Parabolic Induction

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September 28, 2024

We generalize the construction of representations in part 2 and part 4 of Bump, and recognize them both as cases of parabolic induction.

There are analogies between the Archimedean representation theory and the non-Archimedean representation theory in the study of automorphic representations. The Archimedean theory studies representations of $G_\infty =: \mathrm{GL}(2,\mathbb{R})$ or $G_\infty^+ =: \mathrm{GL}(2,\mathbb{R})^+$ and the non-Archimedean theory studies representations of $G_p =: \mathrm{GL}(2,\mathbb{Q}_p)$ for a prime p. Let k be the local field over which G is defined.

Maximal compact subgroup. There exist *maximal compact subgroups* $K_\infty := \mathrm{O}(2) \leq G_\infty$, $K_\infty^+ := \mathrm{SO}(2) \leq G_\infty^+$ and $K_p := \mathrm{GL}(2,\mathbb{Z}_p) \leq G_p$. The maximal compact subgroup of G will be denoted by K.

Group decompositions. Let $B \leq G$ denote the Borel subgroup of upper triangular matrices. The *Iwasawa decomposition* is expressed by the equation

$$G = BK.$$

Let $N \leq G$ denote the subgroup of upper triangular unipotent matrices and $T \leq G$ the subgroup of diagonal matrices. Note that

$$B = TN$$

and this product is semidirect, i.e., T is in the normalizer of N and $T \cap N = 1$. T is called the *Levi subgroup* and is also an example of a *split tours*. N is called the *unipotent radical*.

Let $W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. W is called the Weyl group (though it is not a subgroup of G, it is in the quotient of G by scalars). The *Bruhat decomposition* is expressed by the equation

$$G = \cup_{w \in W} BwB$$

and this union is disjoint.

Topological groups and Haar measure. G is always a Hausdorff locally compact topological group. GL_∞ and GL_∞^+ are *Lie groups*. GL_p is *totally disconnected*, i.e., every neighborhood of the identity contains an open compact subgroup.

There exist left and right-invariant measures on G, called the left and right *Haar measures*, and denoted by $\mathrm{d}_L g$ and $\mathrm{d}_R g$ respectively. They are unique up to scalar multiple. For a Hausdorff locally compact group H, the *modular quasicharacter* δ_H is defined by $\mathrm{d}_R h = \delta_H(h)\mathrm{d}_L h$. We say that H is *unimodular* if $\delta_H = 1$, i.e., the left and right Haar measures coincide.

In any case, G and K are unimodular. We choose $\mathrm{d}g$ so that $\int_K \mathrm{d}g = 1$. The modular quasicharacter of B is $\delta_B \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = |a/b|$.

Smooth functions. Let H be a subgroup of G. If G is Archimedean, we say that a complex-valued function on H is *smooth* if it is smooth as a map between manifolds. If G is non-Archimedean, we say that a complex-valued function on H is *smooth* if it is locally constant. Let $C^\infty(H)$ denote the set of smooth complex-valued maps on H. Let $C_c^\infty(H)$ denote the set of smooth complex-valued maps on H with compact support.

Representations. Let (π, V) be a representation of G. If G is Archimedean, we will study π when it is a (not necessary unitary) Hilbert space representation or when it is a $(\mathfrak{g}, K)$*-module*, where $\mathfrak{g}$ is the Lie algebra of G (c.f. Bump 2.4). Equivalence of $(\mathfrak{g}, K)$-modules is called infinitesimal equivalence. If G is non-Archimedean, we will study π when it is *smooth*. That is, for each $v \in V$, the stabilizer of v in G is open.

Direct sum decomposition and admissible representations. Let (π, V) be representation of G which is a Hilbert space representation if G is Archimedean and smooth if G is non-Archimedean. Let $\hat{K}_\infty$ (resp. $\hat{K}_\infty^+$) be the set of equivalence classes of finite-dimensional unitarizable irreducible representations of K_∞ (resp. K_∞^+). Let $\hat{K}_p$ be the set of equivalence classes of finite-dimensional irreducible representations of K_p whose kernel is open (hence of finite index). Denote this object generally by $\hat{K}$. If $\rho \in \hat{K}$, let $V(\rho)$ be the sum of all K-invariant subspaces of V that are isomorphic as K-modules to ρ. Then,

$$V = \oplus_{\rho \in \hat{K}} V(\rho)$$

If G is Archimedean, this is a Hilbert space direct sum of K-modules. If G is non-Archimedean, this is an algebraic direct sum of K-modules.

