2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Stability of nonlinear systems with slow and fast time variation and switching: the common equilibrium case

Daniel Liberzon and Hyungbo Shim

2024 IEEE 63rd Conference on Decision and Control (CDC) | 979-8-3503-1633-9/24/$31.00 ©2024 IEEE | DOI: 10.1109/CDC56724.2024.10885786

Abstract— **In this paper we consider a class of nonlinear systems with two kinds of inputs: one is slowly-varying, the other is fast-varying and periodic, and both are only piecewise continuous. Under the assumption that the origin is a common equilibrium for all values of the input signals, we provide sufficient conditions under which this equilibrium is semi-globally exponentially stable. Our approach is based on considering the (partial) average system which averages out the fast variation but retains the slow variation, and which can be used to approximate the original system in a certain sense. The stability conditions involve the existence of a suitable Lyapunov function for this average system, along with a bound on the total variation of the slowly-varying input.**

Index Terms— **Switched systems, Averaging, Lyapunov stability**

I. Introduction

In this paper we consider systems with both slow and fast time-varying signals which, moreover, need not be continuous everywhere. A dynamical model for such a system can take the form

$$\dot{x}(t) = f(x(t), u_s(t), u_f(t/\varepsilon)) \tag{1}$$

where $\varepsilon > 0$ is a small parameter, u_s represents a slowly-varying signal, u_f represents a fast-varying signal, and both u_s and u_f are piecewise continuous functions of time.

The most basic question about the system (1) is that of stability. For systems with time-varying parameters, there are well-known sufficient conditions for stability that ask the system to be stable for each frozen value of the parameters and the variation to be sufficiently slow (typically by placing some type of upper bound on the time derivative of the parameters). Such results are by now standard, especially for linear time-varying systems, and appear in textbooks; see, e.g., [6, Section 3.4], [7, Section 9.6] and the references therein.

For *switched systems*, which are characterized by instantaneous switching instead of continuous variation, there exist stability criteria which parallel the ones mentioned above for time-varying systems and which are also well known. They are formulated in terms of stability of each individual mode of the switched system and a slow-switching condition, typically in terms of sufficiently large (average) dwell time; see, e.g., [9] for an introduction to this class of systems and representative basic results.

The recent work [3], [2] made apparently the first attempt to unify these two sets of results. For systems combining continuous variation and switching, these papers utilized the concept of *total variation*. This is the quantity obtained, loosely speaking, by integrating the norm of the derivative of the time-varying parameter vector (or matrix) and adding, at each switching instant, the norm of the jump. For linear systems, it was shown in [3] that exponential stability is preserved if the total variation is suitably small. It was also demonstrated that this approach allows one to recover known results for systems with only continuous variation or only switching, with the results in the latter category actually going beyond the basic ones given in [9]. An extension to nonlinear systems was presented in [2].

For systems with *fast* time variation, a well-known analysis method is based on *averaging*. In its classical formulation (see, e.g., [5, Chapter V], [7, Chapter 10]), it deals with periodic or nearly periodic fast-varying signals by defining an average system and proving, via perturbation arguments, that the behavior of the original system is close to that of the average system. The averaging method has also been applied to other system classes, including switched systems [15]. The above sources assume that the average system is time-invariant. This restriction is relaxed in the works [11], [1] which consider time-varying average systems. In fact, our approach bears quite a close resemblance to the "partial averaging" method of [11], which averages out fast variation but retains slow variation in the average dynamics. Nevertheless, these existing results cannot handle time-varying average systems with discontinuities. The paper [14] did consider averaging for hybrid systems, but of restricted kind in that it was only applied to continuous dynamics and did not alter discrete events. Among other works that address slow and fast—and possibly discontinuous—time variation, it is relevant to mention [13] and [4, Section 7.4]; however, the tools employed in these references and the spirit of the results are quite different from ours.

The overall goal of our current work, therefore, is to develop stability conditions that can handle the presence of both slow and fast time-varying signals or parameters, by suitably combining key features of the averaging theory with those of stability analysis of slowly varying systems, and moreover, to incorporate techniques employed in the

D. Liberzon is with the Coordinated Science Laboratory, University of Illinois Urbana-Champaign, Urbana, IL 61801 USA (email: liberzon@uiuc.edu). His work was supported in part by the NSF CMMI-2106043 and AFOSR MURI FA9550-23-1-0337 grants.

H. Shim is with ASRI and the Department of Electrical and Computer Engineering, Seoul National University, Korea (email: hshim@snu.ac.kr). His work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. RS-2022-00165417).

study of switched systems (in particular, the concept of total variation) to allow the presence of discontinuities in these signals/parameters. We have recently obtained some encouraging preliminary results, confined to the case where the system's dynamics are linear in the state x. The simplest class of systems, which we considered in [10], is given by the dynamics

$$\dot{x} = \big(A(t) + B(t/\varepsilon)\big)x \tag{2}$$

where $x \in \mathbb{R}^n$, $A(\cdot)$ and $B(\cdot)$ are piecewise continuous functions from $[0,\infty)$ to $\mathbb{R}^{n\times n}$, $B(\cdot)$ is periodic with a known period $T > 0$, and $\varepsilon > 0$. For small ε, we think of $A(t)$ as describing slow time-variation and switching in the system, and of $B(t/\varepsilon)$ as describing fast variation and switching.

