Stony Brook University

Conditional Probability

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Topics

- Conditional Probability
- Total Probability Theorem
- Independence
- Several Examples

Example - Dice Roll

Example

Roll a fair dice. Sample space $\Omega = \{1,2,3,4,5,6\}$. What is the probability of the event $\{2\}$.

Since there are a total of 6 outcomes, $\mathrm{P}(\{2\}) = 1/6$.

Suppose, now we are **given the information that the dice came up with an even number**.

What is the probability of the event $\{2\}$ in this case?

First, we obtain a reduced sample space as follows -

we take the sample space Ω and cross out the odd outcomes - $\{\cancel{1},2,\cancel{3},4,\cancel{5},6\}$.

Therefore, we have the **reduced sample space** $\Omega' = \{2,4,6\}$.

Therefore, probability of the event $\{2\}$ given event even occurs, $\mathrm{P}(\{2\}|even) = 1/3$.

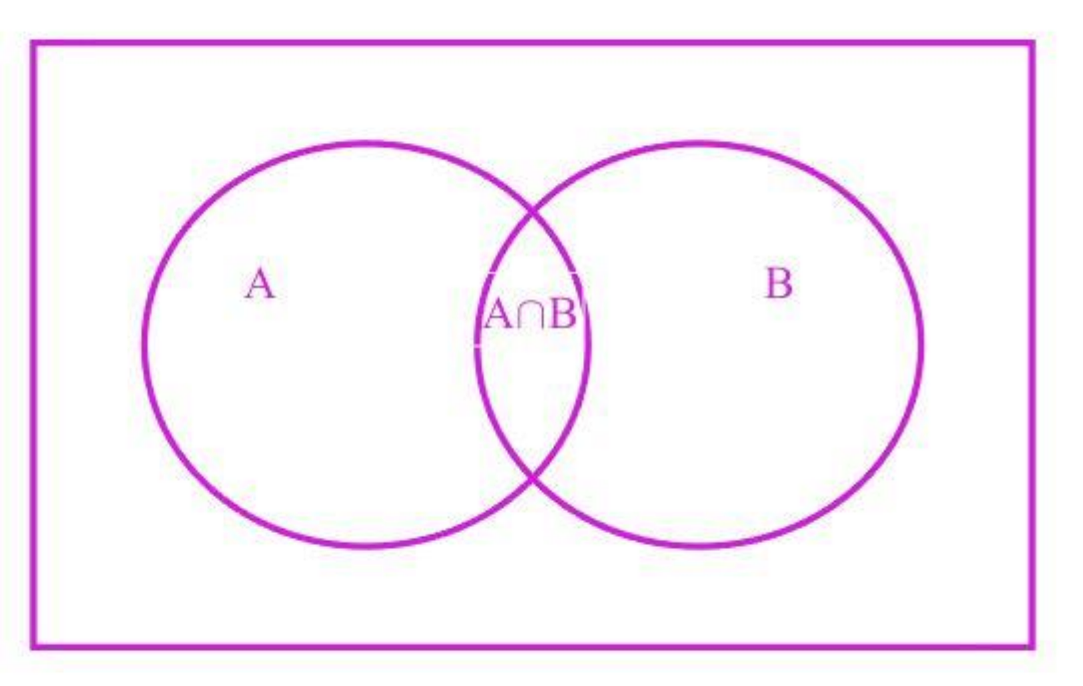
This concept is formalized using conditional probability.

