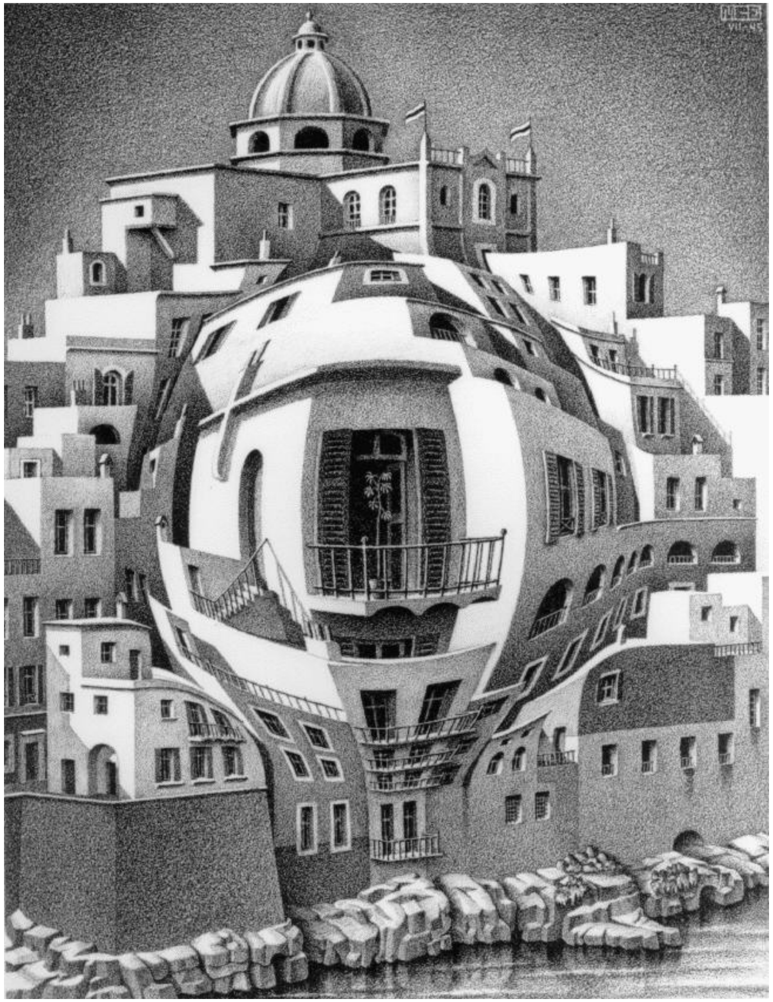
Representing Curves – Part II

Foley & Van Dam, Chapter 11

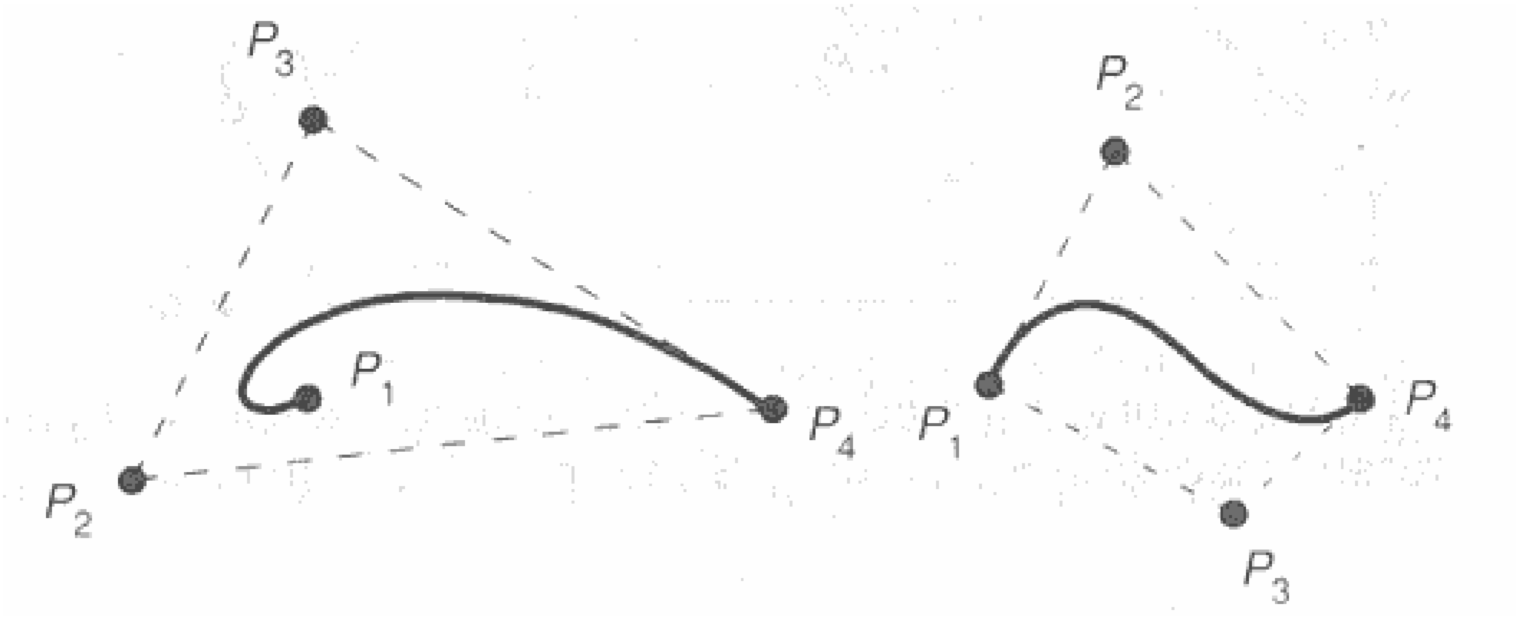


Representing Curves

- Polynomial Splines
  - Bezier Curves
  - Cardinal Splines
  - Uniform, non rational B-Splines
- Drawing Curves
- Applications of Bezier splines

