Math 254A Lecture 12 Notes

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1 Duality: Deriving Properties of s Via Properties of s^*

1.1 Recap

Our setup from last time is a system of n "non-interacting particles." M is the phase space $\mathbb{R}^3 \times \mathbb{R}^3$, $\lambda = m_3 \times m_3$ is a σ-finite but not finite measure, and $\varphi : M \to [0, \infty)$ is $\varphi(r,p) = \varphi_{\mathrm{pot}}(r) + \frac{1}{2}|p|^2$ (potential energy + kinetic energy). We will assume φ is lower bounded and normalize φ so that $\min \varphi = \operatorname{ess\,min} \varphi = 0$. Then, for open interval $I \subseteq \mathbb{R}$,

$$\lambda^{\times n}\left(\left\{(r_1, \dots, r_n, p_1, \dots, p_n) : \frac{1}{n}\Phi_n(r_1, \dots, p_n) := \frac{1}{n}\sum_{i=1}^n \varphi(r_i, p_i) \in I\right\}\right)$$
$$= \exp\left(n \cdot \sup_{E \in I} s(E) + o(n)\right)$$

The intuition is that

$$\lambda^{\times n}\left(\left\{\frac{1}{n}\Phi_n \approx E\right\}\right) \approx e^{n \cdot s(E) + o(n)}.$$

We also have that

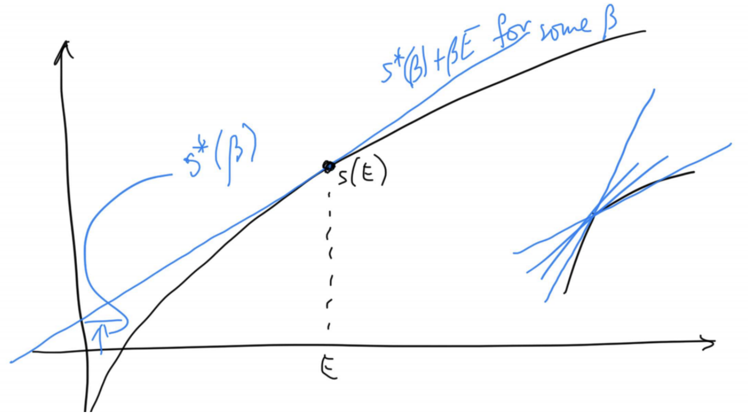
$$s(E) = \inf_{\beta \in \mathbb{R}} \{s^*(\beta) + \beta E\},$$

$$s^*(\beta) = \sup_{E \geq 0} \{s(E) - \beta E\} = \log \int e^{-\beta\varphi} \, d\lambda,$$

which is assumed to be $< \infty$ for all $\beta > 0$. Next, we need to understand where these inf and sup are achieved.

1.2 Supporting tangents and conjugacy between β and E

Definition 1.1. A **supporting tangent** to s at E is a line touching the graph of s at E and bounding from above.

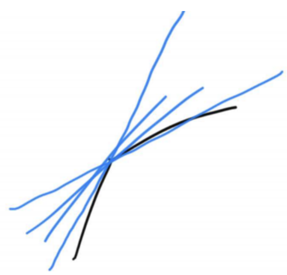


Its slope β must satisfy

$$s(E') \leq s(E) + \beta(E' - E) \qquad \forall E'.$$

Equivalently,

$$D_+ s(E) \leq \beta \leq D_- s(E)$$



or

$$s(E) = s^*(\beta) + \beta E.$$

Up to a sign, this last equation is *symmetric* between "conjugate variables" β and E:

$$s(E) + (-s^*(\beta)) = \beta E.$$

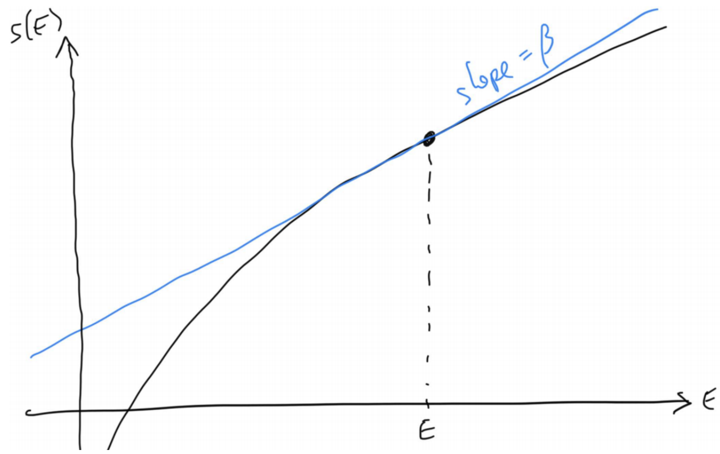
Here, s and $(-s^*)$ are both upper semicontinuous, and they play the same role in this equation. So, by symmetry, β is a slope for a supporting tangent line to s at E iff E is a slope for a supporting tangent line to $-s^*$ at β. That is,

$$D_+ s(E) \leq \beta \leq D_- s(E) \iff D_- s^*(\beta) \leq -E \leq D_+ s^*(\beta).$$

This is the key observation for deriving smoothness and differentiability properties of s from those of s^*.

1.3 Leveraging conjugacy to prove differntiability and strict convexity of s

Here is our picture relating s and s^*:



Here are some main features to be proved about this picture:

Proposition 1.1.

$$s(E) \to \begin{cases} \infty & \textit{as } E \to \infty \\ \log \lambda(\{\varphi = 0\}) & \textit{as } E \downarrow 0. \end{cases}$$

The first case implies s is strictly increasing. Also, it could be in this picture (if $\lambda(\{\varphi = 0\}) = 0$) that the graph gets steeper and steeper and never hits the vertical axis.

Proof. First, we have

$$s(E) = \inf_{\beta > 0} \Big\{ \underbrace{\log \int e^{-\beta \varphi} \, d\lambda}_{s^*(\beta)} + \beta E \Big\}.$$

First, here are some properties of s^*:

$$s^*(\beta) \to \begin{cases} \log \lambda(\{\varphi = 0\}) & \text{as } \beta \to \infty \\ \infty & \text{as } \beta \downarrow 0. \end{cases}$$

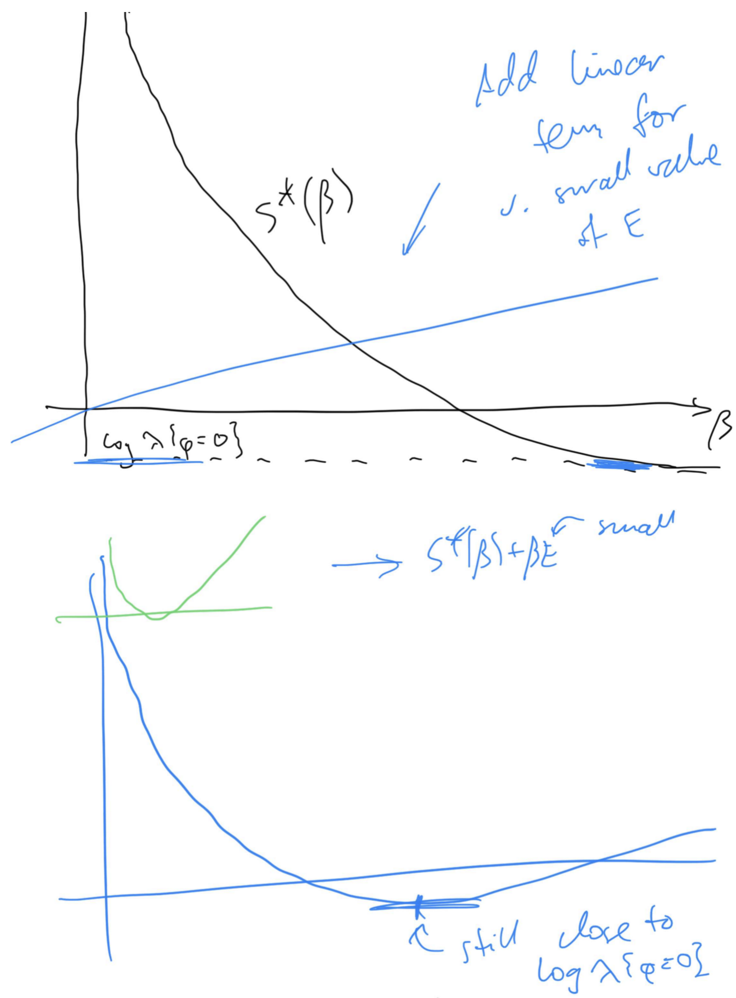
The first of these follows since $\varphi \geq 0$, $\beta_1 > \beta_2 > 0$ implies $e^{-\beta_1 \varphi} \leq e^{-\beta_2 \varphi}$. As $\beta \to \infty$, $e^{-\beta \varphi} \downarrow \mathbb{1}_{\{\varphi = 0\}}$. By the dominated convergence theorem, $s^*(\beta) \to \log \int \mathbb{1}_{\{\varphi = 0\}} \, d\lambda = \log \lambda\{\varphi = 0\}$.

