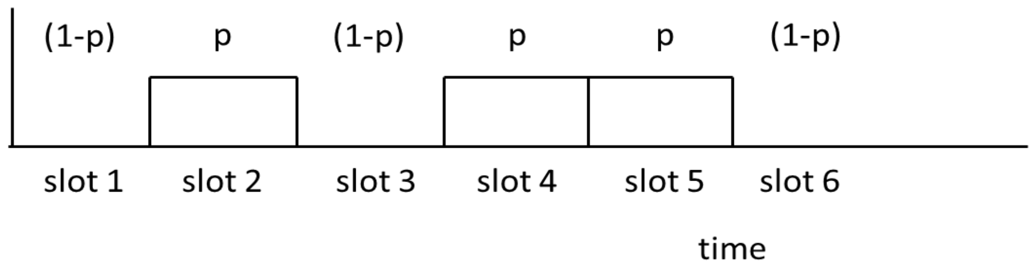

Figure 2.1: Single stream of packets and empty slots

2.1 Probability Problems

In this section a large number of probability problems with a networking flavor will be solved. [1]. More such problems appear as end of chapter exercises

2.1.1 Packet Streams

In these type of problems a stream(s) of packets is modeled as a Bernoulli process. That is one might assume a sequence of time slots of equal width. Here each time slot has a single packet with independent probability p and a time slot has no packet (is empty or idle) with independent probability $1-p$. That is, what happens in each slot is independent of what happens in other slots. Again, mathematically this is a Bernoulli process. There are many real world applications that can be modeled as a Bernoulli process including coin flipping.

Single Packet Stream

What are typical probability questions one can ask about a single Bernoulli stream of packets (see Fig. 2.AA)? Here are some

(a) Write an expression for the probability of the exact sequence in Fig. 2.AA occurring.

One has the probability of an empty slot 1 is $(1-p)$, the probability of a packet in slot 2 is p and so on.... So

$$Prob = (1-p)p(1-p)pp(1-p) = p^3(1-p)^3 \tag{2.1}$$

The overall probability result here is called the "joint" probability. It is the probability of no packet in slot 1 and a packet in slot 2 and no packet in slot 3 and so on... The overall probability involves simply multiplying the event probabilities. This can be done as the event probabilities (i.e. what happens in a time slot) are independent (see any introductory text on probability).

(b) Write an expression of exactly three packets occurring in six slots in any pattern.

Any sequence of 3 packets in 6 slots occurs with the same probability. So one just has to multiply that probability (really say the answer of part (a)) by the number of possible sequences consisting of 3 packets in 6 slots or $\binom{6}{3}$, or

$$\binom{6}{3} p^3(1-p)^3 \tag{2.2}$$

This is a binomial distribution. The "bi" in binomial means two (as in biplane or bifocals) and there are two options in each slot: packet or no packet. So the binomial distribution is the appropriate choice of distribution. If there were more than two choices (say a variety of packet types) we would use the multinomial distribution for this type of problem.

(c) Write an expression for the probability that the first packet appears in slot 10.

The question is answered by the geometric distribution. One has nine empty time slots (each occurring with independent probability $(1-p)$) followed by an arrival in the 10th slot occurring with probability p.

$$Prob(10) = \underbrace{(1-p)(1-p)...(1-p)}_{nine\ empty\ slots} \times \underbrace{p}_{packet\ on\ 10th\ slot} = (1-p)^9 p \qquad (2.3)$$

The individual probabilities can be simply multiplied to find the joint probability since the events in each slot are independent of any event in other slots.

(d) Write an expression for the probability of the first arrival being in slot 10 followed by arrivals in slot 11 and 12.

Since each time slot is independent of every other time slot, the probabilities simply multiply to produce to the joint probability.

$$Prob(10, 11, 12) = ((1-p)^9 \times p) \times p \times p = (1-p)^9 p^3 \qquad (2.4)$$

(e) What is the probability that the 5th packet arrives in the 10th slot?