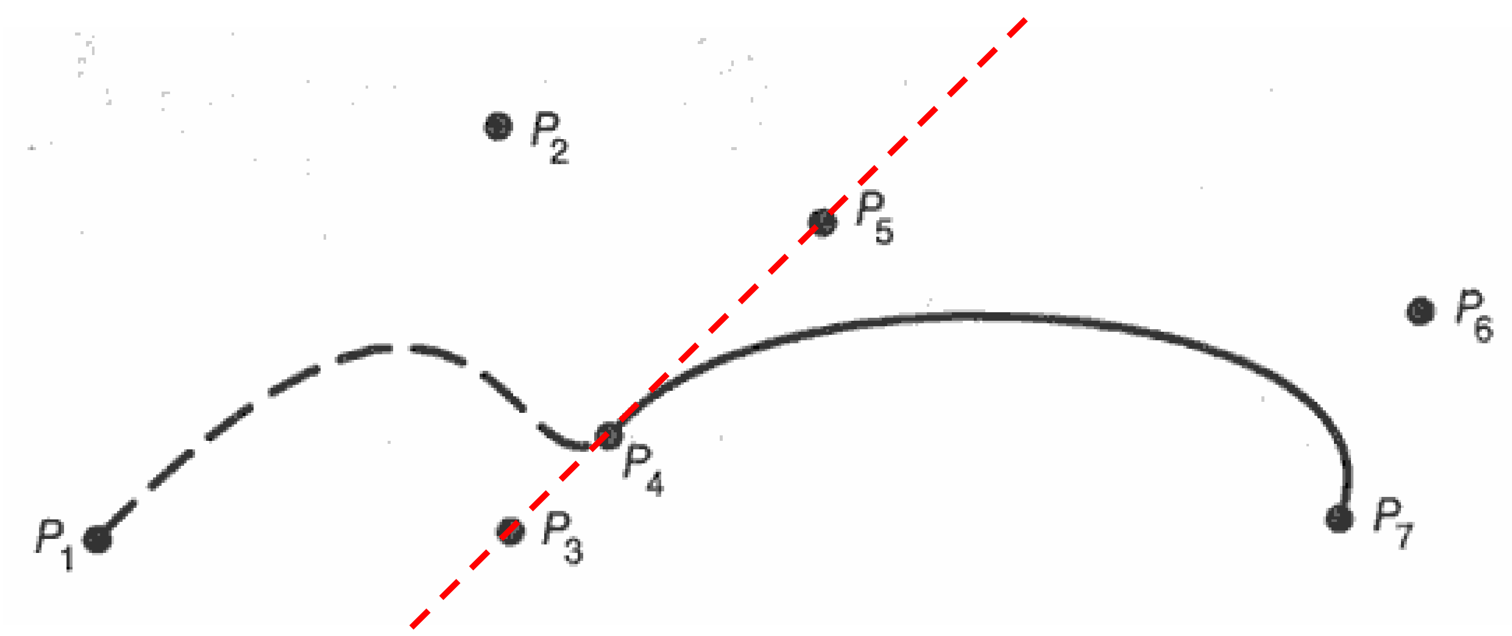
Bezier Curves

- Specify indirectly the tangent with two points that are not on the curve



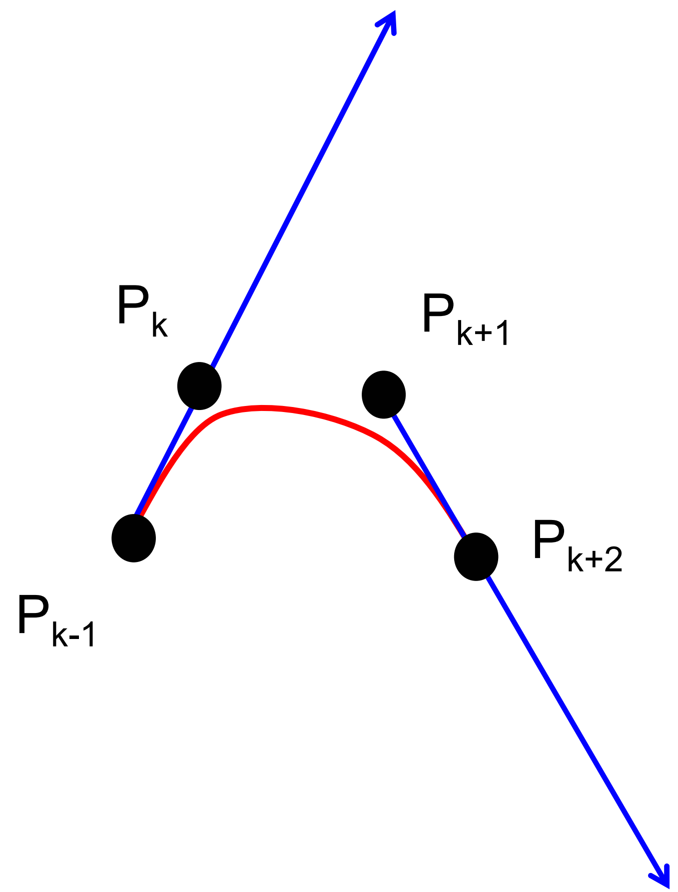
Bezier Curves

- Easy to enforce C^0 and C^1 continuity. If P_{j-1}, P_j and P_{j+1} are collinear then the curve is G^1 in P_j
- Similar to Cardinal Splines



Bezier Curves

- No need to supply tangents
