# Joint Random Variables

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# Topics

- Definition of Joint Random Variables
- Characterization of Discrete Joint Random Variables
  - Cumulative Distribution Function (CDF)
  - Probability Mass Function (PMF)
- Characterization of Continuous Joint Random Variables
  - Cumulative Distribution Function (CDF)
  - Probability Density Function (PDF)
- Marginals
- Independence
- Some Examples

# I. Joint Random Variables

- We are interested in several random variables that are related to each other, for example the length and width of a flower petal or sugar level and diabetic disease in a person.
- Such variables are known as joint random variables or multiple random variables or random vectors.
- We are interested in studying the joint probability distributions of such random variables.
- When we have only two random variables, say $X_1$ and $X_2$, we call this **bivariate probability distributions**.
- In this lecture, we will study the properties of two random variables.
- The same concepts can be extended to multiple random variables $X_1, X_1, X_3, \cdots$

# II. Characterization of Joint Random Variables

- The **discrete joint random variables** are characterized by
  - Cumulative Distribution Function (CDF)
  - Probability Mass Function (PMF)
- The **continuous joint random variables** are characterized by
  - Cumulative Distribution Function (CDF)
  - Probability Density Function (PDF)

# Cumulative Distribution Function

- The **cumulative distribution function (CDF)** of joint random variables $X_1$ and $X_2$ **(discrete or continuous)** is the function

  $F_{X_1,X_2}(x_1, x_2): \mathcal{R}^2 \to [0,1]$, given by

  $F_{X_1,X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$, defined for both discrete and continuous random variables.

- Note that the uppercase $X$ is used for the random variable and the lower case $x$ is used for a specific value.

- For example, $P(X_1 \leq 3, X_2 \leq 5)$ means probability that the random variable $X_1$ is less than or equal to 3 and $X_2$ is less than or equal to 5.

# Properties of CDF

The CDF of the random variables $(X_1, X_2)$ has the following properties:

- $0 \le F_{X_1,X_2}(x_1, x_2) \le 1$.
- $F_{X_1}(x_1) = F_{X_1,X_2}(x_1, \infty)$.
- $F_{X_2}(x_2) = F_{X_1,X_2}(\infty, x_2)$.
- $\lim_{x_1 \to -\infty} F_{X_1,X_2}(x_1, x_2) = 0$.
- $\lim_{x_2 \to -\infty} F_{X_1,X_2}(x_1, x_2) = 0$.

# Properties of CDF, cont.

- If $x_{11} < x_{12}$ and $x_{21} < x_{22}$, then

  $F_{X_1,X_2}(x_{11}, x_{21}) \leq F_{X_1,X_2}(x_{12}, x_{22})$.

- $\lim_{x_1 \to \infty,\, x_2 \to \infty} F_{X_1,X_2}(x_1, x_2) = 1$.

- Note that the above properties are necessary and sufficient conditions for $F_{X_1,X_2}(x_1, x_2)$ to be a joint CDF.

# Joint Probability Mass Function

- The **joint probability mass function (PMF)** of discrete random variables $X_1$ and $X_2$ is defined by

  $$p_{X_1,X_2}(x_1, x_2) = P(X_1 = x_1, X_2 = x_2).$$

- The joint PMF satisfies the property

  $\sum_{x_1,x_2} p_{X_1,X_2}(x_1, x_2) = 1$, that is, the sum of PMF over the support of $X_1$ and $X_2$ is 1.

- **Probability in a region *B*** within the support of $X_1$ and $X_2$ is given by

  $$P\big((X_1, X_2) \in B\big) = \sum_B p_{X_1,X_2}(x_1, x_2).$$

# Marginal Probability Mass Function

- The **marginal** PMFs of $X_1$ and $X_2$ can be obtained from the joint PMF as follows:

  $p_{X_1}(x_1) = \sum_{x_2} p_{X_1,X_2}(x_1, x_2)$, and

  $p_{X_2}(x_2) = \sum_{x_1} p_{X_1,X_2}(x_1, x_2)$.

- Further, the marginals satisfy the property

  $\sum_{x_1} p_{X_1}(x_1) = 1$, and

  $\sum_{x_2} p_{X_2}(x_2) = 1$.