The Pascal (negative binomial) distribution can be used to answer this question. One has

$$Prob(kth\ packet\ in\ Nth\ slot) = \left(\begin{array}{c} N-1 \\ k-1 \end{array} \right) p^k (1-p)^{N-k} \qquad (2.5)$$

$$Prob(5th\ packet\ in\ 10th\ slot) = \left(\begin{array}{c} 10-1 \\ 5-1 \end{array} \right) p^5 (1-p)^{10-5} = \left(\begin{array}{c} 9 \\ 4 \end{array} \right) p^5 (1-p)^5 \quad (2.6)$$

The Pascal distribution is a generalization of the geometric distribution. See the previous section 2.3 for information on the Pascal distribution.

Dual Packet Streams

Suppose now that a switching element has two inputs lines, each of which can be modeled as a Bernoulli process (Figure 2.BB). It is assumed that the two Bernoulli streams are independent of each other and the individual streams consist of independent events.

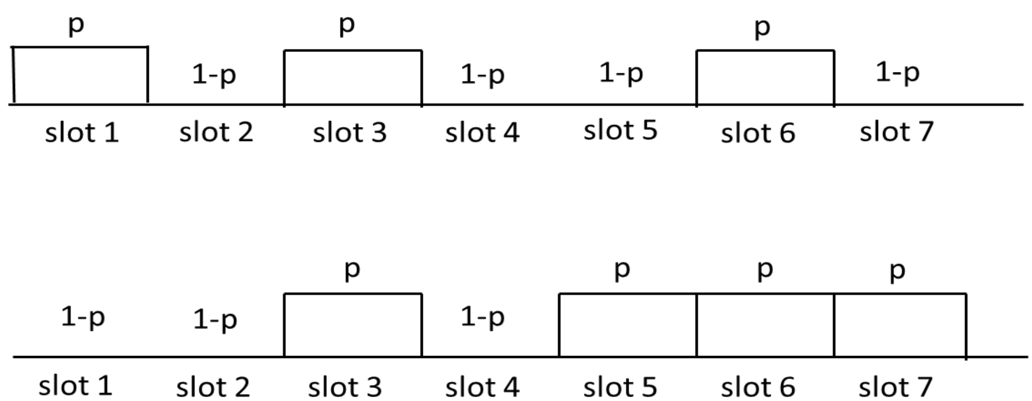

Figure 2.2: Dual streams of packets and empty slots

The following questions are considered.

(a) Write an expression for the probability of no arrivals in both streams in ten consecutive time slots.

Then all 20 (i.e. 2 $\times$ 10) slots are empty and one has the following

$$Prob = \underbrace{(1-p)(1-p)...(1-p)}_{twenty\ times} = (1-p)^{20} \tag{2.7}$$

The individual event probabilities can be multiplied to find the joint probability because of the independence of events (i.e. arrivals and non-arrivals).

(b) Write an expression for the probability of the specific sequences shown in the figure occurring.

There are 7 packets and 7 empty slots. Thus

$$Prob = \underbrace{ppppppp}_{7\ packets}\underbrace{(1-p)(1-p)(1-p)(1-p)(1-p)(1-p)(1-p)}_{7\ empty\ slots} = p^7(1-p)^7 \tag{2.8}$$

(c) Write an expression for the probability of one or two packets arriving in a single time slot (considering both streams).

$$Prob = \underbrace{2p(1-p)}_{1\ arrival} + \underbrace{p^2}_{2\ arrivals} \tag{2.9}$$

The first term represents one packet being on one line (with probability p) and no packet arriving on the other line (with probability $1-p$). The single arriving packet can be on either line so we multiply by a factor of 2 to get the overall probability. The second term is the probability of one arrival for each stream for a total of two arrivals.

(d) Write an expression for the probability of the first arrival of one or two packets in a single time slot occurring in slot 11.