Conditional Probability (1 of 2)

- Let A and B be two events in Ω and let $P(B) > 0$. Then the **conditional probability** of the event A given the occurrence of event B is denoted by $P(A|B)$, and it is defined by:

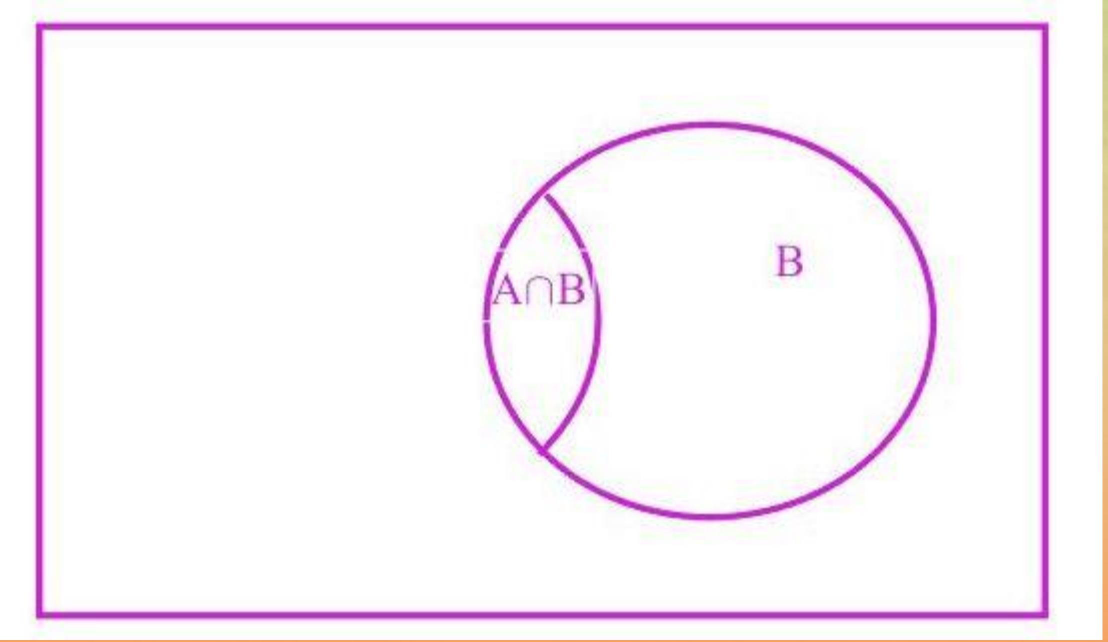
$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

We can "arrive" at this expression using the Venn diagram -
This figure shows two events A and B and their intersection.



Conditional Probability (2 of 2)

- When the event B occurs, we remove any of A that is not in B, as in this figure. Hence the probability $P(A|B)$ is given by $P(A \cap B)$ divided by $P(B)$.



- **Notation**
  - We utilize the vertical bar for conditional probability. $P(A|B)$ is read as probability of A given B.
  - For the intersection of events $P(A \cap B)$, we also use $P(A, B)$.

Example - Three Coin Toss

Example

Toss three coins. What is the probability of exactly two heads, given that the first coin shows a head? Assume that the coins are fair, that is $P(h) = P(t) = {}^1/_2$.

Solution

Method I:

The sample space has $2^3 = 8$ elements given by $\Omega = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$.

Let A be the event that the first coin is a head. Therefore, $A = \{hhh, hht, hth, htt\}$.

Let B be the event of exactly two heads. Therefore, $B = \{hht, hth, thh\}$.

Therefore, we have $A \cap B = \{hht, hth\}$.

Example - Three Coin Toss, cont.

Since the coins are fair, the probability of each outcome is ${}^{1}/_{8}$.

Therefore, $P(A) = {}^{4}/_{8} = {}^{1}/_{2}$

$P(A \cap B) = {}^{2}/_{8} = {}^{1}/_{4}$

The required probability is $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = {}^{1}/_{2}$.

Exercise: Obtain the sample space Ω using the tree diagram.

Method II:

Given that the first coin shows a head, we have the following **reduced sample space:**

$\Omega' = \{\text{hhh}, \text{hht}, \text{hth}, \text{htt}\}$. Therefore, probability of exactly two heads $= {}^{2}/_{4} = {}^{1}/_{2}$.

Example - Defective Fuses

Example

Suppose that five 5 good fuses and 2 defective fuses are mixed up. To find the defective fuses, we test them one by one randomly and **without replacement**. What is the probability that we are lucky and find the defective fuses in the first two trials?

