Math 245B Lecture 25 Notes

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1 Introduction to Hilbert Spaces

1.1 Motivation

Consider $(X, \mathcal{M}, \mu) = (\{1, \dots, n\}, \mathscr{P}(X), \#)$. Then $L^p_{\mathbb{C}}(\mu) = \ell^p(n) = \mathbb{C}^n$. In this case, we are specifying a specific norm:

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

These give different shapes for the unit ball; try drawing the unit ball for different values of p when $n = 2$.

A linear functional φ on $\mathbb{C}^n$ has the form

$$\varphi(x) = \sum_{i=1}^n x_i \overline{y}_i = \langle x, y \rangle$$

for some $y = (y_1, \dots, y_n) \in \mathbb{C}^n$. So $\varphi \in (\ell^p(n))^*$; that is, every linear functional is continuous. The Riesz representation theorem says that

$$\|\varphi_y\|_{(\ell^p(n))^*} = \sup\{|\varphi_y(x)| : \|x\|_p \leq 1\} = \|y\|_{\ell^q},$$

where $1/p + 1/q = 1$.

There is a special case, when $p = 2$. We get that the dual norm is the original norm. So we can think of $\ell^2(n)$ as its own dual.

Definition 1.1. Let H be a vector space over $\mathbb{C}$. An **inner product** on H is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ sending $(x, y) \mapsto \langle x, y \rangle$ such that

1. (bilinearity) $\langle ax + by, z \rangle = a \langle x, y \rangle + b \langle x, z \rangle$ for all $a, b \in \mathbb{C}$, $x, y, z \in H$,

2. (conjugate symmetry) $\langle x, y \rangle = \overline{\langle y, x \rangle}$,

3. $\langle x, x\rangle \in [0, \infty)$ with $\langle x, x\rangle = 0$ iff $x = 0$.

Example 1.1. $\mathbb{C}^n$ is a vector space with the usual inner product.

Example 1.2. $L^2_{\mathbb{C}}(\mu)$ is a vector space with the inner product $\langle f, g\rangle = \int_X f\overline{g}\, d\mu$.

Example 1.3. Let $X = \mathbb{N}$ with counting measure. Then

$$\ell^2 = \ell^2(\mathbb{N}) = \{(x_n)_n : \sum_n |x_n|^2 < \infty\}$$

has the inner product $\langle x, y\rangle = \sum_n x_n \overline{y}_n$.

Definition 1.2. A vector space $(H, \langle \cdot, \cdot \rangle)$ is a **pre-Hilbert space** (or **inner product space**).

1.2 Norms induced by inner products

An inner product space has the associated norm $\|x\| := \sqrt{\langle x, x\rangle}$. First, we have to show that this is actually a norm.

Lemma 1.1 (Cauchy-Bunyakowski-Schwarz inequality[1]). *For all $x, y \in H$,*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Proof. Consider $\langle x - ty, x - ty\rangle$. We get

$$\begin{aligned} 0 &\leq \langle x - ty, x - ty\rangle \\ &= \langle x, x\rangle - t\langle y, x\rangle - t\langle x, y\rangle + t^2\langle x, y\rangle \\ &= \|x\|^2 - 2t\,\mathrm{Re}(\langle x, y\rangle) + t^2\|y\|^2. \end{aligned}$$

This achieves its minimum at $t = \mathrm{Re}(\langle x, y\rangle)/\|y\|^2$. So we get

$$0 \leq \|x\|^2 - \frac{(\mathrm{Re}(\langle x, y\rangle))^2}{\|y\|^2},$$

which gives

$$|\mathrm{Re}(\langle x, y\rangle)| \leq \|x\|\|y\|.$$

Similarly, let $\alpha = \mathrm{sgn}(\langle x, y\rangle)$, an apply this to x and $y' = \alpha y$. Then

$$|\langle x, y\rangle| = |\mathrm{Re}(x, y')| \leq \|x\|\|y'\| = \|x\|\|y\|. \qquad \square$$

Corollary 1.1. *$\|\cdot\|$ is a norm.*

[1] Bunyakowski and Schwarz both knew the general form of this inequality, but, due to geopolitics, there was no way they could have ever met.

