Combinatorics

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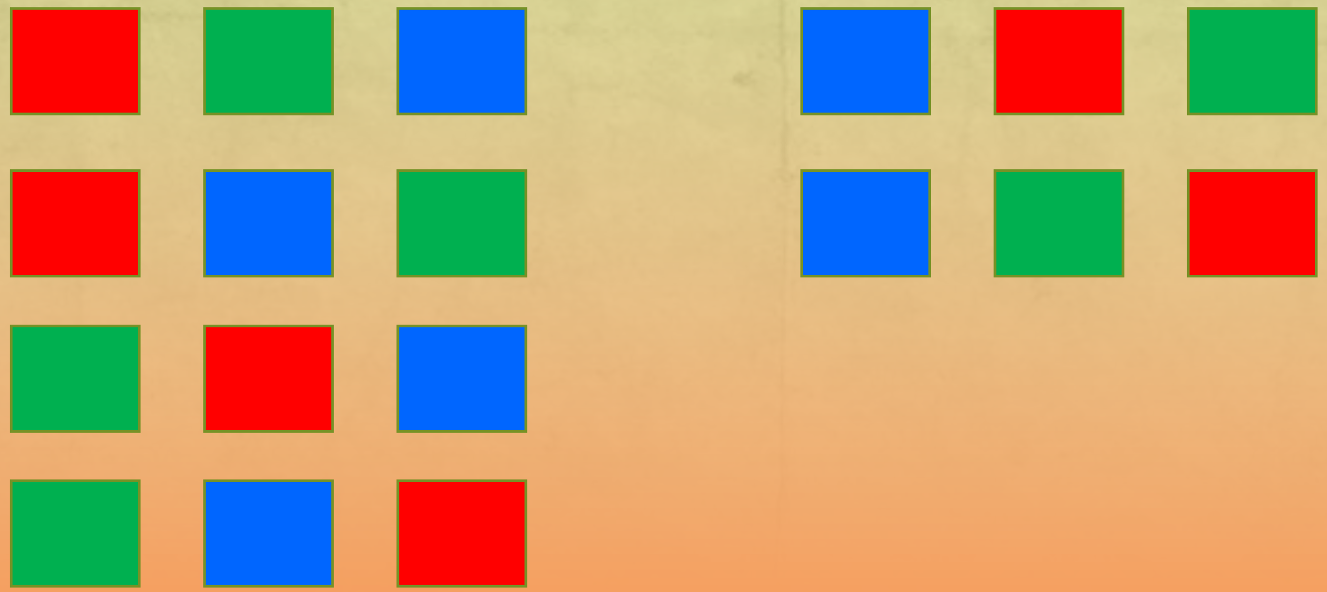
Topics

- Permutations
- Combinations
- Partitions
- Binomial Theorem
- Hypergeometric Probability
- Card Games

Permutations (1 of 4)

- There are $n!$ **permutations**, that is, **ordered arrangements, of n objects**.
- An example:
  - We have 3 objects: $Red\ (R), Green\ (G), Blue\ (B)$. There are a total of $3! = 6$ permutations, as shown here:



Permutations (2 of 4)

- Another Example:
  - For 4 objects, $Yellow(Y), Red\ (R), Green\ (G), Blue\ (B)$, there are a total of $4! = 24$ permutations, as listed here:

    $YRGB, YRBG, YGRB, YGBR, YBRG, YBGR$

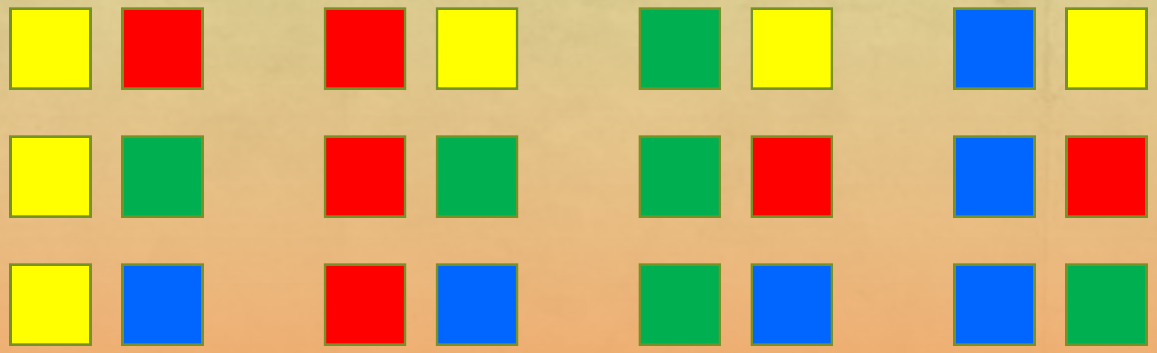
    $RYGB, RYBG, RGYB, RGBY, RBYG, RBGY$