Our stability analysis of the linear system (2) as described in [10] proceeded in the following steps, which we briefly outline here since our approach to tackling the nonlinear system (1) will follow a conceptually similar path. First, the *average system* is defined by integrating the fast dynamics over one period. Assuming for convenience (and without loss of generality) that $B(\cdot)$ has zero average, i.e., $\frac{1}{T}\int_0^T B(s)ds = 0$, the *average system* corresponding to (2) is simply given by $\dot{x} = A(t)x$. Our main technical assumption is that the *total variation* of $A(\cdot)$, as defined in [3], is upper-bounded by a quantity of the form $\mu(t_2 - t_1) + \alpha$, with μ in turn being upper-bounded by suitable system parameters. Second, stability of the average system is established, under the above assumption of sufficiently small total variation of $A(\cdot)$, by invoking the results from [3]. To achieve this, we build a quadratic time-dependent Lyapunov function $V(t,x) = x^T P(t)x$ with $P(t)$ tailored to $A(t)$ for each fixed t. We then show that the time derivative of this Lyapunov function can be upper-bounded in terms of the total variation of $A(\cdot)$, and is negative when the total variation is small enough, yielding exponential stability of the average system. Finally, the original system (2) is represented and analyzed as a perturbation of the average system. To approximate the system (2) by the average system, we consider for (2) the change of variables

$$x = y + \varepsilon \int_0^{t/\varepsilon} B(s)ds \cdot y. \tag{3}$$

We can then show that in the y-coordinates, the dynamics are $\dot{y} = A(t)y + \varepsilon C(t,\varepsilon)y$, where the second term on the right-hand side is a perturbation term (a "vanishing perturbation" in the sense of [7, Section 9.1], i.e., it is 0 at 0). To finish the stability proof, we need to analyze the effect of the perturbation term on the derivative of the Lyapunov function (the same one used to establish stability of the time-varying average system). We can show that the Lyapunov function still decreases exponentially along solutions if ε is sufficiently small, i.e., if the fast variation is sufficiently fast.

In the more recent follow-up paper [12] we studied a larger class of systems—still linear in the state, but with dynamics taking the form

$$\dot{x} = \big(A(t) + B_s(t)B_f(t/\varepsilon)\big)x. \tag{4}$$

This class is more general compared to (2), and in [12] we studied in detail an example of a mechanical system (inverted pendulum on a cart with a moving ball subject to external forces) which conforms to (4) but not to (2). A complication that arises, however, is that the new state variable y, given by a suitable extension of the coordinate transformation (3), now experiences jumps at the discontinuities of B_s with respect to t. Consequently, in the y-coordinates the dynamics are *impulsive* (a combination of continuous flow and jumps). We overcame this challenge in [12] by carrying out a Lyapunov stability analysis of this impulsive system.

The preliminary results outlined above represent just very first steps towards understanding the problem of interest. An obvious next task is to tackle nonlinear dynamics in the form (1), with the objective of deriving general sufficient conditions for their stability. This is the subject of the present paper, in which we will be working under the simplifying assumption that

$$f(0,\cdot,\cdot) \equiv 0. \tag{5}$$

This assumption—which means that $x = 0$ is a common equilibrium for all values of the time-varying signals—helps us mimic, to some extent, the developments in the linear case. Indeed, as we will see, it guarantees that the perturbation term characterizing the difference between the original system and the average system (in appropriate coordinates) can still be represented as a "vanishing perturbation". And similarly to the case of (4), the coordinate change gives rise to an impulsive system. Of course, the nonlinear dynamics also present new technical challenges, overcoming which is the goal of this paper.

The remainder of the paper is organized as follows. In Section II we state our assumptions and the main result. Section III is devoted to proving this result. Section IV concludes the paper.

II. Preliminaries and Statement of Main Result

We now give a more precise description of the system (1) and the technical assumptions that we impose on it.

Assumption 1 (fast-varying signal) The function $u_f : [0,\infty) \to \mathbb{R}^\ell$ is piecewise continuous and periodic with a period $T > 0$.

Assumption 2 (slowly-varying signal) The function $u_s : [0,\infty) \to \Gamma \subset \mathbb{R}^m$ has finitely many discontinuities on any bounded interval, is càdlàg[1], is C^1 between discontinuities, and $\dot{u}_s(\cdot)$ and $|\dot{u}_s(\cdot)|$ are Riemann integrable between discontinuities. Moreover, the set Γ is compact and convex.

Assumption 3 (system's right-hand side) The function $f : \mathbb{R}^n \times \Gamma \times \mathbb{R}^\ell \to \mathbb{R}^n$ satisfies (5), is C^1, and its partial derivatives with respect to x and u_s are C^1 in x.

