

Math 222A Lecture 10 Notes

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1 The Hopf-Lax Solution to Hamilton-Jacobi Equations

1.1 The Hamiltonian in classical mechanics

Last time, we were solving the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, Du) = 0 \\ u(0) = u_0 \end{cases}$$

using the calculus of variations:

$$u(x, t) = \inf_{y(t)=x} \int_0^t L(y(s), \dot{y}(s)) ds + u_0(y(0)).$$

Theorem 1.1. *The function u solves the Hamilton-Jacobi equation for as long as the solutions stay smooth.*

In the proof, we had the convex duality

$$H(x, p) = \max_q p \cdot q - L(x, q)$$

for the Hamiltonian $H(x, p)$ and the Lagrangian $L(x, q)$.

Example 1.1. Here is an example from classical mechanics. Consider the Lagrangian

$$L(x, q) = \frac{1}{2}mq^2 - \phi(x),$$

where $\frac{1}{2}mq^2$ is kinetic energy and $\phi(x)$ is potential energy. Then

$$H(x, p) = \sup_q p \cdot q - \frac{1}{2}mq^2 + \phi(x)$$

Complete the square to get

$$\begin{aligned} &= \sup_q \frac{1}{2m} p^2 - \frac{1}{2m} (p - mq)^2 + \phi(x) \\ &= \frac{1}{2m} p^2 + \phi(x) \end{aligned}$$

In the physical interpretation, the Hamiltonian $H(x, p)$ plays the role of the energy of the system.

1.2 The Hopf-Lax formula

Now we will consider a special case, where $L = L(q)$ does not depend on x (and consequently $H = H(p)$). Assume that L, H are strictly convex and coercive (i.e. $\lim_{q \rightarrow \infty} \frac{L(q)}{|q|} = \infty$). The Euler-Lagrange equation tells us that

$$L_x(y, \dot{y}) + \frac{d}{dt} L_q(y, \dot{y}) = 0.$$

So we get that $L_q(\dot{y})$ is constant. Since L_q is a local diffeomorphism, we get that \dot{y} is constant. That is, the solutions to the Euler-Lagrange equation are linear.

We claim that fixing the endpoints $y(0), y(t)$, the minimum is attained for linear trajectories.

Theorem 1.2 (Hopf-Lax formula¹). *If $L = L(q)$ is convex, then*

$$u(x, t) = \inf_y u_0(y) + tL\left(\frac{x - y}{t}\right).$$

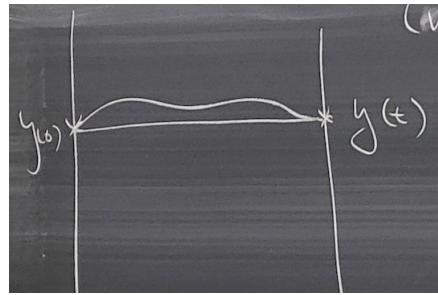
Proof. Since

$$\int_0^t \dot{y}(s) ds = y(t) - y(0),$$

we can average to get

$$\frac{1}{t} \int_0^t \dot{y}(s) ds = \frac{y(t) - y(0)}{t},$$

where the right hand side is the average velocity for a straight path.



¹This is from the 50s or the 60s. Professor Tataru was actually able to meet Lax a few times.

Then

$$\int_0^t L(\dot{y}(s)) ds = t \cdot \frac{1}{t} \int_0^t L(\dot{y}(s)) ds$$

Convexity says that $L\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(L(x) + L(y))$. More generally, we get that $L(hx + (1-h)y) \leq hL(x) + (1-h)L(y)$. If we use n variables, this is $L\left(\frac{x_1+\dots+x_n}{n}\right) \leq \frac{1}{n}(L(x_1) + \dots + L(x_n))$. If we increase the number of variables, this says that $L(\text{avg}(z)) \leq \text{avg}(L(z(s)))$, where we are taking average integrals. This is called **Jensen's inequality**, and it gives us

$$\geq t \cdot L\left(\frac{y(t) - y(0)}{t}\right)$$

In other words, the cost of an arbitrary path is \geq the cost of the straight path. \square

