A Note on Smooth Representations

Nick Pilotti

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Let G be a **l-group**. So, G is a (Hausdorff) topological group and has a neighborhood base of the identity consisting of open compact subgroups.

Let V be a complex normed vector space. We define a function $f : G \to V$ to be **smooth** if it is locally constant, i.e., for each $x \in G$ there is a neighborhood U of x such that $f|_U$ is constant.

Proposition 1. A function $f : G \to V$ is smooth if and only if for all sets $S \subseteq V$, the preimage $f^{-1}(S)$ is open.

Proof. Suppose that f is smooth. Take $S = \{v\}$ to be a singleton. For each $x \in G$ with $f(x) = v$, there is an open neighborhood U of x such that $f|_U \equiv v$. Hence, the set $f^{-1}(\{v\})$ is open. If $S \subseteq V$ is arbitrary, then $f^{-1}(S) = \cup_{v \in S} f^{-1}(\{v\})$ is a union of open sets and therefore open.

Conversely, suppose for all sets $S \subseteq V$, $f^{-1}(S)$ is open. For any $x \in G$, $f^{-1}(f(x))$ is an open neighborhood of x on which f is constant. So f is smooth.

Corollary 2. If $f : G \to V$ is smooth. Then f is continuous.

Definiton. We say that a function $f : G \to V$ is **uniformly locally constant on the right** (resp. on the left) if there exists an open neighborhood U of the identity in G such that $f|_{xU} \equiv f(x)$ (resp. $f|_{Ux} \equiv f(x)$) for all $x \in G$. It is clear that if a function is uniformly locally constant on the left or right, then it is smooth. If G is an abelian group, the notions of uniformly locally constant on the right and uniformly locally constant on the left agree and we simply say that f is uniformly locally constant.

Proposition 3. Suppose that $f : G \to V$ is a group homomorphism. The following are equivalent

$i)$ f is smooth.

$ii)$ f is uniformly locally constant on the right.

$iii)$ f is uniformly locally constant on the left.

iv) f is constant on some open compact subgroup.

v) f is continuous.

Proof. If f is smooth, then f is constant on some open neighborhood U of the identity. Hence, $f|_U \equiv 0$. For all $x \in G$ and $u \in U$, we have

$$f(xu) = f(x) + f(u) = f(x),$$

so f is uniformly locally constant on the right. Similarly, f is uniformly locally constant on the left. Furthermore, since f has an neighborhood base of the identity consisting of open compact subgroups, U must contain some open compact subgroup on which f is constant. This proves that *i)* implies *ii)*, *iii)*, and *iv)*. That *ii)* implies *i)* and *iii)* implies *i)* is trivial.

If f is constant on some open compact subgroup, then as above we show that f is locally constant on the left or right. So *iv)* implies *ii)* and *iii)*.

Finally, we show *v)* implies *i)*. That *i)* implies *v)* was corollary 2. Suppose that f is continuous. Let U be the set of $v \in V$ such that for $|v| < \epsilon$ for some arbitrary $\epsilon > 0$. The set $f^{-1}(U)$ is an open neighborhood of the identity in G and therefore contains an open compact subgroup $K \subseteq G$. The image $f(K) \subseteq U$ is a subgroup of V and therefore must be the trivial subgroup. Note that this ultimately depends on the Archimedean property of $\mathbb{C}$. If there was some $v \in f(K)$ with $v \neq 0$, there is a natural number n large enough so that $|nv| > \epsilon$ but also $nv \in f(K)$.

Definition. A representation of G is the data of a complex vector space E and a group homomorphism $\pi : G \to \mathrm{GL}(E)$. We say that a vector $v \in V$ is **smooth** if the stablizer of v in G contains a compact open subgroup. We say that a π is a **smooth representation** if every vector in V is smooth. By proposition 3, a vector $v \in V$ is smooth if and only if the orbit map $g \mapsto g(v)$ is smooth.

Lemma 4. Let $f : G \to V$ be compactly supported. If f is smooth, then f is uniformly locally constant on the right and left (compare this proposition to Folland, Abstract Harmonic Analysis, 2.6).

Proof. Let $K = \mathrm{Supp}\, f$. For each $x \in K$, there is a neighborhood U_x of 1 such that $f|_{xU_x}$ is constant. There is a symmetric neighborhood V_x of 1 such that $V_x V_x \subseteq U_x$ (c.f. Folland, Abstract Harmonic Analysis, 2.1.b). The sets xV_x cover K, so there exist $x_1, ..., x_n \in K$ such that $K \subseteq \cup_1^n x_j V_{x_j}$. Let $V = \cap_1^n V_{x_j}$.

If $x \in K$ then there is some j for which $x_j^{-1} x \in V_{x_j}$, so that $xy = x_j \left(x_j^{-1} x\right) y \in x_j U_{x_j}$ for all $y \in V$. Hence, $f(xy) = f(x_j)$ for all $y \in V$.

Now take $x \notin K$ and suppose towards contradiction that $xy \in K$ for some $y \in V$, then $f(xy) = f(xyy^{-1}) = f(x)$ by the previous paragraph. But since f is locally constant, K is exactly the set of $x \in G$ such that $f(x) \neq 0$. Therefore, $f(x) = f(xy) \neq 0$, contradicting that $x \notin K$. Hence, $xy \notin K$ for all $y \in V$ and $f|_{xV} \equiv 0$.

Proposition 5. Let E be a vector space of functions $f : G \to V$ and suppose we have a representation of G on E by the right (resp. left) regular action $(gf)(x) = f(xg)$ (resp. $(gf)(x) = f(g^{-1}x)$) for $g \in G$ and $f \in E$. $f \in E$ is a smooth vector if and only if f is uniformly locally constant on the right (resp. left). If $f \in E$ is compactly supported, then f is smooth vector if and only if f is smooth function.

Proof. We show it only for the case of the right regular action, as the case of the left regular action is similar. $f \in E$ is a smooth vector if and only if the stabilizer of orbit map

$$g \mapsto (x \mapsto f(xg))$$

contains a compact open subgroup K. That is

$$f(xg) = f(x)$$

for all $x \in G$ and $g \in K$. Hence, f is a smooth vector if and only if f is uniformly locally constant on the right.

Now, suppose that $f \in E$ is compactly supported. By the above, we only need to show that if f is smooth then f is uniformly locally constant on the right, which was already proved in the lemma.

Example. Here is an example of a function which is locally constant but not uniformly locally constant, i.e., f is a smooth function but not a smooth vector in the regular representation on smooth functions. In $\mathbb{Q}_2$, let $S_N = 2^{-N} + 2^N\mathbb{Z}_2$ for $N \geq 1$. Let $f = \sum_{N=1}^{\infty} \chi_{S_N}$, where χ_{S_N} is the characteristic function of S_N. For $x \in \mathbb{Q}_2$, if $x \in \mathbb{Z}_2$, then $f(x) = 0$. Otherwise, write

$$x = 2^{-N} + a_{-N+1}2^{-N+1} + ...$$

Let M be the smallest integer other than $-N$ such that $a_M \neq 0$ (if such an integer exists). If M does not exist or if $M \geq N$, then $f(x) = 1$. Otherwise, $f(x) = 0$.

f is locally constant, because it is a sum of characterstic functions of disjoint open sets. f is not uniformly locally constant. If U is some open neighborhood of the identity in $\mathbb{Q}_2$, then U contains $2^n\mathbb{Z}_2$ for some $n \geq 1$. For $m > n$, we have $f(2^{-m} + 2^m) = 1$, but $f(2^{-m} + 2^m + 2^n) = 0$ and $2^n \in U$.