. If there is a selfadjoint T with $Tx_i = z_i$, then we can take B to be self-adjoint.*

Proof. Extend $x_1, \dots, x_n, x_{n+1}, \dots, x_m$ to an orthonormal basis for $\mathbb{C}\{x_1, \dots, x_n, z_1, \dots, z_n\}$ ($m < 2n$). Let $\widetilde{B}$ be the matrix induced by splitting up the z_i according to this basis. Then

$$[\widetilde{B}] = \sqrt{\sum |\alpha_{i,j}|^2} \leq (2n \cdot r^2)^{1/2} = \sqrt{2n}r.$$

Extend it by making it 0 on the orthogonal complement. $\square$

Proof. Assume $x_1, \dots, x_n$ are orthonormal, so $x_1, \dots, x_n \xrightarrow{B} y_1, \dots, y_n$. By changing basis and conjugating by change of basis operators, we can get this result for arbitrary sets. Choose B_0 such that $B_0 x_i = y_i$. Take $A_0 \in M$ such that $\|A_0 x_i = y_i\| \leq \frac{1}{2\sqrt{2n}}$; this is possible because M is topologically irreducible. Choose B_1 such that $B_1 x_i = y_i - A_0 x_i$ and $\|B_1\| \leq \frac{1}{2}$. By Kaplansky's density theorem, choose $A_1 \in M$ such that $\|A_1\| \leq \frac{1}{2}$ and $A_1 x_i - B_1 x_i\| \leq \frac{1}{4\sqrt{2n}}$.

Continue recursively: Suppose we have defined B_k such that $\|B_k\| \leq \frac{1}{2^k}$ and $B_k x_i = y_i - A_0 x_i - A_1 x_i - \cdots - A_{k-1} x_i$. Choose $A_k \in M$ such that $\|A_k\| \leq \frac{1}{2^k}$, $\|A_k x_i - B_k x_i\| \leq \frac{1}{2^{k+1}\sqrt{2n}}$. Choose $\|B_{k+1}\| \leq \frac{1}{2^{k+1}}$ with $B_{k+1} x_i = y_i - A_0 x_i - A_1 x_i - \cdots - A_k x_i$. If $T x_i = y_i$, we can choose the B_k and thus the A_k to be self-adjoint by Kaplansky's theorem. Let $A = \sum_{k=0}^\infty A_k$, This converges in norm to an element of M. Moreover,

$$y_i - A x_i = y_i - \sum_{k=0}^\infty A_k x_i = \lim_k (y_i - a_0 x_i - A_1 x_i - \cdots - A_k x_i) = \lim_k (B_{k+1} x_i) = 0$$

because $\|x_i\| = 1$ and $\|B_{k+1}\| \leq 1/2^{k+1}$. This proves n-transitivity and thus Kadison's theorem. □

1.2 Dixmier's averaging theorem

Theorem 1.2 (Dixmier's averaging theorem)**.** *Let M be a von Neumann algebra with center $Z(M)$. For each $x \in M$, denote by $\overline{K(x)}$ the norm closure of the convex hull of $\{uxu^* : u \in U(M)\}$. Then $\overline{K(x)} \cap Z(M) \neq \varnothing$.*

The bulk of the proof is in the following lemma.

Lemma 1.2. *If $x = x^* \in M$, there is a $u \in U(M)$ and $y = y^* \in Z(M)$ such that*

$$\left\| \frac{1}{2}(x + u^* x u) - y \right\| \leq \frac{3}{4} \|x\|.$$

Proof. Suppose $\|x\| = 1$. Define projections $p = \mathbb{1}_{[0,1]}(x)$ and $q = \mathbb{1}_{[-1,0]}(x)$. By the comparison theorem, there exists some $z \in P(Z(M))$ such that $zq \prec zp$ and $(1-z)p \prec (1-z)q$. Take p_1, p_2, q_1, q_2 such that $zq \sim p_1 \leq p_1 + p_2 = 2p$ and $(1 - zp) \sim q - 1 \leq q_1 + q_2 = (1-z)q$.

