Probability Basics

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Topics

- Random Phenomena
- Outcome of an experiment
- Sample Space
- Event
- Sigma Algebra
- Probability Definition
- Properties of Probability
- Some Examples

Experiments with Random Outcomes

- In probability, we study **random** phenomena. Random means unpredictable, that is, it has no pattern or plan.
- Let us conduct the following experiments with random **outcomes**:
  - Roll a dice; possible outcomes are: $\mathbf{1, 2, 3, 4, 5, 6}$
  - Toss a coin; possible outcomes are: **Heads** $(\boldsymbol{H})$**, Tails** $(\boldsymbol{T})$
  - Draw 2 balls from a jar with 10 Red (R) and 10 Blue (B) balls; possible outcomes are: $\boldsymbol{RR, BB, RB, BR}$ Note that the outcome RB (first R, second B) is different from the outcome BR (first B, second R).
  - Conduct CT Scan Test for heart disease: possible outcomes are: **Positive** $(\boldsymbol{P})$**, Negative** $(\boldsymbol{N})$**, Inconclusive** $(\boldsymbol{I})$**.**

Sample Space

- A **sample space Ω** is the set of all possible outcomes of an experiment.
- Sample space for the above experiments:
  - Roll a dice; sample space $\boldsymbol{\Omega = \{1, 2, 3, 4, 5, 6\}}$
  - Toss a coin; sample space $\boldsymbol{\Omega = \{H, T\}}$
  - Draw 2 balls from a jar with 10 Red and 10 Blue balls; sample space $\boldsymbol{\Omega = \{RR, BB, RB, BR\}}$
  - Conduct CT Scan Test for heart disease; sample space $\boldsymbol{\Omega = \{P, N, I\}}$

Event

- An **event** is one or more of the possible outcomes. Thus, an event is a subset of a sample space.
- Some events for the above experiments:
  - Roll a dice; $\{2\}$ is an event; $\{2, 4, 6\}$ is another event, also known as even outcome.
  - Draw 2 balls from a jar with 10 Red and 10 Blue balls; both balls are of the same color is the event $\{RR, BB\}$.
  - CT Scan Test; $\{I\}$ is an event.
- A single outcome, that is, each element of the sample space is called **elementary outcome**.

What is Probability?

- Probability is a way of **assigning every event a value between zero and one**, with the requirement that the event made up of **all possible outcomes**, that is, {1,2,3,4,5,6} for the roll of a dice, **be assigned a value of one**.
- Further, to qualify as a probability, the assignment of the values must satisfy the requirement that for mutually exclusive events, such as {1,3,5} and {2,4,6}, the probability of either event occurring is equal to the sum of the probability of the two events.
- This is formalized in the following definition.

σ-Algebra (read as Sigma Algebra)

- In probability theory, **σ-algebra** on a set Ω is a collection of subsets of Ω, which is closed under complement, closed under countable unions and includes the empty subset.
- That is, a **σ-algebra or σ-field** $\mathcal{F}$ on Ω satisfies the following properties:
  - If $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$.
  - If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.
  - The empty set $\phi \in \mathcal{F}$.
- The above properties imply that
  - $\Omega \in \mathcal{F}$, since $\Omega = \phi^C$.
  - $A \cap B \in \mathcal{F}$ (from De Morgan's Theorem).

σ-Algebra, cont.

- The smallest σ-field associated with the set Ω is the collection $\mathcal{F} = \{\phi, \Omega\}$.
- If A is a subset of Ω, then the smallest σ-field containing A is $\mathcal{F} = \{\phi, A, A^C, \Omega\}$.
- The set of all subsets of Ω is a σ-field. This set is known as the **power set** of Ω.
- Example

  Roll a dice. Sample space $\Omega = \{1,2,3,4,5,6\}$. Let A be the event “even outcome”.

  Therefore, $A = \{2,4,6\} \in \mathcal{F}$,

  $A^C = \{1,3,5\} \in \mathcal{F}$,

  $A \cup A^C = \Omega \in \mathcal{F}$,

  $\Omega^C = \phi \in \mathcal{F}$,

  hence $\mathcal{F} = \{\phi, A, A^C, \Omega\}$

Measure

- In mathematical analysis, a **measure on a set** is a systematic way of assigning a number to each suitable subset of that set, **intuitively interpreted as its size**.
- Technically, a measure is a function that assigns non-negative real numbers to certain subsets of a set.
- It must further be **countably additive**, that is the measure of a “large” subset that can be decomposed into a finite number of “smaller” disjoint subsets, is the sum of the measures of the smaller subsets.