[1] Continuous from the right, has limits from the left; this assumption is made for notational convenience.

The average of f, for each fixed x and u_s, is defined as

$$f_{\text{av}}(x, u_s) := \frac{1}{T} \int_0^T f(x, u_s, u_f(s)) ds.$$

We refer to

$$\dot{x}(t) = f_{\text{av}}(x(t), u_s(t)) \tag{6}$$

as the *average system* associated with (1), and it follows from (5) that

$$f_{\text{av}}(0, \cdot) = 0. \tag{7}$$

Assumption 4 (Lyapunov function for average system) There exists a C^1 function $V : \mathbb{R}^n \times \Gamma \to [0, \infty)$ and positive constants c_1, c_2, c_3, c_4, c_5 such that for all $x \in \mathbb{R}^n$ and $u_s \in \Gamma$ we have

$$c_1|x|^2 \leq V(x, u_s) \leq c_2|x|^2, \tag{8}$$

$$\frac{\partial V}{\partial x}(x, u_s) f_{\text{av}}(x, u_s) \leq -c_3|x|^2, \tag{9}$$

$$\left| \frac{\partial V}{\partial u_s}(x, u_s) \right| \leq c_4|x|^2, \tag{10}$$

and $\dfrac{\partial V}{\partial x}$ is locally Lipschitz and satisfies

$$\left| \frac{\partial V}{\partial x}(x, u_s) \right| \leq c_5|x|. \tag{11}$$

Lemma 9.8 from [7] guarantees that a Lyapunov function V satisfying the conditions (8)–(11) exists if the average system (6) is exponentially stable for fixed values of the input u_s (under suitable regularity conditions on f_{av}). We note that only the conditions (8)–(10) are needed for proving stability of the average system (following [2]), while the last condition (11), along with the Lipschitzness of $\partial V/\partial x$, was not imposed in [2] but is used here for conducting perturbation analysis of the original system.

Suppose that we are given an interval $[t_1, t_2]$, and that $d_1, \ldots, d_m$ are the discontinuities of u_s on this interval with $t_1 < d_1$ and $d_m \leq t_2$. Following [2], we define the *total variation* of u_s on $[t_1, t_2]$ to be[2]

$$\int_{t_1}^{t_2} \|du_s\| := \sum_{i=0}^{m} \int_{d_i}^{d_{i+1}} |\dot{u}_s(t)| dt + \sum_{i=1}^{m} |u_s(d_i^+) - u_s(d_i^-)| \tag{12}$$

where we set $d_0 := t_1$ and $d_{m+1} := t_2$.

Assumption 5 (total variation bound) The total variation of u_s on any time interval $[t_1, t_2]$ satisfies

$$\int_{t_1}^{t_2} \|du_s\| \leq \mu(t_2 - t_1) + \alpha \tag{13}$$

for some real constants μ and α such that

$$\mu < \frac{c_1 c_3}{c_2 c_4} \tag{14}$$

where c_1, c_2, c_3, c_4 come from Assumption 4.

Our main result states that the system (1) is semi-globally exponentially stable for sufficiently small ε.

Theorem 1 *Let Assumptions 1–5 hold. For every $R > 0$ there exist $\varepsilon^*, c, \lambda > 0$ such that all solutions of the system* (1) *with $\varepsilon \in (0, \varepsilon^*]$ and $|x(0)| \leq R$ satisfy $|x(t)| \leq ce^{-\lambda t}|x(0)|$ for all $t \geq 0$.*

III. Proof of the Main Result

The proof of Theorem 1 proceeds by, first, invoking the results from [2] to establish exponential convergence of the Lyapunov function along solutions of the average system (6); next, expressing the original system (1) in suitable coordinates as a perturbation of the average system; and, finally, using perturbation analysis to verify semi-global exponential stability of the original system.

A. Exponential convergence of V along solutions of average system

Under the standing assumptions, Theorem 1 in [2] asserts that the average system (6) is globally exponentially stable. This is established by analyzing the evolution of $V(t) := V(x(t), u_s(t))$ along solutions. While we do not explicitly need this stability result here, we briefly sketch the derivation because our subsequent analysis will directly build on that. First, if an interval $[t_1, t_2)$ contains no discontinuities of u_s, we have in view of (8)–(10) that

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} f_{\text{av}}(x, u_s) + \frac{\partial V}{\partial u_s} \dot{u}_s \leq -c_3|x|^2 + c_4|x|^2|\dot{u}_s| \\ &\leq \Big(-\frac{c_3}{c_2} + \frac{c_4}{c_1}|\dot{u}_s| \Big) V \end{aligned}$$

hence, by the standard comparison principle,

$$V(t_2^-) \leq V(t_1) e^{-\frac{c_3}{c_2}(t_2 - t_1) + \frac{c_4}{c_1} \int_{t_1}^{t_2} |\dot{u}_s(t)| dt}. \tag{15}$$

