Tate's Thesis Notes

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Lemma 2.2.1. "$X(\eta\xi) = 1$, all $\eta \implies k^+\xi \neq k^+ \implies \xi = 0$. Therefore the characters of the form $X(\eta\xi)$ are everywhere dense in the character group."

Let N be the closure of the set $\{X(\eta\xi) : \eta \in k^+\}$ in $\hat{k^+}$. Let ψ be a character of $\hat{k^+}/N$. Let $\pi : \hat{k^+} \to \hat{k^+}/N$ be the projection map. The pullback $\psi' := \pi^*\psi$ is in $\hat{\hat{k^+}}$. By Pontryagin duality, there is $\xi \in G$ such that $\psi'(\chi) = \chi(\xi)$ for all $\chi \in \hat{k^+}$. ψ' is trivial on N, hence $\psi'(X(\eta\ \cdot)) = X(\eta\xi) = 1$ for all $\eta \in k^+$, and by Tate's argument $\xi = 0$. Hence, $(\hat{k^+}/N)\hat{} = 1$. Applying Pontryagin duality once again, $(\hat{k^+}/N)\hat{}\hat{} \cong \hat{k^+}/N = 1$, hence $\hat{k^+} = N$ (we can only do this because N is closed and therefore $\hat{k^+}/N$ is locally compact).

"Local compactness implies completeness and therefore closure..."

Let G be a locally compact group and H a locally compact subgroup. We show that H is closed. Pick $x \notin H$. xH is a locally compact space. Therefore, there exists a compact neighborhood $V \subseteq xH$ of x. V does not intersect H. Since G is Hausdorff, V is closed in the topology of G. Hence, H is open. Since H is a subgroup, H is also closed.

Theorem 2.2.2. "For k real we can take $f(\xi) = e^{-\pi|\xi|^2}$,...."

The formula follows from $\hat{f}(\eta) = f(\eta)$ and $f(\xi) = f(-\xi)$. The ladder is obvious. For the former, we compute

$$\hat{f}(\eta) = \int e^{-\pi|\xi|^2+2\pi i\eta\xi}\ d\xi = e^{-\pi|\eta|^2} \int e^{-\pi(\xi-i\eta)^2}\ d\xi$$

$$= e^{-\pi|\eta|^2} \int e^{-\pi|\xi|^2}\ d\xi = e^{-\pi|\eta|^2} = f(\eta).$$

"... for k complex, $f(\xi) = e^{-2\pi|\xi|}$;..."

Same as in the real case, the formula follows from $\hat{f}(\eta) = f(\eta)$ and $f(\xi) = f(-\xi)$. Write $\eta = u + iv$ and split into two real integrals;

$$\hat{f}(\eta) = \int e^{-2\pi|\xi|+2\pi i(\eta\xi+\overline{\eta}\overline{\xi})}\, d\xi = \int\int e^{-2\pi(x^2+y^2)+4\pi i(xu-yv)} 2\, dx\, dy$$

$$= \left(\int e^{-2\pi x^2+4\pi iux}\sqrt{2}\, dx\right)\left(\int e^{-2\pi y^2-4\pi ivy}\sqrt{2}\, dy\right)$$

$$= e^{-2\pi u^2-2\pi v^2}\left(\int e^{-2\pi(x-iu)^2}\sqrt{2}\, dx\right)\left(\int e^{-2\pi(y+iv)^2}\sqrt{2}\, dy\right)$$

$$= e^{-2\pi(u^2+v^2)}\left(\int e^{-2\pi x^2}\sqrt{2}\, dx\right)\left(e^{-2\pi y^2}\sqrt{2}\, dy\right)$$

$$= e^{-2\pi(u^2+v^2)}$$

"...and for k $\mathfrak{p}$-adic, $f(\xi) =$ the characteristic function of $\mathfrak{o}$,..."

