Math 142 Lecture 25 Notes

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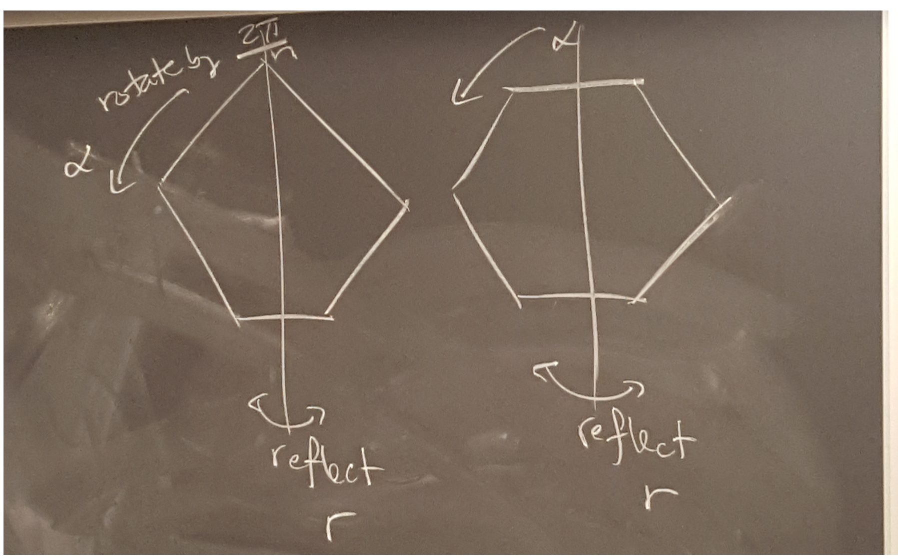
1 Knot Colorings in Algebraic Topology

1.1 Knot groups

Our goal is to relate n-colorings to algebraic topology. We will show that n-colorings of K correspond (almost) to homomorphisms $\pi_1(\mathbb{R}^3 \backslash K) \to D_{2n}$; we will actually be overcounting by n. D_{2n} is the group of symmetries of a regular n-gon,[1]

$$D_{2n} = \left\langle r, \alpha \mid \alpha r = r\alpha^{-1}, r^2 = 1, \alpha^n = 1 \right\rangle.$$

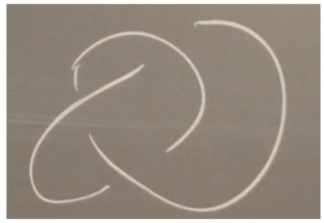


We will work out $\pi_1(\mathbb{R}^3 \setminus K)$ to be the following.

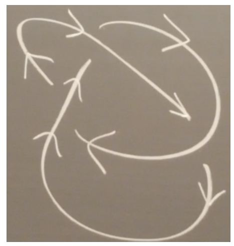
Definition 1.1. The *knot group* of K is constructed by:

[1] Some people call this group D_n. You should always be clear with your notation when discussing this group.

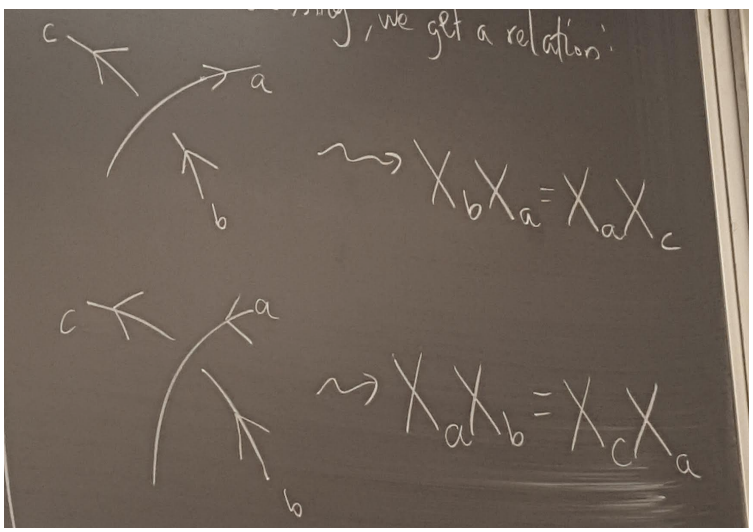
1. Take a nice projection of K.