Second, at a discontinuity t of u_s, the Mean Value Theorem and (10) imply that

$$V(t) - V(t^-) \leq c_4|x(t)|^2|u_s(t) - u_s(t^-)|. \tag{16}$$

Applying the inequality $z \leq e^{z-1}$ to $z = V(t)/V(t^-)$ and using (16) and (8) yields

$$\frac{V(t^+)}{V(t^-)} \leq e^{\frac{V(t^+)}{V(t^-)} - 1} = e^{\frac{V(t^+) - V(t^-)}{V(t^-)}} \leq e^{\frac{c_4}{c_1}|u_s(t) - u_s(t^-)|}$$

or

$$V(t) \leq V(t^-) e^{\frac{c_4}{c_1}|u_s(t) - u_s(t^-)|}. \tag{17}$$

Iteratively combining the two estimates (15) and (17) and recalling the definition of total variation (12), we have on any interval $[t_1, t_2]$ (possibly containing discontinuities of u_s) that

$$V(t_2) \leq V(t_1) e^{-\frac{c_3}{c_2}(t_2 - t_1) + \frac{c_4}{c_1} \int_{t_1}^{t_2} \|du_s\|}$$

[2]The superscripts '+' and '-' denote the right and left limits, respectively. By right-continuity of u_s the former one is actually superfluous, but we occasionally write it for extra clarity.

which in view of the assumed bound (13) implies

$$V(t_2) \le V(t_1) e^{-(\frac{c_3}{c_2} - \frac{c_4}{c_1}\mu)(t_2 - t_1) + \frac{c_4}{c_1}\alpha}.$$

From (14) we see that $V(\cdot)$ decays exponentially fast (implying global exponential stability of the average system).

B. Coordinate transformation and approximation by average system

We want to approximate the system (1) by its average system (6). To this end, define

$$h(x, u_s, u_f) := f(x, u_s, u_f) - f_{\text{av}}(x, u_s). \tag{18}$$

Then $h(0, \cdot, \cdot) \equiv 0$ by (5) and (7). We have that $h(x, u_s, u_f(\cdot))$, as a function of time for x, u_s fixed, is T-periodic (see Assumption 1) and has average 0. Next, define

$$w(x, u_s, t) := \int_0^t h(x, u_s, u_f(s)) ds. \tag{19}$$

We have

$$w(0, \cdot, \cdot) \equiv 0. \tag{20}$$

The function w is also T-periodic in t (its third argument), and so are its partial derivatives (which exist by Assumption 3).

Now, consider the (time-varying) change of variables

$$y = x - \varepsilon w(x, u_s(t), t/\varepsilon) =: \Phi_{t,\varepsilon}(x). \tag{21}$$

This coordinate change is origin-preserving, i.e.,

$$\Phi_{t,\varepsilon}(0) = 0 \qquad \forall\, t, \varepsilon \tag{22}$$

and reduces to identity when $\varepsilon = 0$, i.e., $\Phi_{t,0}(x) = x$ for all x and t. The validity of this coordinate transformation is justified by the following statement. (Its proof is omitted due to page constraints, but we note that it relies on Theorem 1 of [8].)

Lemma 2 *For every compact set $\Omega \subset \mathbb{R}^n$ there exist an $\bar{\varepsilon} > 0$ and numbers $0 < \underline{\Lambda} \le 1 \le \bar{\Lambda}$ (which depend on the choice of $\bar{\varepsilon}$) such that for every $t \ge 0$ and every $\varepsilon \in [0, \bar{\varepsilon}]$ the following properties hold:*

- *The image $\Phi_{t,\varepsilon}(\Omega)$ is compact and the map $\Phi_{t,\varepsilon} : \Omega \to \Phi_{t,\varepsilon}(\Omega)$ is a diffeomorphism. The set $\Omega' := \cup_{t \ge 0, \varepsilon \in [0,\bar{\varepsilon}]} \Phi_{t,\varepsilon}(\Omega)$ is also compact.*
- *Each $x \in \Omega$ and the corresponding $y = \Phi_{t,\varepsilon}(x)$ satisfy $\underline{\Lambda}|y| \le |x| \le \bar{\Lambda}|y|$. Moreover, we can take $\underline{\Lambda}, \bar{\Lambda} \to 1$ as $\bar{\varepsilon} \to 0$.*

The above change of variables (21) is a variation on the one considered in [1] and, modulo time rescaling, in [7, Section 10.4]; a similar coordinate transformation was also used in [15]. The properties stated in Lemma 2 may not hold globally (without additional assumptions), which is one (but not the only) reason why Theorem 1 is only a semi-global result.

