Randoms Processes

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Topics

- Definition of a Random Process
- Types of Random Processes
    - Discrete-time & Continuous-time
    - Discrete-alphabet & Continuous-alphabet
- Realizations of a Random Process
- Examples of Random Processes
    - Sinusoid
    - Bernoulli
    - Binomial

Random Processes Topics, cont.

- Stationarity
  - Strict Sense Stationary (SSS) Process
  - Wide Sense Stationary (WSS) Process
- Expectations of Random Processes
  - Mean
  - Autocorrelation

Definition of a Random Process

- A **random process** $\boldsymbol{X(t)}$ is a collection of random variables indexed by time.
- Each random variable in the collection takes values from a **state space**, for example, integers or real numbers.
- If the time takes values from a countable set, such as $\{0,1,2,\cdots\}$, we call this a **discrete-time** random process.
- If the time takes value in $\mathcal{R}$, the real numbers, we call this a **continuous-time** random process.

Definition of a Random Process, cont.

- **Notation**
  - We use $X[\mathrm{n}]$ or X_n for a discrete-time process.
  - We use $X(t)$ or X_t for a continuous-time process.
  - The index n denotes time for the discrete-time process.
  - The index t denotes time for the continuous-time process.

Discrete-Alphabet & Continuous-Alphabet Process

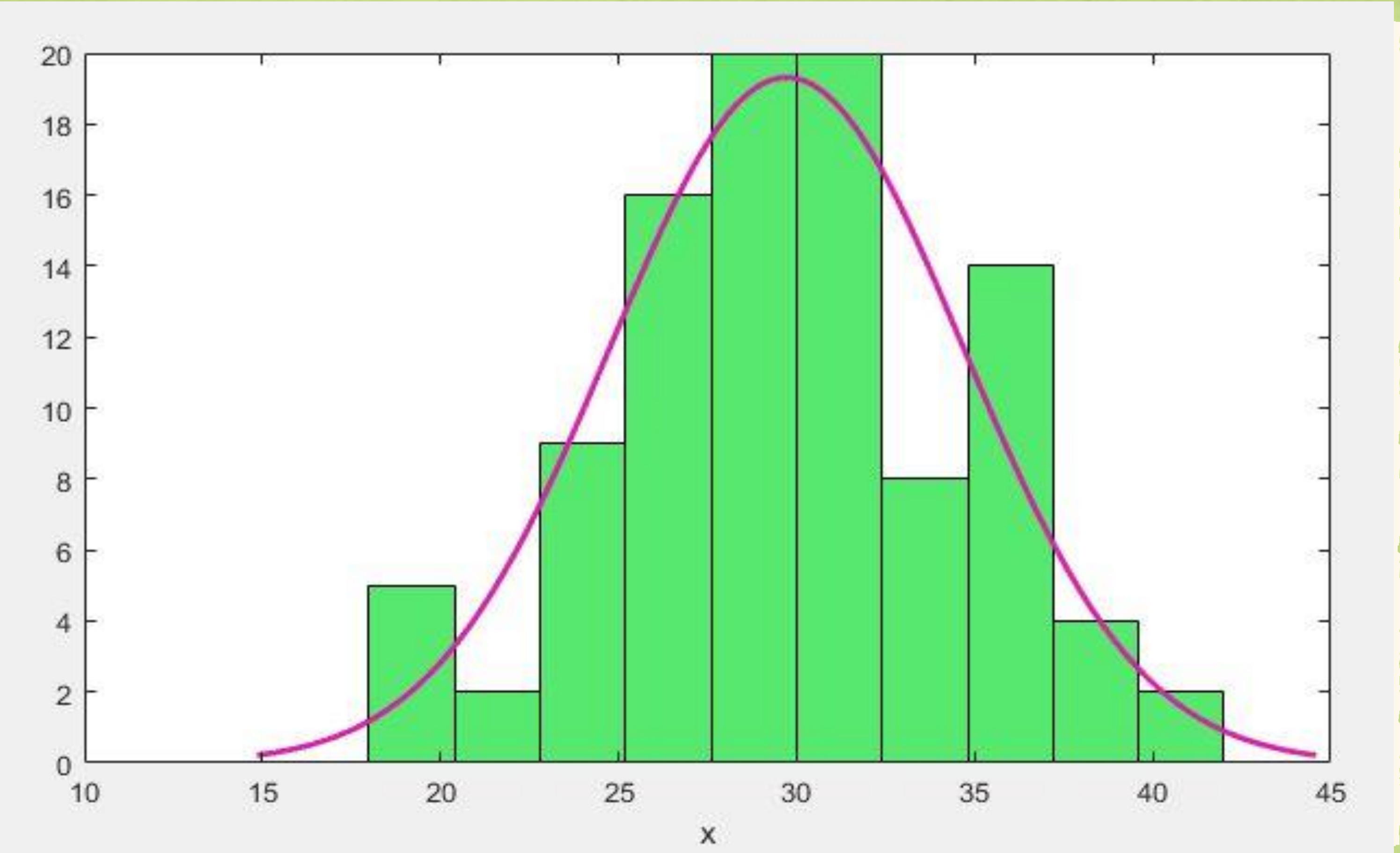
We further classify a random process $X[\mathrm{n}]$ or $X(t)$ as follows:

- **Discrete-Alphabet Process**: The state space of $X[\mathrm{n}]$ or $X(t)$ take discrete values.
- **Continuous-Alphabet Process**: The state space of $X[\mathrm{n}]$ or $X(t)$ take continuous values.

Samples from a Random Variable

- In order to understand random processes, we will first discuss samples from a random variable.
- Let X be a random variable (discrete or continuous), with PDF (or PMF) $f_X(x)$. Generating **samples** from the distribution of X means generating values in the **support of X**, such that the probability of generating the samples is in accordance with the given PDF (or PMF).
- Let X have a binomial distribution, $bin(100{,}0.3)$, with parameters $n = 100$ and $p = 0.3$. The support of X is $\Omega_X = \{0, 1, 2, \cdots 100\}$. Therefore, some example samples for X would be $25, 21, 27, 31, 18$ and so forth.
- If we have a large enough sample, a **histogram** plot of the samples resembles the PDF (or PMF) of X.

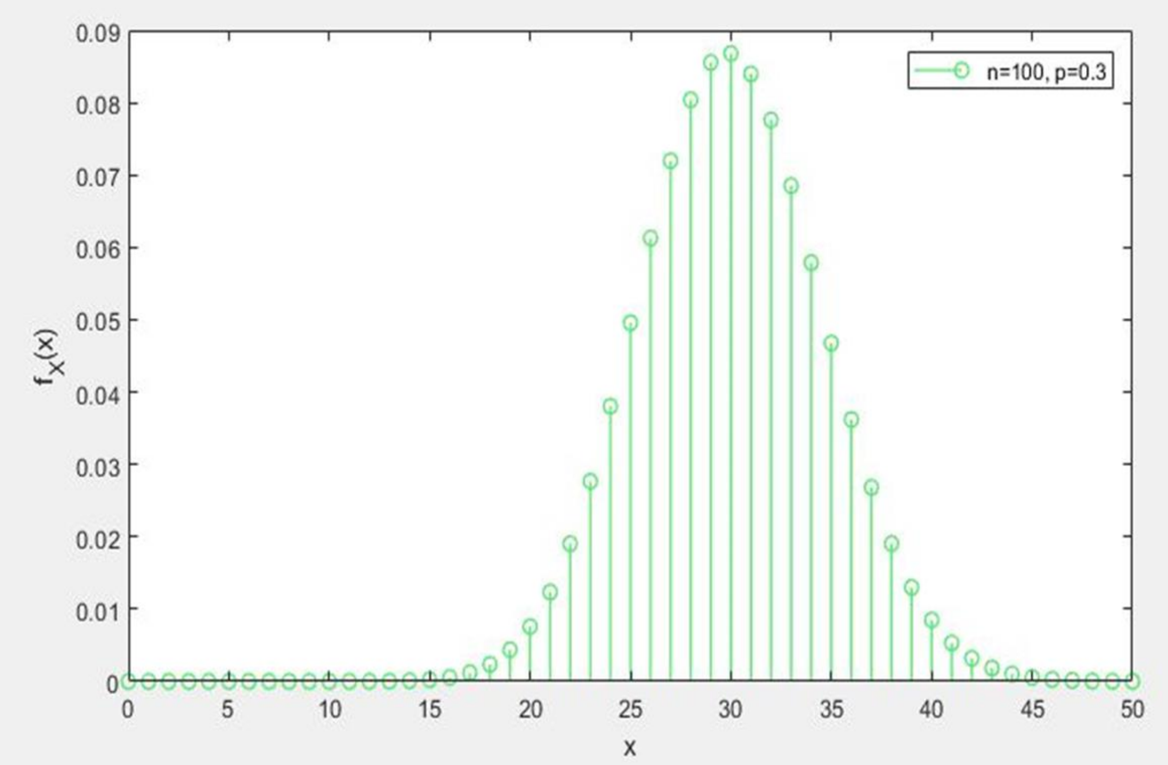
Samples from Binomial Distribution

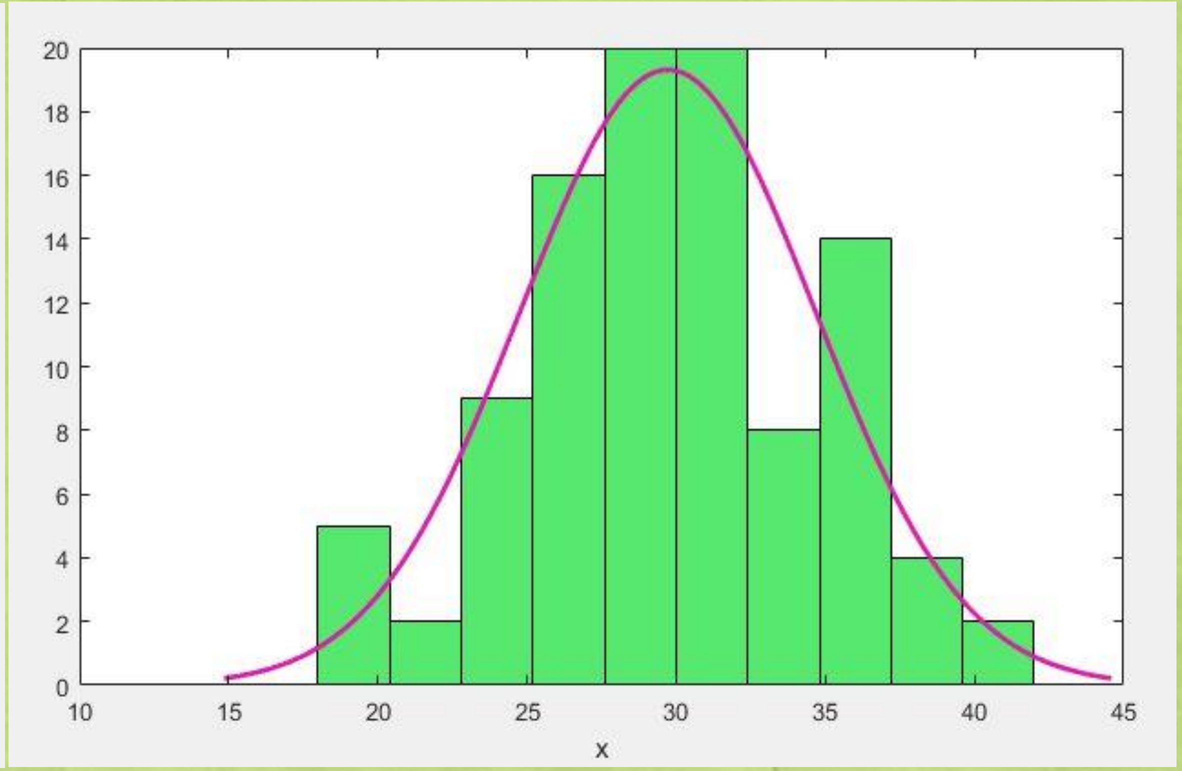


- This plot shows a histogram of binomial distribution samples with **parameters n=100 and p=0.3**
- On the horizontal axis, we plot the range of x.
- The plot has 10 bins, and it shows the count of x in each bin.
- Note that the **probability peaks around x=30**. This means that out of 100 trials, the probability of success is the highest in the range [27, 33].
- The plot in purple is a **smooth density plot** fitted to data.

Binomial Distribution - PMF & Sample Histogram

- The following plots show the PMF of $bin(100,0.3)$ distribution (left) and a histogram of the generated samples (right). The vertical scales are different in the two cases; you can normalize the histogram.