Take two partial isometries $v, w \in M$ with $c^* c = w$ and $vv^* = p$, $w^* w = (1-z)p$, $vv^* = q$. Define $u = v + v^* + w + w^* + p_2 + q_2$. Then

$$\begin{aligned} u &= v^* v + vv^* + w^8 w + ww^* + q_2 + p_2 \\ &= zq + p_2 + (1-z)p + q_1 + q_2 + p_2 \\ &= p + q \\ &= 1. \end{aligned}$$

Also,

$$u^*p_1u = zq, \qquad u^*q_1u = (1-z)p \qquad u^*p_2u = p_2,$$
$$u^*zqu = p_1, \qquad u^*(1-z)pu = q_1, \qquad u^*q_2u = q_2.$$

We have $-zq \le zx \le zp = p_1 + p_2$. So

$$\begin{aligned}
&\implies -p_1 \le zu^*xu \le zq + p_2 \\
&\implies -\frac{1}{2}(zq + p_1) \le \frac{1}{2}(zx + zu^*xu) \le \frac{1}{2}zq + p_1 + p_1 \\
&\implies \frac{1}{2}z \le \frac{1}{2}(2x + zu^*xu) \le z \\
&\implies -\frac{3}{4} \le \frac{1}{2}(2x - zu^*xu) - \frac{1}{4}z \le \frac{3}{4}z.
\end{aligned}$$

Similarly, repeating this with $1 - z$ gives

$$-\frac{3}{4}(1-z) \le \frac{1}{2}((1-z)x + (1-z)u^*xu) + \frac{1}{4}(1-z) \le \frac{3}{4}(1-z).$$

If we add these together, we get

$$\left\| \frac{1}{2}(z + u^*xu) - \frac{2z-1}{4} \right\| \le \frac{3}{4}. \qquad \square$$

Proof. Let K denote the set of maps $\alpha : M \to M$ of the form $\alpha(x) = \sum_{i=1}^n c_i u_i^* x u_i$ with $u_i \in U(M)$, $\sum_i c_i = 1$ and $c_i \ge 0$. For general $x \in M$ denote $a_0 = \mathrm{Re}(x)$ and $b_0 = \mathrm{Im}(z)$. By the lemma, there exist some $u \in U(M)$ and $y_1 = y_1^* \in Z(M)$ with

$$\left\| \frac{1}{2}(a_0 + u^*a_0u) - y_1 \right\| \le \frac{3}{4}\|a_0\|.$$

Denote $\alpha_1(x) = \frac{1}{2}(x + u^*xu)$ and $a_1 = \alpha_1(a_0)$. Use the lemma again on $a_1 - y_1$. Continue inductively.

Given any $\varepsilon > 0$, we can find $\alpha \in K$ and $y \in Z(M)$ for which $\|\alpha(a_0) - y\| < \varepsilon$. Similarly, given this α, we can find $\beta \in K$ and $z \in Z(M)$ for which $\|\beta(\alpha(b_0)) - z\| < \varepsilon$. Thus,

$$\|\beta(\alpha(a_0)) - y\| = \|\beta(\alpha(a_0) - y)\| \le \|\alpha(a_0) - y\| < \varepsilon.$$

Therefore,

$$\|\beta(\alpha(x)) - (y + iz)\| < 2\varepsilon$$

The problem is that $y + iz$ might be dependent on ε. To fix that, we define a sequence $(\Gamma_n) \subseteq K$ and $(z_n) \subseteq Z(M)$ such that if $x_0 = x$ and $x_n = \gamma_n(x_{n-1})$, we have $\|x_n - z_n\| \le \frac{1}{2^n}$. Thus,

$$\|x_{n+1} - x_n\| = \|\gamma_{n+1}(x_n - z_n) - (x_n - z_n)\| \le \|\gamma_{n+1}(x_n - z_n)\| + \|x_n - z_n\| < \frac{1}{2^{n-1}}.$$

Thus, $x_n \to x$ and $z_n \to x$, so $x \in \overline{K(x)} \cap Z(M)$. $\square$

1.3 The Ryll-Nardzewski fixed point theorem

I gave this presentation. See my notes on the subject.

