Discrete-Time LTI Systems

given a system

$$\left.\begin{array}{l} x[n_0] \\ u[n], \quad n \geq n_0 \end{array}\right\} \longrightarrow y[n], \quad n \geq n_0$$

This DT system is linear if

for

$$\left.\begin{array}{l} x_i[n_0] \\ u_i[n], \quad n \geq n_0 \end{array}\right\} \longrightarrow y_i[n], \quad n \geq n_0$$

① $$\left.\begin{array}{l} x_1[n_0] + x_2[n_0] \\ u_1[n] + u_2[n], \quad n \geq n_0 \end{array}\right\} \longrightarrow y_1[n] + y_2[n], \quad n \geq n_0$$

Additivity

② $$\left.\begin{array}{l} \alpha x_1[n_0] \\ \alpha u_1[n], \quad n \geq n_0 \end{array}\right\} \longrightarrow \alpha y_1[n], \quad n \geq n_0$$

Homogeneity

Similar to CT case

total response = zero-state response + zero-input response

DT system is time invariant if its characteristics do not change w/ time.

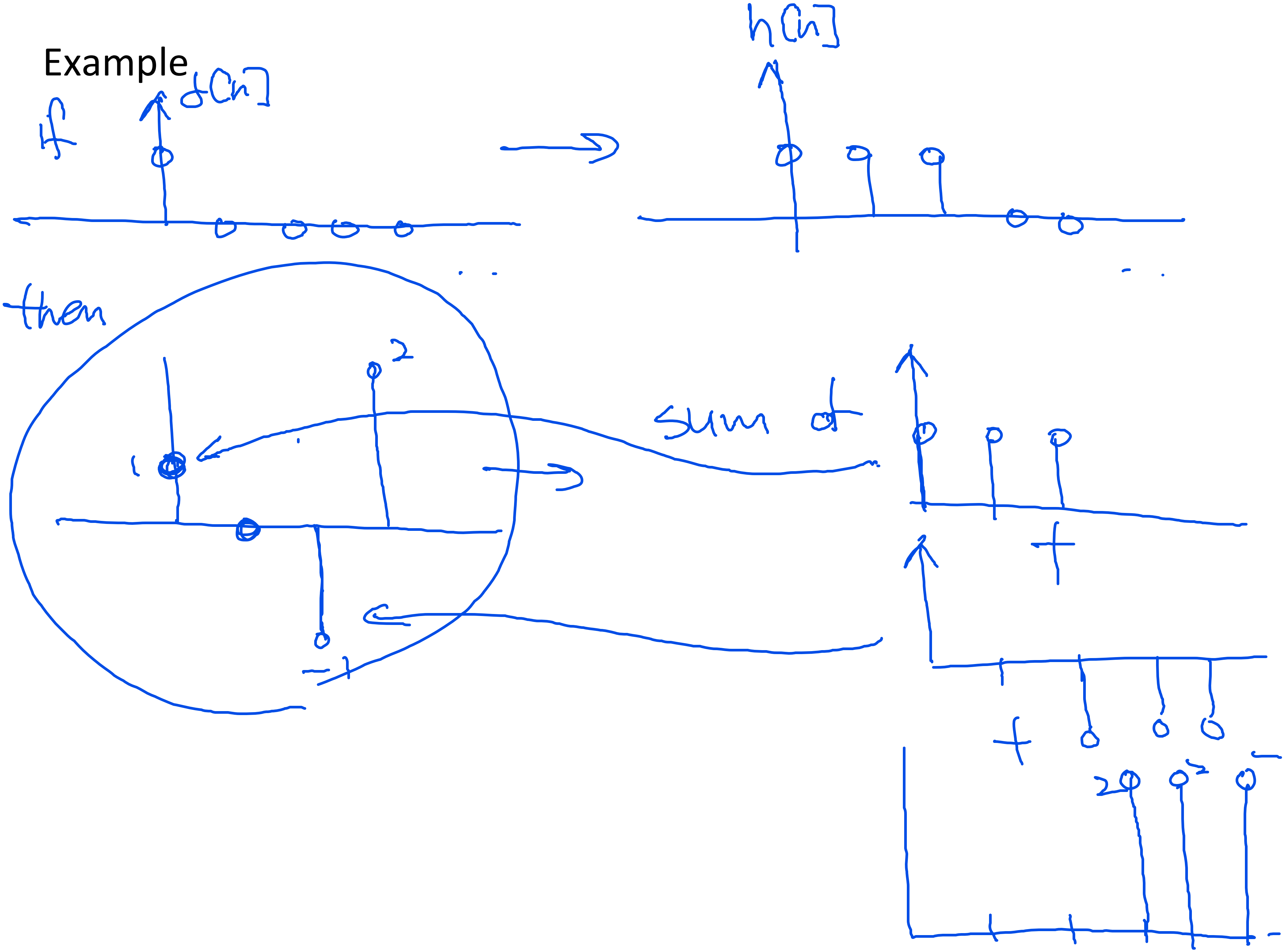
Mathematically,

for $\left.\begin{array}{l} x_i[n_0] = x_0 \\ u_i[n], \quad n \geq n_0 \end{array}\right\} \longrightarrow y_i[n], \quad n \geq n_0$

$\Rightarrow \left.\begin{array}{l} x_i[n_0 + n_1] = x_0 \\ u_i[n - n_1], \quad n \geq n_0 + n_1 \end{array}\right\} \rightarrow y_i[n - n_1] \quad n \geq n_0 + n_1$

time-shifting

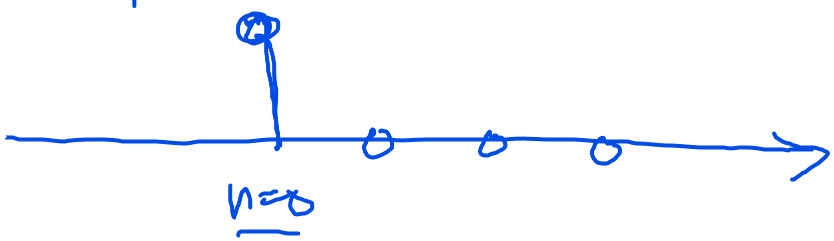
Example



Example (Savings Account)