Probability Space

- A **probability space** is a triplet $(\Omega, \mathcal{F}, P)$:
  - A sample space Ω.
  - A σ-field $\mathcal{F}$.
  - The assignment of probabilities P to events in $\mathcal{F}$.
- A **probability measure** is a real-valued function defined on a probability space that satisfies measure properties such as countable additivity.

Probability (formal definition)

- A probability measure P on $(\Omega, \mathcal{F})$ is a function $P: \mathcal{F} \rightarrow [0,1]$ that satisfies the following axioms:
  - $P(A) \geq 0$
  - $P(\Omega) = 1$
  - If $A_1 A_2, \cdots$ is a collection of **disjoint events** in $\mathcal{F}$, then

    $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

    That is, probability is **countably additive**.

Additional Properties of Probability

From the above definition of probability and set theory, we have the following results:

- For any event A in $\mathcal{F}$, $0 \leq P(A) \leq 1$.
- $P(A^C) = 1 - P(A)$.
- $P(\phi) = 0$.
- If $B \subset A$, then $P(B) \leq P(A)$.
- For any two events A and B in $\mathcal{F}$ (**not necessarily disjoint**),

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;

The events A and B are not necessarily disjoint; therefore, we subtract the term $P(A \cap B)$.

Computing Probabilities

- When all outcomes are **equally likely**, we compute probabilities as follows:
- Probability of an event = $\frac{Number\ of\ favorable\ outcomes\ of\ the\ event}{Number\ of\ possible\ outcomes}$
- That is, probability of an event is the number of favorable outcomes divided by the number of possible outcomes.

Example - Coin Toss

Example

Toss a **fair** coin. What is the probability of the outcome Heads (h)?

Solution:

Sample space $\Omega = \{h, t\}$

Favorable outcome $= \{h\}$

Since it is a fair coin, the heads and tails have equal probability.

Therefore, $P(h) = \frac{1}{2}$.

Compound Experiments

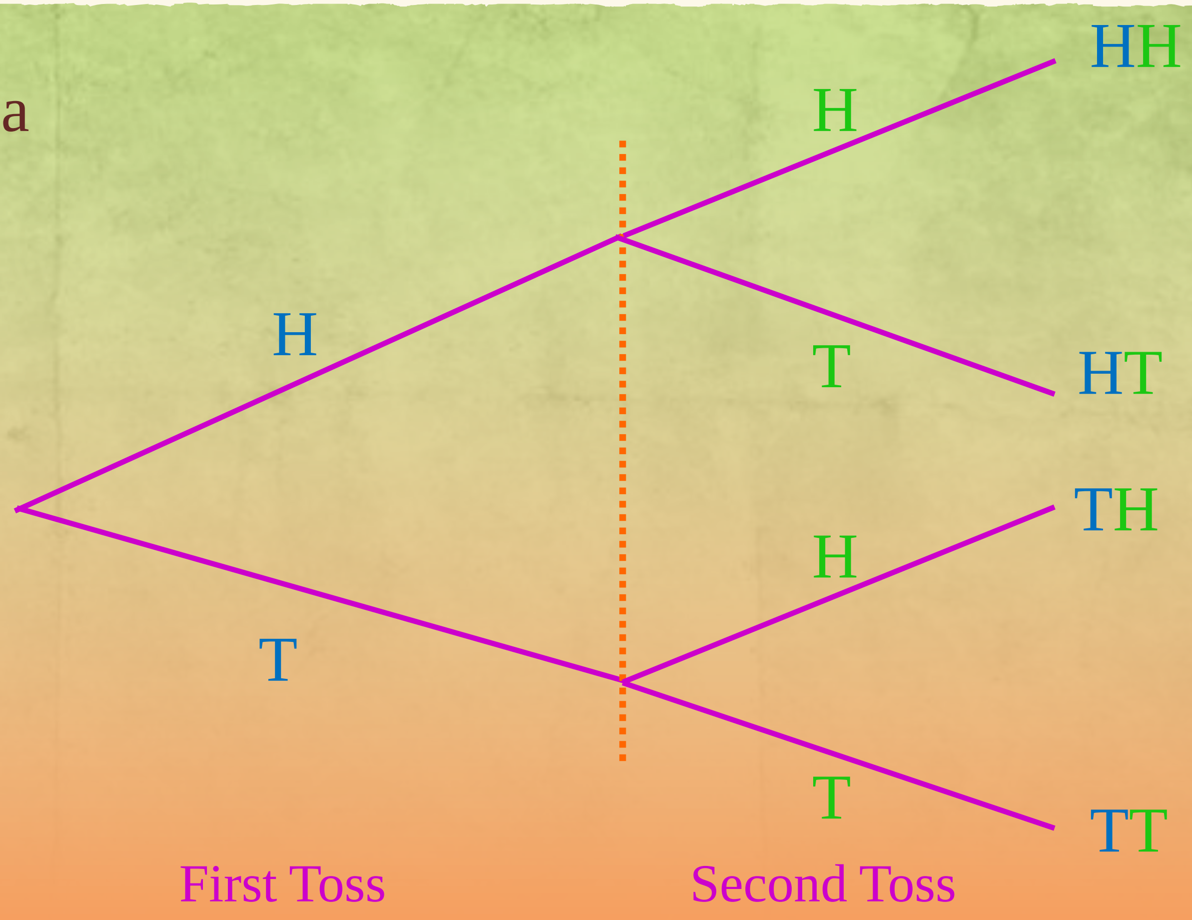
- A compound experiment is a combination of multiple random experiments.
- Suppose we conduct two experiments as follows:
  - The first experiment E_1 has sample space Ω_1.
  - The second experiment E_2 has sample space Ω_2.
- Then the compound experiment $E = E_1 \times E_2$ has the sample space $\Omega = \Omega_1 \times \Omega_2$.