- For each segment curve between P_{k-1} and P_{k+2}, we have:
  - $V(0)=P_{k-1}$
  - $V(1)=P_{k+2}$
  - $V'(0)=3(P_k - P_{k-1})$
  - $V'(1)=3(P_{k+2} - P_{k+1})$



Bezier Curves

- The relation between the Hermite geometry vector and the Bezier geometry vector is:

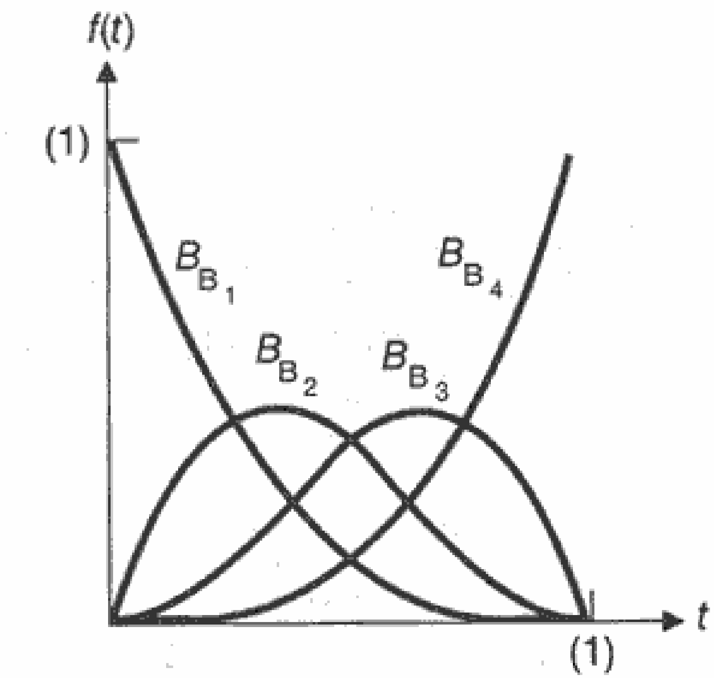
$$G_H = \begin{bmatrix} p_k \\ p_{k+1} \\ T_k \\ T_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} = M_{BH} G_B$$