Away from discontinuities of $u_s(\cdot)$, we can use (21) to rewrite the dynamics of (1) in the new, y-coordinates (suppressing the t-arguments in u_s, as well as in x and y, for convenience):

$$\begin{aligned}\dot{y} = f(x, u_s, u_f(t/\varepsilon)) - \varepsilon w_x(x, u_s, t/\varepsilon) f(x, u_s, u_f(t/\varepsilon))\\ - \varepsilon w_{u_s}(x, u_s, t/\varepsilon)\dot{u}_s - w_t(x, u_s, t/\varepsilon).\end{aligned}$$

By (18) and (19) the last term equals

$$w_t(x, u_s, t/\varepsilon) = f(x, u_s, u_f(t/\varepsilon)) - f_{\text{av}}(x, u_s)$$

which gives

$$\begin{aligned}\dot{y} &= f_{\text{av}}(x, u_s) - \varepsilon w_x(x, u_s, t/\varepsilon) f(x, u_s, u_f(t/\varepsilon))\\ &\quad - \varepsilon w_{u_s}(x, u_s, t/\varepsilon)\dot{u}_s\\ &= f_{\text{av}}(y, u_s) + \big(f_{\text{av}}(x, u_s) - f_{\text{av}}(y, u_s)\big)\\ &\quad - \varepsilon\Big(w_x(x, u_s, t/\varepsilon) f(\Phi_{t,\varepsilon}^{-1}(y), u_s, u_f(t/\varepsilon))\\ &\quad + w_{u_s}(\Phi_{t,\varepsilon}^{-1}(y), u_s, t/\varepsilon)\dot{u}_s\Big)\end{aligned}$$

where the existence of $\Phi_{t,\varepsilon}^{-1}$, on compact sets, is guaranteed by Lemma 2.

For $\theta \in [0, 1]$, define

$$\xi(\theta) := f_{\text{av}}(\theta x + (1 - \theta) y, u_s).$$

Then we have (using differentiability of f_{av}):

$$f_{\text{av}}(x, u_s) - f_{\text{av}}(y, u_s) = \xi(1) - \xi(0) = \int_0^1 \xi'(\theta) d\theta$$

$$= \int_0^1 \frac{\partial f_{\text{av}}}{\partial x}(\theta x + (1-\theta) y, u_s) d\theta \cdot (x - y) =: F(x, y, u_s)(x - y).$$

Using (21), we can rewrite this as

$$\begin{aligned}&f_{\text{av}}(x, u_s) - f_{\text{av}}(y, u_s) =\\ &\qquad \varepsilon F(\Phi_{t,\varepsilon}^{-1}(y), y, u_s) w(\Phi_{t,\varepsilon}^{-1}(y), u_s, t/\varepsilon).\end{aligned}$$

Hence, we see that in the y-coordinates and away from discontinuities of $u_s(\cdot)$ the system (1) can be written in the form

$$\dot{y}(t) = f_{\text{av}}(y(t), u_s(t)) + \varepsilon g(y(t), t, \varepsilon) \tag{23}$$

where the function g is given in detail by the formula

$$\begin{aligned}g(y, t, \varepsilon) &= \left(\int_0^1 \frac{\partial f_{\text{av}}}{\partial x}\left(\theta\, \Phi_{t,\varepsilon}^{-1}(y) + (1 - \theta) y, u_s(t)\right) d\theta\right)\\ &\quad \times w(\Phi_{t,\varepsilon}^{-1}(y), u_s(t), t/\varepsilon)\\ &\quad - w_x(\Phi_{t,\varepsilon}^{-1}(y), u_s(t), t/\varepsilon) f(\Phi_{t,\varepsilon}^{-1}(y), u_s(t), u_f(t/\varepsilon))\\ &\quad - w_{u_s}(\Phi_{t,\varepsilon}^{-1}(y), u_s(t), t/\varepsilon)\dot{u}_s(t).\end{aligned} \tag{24}$$

The function g is a *vanishing perturbation* in the sense that $g(0, t, \varepsilon) = 0$ for all t and ε. A more precise and useful bound on the magnitude of this perturbation is provided by Lemma 3 below.

Now suppose that u_s has a jump at time t. This means that y, given by (21), also has a jump at t (while x is continuous at t). We have

$$\begin{aligned}y(t^+) = x(t) - \varepsilon w(x(t), u_s(t^+), t/\varepsilon) = \Phi_{t^+,\varepsilon}(x(t)),\\ y(t^-) = x(t) - \varepsilon w(x(t), u_s(t^-), t/\varepsilon) = \Phi_{t^-,\varepsilon}(x(t))\end{aligned} \tag{25}$$

hence

$$y(t^+) = \Phi_{t^+,\varepsilon} \circ \Phi^{-1}_{t^-,\varepsilon}(y(t^-)) \tag{26}$$

where the existence of the inverse, on compact sets, is guaranteed by Lemma 2.

Denoting by $d_1, d_2, \ldots$ the discontinuities of u_s, as we did in (12), we can write the overall dynamics of y as the impulsive system

$$\begin{aligned} \dot{y}(t) &= f_{\text{av}}(y(t), u_s(t)) + \varepsilon g(y(t), t, \varepsilon), && t \neq d_i \\ y(t) &= \Phi_{t,\varepsilon} \circ \Phi^{-1}_{t^-,\varepsilon}(y(t^-)), && t = d_i \end{aligned} \tag{27}$$

where we recall our convention that $u_s(d_i^+) = u_s(d_i)$ and so $y(d_i^+) = y(d_i)$ for each i.

