# ADDENDUM TO LESSON 5

## I. INSTANTANEOUS POYNTING VECTOR $\underline{\mathcal{P}}(\underline{r},t)$

$$\underbrace{\underline{\mathcal{P}}(\underline{r},t)}_{\text{Instantaneous Poynting vector}} = \underbrace{\underline{E}(\underline{r},t)}_{V/m} \times \underbrace{\underline{H}(\underline{r},t)}_{A/m}$$

Note: $\frac{V}{m} \times \frac{A}{m} = \frac{W}{m^2}$ power density in $W/m^2$

$$= \underline{i}_E E_0 \cos(\omega t - \underline{k}\cdot\underline{r}) \times \underline{i}_H H_0 \cos(\omega t - \underline{k}\cdot\underline{r})$$

$$= \underbrace{\underline{i}_E \times \underline{i}_H}_{=\underline{i}_k} E_0 H_0 \cos^2(\omega t - \underline{k}\cdot\underline{r}) \quad W/m^2$$

$$= \underline{i}_k E_0 H_0 \cos^2(\omega t - \underline{k}\cdot\underline{r})$$

or

$$\underline{i}_k \frac{E_0^2}{\eta} \cos^2(\omega t - \underline{k}\cdot\underline{r})$$

or

$$\underline{i}_k \eta H_0^2 \cos^2(\omega t - \underline{k}\cdot\underline{r}) \qquad \text{since } \frac{E_0}{H_0} = \eta$$

In the above, the vector $\underline{\mathcal{P}}(\underline{r},t) = \underline{E}(\underline{r},t) \times \underline{H}(\underline{r},t)$ is shown to represent power flow associated with a UPEMW using a simple dimensional analysis.

## II. TIME-AVERAGE POYNTING VECTOR $\underline{P}_{av}(\underline{r})$

By definition, the time-average Poynting vector $\underline{P}_{av}(\underline{r})$ is the average of the instantaneous Poynting vector $\underline{\mathcal{P}}(\underline{r},t)$ over one period $T = \frac{1}{f} = \frac{2\pi}{\omega}$ of the UPEMW, i.e.,

$$\underline{P}_{av}(\underline{r}) = \frac{1}{T}\int_0^T dt\, \underline{\mathcal{P}}_{av}(\underline{r},t)$$

$$= \frac{1}{T}\int_0^T dt\, \underline{i}_k \frac{E_0^2}{\eta} \cos^2(\omega t - \underline{k}\cdot\underline{r})$$

$$= \frac{1}{T}\int_0^T dt\, \underline{i}_k \frac{E_0^2}{\eta} \cos^2(\omega t - \phi) \qquad \text{where } \phi = \underline{k}\cdot\underline{r}$$

$$= \underline{i}_k \frac{E_0^2}{\eta} \frac{1}{T} \int_0^T dt \cos^2(\omega t - \phi)$$

Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ in order to evaluate the integral. Thus, rewrite the expression for $\underline{P}_{av}(\underline{r})$ as

$$\underline{P}_{av}(\underline{r}) = \underline{i}_k \frac{E_0^2}{\eta} \cdot \frac{1}{T}\left\{\int_0^T dt \cdot \frac{1}{2}\cos[2(\omega t - \phi)] + \int_0^T dt\right\}$$

$$\delta = 2\phi$$

$$= \underline{i}_k \frac{E_0^2}{\eta} \cdot \frac{1}{T}\left\{\frac{1}{2}\underbrace{\int_0^T dt \cos(2\omega t - \delta)} + \underbrace{\int_0^T dt}_{T}\right\}$$

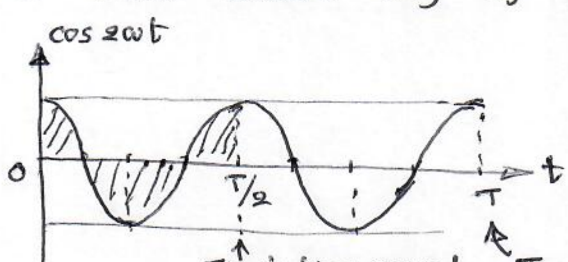
This integral is 0 because the integrand is a double frequency term (the frequency of the integrand is $2\omega$ which is double of the frequency $\omega$ of the UPEMW)

$$\therefore \underline{P}_{av}(\underline{r}) = \begin{cases} \underline{i}_k \frac{1}{2}\frac{E_0^2}{\eta} \\ \text{or} \\ \underline{i}_k \frac{1}{2} E_0 H_0 \\ \text{or} \\ \underline{i}_k \frac{1}{2}\eta H_0^2 \end{cases}$$

← This is the formula to use when calculating the time-average Poynting vector associated with a UPEM wave

## Proving that the integral $\int_0^T dt \cos(2\omega t)$ is equal to 0

Graphical interpretation of integral as area under the curve is the most direct way of proving this result

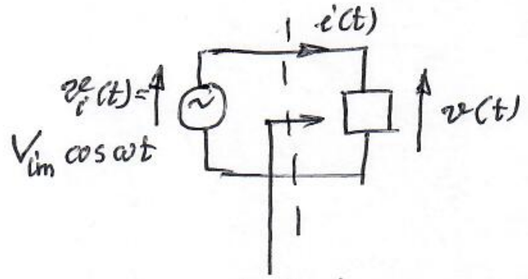



T/2 is time-period of double-freq signal

T is time-period of signal with freq $\omega$

By inspection, the area under the curve for $t$ from 0 to $T/2$ (or for $t$ from 0 to $T$) is zero.

## ANALOGY TO TIME-AVERAGE POWER DELIVERED TO A CIRCUIT ELEMENT UNDER STEADY-STATE SINUSOIDAL CONDITIONS




instantaneous power
$p(t) = v(t)\, i(t)$
delivered to the circuit element

Instantaneous steady-state sinusoidal power delivered to a circuit element is

$$p(t) = v(t)\, i(t)$$

The corresponding time-average powered $P_{av}$ delivered to the circuit element is by definition

$$P_{av} = \frac{1}{T}\int_0^T p(t)\, dt$$

Assuming $v(t) = V_m \cos \omega t$ and $i(t) = I_m \cos(\omega t - \delta)$, show using the integration process used in deriving the expression for the time-average Poynting vector that

$$P_{av} = \begin{cases} \frac{1}{2} V_m I_m \cos \delta \\ \quad \text{or} \\ V_{m,RMS}\, I_{m,RMS} \cos \delta \end{cases}$$

where $\text{RMS value} = \dfrac{\text{Amplitude value}}{\sqrt{2}}$.

The factor $\cos \delta$ in the expression for $P_{av}$ is often referred to as power factor