assume: fixed interest rate 0.01% per d

deposit $ the first day. (n=0)



$$y[0] = 1$$

$$y[1] = y[0] + y[0] \times (0.0001) = (1+0.0001) = (1.0001)^1$$

$$y[2] = y[1] + y[1] \times (0.0001) = y[1](1 + 0.0001)$$

$$= (1.0001)(1.0001) = (1.0001)^2$$

$$h[n] = (1.0001)^n$$

Example (Unit Delay System) $y[n] = u[n-1]$

Consider the system

$y[0] = 0$ C

system input $r[n]$

system output $y[n]$

what is $h[n]$?

$\delta[n] \rightarrow r[n]$

$y[n] = u[n-1]$

$e[n] = r[n] + y[n] = \delta[n] + y[n]$

$u[n] = a\,e[n]$

n	0	1	2	3	4
$r[n]$	1	0	0	0	0
$e[n]$	1	a			
$u[n]$	a	a^2			
$y[n]$	0	a	a^2	a^3	a^4

$h[n] = a^n$, $n = 1, 2, 3 \cdots$

DT LTI system is IIR

(Infinite Impulse Response)

if its impulse response has infinitely

many non-zero entries ($h[n] = \alpha^n$, $n = 1, \ldots$)

otherwise FIR (finite Impulse Response)

Discrete-Time Convolution

Consider a discrete-time LTI

$$u[n] = \sum_{k=0}^{\infty} u[k]\,\delta[n-k]$$ (train of impulse scaled by $u[k]$)

recap: if system is LTI

① $\delta[n] \rightarrow h[n]$ (definition)

② $\delta[n-k] \rightarrow h[n-k]$ time-shifting

③ $u[k]\,\delta[n-k] \rightarrow u[k]\,h[n-k]$ homogeneity

④ $\sum_{k=0}^{\infty} u[k]\,\delta[n-k] \rightarrow \sum_{k=0}^{\infty} u[k]\,h[n-k]$

Additivity.

