SAMPLE PROBLEM SET 2

[1] The electric field in a UPEMW propagating in a lossless dielectric medium with relative permittivity $\epsilon_r = 16$ has the instantaneous expression

$$\underline{E}(x,t) = \underline{i}_z \, 5 \sin(\omega t + 20\pi x + 60^\circ), \ V/m$$

Find:

i) the wavelength λ of the wave

ii) the phase velocity v_{ph} of the wave

iii) the relative refractive index n_r of the wave

iv) the frequency f of the wave

v) the unit vectors $\underline{i}_E$, $\underline{i}_H$ and $\underline{i}_k$

vi) the phasor expression $\overline{\underline{E}}(x)$ of the wave electric field

vii) the instantaneous expression $\underline{H}(x,t)$ of the wave magnetic field

viii) the phasor expression $\overline{\underline{H}}(x)$ of the wave

ix) the instantaneous Poynting vector $\underline{P}(x,t)$, and

x) the time-average Poynting vector $\underline{P}_{av}(x)$.

SOLUTION:

i) By inspection, $k = 20\pi$ rad/m

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{20\pi} = \frac{1}{10} \text{ or } 0.1 \text{ m}$$ ... Answer

ii) $$v_{ph} = \frac{c}{\sqrt{\epsilon_r}} = \frac{3\times10^8}{4} \text{ or } 0.75\times10^8 \text{ m/s}$$ Answer

iii) $$n_r = \sqrt{\epsilon_r} = 4$$ ... Answer

iv) $$f\lambda = v_{ph} \rightarrow f = \frac{v_{ph}}{\lambda} = \frac{3\times10^8}{4\times 1/10} = 0.75\times10^9 \text{ Hz}$$

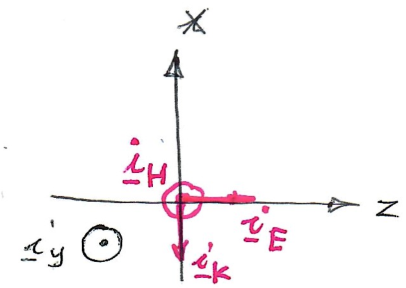
or 0.75 GHz ... Answer

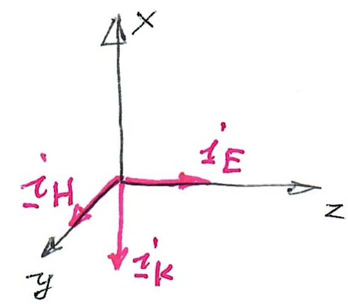
v) $\underline{i}_E = \underline{i}_z$

$\underline{i}_k = -\underline{i}_x$

$$\underline{i}_H = \underline{i}_k \times \underline{i}_E = -\underline{i}_x \times \underline{i}_z$$

$$= \underline{i}_y$$





Answer

vi) By inspection,

$$\bar{E}(x) = -j\,\underline{i}_z\,5\,e^{j(20\pi x + 60^\circ)} \quad V/m$$

Check:

Find $\underline{E}(x,t)$ from $\bar{E}(x) = -j\,\underline{i}_z\,5\,e^{j(20\pi x + 60^\circ)}$.

$$\underline{E}(x,t) = \mathrm{Re}\{\bar{E}(x)\,e^{j\omega t}\}$$

$$= \mathrm{Re}\{-j\,\underline{i}_z\,5\,e^{j(\omega t + 20\pi x + 60^\circ)}\}$$

$$= \underline{i}_z\,5\sin(\omega t + 20\pi x + 60^\circ), \quad V/m$$

This expression checks with the given expression of $\underline{E}(x,t)$.