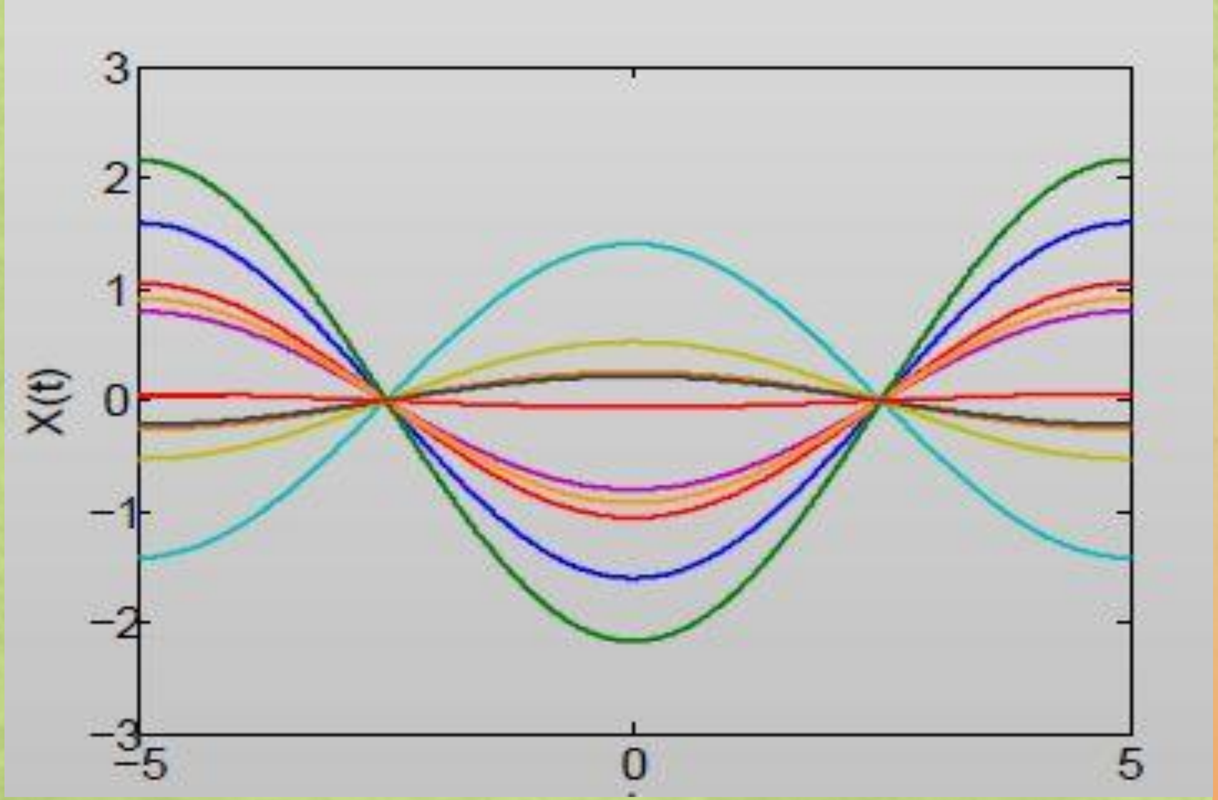




Realizations of a Random Process

- Analogous to samples for random variables, we have realizations for a random process.
- Each sample function of a random process is called a **realization**.
- The collection of all possible realizations is known as the **ensemble** of the random process.
- If we observe the random process at a **specific time instant**, say t_0, then we have a **random variable**, $X(t_0)$.

