LESSON 2

UPW PROPAGATING IN AN ARBITRARY DIRECTION IN SPACE

I. WHAT YOU WILL LEARN IN LESSON 2:

In this lesson, you will learn about how to write or interpret the expression for a UPW propagating in an arbitrary direction in space. In Lesson 1, you learnt about a UPW propagating along a Cartisian co-ordinate axis. Here in Lesson 2, you will learn about a UPW propagating in any arbitrary direction in space.

II. UPW WAVE FUNCTION FOR PROPAGATION IN AN ARBITRARY DIRECTION:

For the same of simplicity and without any loss of generality we will consider a UPW propagating in arbitrary direction in the xz-plane. Let us interpret the wave function

$$\Phi(z,t) = A\cos(\omega t - \mathbf{k}.\mathbf{r}) \qquad (1)$$

where $\mathbf{r} = x\,\mathbf{i_x} + z\,\mathbf{i_z}$ is an arbitrary observation point in the xz-plane, t is the time variable, and A and ω are, respectively, the same amplitude and radian frequency variables as in Lesson 1. The vector $\mathbf{k}$ is the so-called wave-vector expressed as $\mathbf{k} = k\,\mathbf{i_K}$ in the polar co-ordinate system with k being the same entity called wavenumber as in Lesson 1 and $\mathbf{i_K}$ being the unit vector in the direction of $\mathbf{k}$. The vectors $\mathbf{k}$ and $\mathbf{r}$ are illustrated in Fig. 1.The Cartesian expression for $\mathbf{k}$ is $\mathbf{k} = k_x\,\mathbf{i_x} + k_z\,\mathbf{i_z}$ where $k_x = k\cos\theta_x$ and $k_z = k\cos\theta_z$ are, respectively, the components of $\mathbf{k}$ along the x and z directions. Note that the wavenumber is $k = (k_x^2 + k_z^2)^{1/2}$.

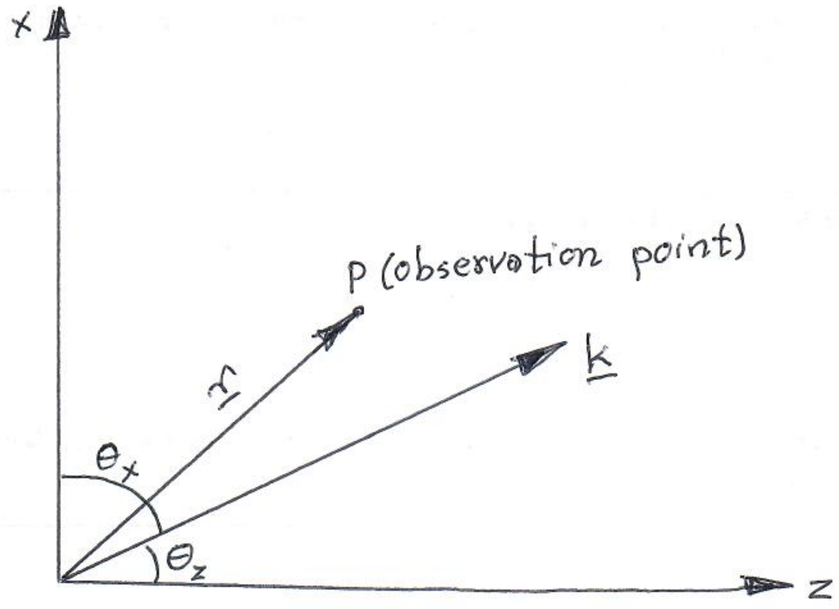


Fig. 1. The vectors **k** and **r** appearing in Eq. (1) are illustrated here

Now the question arises as to how to interpret the physical meaning of the vector dot product **k.r** in Eq. (1). By definition, the dot product of two vectors is simply the product of the magnitude of one vector with the projection of the other vector on to the first vector.

Let us introduce a z' axis passing through the origin and lying along the vector **k** (see Fig. 2).

