

Math 254A Lecture 10 Notes

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1 Introduction to Statistical Physics

1.1 Recap

Last time, we mentioned that in the case $X = M(K)$, λ is finite on K , and $\varphi : K \rightarrow X$ sends $p \mapsto \delta_p$ for a compact metric space K , we get

$$s(\mu) = \begin{cases} -\int \frac{d\mu}{d\lambda} \log \frac{d\mu}{d\lambda} d\lambda & \mu \in P, \mu \ll \lambda \\ -\infty & \text{else.} \end{cases}$$

For example, if $K = A$ is finite and $\lambda \in P(A)$, we get

$$\lambda^{\times n}(\{a \in A^n : p_a \approx \mu\}) e^{-nD(\mu||\lambda)+o(n)},$$

where

$$D(\mu||\lambda) ::= \sum_a \frac{\mu(a)}{\lambda(a)} \log \frac{\mu(a)}{\lambda(a)}.$$

1.2 Quick intro to statistical physics

Imagine n point particles located in space with positions $r_i(t)$ for $1 \leq i \leq n$. The laws of motion give

$$\frac{d^2 r_i(t)}{dt^2} = \frac{1}{m_i} F_i,$$

where the F_i are forces. A typical simple example of forces is

$$F_i = -\nabla V(r_i(t))$$

for some **potential energy** $V : \mathbb{R}^3 \rightarrow \mathbb{R}$.

How big is n ? Avogadro's number, $\approx 6 \times 10^{23}$, is roughly the number of Carbon atoms in the graphite in a pack of pencils. This tells us that the number of particles is way too large to be solved in the usual way. The founding ansatz in statistical physics is that in

a macroscopic physical system, once a few key quantities are fixed, “almost” all possible microscopic states of the system look macroscopically the same.

In an example of classical particles, the total energy is conserved:¹

$$\Phi(r_i, p_i) = \underbrace{\sum_{i=1}^n V(r_i)}_{\text{potential energy}} + \underbrace{\sum_{i=1}^n \frac{1}{2m_i} |p_i|^2}_{\text{kinetic energy}},$$

where $p_i = m_i \frac{dr_i}{dt}$ is the momentum of a particle, and m_i is the mass.

The **phase space** is $(\mathbb{R}^3)^n \times (\mathbb{R}^3)^n$, which keeps track of the position and momentum of every particle. The **micro-state** moves around in the level set

$$\Omega(E) = \{(r_1, \dots, r_n, p_1, \dots, p_n) : \Phi(r_i, p_i) = E\}$$

if the system is isolated, i.e. no energy exchange occurs with the surroundings.

We want to restrict to cases where these sets are bounded, which we can do if we impose a restriction on the set of potential energies (for example, if the particles occupy a **potential well**). Any other macroscopic observable quantity will be a function of $(r_1, \dots, r_n, p_1, \dots, p_n)$. We want the functions that arise this way to stay close to some constant value on most of $\Omega(E)$. By “most,” we mean most in the sense of the measure on $\Omega(E)$; instead of using $6n - 1$ dimensional Hausdorff measure on $\Omega(E)$ (which is a $6n - 1$ dimensional manifold), it is easier in practice to thicken the manifold (loosen the restriction on the exact energy E) and use $6n$ dimensional Lebesgue measure.

In the previous formula for the energy, these particles are not interacting. This is the easiest case where we can solve things explicitly. In reality, molecules have pairwise interactions, so we could include a term like $\sum_{i,j} V_{i,j}(r_i, r_j)$. We will first deal with the noninteracting particles case, and once we have developed the tools to talk about these things, we will then deal with systems with interactions.

We call these observables **concentrated** (probability terminology) or **self-averaging** (physics terminology).

¹In practice, you can’t directly measure or control the amount of total energy in a system. You have to control factors that would affect the energy in the system, such as temperature and pressure.