Math 245C Lecture 2 Notes

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1 Integration With Push-Forward Measures and Distribution Functions

1.1 Integration with push-forward measures

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $0 < p < \infty$.

Definition 1.1. Let $(Y, \mathcal{N}, \nu)$ be another measure space, and let T be a measurable map. We say that T **pushes** μ **forward to** ν if $\nu(B) = \mu(T^{-1}(B))$ for all $B \in \mathcal{N}$.

Proposition 1.1. *T pushes μ forward to ν if and only if*

$$\int_Y f\,d\nu = \int_X f \circ T\,d\mu,$$

for all $f \in L^1(\nu)$.

Proof. We can restate the condition in the definition as

$$\int_Y f\,d\nu = \int_X f \circ T\,d\mu,$$

where $f = \mathbb{1}_B$. By linearity, this holds for when f is a simple function. This means that if $f : Y \to [0, \infty]$ is ν-measurable, then $\int_Y f\,d\nu = \int_X f \circ T\,d\mu$. By linearity, this holds for all $f \in L^1$. $\square$

Recall that if $F \in \mathrm{NBV}(\mathbb{R})$, there exists a unique Borel complex measure such that $\mu_F((-\infty, x]) = F(x)$.

Proposition 1.2. *Assume $f : X \to \mathbb{C}$ is measurable and $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$. If $\phi : (0, \infty) \to \mathbb{R}$ is Borel, then*

$$\int_X \phi \circ |f|\,d\mu = \int_0^\infty \phi(\alpha)\,d\mu_{-\lambda_f}(\alpha).$$

In other words, $|f|_*\mu = \mu_{-\lambda_f}$.

Proof. It suffices to show the proposition when $\phi = \mathbb{1}_E$ and $E \subseteq (0, \infty)$ is Borel. In fact, it is not a loss of generality to further assume $E = (a, b]$, where $-\infty < a < b < \infty$. We need to check that $\mu_{-\lambda_f}(E) = \mu(x : \{|f(x)| \in E\})$. We have

$$\begin{aligned} \mu(\{x : |f(x)| \in E\}) &= \mu(\{x : a < |f(x)| \leq b\}) \\ &= \mu(\{x : a < |f(x)|\}) - \mu(\{x : b < |f(x)|\}) \\ &= \lambda_f(a) - \lambda_f(b) \\ &= \mu_{-\lambda_f}((a, b]) \\ &= \mu_{-\lambda_f}(E). \end{aligned}$$

□

1.2 Integration with respect to distribution functions

Proposition 1.3. *Let $f : X \to \mathbb{C}$ be a simple function such that $f \in L^p(\mu)$.*

1. *For all $0 < \varepsilon_1 < \varepsilon_2$, $\lambda_f \in \mathrm{BV}([\varepsilon_1, \varepsilon_2])$.*
2. $$\int_X |f|^p \, d\mu(x) = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.$$

Here is a wrong proof: Let $\phi(t) = |t|^p$. Then, using integration by parts,

$$\int_X \phi(|f|) \, d\mu = \int_0^\infty \phi(\alpha) \, d(-\lambda_f) = -\int_0^\infty \underbrace{\phi'(\alpha)}_{p\alpha^{p-1}}(-\lambda_f) \, d\alpha + -\lambda_f \phi|_0^\infty .$$

Proof. Write $f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$, where the A_i are measurable and pairwise disjoint. We can also assume a_i are distinct. We have $|f|^p = \sum_{i=1} |a_i|^p \mathbb{1}_{A_i}$, so

$$\sum_{i=1}^n |a_i|^p \mu(A_i) = \int_X |f|^p \, d\mu < \infty.$$

Let $I = \{a_i : a_i \neq 0\}$. Then $\|f\|_{L^p}^p \geq |a_i|^p \mu(A_i)$ for all $a_i \in I$. So

$$\mu\left(\bigcup_{a_i \in I} A_i\right) \leq \|f\|_{L^p}^p \sum_{a_I \in I} \frac{1}{|a_i|^p} =: \gamma.$$

If $\alpha > \max_{i=1,\dots,n} |a_i| := \overline{\gamma}$, then $\lambda_f(\alpha) = 0$. If $\alpha > 0$, $\{|f| < \alpha\} \subseteq \bigcup_{a_i \in I} A_i$, so $\lambda_f(\alpha) \leq \gamma$. If $\varepsilon_1 < \varepsilon_2 < \infty$, then $\lambda_f|_{[\varepsilon_1, \varepsilon_2]}$ has range contained in $[0, \gamma]$. This proves that $\lambda_f \in \mathrm{BV}([\varepsilon_1, \varepsilon_2])$.

Let $b < \overline{\gamma}$. Then by the previous proposition,

$$\int_X |f|^p \, d\mu = \int_0^\infty \alpha^p \, d\mu_{-\lambda_f}(\alpha)$$

$$
\begin{aligned}
&= \int_0^b \alpha^p \, d\mu_{-\lambda_f}(\alpha) \\
&= \lim_{\varepsilon_1 \to 0} \int_{\varepsilon_1}^b \alpha^p \, d(-\lambda_f) \\
&= \lim_{\varepsilon_1 \to 0} - \int_{\varepsilon_1}^b \alpha^{p-1}(-\lambda_f)(\alpha) \, d\alpha + \cancel{[-\alpha^p \lambda_f(\alpha)]_{\varepsilon_1}^b} \\
&= p \int_0^b \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.
\end{aligned}
$$

Indeed, since λ_f is bounded, $\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0$. □

Corollary 1.1. *Let $f \in L^p(\mu)$. Then*

$$\int_X |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.$$

Proof. Let $f_n : X \to \mathbb{C}$ be a sequence of simple functions such that $|f_n| \leq |f_n| \leq |f|$ for all n and $|lim_n |f_n| = |f|$. By the previous proposition,

$$\int_X |f_n|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_{f_n}(\alpha) \, d\alpha.$$

Since $\lambda_{f_n} \leq \lambda_{f_{n+1}} \leq \lambda_f$ and $\lim_n \lambda_{f_n} = \lambda_f$, we apply the dominated convergence theorem to conclude the proof, □