Math 245B Lecture 16 Notes

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1 Fréchet Spaces, Weak Topologies, and The Weak* Topology

1.1 Fréchet spaces

Proposition 1.1. *Let $(\mathcal{X}, (p_\alpha)_\alpha, \mathcal{T})$ be a locally convex topological vector space generated by the seminorms p_α.*

1. *$\mathcal{T}$ is Hausdorff iff for all $x \in \mathcal{X} \setminus \{0\}$, there exists some α such that $p_\alpha(x) \neq 0$.*
2. *If $\mathcal{T}$ is Hausdorff and A is countable, then $(\mathcal{X}, \mathcal{T})$ is metrizable with a translation invariant metric: $\rho(x+z, y+z) = \rho(x,y)$ for all z.*

Proof. Proving the first statement is easiest done with the left implication and the contrapositive of the right implication.

1. ($\Longleftarrow$): Let $x, y \in \mathcal{X}$ such that $x \neq y$. Then there exists α such that $p_\alpha(y-x) > 0$. Consider $U_{x,\alpha,\varepsilon}, U_{y,\alpha,\varepsilon}$ for $\varepsilon < p_\alpha(y-x)/2$.

   ($\Longrightarrow$): Otherwise, there exists $x \neq 0$ such that $p_\alpha(x) = 0$ for all α. Then $x \in U_{0,\alpha,\varepsilon}$ for all α, ε. So x lies in any neighborhood of 0.

2. Given $(p_n)_{n \in \mathbb{N}}$, define
$$\rho(x,y) = \max\{2^{-n} \min\{\rho(x-y), 1\} : n \in \mathbb{N}\}.$$
   The min inside satisfies the triangle inequality, and taking maxes preserves the triangle inequality. So this is a pseudometric. Since ρ is a function of $x-y$, it is translation invariant. Lastly, if $x \neq y$, then $p_x(x-y) \neq 0$ for some n, so $\rho(x,y) > 0$. □

Definition 1.1. A **Fréchet space** a locally convex topological vector space with the above metric such that ρ can be chosen to be complete.

Example 1.1. $\mathbb{R}^{\mathbb{N}}$ with the product topology, $C(\mathbb{R}^n)$ with the topology of local uniform convergence, and L^1_{loc} are all Fréchet spaces.

1.2 Weak topologies

Definition 1.2. Let $T_\alpha : \mathcal{X} \to (\mathcal{Y}_\alpha, \|\cdot\|_\alpha)$ be a collection of linear maps with the resulting family of seminorms $p_\alpha(x) = \|T_\alpha x\|_\alpha$. These generate the **weak topology generated by** $(T_\alpha)_\alpha$.

Example 1.2. Let $T_m : C(\mathbb{R}^n) \to C([m, m]^d)$ send $f \mapsto f|_{[-m,m]^d}$. Then the topology of local uniform convergence is the weak topology generated by these maps.

Example 1.3. On $C^\infty([0,1])$, for each k, consider $(d/dx)^k : C^\infty([0,1]) \to C([0,1])$. Now take the weak topology generated by these.

Usually in the setting of normed spaces, we refer to a very specific weak topology.

Definition 1.3. The **weak topology** on $(\mathcal{X}, \|\cdot\|)$ is the topology generated by $\mathcal{X}^*$, the set of continuous linear functionals.

Remark 1.1. In general, $\mathcal{T}_{\text{weak}} \subseteq \mathcal{T}_{\text{norm}}$. These are equal iff $\dim(\mathcal{X}) < \infty$. If $f \in \mathcal{X}^*$, show that $U_{x,f,\varepsilon} = \{y : |f(y-x)| < \varepsilon\}$ is contained in a ball around x.

Remark 1.2. Convergence in the weak topology means the following:

$$x_n \to x \iff f(x_n) \to f(x) \ \forall f \in \mathcal{X}^*.$$

In norm topologies, we have $|f(x_n) - f(x)| \leq \|f\|\|x_n - x\|$. So the weak topology is weaker particularly because it does not give this uniformity of convergence.

1.3 The weak* topology

If $(\mathcal{X}^*, \|\cdot\|)$ is a Banach space, then we have the dual space $(\mathcal{X}^*, \|\cdot\|_*)$, This has its own dual $(\mathcal{X}^{**}, \|\cdot\|_{**})$. We have 2 choices for the weak topology on $\mathcal{X}^*$: we can take the usual weak topology, or we can restrict to the even weaker topology generated by $\mathcal{X}$ embedded into $\mathcal{X}^{**}$.

Definition 1.4. The **weak* topology** on $\mathcal{X}^*$ is generated by the family of maps $\hat{x} : f \mapsto f(x) \in \mathcal{K}$ where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$.

Theorem 1.1 (Alaoglu)**.** *$B^* = \{f \in \mathcal{X}^* : \|f\|_* \leq 1\}$ is compact for the weak* topology.*

Proof. Say $K = \mathbb{C}$.

$$\begin{aligned} B^* &= \{f : \mathcal{X} \to \mathbb{C} \mid f(x + \lambda y) = f(x) + \lambda f(y), |f(x)| \leq \|x\|\} \\ &= \{f : \mathcal{X} \to \mathbb{C} : f \text{ is linear}, f(x) \in \overline{B_{\mathbb{C}}(0, \|x\|)} \ \forall x\}. \end{aligned}$$

That is, $f \in \prod_{x \in \mathcal{X}} \overline{B_{\mathbb{C}}(0, \|x\|)}$.

$$= \{f \in \prod_{x \in \mathcal{X}} \overline{B_{\mathbb{C}}(0, \|x\|)} : f(x+y) - g(x) = \lambda f(y) = 0 \ \forall x, y, \lambda\}$$

$$= \bigcap_{x,y,\lambda} \{f \in \prod_{x \in \mathcal{X}} \overline{B_{\mathbb{C}}(0, \|x\|)} : f(x+y) - g(x) = \lambda f(y) = 0\}.$$

By Tychonoff's theorem, we need only show that $\mathcal{T}_{\text{weak}^*}|_{B^*} = \mathcal{T}_{\text{prod}}|_{B^*}$. These are weak topologies generated by the same family of maps. □

Proposition 1.2. *Let* $(\mathcal{X}, \|\cdot\|)$ *be separable. Then* $\mathcal{T}_{\text{weak}^*}|_{B^*}$ *is metrizable.*

Proof. Let $(x_n)_n$ be a dense sequence in $\mathcal{X}$. Then define ρ on B^* by

$$\rho(f,g) = \max\{2^{-n}/\|x_n\||f(x_n) - g(x_n)| : n \in \mathbb{N}\}.$$

This generates $\mathcal{T}_{\text{weak}^*}|_{B^*}$. For all $x \in \mathcal{X}$, there exists $x_{n_i} \to x$, and therefore $\hat{x}_{n_i}|_{B^*} \to \hat{x}|_{B^*}$ uniformly. □