Example - Sinusoid with Random Amplitude



Example

Consider the random process

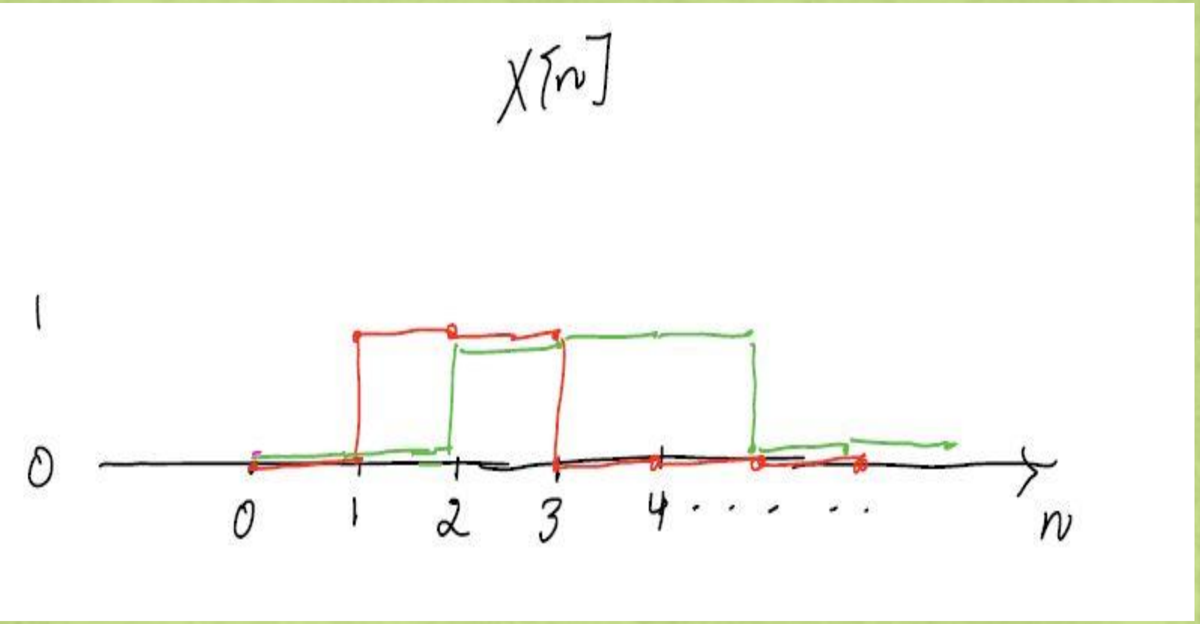
$X(t) = \boldsymbol{A} \cos(0.2\pi t)$,

where the amplitude $\boldsymbol{A}$ is a random variable drawn from a standard normal distribution $\mathcal{N}(0,1)$.

This figure shows several **realizations** of the process (one color per realization). For each realization, we sample for $\boldsymbol{A}$ from the given distribution.

Hence the value for $\boldsymbol{A}$ is different for each realization. The frequency of the random process is constant.

Example - Bernoulli Process



Example

A **Bernoulli process** X_n is a random process with a sequence of independently and identically distributed random variables, where each random variable takes a value 1 or 0, with probabilities p and $1 - p$, respectively. We denote this process as $X_n \sim Ber(p)$.