We say that π is *admissible* if for each $\rho \in \hat{K}$, $V(\rho)$ is finite dimensional. If G is Archimedean, an irreducible unitary representation is always admissible. If G is non-Archimedean, then π is admissible if and only if for any open subgroup $U \leq G$, the space V^U of vectors stabilized by U is finite dimensional.

Multiplicative characters. A *quasicharacter* of a group H is a homomorphism χ from H to the multiplicative group $\mathbb{C}^\times$ of complex numbers. We say that χ is a *character* (or *unitary character*) if $|\chi| = 1$.

Tate's thesis (section 2.3) provides a useful description of the quasicharacters on $k^\times$. Let u denote the kernel of the absolute value map $|\cdot| : k \to \mathbb{R}_{\geq 0}$ (so if $F = \mathbb{R}$, $U = \{\pm 1\}$ and if $F = \mathbb{Q}_p$, $U = \{x \in \mathbb{Z}_p : p \nmid x\}$). We call a quasicharacter *unramified* if it is trivial on u. The unramified quasicharacters are the maps of the form $c(\alpha) = |\alpha|^s$, where s is any complex number, s is determined by c if k is Archimedean, and s is determined mod $2\pi i$. The quasicharacters are maps of the form $c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s$, where $\tilde{c}$ is any character of u, $\tilde{c}$ is uniquely determined by c, and s is determined as in the case of unramified quasicharacters. In particular, the quasicharacters on $\mathbb{R}$ are maps of the form $c(\alpha) = \mathrm{sgn}(\alpha)^\epsilon |\alpha|^s$, where ϵ is an equivalence class of $\mathbb{Z}$ modulo 2.

If χ_1 and χ_2 are quasicharacters of k, then

$$\chi \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \chi_1(a)\chi_2(c)$$

defines a quasicharacter of B.

Induced Representation for a quasicharacter of B. Let χ be a quasicharacter of B. Let V^∞ be the set of $f \in C^\infty(G)$ such that

$$f(bg) = \delta_G(b)^{-1/2}\delta_B(b)^{1/2}\chi(b)f(g)$$

for all $b \in B$ and $g \in G$. By the Iwasawa decomposition, functions $f \in V^\infty$ are uniquely determined by their restriction to K. G acts on V^∞ by right translation, i.e.,

$$(\pi(g)f)(x) = f(xg)$$

If G is Archimedean, we define a Hermitian innner product on V^∞ by

$$\langle f_1, f_2 \rangle = \int_K f_1(k)\overline{f_2(k_\theta)}\, \mathrm{d}k$$

Let V be the Hilbert space completion of V^∞ in $C^\infty(G)$. The action of G on V^∞ can be extended to V, and V^∞ is the space of smooth vectors for V (c.f. Bump 2.4).

If G is non-Archimedean, let $V = V^\infty$. Let $V_c = V \cap C_c^\infty(G)$ be the subspace of V which consists of compactly supported smooth functions. V_c is a subrepresentation of V.

V is called the *induced representation* of χ in G, also denoted $\mathrm{Ind}_B^G \chi$. If G is non-Archimedean, we call V_c the *compact induced representation*.

Classification of $(\mathfrak{g}, K)$-modules for $\mathbf{GL}(2, \mathbb{R})^+$. For the following, $G = \mathrm{GL}_\infty^+$. Note that

$$\kappa_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix},\ \theta \in [0, 2\pi)$$

parameterizes elements of K. Since $K \cong \mathbb{R}/\mathbb{Z}$, the irreducible unitary representations of K are one dimensional and are parameterized by integers. They are the characters $\sigma_k(\kappa_\theta) = e^{ik\theta}$ with $k \in \mathbb{Z}$. We will denote the isotypic component $V(\sigma_k)$ as simply $V(k)$. Thus,

$$V = \oplus_{k \in \mathbb{Z}} V(k)$$

Each $V(k)$ is one dimensional. Let Σ be the set of all integers such that $V(k) \neq 0$. We call Σ the *set of K-types of V*. If V is an irreducible admissible $(\mathfrak{g}, K)$-module, Σ is either all even or all odd. We say the *parity* of V is even or odd accordingly.

