Expectations - One Random Variable

Vibha Mane
The COSINE Lab
Department of Electrical and Computer Engineering

Topics

- Definition of Expected Value
- Mean and Variance of a Random Variable
- Properties of expectations
  - Linearity
  - Mean and variance of a constant
- Some Examples of Expectations
  - Bernoulli
  - Binomial
  - Poisson
  - Geometric
  - Normal

Definition

- The **expected value of a discrete random** variable X is defined by

  $E\{X\} = \sum_x x\, p_X\,(x)$ and it is **denoted by μ_X**.

- The **expected value of a continuous random** variable X is defined by

  $E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x) dx$ and it is denoted by μ_X.

- μ_X is also known as the **mean** of X.

Moments

- In general, the kth moment of a discrete random variable X is defined by

  $E\{X^k\} = \sum_x x^k \, p_X(x).$

- And the kth moment of a continuous random variable X is defined by

  $E\{X^k\} = \int_{-\infty}^{\infty} x^k \, f_X(x) dx.$

- We will focus on computing and working with the **first and second moments**.

Variance

- The **variance $\boldsymbol{\sigma_X^2}$** of a random variable X (discrete or continuous) is defined by

$$\sigma_X^2 = E\{X^2\} - (E\{X\})^2$$

$$= E\{X^2\} - \mu_X^2.$$

- The **standard deviation $\boldsymbol{\sigma_X}$** is the positive square root of σ_X^2.
- In general,

$$-\infty < \mu_X < \infty,$$

$$0 < \sigma_X^2 < \infty.$$

Some Properties of Expectations

- **Linearity** of the expectation operator

  $E\{a_1X_1 + a_2X_2\} = a_1E\{X_1\} + a_2E\{X_2\}$,

  where a_1 and a_2 are some constants (meaning, not random).

- If ***c*** **is a constant**, then

  $E\{c\} = c$ and

  $var\{c\} = 0$, where $var\{.\}$ indicates the variance.

- We also have

  $var\{X + c\} = var\{X\}$ and

  $var\{cX\} = c^2 var\{X\}$.

Some Properties of Expectations, cont.

- Let $X_1, X_2, \cdots X_n$ be **independent random variables**. Then

$var\{X_1 + X_2 + \cdots X_n\} = var\{X_1\} + var\{X_2\} + \cdots var\{X_n\},$ (Proof in Appendix).

For dependent random variables, there will be cross terms in this expression.

Example - Bernoulli Random Variable (1 of 2)

Example

X is a Bernoulli random variable, with PMF defined by

$P(X = 0) = 1 - p$ and

$P(X = 1) = p.$

The mean of X is given by

$$E\{X\} = \sum_x x\, p_X(x)$$
$$= 0. P(X = 0) + 1. P(X = 1)$$
$$= 0. +1. p$$
$$= p \equiv \mu_X.$$

Example - Bernoulli Random Variable (2 of 2)

The second moment of X is given by

$$E\{X^2\} = \sum_x x^2\, p_X(x)$$

$$= 0.\, P(X = 0) + 1^2.\, P(X = 1)$$

$$= p.$$

The variance of X is given by

$$\sigma_X^2 = E\{X^2\} - \mu_X^2$$

$$= p - p^2$$

$$= p(1 - p).$$

In order to compute the mean and the variance, we need to compute two different sums (discrete random variables) or two different integrals (continuous random variables).

Example - Binomial Random Variable (1 of 2)

Example

Y is a binomial random variable, $bin(n, p)$. A binomial random variable is a sum of n independent Bernoulli random variables. Therefore, we can utilize the results of Bernoulli random variable to compute the mean and variance of binomial random variable.

Let $X_1, X_2, \cdots X_n$ be n independent Bernoulli random variables.

Let $Y = X_1 + X_2 + \cdots X_n$.

$$
\begin{aligned}
E\{Y\} &= E\{X_1 + X_2 + \cdots X_n\} \\
&= E\{X_1\} + E\{X_2\} + \cdots E\{X_n\}, \quad \text{using the linearity property} \\
&= p + p + \cdots p \ \ (n \text{ times}), \quad \text{using the expression from Bernoulli random variable} \\
&= np.
\end{aligned}
$$

Example - Binomial Random Variable (2 of 2)

The variance of Y is given by

$$var\{Y\} = var\{X_1 + X_2 + \cdots X_n\}$$

$$= var\{X_1\} + var\{X_2\} + \cdots var\{X_n\}, \quad \text{as } X_1, X_2, \cdots X_n \text{ are independent.}$$

$$= p(1-p) + p(1-p) + \cdots p(1-p) \ \ (n \text{ times})$$

$$= np(1-p).$$

Example - Poisson Random Variable (1 of 2)