vii) $\underline{H}(\cancel{x,t}) = \underline{i}_H\,H_o\sin(\omega t + 20\pi x + 60^\circ)$, A/m

where $\underline{i}_H = \underline{i}_y$

and $H_o = \dfrac{E_o}{\eta}$

Since $\eta = \dfrac{\eta_o}{\sqrt{\epsilon_r}} = \dfrac{377}{4}$,

$$\underline{H}(x,t) = \underline{i}_y\,\frac{4\times5}{377}\sin(\omega t + 20\pi x + 60^\circ) \quad \text{--- Answer}$$

viii) $$\bar{H}(x) = -j\,\underline{i}_y\,\frac{20}{377}\,e^{j(20\pi x + 60^\circ)} \quad A/m \quad \text{--- Answer}$$

ix) $\underline{P}(x,t) = \underline{E}(x,t) \times \underline{H}(x,t)$

$$= \underline{1}_z\, 5 \sin(\omega t + 20\pi x + 60^\circ) \times \underline{1}_y \frac{20}{377} \sin(\omega t + 20\pi x + 60^\circ)$$

$$= \underbrace{-\underline{1}_x}_{=\underline{1}_k} \frac{100}{377} \sin^2(\omega t + 20\pi x + 60^\circ) \,, \quad W/m^2$$

<u>Answer</u>

x) $\underline{P}_{av}(x) = \frac{1}{2} \frac{E_o^2}{\eta} \underline{1}_k$

$$= \frac{1}{2} \frac{25}{377/4} \underline{1}_k = \frac{50}{377} \underline{1}_k \, W/m^2$$

<u>Answer</u>

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[2] Identify the polarization of the four UPEMWs whose electric fields have the following phasor expressions:

i) $\bar{E}(z) = (\underline{1}_x + j\,\underline{1}_y)\, e^{-jkz}$

ii) $\bar{E}(z) = (\underline{1}_x - j\,\underline{1}_y)\, e^{-jkz}$

iii) $\bar{E}(z) = (\underline{1}_x + j\,\underline{1}_y)\, e^{jkz}$

iv) $\bar{E}(z) = (\underline{1}_x - j\,\underline{1}_y)\, e^{jkz}$

NOTE: i) The first two waves are $+z$ propagating while the last two waves are $-z$ propagating

ii) If the wave polarization is circular (or more generally elliptical), its sense of rotation must be referenced relative to the direction of propagation.

<u>SOLUTION:</u>

i) $\underline{E}(z,t) = Re\{\bar{E}(z)\, e^{j\omega t}\}$

$$= Re\{(\underline{1}_x + j\underline{1}_y)\, e^{j(\omega t - kz)}\}$$

$$= \underline{1}_x \cos(\omega t - kz) - \underline{1}_y \sin(\omega t - kz)$$

The polarization of the UPEMW is readily identified by plotting the locus (or trajectory) of the tip of the wave electric field $\underline{E}(z,t)$ as $t \uparrow$. For simplicity let the observation point be fixed at $z=0$ for all values of t.

