LESSON 4

PHASOR WAVEFUNCTIONS

I. WHAT YOU WILL LEARN IN THIS LESSON:

In the previous lessons, we treated wavefunctions of both the scalar and vector varieties as instantaneous functions of space and time. In the present lesson, you will learn about phasors which provide a powerful tool based on the use of complex numbers in handling steady-state sinusoidal signals. The fact that phasors enormously simplify the calculational processes such as addition, subtraction, multiplication and division of steady-state sinusoidal quantities is illustrated.

II. ILLUSTRATING THE SIMPLIFYING POWER OF PHASORS IN STEADY-STATE SINUSOIDAL ANALYSIS:

Consider the simple low-frequency series LR circuit shown in Fig. 1 which will be used to illustrate the power of phasors in finding the instantaneous expression for the steady-state sinusoidal current i(t) in the circuit.

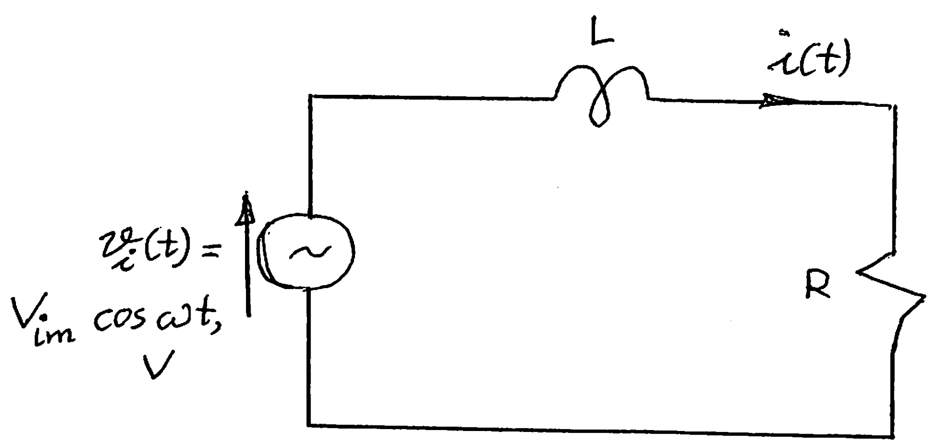

Fig. 1. Sinusoidal voltage source driving a series LR circuit

First let us clarify how electromagnetic waves tie in to this low-frequency circuit. Assuming that the frequency of operation is f = 100 Hz, the UPEMW frequency-wavelength relationship which we derived earlier gives

$$f.\lambda = c \rightarrow \quad \lambda = (3.10^8\,\text{m/s})/(100\ 1/\text{s}) = 3.10^6\ \text{m}$$

Clearly the wavelength value of λ of three million meters is very large indeed compared to the dimension in practice of the series LR circuit. The implication of this value of λ is that to all intents and purposes the current and voltage in the circuit do not change with position on the circuit. In other words, the current, for instance, in a low-frequency circuit may be treated as constant around the circuit.

In order to find an instantaneous expression for the steady-state sinusoidal current i(t), we must apply the Kirchhoff voltage law (KVL) around the circuit which yields the ordinary differential equation of the first order

$$L\, di(t)/dt + Ri(t) = V_{im} \cos \omega t \tag{1}$$

where the right hand side represents the driving term and the left hand side represents the driven entity i(t). Finding the instantaneous expression for the steady-state sinusoidal current i(t) of this equation may be achieved by solving this differential equation (1) using one of the following two methods:

(i) a trigonometric solution; and

(ii) the method of phasors.