C. Stability of original system by perturbation analysis

With the impulsive system (27) in place, we need to analyze the evolution of $V(t) := V(y(t), u_s(t))$ along continuous dynamics and along the jumps separately. In between the jumps, we have

$$\dot{V} = \frac{\partial V}{\partial y} f_{\text{av}}(y, u_s) + \frac{\partial V}{\partial u_s}\dot{u}_s + \frac{\partial V}{\partial y}\varepsilon g(t, y, \varepsilon). \tag{28}$$

To proceed, we need to characterize the perturbation g given by (24) in more detail. (Proofs of this and the next lemma are omitted due to page constraints.)

Lemma 3 *Let a compact set $\Omega \subset \mathbb{R}^n$ be arbitrary, and let $\bar{\varepsilon} > 0$ be as in Lemma 2. There exist $\delta_1, \delta_2 > 0$ such that for every $t \geq 0$ at which $\dot{u}_s(t)$ exists, every $\varepsilon \in (0, \bar{\varepsilon}]$, and every $y \in \Phi_{t,\varepsilon}(\Omega)$, we have*

$$|g(y, t, \varepsilon)| \leq \delta_1 |y| + \delta_2 |y| \cdot |\dot{u}_s(t)|. \tag{29}$$

Combining (28) and (29) with (8)–(11) yields the following bound valid on compact sets (in the same sense as in Lemma 3):

$$\begin{aligned} \dot{V} &\leq -c_3|y|^2 + c_4|y|^2|\dot{u}_s| + c_5|y|^2\varepsilon\delta_1 + c_5|y|^2\varepsilon\delta_2|\dot{u}_s| \\ &\leq \Big(-\frac{c_3}{c_2} + \frac{c_5}{c_1}\varepsilon\delta_1 + \big(\frac{c_4}{c_1} + \frac{c_5}{c_1}\varepsilon\delta_2\big)|\dot{u}_s|\Big)V \end{aligned}$$

hence, for any interval $[t_1, t_2)$ not containing any discontinuities of u_s, we have

$$\begin{aligned} &V(t_2^-) \leq \\ &e^{\big(-\frac{c_3}{c_2} + \frac{c_5}{c_1}\varepsilon\delta_1\big)(t_2 - t_1) + \big(\frac{c_4}{c_1} + \frac{c_5}{c_1}\varepsilon\delta_2\big)\int_{t_1}^{t_2}|\dot{u}_s(t)|dt} \\ &\qquad\qquad\qquad \times V(t_1). \end{aligned} \tag{30}$$

Next, we analyze the behavior of $V(t)$ during the jumps in (27), for which we need the following preliminary result.

Lemma 4 *Let a compact set $\Omega \subset \mathbb{R}^n$ be arbitrary, and let $\bar{\varepsilon} > 0$ be as in Lemma 2. There exists a $\phi > 0$ such that whenever $x(t) \in \Omega$ and y jumps at time t according to (26) with $\varepsilon \in (0, \bar{\varepsilon}]$, we have*

$$|y(t) - y(t^-)| \leq \varepsilon\phi|u_s(t) - u_s(t^-)| \cdot |y(t^-)|.$$

Let us now inspect the difference in the values of $V(t)$ before and after a jump, which we can write as

$$\begin{aligned} V(t) - V(t^-) &= V(y(t), u_s(t)) - V(y(t^-), u_s(t^-)) \\ &= \big(V(y(t), u_s(t)) - V(y(t^-), u_s(t))\big) \\ &\quad + \big(V(y(t^-), u_s(t)) - V(y(t^-), u_s(t^-))\big). \end{aligned} \tag{31}$$

Assume that we are in a situation described by Lemma 4. To analyze the first difference in (31), we can first apply the Mean Value Theorem to V and then use Lemma 4 to obtain