This is just the probability of no packets in the first ten individual time slots multiplied by the result of part **(c)** (the probability of one or two packets arriving in a single time slot).

$$Prob = (1-p)^{20}\left(2p(1-p) + p^2\right) \tag{2.10}$$

A Stream of Different Packet Types

When there are different types of packets the multinomial distribution is often useful. Consider the nearby figure of a sequence with 2 voice packets (packets carrying voice information), 1 data packet (a packet carrying data) and 3 idle time slots with no packets, all in 6 time slots. It is assumed that the probability of a voice packet arrival is p_v, the probability of a data packet arrival is p_d and the probability of no packet in a slot is p_{idle}. We assume all arrivals are independent of each other. Naturally $p_v + p_d + p_{idle} = 1$. That is, in a time slot one of the three events occur with probability of 1. Also let n be the total number of arrivals of all types ($n = n_v + n_d + n_{idle}$).

(a) Suppose now we wish to determine the probability of 2 voice packets, 1 data packet and 3 idle slots occurring in six slots *in any order*. The probability of such a particular sequence is $(p_v^2)(p_d^1)(p_{idle}^3)$. Multiplying by the number of such

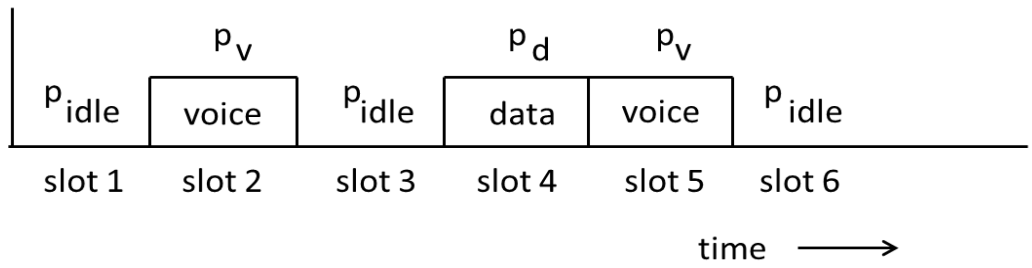

Figure 2.3: A packet stream with packets of different types

sequences/patterns, which is the multinomial coefficient, yields the multinomial distribution. So for any such order/pattern

$$Prob(n_v, n_d, n_{idle}) = \frac{n!}{n_v! n_d! n_{idle}!} (p_v{}^{n_v})(p_d{}^{n_d})(p_{idle}{}^{n_{idle}}) \tag{2.11}$$

For the parameters being considered one has

$$Prob(n_v = 2, n_d = 1, n_{idle} = 3) = \frac{6!}{2!1!3!} (p_v{}^2)(p_d{}^1)(p_{idle}{}^3) \tag{2.12}$$

The individual probabilities can be substituted on the right side of the equations to find the joint probability at the left side of the equation.

(b) Another question for this situation is what is the average number of voice packets in 6 slots? Of data packets? Of voice or data packets?

The average number of voice packets is $np_v = 6p_v$.

The average number of data packets is $np_d = 6p_d$.

The average number of voice or data packets is $(6p_v + 6p_d) = 6(p_v + p_d)$.

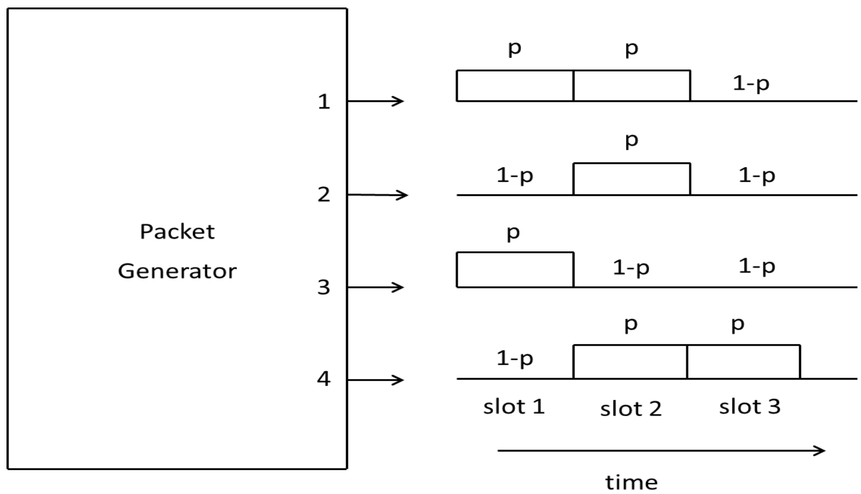

Figure 2.4: Bernoulli packet generator with four output streams

Packet Generators

Finally, consider multiple Bernoulli streams from the outputs of a packet generator (see the nearby figure). Here a "packet generator" is a device creating packet streams, perhaps for testing purposes. It is assumed that both the packets within a specific stream and the streams with respect to each other are independent of each other.

