Vector Calculus

Foley & Van Dam, Appendix

Vector Calculus

- Scalars, points and vectors
- Fundamental operations
- Linear combination of vectors
- Dot product
- Cross product
- Vector representation in 3D
- Vector operations in orthonormal basis
- Parametric line representation
- Parametric plane representation

Mathematical Entities

In computer graphics most geometric objects can be defined using three basic entities: *Scalars, Points, and Vectors*

Scalar:
Serves as unit of measurements, such as length or degrees

Point:
Its attributes define a location in space (representation can be Cartesian, polar, cylindrical,…)
Points exist in space regardless of any coordinate system

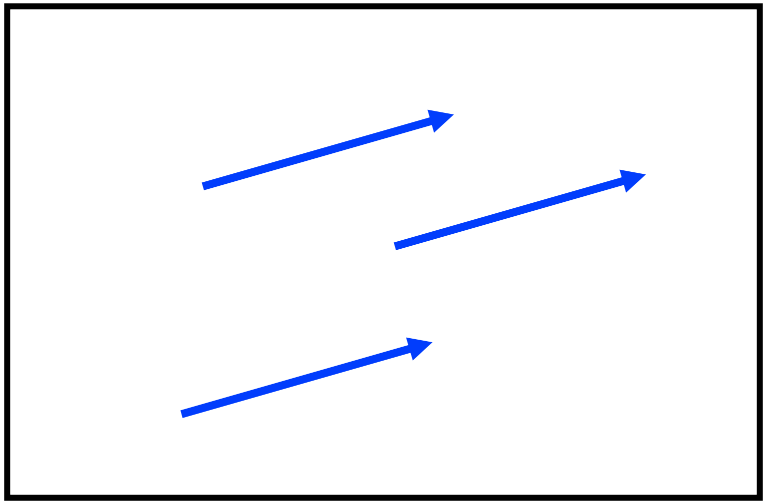
Mathematical Entities

Vector:

A quantity with direction and magnitude (such as force, velocity, ..)

Vectors exist regardless of any coordinate system

A vector does not have a fixed position

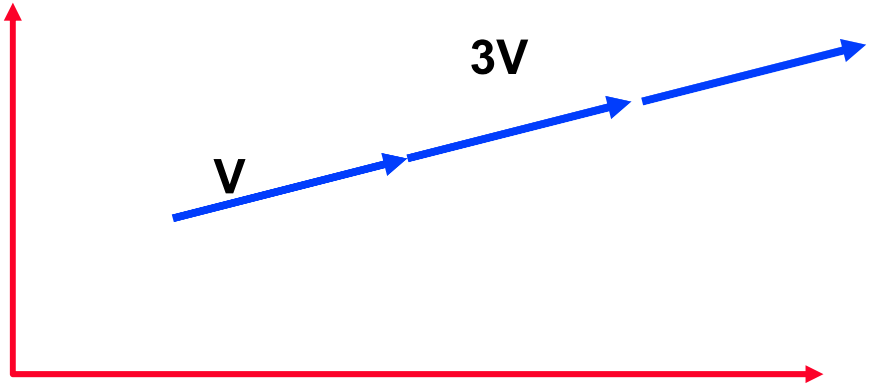


Identical vectors

Fundamental Operations

Vector-scalar multiplication:

- A multiplication of scalar α with a vector v gives a vector
- The vectors αv and v have same direction
- Magnitude $|\alpha\, v| = \alpha\, |v|$



Fundamental Operations

Point-vector addition:

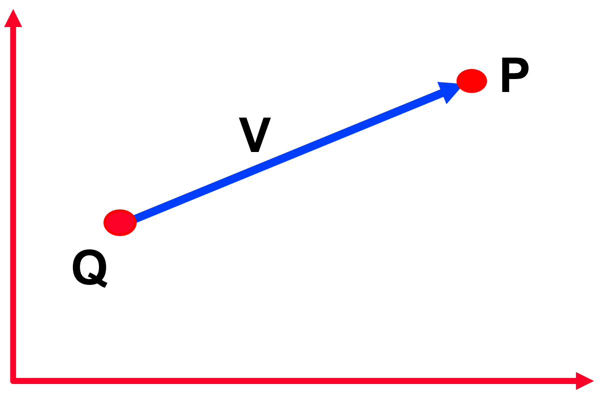
- Subtraction of two points P and Q gives a vector v:

$$v = P - Q$$

- Adding a vector v to point Q gives a point P:

$$P = Q + v$$

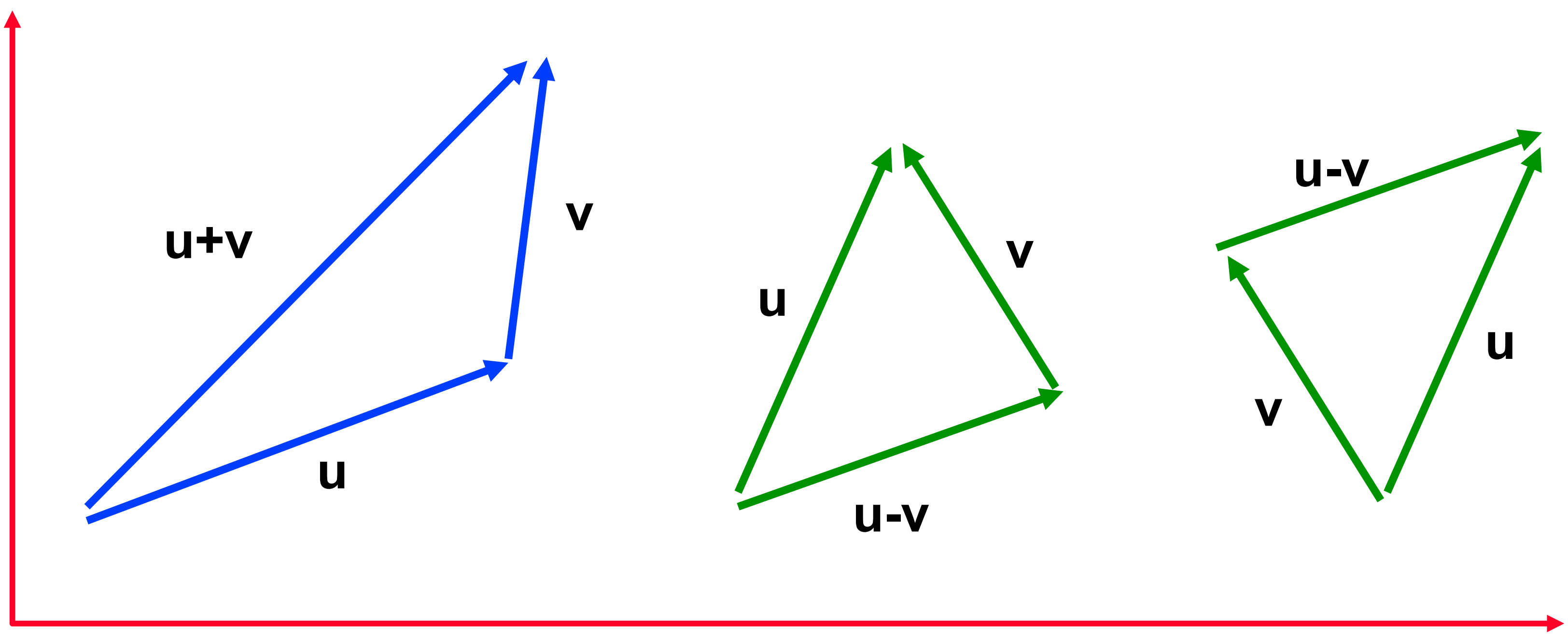
- If Q is located at the origin then $P = v$



Fundamental Operations

Vector-vector addition:

- Adding two vectors u and v gives a new vector
- The resulting vector follows the **head-to-tail rule**



Linear Combination of Vectors

A linear combination of m vectors:

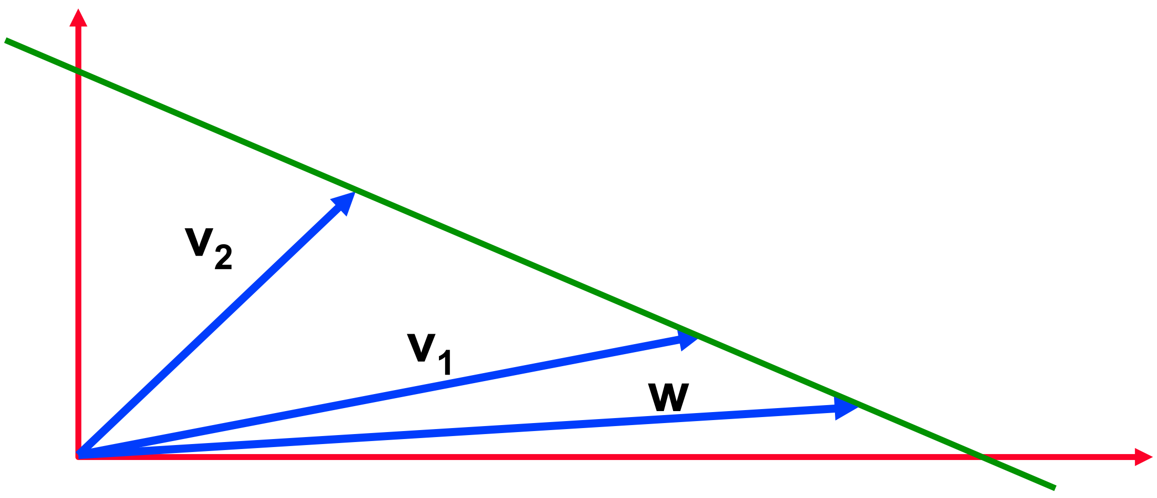
$$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$$

The combination is *Affine* if the coefficients sum to 1

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$$

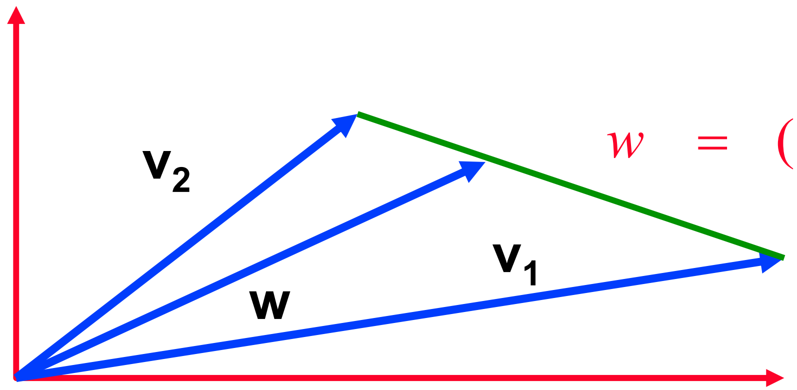
Example:

$$w = (1 - t) v_1 + t v_2 = v_1 + t\left(v_2 - v_1\right)$$



Linear Combination of Vectors

The combination is *Convex* if the coefficients sum to 1, and are not negative: $\alpha_i \geq 0 \quad , i = 1..m$