# Joint Probability Density Function

- The **joint probability density function (PDF)** of continuous random variables $X_1$ and $X_2$ satisfies the following property

$$F_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(u_1, u_2)\, du_1\, du_2,$$

$u_1$ and $u_1$ are dummy variables of integration.

for some integrable function f: $\mathcal{R}^2 \to [0, \infty)$.

- The joint PDF satisfies the property

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 = 1,$ that is, the integral of the PDF over the support of $X_1$ and $X_2$ is 1.

# Joint Probability Density Function, cont.

- **Probability in a region *B*** within the support of $X_1$ and $X_2$ is given by

  $P\big((X_1, X_2) \in B\big) = \int_B f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2.$

# Marginal Probability Density Function

- The **marginal** PDFs of $X_1$ and $X_2$ can be obtained from the joint PDF as follows:

$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) dx_2,$ and

$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) dx_1.$

- Further, the marginals satisfy the property

$\int_{-\infty}^{\infty} f_{X_1}(x_1) dx_1 = 1$, and

$\int_{-\infty}^{\infty} f_{X_2}(x_2) dx_2 = 1.$

# Notation

- **Notation**
  - We use uppercase $F_{X_1,X_2}(x_1, x_2)$ for joint CDF.
  - We use lowercase $p_{X_1,X_2}(x_1, x_2)$ for joint PMF.
  - We use lowercase $f_{X_1,X_2}(x_1, x_2)$ for joint PDF.

# Example - Discrete Joint Random Variables (1 of 8)

## Example

The joint PMF $p_{X_1,X_2}(x_1, x_2)$ of discrete random variables $X_1$ and $X_2$ is given by the following table:

| $X_1 \backslash X_2$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $1/12$ | $1/12$ | $1/6$ |
| $0$ | $1/6$ | $1/6$ | $0$ |
| $1$ | $1/12$ | $1/6$ | $1/12$ |

# Example - Discrete Joint Random Variables (2 of 8)

(a) Determine the marginal PMF $p_{X_1}(x_1)$.

The marginal PMF is given by the following sum formula

$p_{X_1}(x_1) = \sum_{x_2} p_{X_1,X_2}(x_1, x_2)$.

We compute this marginal for each value of $X_1$.

$$p_{X_1}(X_1 = -1) = p_{X_1,X_2}(X_1 = -1, X_2 = -1) + p_{X_1,X_2}(X_1 = -1, X_2 = 0)$$
$$+ p_{X_1,X_2}(X_1 = -1, X_2 = 1).$$

That is, we sum up over all values of $X_2$, with $X_1 = -1$. This is the **row sum** in the above table.

Therefore, $p_{X_1}(X_1 = -1) = \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{1}{3}$.

# Example - Discrete Joint Random Variables (3 of 8)

Continuing in a similar manner, we obtain

$$p_{X_1}(X_1 = 0) = p_{X_1,X_2}(X_1 = 0, X_2 = -1) + p_{X_1,X_2}(X_1 = 0, X_2 = 0)$$

$$+ p_{X_1,X_2}(X_1 = 0, X_2 = 1).$$

$$= \frac{1}{6} + \frac{1}{6} + 0 = \frac{1}{3}.$$

And,

$$p_{X_1}(X_1 = 1) = p_{X_1,X_2}(X_1 = 1, X_2 = -1) + p_{X_1,X_2}(X_1 = 1, X_2 = 0)$$

$$+ p_{X_1,X_2}(X_1 = 1, X_2 = 1).$$

$$= \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}.$$

# Example - Discrete Joint Random Variables (4 of 8)

The marginal $p_{X_1}(x_1)$ is shown in the table below:

| $X_1 \backslash X_2$ | $-1$ | $0$ | $1$ | $p_{X_1}(x_1)$ |
|---|---|---|---|---|
| $-1$ | $1/12$ | $1/12$ | $1/6$ | $1/3$ |
| $0$ | $1/6$ | $1/6$ | $0$ | $1/3$ |
| $1$ | $1/12$ | $1/6$ | $1/12$ | $1/3$ |

# Example - Discrete Joint Random Variables (5 of 8)

(b) Determine the marginal PMF $p_{X_2}(x_2)$.

The marginal PMF is given by the following sum formula

$p_{X_2}(x_2) = \sum_{x_1} p_{X_1,X_2}(x_1, x_2)$.