III. TRIGONOMETRIC SOLUTION:

The general solution to Eq. (1) consists of a superposition of a transient solution and a forced (or steady-state) solution. Here we are interested only in a steady-state solution which must have the same time dependency as the forcing function. Hence the steady-state solution we are looking for must be of the form

$$i(t) = I_m \cos(\omega t - \Phi) \tag{2}$$

where the amplitude I_m and the phase angle Φ are two unknowns but the frequency ω of the forced solution is the same as that of the forcing function. Plugging Eq. (2) into Eq. (1) yields

$$-\omega L I_m \sin(\omega t - \Phi) + R I_m \cos(\omega t - \Phi) = V_{im} \cos \omega t \tag{3}$$

which may be divided by $[R^2 + \omega^2 L^2]^{1/2}$ and then written as

$$I_m [\cos\psi \cos(\omega t - \Phi) - \sin\psi \sin (\omega t - \Phi)] = [V_{im}/[R^2 + \omega^2 L^2]^{1/2}] \cos \omega t \tag{4}$$

where

$$\psi = \cos^{-1}[R/(R^2 + \omega^2 L^{2)1/2}] \tag{5}$$

Now invoking the trigonometric identity cos(A+B) = cosA cosB -sinA sinB, one may rewrite Eq. (3) as

$$I_m \cos[\psi + (\omega t - \Phi)] = [V_{im}/[R^2 + \omega^2 L^2]^{1/2}] \cos \omega t \tag{6}$$

By equating the amplitudes on both sides of Eq. (6) and the arguments of the cosine function on both sides of the same equation, one obtains the desired expressions for the unknowns, i.e.,

$$I_m = V_{im}/(R^2 + \omega^2 L^2)^{1/2} \tag{7a}$$

and

$$\Phi = \psi = \tan^{-1}(\omega L/R) \tag{7b}$$

Finally from Eq. (2) one gets the desired expression

$$i(t) = \{V_{im}/(R^2 + \omega^2 L^2)^{1/2}\} \cos[\omega t - \tan^{-1}(\omega L/R) \tag{8}$$

IV. *PHASOR SOLUTION*:

The equation to be solved is the differential equation in Eq. (1) where i(t) is a real steady-state sinusoidal function which is unknown and is to be found. Consider the following new equation which is a variant of Eq. (1):

$$L\, di_x(t)/dt + Ri_x(t) = V_{im} \sin \omega t \tag{9}$$

In Eq. (9), $i_x(t)$ is another real function which represents instantaneous sinusoidal steady-state current current in the same series LR circuit but now the driving source is $V_{im} \sin \omega t$. If one adds to Eq. (1) j times Eq. (9), one gets

$$L\, di(t)/dt + Ri(t) + j\,\{ L\, di_x(t)/dt + Ri_x(t)\} = V_{im} \cos \omega t + j\,(V_{im} \sin \omega t)$$

or equivalently combining terms and using the Euler's theorem one gets

$$L\, di_1(t)/dt + Ri_1(t) = V_{im}\, e^{j\omega t} \tag{10}$$

It should be evident that, if we find the steady-state solution $i_1(t) = i(t) + ji_x(t)$ to Eq. (10), then finding the originally sought for function i(t) is simply given by

$$i(t) = \text{Re}\,\{i_1(t)\} \tag{11}$$

Finding the forced steady-state solution $i_1(t)$ to Eq. (10) simply requires $i_1(t)$ to be a complex function of the form

$$i_1(t) = \overline{I}\, e^{jwt}, \tag{12}$$

i.e., with the same time variation e^{jwt} as the forcing function on the right-hand side of Eq. (10).

Plugging Eq. (12) into Eq. (10) algebraizes the differential equation to

$$j\omega L\, \overline{I}\, e^{jwt} + R\, \overline{I}\, e^{jwt} = V_{im}\, e^{j\omega t}$$

or to the algebraic equation

$$(R + j\omega L)\, \overline{I} = V_{im} \tag{13}$$

Eq. (13) yields

$$\bar{I} = V_{im}/(R+j\omega L) \tag{14}$$

which may be written in the polar form

$$\bar{I} = V_{im}/[(R^2 + \omega^2 L^2)^{1/2} e^{j\mathrm{atan}(\omega L/R)}] = \frac{V_{im}}{(R^2+\omega^2L^2)^{1/2} \angle \tan^{-1}(\frac{\omega L}{R})} \tag{15}$$

or

$$\bar{I} = [V_{im} e^{-j\mathrm{atan}(\omega L/R)}] / (R^2 + \omega^2 L^2)^{1/2} \tag{15}$$

or

$$\bar{I} = \frac{V_{im} \angle \tan^{-1}(\frac{\omega L}{R})}{(R^2+\omega^2L^2)^{1/2}} \tag{16}$$

Finally, from Eqs (11) and (12), one gets the desired instantaneous current expression

$$i(t) = \{V_{im}/(R^2 + \omega^2 L^2)^{1/2}\} \cos[\omega t - \tan^{-1}(\omega L/R)$$

which, as expected, is the same as the expression in Eq. (8) obtained by the trigonometric method.