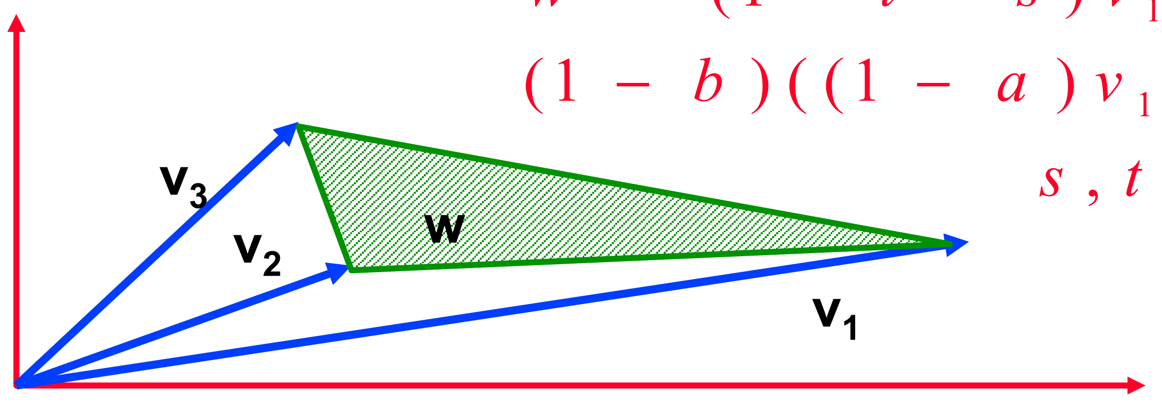


$$w = (1-t)v_1 + tv_2 \qquad t \in [0,1]$$

$$w = (1-t-s)v_1 + tv_2 + sv_3 =$$

$$(1-b)((1-a)v_1 + av_2) + bv_3$$

$$s,t,a,b \in [0,1]$$



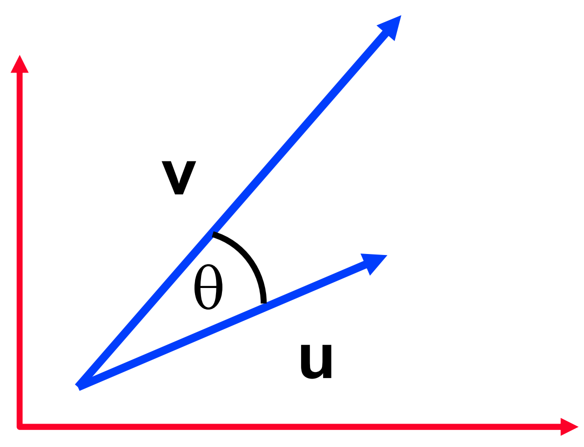
Fundamental Operations

Vector-vector dot product:

- Dot product between vector v and vector u gives a **scalar**:

$$v \cdot u = |v|\,|u| \cos\theta$$

- If u and v are ***orthogonal*** $v \cdot u = 0$
- The ***magnitude*** $|v|$ of a vector can be given using a dot product: $|v| = \sqrt{v \cdot v}$
- The vector $\hat{v} = v / |v|$ is a unit vector having the same direction as *v*



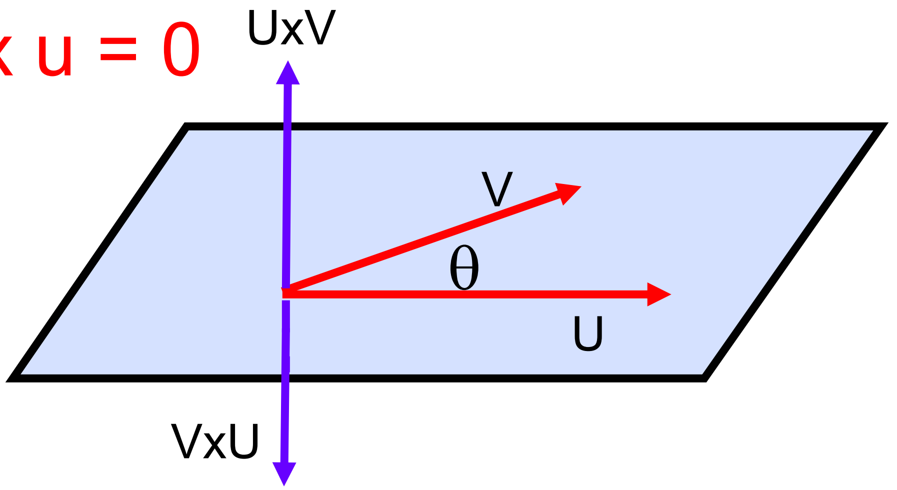
Fundamental Operations

Vector-vector cross product:

- Cross product between vector v and vector u gives a **vector**:

$$v \times u = \hat{n} \, |v| \, |u| \sin \theta$$

- n is a unit vector perpendicular to both u and v whose direction follows the right-hand rule
- If u and v are **parallel** v x u = 0



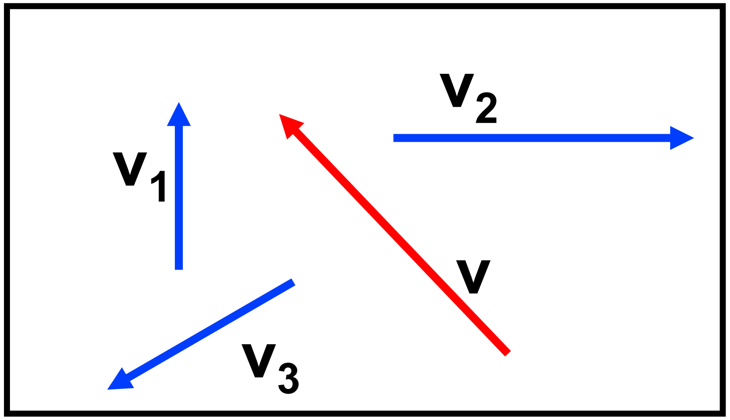
Vector Representation in 3D

Given 3 linearly independent vectors in 3D: v_1, v_2, and v_3, we can represent any vector v in 3D as a linear sum:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

The scalars $\alpha_1, \alpha_2, \alpha_3$ are the components of v with respect to the **basis** $[v_1, v_2, v_3]$

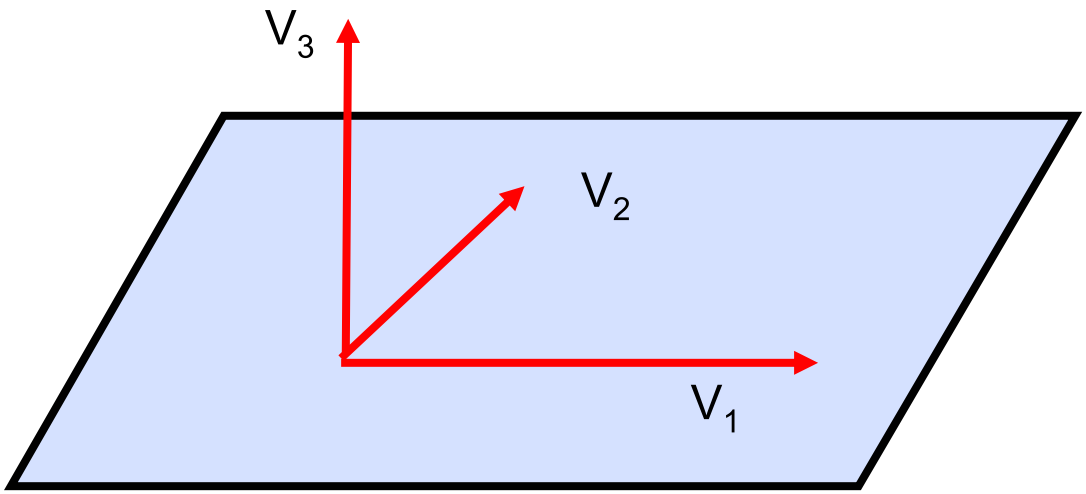
We write this as the column matrix: $v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$



Vector Representation in 3D

- If the vectors v_1, v_2, v_3 are mutually orthogonal, i.e. $v_1 \cdot v_2 = v_1 \cdot v_3 = v_2 \cdot v_3 = 0$ and $|v_1| = |v_2| = |v_3| = 1$ the basis $[v_1, v_2, v_3]$ forms an orthonormal basis

