Lesson 2: UPW Propagating in an Arbitrary Direction in Space

1 Key Learning Objectives

- Understand how to write and interpret the expression for a UPW propagating in an arbitrary direction.
- Extend the understanding from Lesson 1, which covered waves along Cartesian coordinate axes, to waves in any direction.

2 UPW Wave Function for Propagation in an Arbitrary Direction

For simplicity, consider a uniform plane wave (UPW) propagating in an arbitrary direction in the xz-plane. The wave function is given by:

$$\Phi(x, z, t) = A \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \tag{1}$$

where:

- $\mathbf{r} = x\hat{i}_x + z\hat{i}_z$ is the position vector.
- A is the amplitude.
- ω is the radian frequency.
- $\mathbf{k}$ is the wave vector, given by:

$$\mathbf{k} = k_x\hat{i}_x + k_z\hat{i}_z \tag{2}$$

  where:

$$k_x = k \cos\theta_x \tag{3}$$
$$k_z = k \cos\theta_z \tag{4}$$

- The wavenumber is given by:

$$k = \sqrt{k_x^2 + k_z^2} \tag{5}$$

2.1 Physical Interpretation of the Dot Product $\mathbf{k} \cdot \mathbf{r}$

By definition, the dot product of two vectors is:

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_z z \tag{6}$$

which represents the projection of the position vector onto the direction of wave propagation. If we introduce a new coordinate axis z' along the wave vector direction, we can rewrite the wave function as:

$$\Phi(z', t) = A \cos(\omega t - kz') \tag{7}$$

This confirms that the wave propagates along the direction of $\mathbf{k}$.

3 Example: Dot Product of Two Vectors

Problem: What is $\mathbf{r} \cdot \hat{i}_x$ equivalent to?

Solution: Since the magnitude of $\hat{i}_x$ is 1 and the projection of $\mathbf{r} = x\hat{i}_x + y\hat{i}_y + z\hat{i}_z$ onto the x-axis is x, we get:

$$\mathbf{r} \cdot \hat{i}_x = x \tag{8}$$

4 Finding the Direction of Propagation and Phase Velocity

Given the instantaneous expression:

$$\Phi(x, y, t) = 5\cos(6\pi \times 10^9 t - 10^4 x - 10^4 y) \tag{9}$$

we determine the direction of propagation as follows:

- Comparing with the general form $\Phi(x, y, t) = A\cos(\omega t - k_x x - k_y y)$, we get:

$$k_x = 10^4 \tag{10}$$

$$k_y = 10^4 \tag{11}$$

- The propagation direction angle is:

$$\theta = \tan^{-1}\left(\frac{k_y}{k_x}\right) = 45^\circ \tag{12}$$

- The total wavenumber is:

$$k = \sqrt{k_x^2 + k_y^2} = 2 \times 10^4 \text{ rad/m} \tag{13}$$

- Given $\omega = 6\pi \times 10^9$ rad/s, the phase velocity is:

$$v_p = \frac{\omega}{k} = 3 \times 10^8 \text{ m/s} \tag{14}$$

  indicating that the wave is an electromagnetic wave propagating in vacuum.

5 Conclusion

- The UPW wave equation can be generalized for arbitrary propagation directions using the wave vector $\mathbf{k}$.
- The dot product $\mathbf{k} \cdot \mathbf{r}$ helps determine the phase variation along the propagation direction.
- The phase velocity and direction of propagation can be computed using vector components of the wavenumber.