Let s_1 and s_2 be complex numbers and ϵ a equivalence class of $\mathbb{Z}$ modulo 2. Let χ_1 and χ_2 be the quasicharacters of $\mathbb{R}^\times$ given by $\chi_1(\alpha) = \mathrm{sgn}(\alpha)^\epsilon |\alpha|^{s_1}$ and $\chi_2(\alpha) = |\alpha|^{s_2}$. Let χ be the quasicharacter on B given by χ_1 and χ_2. Let V be the induced representation of χ in G. V^∞ consists of $f \in C^\infty(G)$ such that

$$f\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} g \right) = a^{s_1 + 1/2} b^{s_2 - 1/2} f(g),\ g \in G,\ a, c > 0$$

and

$$f\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} g \right) = (-1)^\epsilon f(g),\ g \in G.$$

The restriction of f to K can be any smooth function, subject to the condition

$$f(\kappa_{\theta+\pi}) = (-1)^\epsilon f(\kappa_\theta)$$

for all θ.

Now let,

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \ L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \ H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be elements of the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$, which is the complexification of $\mathfrak{g}$. Define Δ by

$$-4\Delta = H^2 + 2RL + 2LR.$$

If $\ell \equiv \epsilon \bmod 2$, there exists a unique element f_ℓ of V whose restriction satisfies

$$f_\ell(\kappa_\theta) = e^{i\ell\theta}.$$

We have

$$Hf_\ell = \ell f_\ell, Rf_\ell = (s + 1/2)f_{\ell+2}, \ Lf_\ell = (s - 1/2)f_{\ell-2}$$

and

$$\Delta f_\ell = \lambda f_\ell, \ Zf_\ell = \mu f_\ell$$

where

$$\lambda = s(1 - s), \ \mu = s_1 + s_2, \ s = \frac{1}{2}(s_1 - s_2 + 1)$$

Let V_{fin} be space of K-finite vectors of V (c.f. Bump 2.4). Then Δ and Z act as scalars on V_{fin}, with eigenvalues λ and μ, respectively. The set of K-types of V_{fin} consists of all integers congruent to ϵ modulo 2.

(i) Suppose that s is not of the form $\frac{k}{2}$, where k is an integer congruent to ϵ mod 2. Then V_{fin} is irreducible.

(ii) Suppose that $s \geq \frac{1}{2}$ and that $s = \frac{k}{2}$, where $k \geq 1$ is an integer congruent to ϵ module 2. Then V_{fin} has two irreducible invariant subspace V_{fin}^+ and V_{fin}^-. The set of K-types of V_{fin}^+ is

$$\Sigma^+(k) = \{\ell \in \mathbb{Z} | \ell \equiv k \bmod 2, \ell \geq k\}$$

and the set of K-types of V_{fin}^- is

$$\Sigma^-(k) = \{\ell \in \mathbb{Z} | \ell \equiv k \bmod 2, \ell \leq -k\}$$

The quotient $V_{\mathrm{fin}}^0 = V_{\mathrm{fin}}/(V_{\mathrm{fin}}^+ \oplus V_{\mathrm{fin}}^-)$ is irreducible, unless $k = 1$, in which case it is zero. Its set of K-types is

$$\Sigma^0(k) = \{\ell \in \mathbb{Z} | \ell \equiv k \bmod 2, -k < \ell < k\}$$

(iii) Suppose that $s \leq \frac{1}{2}$ and that $s = 1 - \frac{k}{2}$, where $k \geq 1$ is an integer congruent to ϵ mod 2. Then V_{fin} has an invariant subspace V_{fin}^0 whose set of K-types is $\Sigma^0(k)$. Here V_{fin}^0 is irreducible, unless $k = 1$, in which case it is zero. The quotient $V_{\text{fin}}/V_{\text{fin}}^0$ decomposes into two irreducible invariant subspaces V_{fin}^+ and V_{fin}^-. The set of K-types of $V_{\text{fin}}^\pm$ is $\Sigma^\pm(k)$.