V. *DEFINING STATEMENT FOR A PHASOR QUANTITY*:

Combining Eqs. (11) and (12) yields

$$i(t) = \mathrm{Re}\,\{\bar{I}\, e^{j\omega t}\} \tag{17}$$

Eq. (17) provides a defining statement of a phasor which here is $\bar{I}$. In general, a phasor is that quantity, complex in general, which when multiplied by $e^{j\omega t}$ produces another complex number whose real part is the corresponding instantaneous steady-state sinusoidal quantity.

VI. TIME DOMAIN AND FREQUENCY DOMAIN:

A steady-state time-varying sinusoidal signal may be specified either as an instantaneous quantity or as a phasor quantity. The instantaneous representation implies specification of the signal in time domain while the phasor representation implies specification of the signal in frequency domain. In converting an instantaneous expression of a steady-state sinusoidal signal to a phasor expression one is simply moving the signal from time domain to frequency domain. Some operations are more efficiently or easily performed in the frequency domain, hence it behooves to convert the instantaneous signal to its phasor format. In short, it is quite common to start in time domain, then move to frequency domain where the necessary operation is performed and then translate back to the time domain.

VII. SAMPLE PROBLEMS:

[1] Convert the phasor wavefunction

$$\bar{\Phi}(z) = 2\angle 37^\circ\, e^{-jkz} - j\angle{-50^\circ}\, e^{jkz}$$

into its corresponding instantaneous expression $\phi(z,t)$.

SOLUTION:

$$\phi(z,t) = \text{Re}\{\bar{\Phi}(z)\, e^{j\omega t}\} \quad \text{from definition of phasor}$$

$$= \text{Re}\{2e^{j37^\circ} \cdot e^{-jkz} \cdot e^{j\omega t} - j e^{-j50^\circ} \cdot e^{jkz} \cdot e^{j\omega t}\}$$

$$= \text{Re}\{2\, e^{j(\omega t - kz + 37^\circ)} - j e^{j(\omega t + kz - 50^\circ)}\}$$

$$= 2\cos(\omega t - kz + 37^\circ) + \sin(\omega t + kz - 50^\circ) \quad \text{Answer}$$

[2] Convert the instantaneous wave function

$$\underline{E}(z,t) = 3\sin(\omega t - kz + 60^\circ)\,\underline{i}_x - 2\cos(\omega t + kz + 20^\circ)\,\underline{i}_x$$

into its corresponding phasor wavefunction.

SOLUTION:

The required answer is found indirectly from an inspection of the given instantaneous quantity $\underline{E}(z,t)$ as

$$\underline{\bar{E}}(z) = -j3\, e^{-j(kz - 60^\circ)}\,\underline{i}_x - 2\, e^{j(kz + 20^\circ)}\,\underline{i}_x$$

Check if the above expression is correct by converting this phasor $\underline{\bar{E}}(z)$ into its corresponding instantaneous expression by invoking the definition of phasors:

$$\underline{E}(z,t) = \text{Re}\{\underline{\bar{E}}(z)\, e^{j\omega t}\}$$

$$= \text{Re}\{-j3\, e^{j(\omega t - kz + 60^\circ)}\,\underline{i}_x - 2\, e^{j(\omega t + kz + 20^\circ)}\,\underline{i}_x\}$$

$$= 3\sin(\omega t - kz + 60^\circ)\,\underline{i}_x - 2\cos(\omega t + kz + 20^\circ)\,\underline{i}_x$$

This checks with the original expression for $\underline{E}(z,t)$