Example - Two Coin Toss

Example: Toss 2 fair coins, a blue coin and a green coin. What is the probability of one H and one T?

Solution:

We utilize a **tree diagram** to list the outcomes -



Example - Two Coin Toss, cont.

Sample space = {H H, H T, T H, T T}, total = 4

Note that H T and T H are two different outcomes.

Favorable outcomes = {H T, T H}, total = 2

Therefore, probability of one H and one T = $\frac{2}{4} = \frac{1}{2}$

Example - Two Rolls of a Dice

Example

A fair dice is rolled twice. What is the probability that the sum is 7?

Solution: The sample space of each role of the dice is $\Omega_1 = \Omega_2 = \{1,2,3,4,5,6\}$. Therefore, the sample space for two rolls is $\Omega = \Omega_1 \times \Omega_2$, with $\mathbf{6 \times 6 = 36}$ **possible outcomes**:

$(1,1), (2,1) \cdots \cdots (6,1)$

$(1,2), (2,2) \cdots \cdots (6,2)$

$(1,6), (2,6) \cdots \cdots (6,6)$

Example - Two Rolls of a Dice, cont.

Of these, the following 6 outcomes give a sum of 7:

(1,6), (2,5), (3,4), (4,3), (5,2), (6,1).

Therefore, the required probability

$$P(sum = 7) = \frac{Favorable\ Outcomes}{Total\ Outcomes} = \frac{6}{36} = \frac{1}{6}$$

Example - Two Coin Toss, Unequal Probabilities

Example

Toss two coins. Let $P(h) = p$ for each coins. Therefore, $P(t) = 1 - p$. Note that p can take any value between 0 and 1. What is the probability of landing two heads?

Solution:

The sample space of each toss of the coin is $\Omega_1 = \Omega_2 = \{h, t\}$.

Therefore, the sample space for two tosses is $\Omega = \Omega_1 \times \Omega_2$. It has $2 \times 2 = 4$ outcomes given by: $\{hh, ht, th, tt\}$.

Required probability = $\mathrm{P}(hh) = P(h)P(h) = p^2$.

As an example, if $p = 2/3$, then $\mathrm{P}(hh) = (^2/_3)^2 = {}^4/_9$.

We have factored $\mathrm{P}(hh) = P(h)P(h)$. This is because each coin toss is independent of the other. The concept of independence will be covered in a later lecture.

Example - Balls and Urns

Example

Two balls are drawn randomly from an urn containing 6 red (R) and 5 blue (B) balls. What is the probability that one of the drawn balls is red and the other is blue?

The problem of "balls and urns" is paradigmatic; that is, many probability problems can be recast as balls and urns problems. Hence, there are many examples.

Example - Balls and Urns, cont.

Solution:

$$P(1st\ R, 2nd\ B) = \frac{6}{11} \times \frac{5}{10}$$

6 R out of 11, followed by 5 B out of the remaining 10.

$$P(1st\ B, 2nd\ R) = \frac{5}{11} \times \frac{6}{10}$$

$$P(1\ R, 1\ B) = \frac{6}{11} \times \frac{5}{10} + \frac{5}{11} \times \frac{6}{10} = \frac{6}{11}$$

Here the required probability of $(1\ R, 1\ B)$ is the sum of the first two probabilities.

References

1. Charles Boncelet, Probability, Statistics and Random Signals, Oxford University Press, 2016.
2. Sheldon Ross, A First Course in Probability, Macmillan Publishing Company, 1988.
3. R. D. Yates, et al., Probability and Stochastic Processes, John Wiley, 2005.