- Combining with the Hermite interpolation:

$$V(u) = U(u) M_H M_{BH} G_B = \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

Bezier Curves

$$V(u) = U(u) M_H M_{BH} G_B = \underbrace{\begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\text{Blending functions}} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$



$U(u)M_H M_{BH}$: Bernstein polynomials:

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

with

$$\binom{n}{i} = \frac{n!}{(n-i)!\,i!}$$

Note: Assume values between 0 and 1

Bezier Curves

- Bezier curves produces C^1 continuous curves
- Linear (convex) combination of 4 basis functions.

Alternatively, it is a convex combination of 4 control points

- **Advantage:**
  - No need for tangents
  - Curve always contained in the convex hull
- **Disadvantage:**
  - Tangent approximation might be imprecise

- NOTE: The constant 3 is obtained by assuming that

$$V'(0)=\beta(P_k-P_{k-1}) \text{ and } V'(1)=\beta(P_{k+2}-P_{k+1})$$

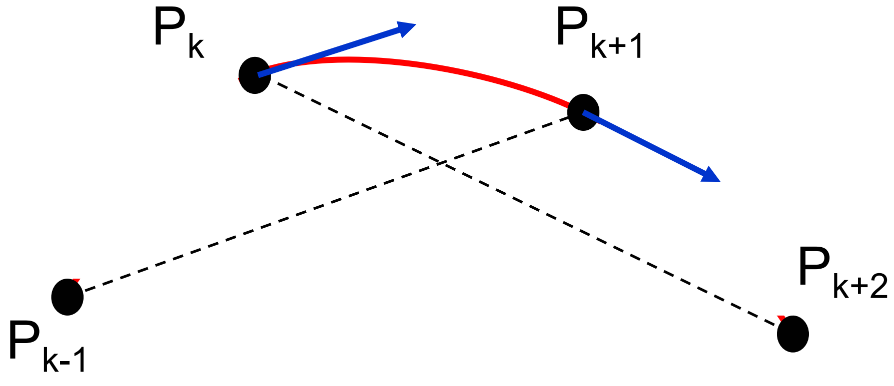
Then deriving β such that the Bezier curve between

$$P_{k-1}=(0,0),\ P_k=(1,0),\ P_{k+1}=(2,0),\ P_{k+2}=(3,0)$$

has constant velocity between P_{k-1} and P_{k+2}

Cardinal Splines

- No need to supply tangents
- For each segment curve between P_k and P_{k+1}, we have:
  - $V(0)=P_k$
  - $V(1)=P_{k+1}$
  - $V'(0)=s(P_{k+1}- P_{k-1})$ (s = Tension Parameter)
  - $V'(1)=s(P_{k+2}- P_k)$



Cardinal Splines

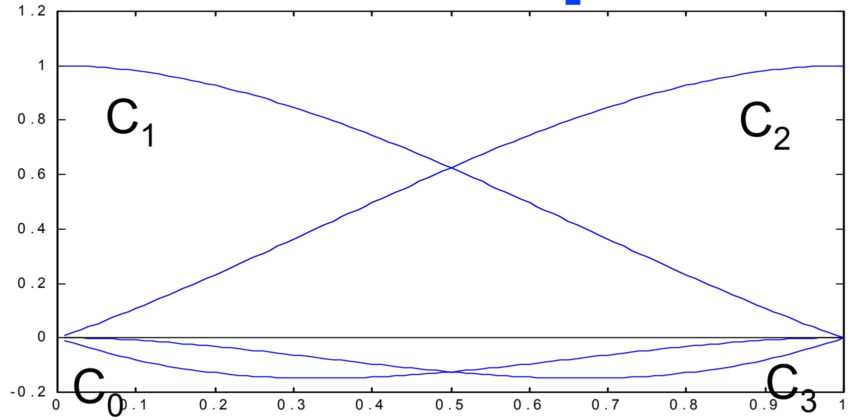
- The relation between the Hermite geometry vector and the Cardinal geometry vector is:

$$G_H = \begin{bmatrix} p_k \\ p_{k+1} \\ T_k \\ T_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix} = M_{HC} G_C$$