Possible problems involving a packet generator providing four streams of packets include the following.

(a) Write the probability in one slot of a packet on both outputs 1 and 2 and no packets on outputs 3 and 4.

Since the outputs are independent of each other the probability of this particular output pattern is:

$$p^2(1-p)^2 \tag{2.13}$$

(b) What is the probability in a time slot of exactly two packets at the outputs in any pattern?

Each output pattern with exactly two output packets and two idle outputs has identical probability $p^2(1-p)^2$. This is simply multiplied by the number of such patterns (i.e. the binomial coefficient).

$$Prob(exactly\, 2\, packets\, in\, any\, pattern) = \left(\begin{array}{c} 4 \\ 2 \end{array} \right) p^2(1-p)^2 \tag{2.14}$$

This is the binomial distribution.

(c) Write an expression for the average number of packets in a slot across all four outputs.

There are two possible answers. We are really asking for the mean of the binomial distribution. One answer is simply the weighted average definition of the mean.

$$\overline{number\ of\ output\ packets} = \sum_n np(n) = \sum_{n=1}^{4} n \left(\begin{array}{c} 4 \\ n \end{array} \right) p^n(1-p)^{4-n} \tag{2.15}$$

Here the over lined quantity is simply the average of that quantity. The second, simpler answer, is just to recognize that the binomial mean is simply Np where N is the number of output streams or just $4p$. Since p is a number between 0 and 1, $4p$ is a number between 0 and 4 which makes sense since there are four outputs. That is there are most 4 output packets in a slot.

(d) Write an expression for the probability of three or more packets appearing in a slot across all four outputs. Call the answer q.

$$Prob(3\, or\, more\, output\, packets) = q = \underbrace{\left(\begin{array}{c} 4 \\ 3 \end{array} \right) p^3(1-p)^1}_{3\, packets} + \underbrace{\left(\begin{array}{c} 4 \\ 4 \end{array} \right) p^4(1-p)^0}_{4\, packets} \tag{2.16}$$

There are two terms above, one for exactly three packets at the outputs in any pattern and one for exactly four packets at the outputs (there is only one pattern for this second case).

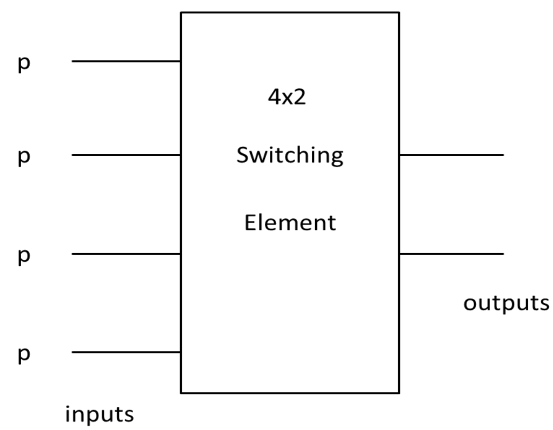

Figure 2.5: A four input, two output, switching element

(e) Write an expression in terms of q of part **(d)** for the probability that the ith slot is the first slot to have three or more packets appearing across all the outputs.

This is just a geometric distribution with parameter q.

$$Prob = (1-q)^{i-1}q \tag{2.17}$$

2.1.2 Switching Elements

"Switching elements" are devices that connect signal inputs to signal outputs according to some rules in a structured manner. That is, packets or calls are routed from input to outputs. Interconnection networks are patterned connections of relatively simple switching elements into an overall switching network. Such interconnection networks are often implemented on chips (Robertazzi 1993a, 2017).

