EEO 306
Faculty: Vibha Mane

Name:________________________

Student ID:________________________

Sample Final
Total Score: 36 points
All problems carry equal weight of 6 points each
Please show your work and justify your answers

1. In a game of Blackjack, the tens and face cards (10's, Jacks, Queens and Kings) count as 10 points, and Aces count as either 1 or 11 points. A blackjack occurs if the sum of the two cards is 21 (counting Ace as 11 points). A player is dealt two cards. What is the probability that the player has Blackjack?

The player has a Blackjack if the sum of his two cards is 21.

This would require

1 Ace

and 1 card among 10's, Jacks, Queens and Kings.

Number of ways to get 1 Ace = ${}^{4}C_1$.
Number of ways to get one 10 = ${}^{16}C_1$.
Number of ways of selecting 2 cards out of 52 = ${}^{52}C_2$.

Therefore, the required prob. = $\dfrac{{}^{4}C_1 \cdot {}^{16}C_1}{{}^{52}C_2}$.

c

2. Let X be a Poisson random variable with parameter λ.

   (a) What is the probability that X is even?

   (b) Simplify the above expression (X is even), by utilizing the following expansions for e^{λ} and $e^{-\lambda}$:

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} \cdots \quad \text{(i)}$$

and

$$e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} \cdots \quad \text{(ii)}$$

(a) PMF of Poisson R.V. is

$$p_X(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$P(X = \text{even}) = P(X \in \{0, 2, 4, 6, \ldots\})$$

$$= \frac{e^{-\lambda} \lambda^0}{0!} + \frac{e^{-\lambda} \lambda^2}{2!} + \frac{e^{-\lambda} \lambda^4}{4!} + \cdots$$

$$= e^{-\lambda} \left\{ 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots \right\} \quad \text{(iii)}$$

Adding the given expressions (i) and (ii) gives

$$e^{d} + e^{-d} = 1 + \cancel{d} + \frac{d^2}{2!} + \cancel{\frac{d^3}{3!}} + \cdots$$

$$+ 1 - \cancel{d} + \frac{d^2}{2!} - \cancel{\frac{d^3}{3!}} + \cdots$$

$$= 2\left\{1 + \frac{d^2}{2!} + \frac{d^4}{4!} + \cdots\right\}$$

Substituting in (iii), we obtain

$$P(X = \text{even}) = e^{-d} \cdot \frac{1}{2}\left\{e^{d} + e^{-d}\right\}$$

$$= \frac{1}{2}\left\{1 + e^{-2d}\right\}.$$

3. A plane is missing, and it is assumed that it is equally likely to have gone down in any of three possible regions. Let $1-\alpha_i$ denote the probability that the plane will be found upon a search of the ith region when the plane is, in fact, in that region, $i = 1, 2, 3$. (The constants α_i are called overlook probabilities because they represent the probabilities of overlooking the plane.) What is the conditional probability that the plane is in the ith region, given that a search of region 1 is unsuccesful?

Let R_i be the event that the plane is in region i, $i = 1, 2, 3$.

Let E be the event that a search of region 1 is unsuccessful

Let $P(R_i | E)$ be the prob. that the plane is in ith region, given that a search of region 1 is unsuccessful.

$i=1$ Using Bayes Theorem

$$P(R_1 | E) = \frac{P(E | R_1) \cdot P(R_1)}{P(E)}$$

From total prob. theorem

$$P(E) = P(E|R_1)\cdot P(R_1) + P(E|R_2)\cdot P(R_2) + P(E|R_3)\cdot P(R_3)$$

$$P(R_1) = P(R_2) = P(R_3) = 1/3$$

$P(E|R_1) = \alpha_1$ overlook prob. for R_1

$P(E|R_2) = 1$
$P(E|R_3) = 1$
} if the plane is in R_2 or R_3, the search in R_1 will be unsuccful.