The Fourier transform of f is

$$\hat{f}(\eta) = \int_{\mathfrak{o}} e^{-2\pi i\Lambda(\eta\xi)} d\eta.$$

By lemma 2.2.3, the integrand is a trivial character if $\eta \in \mathfrak{d}^{-1}$ and otherwise the integrand is a nontrivial character on the additive subgroup $\mathfrak{o}$. In the ladder case, the integral evaluates to zero by the Schur orthgonality relations, since $\mathfrak{o}$ is compact. Hence, $\hat{f}$ is the characteristic function of $\mathfrak{d}^{-1}$ multiplied by $(N\mathfrak{d})^{-1/2}$. The double fourier transform of f is

$$\hat{\hat{f}}(\xi) = (N\mathfrak{d})^{-1/2}\int_{\mathfrak{d}^{-1}} e^{-2\pi i\Lambda(\xi\eta)}\, d\xi.$$

Since $\mathfrak{d}^{-1}$ is a fractional ideal which contains $\mathfrak{o}$, we have $\mathfrak{d}^{-1} = \alpha^{-1}\mathfrak{o}$ for some $\alpha \in \mathfrak{o}$. Note that $\mathfrak{d} = \alpha\mathfrak{o}$, so $|\alpha| = (N\mathfrak{d})^{-1}$. Hence,

$$\hat{\hat{f}}(\xi) = (N\mathfrak{d})^{-1/2}\int_{\alpha^{-1}\mathfrak{o}} e^{-2\pi i\Lambda(\xi\alpha^{-1}\alpha\eta)}\, d\xi.$$

By lemma 2.2.5.,

$$\hat{\hat{f}}(\xi) = (N\mathfrak{d})^{1/2}\int_{\mathfrak{o}} e^{-2\pi\Lambda(\xi\alpha^{-1}\eta)}\, d\xi.$$

By lemma 2.2.3., the integrand is a trivial character if $\alpha^{-1}\eta \in \mathfrak{d}^{-1} = \alpha^{-1}\mathfrak{o}$ or equivalently $\eta \in \mathfrak{o}$ and otherwise the integrand is a nontrivial character on the additive subgroup $\mathfrak{o}$. Hence, $\hat{\hat{f}}(\xi)$ is the characteristic function of $\mathfrak{o}$. We have $\hat{\hat{f}}(\xi) = f(\xi) = f(-\xi)$, which completes the computation.

Lemma 2.4.1. "Using the fact that the integral is absolutely convergent for s near 0 to make estimates, it is a routine matter to show that the function has a derivative for s near 0. The derivative can in fact be computed by 'differentiating under the integral sign'."

By theorem 2.3.1., $c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^t$ where $\tilde{c}$ is a character of u and t is a complex number. We assume that $\tau = \mathrm{Re}(t) > 0$. fix $0 < \sigma < \tau$ and suppose $|s| \leq \sigma$. Then,

$$|f(\alpha)c(\alpha)|\alpha|^s| = |f(\alpha)||\alpha|^{\tau+\mathrm{Re(s)}} \leq |f(\alpha)|(|\alpha|^{\tau-\sigma} + |\alpha|^{\tau+\sigma}) \in L^1(k^*)$$

by $\mathfrak{z}_2$. Hence, the derivative for s near 0 may be computed by differentiating under the integral sign;

$$\frac{\mathrm{d}}{\mathrm{d}s} \int f(\alpha)c(\alpha)|\alpha|^s \,\mathrm{d}\alpha = \int f(\alpha)c(\alpha)\log(\alpha)|\alpha|^s \,\mathrm{d}\alpha.$$

Lemma 2.4.3. Errata: Line 2. The final step expression should be $c(-1)\rho(c)\rho(\hat{c})\zeta(f, c)$.

2.5 Computation of $\rho(c)$ by Special ζ-functions

k Real

We check the identities

$$\hat{(}\xi) = f(\xi) \text{ and } \hat{f}_{\pm}(\xi) = if_{\pm}(\xi)$$

The first was already proved in theorem 2.2.2. We prove the second;

$$\hat{f}_{\pm}(\xi) = \int \eta e^{-\pi\eta^2 + 2\pi i\xi\eta} \, d\eta = e^{-\pi\xi^2} \int \eta e^{-\pi(\eta - i\xi)^2} \, d\eta$$

$$= e^{-\pi\xi^2} \int (\eta + i\xi) e^{-\pi\eta^2} \, d\eta = e^{-\pi\xi^2} \left[\int \eta e^{-\pi\eta^2} \, d\eta + i\xi \int e^{-\pi\eta^2} \, d\eta \right]$$

$$= i\xi e^{-\pi\xi^2} = if_{\pm}(\xi)$$

Explicit Expressions for $\rho(c)$:

"... the second form follows from elementary Γ-functions identities."