That is, now we sum over $X_1$, for each value of $X_2$. This is the **column sum** in the above table.

$$p_{X_2}(X_2 = -1) = p_{X_1,X_2}(X_1 = -1, X_2 = -1) + p_{X_1,X_2}(X_1 = 0, X_2 = -1)$$
$$+ p_{X_1,X_2}(X_1 = 1, X_2 = -1).$$

Here we sum up over all values of $X_1$, with $X_2 = -1$.

Therefore, $p_{X_2}(X_2 = -1) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{4}{12}$.

# Example - Discrete Joint Random Variables (6 of 8)

The marginal $p_{X_2}(x_2)$ is also shown in the table below:

| $X_1 \backslash X_2$ | $-1$ | $0$ | $1$ | $p_{X_1}(x_1)$ |
|---|---|---|---|---|
| $-1$ | $1/12$ | $1/12$ | $1/6$ | $1/3$ |
| $0$ | $1/6$ | $1/6$ | $0$ | $1/3$ |
| $1$ | $1/12$ | $1/6$ | $1/12$ | $1/3$ |
| $p_{X_2}(x_2)$ | $4/12$ | $5/12$ | $3/12$ | $1$ |

Each of the marginal satisfies $\sum_{x_1} p_{X_1}(x_1) = 1$, and $\sum_{x_2} p_{X_2}(x_2) = 1$.

# Example - Discrete Joint Random Variables (7 of 8)

(c) What is $P(X_1 = X_2)$?

$$P(X_1 = X_2) = p_{X_1,X_2}(X_1 = -1, X_2 = -1) + p_{X_1,X_2}(X_1 = 0, X_2 = 0)$$

$$+ p_{X_1,X_2}(X_1 = 1, X_2 = 1).$$

$$= \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}.$$

# Example - Discrete Joint Random Variables (8 of 8)

(c) Are $X_1$ and $X_2$ independent?

For $X_1$ and $X_2$ to be independent, the joint must factorize into marginals for all values of $X_1$ and $X_2$.

That is, $p_{X_1,X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$.

Let us check this for some values.

$p_{X_1,X_2}(X_1 = -1, X_2 = -1) = \frac{1}{12}$ (from the above table).

$p_{X_1}(X_1 = -1)\, p_{X_2}(X_2 = -1) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$ (from the above computations).

Since $p_{X_1,X_2}(x_1, x_2) \neq p_{X_1}(x_1)p_{X_2}(x_2)$, therefore $X_1$ and $X_2$ are not independent.

# Example - Continuous Joint Random Variables (1 of 6)

Example

The joint PDF $f_{X_1,X_2}(x_1, x_2)$ of continuous random variables $X_1$ and $X_2$ is given by

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 \le x_1 \le 1, \quad 0 \le x_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the marginal PDF $f_{X_1}(x_1)$.

The marginal PDF is given by the following expression

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)dx_2,$$

$$= \int_0^1 (x_1 + x_2)dx_2$$

# Example - Continuous Joint Random Variables (2 of 6)

$$= \left| x_1 x_2 + \frac{x_2^2}{2} \right|_0^1$$

$$= x_1 + \frac{1}{2}, \qquad 0 \leq x_1 \leq 1.$$

The marginal of $X_1$ is a function of $x_1$.

(b) Determine the marginal PDF $f_{X_2}(x_2)$.

The marginal PDF is given by the following expression

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)dx_1,$$

$$= \int_0^1 (x_1 + x_2)dx_1$$

# Example - Continuous Joint Random Variables (3 of 6)

$$= \left| \frac{x_1^2}{2} + x_1 x_2 \right|_0^1$$

$$= \frac{1}{2} + x_2, \qquad 0 \le x_2 \le 1.$$

The marginal of $X_2$ is a function of $x_2$.

(c) Determine the probability $P(X_1 \le 1/2)$.

The required probability is given by

$$P(X_1 \le 1/2) = \int_0^{1/2} f_{X_1}(x_1) dx_1,$$

$$= \int_0^{1/2} (x_1 + \tfrac{1}{2}) dx_1$$

Here we use the marginal of $X_1$.

# Example - Continuous Joint Random Variables (4 of 6)

$$= \left| \frac{x_1^2}{2} + \frac{x_1}{2} \right|_0^{1/2}$$

$$= \frac{3}{8}.$$

(d) Determine the probability $P(X_1 + X_2 \leq 1)$.