This plot shows some realizations of the process.

Example - Bernoulli Process, cont.

- At each time instant n, X_n is a random variable. That is, $X_0, X_1, X_2, \cdots$ are all random variables with $Ber(p)$ distribution.

- Since the mean of a Bernoulli random variable with parameter p is p, therefore

$$E(X_0) = p$$

$$E(X_1) = p$$

$$E(X_2) = p$$

$$\vdots$$

Example - Binomial Process

Example

We construct a new random process Y_n from the Bernoulli process X_n, as follows:

$Y_n = \sum_{u=0}^{n} X_u,$

where $X_n \sim Ber(p)$. Determine the PMF of Y_n.

Solution

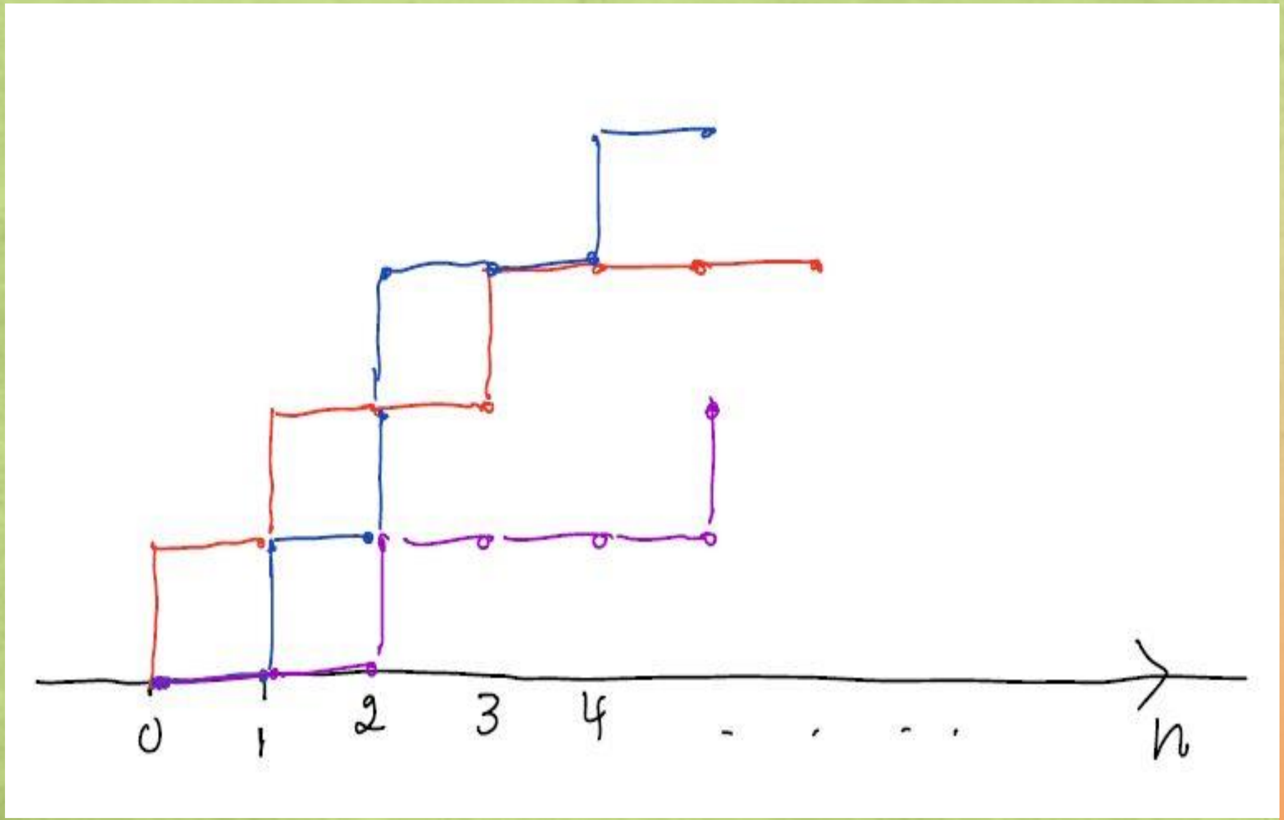
Note that, at each time instant n, Y_n is a random variable. We want to determine the PMF at each of those time instants. The sample space for Y_n is $\{0, 1, 2, \cdots n+1\}$. That is,