Solution

Let D_1 be the event defective fuse in the first trial

and D_2 be the event defective fuse in the second trial.

Then $D_1 \cap D_2$ is the event defective fuses in **trials one and two**.

Example - Defective Fuses, cont.

From the **conditional probability** definition,

$$P(D_1 \cap D_2) = P(D_1)\, P(D_2|D_1).$$

Now $P(D_1) = {}^2/_7$, since 2 out of the 7 fuses are defective,

and $P(D_2|D_1) = {}^1/_6$ since after we pull out the first defective fuse, there are 6 fuses left of which 1 would be defective.

Hence, the required probability is $P(D_1 \cap D_2) = {}^2/_7 \times {}^1/_6 = {}^1/_{21}$.

Example - Two Dice Roll

Example

On the roll of two fair dice, if at least one face is known to be an even number, what is the probability that the sum is 8?

Solution

For the roll of two dice, the total number of events is $6 \times 6 = 36$.

Let A be the event that the sum is 8. Therefore, $A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$.

Let B be the event that at least one face is even. To determine the number of events in B, we first count the number of events with both odd faces; call this B^C.

$B^C = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)\}$.

Example - Two Dice Roll, cont.

Event B^C has 9 outcomes. Therefore, the event B has $36 - 9 = 27$ outcomes.

Also, we have $A \cap B = \{(2,6), (4,4), (6,2)\}$.

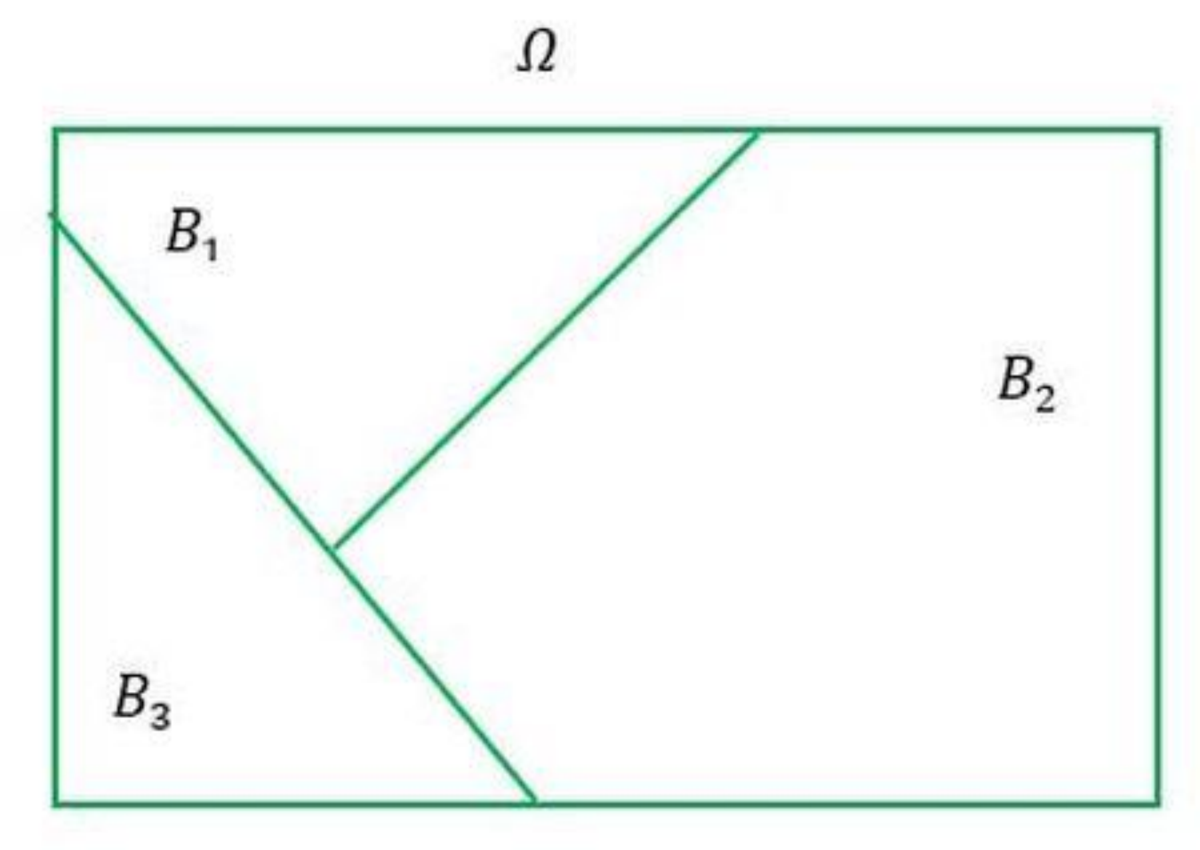
Since the dice are fair, the probability of each event is ${}^{1}/_{36}$.