Thus, $y[n] = \sum_{k=0}^{\infty} u[k]\, h[n-k]$

$\Rightarrow$ the current value of $y[n]$ depends on the input $u[k]$ $\underline{0 \le k \le n}$

$\Rightarrow$ Discrete-time convolution

$$y[n] = \sum_{k=0}^{\infty} u[n-k]\, h[k]$$

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \quad \equiv \quad h[n] \longrightarrow \boxed{u[n]} \longrightarrow y[n]$$

For FIR System of Length N

$\Rightarrow h[n] = 0$ for $n \geq N$

$\Rightarrow h[n-k] = 0$ for $n-k \geq N$ or

$k \leq n-N$

thus $y[n] = \sum_{k=n-N+1}^{n} h[n-k]\, u[k]$

$= \sum_{k=0}^{N-1} h[k]\, u[n-k]$ ⇐ only evaluate

N values

From Previous Example (Savings Account)

$h[n] = a^n$, $n = 1, 2, 3 \ldots$

let's deposit

$u[0] = \$100$ $n=0$

$u[1] = -\$50$ $n=1$

$\rightarrow u[2] = \$200$

$u[10] = \$50$

Q: what is the balance when $n = 10$, 100

$$y[n] = \sum_{k=0}^{n} h[n-k]\,u[k] = \sum_{k=0}^{n} (1.0001)^{n-k}\,u[k]$$

$$y[10] = \sum_{k=0}^{10} (1.0001)^{10-k}\,u[k] = (1.0001)^{10-0}u[0] +$$

$$(1.0001)^{10-1}u[1] + (1.0001)^{10-2}u[2] + (1.0001)^{10-10}u[10]$$

A System with Difference Equation

order = 2

Consider

$$y[n+2] - 0.1\,y[n+1] - 0.06\,y[n] = u[n+1] + 2u[n]$$

Assume the system is initially relaxed.

$\Rightarrow$ $y[n] = u[n] = 0$ for $n < 0$

then impulse response.

$$y[n+2] = 0.1\,y[n+1] + 0.06\,y[n] + \delta[n+1] + 2\delta[n]$$

start

$n = -2$

$$y[0] = 0.1\,y[-1] + 0.06\,y[-2] + \delta[-1] + 2\delta[-2]$$

$= 0$

$$y[1] = 0.1\,y[0] + 0.06\,y[-1] + \delta[0] + 2\delta[-1]$$

$= 1$

order = 3

Example consider

$$y[n] - y[n-1] = 0.2\,(u[n] - u[n-3])$$

$$y[n+1] = u[n-1]$$

order = 2

Q: Is this FIR or IIR ?