2. Put a direction on the knot.



3. Number the arcs $1, \ldots, n$.
4. $\pi_1(\mathbb{R}^3 \setminus K)$ is generated by $x_1, \ldots, x_n$ (one for each arc).
5. For each crossing, we get a relation:



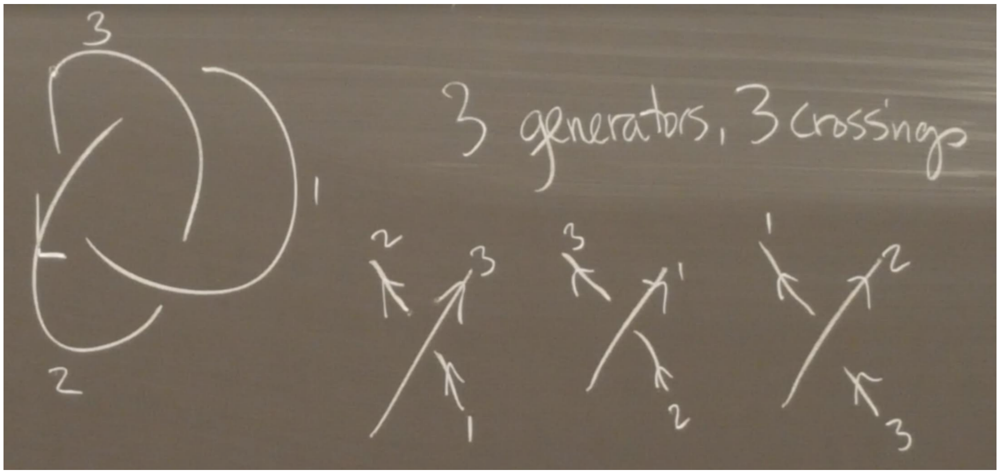
This says that x_c is a conjugate of x_b.

We can also define this for oriented links, but we will not prove that here.

Example 1.1. Let K be the unknot. K has one arc and no crossings, so

$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1 \rangle \cong \mathbb{Z}.$$

Example 1.2. Let K be the trefoil knot. This has 3 arcs and 3 crossings.



$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1, x_2, x_3 \mid x_1x_3 = x_3x_2, x_2x_1 = x_1x_3, x_3x_2 = x_2x_1 \rangle$$

Note that $x_3 = x_2x_1x_2^{-1}$, so x_3 is a redundant generator.

$$\cong \left\langle x_1, x_2 \mid x_1x_2x_1x_2^{-1} = x_2x_1x_2^{-1}x_2, x_2x_1 = x_1x_2x_1x_2^{-1} \right\rangle$$

Write $a = x_1$, $b = x_2$, and simplify.

$$\cong \langle a, b \mid aba = bab, bab = aba \rangle$$

These relations are redundant.

$$\cong \langle a, b \mid aba = bab \rangle .$$

In general, it is hard to tell apart groups like this by their generators. Before, we used Abelianization to tell apart fundamental groups. However, that approach doesn't work here.

Proposition 1.1. *Let K be a knot. Then*

$$\mathrm{Ab}(\pi_1(\mathbb{R}^3 \setminus K)) \cong \mathbb{Z}.$$

Proof. Every relation is $x_a x_b = x_b x_c$. This reorders to $x_a x_b = x_c x_b$, which gives us that $x_a = x_c$. Check that all generators are identified this way. $\square$

Regardless, we are looking at the right object.

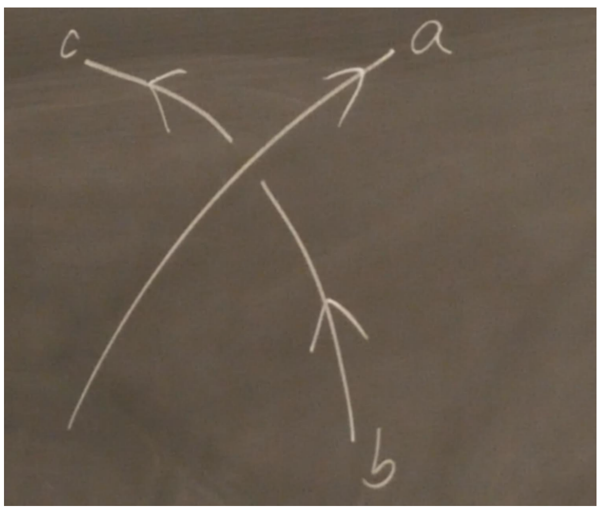
Theorem 1.1 (Gordon-Luecke,1989)**.** *$\pi_1(\mathbb{R}^3 \setminus K_1) \cong \pi_1(\mathbb{R}^3 \setminus K_2)$ iff K_1 and K_2 are equivalent.*

So this fundamental group determines the knot up to isotopy or mirroring.

