Rankin-Selberg Method

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Let $f(z), g(z)$ be modular forms and let $A(n), B(n)$ be their Fourier coeficients. Define the Eisenstein series for $SL(2,\mathbb{Z})$ by

$$E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)\sum_{\overline{\Gamma_\infty}\backslash\overline{\Gamma(1)}} \mathrm{im}(\gamma(z))^s$$

Also, define

$$\Lambda(s) = 4^{-s-k+1}\pi^{-2s-k+1}\Gamma(s)\Gamma(s+k-1)\zeta(2s)\sum_{n=1}^{\infty} A(n)B(n)n^{-s-k+1}$$

We show (*)

$$\Lambda(s) = \int_{\Gamma(1)\backslash\mathcal{H}} f(z)\overline{g}(z)E(z,s)y^k\,\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

First note that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ we have

$$f(\gamma(z))g(\gamma(z))\mathrm{im}(\gamma(z))^k = (cz+d)^k(c\overline{z}+d)^k f(z)g(z)\frac{y^k}{|cz+d|^{2k}} = f(z)g(z)y^k$$

hence $f(z)\overline{g}(z)y^k$ is invariant under fractional linear transformations. Also, $E(z,s)$ is automorphic in z. Finally, it is easy to check that the differential $\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$ is invariant under linear fractional transformations by checking it for the generators S and T of $\Gamma(1)$. Hence, the integral is well-defined.

The right hand side of $(*)$ is equal to

$$\pi^{-s}\Gamma(s)\zeta(2s)\sum_{\gamma\in\overline{\Gamma_\infty}\backslash\overline{\Gamma(1)}}\int_{\overline{\Gamma(1)}\backslash\mathcal{H}} f(\gamma(z))\overline{g}(\gamma(z))\mathrm{im}(\gamma(z))^{s+k}\,\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

$$= \pi^{-s}\Gamma(s)\zeta(2s) \int_{\Gamma_\infty \backslash \mathcal{H}} f(z)\overline{g}(z) y^{s+k} \ \frac{\mathrm{d}x \ \mathrm{d}y}{y^2}$$

Now,

$$\int_{\Gamma_\infty \backslash \mathcal{H}} f(z)\overline{g}(z) y^{s+k} \ \frac{\mathrm{d}x \ \mathrm{d}y}{y^2}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A(n)\overline{B}(m) \int_{\Gamma_\infty \backslash \mathcal{H}} e^{2\pi i(n-m)x} e^{-2\pi(n+m)y} y^{s+k} \ \frac{\mathrm{d}x \ \mathrm{d}y}{y^2}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A(n)\overline{B}(m) \int_0^\infty \int_0^1 e^{2\pi i(n-m)x} e^{-2\pi(n+m)y} y^{s+k-1} \ \mathrm{d}x \ \frac{\mathrm{d}y}{y}$$

$$= \sum_{n=1}^{\infty} A(n)\overline{B}(n) \int_0^\infty e^{-4\pi n y} y^{s+k-1} \frac{\mathrm{d}y}{y}$$

Making the substitution $y = \frac{t}{4\pi n}$ this is

$$= (4\pi)^{-s-k+1}\Gamma(s+k-1) \sum_{n=1}^{\infty} A(n)\overline{B}(n) n^{-s-k+1}$$

Finally, $B(n) = \overline{B}(n)$ because the $B(n)$ are eigenvalues of the Hecke operators, which are self adjoint. This proves $(*)$.