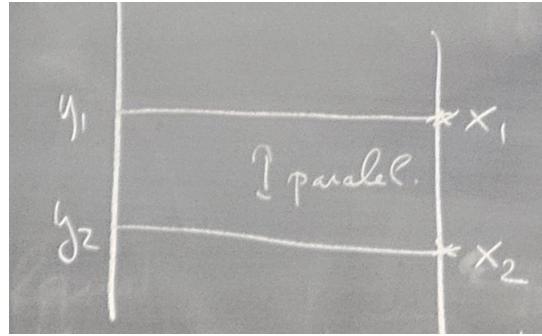
We are not done yet. We still need to minimize $u_0(y(0))$ over the choice of $y(0)$.

1.3 Properties of the Hopf-Lax solution

Assume L is convex and coercive. For simplicity, also assume that u_0 is bounded. Observe that if $t > 0$, then we can restrict $q = \frac{x-y}{t}$ to a compact set. So if u_0 is also continuous, then the infimum is attained.

Proposition 1.1. *If $u_0 \in \text{Lip}$, then $u \in \text{Lip}$.*

Proof. Here is a proof by picture. Suppose we have points x_1, x_2 , and we want to compare $u(x_1)$ and $u(x_2)$. It is enough to consider parallel trajectories with y_1, y_2 .



Take $x_1 - y_1 = x_2 - y_2$. Then $y_1 - y_2 = x_1 - x_2$. We have

$$u(x_1, t) = \inf_{y_1} u_0(y_1) + tL\left(\frac{x_1 - y_1}{t}\right),$$

$$u(x_2, t) = \inf_{y_2} u_0(y_2) + tL\left(\frac{x_2 - y_2}{t}\right).$$

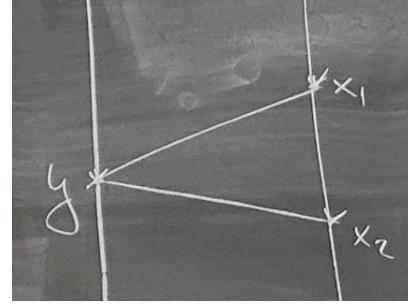
Using the Lipschitz condition, $|u_0(y_1) - u_0(y_2)| \leq L|y_1 - y_2| = L|x_1 - x_2|$. So the conclusion is that

$$|u(x_1, t) - u(x_2, t)| \leq L|x_1 - x_2|. \quad \square$$

What if we don't assume u is Lipschitz? Can we still conclude that u is Lipschitz?

Proposition 1.2. *If u_0 is continuous, then $u(t)$ is Lipschitz.*

Proof. In this case, compare x_1 and x_2 to the same y :



We have

$$u(x_1) = \inf_y u_0(y) + L \left(\frac{x_1 - y}{t} \right),$$

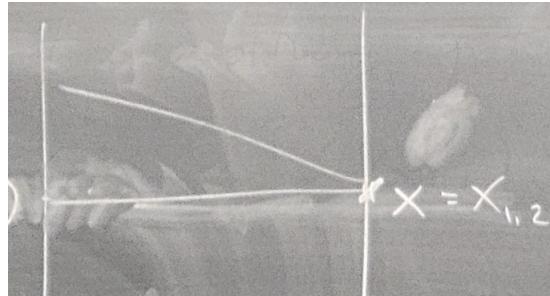
$$u(x_2) = \inf_y u_0(y) + L \left(\frac{x_2 - y}{t} \right).$$

The difference

$$\left| L \left(\frac{x_1 - y}{t} \right) - L \left(\frac{x_2 - y}{t} \right) \right| \leq C \cdot \frac{|x_1 - x_2|}{t},$$

where the Lipschitz constant $C = C(t)$ in the set where $\frac{x_1 - y}{t}$ and $\frac{x_2 - y}{t}$ live.

Where should we look? $\frac{y - x_1}{t}, \frac{y - x_2}{t}$ cannot be too large. Let $x = x_1 = x_2$, and compare the straight trajectory to an arbitrary trajectory.