Then

$$\underline{E}(0,t) = \underline{i}_x E_x(0,t) + \underline{i}_y E_y(0,t)$$

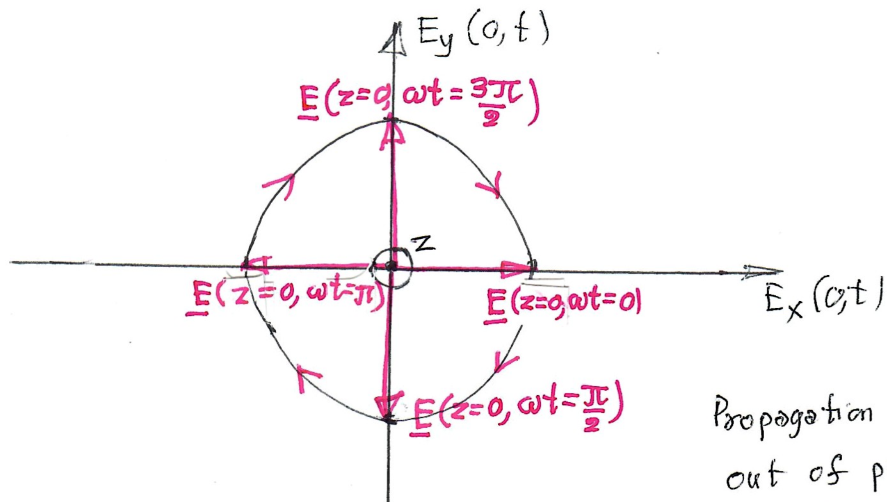
where

$$E_x(0,t) = \cos \omega t \quad \text{and}$$

$$E_y(0,t) = -\sin \omega t$$

The next step is to plot the locus of the tip of the $\underline{E}(0,t)$ vector in the $E_y(0,t)$ vs $E_x(0,t)$ plane as $t \uparrow$. This is readily done from the following table:

ωt rad	$E_x(0,t) = \cos \omega t$	$E_y(0,t) = -\sin \omega t$	$\underline{E}(0,t) = \underline{i}_x E_x(0,t) + \underline{i}_y E_y(0,t)$
0	1	0	$\underline{i}_x$
$\pi/2$	0	-1	$-\underline{i}_y$
π	-1	0	$-\underline{i}_x$
$3\pi/2$	0	1	$\underline{i}_y$

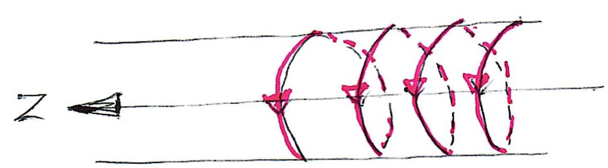


Propogation dir^n is out of paper in the +z direction

CONCLUSION: The polarization is left circular about the $+z$ dir^n of wave propogation

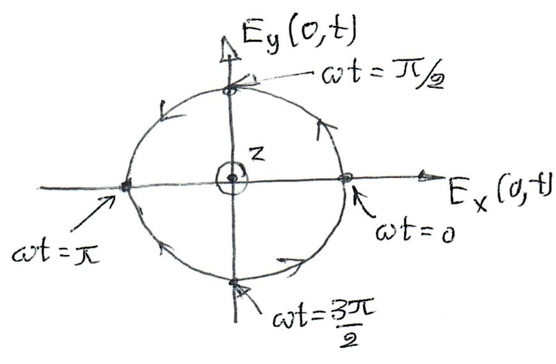
Answer

NOTE: The locus of the tip of the $\underline{E}(z,t)$ vector as $t\uparrow$ turned out to be a closed circle because the observation point z was assumed to be fixed at $z=0$ as t changes. If now take into account that the wave is actually moving, i.e., z increases in the direction of propagation of the wave, the locus of the tip of the $\underline{E}(z,t)$ vector executes a helical path instead of a circular path. This feature is qualitatively illustrated in the sketch below:



Left-handed helical trajectory of the tip of the $\underline{E}(z,t)$ vector as $t\uparrow$

ii) The wave corresponding to the electric field $\underline{E}(z) = (\underline{1}_x - j\underline{1}_y)\, e^{-jky}$ is readily shown as above to represent a right-circularly polarized wave about the +z dirn of propagation. The proof is left as an exercise for the student.



iii) Show that the wave specified by its electric field vector $\underline{E}(z) = (\underline{1}_x + j\underline{1}_y)\, e^{jkz}$ has the same expressions for the $E_x(z=0, \omega t)$ and $E_y(z=0, \omega t)$ components but now the propagation of the wave is in the $-z$

direction. Hence the wave is right-circularly polarized about the $-z$ direction of propagation.

iv) Show that $E_x(0,t) = \cos \omega t$ and $E_y(0,t) = \sin \omega t$ while the propogation is in the $-z$ direction. This gives the result that the wave is left-circularly polarized about the $-z$ dir^n of propagation.