ESE/EEO 306

Probability Basics

1. The set of all possible outcomes of an experiment is known as **sample space**, and it is denoted by Ω.

2. An **event** is a subset of Ω satisfying the properties of σ-algebra.

3. Let $A_1, A_2, \ldots$ be **disjoint** events in Ω. Then we have,
$$\mathbb{P}(A_1 \cup A_2 \cup \ldots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots$$
That is, probability is countably additive.

4. For two events A and B in Ω, not necessarily disjoint, we have
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

5. The probability of the intersection of two events $\mathbb{P}(A \cap B)$ is also denoted by $\mathbb{P}(A, B)$.

Conditional Probability

6. The conditional probability of an event A given the occurrence of another event B is defined by
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A, B)}{\mathbb{P}(B)},$$
where $\mathbb{P}(B) > 0$.

7. Total Probability Theorem: Let $B_1, B_2 \ldots B_n$ be a partition of the sample space Ω, that is
   (a) $B_1 \cup B_2 \cup \ldots B_n = \Omega$ and
   (b) $B_i \cap B_j = \emptyset, \quad i \neq j$.

   Then
$$\mathbb{P}(A) = \sum_{j=1}^{n} \mathbb{P}(A \mid B_j)\, \mathbb{P}(B_j).$$

8. Bayes' theorem:
$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\, \mathbb{P}(B)}{\mathbb{P}(A)}.$$

Independence

9. Two events A and B are independent if
$$\mathbb{P}(A, B) = \mathbb{P}(A)\, \mathbb{P}(B).$$

Combinatorics

10. Permutations represent the number of **ordered arrangements** of n **distinct** objects. It is given by
$$n! = n(n-1)\cdots 1.$$

11. Combinations represent the number of ways of selecting r objects from a group of n distinct objects, with the selection of the objects being **unordered**. It is given by
$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

12. A **partitioning** of n objects into k groups of sizes $r_1, r_2, \ldots r_k$, such that $r_1 + r_2 + \ldots r_k = n$ can be done in
$$\frac{n!}{r_1!\, r_2! \ldots r_k}$$
different ways.

Geometric Series

13. The sum of the first $N+1$ terms of a geometric series is given by
$$\sum_{j=0}^{N} r^j \;=\; \frac{1 - r^{N+1}}{1 - r}, \quad r \neq 1.$$

14. For $N \to \infty$, the sum is given by
$$\sum_{j=0}^{\infty} r^j \;=\; \frac{1}{1 - r}, \quad |r| < 1.$$

Integration

15. Integration by parts:
$$\int_a^b u(x)v'(x)dx \;=\; \Big[u(x)\, v(x)\Big]_a^b - \int_a^b u'(x)v(x)dx.$$

Characterization of Discrete Random Variables

16. The cumulative distribution function (CDF) of a random variable X (discrete or continuous) is the function $F_X(x)$ defined by
$$F_X(x) = \mathbb{P}(X \leq x).$$

17. The probability mass function (PMF) of a discrete random variable X is defined by

$$p_X(x) = \mathbb{P}(X = x).$$

18. For a discrete random variable which takes values $\{x_1, x_2, \ldots\}$, we have

$$\sum_{i=1}^{\infty} p_X(x_i) = 1.$$

That is, the sum over its support is 1.

19. The relationship between the CDF and PMF of a discrete random variable X is given by

$$F_X(x) = \sum_{X=-\infty}^{x} \mathbb{P}(X = x).$$

20. The PMF $p_X(x)$ of X can be obtained from CDF $F_X(x)$ by

$$p_X(x) = \mathbb{P}(X = x) = F_x(x) - F_x(x^-).$$

21. The CDF of random variable X (discrete or continuous) has the following properties:

    (a) If $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$, that is, CDF is non-decreasing.

    (b) $F_X(x^+) = F_X(x)$, that is, CDF is right continuous).

    (c) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.

    (d) $\mathbb{P}(X > x) = 1 - F_X(x)$.

    (e) $\mathbb{P}(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$.

Examples of Discrete Random Variables

22. A binomial random variable with parameters n and p has a PMF given by

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$x \in \{0, 1, 2, \cdots, n\}$.

23. A geometric random variable with parameter p has a PMF given by

$$p_X(x) = (1-p)^{x-1} p, \quad x \in \{1, 2, \cdots\}.$$

24. A hypergeometric random variable with parameters N, K and n has a PMF given by

$$p_X(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}},$$

$x \in \{0, 1, \cdots, n\}$, $0 \leq K \leq N$, $N > 0$, and $1 \leq n \leq N$.

25. A Poisson random variable with parameter λ has a PMF given by

$$p_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad \lambda > 0, \ \ x \in \{0, 1, \cdots\}.$$

Characterization of Continuous Random Variables

26. The relationship between the CDF and PDF of a continuous random variable X is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(u) du.$$

27. Additional properties of CDF are given in **Discrete Random Variables**.

28. For a continuous random variable,

$$\int_{-\infty}^{\infty} f_X(x) = 1.$$

That is, the integral over its support is 1.

29. For a continuous random variable,

$$\mathbb{P}(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1).$$

Examples of Continuous Random Variables

30. The PDF of a uniform random variable $\mathcal{U}(a, b)$ is given by

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$$

31. An exponential random variable with parameter λ has a PDF given by

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \ \ \lambda > 0.$$

32. The PDF of a normal distribution $\mathcal{N}(\mu, \sigma^2)$ is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$
$$-\infty < x < \infty.$$

33. We transform the normal distribution $X \sim \mathcal{N}(\mu, \sigma^2)$ to a standard normal distribution $Z \sim \mathcal{N}(0, 1)$ as follows

$$Z = \frac{X - \mu}{\sigma}.$$

34. In the table of standard normal distribution, the CDF is denoted by $\Phi(z)$.