$$h[n] = h[n-1] + 0.2\,(\delta[n] - \delta[n-3])$$

$$h[0] = 0.2$$

$$h[1] = 0.2$$

$$h[2] = 0.2$$

$$h[3] = 0.2 + 0.2\,(0 - 1) = 0$$

$\vdots$

0 $\Rightarrow$ FIR

order = 1

Example $y[n+1] - 0.5\,y[n] = 0.2\,u[n]$

Q: Is this FIR or IIR

$h[n+1] = 0.5\,h[n] + 0.2\,\delta[n]$

$n=-1$ $\quad h[0] = 0$

$n=0$ $\quad h[1] = 0.2$

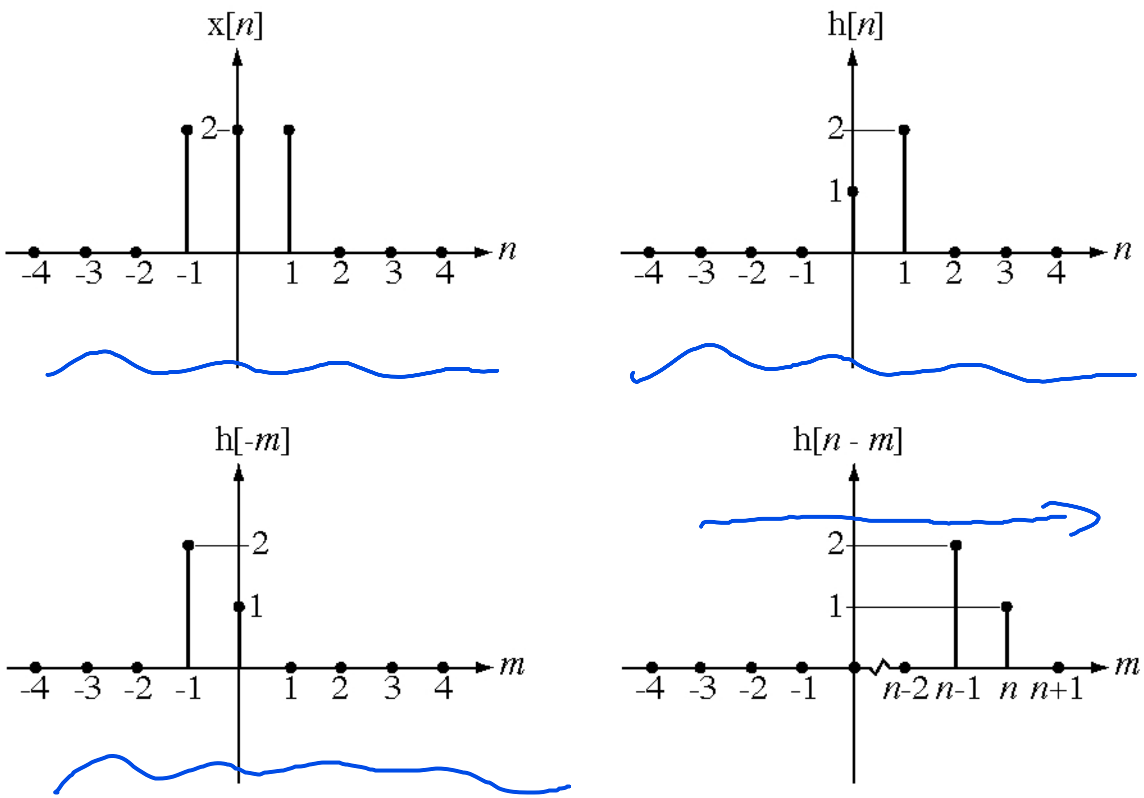
$h[2] = (0.5)(0.2)$

$h[3] = (0.5)(0.2)(0.5)$

$\vdots$

$h[n] = (0.2)(0.5)^{n-1}$ $\Rightarrow$ IIR

Graphical DT Convolution



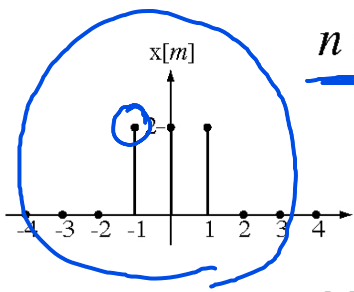
flip

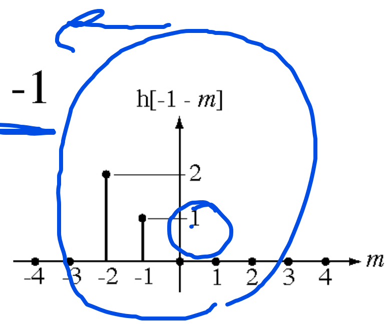
slide

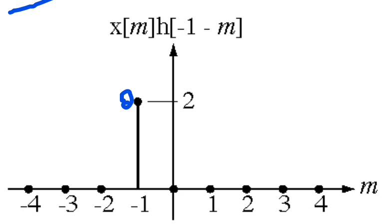
multiply

sum

$n = -1$

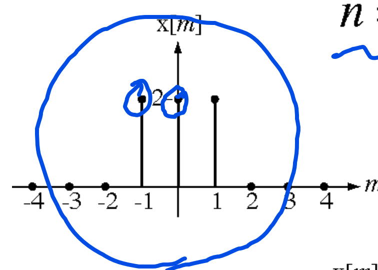