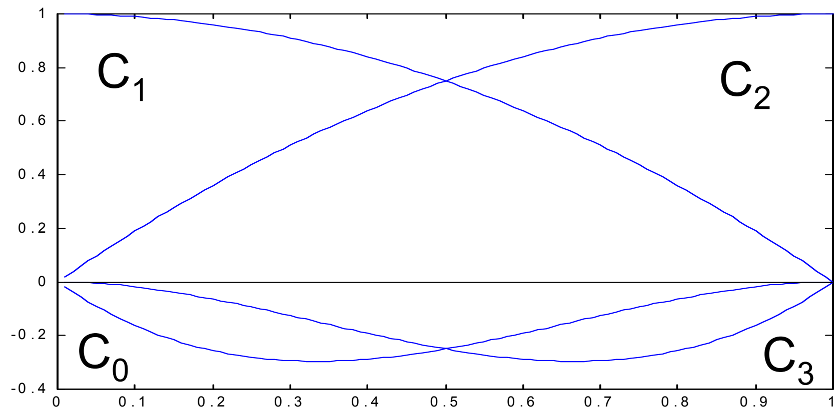
- Combining with the Hermite interpolation:

$$V(u) = U(u) M_H M_{CH} G_C = \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

Cardinal Splines



Cardinal spline blending functions for s=1

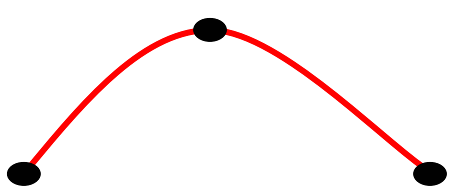


Cardinal spline blending functions for s=2

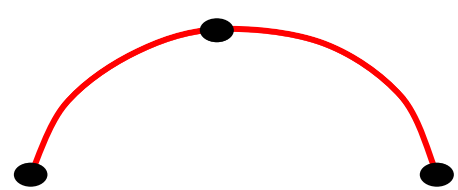
$$V(u) = C_0(u)P_{k-1} + C_1(u)P_k + C_2(u)P_{k+1} + C_3(u)P_{k+2}$$

Cardinal Splines

- Cardinal spline produces C^1 continuous curve
- Linear combination of 4 basis functions. Alternatively, it is a linear combination of 4 control points
- **Advantage:** No need for tangents
- **Disadvantage:** Tangent approximation might be imprecise



Small tension



Big tension

B-Spline Approximation

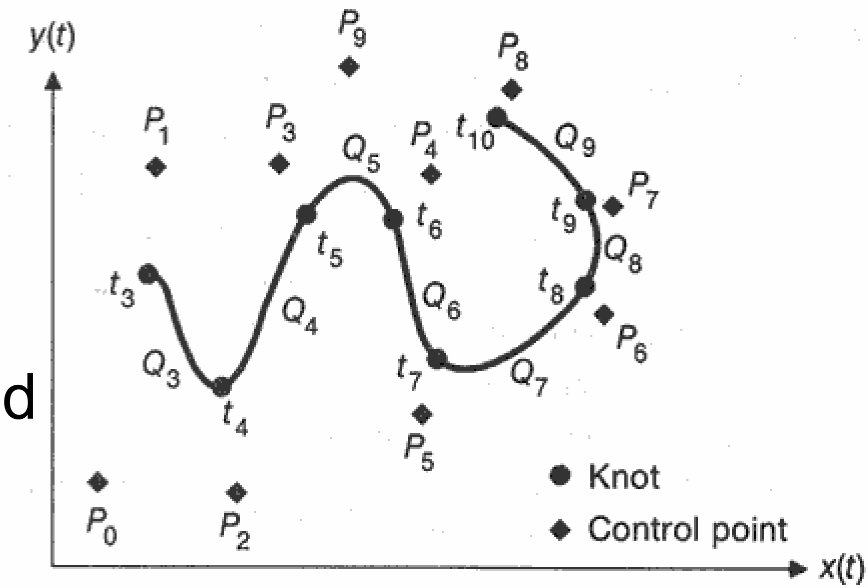
- A cubic B-spline (Basis Spline) approximates $m \geq 3$ points $P_0, P_1, \ldots, P_m$ with a curve consisting of m-2 cubic polynomial curve segments $Q_3, Q_4, \ldots, Q_m$