    $GYRB, GYBR, GRYB, GRBY, GBYR, GBRY$

    BYR$G, BYGR, BRYG, BRGY, BGYR, BGRY$

Permutations (3 of 4)

- There are ${}^{n}_{r}P = \frac{n!}{(n-r)!}$ **permutations of r objects out of n objects**.
- An example:
  - Select 2 objects out of 4 objects: $Yellow (Y), Red\ (R), Green\ (G), Blue\ (B)$. There are ${}^{n}_{r}P = \frac{4!}{2!} = 4 \times 3 = 12$ ways of doing this, as shown here:



  - Note that **this is an ordered selection**.

Permutations (4 of 4)

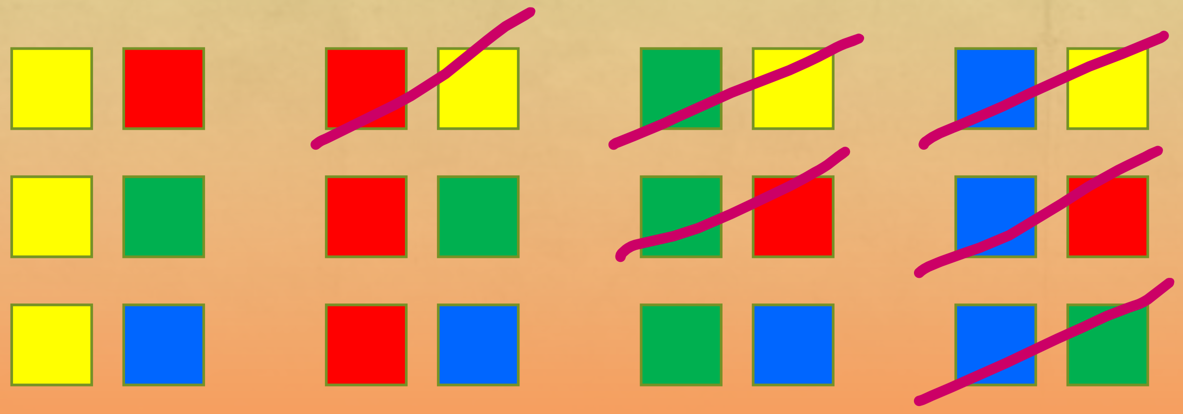
- Another example:
  - Select 3 letters (distinct letters) out of 26 letters. There are 26 ways of selecting the first letter, 25 ways of selecting the second letter out of the remaining letters, and 24 ways of selecting the third letter. Therefore, we have $\mathbf{26 \times 25 \times 24}$ ways of doing so.
  - Using the above expression, with $n = 26$ and $r = 3$ gives ${}^n_rP = \frac{26!}{23!} = 26 \times 25 \times 24$.

Combinations (1 of 3)

- There are ${}^{n}_{r}C = \frac{n!}{r!(n-r)!}$ **combinations** of r objects out of n objects.
- It is read as **"n choose r"**.
- This is the number of ways of **selecting r objects out of n objects**. Note that **this is an unordered selection**.

Combinations (2 of 3)

- An example:
  - Select 2 objects out of 4 objects: $Yellow(Y), Red\ (R), Green\ (G), Blue\ (B)$.
  - There are ${}^4_2C = \frac{4!}{2!2!} = 6$ ways off doing so.
  - Note that since **this is an unordered selection**, that is, YR and RY are considered duplicates.
  - We take 12 ways from the previous example and remove the duplicates. This gives a total of 6 ways.



Combinations (3 of 3)

- Another example:
  - A committee of 3 people is to be selected from a group of 20 people. How many different committees are possible?
  - There are ${}^{20}_{3}C = \frac{20!}{3!17!} = 1140$ ways off doing so.
- **Notation:**
  - We also write ${}^{n}_{r}C$ as $\binom{n}{r}$ or $C(n, r)$.
  - We also write ${}^{n}_{r}P$ as $P(n, r)$

Partitions

- A **partitioning** of n objects into, say, 3 distinct groups of sizes r_1, r_2, and r_3, such that $r_1 + r_2 + r_3 = n$, can be done in $\frac{n!}{r_1!r_2!r_3!}$ different ways.

- Generalizing the above expression, a partitioning of n objects into k groups of sizes $r_1, r_2, \cdots r_k$, such that $r_1 + r_1 + \cdots r_k = n$ can be done in $\frac{n!}{r_1!r_2!\cdots r_k!}$ different ways.

- An example:
  - 10 students are to be assigned to two teams, a lighting team consisting of 7 students and a sound team consisting of 3 students. In how many ways can this be done?
  - There are $\frac{10!}{7!3!} = 120$ ways of doing so. Note that the two teams are distinct here.

