Math 255A Lecture 5 Notes

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October 8, 2018

1 Müntz's Theorem and the Poisson Equation

1.1 Müntz's theorem

First, let's finish our proof of Runge's theorem.

Theorem 1.1 (Runge)**.** *Let $K \subseteq \mathbb{C}$ be a compact set with $K^c = \mathbb{C} \setminus K$ connected. Let f be a function which is holomorphic is a neighborhood of K. Then for any $\varepsilon > 0$, there exists a holomorphic polynomial g such that $|f(z) - g(z)| \leq \varepsilon$ for all $z \in K$.*

Proof. We had a measure μ on K such that $\int_K z^n \, d\mu(z) = 0$ for all $n \in \mathbb{N}$, and we got was

$$\int_K f(z) \, d\mu(z) = -\frac{1}{\pi} \iint_{\mathbb{C} \setminus K} \frac{\partial \psi}{\partial \overline{\zeta}} f(\zeta) M(\zeta) \, L(d\zeta),$$

where

$$M(\zeta) = \int_K \frac{1}{\zeta - z} \, d\mu(z).$$

To finish the proof, it suffices to show that $M = 0$ on $\mathbb{C} \setminus K$. Consider the Laurent expansion of M at ∞:

$$M(\zeta) = \sum_{j=0}^{\infty} \frac{1}{\zeta^{j+1}} \int_K z^j \, d\mu(z) = \sum_{j=0}^{\infty} \frac{1}{\zeta^{j+1}} 0 = 0.$$

Then $M = 0$ for large $|\zeta|$, and hence $M = 0$ in all of $\mathbb{C} \setminus K$ because $\mathbb{C} \setminus K$ is connected. $\square$

Theorem 1.2 (Müntz)**.** *Let $(\lambda_j)_{j \in \mathbb{N}}$ be a sequence of distinct positive real numbers such that $\lambda_j \to \infty$ as $j \to \infty$. Then the closed linear span of the functions $1, t^{\lambda_1} t^{\lambda_2}, \dots\}$ in $C([0,1])$ is equal to $C[0,1])$ if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Proof. We shall only prove the sufficiency of the series condition. By the spanning criterion, we have to show the following: if μ is a finite complex Borel measure on $[0,1]$ such that $\int_{[0,1]} 1\,d\mu(t) = \int_{[0,1]} t^{\lambda_j}\,d\mu(t) = 0$ for all j, then for all $f \in C[0,1]$, $\int f\,d\mu = 0$. We claim that if $\int_{[0,1]} t^{\lambda_j}\,d\mu(t) = 0$ for all j, then $\int_{[0,1]} t^k\,d\mu(t) = 0$ for all $k = 1, 2, \ldots$. The claim implies the result by the Weierstrass approximation theorem.

We may assume that μ is concentrated on $(0,1]$ since the integrands t^k all vanish at $t = 0$. Consider the function $F(\zeta) = \int_{[0,1]} t^\zeta\,d\mu(t)$, where $\zeta \in \mathbb{C}$ with $\operatorname{Re}(\zeta) > 0$. Then F is bounded and holomorphic in $\operatorname{Re}(\zeta) > 0$. We have $F(\lambda_j) = 0$ for all j. Map the right half plane onto the disc: $G(z) = F(\zeta)$, where $\zeta = (1+z)/(1-z)$ for $|z| < 1$. Then $G \in \operatorname{Hol}(|z| < 1)$ is bounded, and $G(\alpha_j) = 0$, where $\alpha_j = (\lambda_j - 1)/(\lambda_j + 1) \to 1$.

Recall now Jensen's formula, which says that if $f \in \operatorname{Hol}(|z| < 1)$ such that $f(0) \neq 0$, and $(\alpha_k)_{j=1}^N$ are the zeros of f (counting multiplicities) such that $|\alpha_j| \leq r < 1$, then

$$\sum_{|\alpha_j| \leq r} \log \frac{r}{|\alpha_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\varphi})|\,d\varphi - \log|f(0)|.$$

So if f is bounded, the right hand side is $O(1)$ as $r \to 1$. Using that $\log(t) \geq 1 - t$ for $t \geq 0$, we get

$$\sum_{|\alpha_j| \leq r} (r - |\alpha_j|) \leq C$$

for $r < 1$. Letting $r \to 1$, we get that if $f \in \operatorname{Hol}(|z| < 1)$ is bounded and not identically 0, the zeros (α_j) of f satisfy $\sum (1 - |\alpha_j|) < \infty$.

In our case, $\alpha_j = (\lambda_j - 1)/(\lambda_j + 1)$, and we may assume that $\alpha_j > 0$. Then

$$\sum (1 - |\alpha_j|) = \sum (1 - \frac{\lambda_j - 1}{\lambda_j + 1}) = \sum \frac{2}{\lambda_{j+1}} = \infty.$$

Thus, $G = 0$, so $F(\zeta) = \int_{[0,1]} t^3\,d\mu(t) = 0$ for $\operatorname{Re}(\zeta) > 0$. □

1.2 Solving the Poisson equation using Hahn-Banach

We will try to solve the Poisson equation. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $f \in L^2(\Omega)$ be real-valued. Let $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ be the Laplacian. We would like to solve the equation $\Delta u = f$ in some sense. The existence of solutions to this equation can be reduced to the proof of an inequality.

Proposition 1.1. *There exists a constant $A > 0$ such that for any $\varphi \in C_0^2(\Omega)$ (C^2 functions on Ω with compact support), we have*

$$\|\varphi\|_{L^2(\Omega)} \leq A \|\Delta\varphi\|_{L^2(\Omega)}.$$

We will prove this next time.

Remark 1.1. An inequality of this form holds for all differential operator with constant coefficients, in place of Δ.