- Segment $Q_i(t)$ is defined for $t \in [0,1]$, but with the variable substitution t=t+k we can make the domains of the segments sequential so that $Q_i(t)$ is defined for $t_i \leq t < t_{i+1}$

- The values of the curve for t_i, $i \geq 3$ are called knots (there are m-1 knots)

- The segment Q_i, $i \geq 3$ is defined by 4 control points:

  $P_{i-3}, P_{i-2}, P_{i-1}, P_i$



B-Spline Approximation

For Hermite and Bezier curves we have:

$$Q(t) = T \cdot M \cdot G, \quad t \in [0,1]$$

Define

$$T_i = [(t - t_i)^3, (t - t_i)^2, (t - t_i), 1]$$

The B-spline formulation for the segment $Q_i(t)$ is:

$$\begin{aligned} Q_i(t) &= T_i \cdot M_{Bs} \cdot G_{Bs} \\ &= \underbrace{\begin{bmatrix}(t - t_i)^3 & (t - t_i)^2 & (t - t_i) & 1\end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}}_{\text{Blending functions}} \cdot \underbrace{\begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}}_{\text{Geometry matrix}} \end{aligned}$$

B-Spline Approximation

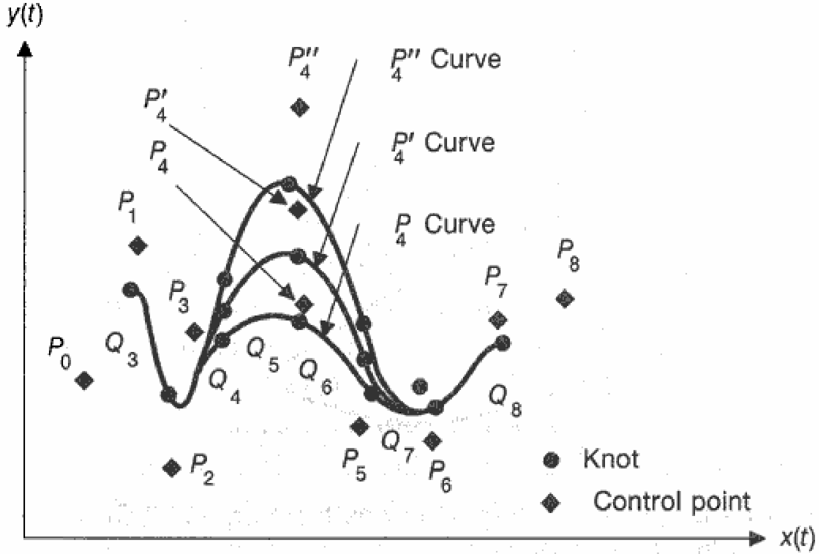
- B-splines are uniform when the knots are spaced at equal intervals of t

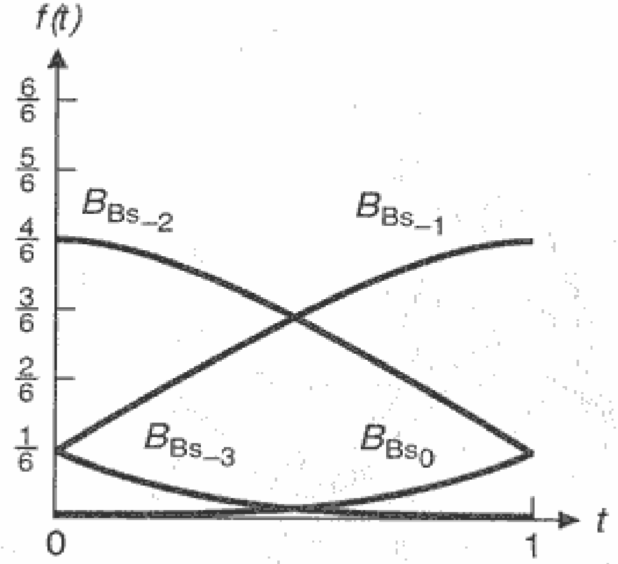
- Uniform B-splines use the same blending function for each $Q_i(t)$