Example - Dealing Cards

Example

(a) In how many ways can you shuffle, that is, arrange a deck of 52 cards?

Solution:

There are $n!$ ordered arrangements of n objects. Therefore, there are $52!$ ways of doing so.

(b) In how many ways can you deal (or divide) 52 cards among 4 players?

Solution:

Here we are **partitioning** 52 cards into 4 groups with 13 cards in each group. Hence, there are $\frac{52!}{(13!)^4}$ ways of doing so.

Interpretation of Combinations

- One interpretation of the binomial coefficients $\binom{n}{r}$ is that it is the number of ways n objects can be **divided into two piles**, with r objects in the first pile and $n-r$ objects in the second pile.

- As an example, let us say that we have 5 balls labeled R, B, G, Y, M. We want to split these objects into two piles: $Pile$ 1 with 3 objects and $Pile$ 2 with 2 objects.

- There are $\binom{5}{3} = \frac{5!}{3!2!} = 10$ ways of doing so, as listed below:

Interpretation of Combinations, cont.

Pile 1	***Pile 2***
R, B, G	Y, M
R, B, Y	G, M
R, B, M	Y, G
R, Y, G	B, M
R, M, G	B, Y
Y, B, G	R, M
M, B, G	R, Y
R, Y, M	B, G
B, Y, M	R, G
G, Y, M	R, B

Combinations and the Binomial Coefficients

- Binomial coefficients obey the following **recursive relation**:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

- The above expression is related to Pascal's triangle (Boncelet, Chapter 3).

Binomial Theorem

- For any integer $n \geq 0$,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r\, y^{n-r}$$

- As an example, for $n = 4$,

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

- The theorem can be proved by induction.
- As a special case, for $\boldsymbol{x = y = 1}$, we have,

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

Example - Binomial Theorem

Example

Show that

$$3^n = \sum_{r=0}^{n} \binom{n}{r} 2^r$$

Solution:

In the binomial theorem, set $x = 2$ and $y = 1$; this will give the above expression.

Hypergeometric Probability

- Consider the task of **selecting, without replacement**, r objects out of n objects. The number of ways this can be done is given by

$$\binom{n}{r}.$$

- Next, consider the task of **selecting, without replacement**, r_1 objects from a group of n_1 objects, r_2 objects from a group of n_2 objects, and so forth, up to r_m objects from a group of n_m objects.
- The number of ways this can be done is given by the product of m binomials

$$\binom{n_1}{r_1}\binom{n_2}{r_2}\cdots\binom{n_m}{r_m}.$$

Hypergeometric Probability, contd.

- If all the selections are **equally likely**, then the probability of the above selection of $r_1, r_2, \cdots r_m$ objects is given by

$$\frac{\binom{n_1}{r_1}\binom{n_2}{r_2}\cdots\binom{n_m}{r_m}}{\binom{n}{r}}$$

- This is known as **hypergeometric probability**.

Example - Card Probabilities

Example

We draw 3 cards at random from a deck of 52 cards. What is the probability that the drawn cards are all red? Note that the cards are drawn here without replacement.

Solution:

Number of ways of drawing 3 cards out of 52 cards is $\binom{52}{3} = \frac{52!}{3!49!}$

Since there are 26 red cards in the deck, number of ways of selecting 3 red cards out of 26 red cards is $\binom{26}{3} = \frac{26!}{3!23!}$

Therefore, the required probability = $\frac{Number\ of\ favorable\ outcomes}{Number\ of\ possible\ outcomes} = \frac{\binom{26}{3}}{\binom{52}{3}}$.

Example - More Card Probabilities

Example

We select 13 cards at random from a deck of 52 cards. What is the probability that the drawn cards contain **exactly** 2 kings and 1 ace? Note that the cards are drawn without replacement.

Solution:

Number of ways of drawing 13 cards out of 52 cards is $\binom{52}{13}$.

Number of ways of selecting 2 kings is $\binom{4}{2}$.

Number of ways of selecting 1 ace is $\binom{4}{1}$.

Example - More Card Probabilities, cont.

Number of ways of selecting remaining 10 cards, such that there are no kings or aces here is given by $\binom{44}{10}$.

Therefore, the required probability = $\frac{Number\ of\ favorable\ outcomes}{Number\ of\ possible\ outcomes} = \frac{\binom{4}{2}\binom{4}{1}\binom{44}{10}}{\binom{52}{13}}$.

References

1. Charles Boncelet, Probability, Statistics and Random Signals, Oxford University Press, 2016.
2. Sheldon Ross, A First Course in Probability, Macmillan Publishing Company, 1988.
3. R. D. Yates, et al., Probability and Stochastic Processes, Wiley, 2005.