Similitude Siegel-Weil Formula

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January 31, 2025

Let $k = \mathbb{Q}$. Let V be a m dimensional quadratic space over k, and W a $2n$ dimensional symplectic space over k. Let $H = \mathrm{GO}(V)$ and let $H_1 = \mathrm{O}(V)$ be the kernel of the similitude map λ. If m is even, let $G' = \mathrm{GSp}(W)$. If m is odd, let G' be $\widehat{\mathrm{GSp}(W)}$, a certain the two-fold cover of $\mathrm{GSp}(W)$. For $g \in G'$, let $\lambda(g)$ be the similitude factor of the projection of g to $\mathrm{GSp}(W)$. Let G subgroup of $g \in G'$ such that $\lambda(g) \in \lambda(H)$. Let G_1 be the subgroup of $g \in G$ such that $\lambda(g) = 1$.

Let $\mathbb{W} = V \otimes_k W$. $\mathrm{GMp}(\mathbb{W})$ is a certain extension of $\mathrm{GSp}(\mathbb{W})$ by $\mathbb{C}^1$, called the metaplectic cover of $\mathrm{GSp}(\mathbb{W})$. The typical element of $\mathrm{GMp}(\mathbb{W})$ will be denoted by (g, ϵ), where $g \in \mathrm{GSp}(\mathbb{W})$ and $\epsilon \in \mathbb{C}^1$. p is the projection from $\mathrm{GMp}(\mathbb{W})$ to $\mathrm{GSp}(\mathbb{W})$. There are inclusions

$$H \hookrightarrow \mathrm{GSp}(\mathbb{W}),\ G \hookrightarrow \mathrm{GSp}(\mathbb{W}).$$

These are inclusions are selected so that their preimages in $\mathrm{GMp}(\mathbb{W})$ are nice. $p^{-1}(H)$ is trivial as an extension of H. If m is even, then $p^{-1}(G)$ is trivial as an extension of G. If m is odd, then $p^{-1}(G)$ is the metaplectic cover of $\mathrm{GSp}(Y)$.

Let ψ be a fixed additive character of k and $\omega = \omega_\psi$ the standard Weil representation of $H_1(\mathbb{A}) \times G_1(\mathbb{A})$ on the Schwartz space $S(V(\mathbb{A})^n)$. Let

$$R = \{(h, g) \in H \times G \mid \nu(h) = \nu(g)\}.$$

There is a representation of $R(\mathbb{A})$ on $S(V(\mathbb{A})^n)$ given by

$$\omega(h,g)\varphi(x) = |\nu(h)|^{-\frac{nm}{4}} (\omega(g_1)\varphi)\left(h^{-1}x\right).$$

We call ω the extended Weil representation.

The theta kernel, defined for $(h,g) \in R(\mathbb{A})$ by

$$\theta(h,g;\varphi) = \sum_{x \in V(k)^n} \omega(h,g)\varphi(x)$$

is left $R(k)$ invariant. The theta integral is defined (when convergent) by

$$I(g,\varphi) = \int_{H_1(k)\backslash H_1(\mathbb{A})} \theta(h_1 h, g; \varphi)\, \mathrm{d}h_1$$

where $g \in G(\mathbb{A})$, $h \in H(\mathbb{A})$ with $\lambda(h) = \lambda(g)$, and $\varphi \in S(V(\mathbb{A})^n)$. It does not depend on the choice of h. A technical procedure is needed to define the theta integral in all cases.

λ_s is a certain character of Siegel parabolic P in G. Let $I(s) = I_P^G(\lambda_s)$ be the normalized induced representation of $G(\mathbb{A})$. The Eisenstein series associated to a section $\Phi_s \in I(s)$ is defined for $\mathrm{Re}(s) > 1$ by

$$E(g,s,\Phi_s) = \sum_{\gamma \in P(k)\backslash G(k)} \Phi_s(\gamma g),$$

and the normalized Eisenstein series is

$$E^*(g,s,\Phi_s) = b_G(s) \cdot E(g,s,\Phi_s)$$

where $b_G(s)$ is a certain function of s.

The Siegel-Weil formula says that for a section $\Phi_s \in I(s)$ with $\Phi_0 \in \Pi(V)$

so that $\Phi_0 = [\varphi]$ for some $\varphi \in S\left(V\left(\mathbb{A}\right)^n\right)$,

$$E\left(g, 0, \Phi_s\right) = 2I\left(g, \varphi\right).$$