The identities are

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \text{ and } 2^{2s-1}\Gamma(s)\Gamma(s+1/2) = \pi^{1/2}\Gamma(2s)$$

The second formula can be written as

$$\Gamma\left(\frac{s}{2}\right) = \frac{2^{1-s}\pi^{1/2}\Gamma(s)}{\Gamma(\frac{s+1}{2})}$$

Hence,

$$\rho(||^s) = \frac{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})}{\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})} = \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(s)}{\pi^{-\frac{1-s}{2}}\Gamma(\frac{s+1}{2})\Gamma(\frac{1-s}{2})}$$

$$= \frac{2^{1-s}\pi^{1-s}\Gamma(s)}{\Gamma(\frac{s+1}{2})\Gamma(1-\frac{s+1}{2})} = 2^{1-s}\pi^{-s}\sin\left(\frac{\pi(s+1)}{2}\right)\Gamma(s)$$

$$= 2^{1-s}\pi^{-s}\cos\left(\frac{\pi}{2} - \frac{\pi(s+1)}{2}\right)\Gamma(s)$$

$$= 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s).$$

For $\rho(\pm||^s)$, we use

$$\Gamma\left(\frac{s+1}{2}\right) = \frac{2^{1-s}\pi^{1/2}\Gamma(s)}{\Gamma(\frac{s}{2})}$$

Hence,

$$\rho(\pm||^s) = -i\frac{\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})}{\pi^{-\frac{(1-s)+1}{2}}\Gamma(\frac{(1-s)+1}{2})} = -i\frac{2^{1-s}\pi^{-s/2}\Gamma(s)}{\pi^{-\frac{(1-s)+1}{2}}\Gamma(\frac{s}{2})\Gamma(\frac{(1-s)+1}{2})}$$

$$= -i\frac{2^{1-s}\pi^{1-s}\Gamma(s)}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})} = -i2^{1-s}\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)$$

Errata: In the first form of $\rho(\pm||^s)$, the minus sign out front is missing.

k Complex

Errata: In the first equation on page 319, the first plus sign should be a minus.

Errata (?): In the expressions for $\zeta(f_n, c_n||^s)$, the exponent of 2π should be $(1-s) - \frac{|n|}{2}$. This term cancels out, so the expression for $\rho(c_n||^s)$ is still correct.

k $\mathfrak{p}$-adic

Root numbers and Gauss sums. In the case of $\mathbb{Q}_p$, the roots numbers are the "signed" part of a Gauss sum. For each $n \geq 1$, there is a ring homomorphism $\mathbb{Q}_p \to \mathbb{Z}/p^n\mathbb{Z}$ given by reduction modulo p^n. Restricting to the group of units u, this gives group homomorphism $u \to (\mathbb{Z}/p^n)^\times$. The kernel is $(1 + p^n\mathbb{Z}_p)$, so we have an isomorphism $u/(1 + p^n\mathbb{Z}_p) \cong (\mathbb{Z}/p^n)^\times$. Hence, a character on u with conductor n is lifted from a character on $(\mathbb{Z}/p^n)^\times$. Conversely, if χ is a Dirichlet character with conductor p^n, then χ defines a character on u with conductor n.

Suppose that c is a character on $\mathbb{Z}_p^\times$ and χ its corresponding Dirichlet character. Tate's formula for the roots number of c gives

$$\rho_0(c) = p^{-n/2} \sum_{\substack{\epsilon \bmod p^n \\ (\epsilon, p^n) = 1}} \chi(\epsilon) e^{2\pi i \epsilon / p^n}$$

which is a Gauss sum times $p^{-n/2}$. Since we know the root number lies on the unit circle, this is the "signed" part of the Gauss sum and $p^{n/2}$ is the modulus of the Gauss sum. Therefore, the root numbers are a generalization of Gauss sums.