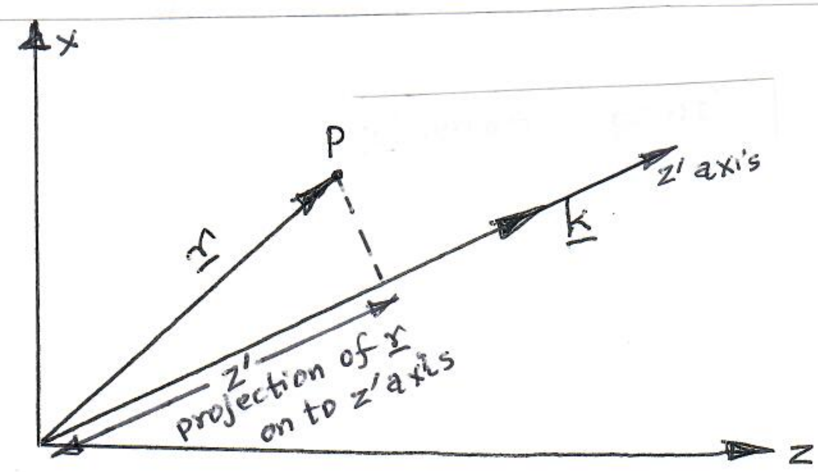


Fig. 2. This figure shows Fig. 1 together with a superimposed z' axis.

Applying the definition of the vector dot product now yields **k.r** = kz'. This in turn means that Eq. (1) may be rewritten as

$$\Phi(z,t) = A\cos(\omega t - kz') \qquad (2)$$

which is an expression for a wavefunction representing a UPW propagating in the z' direction or equivalently in the direction of the vector **k**. Since **k** = k $\mathbf{i}_k$, it follows that one might also label $\mathbf{i}_k$ as equivalently representing the direction of propagation.

SAMPLE EXAMPLE ON DOT PRODUCT OF TWO VECTORS:

What is $\mathbf{r}.\mathbf{i}_x$ equivalent to?

Answer: x

How?

The magnitude of $\mathbf{i}_x$ is 1, and the projection of $\mathbf{r} = x\,\mathbf{i}_x + y\,\mathbf{i}_y + \mathbf{i}_z$ on to the x-axis is x, i.e., $\mathbf{r}.\mathbf{i}_x = x$.

[Ex] A UPEMW (uniform plane electromagnetic wave) propagates in the xy plane at an angle of 30° to the x direction. You are given the following information:

i) E_o (the amplitude of the electric field) = 2 V/m
ii) the wave is linearly polarized in the z direction, i.e, $\dot{\underline{i}}_E = \underline{i}_z$
iii) $f = 1$ GHz
iv) the propagation is in vacuum which means $v_p = c$, where $c = 3 \cdot 10^8$ m/s.
v) $\underline{E}(\underline{r}=0, t=0) = \underline{0}$

Derive the instantaneous expression for the wave electric field.

Solution:

The instantaneous expression for the electric field must be of the general form

$\underline{E}(\underline{r}, t) = \dot{\underline{i}}_E \, E_o \cos(\omega t - \underline{k} \cdot \underline{r} + \delta)$

where $\dot{\underline{i}}_E$, E_o, ω, $\underline{k}$ and δ are quantities to be found

$\underline{E}(\underline{r}=0, t=0) = 0 \longrightarrow \dot{\underline{i}}_E E_o \cos\delta = 0 \rightarrow \cos\delta = 0$ or $\delta = 90°$

$E_o = 2$

$\omega = 2\pi f = 2\pi \cdot 10^9$ rad/s.