- If $v_1 \times v_2 = v_3$ the basis is right-handed



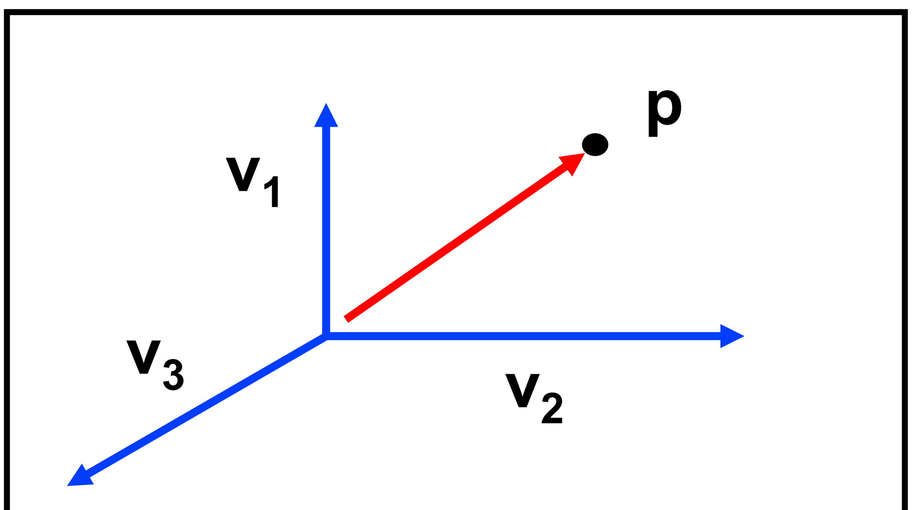
Point Representation in 3D

- To represent a point P in 3D we have to add to the basis $[v_1,v_2,v_3]$ an additional point P_0, such that:

$$P = P_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

- A basis set of vectors and a particular point P_0 define a *frame*
- The point P_0 is the origin of the frame
- We write this as a column matrix:

$$P = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$



Vector Operations in Orthonormal Basis

- Scalar multiplication:

$$\alpha \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \alpha\beta_3 \end{pmatrix}$$

- Vector addition:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_3 \end{pmatrix}$$

- Dot product:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \bullet \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}^{T} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$$

Vector Operations in Orthonormal Basis

- Cross product:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \times \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \end{pmatrix} = \begin{vmatrix} \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}$$

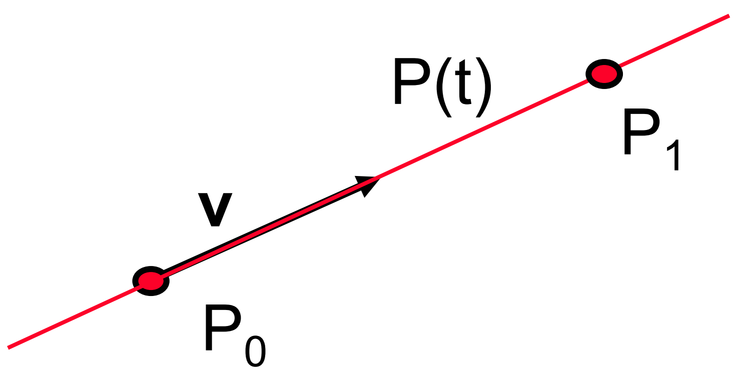
Parametric Lines

- A set of points of the form

$$P(t)=P_0 + t\,v$$

defines a line passing through point P_0 and parallel to vector v

- A line segment from P_0 to P_1 can be defined as $P(t)=P_0 + t\,(P_1 - P_0)$, where $t = 0..1$
- **Convex sum**: The above definition can be rewritten as: $P(t) = (1 - t)\,P_0 + t\,P_1$



Parametric Planes

- A plane is defined by 3 points P, Q, and R
- The line segment between P and Q is defined by the convex sum:

$$S(\alpha) = (1 - \alpha)\,P + \alpha\,Q\,, \quad \alpha = 0..1$$

- Similarly, the line segment between $S(\alpha)$ and R is:

$$T(\alpha, \beta) = (1 - \beta)\,S(\alpha) + \beta\,R\,, \quad \beta = 0..1$$

- $T(\alpha, \beta)$ defines all the points in the triangle PQR