Example

The PMF of Poisson random variable with parameter λ is given by

$$p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \cdots, \quad \lambda > 0.$$

$$E\{X\} = \textstyle\sum_x x\, p_X(x)$$

$$= \textstyle\sum_{x=0}^{\infty} x\, e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= \lambda e^{-\lambda} \textstyle\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}, \qquad \text{using } \frac{x}{x!} = \frac{1}{(x-1)!}$$

$$= \lambda e^{-\lambda} \textstyle\sum_{x'=0}^{\infty} \frac{\lambda^{x'}}{x'!}, \qquad \text{using change of variables, } x' = x - 1$$

Example - Poisson Random Variable (2 of 2)

$= \lambda e^{-\lambda} e^{\lambda},$ using the Taylor series expansion for e^{λ}.

$= \lambda \equiv \mu_X.$

The variance of X is given by

$\sigma_X^2 = \lambda,$ (Proof in Appendix).

The expression for both mean and variance is λ.

Example - Geometric Random Variable

Example

The PMF of a geometric random variable X, with parameter p, is given by

$$p_X(x) = (1-p)^{x-1}p, \quad x = 1,2,3,\cdots$$

- Its mean is given by

$$\mu_X = \frac{1}{p}.$$

- Its variance is given by

$$\sigma_X^2 = \frac{1-p}{p^2}, \qquad \text{(Proof in Appendix).}$$

Example - Normal Random Variable (1 of 3)

Example

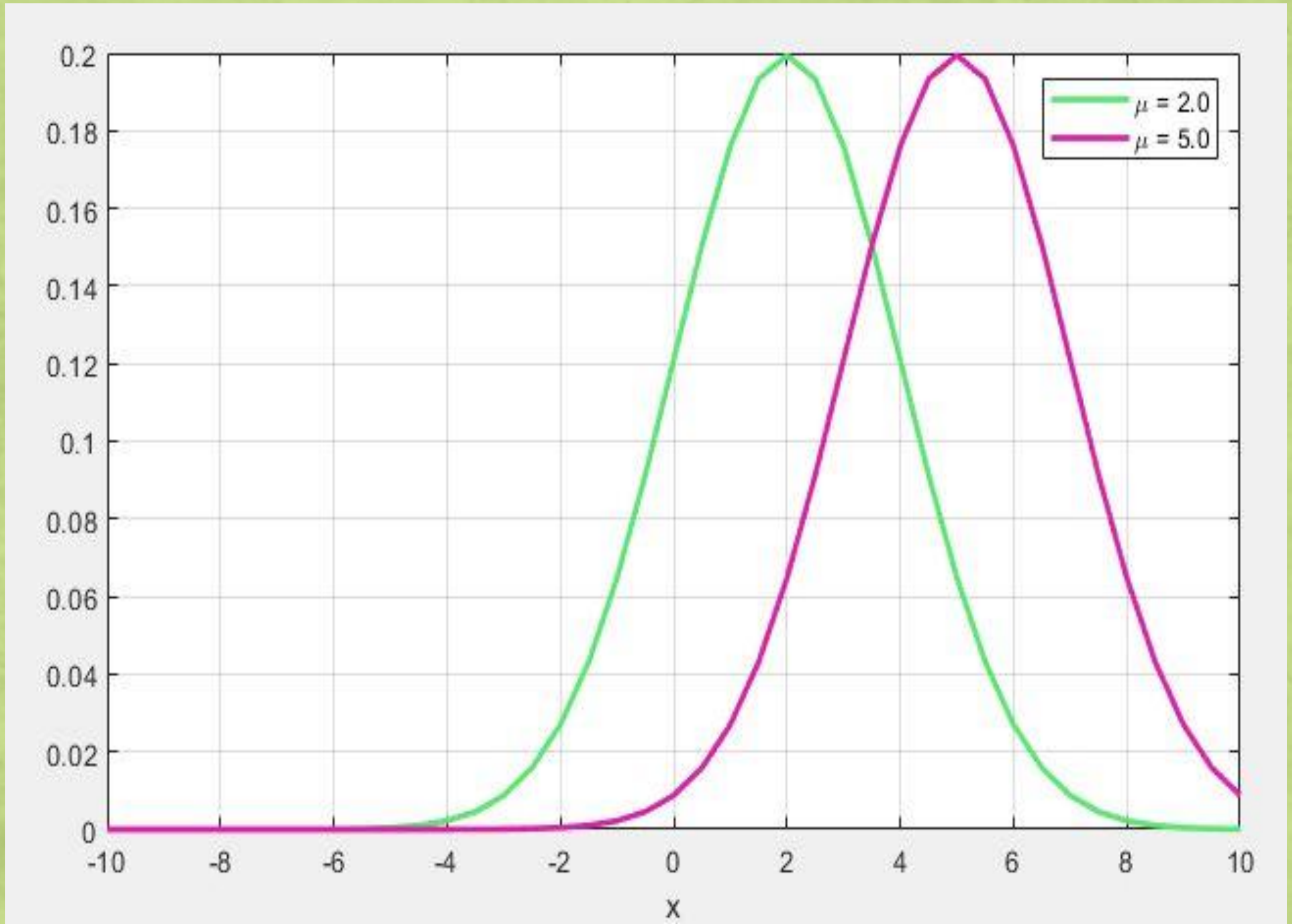
The PDF of a normal distribution with parameters μ_X and σ_X^2 is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}.$$