Switching Element Inputs

Consider a switching element in the nearby figure that has 4 inputs and 2 outputs.

Time is slotted into slots of equal width. In a time slot a packet arrives at each input with independent probability p and a packet does not arrive at each input in a time slot with independent probability $1-p$. In the following consider only the inputs.

(a) What is the probability of exactly two incoming packets in a slot across all 4 inputs in any pattern?

There are two options at each input in each slot, packet or idle slot (i.e. no packet). This is a binomial distribution problem.

$$Prob(exactly\, 2\, incoming\, packets) = \binom{4}{2} p^2(1-p)^2 \tag{2.18}$$

That is, there are two inputs with packets (occurring with probability p^2), two inputs without packets (occurring with probability $(1-p)^2$). The number of ways this can happen is $\binom{4}{2} = 6$ so the probability of two arriving packets and 2 idle slots in some specific pattern is multiplied by this amount.

(b) What is the probability of at least two incoming packets in a slot across all inputs?

One can write this as

$$Prob(at\, least\, 2\, incoming\, packets)$$

$$= Prob(2\, packets) + Prob(3\, packets) + Prob(4\, packets) \tag{2.19}$$

$$= \binom{4}{2} p^2(1-p)^2 + \binom{4}{3} p^3(1-p)^1 + \binom{4}{4} p^4(1-p)^0 \tag{2.20}$$

Or

$$Prob(at\, least\, 2\, incoming\, packets)$$

$$= 1 - Prob(0\,packets) - Prob(1\,packets) \tag{2.21}$$

$$= 1 - \begin{pmatrix} 4 \\ 0 \end{pmatrix} p^0(1-p)^4 - \begin{pmatrix} 4 \\ 1 \end{pmatrix} p^1(1-p)^3 \tag{2.22}$$

$$= 1 - (1-p)^4 - 4p(1-p)^3 \tag{2.23}$$

(c) What is the average number of packets at the inputs in a time slot?

This is the mean value of the binomial distribution or $4p$. More formally

$$\overline{number\ of\ packets} = \sum_{n=1}^{4} n \begin{pmatrix} 4 \\ n \end{pmatrix} p^n(1-p)^{4-n} = 4p \tag{2.24}$$

In the above, the over lined quantity means the average of that quantity.

2x1 Switching Element

Now consider a switching element that has two inputs and one output. Time is slotted again. In each slot either 0 or 1 packets arrive at each input. A packet arrives to an input in a slot with independent probability p and doesn't arrive to an input in a slot with independent probability $1-p$. An output packet is produced in a slot if at least one packet arrives to the inputs. If two packets arrive one packet is forwarded to the output and other packet is dropped/cleared from the system. A higher level protocol can resend the dropped packet if desired. Switching networks are sometimes designed to handle traffic on a statistical basis in this manner.

(a) Write an expression for the probability of a packet at the output in a time slot. Call this answer q.

The probability that there is a packet at the output is equal to the probability that there is at least one packet at the inputs. This is equal to one minus the probability there are no packets at the inputs.

$$Prob(one\ output\ packet) = Prob(at\ least\ one\ input\ packet) \tag{2.25}$$

$$Prob(at\ least\ one\ input\ packet) = 1 - Prob(no\ input\ packets) \tag{2.26}$$

$$Prob(one\ output\ packet) = 1 - (1-p)^2 = q \tag{2.27}$$

(b) Suppose that the process starts with slot 1. What is an expression for the probability that the first output packet appears in slot i?

The answer to this question is a geometric distribution. There is no packet for $i-1$ slots with probability $(1-q)\times(1-q)\times ...(1-q) = (1-q)^{i-1}$ and in the ith slot there is an output packet with probability q. To find the overall joint probability these probabilities multiply since events in each slot are independent.

$$Prob(1st\ output\ packet\ in\ slot\ i) = (1-q)^{i-1}q \tag{2.28}$$

(c) What is the probability that i consecutive slots all have output packets?