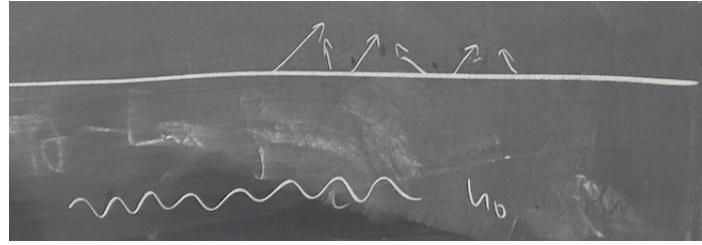


The oblique trajectory loses if $u_0(x) + tL(0) \leq u_0(y) + tL\left(\frac{x-y}{t}\right)$. This is when $\frac{2M}{t} \leq L\left(\frac{x-y}{t}\right)$. So we can restrict to y such that $L\left(\frac{x-y}{t}\right) \leq \frac{2M}{t}$. So $\frac{x-y}{t}$ is in a compact set depending on t . Then the conclusion is that

$$|u(x_1, t) - u(x_2, t)| \leq C(t) \cdot \frac{|x_1 - x_2|}{t},$$

where $C(t)$ is the Lipschitz constant for L in the region $L(q) \leq \frac{C}{t}$. This Lipschitz constant goes to ∞ as $t \rightarrow 0$. \square

In terms of the Hamilton-Jacobi equation, there will be lots of velocities with different speeds. So there is only an average velocity that survives.



We say that this PDE has a mild **regularizing effect**.

1.4 Almost everywhere solvability of the Hamilton-Jacobi equation

Recall the following theorem from real analysis (which requires measure theory).

Theorem 1.3. *If u is a Lipschitz function, then u is differentiable almost everywhere.*

So we get the following conclusion.

Corollary 1.1. *The solution u is differentiable almost everywhere.*

Proposition 1.3. *Let (x, t) be a differentiability point for u . Then the Hamilton-Jacobi equation holds at (x, t) .*

Corollary 1.2. *The function u solves the Hamilton-Jacobi equation almost everywhere.*

Let's prove the proposition.

Proof. We can think of the Hamilton-Jacobi equation as proving two separate inequalities. If our trajectory is optimal, then it is optimal if we only look at the trajectory at a shorter

length of time. Look at the optimal trajectory, ending at y and with slope $\frac{x-y}{t}$.



Then

$$u(x, t) = u_0(y) + tL\left(\frac{x-y}{t}\right),$$

so

$$u\left(x - h\frac{x-y}{t}, t-h\right) = u_0(y) + (t-h)L\left(\frac{x-y}{t}\right)$$

The first equation tells us that y is the optimal trajectory for (x, t) , and the second says that y is optimal for $(x \cdot h\frac{x-y}{t}, t-h)$.

Let $q = \frac{x-y}{t}$. Then dividing by h gives

$$\frac{u(x, t) - h(x - hq, t - h)}{h} = hL(q).$$

Letting $h \rightarrow 0$ gives

$$\partial_x u \cdot q + \partial_t u = L(q).$$

So for this special q we have chosen,

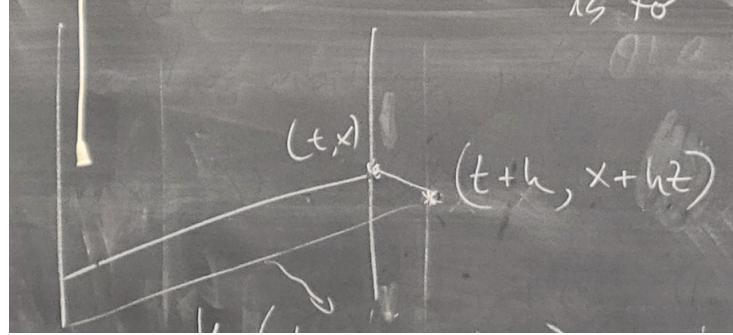
$$\partial_t u + \partial_x u \cdot q - L(q) = 0.$$

We want to think of this in terms of the Legendre transform. Since $H(p) = \sup p \cdot q - L(q)$ the latter half of our equation, $\partial_x u \cdot q - L(q)$, is $\leq H(\partial_x u)$. So we get

$$\partial_t u + H(\partial_x u) \geq 0.$$

Now we want to produce the other inequality. Notice that for the previous inequality, it was enough to work with a specific value of q , whereas for this direction, we will need to look at all values of q . Instead of looking at the past of (t, x) , look at the future of (t, x) .

