Math 206A Lecture 4 Notes

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1 Bárány's Theorem and Equipartition

1.1 Statement of Bárány's theorem

Theorem 1.1 (Bárány)**.** *For every d, there exists a constant $\alpha_d > 0$ such that for every $Z = \{z_1, \dots, x_n\} \subseteq \mathbb{R}^d$, there exists $x \in \mathbb{R}^d$ such that $x \in \mathrm{Con}(Z_I)$, $|I| = d+1$ for at least $\alpha_d \binom{n}{d+1}$ subsets I.*

What is this saying? In $d = 2$, there is some point that lies in a constant proportion of all the subsets you can make as the convex hulls of 3 points. In $d = 1$, we can do this by picking the middle of the z_i. Then x is contained in $(n/2)^2 \sim \binom{n}{2}/2$ of the Z_I.

1.2 Equipartition

Theorem 1.2. *Suppose $\mathbb{Q} \subseteq \mathbb{R}^2$ is a convex polgyon. Then there exist perpendicular lines ℓ_1, ℓ_2 that partition Q into 4 parts of equal area.*

Proof. Fix a line ℓ in the plane, and consider ℓ_1 parallel to ℓ such that the area of Q_+ and Q_- are the same. Do the same with ℓ_2 perpendicular to ℓ. The diagonal pieces (when Q is split into 4) have the same area, but we may have adjacent areas $a \neq b$. Take this construction, and rotate ℓ up to $\pi/2$. There exists a rotation θ such that $a_\theta = b_\theta$. □

Theorem 1.3. *Let $Q \subseteq \mathbb{R}^2$ be a convex polygon. Then there exist ℓ_1, ℓ_2, ℓ_3 that intersect at 1 point such that Q is partitioned into 6 parts of equal area.*

Proof. Fix $\ell \subseteq \mathbb{R}^2$ be a line that splits Q into two parts of equal area. Pick x on the line, and let 4 rays pass out of it. We rotate ℓ and the rays separately. Let ℓ_θ be the rotation of $|ell$ be θ, where $\theta \in [0, \pi]$. Let β_θ be the angle between the actual ray and the extension of the opposite ray. By convexity, the point x is uniquely determined by the rays. □

Corollary 1.1. *For all $Z = \{z_1, \dots, z_{6k}\} \subseteq \mathbb{R}^2$ with no 3 points on the same line, there exist lines ℓ_1, ℓ_2, ℓ_3 which separate Z into 6 groups of equal size.*

Proof. The same proof works. □

Theorem 1.4 (Boros-Füredi)**.** *For every $Z = \{z_1, \ldots, z_{6k}\} \subseteq \mathbb{R}^2$ with no 3 points on the same line, there exists $x \in \mathbb{R}^2$ such that x is in at least $8k^3$ triangles $z_i z_j z_r$.*

Note that $\binom{6k}{3} \sim 36k^3$, so $\alpha_2 \geq 8/36 = 2/9$.

Proof. Let $x, \ell_1, \ell_2, \ell_3$ be as given by the previous corollary. Note that if you take 3 points from every other portion of the 6 portions of the plane, x is in their convex hull (a triangle). This gives us $2k^3$ triangles. Now, if we pick two points in opposite portions, there are 2 portions (on the side) where picking a point in them will make x in the convex hull of the 3 points. So we get $3k^2 \cdot (2k) = 6k^3$ more triangles. □

These authors claimed that 2/9 was optimal, but their proof had a mistake in it. The result was true, but this was not corected until about 30 years later by Bukh.