Therefore, $P(B) = {}^{27}/_{36}$

$P(A \cap B) = {}^{3}/_{36}$

The required probability is $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/36}{27/36} = {}^{1}/_{9}$.

Total Probability Theorem

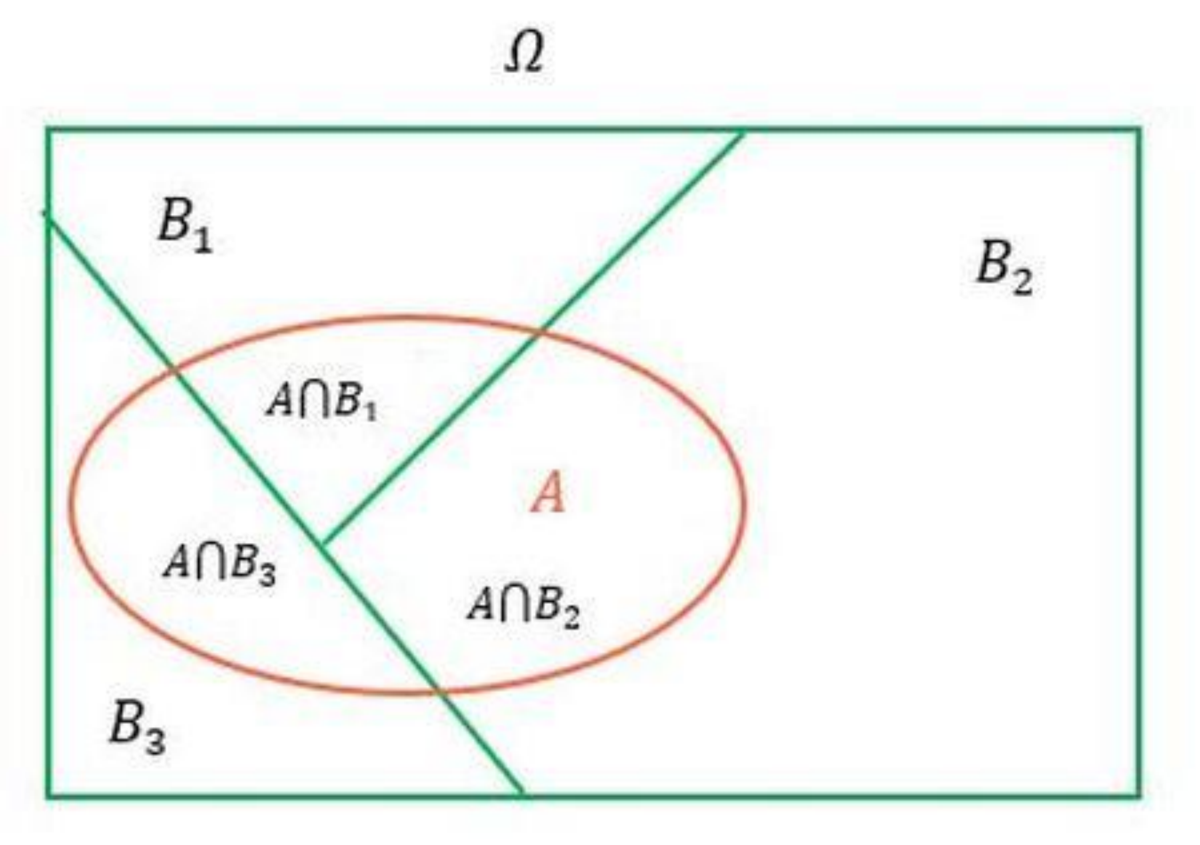


- Let $B_1, B_2, \cdots B_n$ be a **partition** of the sample space Ω, that is,
  - $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$ (the **sets cover** Ω).
  - $B_i \cap B_j = \phi$, for all $i, j = 1,2 \cdots n; i \neq j$ (**disjoint sets**).
- Then for any event A in Ω,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \cdots + P(A|B_n)P(B_n)$$

This result is known as the **total probability theorem**.

Total Probability Theorem, cont.



Proof: $P(A) = P(A \cap \Omega)$, since $A = A \cap \Omega$.

Further, $A \cap \Omega = A \cap (B_1 \cup B_2 \cup \cdots B_n)$

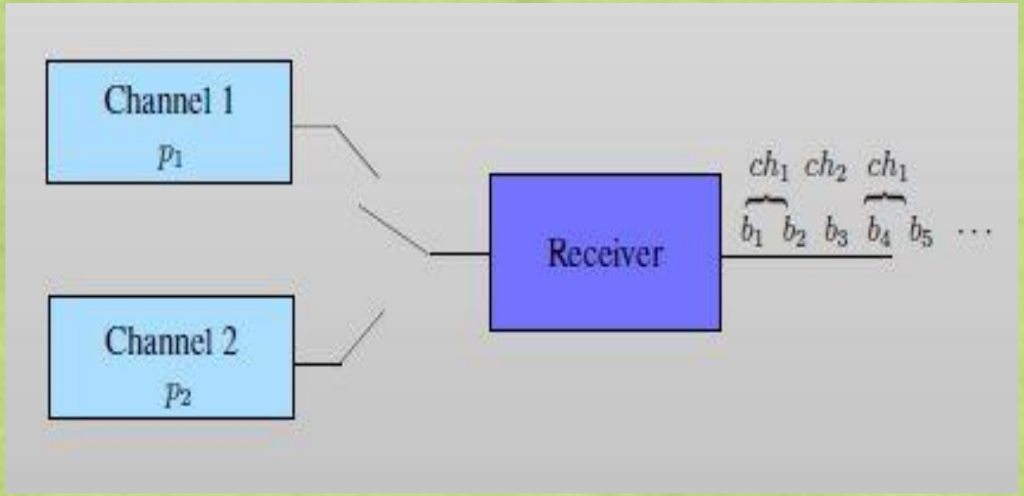
$$= (A \cap B_1) \cup (A \cap B_2) \cup \cdots (A \cap B_n).$$

Therefore,

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_n),$$

From the additive property of probability.

Example - Bit Error Rate (1 of 3)



Example

At a receiver, signals are coming through two separate channels. The signal bits from each channel at the receiver are interleaved in the following way: two bits are taken from the first channel, then one bit from the second channel, again two bits from the first, one bit from the second and so on. The probability of a bit error from the first channel is p_1 and the probability of a bit error from the second channel is p_2. If a bit is randomly selected at the receiver, what is the probability of its error $P(E)$?

Example - Bit Error Rate (2 of 3)

Solution

Let A be the event that the selected bit is from channel 1.

Then A^C is the event that the selected bit is from channel 2.

Since we do not know which channel the given bit comes from, we compute the probability of error from each channel, and then we combine the result using **total probability theorem.**

Given that the bit is selected from channel 1, the probability of error = $P(E|A) = p_1$.

Given that the bit is selected from channel 2, the probability of error = $P(E|A^C) = p_2$.