This is just the probability of there being a packet at the output in a slot, q, multiplied i times or simply q^i.

4x2 Switching Element

Consider a 4 input 2 output (4x2) switching element as in the previous figure.

There is at most one packet per input per time slot. If one or two packets arrive at the inputs in a time slot then the packets go to the outputs. If 3 or 4 packets arrive, two packets are randomly chosen to go the outputs and the remaining packet(s) is erased/dropped. Let p be the independent probability of a packet arrival on an input in a time slot and let $1-p$ be the independent probability of no arriving packet at an input in a time slot.

(a) What is the probability that exactly one packet arrives in one slot across all inputs and exactly two packets arrive in the next slot across all inputs?

Arrivals from slot to slot are independent. So the requested probability is the product of two binomial probabilities.

$$Prob(1,\ then\ 2\ arrivals) = P(1\ arrival)Prob(2\ arrivals) \tag{2.29}$$

$$Prob(1,\ then\ 2\ arrivals) = \left[\begin{pmatrix} 4 \\ 1 \end{pmatrix} p^1(1-p)^3\right]\left[\begin{pmatrix} 4 \\ 2 \end{pmatrix} p^2(1-p)^2\right] \tag{2.30}$$

(b) What is the probability that there are two packets at the outputs?

The probability of two packets at the outputs equals the probability of 2 to 4 arrivals at the inputs.

$$Prob(2\ output\ packets) = \sum_{n=2}^{4} \begin{pmatrix} 4 \\ n \end{pmatrix} p^n(1-p)^{4-n} \tag{2.31}$$

(c) What is the mean thruput (the average number of packets at the outputs in a time slot)?

$$\overline{Thruput} = 1 \cdot Prob(1\ output\ packet) + 2 \cdot Prob(2\ output\ packets) \tag{2.32}$$

The average thruput is a weighted sum. It is one times the probability of one packet at the inputs (leading to a single output packet) plus 2 times the probability of 2 to 4 packets at the inputs (which leads to two packets at the outputs). Thus

$$\overline{Thruput} = 1 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} p^1(1-p)^3 + 2 \cdot \sum_{n=2}^{4} \begin{pmatrix} 4 \\ n \end{pmatrix} p^n(1-p)^{4-n} \tag{2.33}$$

Concatenator

Consider a switching element serving as a "concatenator". There are four inputs and four outputs. Time is slotted. The independent probability of there being a packet at each input in a time slot is p. The independent probability of there being no packet at an input in a time slot is $1-p$.

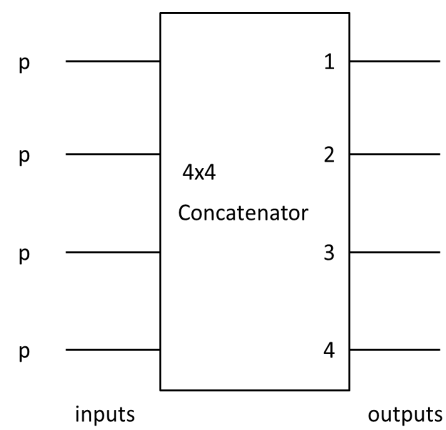

Figure 2.6: A four input, four output, concatenator

Referring to the figure, if one input packet arrives across all inputs in a time slot, it is sent to the top output. If two packets arrive in a time slot, the packets are sent to the top two outputs. If three packets arrive in a time slot, the packets are sent to the top three outputs. If four packets arrive at the inputs, each output gets a packet.

(a) What is the probability that there is a packet at the top output in a time slot?

One must be careful - the answer is not 1 as no packet may arrive to the inputs in a time slot. The probability in question is equal to one minus the probability of no arrivals or the probability of at least one arrival.

$$Prob(packet\ at\ top\ output) = 1 - (1-p)^4 \tag{2.34}$$

Or

$$Prob(packet\ at\ top\ output) = \sum_{n=1}^{4} \binom{4}{n} p^n (1-p)^{4-n} \tag{2.35}$$

(b) What is the probability that in a time slot, output 3 has no packet?

