Similitude Groups

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1 Basic Definitions

Let F be a field. Let W be a $2n$-dimensional nondegenerate symplectic space over F with symplectic form $b(\cdot,\cdot)$. We definite the **symplectic similitude group** of the vector space W to be the set of $g \in \mathrm{GL}(W)$ such that there exists $\lambda \in F^\times$ with $b(gx, gy) = \lambda b(x,y)$ for all $x, y \in W$. The factor λ is uniquely determined by g and is called the **factor of similitude** for g. The symplectic similitude group of W is denoted by $\mathrm{GSp}(W)$. Let $s : \mathrm{GSp}(W) \to F^\times$ be the map which takes a linear transformation to its factor of similitude. $\mathrm{GSp}(W)$ is a subgroup of $\mathrm{GL}(W)$ and s is a homomorphism. $\mathrm{Sp}(W)$ is a normal subgroup of $\mathrm{GSp}(W)$, being the kernel of s.

There exists a basis for W relative to which $\hat{b}$, the matrix representation for b, takes the form

$$\hat{b} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. For $g \in \mathrm{GL}(W)$, let $\hat{g}$ denote the matrix representation of g relative to the chosen basis for W. $g \in \mathrm{GSp}(W)$ if and only if

$$g^t b g = \lambda b.$$

for some $\lambda \in F^\times$. Writing $\hat{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are $n \times n$ matrices with entries in F, the above is equivalent to the relations

$$A^t B = B^t A,\ C^t D = D^t C,\ A^t D - B^t C = \lambda I_n.$$

Henceforth, we identify $\mathrm{GSp}(W)$ with its realization as a matrix group. For $\lambda \in F^{\times}$, the matrix

$$S_{\lambda} := \begin{pmatrix} \lambda I_n & 0 \\ 0 & I_n \end{pmatrix}$$

is an element of $\mathrm{GSp}(W)$ with factor of similitude λ. Hence, the map s is surjective and we have a short exact sequence

$$1 \longrightarrow \mathrm{Sp}(W) \longrightarrow \mathrm{GSp}(W) \xrightarrow{s} F^{\times} \longrightarrow 1.$$

Let Z denote the group of scalar matrices. Z is a subgroup of $\mathrm{GSp}(W)$ and is isomorphic to $F^{\times}$. Suppose that $g \in \mathrm{GSp}(W)$ with $s(g) = \lambda$ and $\mu \in F^{\times}$. Observe that $s(\mu g) = \mu^2 s(g)$. Now consider reduction of s modulo $F^{\times,2}$. Call the kernel of this map $\mathrm{GSp}^2(W)$. It is the subgroup of $\mathrm{GSp}(W)$ which consists of elements with a factor of similitude that is square in $F^{\times}$. We have another short exact sequence

$$1 \longrightarrow \mathrm{GSp}^2(W) \longrightarrow \mathrm{GSp}(W) \xrightarrow{s} F^{\times}/F^{\times,2} \longrightarrow 1\ .$$

We have $\mathrm{GSp}^2(W) = Z \cdot \mathrm{Sp}(W)$. However, if F has characteristic over than 2 the product is not quite direct. If $g = \mu g'$ for some $\mu \in F^{\times}$, then $g = (-\mu)(-g')$ and $-\mu \in F^{\times}$, $-g' \in \mathrm{Sp}(W)$. In order to make a direct product when the characteristic is other than 2, for each pair $\{\mu, -\mu\}_{\mu \in F^{\times}}$, we arbitrarily select of the elements to be "positive" and call the other element "negative." Denote the set of positive elements Z^+. For instance, if F is an ordered field, we can let Z^+ be the set of $\mu > 0$. Let $\{\lambda_i\}$ be a set of representatives for $F^{\times}/F^{\times,2}$. Each $g \in \mathrm{GSp}(W)$ can be represented in the form

$$g = \mu s_{\lambda_i} g'$$

for some uniquely chosen $\mu \in Z^+$, λ_i, and $g' \in \mathrm{Sp}(W)$. In case Z^+ is a group, we have a direct product of groups $\mathrm{GSp}^2(W) = Z^+ \cdot \mathrm{Sp}(W)$. Note that $s(g) = \mu^2 \lambda_i$. The set $S := \{s_{\lambda_i}\}$ is an abelian group modulo Z and isomorphic to $F^{\times}/F^{\times,2}$. $\mathrm{GSp}(W)/Z$ is the semidirect product $S \cdot \mathrm{Sp}(W)$. The adjoint action of S on $\mathrm{Sp}(W)$ is

$$\begin{pmatrix} \lambda I_n & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \lambda^{-1} I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & \lambda B \\ \lambda^{-1} C & D \end{pmatrix}.$$