The mean is μ_X and the variance is σ_X^2 (Proof in Appendix).

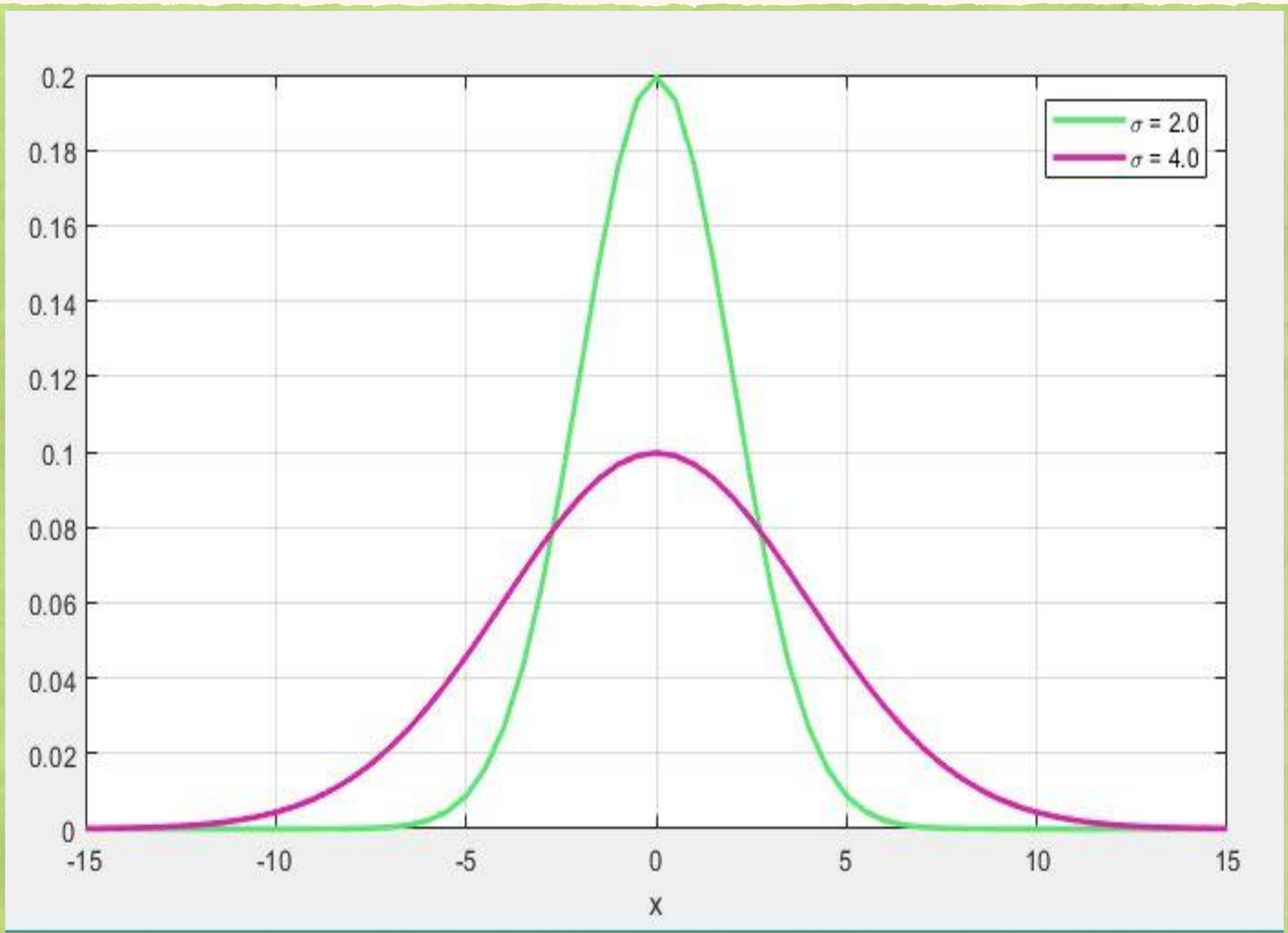
- The **mean** of a distribution **signifies its location**.
- The **variance** of a distribution signifies its **spread or uncertainty**.

