Math 254A Lecture 29 Notes

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1 Equilibrium Measures, the D-L-R Equations, and Uniqueness vs Non-uniqueness

1.1 Second variational principle and equilibrium measures

Here is the main variational principle we proved last time: For $f \in C(A^{\mathbb{Z}^d})$, let

$$p(f) := \lim_{B \uparrow \mathbb{Z}^d} \frac{1}{|B|} \log \sum_{x \in A^B} \exp(-s_B f(x)),$$

where

$$s_B f(x) = \sum_{v \in B} f(T^v \widehat{x}),$$

and $\widehat{x}$ is an extension of x to an element of $A^{\mathbb{Z}^d}$. Then

$$p(f) = \sup\{h(\mu) - \langle f, \mu \rangle : \mu \in P^T(A^{\mathbb{Z}^d}) = h^*(f)\}.$$

Here, p is called the **pressure** of f.[1] (Recall that $h : M(A^{\mathbb{Z}^d}) \to [-\infty, \log |A|]$ is concave and upper semicontinuous.

Proof. Here is a sketch: Note that

$$\begin{aligned} s_B f(x) &= \sum_{v \in B} f(T^v \widehat{x}) \\ &= |B| \frac{1}{|B|} \sum_{v \in B} f(T^v \widehat{x}) \\ &= |B| \langle f, \underbrace{\frac{1}{|B|} \sum_{v \in B} \delta_{T^v \widehat{x}}}_{=: P_x^B} \rangle, \end{aligned}$$

[1]This is not always the same thing as pressure in physics. Sometimes, physicists will call this the **Massieu function**.

. where $P_x^B = P_x^W + o_{B\uparrow\mathbb{Z}^d}(1)$. So in the limit, it is enough to prove the formula for

$$\sum_{x\in A^B} \exp(-|B|\langle f, P_x^B\rangle)$$

Proof of ($\geq$): Fix $\mu \in P^T$, fix $\varepsilon > 0$, and let $U = \{\nu \in P^T : \langle f, \nu\rangle < \langle f, \mu\rangle + \varepsilon\}$. This is a weak* open, concave neighborhood of μ. Now

$$\begin{aligned}
\sum_{x\in A^B} \exp(\cdots) \geq \sum_{x:P_x^B\in U} \exp(-|B|\langle f, P_x^B\rangle) \\
&\geq \sum_{x:P_x^B\in U} \exp(-|B|(\langle f,\mu\rangle + \varepsilon)) \\
&\geq \underbrace{|\Omega_B|}_{\geq \exp(|B|h(\mu)+o(|B|))} \exp(-|B|\langle f,\mu\rangle - \varepsilon|B|).
\end{aligned}$$

So we get

$$\frac{1}{|B|}\log(\cdots) \geq h*\mu) - \langle f,\mu\rangle - \varepsilon - o(1).$$

Since ε is arbitrary, we get ($\geq$).

Proof of ($\leq$): Let $\varepsilon > 0$, and pick a finite cover $P \subseteq U_1 \cup \cdots \cup U_r$ such that

$$\sup_{\mu\in U_i} h(\mu) + \sup_{\mu\in U_i}(-\langle f,\mu\rangle) \leq h + \varepsilon,$$

where h is the desired right hand side. Now

$$\begin{aligned}
\sum_x \exp(-s_B f(x)) &\leq \sum_{i=1}^n \sum_{x:P_x^B\in U_i} \exp(-s_B f(x)) \\
&\leq \sum_{i=1}^r \exp(s(U_i)\cdot|B| + o(|B|) + \sup_{\mu\in U_i}(-\langle f,\mu_i\rangle) + \varepsilon \\
&\leq \sum_{i=1}^r \exp\left(\sup_{\mu\in U_i}(h(\mu) - \langle f,\mu\rangle) + 2\varepsilon|B|\right) \\
&\leq r\cdot\max_i(\cdots).
\end{aligned}$$

Apply this to $\frac{1}{|B|}\log(\cdots)$ to get that this is $\leq RHS + 2\varepsilon + o(1)$. $\square$

We have a max, rather than a sup:

$$p(f) = \max\{h(\mu) - \langle f,\mu\rangle : \mu \in P^T\}.$$

Definition 1.1. $\mu \in P^T$ is an **equilibrium measure** for f if $h(\mu) - \langle f,\mu\rangle = p(f)$.