This probability is just the probability that two or less packets arrive to the inputs in a time slot. One has

$$Prob(no\, packet\, at\, output\; 3) =$$

$$= Prob(0\; input\, packets) + Prob(1\; input\, packet) + Prob(2\; input\, packets) \quad (2.36)$$

$$= (1-p)^4 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} p(1-p)^3 + \begin{pmatrix} 4 \\ 2 \end{pmatrix} p^2(1-p)^2 \quad (2.37)$$

(c) What is the probability that output 4 has a packet in a time slot?

This is simply the probability of four input packets.

$$Prob(packet\, at\, output\; 4) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} p^4(1-p)^0 = p^4 \quad (2.38)$$

2.1.3 Clusters of Computers

Clusters (i.e. groups) of computers provide parallel processing power to solve problems in scientific, engineering and business computing and in data centers.

A Cluster in a Rack

Consider an equipment rack in a data center housing 96 computers. Let p be the independent probability that a computer is down (i.e. not working). Let $1-p$ be the independent probability that a computer is up (i.e. working).

(a) Write an expression for the probability that exactly one server is down.

This is just a binomial distribution.

$$Prob(exactly\ 1\ server\ down) = \begin{pmatrix} 96 \\ 1 \end{pmatrix} p(1-p)^{96-1} = 96p(1-p)^{95} \tag{2.39}$$

(b) Write an expression for the probability that two or less computers are down.

This is

$$Prob(2\ or\ less\ down) = Prob(0\ down) + Prob(1\ down) + Prob(2\ down) \tag{2.40}$$

$$= \begin{pmatrix} 96 \\ 0 \end{pmatrix} p^0(1-p)^{96} + \begin{pmatrix} 96 \\ 1 \end{pmatrix} p(1-p)^{95} + \begin{pmatrix} 96 \\ 2 \end{pmatrix} p^2(1-p)^{94} \tag{2.41}$$

Simplifying

$$= (1-p)^{96} + 96p(1-p)^{95} + 4560p^2(1-p)^{94} \tag{2.42}$$

(c) As the number of computers is increased, does the answer of **(b)** increase or decrease? Why?

The more computers, the more likely that there are failed computers. Thus the probability of part **(b)** decreases.

Clusters

Let there be three computers in each of four clusters (see nearby figure).

Let p be the independent probability that a computer is up (i.e. working) and and $1-p$ be the independent probability that a computer is down (i.e. not working). Computers and clusters are independent with respect to each other.

(a) A cluster is "functioning" if at least one computer in it is working. Find an expression for the probability that all four clusters are functioning.

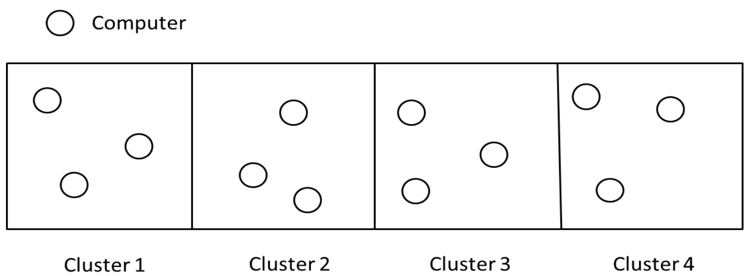

Figure 2.7: Four clusters consisting of three computers each

The probability that that at least one computer is up in a specific cluster is

$$Prob(at\, least\, 1\, computer\, up\, in\, a\, cluster) = r = 1 - (1-p)^3 \tag{2.43}$$

This is one minus the probability that all three computers in a specific cluster are down (or overall the probability that at least one computer in a specific cluster is up).

Now, across all clusters the probability that each cluster is functioning is:

$$Prob(every\, cluster\, is\, up) = (1-(1-p)^3)^4 = r^4 \tag{2.44}$$

The individual cluster probabilities multiply since the clusters are independent with respect to each other.

(b)) Find an expression for the expected (i.e. average) number of clusters that are functioning.