Therefore,

$$P(E) = \alpha_1 \cdot \frac{1}{3} + 1\cdot\frac{1}{3} + 1\cdot\frac{1}{3}$$

$$= (\alpha_1 + 2)\frac{1}{3}.$$

$$P(R_1|E) = \frac{\alpha_1 \cdot \frac{1}{3}}{(\alpha_1+2)\cdot\frac{1}{3}} = \frac{\alpha_1}{\alpha_1+2}.$$

$i = 2$

$$P(R_2 | E) = \frac{P(E | R_2) \cdot P(R_2)}{P(E)}$$

$$= \frac{1 \cdot \frac{1}{3}}{(\alpha_1 + 2) \cdot \frac{1}{3}}$$

$$= \frac{1}{\alpha_1 + 2}$$

$i = 3$

$$P(R_3 | E) = \frac{1}{\alpha_2 + 2},$$

using similar argument as in $i = 2$.

4. Random variables X_1 and X_2 have the joint PMF $p_{X_1,X_2}(x_1, x_2)$ given by the following table:

$p_{X_1,X_2}(x_1, x_2)$	$x_2 = -1$	$x_2 = 0$	$x_2 = 1$
$x_1 = -1$	0	0	1/3
$x_1 = 0$	0	1/3	0
$x_1 = 1$	1/3	0	0

(a) Compute the marginal PMF $p_{X_1}(x_1)$.

(b) Compute the marginal PMF $p_{X_2}(x_2)$.

(c) Compute the probability $P(X_1 < X_2)$.

(d) Are X_1 and X_2 independent?

(a) Marginal $p_{X_1}(x_1)$ of X_1 is given by -

$$p_{X_1}(x_1) = \sum_{x_2} p_{X_1, X_2}(x_1, x_2).$$

This is the row sum for each value of x_1.

$$P(X_1 = -1) = P(X_1 = -1, X_2 = -1) + P(X_1 = -1, X_2 = 0) + P(X_1 = -1, X_2 = 1)$$

$$= 0 + 0 + 1/3 = 1/3.$$

Continuing in this manner, we obtain

$P(X_1 = 0) = 0 + 1/3 + 0 = 1/3.$

$P(X_1 = 1) = 1/3 + 0 + 0 = 1/3.$

(b) Marginal $p_{X_2}(x_2)$ of X_2 is given by.

$$p_{X_2}(x_2) = \sum_{x_1} p_{X_1,X_2}(x_1, x_2).$$

This is the column sum for each value of x_2.

$P(X_2 = -1) = 0 + 0 + 1/3 = 1/3.$

$P(X_2 = 0) = 0 + 1/3 + 0 = 1/3.$

$P(X_2 = 1) = 1/3 + 0 + 0 = 1/3.$

(c) $$\begin{aligned} P(X_1 < X_2) &= P(X_1 = -1, X_2 = 0) \\ &\quad + P(X_1 = -1, X_2 = 1) \\ &\quad + P(X_1 = 0, X_2 = 1) \\ &= 0 + 1/3 + 0 = 1/3 \end{aligned}$$

(d) X_1 and X_2 are said to be independ if the following relationship holds for all value of x_1 and x_2.

$$p_{X_1, X_2}(x_1, x_2) = p_{X_1}(x_1) \cdot p_{X_2}(x_2).$$

For $X_1 = -1$, $X_2 = -1$,

$p_{X_1, X_2}(-1, -1) = 0$, from the table.

$p_{X_1}(-1) = 1/3$, from (a)

$p_{X_2}(-1) = 1/3$, from (b)

$$p_{X_1, X_2}(-1, -1) \neq p_{X_1}(-1) \cdot p_{X_2}(-1)$$

Therefore, X_1 and X_2 are not independent.

5. (a) An Urn contains N white balls and M black balls. Draw a ball **with replacement** until a black ball is selected. What is the probability that **exactly** k draws are needed?

(b) What is the expected number of draws needed to observe a black ball?

(a) Let X denote the number of draws needed until we observe a black ball.

X is a geometric random variable, with prob. of success in one trial

$$p = \frac{M}{M+N}.$$

$$P(X=k) = (1-p)^{k-1} \cdot p,$$ using PMF of a geometric R.V.