Example - Normal Random Variable (2 of 3)



This plot shows the PDF of a normal distribution with two different values of mean μ and same variance σ^2.

Example - Normal Random Variable (3 of 3)



This plot shows the PDF of a normal distribution with the same mean μ and two different values of variance σ^2.

Appendix A - Variance of a Poisson Random Variable (1 of 3)

- The PMF of Poisson random variable with parameter λ is given by

$$p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \cdots, \quad \lambda > 0.$$

- Its mean, as shown earlier, is given by

$$E\{X\} \equiv \mu_X = \lambda.$$

- Next, we compute the variance.

$$E\{X(X-1)\} = \sum_{x=0}^{\infty} x(x-1)\, e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^x}{(x-2)!}, \qquad \text{using } \frac{x(x-1)}{x!} = \frac{1}{(x-2)!}$$

Appendix A - Variance of a Poisson Random Variable (2 of 3)

$= e^{-\lambda} \sum_{x'=0}^{\infty} \frac{\lambda^{x'+2}}{x'!}$, using change of variables, $x' = x - 2$

$= e^{-\lambda} \lambda^2 \sum_{x'=0}^{\infty} \frac{\lambda^{x'}}{x'!}$

$= \lambda^2 e^{-\lambda} e^{\lambda}$, using the Taylor series expansion for e^{λ}.

$= \lambda^2$.

Appendix A - Variance of a Poisson Random Variable (3 of 3)

$$var\{X\} = E\{X^2\} - E^2\{X\}$$

$$= E\{X(X-1)\} + E\{X\} - E^2\{X\}$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda.$$

Appendix B - Mean and Variance of a Geometric Random Variable (1 of 5)

- The PMF of a geometric random variable X, with parameter p, is given by

$$p_X(x) = (1-p)^{x-1}p, \quad x = 1,2,3,\cdots$$

- Its mean is given by

$$\mu_X = \frac{1}{p}.$$

- Its variance is given by

$$\sigma_X^2 = \frac{1-p}{p^2}.$$

Appendix B - Mean and Variance of a Geometric Random Variable (2 of 5)

Proof:

$$E\{X\} = \sum_x x\, p_X\,(x)$$

$$= \sum_{x=1}^{\infty} x\, p(1-p)^{x-1}$$

$$= \frac{p}{1-p} \sum_{x=1}^{\infty} x(1-p)^{x}$$

$$= \frac{p}{1-p}\, \frac{1-p}{\{1-(1-p)\}^2}$$

$$= \frac{1}{p}.$$

Appendix B - Mean and Variance of a Geometric Random Variable (3 of 5)

In the above derivation, we have utilized the following relation -

$$\sum_{x=1}^{\infty} x a^x = \frac{a}{(1-a)^2}, \quad |a| < 1.$$

This is obtained by differentiating the geometric series -

$$\sum_{x=0}^{\infty} a^x = \frac{1}{1-a}$$

Differentiating with respect to a,

$$\sum_{x=1}^{\infty} x a^{x-1} = \frac{1}{(1-a)^2}$$

$$\Longrightarrow \quad \sum_{x=1}^{\infty} x a^x = \frac{a}{(1-a)^2}$$

Appendix B - Mean and Variance of a Geometric Random Variable (4 of 5)

Next, we compute the variance -

We will make use of the following relation, obtained by differentiating the geometric series twice,

$$\sum_{x=1}^{\infty} x(x+1)a^{x} = \frac{2a}{(1-a)^{3}}, \quad |a| < 1.$$

Therefore,

$$E\{X(X+1)\} = \sum_{x=1}^{\infty} x(x+1)p(1-p)^{x-1}$$

$$= \frac{p}{1-p}\sum_{x=1}^{\infty} x(x+1)(1-p)^{x}$$

$$= \frac{p}{1-p}\,\frac{2(1-p)}{\{1-(1-p)\}^{3}} = \frac{2}{p^{2}}.$$

Appendix B - Mean and Variance of a Geometric Random Variable (5 of 5)

$$var\{X\} = E\{X^2\} - E^2\{X\}$$

$$= E\{X^2 + X\} - E\{X\} - E^2\{X\}$$

$$= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1-p}{p^2}.$$

References

1. Charles Boncelet, Probability, Statistics and Random Signals, Oxford University Press, 2016.
2. Sheldon Ross, A First Course in Probability, Macmillan Publishing Company, 1988.
3. R. D. Yates, et al., Probability and Stochastic Processes, John Wiley, 2005.