We can express the decomposition above by the formula

$$\mathrm{GSp}(W) = Z^{+} \cdot S \cdot \mathrm{Sp}(W) = S \cdot \mathrm{GSp}^{2}(W).$$

2 Examples

If $\dim W = 2$, then $\mathrm{Sp}(W) = \mathrm{SL}(2, F)$ and $\mathrm{GSp}(W) = \mathrm{GL}(2, F)$.

If $F^{\times}/F^{\times,2} = 1$, then $\mathrm{GSp}(W) = \mathrm{GSp}^{2}(W) \cong F^{\times} \times \mathrm{Sp}(W)$. This case occurs if F is algebraically closed or of characteristic 2. For instance, if $F = \mathbb{C}$ or if F is the finite field of order 2^{k}.

If $|F^{\times}/F^{\times,2}| = 2$, then $\{\pm 1\}$ is a set of representative for $|F^{\times}/F^{\times,2}|$ and is also a group. In this case the decompositions $S \cdot \mathrm{Sp}(W)$ and $S \cdot \mathrm{Sp}^{2}(W)$ are semidirect products. The former consists of elements

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} -A & -B \\ C & D \end{pmatrix}$$

with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(W)$. This case occurs if $F = \mathbb{R}$ or F is a finite field of characteristic not equal to 2. Observe that if $|F^{\times}/F^{\times,2}| > 2$, we can not choose $\{\lambda_i\}$ to form a group. $F^{\times}/F^{\times,2}$ is an elementary abelian 2-group. Therefore, if $\{\lambda_i\}$ was a group, then $\lambda_i^{2} = 1$ for all λ_i, but the only square roots of 1 in a field are ± 1.

If F is a finite field of order q and $\dim W = 2n$, then

$$|\mathrm{GSp}(W)| = (q-1)\,|\mathrm{Sp}(W)|.$$

It is known that

$$|\mathrm{Sp}(W)| = q^{n^{2}} \prod_{i=1}^{n} \left(q^{2i} - 1\right).$$

3 The Heisenberg Group

Let F be a nonarchimedian local field of characteristic not equal to 2. Let $H(W)$ be the Heisenberg group of W. $\mathrm{GSp}(W)$ acts as a group of automorphisms on $H(W)$ by

$$g\left(w,t\right)=\left(gw,s\left(g\right)t\right).$$

This is clearly a left group action. We check it is an automorphism of $H\left(W\right)$;

$$g\left[\left(w_1,t_1\right)\left(w_2,t_2\right)\right]=g\left(w_1+w_2,t_1+t_2+\frac{1}{2}\langle w_1,w_2\rangle\right)$$

$$=\left(gw_1+gw_2,s\left(g\right)t_1+s\left(g\right)t_2+\frac{s\left(g\right)}{2}\langle w_1,w_2\rangle\right)$$

$$=\left(gw_1+gw_2,s\left(g\right)t_1+s\left(g\right)t_2+\frac{1}{2}\langle gw_1,gw_2\rangle\right)$$

$$\left(gw_1,s\left(g\right)t_1\right)\left(gw_2,s\left(g\right)t_2\right)$$

$$\left[g\left(w_1,t_1\right)\right]\left[g\left(w_2,t_2\right)\right].$$

Let ψ be a nontrivial character of F. Let $\psi_a\left(t\right)=\psi\left(at\right)$. Let ρ_ψ be the unique smooth irreducible representation of $H\left(W\right)$ with central chracter ψ. Then $\rho_\psi^g\left(h\right)=\rho_\psi\left(g^{-1}h\right)$ is a smooth irreducible representation of $H\left(W\right)$ with central chracter $\psi_{s(b)}$. Hence, $\rho_\psi^g\cong\rho_{\psi_{s(b)}}$.