$k = \dfrac{\omega}{v_p} = \dfrac{2\pi \cdot 10^9}{3 \times 10^8} = \dfrac{20\pi}{3} \dfrac{\text{rad}}{\text{m}}$

$\underline{k} = k \dot{\underline{i}}_E$ but $\underline{i}_k = \underline{i}_x \cos 30° + \underline{i}_y \cos 60° = \underline{i}_x \frac{\sqrt{3}}{2} + \underline{i}_y \frac{1}{2}$

$\underline{k} \cdot \underline{r} = \frac{20\pi}{3} \cdot \left(\underline{i}_x \frac{\sqrt{3}}{2} + \underline{i}_y \frac{1}{2}\right) \cdot (x \underline{i}_x + y \underline{i}_y + z \underline{i}_z) = \frac{20\pi}{3}\left(\frac{\sqrt{3}}{2} x + \frac{1}{2} y\right)$

$= \dfrac{10\pi}{\sqrt{3}} x + \dfrac{10\pi}{3} y$

$\therefore \underline{E}(\underline{r}, t) = \underline{i}_z \, 2 \cos\left(2\pi \cdot 10^9 t - \frac{10\pi}{\sqrt{3}} x - \frac{10\pi}{3} y + 90°\right), \frac{V}{m}$ Answer

which, since $\cos(A+B) = \cos A \cos B - \sin A \sin B$, may also be written as

$\underline{E}(\underline{r}, t) = \underline{i}_z \, {-2} \sin\left(2\pi \cdot 10^9 t - \frac{10\pi}{\sqrt{3}} x - \frac{10\pi}{3} y\right), \frac{V}{m}$ Alternate Answer

SAMPLE PROBLEM ON FINDING THE DIRECTION OF PROPAGATION OF A UPW FROM A GIVEN INSTANTANEOUS EXPRESSION FOR THE WAVEFUNCTION:

The instantaneous expression for a UPW is given to be

$\Phi(\mathbf{r},t) = 5\cos(6\pi.10^9 t - 10\sqrt{2}\,\pi x - 10\sqrt{2}\,\pi y)$.

Find the direction of propagation of the wave and the phase velocity v_p.

SOLUTION:

Comparing the given expression for the wave to the standard expression

$\Phi(\mathbf{r},t) = 5\cos(\omega t - k_x x - k_y y)$,

one gets

$k_x = k_y = 10\sqrt{2}\,\pi$

and thus

$\theta_x = \tan^{-1}(k_y/k_x) = 45^0$ and $\underline{i}_k = \underline{i}_x \cos\theta_x + \underline{i}_y \sin\theta_x = \frac{1}{\sqrt{2}}(\underline{i}_x + \underline{i}_y)$

$k = (k_x^2 + k_y^2)^{1/2} = 20\pi$ rad/m

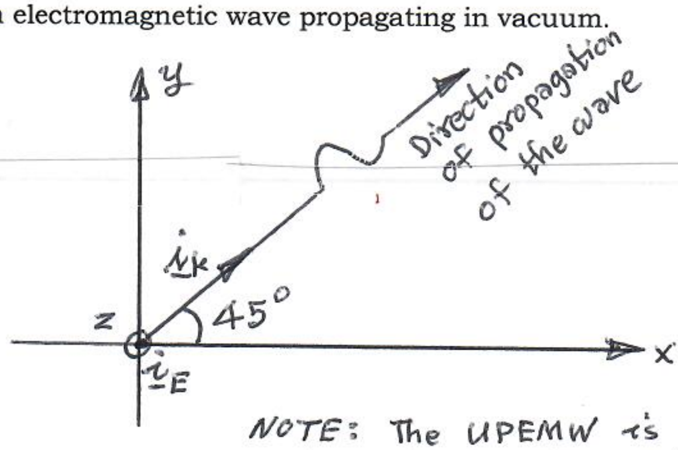
Thus, the wave propagates in the xy plane at an angle of 45° to the x or y direction *Answer*

Also, $\omega = 6\pi.10^9$ rad/s,

hence

$v_p = \omega/k = 3.10^8$ m/s, *Answer*

i.e., the wave is an electromagnetic wave propagating in vacuum.



NOTE: The UPEMW is linearly polarized in the z direction, i.e., $\underline{i}_E = \underline{i}_z$