$Y_0 \in \{0,1\}$

$Y_1 \in \{0,1,2\}$

$Y_2 \in \{0,1,2,3\}$

Example - Binomial Process, cont.



Y_n has a binomial distribution given by

$$f_{Y_n}(y) = \binom{n+1}{y} p^y (1-p)^{n+1-y},$$

$$y = 0,1,2 \cdots n+1$$

This plot shows several realizations of the process (one color per realization).

Stationary Processes

We study the following two types of stationary processes:

- Strict Sense Stationary (SSS)
- Wide Sense Stationary (WSS)

Strict Sense Stationary Process

- Let $X(t)$ be a random process. Let $X(t_1), X(t_2), \cdots X(t_k)$ denote **random variables** obtained by sampling the process $X(t)$ at time instants $t_1, t_1, \cdots t_k$, respectively.

- Let the joint PDF of this set of random variables be given by

$$f_{X(t_1),X(t_2),\cdots X(t_k)}(x_1, x_2, \cdots x_k)$$

- Suppose next, we shift all the sampling times by a **fixed amount τ**. The joint PDF for this new set of variables is

$$f_{X(t_1+\tau),X(t_2+\tau),\cdots X(t_k+\tau)}(x_1, x_2, \cdots x_k)$$

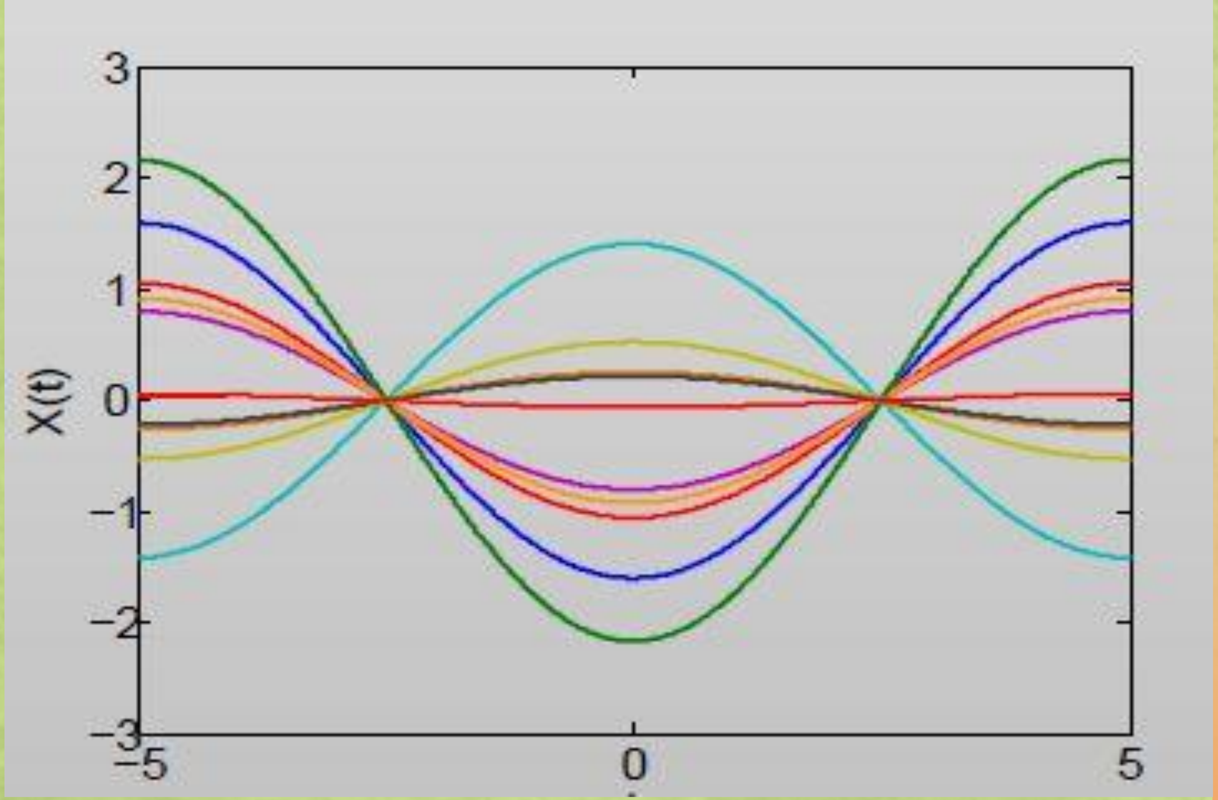
Strict Sense Stationary Process, cont.