x[m]
2
m
-4 -3 -2 -1 1 2 3 4

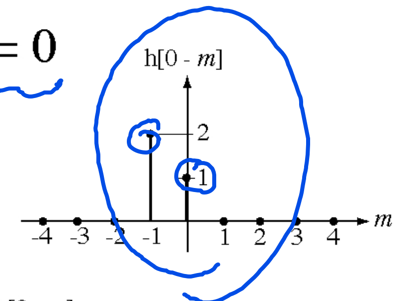

h[-1 - m]
2
1
m
-4 -3 -2 -1 1 2 3 4

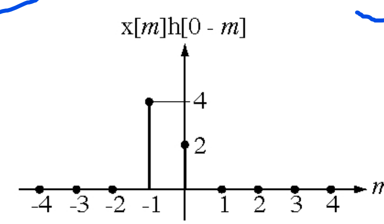

x[m]h[-1 - m]
2
m
-4 -3 -2 -1 1 2 3 4

$y[-1] = 2$

$n = 0$

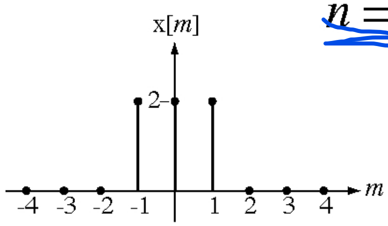

x[m]
2
m
-4 -3 -2 -1 1 2 3 4

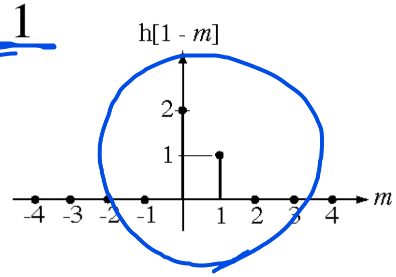

h[0 - m]
2
1
m
-4 -3 -2 -1 1 2 3 4

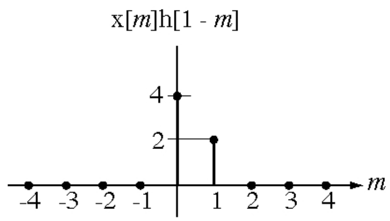

x[m]h[0 - m]
4
2
m
-4 -3 -2 -1 1 2 3 4

$y[0] = 6$

$n = 1$

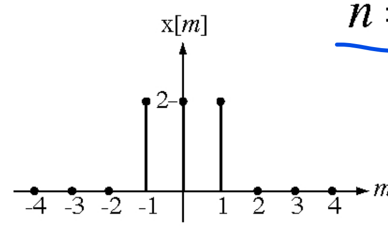

x[m]
2
-4 -3 -2 -1 1 2 3 4
m

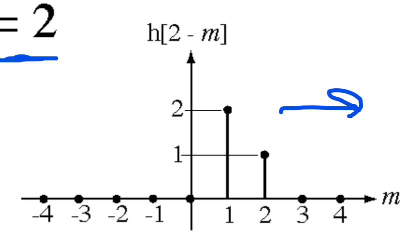

h[1 - m]
2
1
-4 -3 -2 -1 1 2 3 4
m

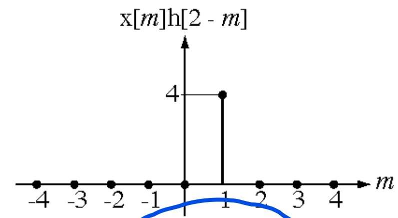

x[m]h[1 - m]
4
2
-4 -3 -2 -1 1 2 3 4