Example - Bit Error Rate (3 of 3)

Since two out of three bits are from channel 1 and one out of three bits is from channel 2,

we have the probability of selecting the bit from channel 1, $P(A) = {}^{2}/_{3}$,

and the probability of selecting the bit from channel 2, $P(A^C) = {}^{1}/_{3}$.

Therefore, from total probability theorem, the probability of error is given by:

$P(E) = P(E|A)P(A) + P(E|A^C)P(A^C)$.

That is, $P(E) = \frac{2}{3}\,p_1 + \frac{1}{3}\,p_2$.

Example

Urn I contains 2 red (R) and 8 blue (B) balls. Urn II contains 9 red and 6 blue balls. We **select at random one of the two urns** and choose a ball at random from it. Find the probability that the selected ball is red.

Urn I

Urn II

Which Urn is selected?

Example - Balls and Urns (2 of 3)

Solution

Here we do not know which Urn was selected. Therefore, we first compute the probability **conditioned on each Urn separately -**

Let U_I and U_{II} denote Urns I & II, respectively.

Probability of selecting a red ball from Urn I = $P(R|U_I) = {}^{2}/_{10}$,

Probability of selecting a red ball from Urn II = $P(R|U_{II}) = {}^{9}/_{15}$,

Example - Balls and Urns (3 of 3)

Next, we combine the above results using **total probability theorem**. The probability of selecting a red ball =

$$P(R) = P(R|U_I)P(U_I) + P(R|U_{II})P(U_{II}).$$

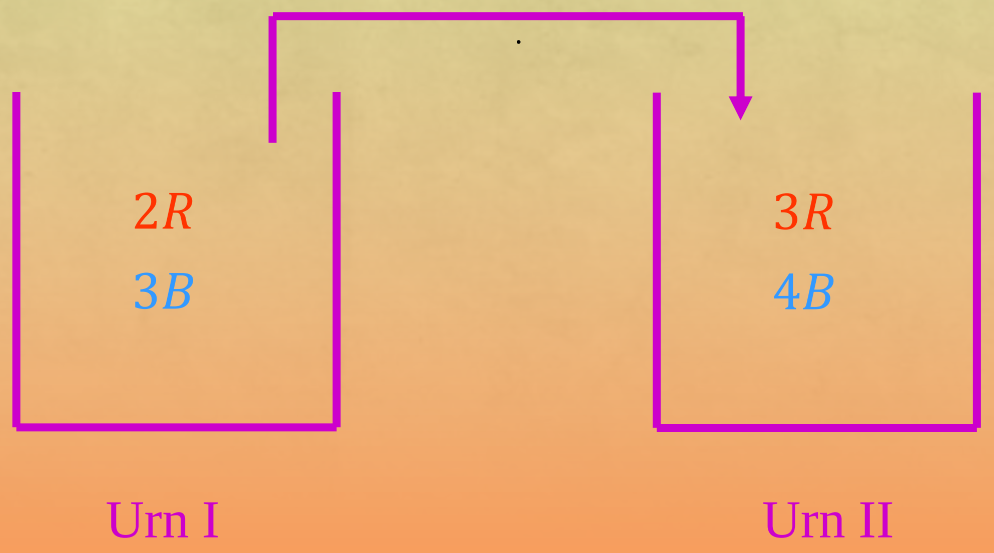
We are also given that the probability of selecting either Urn is $1/2$.

Therefore, $P(R) = \frac{2}{10} \times \frac{1}{2} + \frac{9}{15} \times \frac{1}{2} = \frac{2}{5}$.

Example - More Balls and Urns (1 of 3)

Example

Urn I contains 2 red (R) and 3 blue (B) balls. Urn II contains 3 red and 4 blue balls. **A ball is selected at random from Urn I** and transferred to Urn II. Then a ball is drawn from Urn II. What is the probability that the selected ball is blue?



Which color ball is transferred from Urn I to Urn II?