1.2 Correspondence between knot colorings and fundamental group homomorphisms

Theorem 1.2. *Let K be a knot. Then there is a correspondence between n-colorings of K and homomorphisms $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ (except for n homomorphisms).*

Proof. Given an n-coloring that sends arc i to color $\ell_i \in \{1, \ldots, n\}$, construct the homomorphism $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$, via $x_i \mapsto r\alpha^{\ell_i}$. We need to check the relations. At a crossing



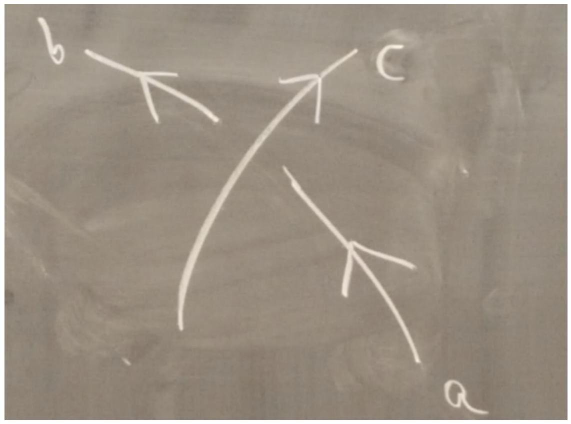
we know that $x_b x_a = x_a x_c$. We get

$$x_b x_a \mapsto r\alpha^{\ell_b} r\alpha^{\ell_a} = rr\alpha^{-\ell_b}\alpha^{\ell_a} = r^2\alpha^{\ell_a - \ell_b} = \alpha^{\ell_a - \ell_b},$$

$$x_a x_c \mapsto r\alpha^{\ell_a} r\alpha^{\ell_c} = rr\alpha^{-\ell_a}\alpha^{\ell_c} = r^2\alpha^{\ell_c - \ell_a} = \alpha^{\ell_c - \ell_a}.$$

We want $\alpha^{\ell_a - \ell_b} = \alpha^{\ell_c - \ell_a}$; i.e. we need $\ell_a - \ell_b \equiv \ell_c - \ell_a \pmod n$. This is equivalent to $2\ell_a \equiv \ell_b + \ell_c \pmod n$, which is the requirement for an n-coloring.

The argument is similar for the other crossing type. So we have a homomorphism. We can also go backward (homomorphism to coloring), but only if for all i, $\phi(x_i) = r\alpha^{\ell_i}$ for some ℓ_i; ℓ_i will be the color of arc i.If $\phi(x_a) = \alpha^{\ell_i}$ for some a, we need $\phi(x_a x_c) = \phi(x_c x_b)$.



Count how many reflections we have on the left hand side and on the right.

LHS # refls.	$\phi(x_c)$
1	$r\alpha^{\ell_c}$
0	α^{ℓ_c}

RHS # refls.	$\phi(x_c)$	$\phi(x_b)$
0	$r\alpha^{\ell_c}$	$r\alpha^{\ell_b}$
1	$r\alpha^{\ell_c}$	α^{ℓ_b}
1	α^{ℓ_c}	$r\alpha^{\ell_b}$
0	α^{ℓ_c}	α^{ℓ_b}

In either case, $\phi(x_b) = \alpha^{\ell_b}$ for some ℓ_b. Follow our knot around, doing the same analysis at every crossing. Then $\phi(x_i) = \alpha^{\ell_i}$ for all i.

Now $\phi(x_a x_b) = \phi(x_c x_b)$ iff $\alpha^{\ell_a}\alpha^{\ell_c} = \alpha^{\ell_c}\alpha^{\ell_b}$. This is the condition that $\ell_a \equiv \ell_b \pmod{n}$. Check that this makes $\phi(x_i) = \phi(x_j)$ for al i, j. So ignore these homomorphisms (there are n of them). Hence,

$$|\{n\text{-colorings of } K\}| = |\{\text{homomorphisms } \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}\}| - n. \qquad \square$$

Here is an application of this result.

Example 1.3. Let K be a knot siting on a torus that gives the element $3, 4 \in \pi_1(T^2)$; this goes 3 times around the torus and 4 times through the center hole. We can show that $\pi_1(\mathbb{R}^3 \setminus K) \cong \langle a, b, a^3 = b^4 \rangle)$.

K is not n colorable for any n. If $\phi : \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ is a homomorphism such that $\phi(a) = r\alpha^i$ and $\phi(b) = r\alpha^j$, we need that $\phi(a)^3 = \phi(b)^4$. But $\phi(a)^3 = r\alpha^i r\alpha^i r\alpha^i = r\alpha^i$ and $r\alpha^j r\alpha^j r\alpha^j r\alpha^j = 1$. So we have a contradiction.