m

$y[1] = 6$

$n = 2$

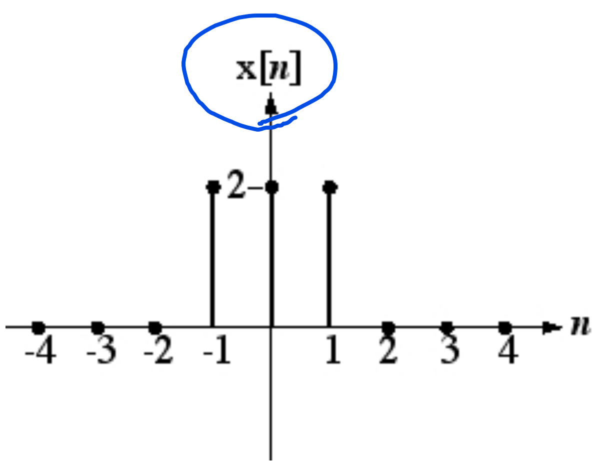

x[m]
2
-4 -3 -2 -1 1 2 3 4
m

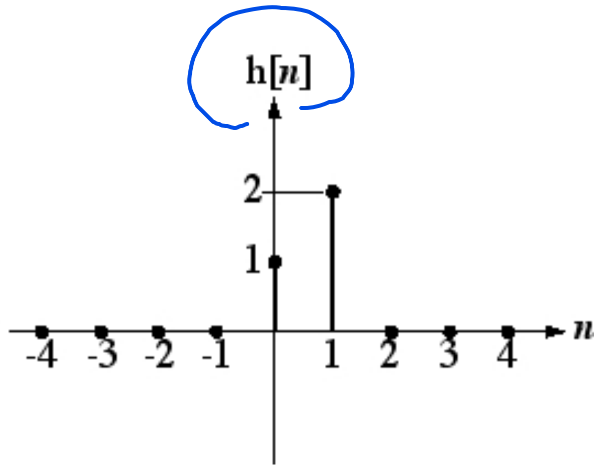

h[2 - m]
2
1
-4 -3 -2 -1 1 2 3 4
m

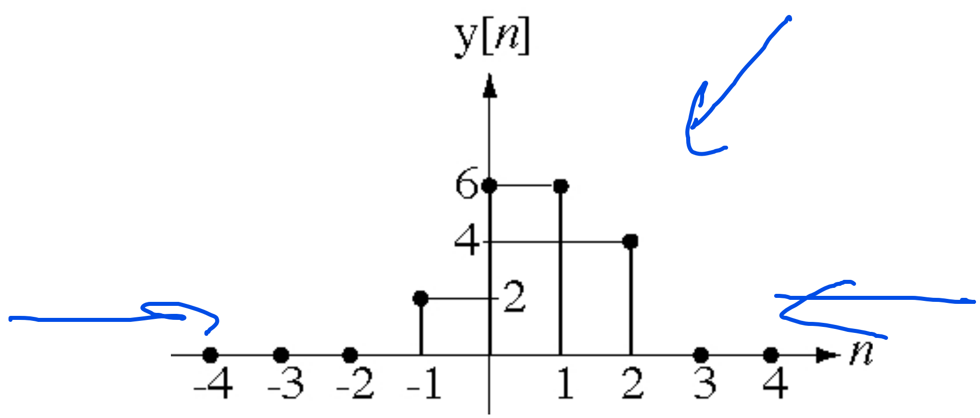

x[m]h[2 - m]
4
-4 -3 -2 -1 1 2 3 4
m

$y[2] = 4$

Answer







Convolution Sum Properties

* $y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$

* $x[n] * A\delta[n-n_0] = Ax[n-n_0]$

Let $y[n] = x[n] * h[n]$ then

* $y[n-n_0] = x[n] * h[n-n_0] = x[n-n_0] * h[n]$

* $y[n] - y[n-1] = x[n] * (h[n] - h[n-1])$

$= [x[n] - x[n-1]] * h[n]$

$$x[n] * y[n] = y[n] * x[n]$$

$$(x[n] * y[n]) * z[n]$$
$$= x[n] * (y[n] * z[n])$$

$$(x[n] + y[n]) * z[n] =$$
$$x[n] * z[n] + y[n] * z[n]$$