There exists exactly one isomorphism class of $(\mathfrak{g}, K)$-module with each given Σ. Every irreducible admissible $(\mathfrak{g}, K)$-module is realized in one of the above cases.

Let $\mathcal{P}_\mu(\lambda, \epsilon)$ denote the infinitesimal equivalence class of irreducible admissible $(\mathfrak{g}, K)$-modules of parity ϵ on which Δ and Z have eigenvalues values λ and μ, respectively. If $\mu = 0$, we denote this infinitesimal equivalence class of representations by $\mathcal{P}(\lambda, \epsilon)$. These representations are called *principal series representations*, and are exactly those representations which are equivalent to the irreducible admissible representation in case (i). We call a representations which is equivalent to one of the two infinite dimensional irreducible admissible representations in case (ii) or (iii) a reprsentation of the *discrete series*. They are denoted by $\mathcal{D}^\pm(k)$.

Classification of $(\mathfrak{g}, K)$-modules for $\mathbf{GL}(2, \mathbb{R})$. For the following, $G = \mathrm{GL}_\infty$. Let s_1, s_2, ϵ, and χ be as in the previous section. χ is a quasicharacter on B^+, the Borel subgroup of $\mathrm{GL}(2, \mathbb{R})^+$. Let ϵ_1, ϵ_2 be congruence classes of $\mathbb{Z}$ mod 2 such that $\epsilon_1 + \epsilon_2 \equiv \epsilon$ mod 2. There are two possible choices for ϵ_1 and ϵ_2. Let $\chi_i : \mathbb{R}$ be the quasicharacters on $\mathbb{R}^\times$ given by $\chi_i(y) = \mathrm{sgn}(y)^{\epsilon_i}|y|^{s_i}$ and extend χ to all of B by $\chi \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \chi_1(a)\chi_2(b)$.

Let V be the induced representation of χ in G, V^∞ the space of smooth vectors, and V_{fin} the space of K-finite vectors. The restriction of G to G^+ is the same representation as in the previous paragraph (depending only on s_1, s_2, ϵ).

This construction is used to show that every irreducible admissible $(\mathfrak{g}, K)$-module may be realized as the space of K-finite vectors is an admissible representation of $G = \mathrm{GL}(2, \mathbb{R})$ on a Hilbert space.

(i) The finite-dimensional representations are obtained by tensoring the symmetric powers of the standard representation with the one-dimensional representation of the form $\chi \circ \det$, where χ is a character of $\mathbb{R}^\times$.

(ii) If χ_1 and χ_2 are quasicharacters of $\mathbb{R}^\times$ such that $\chi_1\chi_2^{-1}$ is not a quasicharacter of the form $y \mapsto \mathrm{sgn}(y)^\epsilon|y|^{k-1}$, where ϵ is an equivalence class of $\mathbb{Z}$ mod 2 and k is an integer of the same parity of ϵ, then V_{fin} is irreducible.

(iii) If the hypothesis of case *(ii)* does not hold, then $V_{\text{fin}}/V_{\text{fin}}^0 = \mathcal{D}_\mu^+(k) \oplus$

$\mathcal{D}_\mu^-(k)$. The $\mathcal{D}_\mu^\pm(k)$ are irreducible.

Every irreducible admissible $(\mathfrak{g}, K)$-module is isomorphic to one of the above cases.

The Jacquet module, Jacquet functor, and supercuspidal representations. For the following, G is non-Archimedean. Let (π, V) be a smooth representation B. Define V_N to be the vector subspace of V generated by elements of the form $\pi(u)v - v$ for $u \in N$, $v \in V$. The quotient $J(V) = V/V_N$ is a smooth representation of T called the *Jacquet module* of V. This defines a functor, called the *Jacquet functor*, from the category of smooth representations of B to the category of smooth representations of T. This functor is exact. An irreducible admissible representation of G is called *supercuspidal* if $J(V) = 0$.