Secondly, we have $\lambda(\{\varphi \leq M\}) \to \infty$ as $M \to \infty$, so for all $K > 0$, pick M so that $\lambda(\{\varphi \leq M\}) \geq K$. Now pick β so small that $e^{-\beta M} \geq 1/2$, so now

$$s^*($$

$$\begin{aligned}
beta) &= \log \int e^{-\beta\varphi}\, d\lambda \\
&= \log \int_{\{\varphi \le M\}} e^{-\beta M}\, d\lambda \\
&\ge \log \left(\frac{1}{2} \lambda(\{\varphi \le M\}) \right) \\
&\ge \log \left(\frac{K}{2} \right) \\
&\xrightarrow{K \to \infty} \infty.
\end{aligned}$$

For the rest, here are some pictures (which can be justified with some εs and δs):



So $s(E) = \min_{\beta>0}\{s^*(\beta) + \beta E\}$ is close to $\inf_{\beta>0} s^*(\beta) = \log \lambda(\{\varphi = 0\}) = \lim_{E\downarrow 0} s(E)$ if E is small enough. Similarly, if E is very big,

$$s(E) = \min_{\beta>0}\{s^*(\beta) + \beta E\} \to \infty.$$

as $E \to \infty$. $\square$

Lemma 1.1. *s is differentiable on $(0, \infty)$ (i.e. no corners).*

Proof. s is differentiable at E iff $D_+s(E) = D_-s(E) = s'(E)$. By our previous discussion, this is equivalent to if there is only one slope β for a supporting tangent at E. This is equivalent to if for this E, the solution to $s(E)+(-s^*(\beta)) = \beta E$ in β is unique. Equivalently, this is when $\inf_{\beta>0}\{s^*(\beta) + \beta E\}$ is achieved at exactly one β. This occurs precisely when $s^*(\cdot) + E(\cdot)$ is strictly concave where the minimum is achieved. Quantifying over E this tells us that s is differentiable if and only if s^* is strictly convex.

Now let's show that s^* is strictly convex: Suppose $\alpha > \beta > 0$ and $0 < t < 1$. Then

$$s^*(t\alpha + (1-t)\beta) = \log \int e^{(-t\alpha-(1-t)\beta)\varphi}\, d\lambda$$

Apply Hölder's inequlity with exponents $1/t$ and $1/(1-t)$:

$$\leq t \log \int e^{-\alpha\varphi}\, d\lambda + (1-t) \log \int e^{-\beta\varphi}\, d\lambda,$$

with equality iff $e^{-\alpha\varphi}$ is a constant multiple of $e^{-\beta\varphi}$. This is possible only if φ is constant a.e., which is not true. $\square$

Proposition 1.2. *s is strictly concave on $[0, \infty)$.*

Proof. As before, this is equivalent to $s^*(\beta) = \log \int e^{-\beta\varphi}\, d\lambda$ being differentiable. This holds by differentiating under the integral. $\square$