$$= \left(1 - \frac{M}{M+N}\right)^{k-1} \cdot \left(\frac{M}{M+N}\right).$$

(b) Expected number of draws needed to observe a black ball -

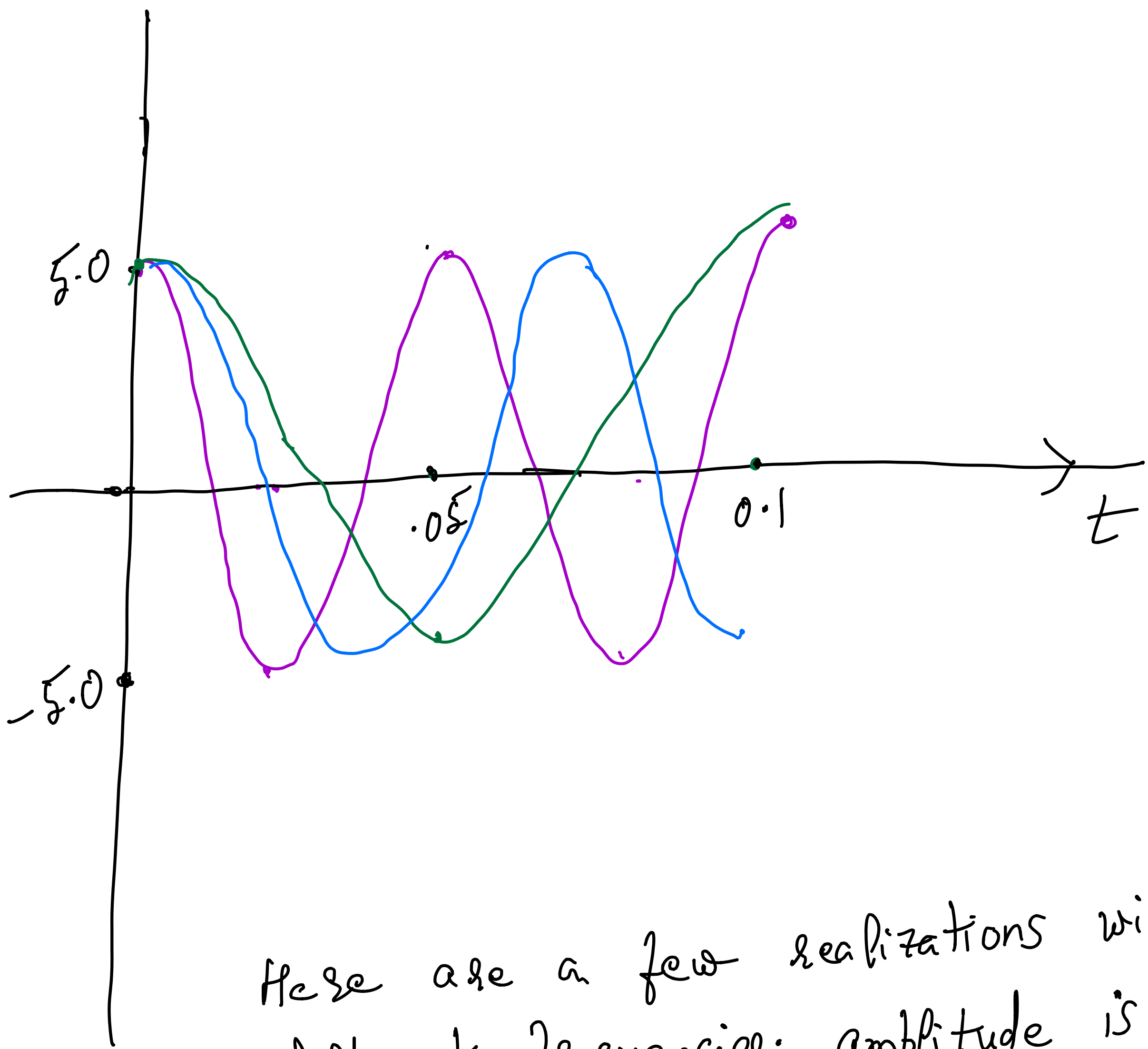
$E\{X\} = \frac{1}{p}$, using expected value of a geometric R.V.

$$= \frac{1}{M/(M+N)} = \frac{M+N}{M}.$$

6. Sketch the ensemble, that is, realizations of the random process

$$X(t) \;=\; Acos(2\pi \boldsymbol{f} t),$$

where $\boldsymbol{f}$ is a uniform random variable $\mathcal{U}(10, 20)$. That is, $\boldsymbol{f}$ is uniformly distributed in the range $[10, 20]$ Hz and $A = 5$ is a constant.



Here are a few realizations with different frequencies; amplitude is always 5.0.