Stony Brook University

ESE/EEO 306
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Sample Midterm Solution
Total Score: 36 points
All problems carry equal weight of 6 points each
Please show your work and justify your answers

1. A 5-card poker hand is said to be a full house if it consists of 3 cards of the same rank, and the other 2 cards of another same rank. What is the probability of being dealt a full house? Note that here the cards are drawn **without replacement**.

   Solution:

   There are $\binom{52}{5}$ ways of selecting 5 cards out of 52. This is the denominator in the probability calculation.

   For the numerator, there are $\binom{4}{3}$ ways of selecting 3 cards out of 4, of the same rank.

   And there are $\binom{4}{2}$ ways of selecting the remaining 2 cards out of 4, of another same rank.

   Further, there are 13×12 ways of selecting two different ranks out of 13.

   Therefore, the required probability is

$$P\left(3 \text{ cards of the same rank, } 2 \text{ cards of another same rank}\right) = \frac{(13)(12)\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}.$$

2. Urn I contains 10 white and 5 red balls. Urn II contains 3 white and 6 red balls. We select an Urn at random, draw a ball from it and find that it is red. What is the probability that it is drawn from Urn II? Note that the probability of selecting either Urn is 1/2.

   Solution:

   We know that the outcome of the experiment is the red ball. Therefore, we utilize **Bayes' theorem**:

$$P\left(U_{II}|\text{Red}\right) = \frac{P\left(\text{Red}|U_{II}\right)\, P\left(U_{II}\right)}{P\left(\text{Red}\right)}.$$

   Further, $P\left(\text{Red}|U_{II}\right) = \frac{6}{9}$, as Urn II has 6 red balls out of total 9 balls.

   $P\left(U_{II}\right) = 1/2$; this is the probability of selecting either Urn.

   Next, we want to compute the denominator $P\left(\text{Red}\right)$. This is done using **total probability theorem**:

$$\begin{aligned} P\left(\text{Red}\right) &= P\left(\text{Red}|U_I\right) P\left(U_I\right) + P\left(\text{Red}|U_{II}\right) P\left(U_{II}\right) \\ &= \left(\frac{5}{15}\right)\left(\frac{1}{2}\right) + \left(\frac{6}{9}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{2}. \end{aligned}$$

   Substituting in Bayes' theorem, we obtain,

$$P\left(U_{II}|\text{Red}\right) = \frac{\left(\frac{2}{3}\right)\,\left(\frac{1}{2}\right)}{\frac{1}{2}} = \frac{2}{3}$$

3. Four fair coins are flipped. If the outcomes are assumed independent, find the probability mass function (PMF) of the number of heads obtained.

   Solution:

   Let X be the number of heads obtained. Since each coin flip is independent of the others, X has a binomial distribution with parameters $n = 4$ and $p = 0.5$.

   Note that n is the total number of trials and p is the probability of success in one trial.

   The PMF of X is given by

$$P(X = x) \;=\; \binom{4}{x} (0.5)^x \, (0.5)^{4-x}, \quad x = 0, 1, 2, 3, 4.$$

4. Phone calls arrive at a call center at a rate of 3 calls per minute. The number of calls is modeled as a Poisson random variable X.

   (a) Find the probability that the number of calls received in a 1-minute interval is **exactly** 2.

   (b) Find the probability that the number of calls received in a 1-minute interval is between 2 and 4, inclusive.

   <u>Solution:</u>

   The number of calls X is a Poisson random variable, with the PMF given by:

$$p_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots,$$

   and the rate of calls per minute, $\lambda = 3$

   (a)

$$P(X = 2) = e^{-3} \frac{3^2}{2!}$$

   (b)

$$P(2 \leq X \leq 4) = e^{-3} \frac{3^2}{2!} + e^{-3} \frac{3^3}{3!} + e^{-3} \frac{3^4}{4!}.$$

5. Buses arrive at a specified stop at 15-minute intervals starting at 7 am. That is, they arrive at 7: 00, 7: 15, 7: 30, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7: 00 and 7: 15, find the probability that he waits less than 5 minutes for a bus.

   Solution:

   The arrival time of the customer is a uniform random variable X between 7: 00 and 7: 15. Its PDF is given by:

$$f_X(x) = \frac{1}{15}, \quad 7{:}00 \leq x \leq 7{:}15.$$

   If the passenger arrives between 7: 10 and 7: 15, his wait time will be less than 5 minutes, as the next bus arrives at 7: 15.

   Therefore, probability that the passenger waits less than 5 minutes is

$$\int_{7:10}^{7:15} \frac{1}{15}\, dx = \frac{5}{15} = \frac{1}{3}.$$

6. A continuous random variable X has the probability density function(PDF)

$$f_X(x) = \begin{cases} cx^2, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find

(a) the constant c,

(b) the probability $P(1/3 \leq X \leq 2/3)$, and

(c) the cumulative distribution function (CDF) $F_X(x)$.

Solution:

(a) The constant c is obtained from the following condition for PDF:

$$\int_0^1 f_X(x)\,dx = \int_0^1 cx^2 = 1.$$

Note that the limits of the integral go from 0 to 1, as this is the support (range) of X. This gives, $c = 3$.

(b)

$$\begin{aligned} P(1/3 \leq X \leq 2/3) &= \int_{1/3}^{2/3} f_X(x)\,dx \\ &= \int_{1/3}^{2/3} 3x^2 dx \\ &= x^3\Big|_{1/3}^{2/3} \\ &= 0.259 \end{aligned}$$

(c) In the range $[0, 1]$

$$\begin{aligned} F_X(x) &= \int_0^x f_X(u)\,du \\ &= \int_0^x 3u^2 du \\ &= u^3\Big|_0^x \\ &= x^3. \end{aligned}$$

Therefore,

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^3, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$