$$\begin{aligned} &V(y(t), u_s(t)) - V(y(t^-), u_s(t)) \\ &= \frac{\partial V}{\partial y}(\bar{y}, u_s(t))\big(y(t) - y(t^-)\big) \\ &\qquad \text{(where } \bar{y} = \rho y(t) + (1-\rho)y(t^-) \text{ for some } \rho \in [0,1]) \\ &= \Big(\frac{\partial V}{\partial y}(\bar{y}, u_s(t)) - \frac{\partial V}{\partial y}\big(y(t^-), u_s(t)\big)\Big)(y(t) - y(t^-)) \\ &\quad + \frac{\partial V}{\partial y}(y(t^-), u_s(t))\big(y(t) - y(t^-)\big) \\ &\leq L_V|\bar{y} - y(t^-)| \cdot |y(t) - y(t^-)| + c_5|y(t^-)| \cdot |y(t) - y(t^-)| \\ &\qquad \text{(where } L_V \text{ is a Lipschitz constant for } \partial V/\partial y \text{ over} \\ &\qquad \Omega' \times \Gamma\text{, which exists by Assumption 4)} \\ &\leq \big(L_V|y(t) - y(t^-)| + c_5|y(t^-)|\big) \cdot |y(t) - y(t^-)| \\ &\qquad \text{(because } |\bar{y} - y(t^-)| \leq |y(t) - y(t^-)|) \\ &\leq \big(L_V\varepsilon\phi|u_s(t) - u_s(t^-)| + c_5\big)|y(t) - y(t^-)| \cdot |y(t^-)| \\ &\qquad \text{(using Lemma 4)} \\ &\leq \big(L_V\varepsilon\phi\,\text{diam}(\Gamma) + c_5\big)|y(t) - y(t^-)| \cdot |y(t^-)| \\ &\qquad \text{(where diam}(\Gamma) \text{ is the diameter of the compact set } \Gamma) \\ &\leq \big(L_V\varepsilon\phi\,\text{diam}(\Gamma) + c_5\big)\varepsilon\phi|u_s(t) - u_s(t^-)| \cdot |y(t^-)|^2 \\ &\qquad \text{(using Lemma 4 again)} \\ &\leq \big(L_V\varepsilon\phi\,\text{diam}(\Gamma) + c_5\big)\frac{\varepsilon\phi}{c_1}|u_s(t) - u_s(t^-)|V(y(t^-), u_s(t^-)). \end{aligned}$$

As for the second difference in (31), we have

$$\begin{aligned} V(y(t^-), u_s(t)) &- V(y(t^-), u_s(t^-)) \\ &= \frac{\partial V}{\partial u_s}(y(t^-), \bar{u})\big(u_s(t) - u_s(t^-)\big) \\ &\leq c_4|y(t^-)|^2 \cdot |u_s(t) - u_s(t^-)| \\ &\leq \frac{c_4}{c_1}|u_s(t) - u_s(t^-)|V(y(t^-), u_s(t^-)) \end{aligned}$$

where $\bar{u} = \rho u_s(t) + (1-\rho)u_s(t^-)$ for some $\rho \in [0,1]$. Combining these two bounds, we can write

$$V(t) - V(t^-) \leq \ell(\varepsilon)|u_s(t) - u_s(t^-)|V(t^-)$$

where $\ell(\varepsilon)$ takes the form $\ell_2\varepsilon^2 + \ell_1\varepsilon + \ell_0$ with $\ell_0 := \frac{c_4}{c_1}$, $\ell_1 := \frac{c_5}{c_1}\phi$, and $\ell_2 := \frac{L_V\phi^2 \text{diam}(\Gamma)}{c_1}$. Proceeding similarly to how we derived (17), we have

$$\begin{aligned} \frac{V(t)}{V(t^-)} &\leq e^{\frac{V(t)}{V(t^-)} - 1} = e^{\frac{V(t) - V(t^-)}{V(t^-)}} \\ &\leq e^{\ell(\varepsilon)|u_s(t) - u_s(t^-)|} \end{aligned}$$

hence

$$V(t) \le e^{\ell(\varepsilon)|u_s(t)-u_s(t^-)|} V(t^-). \tag{32}$$

Iteratively combining the bounds (30) and (32) and recalling (12) and (13), we arrive at

$$\begin{aligned} V(t_2) &\le \exp\Big[\big(-\tfrac{c_3}{c_2}+\tfrac{c_5}{c_1}\varepsilon\delta_1\big)(t_2-t_1) \\ &\qquad +\big(\tfrac{c_4}{c_1}+\ell_2\varepsilon^2+\tfrac{c_5}{c_1}\varepsilon\max\{\delta_2,\phi\}\big)\textstyle\int_{t_1}^{t_2}\|du_s\|\Big]V(t_1) \\ &\le \exp\Big[\big(-\tfrac{c_3}{c_2}+\tfrac{c_5}{c_1}\varepsilon\delta_1\big)(t_2-t_1) \\ &\quad +\big(\tfrac{c_4}{c_1}+\ell_2\varepsilon^2+\tfrac{c_5}{c_1}\varepsilon\max\{\delta_2,\phi\}\big)(\mu(t_2-t_1)+\alpha)\Big]V(t_1) \end{aligned}$$

for an arbitrary interval $[t_1,t_2]$, and exponential convergence follows if

$$\big(-\frac{c_3}{c_2}+\frac{c_5}{c_1}\varepsilon\delta_1\big)+\mu\big(\frac{c_4}{c_1}+\ell_2\varepsilon^2+\frac{c_5}{c_1}\varepsilon\max\{\delta_2,\phi\}\big)<0. \tag{33}$$

For $\varepsilon=0$ this reduces to (14) which we assumed to be true; therefore, (33) also holds for $\varepsilon>0$ small enough.