- Non rational when x(t), y(t) and z(t) are not defined as ratio between two polynomial

B-Spline Approximation

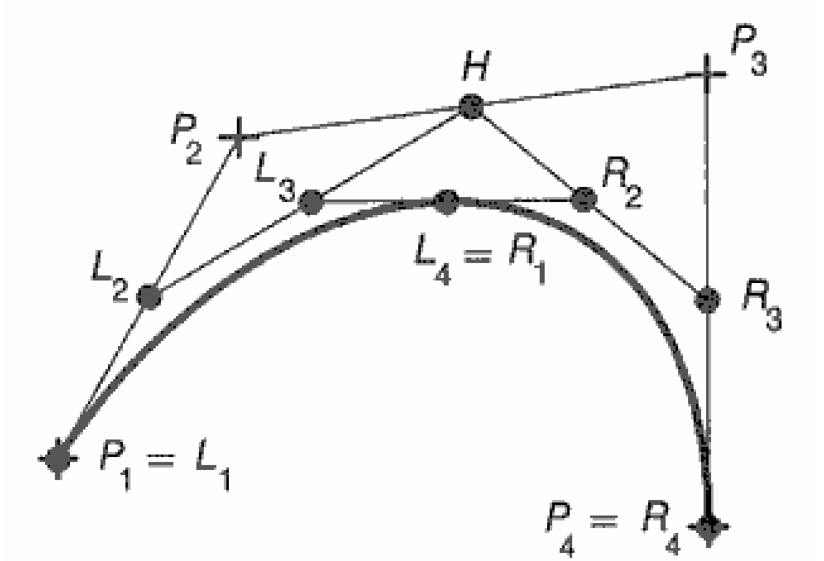
- **Properties:**
  - Changing the control points has a local effect
  - B-splines are C^0, C^1 and C^2 continuous
  - The curve is contained in the convex hull defined by the control points
  - The curve can be closed by repeating the first three control points: $P_0, P_1, P_2, \ldots, P_m, P_0, P_1, P_2$
  - Little control on where the spline goes (drawback)

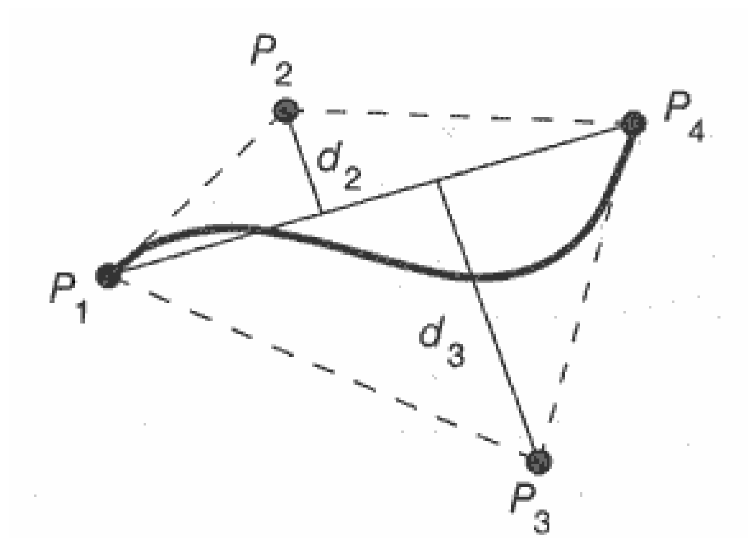




Drawing Curves

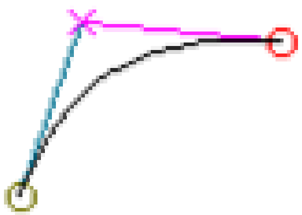
- Direct evaluation of the parametric polynomial (Horner’s rule)
- Forward finite differences (explained in class)
- Recursive subdivision (stop dividing and draw a line when segment is flat)





Applications of Bezier splines

- A quadratic Bezier curve is defined in terms of three control points:

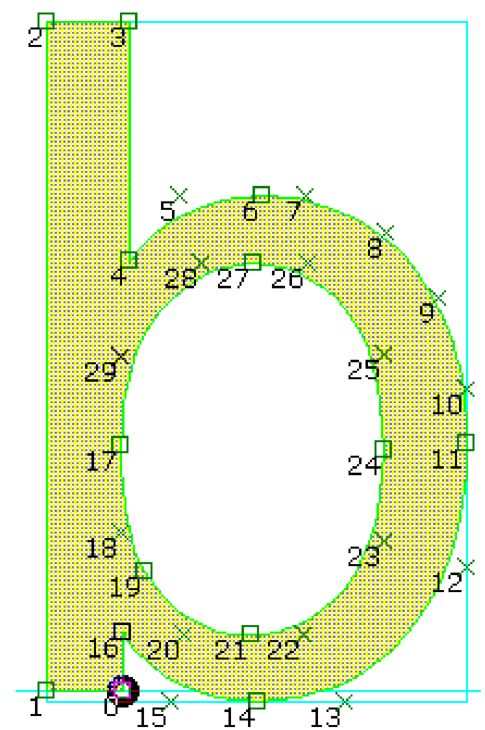


$$V(u) = U(u)MG = \begin{bmatrix} u^2 \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \end{bmatrix}$$

- It is always possible to represent a quadratic Bezier curve with a cubic Bezier curve (just set to zero u^3 coefficient)
- The inverse is not true: most cubic curves cannot be represented exactly by quadratic curves. Sometimes, not even by series of quadratic curves

Applications of Bezier splines

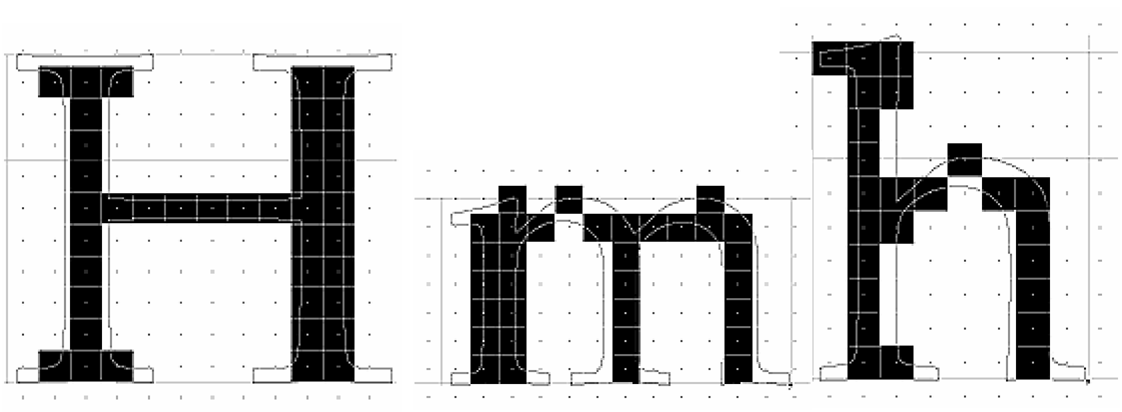
- Outlines of Postscript and TrueType characters are defined in terms of Bezier curves
- Postscript uses cubic forms and TrueType uses quadratic forms
- Converting TrueType to Postscript is trivial; the opposite can be done only with approximations

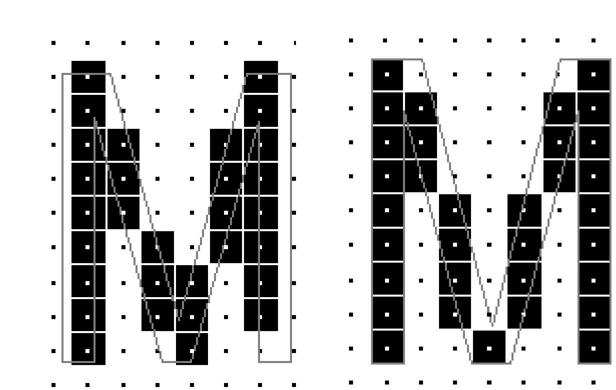


Control points defining the outline of the letter 'b' of Monotype Arial.
On-curve points are indicated with a square and off-curve points with crosses

Applications of Bezier splines

- This representation can be scaled to arbitrary sizes, rotated, etc..
- Finally the outline is rasterized and filled
- The final quality can be improved with antialiasing
- When the font size is small, “hinting” may be necessary to improve symmetry and readability





Before hinting After hinting