Lemma 4.2.2. "...the inversion formula holds." Why is the counting measure on k dual to the measure which gives V mod k volume 1? If we let $\varphi \equiv 1$ on V mod k, we see that its Fourier transform is the charactertic function of 0 by the Schur orthgonality relations. Hence,

$$\hat{\hat{\varphi}}(-\mathfrak{X}) = \sum_{\xi \in k} \hat{\varphi}(\xi) e^{2\pi i \Lambda} = \hat{\varphi}(0) e^0 = 1 = \varphi(\mathfrak{X}).$$

This is easily generalized to any compact abelian group. The counting measure on its character group is always dual to measure which gives G volume 1.

Theorem 4.2.1. How is this equivalent to geometric Riemann-Roch theorem?

k in the large!

Riemann zeta function. In the case $k = \mathbb{Q}$ and $c = 1$ we get the (completed) Riemann zeta function. We pick $f_\infty(t) = e^{-\pi t^2}$ and f_p to be the characteristic function of $\mathbb{Z}_p$ for each prime p. We showed above that each local function is its own Fourier transform, so in this case $\hat{f} = f$. The zeta function is

$$\zeta(f, ||^s) = \int f(\mathfrak{a})|\mathfrak{a}|^s \, \mathrm{d}\mathfrak{a}$$
$$= \left(\int_{\mathbb{R}^\times} e^{-\pi t^2}|t|^s \, \mathrm{d}t\right)\left(\prod_{p\nmid\infty}\int_{\mathbb{Z}_p} |\alpha|^s \, \mathrm{d}\alpha\right).$$

At the Archimedean place, the integral is

$$2\int_0^\infty e^{-\pi t^2} t^s \, \frac{dt}{t} = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).$$

At the finite places, we evaluate by summing over each annulus $p^v u$ with u the group of units in $\mathbb{Z}_p$. Since $\mathrm{d}\alpha$ is a multiplicative Haar measure, $\int_{p^v u} \mathrm{d}\alpha = \int_u \mathrm{d}\alpha = 1$. We have,

$$\int_{\mathbb{Z}_p} |\alpha|^s \, \mathrm{d}\alpha = \sum_{v=0}^{\infty}\int_{p^v u} |\alpha|^s \, \mathrm{d}\alpha = \sum_{v=0}^{\infty} p^{-vs} = \frac{1}{1-p^{-s}}.$$

Hence,

$$\zeta(f, ||^s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\prod_p \frac{1}{1-p^{-s}}$$

and the functional equation

$$\zeta(f, ||^s) = \zeta(\hat{f}, ||^{1-s})$$

is the usual functional equation for the Riemann zeta function.

This is often written in the form

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$
$$= \frac{\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})}{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})}\zeta(1-s)$$

where ζ is the usual (non-completed) Riemann zeta function. The factor out front is simply the inverse of the local factor for $\mathbb{R}$.

Characters on $\mathbb{Q}$. Remaining in the case $k = \mathbb{Q}$ but allowing c to be non-trivial, we get the (completed) Dirichlet L-functions. Recall from above that each equivalence class of quasi-characters on a p-adic field $\mathbb{Z}_p$ with conductor $n \geq 0$ is represented by Dirichlet character with conductor p^n. Equivalence classes of characters on $\mathbb{R}$ are represented by the characters $\operatorname{sgn}(x)^\epsilon$ with ϵ an equivalence class of $\mathbb{Z}$ mod 2. Let $c(\mathfrak{a})$ be a general character of the ideles. Let S be a finite set which contains the Archimedean prime and all primes where c_p is ramified. S' will denote $S \setminus \{\infty\}$. $c(\mathfrak{a})$ is of the form

$$c(\mathfrak{a}) = \prod_{p \in S} \tilde{c}_p(\tilde{a}_p) \cdot \prod_{p \in S} |a|_p^{it_p} \cdot \chi(\varphi_S(\mathfrak{a}))$$