1.4 $\ell^1(\mathbb{Z})$ is not a C^*-algebra

Theorem 1.3. *$\ell^1(\mathbb{Z})$ is not a C^*-algebra.*

Theorem 1.4. *Let $\varphi \in C(S^1)$ with $\varphi(z) = 0$ for all $z \in S^1$. Then $\widehat{\varphi} \in \ell^\infty(\mathbb{Z})$, $\widehat{x\varphi} \in \ell^\infty(\mathbb{Z})$.*

These are consequences of the following fact.

Theorem 1.5. *Let $\Omega(\ell^1(\mathbb{Z}))$ be the maximal ideal space of $\ell^1(\mathbb{Z})$. $\Omega(\ell^1(\mathbb{Z})) \cong S^1$, where $\Omega(\ell^1(\mathbb{Z}))$ is equipped with the weak topology in $(\ell^1(\mathbb{Z}))^* \cong \ell^\infty(\mathbb{Z})$.*

Proof. Let i denote the natural isomorphism from $(\ell^1(\mathbb{Z}))^* \to \ell^\infty$. We claim that $i(\Omega(\ell^1(\mathbb{Z}))) = \{\alpha \in \ell^\infty(\mathbb{Z}) : \alpha(m+n) = \alpha(m)\alpha(n)\}$.

For any $\varphi \in \Omega(\ell^1(\mathbb{Z}))$ with $i(\varphi) = \alpha\varphi$,

$$\alpha\varphi(m+n) = \sum \delta_{m+n}\alpha_\varphi = \varphi(\delta_{m+n}) = \varphi(\delta_m * \delta_n) = \varphi(\delta_m)\varphi(\delta_n) = \alpha_\varphi(m) \cdot \alpha_\varphi(n).$$

On the other hand, if $\alpha(m+n) = \alpha(m) \cdot \alpha(n)$, then

$$\begin{aligned}
i^{-1}(\alpha)(f * g) &= \sum (f * g)\alpha \sum_i \sum_j f(i-j)g(j)\alpha(i) \\
&= \sum_j \sum_i f(i-j)g(j)\alpha(i-j)\alpha(j) \\
&= \langle g, \alpha \rangle \langle f, \alpha \rangle \\
&= i^{-1}(\alpha(f)) \cdot i^{-1}(\alpha(g)).
\end{aligned}$$

Now observe that $\alpha(m) = (\alpha(1))^m$, which gives a bijection $\widehat{\mathbb{Z}} \to A^1$ by $\alpha \mapsto \alpha(1)$. These spaces are compact, so we only need to check continuity of the map to get a homeomorphism. If $\alpha_i \xrightarrow{wk} \alpha$, then

$$\alpha_i(1) = \sum \delta_1 \alpha_i \to \sum \delta_1 \alpha = \alpha(1).$$

So we get that $S^1 \cong \widehat{\mathbb{Z}} \cong \Omega(\ell^1(\mathbb{Z}))$. □

Now we can show that $\ell^1(\mathbb{Z})$ is not a C^*-algebra.

Proof. Assume $\ell^1(\mathbb{Z})$ is a C^*-algebra. Then by the Gelfand transform, $\ell^1(\mathbb{Z}) \cong C(S^1)$. Then $\Gamma(\ell^1(\mathbb{Z})) = \{\varphi \in C(S^1) : \widehat{\varphi} \in \ell^1(\mathbb{Z})\}$.

We claim that $\widehat{\Gamma}(f) = f$, where $f \in \ell^1(\mathbb{Z})$. If $\Gamma(f) \in C^1(S^1)$, then $\Gamma(f)(z) = \langle f, z^n \rangle = \sum f(n) z^n$. We check

$$\widehat{\Gamma(f)}(n) = \frac{1}{2\pi} \int_0^{2\pi} \sum f(n) e^{inx}) e^{inx}\, dx = f(n).$$

We now claim that if $\varphi \in C(S^1)$ then $\widehat{\varphi} \in \ell^1(\mathbb{Z})$. We have

$${}^\wedge(\Gamma(\widehat{\varphi}) - \varphi) - \widehat{\Gamma(\widehat{\varphi})} - \widehat{\varphi} = 0$$

by the first claim. $\square$

Here is the proof of the other result.

Proof. $\Gamma(f)$ is invertible if and only if f is invertible. Then if $\varphi = \Gamma(f)$, then $1/\varphi = \Gamma(f^{-1})$. $\square$