The required probability is given by

$$P(X_1 + X_2 \leq 1) = \int_0^1 \int_0^{1-x_1} (x_1 + x_2) dx_1 \, dx_2$$

This comes from the expression on slide 11 on probability in a region.

The limits of the integral are tricky to determine and are explained shortly.

# Example - Continuous Joint Random Variables (5 of 6)

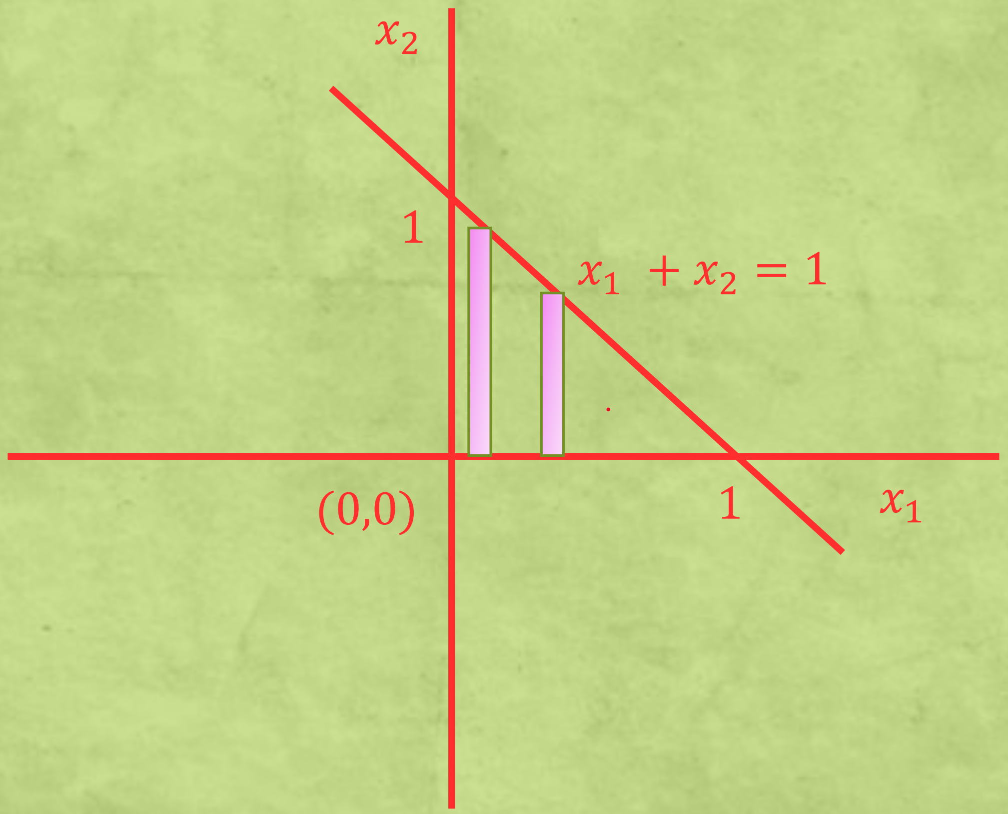
$$= \int_0^1 \left[x_1(1 - x_1) + \frac{(1-x_1)^2}{2}\right] dx_1$$

After doing the $x_2$ integral.

$$= \int_0^1 \left[\frac{1}{2} - \frac{{x_1}^2}{2}\right] dx_1$$

$$= \frac{1}{3}.$$

# Example - Continuous Joint Random Variables (6 of 6)




**Determining the limits of integration** -

The triangular region is the region $X_1 + X_2 \leq 1$.

This is bound by the line $x_1 + x_2$=1, the $x_1$-axis and the $x_2$-axis.

It includes the following constraints from the support of $X_1$ and $X_2$:

$0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$.

Sliding the purple vertical strip over the triangular region **"covers"** this region. For this strip, $x_1 : 0 \rightarrow 1$ and $x_2 : 0 \rightarrow 1 - x_1$.

# References


1. Charles Boncelet, Probability, Statistics and Random Signals, Oxford University Press, 2016.
2. Sheldon Ross, A First Course in Probability, Macmillan Publishing Company, 1988.
3. R. D. Yates, et al., Probability and Stochastic Processes, John Wiley, 2005.