Our picture looks like



One trajectory from $(t + h, x + hz)$ is to go through x , but this may not be optimal. So

$$u(t + h, x + hz) \leq u(t, x) + \underbrace{hL(z)}_{=\int_t^{t+h} L(z) ds}$$

As before, subtract the right hand side, divide by h , and let $h \rightarrow 0$. Then we get

$$\frac{h(t + h, x + hz) - u(t, x)}{h} \leq L(t) \implies \partial_t u + \partial_x u z \leq L(z).$$

So we have proven that for all z ,

$$\partial_t u + \partial_x u \cdot z \leq 0.$$

Taking the supremum over all z , we get

$$\partial_t u + H(\partial_x u) \leq 0. \quad \square$$

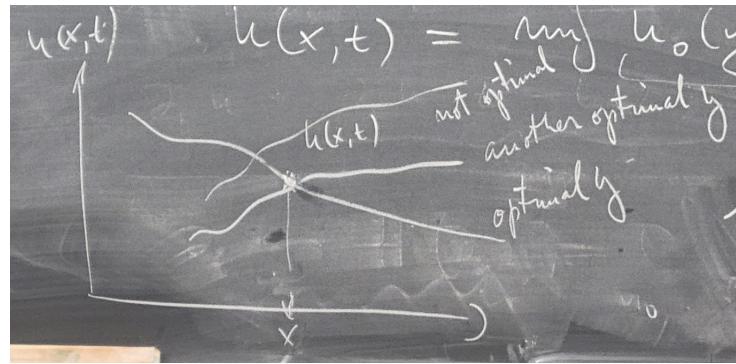
Now we will tell a story. The details are in Evans' book, but the overall story is more important. We want to ask a question: Does solving the Hamilton-Jacobi equation almost everywhere suffice to guarantee uniqueness for Hamilton-Jacobi? Equivalently, does this guarantee that u is the minimal value function? The answer is no.

Are there other interesting properties for the function u ? Look at the Hopf-Lax formula

$$u(x, t) = \inf u_0(y) = tL\left(\frac{x-y}{t}\right).$$

Observe that this is an infimum of functions which are smooth in x . We can compare what

this looks like for different optimal/nonoptimal y :



Since we are taking a minimum, we can see that our curve could have a corner pointing upwards, but a corner pointing downwards is not possible. This points to a concavity property of our solution.

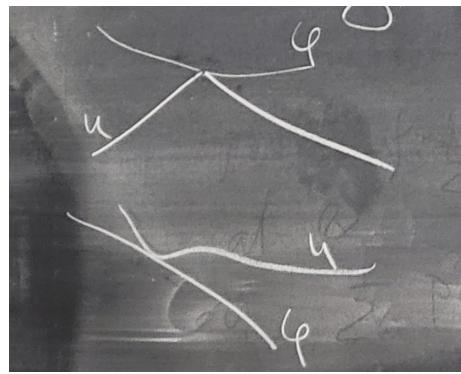
Proposition 1.4. u is semiconcave.

Concave means that $u(t,x) \geq \frac{u(t,x+y) + u(t,x-y)}{2}$. **Semiconcave** means that

$$u(t,x) \geq \frac{u(t,x+y) + u(t,x-y)}{2} - c \cdot (x-y)^2.$$

Theorem 1.4. The optimal value function u is the unique semiconcave solution to the Hamilton-Jacobi equation.

The proof is in Evans, but it is a little hard to follow. There is a better way to do things! Instead of plugging in u to check whether it satisfies the equation, if we have a corner, draw a tangent test function φ with $\varphi_t + H(\partial_x \phi) \geq 0$ or $\varphi_t + H(\partial_x \phi) \leq 0$.



These are called **viscosity solutions** for Hamilton-Jacobi equations.