Using r from part **(a)** one has the binomial mean which can be written two ways.

$$\overline{number\, of\, up\, clusters} = 4r = \sum_{i=1}^{4} i \begin{pmatrix} 4 \\ i \end{pmatrix} r^i(1-r)^{4-i} \tag{2.45}$$

Here r is a probability between 0 and 1 so that the expected number of clusters

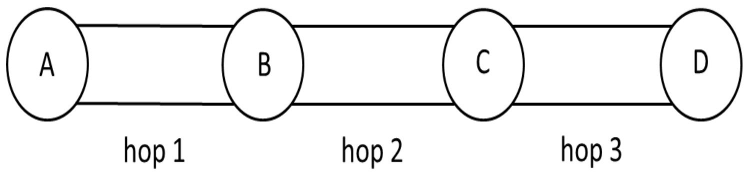

Figure 2.8: A linear network with two links between each pair of consecutive nodes

that are functioning, $4r$, is a number between 0 and 4. This makes intuitive sense.

2.1.4 Linear Networks

Linear networks have a simple topological structure that allows for interesting problems.

Idle Paths and Blocking

Consider the linear network in the nearby figure.

In this academic exercise each link can support at most one call (actual telecommunication links support many calls). A new call can only use a link if it is "idle" (not busy with an existing call). The independent probability that a link is busy with an existing call is p. The independent probability that a link is idle and can accommodate a new call is $1-p$.

(a) Write an expression for the probability that there is at least one idle path for a call from A to D (see the figure).

$$Prob(1\; or\; 2\; idle\; paths\; from\; A\; to\; D) = \tag{2.46}$$

$$= (Prob(1\; or\; 2\; idle\; links\; from\; A\; to\; B))^3 = q^3 = \tag{2.47}$$

Here q is the probability of there being at least one idle link from A to B.

$$q^3 = \left(\underbrace{\binom{2}{1}(1-p)p}_{1\ idle\ link\ A\ to\ B} + \underbrace{\binom{2}{2}(1-p)^2 p^0}_{2\ idle\ links\ A\ to\ B} \right)^3 = \left(2(1-p)p + (1-p)^2\right)^3 \qquad (2.48)$$

Alternately the probability that at least one of the two AB links is idle is the same as one minus the probability both AB links are busy. So

$$= (1 - Prob(2\ AB\ links\ busy))^3 = q^3 = \qquad (2.49)$$

$$= \left(1 - \binom{2}{2} p^2 (1-p)^0 \right)^3 = (1-p^2)^3 \qquad (2.50)$$

(b) Write an expression for the probability a path from A to D being blocked between nodes B and C.

This is simply the probability both links between B and C are busy or p^2.

(c) Let the linear network in the figure go on to the right forever. Write an expression for the probability that a new call originating at node A is blocked at hop i.

This is an application of the geometric distribution. Each hop is successfully transited with probability q. It is blocked eventually at hop i with probability $(1-q)$. Thus

$$Prob(call\ is\ blocked\ at\ i) = q^{i-1}(1-q) \qquad (2.51)$$

(d) What is the average number of hops transited by a new call for part **(c)**?

$$\overline{number\ of\ successful\ hops} = \sum_i (i-1) q^{i-1}(1-q) \qquad (2.52)$$

$$= \sum_i i q^{i-1}(1-q) - \sum_i q^{i-1}(1-q) \qquad (2.53)$$

$$= \frac{1}{1-q} - 1 = \frac{1-(1-q)}{1-q} = \frac{q}{1-q} \tag{2.54}$$

If the call is stopped at the ith hop it has successfully transited $i-1$ hops. This accounts for the (i-1) term in the summation of equation (2.52). The last equation is found by some algebraic manipulation to use an infinite summation formula one can look up in a mathematical handbook. See question 1 in section 2.3 for a similar derivation.

2.1.5 Now What?

We will look at more complex switching problems in following sections. But this section has given us a good foundation of probability problem solution techniques. In practicing such problems, say for example for an exam, the authors recommend solving each problem to completion and only then looking at the answers. Similar problems as in this section also appear in the end of the chapter exercises.