Proof.

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \|x\|^2 + 2\operatorname{Re}(\langle x,y\rangle + \|y\|^2 \\ &\le \|x^2\| + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

□

Definition 1.3. A **Hilbert space** is a complete pre-Hilbert space.

Example 1.4. All the previous examples are complete.

Proposition 1.1. $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}$ *is continuous for the norm topology on* H*.*

Proof. Suppose that $x_n \to x$ in norm and $y_n \to y$ in norm. Then

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x, y\rangle| &\le |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \\ &\le \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\ &\to 0.\end{aligned}$$

□

Proposition 1.2 (Parallelogram law)**.** *For all* $x, y \in H$*,*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof. Expand out $\langle x+y, x+y\rangle$ and cancel terms. □

1.3 Orthogonality

Definition 1.4. Elements $x, y \in H$ are **orthogonal** if $\langle x, y\rangle = 0$.

Definition 1.5. If $E \subseteq H$, its **orthogonal complement** is

$$E^\perp = \{x \in : \langle x, y\rangle = 0 \, \forall y \in E\}.$$

Theorem 1.1 (Pythagorean theorem[2])**.** *If* $x_1, \dots, x_n \in H$ *are pairwise orthogonal, then*

$$\left\|\sum_i x_i\right\| = \sum_i \|x_i\|^2.$$

Proof. Expand $\left\langle \sum_i x_i, \sum_j x_j \right\rangle$ and cancel terms. □

[2]A person named Pythagoras probably didn't exist. Nevertheless, the Pytagoreans almost surely did not know what a Hilbert space is.

Theorem 1.2. *Let H be a Hilbert space, and let M be a closed subspace. Then any $x \in H$ can be written uniquely as $x = y + z$, where $y \in M$ and$z \in M^{\perp}$. We write $H = M \oplus M^{\perp}$.*

Proof. Let $\delta = \inf\{\|x - y\| : y \in M\}$. Pick $(y_n)_n$ in M such that $\|x - y_n\| \to \delta$. We claim that (y_n) is Cauchy. We have

$$\|y_n - y_m\|^2 + \|y_n + y_m - 2x\|^2 = 2(\|y_n - x\|^2 + \|y_m - x\|^2).$$

Rewrite this as

$$\|y_n - y_m\|^2 + 4 \underbrace{\left\| \frac{y_n + y_m}{2} - x \right\|^2}_{\to \delta^2} = 2(\underbrace{\|y_n - x\|^2}_{\to \delta^2} + \underbrace{\|y_m - x\|^2}_{\to \delta^2}).$$

This is only possible if $\|y_n - y_m\| \to 0$.

So the limit $y = \lim_n y_n$ exists. This is the unique closest point in M to x. □

1.4 Isomorphisms of Hilbert spaces

Definition 1.6. A **unitary operator** $U : H_1 \to H_2$ is linear operator such that $U \in \mathcal{L}(H_1, H_2)$ is an isomorphism and $\langle Ux, Uy \rangle_2 = \langle x, y \rangle_1$.

This is the true notion of isomorphism for inner product spaces. Next time, we will prove the following theorem:

Theorem 1.3. *Let H be a Hilbert space over $\mathbb{C}$.*

1. *If $\dim(H) = n < \infty$, then $H \cong \mathbb{C}^n$.*
2. *If $\dim(H) = \infty$ and H is separable, then $H \cong \ell^2(\mathbb{N})$.*

Example 1.5. $L^2(\mathbb{R})$ is separable, so $L^2(\mathbb{R}) \cong \ell^2(\mathbb{N})$.

Example 1.6. The **Fourier transform** is the unitary equivalence $L^2([0,1]) \cong \ell^2(\mathbb{Z})$.