where for $p \in S'$ the $\tilde{c}_p$ are ramified characters with conductor $n_p \geq 1$ on $\mathbb{Z}_p^\times/(1+p^{n_p}\mathbb{Z}_p) \cong (\mathbb{Z}/p^{n_p}\mathbb{Z})^\times$, t_p are real numbers, and χ is a character on the subgroup of nonzero rational numbers which are in $\mathbb{Z}_p^\times$ for each $p \in S'$. The map φ_S takes $\mathfrak{a}$ to its ideal in $\mathbb{Q}$ and removes the primes in S' from its factorization. Let $n = \prod_{p \in S'} p_p^n$. For an idele $\mathfrak{a}$, let $\tilde{\mathfrak{a}} = \prod_{p \in S'} \tilde{a}_p$ and define $\psi(\tilde{\mathfrak{a}}) = \prod_{p \in S'} \tilde{c}_p(\tilde{a}_p)$. ψ is a character on the group $\prod_{p \in S'} \mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p) \cong \prod_{p \in S'}(\mathbb{Z}/n_p\mathbb{Z})^\times \cong \mathbb{Z}/n\mathbb{Z}^\times$. Hence, ψ is associated to a Dirichlet character with conductor n.

We are only considering characters where $c(x) = 1$ for all $x \in \mathbb{Q}^\times$. A great deal of simplication occurs because $\mathbb{Q}$ has class number 1. For $p \in S'$ we have

$$1 = c(p^k) = p^{ik(t_\infty - t_p)}$$

for all $k \in \mathbb{Z}$. Hence, the t_p are all equal for $p \in S$. Since we are only concerned with equivalence classes of characters, we may suppose that $t_p = 0$ for all $p \in S$. We have $\tilde{c}_\infty(\tilde{a}_\infty) = \operatorname{sgn}(a_\infty)^\epsilon$ for ϵ and equivalence class of $\mathbb{Z}$ mod 2. Plugging in -1 gives

$$1 = c(-1) = (-1)^\epsilon \psi(-1)$$

Thus, ϵ is determined by the sign of the character $\psi(-1)$. Our formula for $c(\mathfrak{a})$ simplifies to

$$c(\mathfrak{a}) = \operatorname{sgn}(a_\infty)^\epsilon \psi(\tilde{a}) \chi(\varphi_S(\mathfrak{a})).$$

Finally, since $\mathbb{Q}$ has class number 1, χ is determined by its values $\chi(\varphi_S(x))$ for $x \in \mathbb{Q}^\times$. These are just

$$\chi(\varphi_S(x)) = \operatorname{sgn}(x)^\epsilon \overline{\psi}(\tilde{x})$$

Dirichlet L-functions. With $c(\mathfrak{a})$ selected as in the remaining paragraph, pick $f_\infty(t) = e^{-\pi t^2}$ if $\epsilon \equiv 0$ and $f_\infty(t) = te^{-\pi t^2}$ if $\epsilon \equiv 1$. For $p \in S'$, pick f_p to be the characteristic function of $p^{-n_p}\mathbb{Z}_p$. For $p \notin S$, pick f_p to be the characteristic function of $\mathbb{Z}_p$. The zeta function of f is

$$\zeta(f, c||^s) = \int f(\mathfrak{a})|\mathfrak{a}|^s \, \mathrm{d}\mathfrak{a}$$

$$\left(\int_{\mathbb{R}^\times} \operatorname{sgn}(t)^\epsilon e^{-\pi t^2}|t|^s \, \mathrm{d}t\right)\left(\prod_{p\in S'}\int_{p^{-n_p}\mathbb{Z}_p} \tilde{c}_p(\tilde{\alpha})|\alpha|^s \, \mathrm{d}\alpha\right)\left(\prod_{p\notin S}\int_{\mathbb{Z}_p} |\alpha|^{s+it_p} \, \mathrm{d}\alpha\right)$$

The t_p are real numbers defined by $\chi(\varphi_S(\mathfrak{a})) = \prod_{p\notin S}|a_p|^{it_p}$, since we recall that $c_p(\mathfrak{a})$ is an unramified character for $p \notin S$. The integral at the Archimedean place is either