Observe that

$$\sum_{x\in A^B} \exp(-|B|\cdot\langle f, P_x^B\rangle) = \exp(p(f)\cdot|B| + o(|B|)).$$

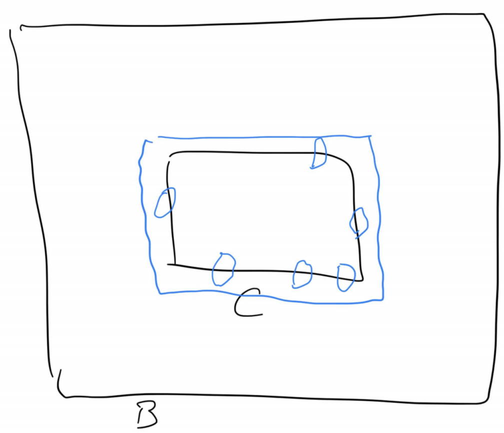
Given $\mu \in P^T$ and a small enough neighborhood U, we have

$$\sum_{x:P_x^B\in U} \exp(-|B|\langle f, P_x^B\rangle)\exp\left((h(\mu) - \langle f,\mu\rangle \pm \varepsilon)|B| + o(|B|)\right)$$

Microscopic states will look very similar to equilibrium measures. Equilibrium measures measures always exist. If an equilibrium measure is unique, then we can integrate against it to predict values of any other observable. If there are multiple equilibrium measures, they describe a system with several possible phases of a given temperature.

1.2 The D-L-R equations and uniqueness vs non-uniqueness

The next stage in the story is how to characterize equilibrium measures. For simplicity, we will study a finite-range interaction $\varphi = \varphi_F$ for some fixed finite $F \subseteq \mathbb{Z}^d$. Consider $C \subseteq B$:



Look at

$$\begin{aligned}\Phi_B(x) = \sum_{v\in F\subseteq B} \varphi(x_{v+F}) \\ &= \sum_{v+F\subseteq C}\varphi(x_{v+F}) + \sum_{v+F\subseteq B\setminus C}\varphi(x_{v+F}) + \sum_{\substack{v+F\subseteq B\\ (v+F)\cap C\neq\varnothing\\ (v+F)\cap(BV)\neq\varnothing}}\varphi(x_{v+F}) \\ &= \Phi_C(x) + \Phi_{B\setminus C}(x) + \Phi^{\text{int}}_{B,C}(x).\end{aligned}$$

The canonical distribution on A^B is

$$\begin{aligned} d\gamma(x) &= \frac{1}{Z}\exp(-\Phi_B(x)) \\ &= \frac{1}{Z}\exp(-\Phi_C - \Phi_{B\setminus C} - \Phi^{\text{int}}_{B,C}). \end{aligned}$$

Use this to write

$$\gamma(x_C = y \mid x_{B\setminus C} = z) = \frac{\exp(-\Phi_C(y) - \cancel{\Phi_{B\setminus C}(x)} - \Phi^{\text{int}}(y,z))}{\sum_{y'\in A^C}\exp(-\Phi_c(y') - \cancel{\Phi_{B\setminus C}(z)} - \Phi^{\text{int}}(y',z)}$$

This depends on z on through $z_{\partial_F C}$, where $\partial_F C = [\bigcup_{(v+F)\cap C\neq\varnothing}(v+F)]\setminus C$. So we can define a family of conditional measures for $y \in A^c$ and $z \in A^{\mathbb{Z}^d\setminus C}$:

$$\gamma_{C,z}(dy) = \frac{\exp(-\Phi_C(y) - \Phi^{\text{int}}(y, z_{\partial_F C})}{\sum_{y'\in A^C}\exp(-\Phi_C(y') - \Phi^{\text{int}}(y', z_{\partial_F C}))}$$

Definition 1.2. The family $\gamma_{C,z}$ for finite $C \subseteq \mathbb{Z}^d$ and $z \in A^{\mathbb{Z}^d\setminus C}$ is called the **specification** associated to φ.

The specification describes the conditional behavior inside C under a canonical distribution. A $\mu \in P(A^{\mathbb{Z}^d})$ satisfies the D-L-R equations with respect to φ if for all finite $C \subseteq \mathbb{Z}^d$, we have

$$\mu(x_c = y \mid x_{\mathbb{Z}^d\setminus C} = z) = \gamma_{C,z}(y)$$

for all C, z. Equivalently, for all $f \in C(A^{\mathbb{Z}^d})$, we have

$$\int f\, d\mu = \iint f(y, x_{\mathbb{Z}^d\setminus C})\gamma_{C,x_{\mathbb{Z}^d\setminus C}}(dz)\, d\mu(x).$$

Theorem 1.1. *μ is an equilibrium measure for φ if and only if $\mu \in P^T$ and satisfies all D-L-R equations.*

This is much easier to analyze. This is the gateway to theorems such as the following:

Theorem 1.2. *For any local interaction φ, $\beta\varphi$ has a unique equilibrium state for all sufficiently small β. That is, there is a critical $\beta > 0$ such that for all $\beta < \beta_c$, $T > T_c$.*

The above is a corollary of Dobrushin's uniqueness theorem. In general, things are easier at high temperatures because of techniques like that theorem.

On the other hand, if $A = \{-1, 1\}$, $F = \{0, c_1, c_2, \ldots, c_d\}$, then

$$\varphi(x_0, x_{e_1}, \ldots, x_{e_d}) = -\sum_{i=1}^{d} x_0 x_{e_i}.$$

This is the basis for what is called the **Ising model**.

Theorem 1.3 (Peierls)**.** *If β is high enough and $d \geq 2$, then $\beta\varphi$ has multiple equilibrium states.*