To finish the proof of the theorem, take arbitrary numbers $R>0$ as in the theorem statement and $0<\underline{\Lambda}\le 1\le\bar{\Lambda}$ as in Lemma 2. Define the set Ω to be the closed ball centered at the origin with radius $\bar{c}R\bar{\Lambda}/\underline{\Lambda}$, where $\bar{c}$ satisfies

$$\bar{c} > e^{\frac{1}{2}\frac{c_4}{c_1}\alpha}\sqrt{c_2/c_1}. \tag{34}$$

Relative to this Ω, use Lemmas 2–4 to compute the quantities $\bar{\varepsilon},\delta_1,\delta_2,\phi$, as well as the Lipschitz constant L_V from the proof of Lemma 4, ensuring (by decreasing $\bar{\varepsilon}$ if needed) that the second statement of Lemma 2 holds with the selected $\underline{\Lambda},\bar{\Lambda}$. Find a positive $\varepsilon\le\bar{\varepsilon}$ that satisfies (33) and call it ε^*. In view of the bounds $c_1|y(t)|^2\le V(t)$ and $V(0)\le c_2|y(0)|^2$, which follow from (8), we see that every solution with $|x(0)|\le R$ (hence $|y(0)|\le R/\underline{\Lambda}$) satisfies, in the y-coordinates, the bound

$$|y(t)| \le e^{\frac{1}{2}(\frac{c_4}{c_1}+\ell_2\varepsilon^2+\frac{c_5}{c_1}\varepsilon\max\{\delta_2,\phi\})\alpha}\sqrt{\frac{c_2}{c_1}}R/\underline{\Lambda}.$$

Therefore, by further decreasing ε^* if necessary, we can ensure that $x(t)$ remains in the interior of Ω for all $t\ge 0$ when $\varepsilon\le\varepsilon^*$. Thus the above analysis is applicable and establishes exponential convergence of $|y(t)|$ to 0, and consequently exponential convergence of $|x(t)|$ to 0, as claimed in the theorem—with $c:=\bar{c}\bar{\Lambda}/\underline{\Lambda}$ and $-\lambda$ being the left-hand side of (33) with $\varepsilon=\varepsilon^*$. □

IV. Conclusions

We considered nonlinear systems with two kinds of input signals: a slowly-varying one and a fast-varying periodic one, with both being allowed to have discontinuities. Under the assumption that the origin is a common equilibrium for all values of these input signals, our main result (Theorem 1) provided sufficient conditions for this equilibrium to be semi-globally exponentially stable. Our approach relied on constructing a (partial) average system which retains the slow variation but not the fast one. The stability conditions were formulated in terms of a suitable Lyapunov function for this average system (for frozen values of the slowly-varying input), along with a bound on the total variation of this input which guarantees stability of the average system. The proof involved expressing the original system in suitable coordinates as a perturbation of the average system, and conducting perturbation analysis to establish stability of the former. Future work will focus on lifting the common equilibrium assumption and on addressing more general classes of nonlinear systems.

References

[1] Y. U. Choi, H. Shim, and J. H. Seo. A stability property of nonlinear systems with inputs having slowly varying average and its application to HIV problem. In *Proc. 2005 American Control Conf.*, pages 3025–3029, 2005.

[2] X. Gao, D. Liberzon, and T. Başar. On stability of nonlinear slowly time-varying and switched systems. In *Proc. 57th IEEE Conf. on Decision and Control*, pages 6458–6463, 2018.

[3] X. Gao, D. Liberzon, J. Liu, and T. Başar. Unified stability criteria for slowly time-varying and switched linear systems. *Automatica*, 96:110–120, 2018.

[4] R. Goebel, R. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems*. Princeton University Press, 2012.

[5] J. K. Hale. *Ordinary Differential Equations*. John Wiley, 1969.

[6] P. A. Ioannou and J. Sun. *Robust Adaptive Control*. Prentice Hall, New Jersey, 1996.

[7] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, 3rd edition, 2002.

[8] S. R. Kou, D. L. Elliott, and T. J. Tarn. Observability of nonlinear systems. *Information and Control*, 22:89–99, 1973.

[9] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.

[10] D. Liberzon and H. Shim. Stability of linear systems with slow and fast time variation and switching. In *Proc. 61st IEEE Conf. on Decision and Control*, pages 674–678, 2022.

[11] J. Peuteman and D. Aeyels. Exponential stability of nonlinear time-varying differential equations and partial averaging. *Mathematics of Control, Signals and Systems*, 15(1):42–70, 2002.

[12] H. Shim and D. Liberzon. Further results on stability of linear systems with slow and fast time variation and switching. In *Proc. 27th ACM Int. Conf. Hybrid Systems: Computation and Control (HSCC 2024)*, pages 1–7, 2024.

[13] A. R. Teel, L. Moreau, and D. Nesic. A unified framework for input-to-state stability in systems with two time scales. *IEEE Trans. Automat. Control*, 48:1526–1544, 2003.

[14] A. R. Teel and D. Nešić. Averaging theory for a class of hybrid systems. *Dynamics of Continuous, Discrete and Impulsive Systems A: Mathematical Analysis*, 17:829–851, 2010.

[15] W. Wang and D. Nesic. Input-to-state stability and averaging of linear fast switching systems. *IEEE Transactions on Automatic Control*, 55:1274–1279, 2010.