- The process $X(t)$ is **strict sense stationary (SSS)** if the **invariance condition**

  $$f_{X(t_1),X(t_2),\cdots X(t_k)}(x_1,x_2,\cdots x_k) = f_{X(t_1+\tau),X(t_2+\tau),\cdots X(t_k+\tau)}(x_1,x_2,\cdots x_k)$$

  holds for all time shifts τ and all $k = 1,2,3,4,\cdots$
- The above definition also holds for discrete-time process $X[\mathrm{n}]$, with the PDF replaced by PMF.

Example - Sinusoid with Random Amplitude (1 of 2)



Example

Consider again the random process

$X(t) = \boldsymbol{A}\cos(0.2\pi t)$,

where $\boldsymbol{A}$ is a random variable drawn from a standard normal distribution $\mathcal{N}(0,1)$.

Next, we observe the random variables at two different times, $t_1 = 0.0$ and $t_2 = 2.5$.

At $t_2 = 2.5$, the random variable $X(t_2)$ always takes the value zero.

At $t_1 = 0.0$, the random variable $X(t_1)$ takes values from $\mathcal{N}(0,1)$, such as $0.09, 1.12, -0.032$ and so forth.

Example - Sinusoid with Random Amplitude (2 of 2)

We observe that $X(t)$ has two different distributions (or PDF) at two different times. Therefore, $X(t)$ is not invariant to a shift in time, and hence it is not an SSS process.

Some Examples

- The Bernoulli process defined above has the same distribution for all values of time n. Hence, it is an SSS process.
- Is the Binomial process defined above SSS?
- Consider the process $X(t) = A\sin(2\pi \boldsymbol{f} t)$, where $\boldsymbol{f}$ is a random variable distributed uniformly in the range $[1,10]$ Hz, and A is a constant. Is this process SSS?
- Consider the process $X(t) = A\sin(2\pi f t + \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a random variable distributed uniformly in the range $[0,2\pi]$, and A and f are constants. Is this process SSS?

In the random process, we use boldface to denote the variable that is random.

Expectations of Random Processes

We will study the following expectations of a random process

- Mean
- Autocorrelation

Mean of a Random Process

- The **mean** of a random process is defined by

$$E(X[n]) \equiv \mu_X[n] = \sum_{-\infty}^{\infty} x\, f_{X[n]}(x), \qquad \text{for discrete-time discrete-alphabet process}$$

$$E(X[n]) \equiv \mu_X[n] = \int_{-\infty}^{\infty} x\, f_{X[n]}(x)dx, \qquad \text{for discrete-time continuous-alphabet process}$$

$$E(X(t)) \equiv \mu_X(t) = \sum_{-\infty}^{\infty} x\, f_{X(t)}(x), \qquad \text{for continuous-time discrete-alphabet process}$$

$$E(X(t)) \equiv \mu_X(t) = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)dx, \qquad \text{for continuous-time continuous-alphabet process}$$

The mean is in general a function of time n (or t).