Example - More Balls and Urns (2 of 3)

Solution:

Let U_I and U_{II} denote Urns I & II, respectively. We don't know which color ball was transferred from Urn I to Urn II. Therefore, we compute the **conditional probability** for each of the following cases, and then combine the two cases:

- A blue ball is transferred from Urn I to Urn II, and then a ball is drawn from Urn II.
- A red ball is transferred from Urn I to Urn II, and then a ball is drawn from Urn II.

$$P(B\ from\ U_{II}|B\ from\ U_I \rightarrow U_{II}) = {}^5\!/_8,$$

If a blue ball is transferred from U_I to U_{II}, then there will be $5B$ balls in U_{II} and total 8 balls in U_{II}. Therefore, the probability is $5/8$.

$$P(B\ from\ U_{II}|R\ from\ U_I \rightarrow U_{II}) = {}^4\!/_8,$$

Same argument, if a red ball is transferred from U_I to U_{II}.

Example - More Balls and Urns (3 of 3)

Further, we have

$\mathrm{P}(B\ from\ U_I \to U_{II}) = {}^3/_5$

This is the probability of drawing a blue ball from Urn I.

$P(R\ from\ U_I \to U_{II}) = {}^2/_5$

This is the probability of drawing a red ball from Urn I.

Therefore, from **total probability theorem**,

$$P(B\ from\ U_{II}) = P(B\ from\ U_{II} | B\ from\ U_I \to U_{II})\ \mathrm{P}(B\ from\ U_I \to U_{II})$$

$$+ P(B\ from\ U_{II} | R\ from\ U_I \to U_{II})\ \mathrm{P}(\mathrm{R}\ from\ U_I \to U_{II})$$

$$= \frac{5}{8} \times \frac{3}{5} + \frac{4}{8} \times \frac{2}{5} = \frac{23}{40}.$$

Independence

- In general, the occurrence of an event B changes the probability that the event A occurs, the original probability being replaced by $P(A|B)$. If the probability remains unchanged, then

$$P(A|B) = P(A),$$

and we say that A and B are **independent**.

- Since $P(A|B) = \frac{P(A \cap B)}{P(B)}$, for **independent events**, we have

$$P(A \cap B) = P(A)P(B).$$

- That is, the joint probability is the product of the probability of each event.

Independence, cont.

- If events A and B are independent, then so are A and B^C.
- Note that $A \cap B = \phi$ does not imply that A and B are independent.

Example - Three Coin Toss

Example

Toss three fair coins.

(a) Are the event "first coin is a head" and the event "second coin is a tail" independent?

Solution:

Yes. Let us show this.

Let A be the event that the first coin is a head; $\mathrm{P}(A) = {}^{1}/_{2}$.

Let B be the event that the second coin is a tail; $\mathrm{P}(B) = {}^{1}/_{2}$.

The sample space for the three-coin toss has 8 outcomes, each occurring with probability ${}^{1}/_{8}$.

The event $A \cap B$ has two outcomes, given by $\{hth, htt\}$.

Example - Three Coin Toss, cont.

$P(A \cap B) = {}^2/_8 = {}^1/_4$.

And $P(A)P(B) = {}^1/_2 \times {}^1/_2 = {}^1/_4$.

Since $P(A \cap B) = P(A)P(B)$, therefore the two events A and B are **independent**.

(b) Are the event "first coin is a head" and the event "first coin is a tail" independent?

No. Let us show this.

Let A be the event that the first coin is a head; $\mathrm{P}(A) = {}^1/_2$.

Let B be the event that the first coin is a tail; $\mathrm{P}(B) = {}^1/_2$.

The event $A \cap B = \phi$, an empty set. Therefore, $\mathrm{P}(A \cap B) = 0$.

Since $P(A \cap B) \neq P(A)P(B)$, therefore the two events A and B are **not independent**.

References

1. Charles Boncelet, Probability, Statistics and Random Signals, Oxford University Press, 2016.
2. Sheldon Ross, A First Course in Probability, Macmillan Publishing Company, 1988.
3. R. D. Yates, et al., Probability and Stochastic Processes, John Wiley, 2005.