Classification of irreducible admissible non-supercuspidal representation of $\mathbf{GL}_p$. For the following, G is non-Archimedean. Let χ_1 and χ_2 be quasicharacters of $F^\times$. Fix two characters ξ_1 and ξ_2 of u such that $\chi_i(y) = |y|^{s_i}\xi(y)$ $(i = 1, 2)$. Let χ be the quasicharacter on B given by χ_1 and χ_2. Let $\mathcal{B}(\chi_1, \chi_2)$ be the induced representation of χ in G.

(i) Suppose we don't have $(\chi_1\chi_2^{-1})(y) = |y|^{-1}$ for all $y \in F^\times$ or $(\chi_1\chi_2^{-1})(y) = |y|$ for all $y \in F^\times$. Then $\mathcal{B}(\chi_1, \chi_2)$ is irreducible and $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi_2, \chi_1)$. If μ_1 and μ_2 are quasicharacters of $F^\times$, we have $\mathcal{B}(\chi_1, \chi_2) = \mathcal{B}(\mu_1, \mu_2)$ if and only if either $\chi_1 = \mu_1$ and $\chi_2 = \mu_2$ or $\chi_1 = \mu_2$ and $\chi_2 = \mu_1$.

(ii) If $(\chi_1\chi_2^{-1})(y) = |y|^{-1}$ for all $y \in F$, then $\mathcal{B}(\chi_1, \chi_2)$ has a one dimensional invariant subspace and the quotient representation is irreducible.

(iii) If $(\chi_1\chi_2^{-1})(y) = |y|$, then $\mathcal{B}(\chi_1, \chi_2)$ has an irreducible invariant subspace of codimension one.

Each admissible irreducible non-supercuspidal representation of G is isomorphic to an irreducible representation covered by one of the above cases.

An representation which is isomorphic to one of the irreducible representations covered by case (i) is called a *principal series represntation*. A representaiton which is isomorphic to one of the infinite dimensional irreducible representations covered by cases (ii) or (iii) is called a *special representation* or *Steinberg representation*. Thus each admissible irreducible non-supercuspidal representation of G is either a principal series representation, a special representation, or one dimensional.

Uniqueness of the K-fixed vector. Suppose that G is Archimedean. Assume that π is an irreducible unitary representation of G and for $\phi \in C_c^\infty(K\backslash G/K)$, the operator $\pi(\phi)$ is a compact operator on V^K. Then the dimension of V^K is at most one.

Suppose that G is non-Archimedean. Assume that π is an irreducible admissible representation of G_p. Then V^K is at most one dimensional, and the space of linear functionals L on V satisfying

$$L(\pi(k)v) = L(v),\ k \in K, v \in V$$

is at most one dimensional.

TODO: **Intertwining integrals.**

Summary. In both theories we are dealing with Hausdorff locally compact topological groups of matrices. We restrict our attention to those representations which are compatible with the topology of the group in a certain sense and which are admissible. These groups have a similar structure, in that they both contains a maximal compact subgroup K and a Borel subgroup B of upper triangular matrices. Representations are obtained by picking two multiplicative quasicharacters χ_1 and χ_2 on $F^\times$ and using them to define a quasicharacter χ on B. From χ we induce a representation of G. The unramified parts of χ_1 and χ_2 determine complex numbers s_1 and s_2, which control important properties of the representation. It is also important to consider the case where χ_1 and χ_2 are unramified. As long as the function $\chi_1\chi_2^{-1}$ is not of a certain form, these representation are irreducible. Representations of this form are called the principal series representations. If the representations are reducible, they contain in their composition series one or two infinite dimensional irreducible representations and a finite dimensional representation. The infinite dimensional representations are called discrete series in the Archimedean case and special in the non-Archimedean case. All irreducible admissible representations, besides the supercuspidal ones, are obtained in this way. Intertwining intergrals are used to defined G-equivariant maps between representations.