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$$

or

$$\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)$$

according to whether $\epsilon = 0$ or $\epsilon = 1$. We can write both conditions together as

$$\pi^{-\frac{s+\epsilon}{2}}\Gamma\left(\frac{s+\epsilon}{2}\right)$$

The local factors $p \in S'$ are computed by Tate in section 2.5 to be

$$p^{n_p s}\tau(\tilde{c}_p)\int_{1+p^{n_p}\mathbb{Z}_p} \mathrm{d}\alpha = \frac{p^{n_p s}\tau(\tilde{c}_p)}{p^{n_p} - p^{n_p-1}}$$

where τ denotes the Gauss sum, according to our observation above. The group of units $\mathbb{Z}_p^\times$ has measure 1 in the multiplicative Haar measure and $\mathbb{Z}_p^\times/(1+p^{n_p}\mathbb{Z}_p) \cong \mathbb{Z}_{p^n}^\times$ has cardinality $p^{n_p} - p^{n_p-1}$, from which we get the measure of $1+p^{n_p}\mathbb{Z}_p$ (in the functional equation these terms will cancel out, since they do not depend on s). Finally, the factors at $p \notin S$ are

$$\frac{1}{1-p^{-s+it_p}}.$$

Their product is the Dirichlet L-function associated to $\overline{\chi}$;

$$\prod_{p\notin S}\frac{1}{1-p^{-s+it_p}} = \sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^s}.$$

This can be seen by writing

$$\prod_{p\notin S}\frac{1}{1-p^{-s+it_p}}=\prod_{p\notin S}\sum_{v=0}^{\infty}p^{ivt_p}p^{-vs}=\sum_{\substack{n=1\\ p\nmid n,\forall p\in S'}}^{\infty}\left(\prod_{p\notin S}|n|_p^{-it_p}\right)n^{-s}=\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n^s}$$

Recall we extend χ to $\mathbb{Z}$ by writing $\chi(n)=0$ if $p\nmid n$ for some $p\in S'$. Now it is clear that $p^{it_p}=\overline{\chi(p)}$. The product of the local factors for $p\in S'$ is

$$\prod_{p\in S'}\frac{p^{n_p s}\tau(\tilde{c}_p)}{p^{n_p}-p^{n_p-1}}=\frac{n^s\tau(\tilde{c}_p)}{C}$$

where C is a constant not depending on s. A formula for C is

$$\sum_{k=0}^{|S'|}\sum_{p_1,\ldots,p_k\in S'}(-1)^k\frac{n}{p_1\ldots p_k}$$

although I don't think this is important.

Putting all of this together, the zeta function is

$$\zeta(f,\chi||^s)=C\pi^{-\frac{s+\epsilon}{2}}\Gamma\left(\frac{s+\epsilon}{2}\right)n^{s+1}(-1)^{\epsilon}\tau(\overline{\chi})\prod_{p\nmid n}\frac{1}{1-\chi(p)p^{-s}}$$

The zeta function of the Fourier transform is the same (replacing s with $1-s$ and χ with $\overline{\chi}$) except when $p\in S'$ where the local factors are

$$\frac{p^{n_p}}{p^{n_p}-p^{n_p-1}}$$

and the real place you must multiply by i^{ϵ}. The zeta function of the transform is

$$\zeta(\hat{f},\chi||^s)=Ci^{\epsilon}\pi^{-\frac{(1-s)+\epsilon}{2}}\Gamma\left(\frac{(1-s)+\epsilon}{2}\right)n\prod_{p\nmid n}\frac{1}{1-\overline{\chi}(p)p^{-(1-s)}}$$

Now, letting $L(s,\chi)$ be the usual Dirichlet zeta function, the functional equation is

$$L(s,\chi)=(-i)^{\epsilon}\tau(\chi)n^{-s}\frac{\pi^{-\frac{(1-s)+\epsilon}{2}}\Gamma(\frac{(1-s)+\epsilon}{2})}{\pi^{-\frac{s+\epsilon}{2}}\Gamma(\frac{s+\epsilon}{2})}L(1-s,\overline{\chi})$$

$$=\frac{\tau(\chi)}{i^{\epsilon}\sqrt{n}}n^{1/2-s}2^s\pi^{s-1}\sin\left(\frac{\pi}{2}(s+\epsilon)\right)\Gamma(1-s)L(1-s,\overline{\chi})$$