Autocorrelation of a Random Process

- The **autocorrelation** of a random process is defined by

The autocorrelation is in general a function of two different time instants n_1 and n_2 (or t_1 and t_2).

$$E(X[n_1]X[n_2]) \equiv R_{XX}[n_1, n_2] = \sum_{-\infty}^{\infty} x_1 x_2 f_{X[n_1]X[n_2]}(x_1, x_2),$$

for discrete-time discrete-alphabet process

$$E(X[n_1]X[n_2]) \equiv R_{XX}[n_1, n_2] = \int_{-\infty}^{\infty} x_1 x_2 f_{X[n_1]X[n_2]}(x_1, x_2) dx_1 dx_2,$$

for discrete-time continuous-alphabet process

$$E(X(t_1)X(t_1)) \equiv R_{XX}(t_1, t_2) = \sum_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1, x_2),$$

for continuous-time discrete-alphabet process

$$E(X(t_1)X(t_1)) \equiv R_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1, x_2) dx_1 dx_2,$$

for continuous-time continuous-alphabet process

Wide Sense Stationary Process

- The discrete-time random process $X[\mathrm{n}]$ is **wide sense stationary (WSS)** if and only if
  - *i.* $E(X[n]) \equiv \mu_X$, that is, the mean is independent of time.
  - *ii.* $E(X[n_1]X[n_2]) \equiv R_{XX}[n_1 - n_2] = R_{XX}[m]$, that is, the autocorrelation is a function of time difference only.
    Here $m = n_1 - n_2$.

- The continuous-time random process $X(t)$ is **wide sense stationary (WSS)** if and only if
  - *i.* $E(X(t)) = \mu_X$, that is, the mean is independent of time.
  - *ii.* $E(X(t_1)X(t_1)) \equiv R_{XX}(t_1, t_2) = R_{XX}(\tau)$, that is, the autocorrelation is a function of time difference only.
    Here $\tau = t_1 - t_2$.

Example - Sinusoid with Random Amplitude (1 of 3)

Example

Consider again the random process

$X(t) = \boldsymbol{A}\sin(0.2\pi t),$

where $\boldsymbol{A}$ is a random variable drawn from a standard normal distribution $\mathcal{N}(0,1)$.

Mean

$$\begin{aligned}\mu_X(t) &= E(X(t)) \\ &= E(\boldsymbol{A}\sin(0.2\pi t)) \\ &= E(\boldsymbol{A})\sin(0.2\pi t),\end{aligned}$$

we pull $\sin(0.2\pi t)$ out of the expectation integral, as this is not random.

Example - Sinusoid with Random Amplitude (2 of 3)

Therefore,

$\mu_X(t) = 0$, as $E(\boldsymbol{A}) = 0$, for the standard normal distribution $\mathcal{N}(0,1)$.

Hence the mean is independent of time.

Autocorrelation

$$\begin{aligned} R_{XX}(t_1, t_2) &= E(X(t_1)X(t_1)) \\ &= E(\boldsymbol{A}\sin(0.2\pi t_1)\,\boldsymbol{A}\sin(0.2\pi t_2)) \\ &= E(\boldsymbol{A}^2)\sin(0.2\pi t_1)\sin(0.2\pi t_2), \end{aligned}$$

again only $\boldsymbol{A}^2$ is in the expectation integral, as this is random, and the other terms are not.

For the standard normal distribution, $E\left(\boldsymbol{A}^2\right) = \sigma^2 = 1$, as the mean is zero.

Example - Sinusoid with Random Amplitude (3 of 3)

And $\sin(0.2\pi t_1)\sin(0.2\pi t_2) = \frac{1}{2}[\cos(0.2\pi(t_1 - t_2)) - \cos(0.2\pi(t_1 + t_2))]$

$$= \frac{1}{2}[\cos(0.2\pi\tau)) - \cos(0.2\pi(t_1 + t_2))], \text{ where } \tau = (t_1 - t_2).$$

Therefore,

$R_{XX}(t_1, t_2) = \frac{1}{2}[\cos(0.2\pi\tau)) - \cos(0.2\pi(t_1 + t_2))].$

The second term is a function of time instants t_1 and t_2. Hence, this is not a WSS process.

References

1. Charles Boncelet, Probability, Statistics and Random Signals, Oxford University Press, 2016.
2. Sheldon Ross, A First Course in Probability, Macmillan Publishing Company, 1988.
3. R. D. Yates, et al., Probability and Stochastic Processes, John Wiley, 2005.