Notes on "On a conjecture of Jacquet" by Harris and Kudla

Nick Pilotti

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Theorem to be proved. Let k be a number field and let $\pi_i,\ i=1,2,3$ be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ such that the product of their central characters is trivial. Jacquet conjectured that the central value $L\left(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3\right)$ of the triple product L-function is nonzero if and only if there exists quaternion algebra B over k and automorphic forms $f_i^B\in\pi_i^B$ such that the integral

$$I\left(f_1^B,f_2^B,f_3^B\right)=\int_{Z(\mathbb{A})B^\times(k)\backslash B^\times(\mathbb{A})} f_1^B(b)\,f_2^B(b)\,f_3^B(b)\ \mathrm{d}^\times b\neq 0$$

where π_i^B is the representation of $B^\times(\mathbb{A})$ corresponding to π_i.

Summary. The proof comes down to the equality

$$L\left(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3\right)\cdot Z^*(F,\Phi)$$

$$=\sum_r\int_{\mathbb{A}^\times B^\times(k)\backslash B^\times(\mathbb{A})} I^{1,r}(b_1,\varphi;F)\ \mathrm{d}b_1\cdot\int_{\mathbb{A}^\times B^\times(k)\backslash B^\times(\mathbb{A})} I^{2,r}(b_2,\varphi;F)\ \mathrm{d}b_2$$

The integrals in the second expression are finite linear combinations of the quantities $I\left(f_1^B,f_2^B,f_3^B\right)$ and every such quantity can be obtained as an integral $\int_{\mathbb{A}^\times B^\times(k)\backslash B^\times(\mathbb{A})} I^{1,r}(b_1,\varphi;F)\ \mathrm{d}b_1$ for some φ,F and r.

Section 1 introduces a representation of the triple product L-function in terms of a Zeta integral and the integral of an Eisenstein series. Section 2 shows how the integral representation should be modified by changing the Zeta integral at the local factors. Section 3 constructs

1 The integral representation of the triple product L-function.

- Let $G = \mathrm{GSp}_6$ be the group of similitudes of the standard 6 dimensional symplectic vector space over k, and let $P = MN$ be the Siegel parabolic.

- Let $K_G = K_{G,\infty} \cdot K_{G,f}$ be the standard maximal compact subgroup of $G(\mathbb{A})$.

- λ_s denotes a character of $P(\mathbb{A})$ parameterized by $s \in \mathbb{C}$.

- Let $I(s) = I_P^G(\lambda_s)$ be the normalized induced representation of $G(\mathbb{A})$.

- The Eisenstein series associated to a section $\Phi_s \in I(s)$ is defined for $\mathrm{Re}(s) > 1$ by

$$E(g, s, \Phi_s) = \sum_{\gamma \in P(k) \backslash G(k)} \Phi_s(\gamma g),$$

  and the normalized Eisenstein series is

$$E^*(g, s, \Phi_s) = b_G(s) \cdot E(g, s, \Phi_s),$$

  where $b_G(s)$ is a function of s. These functions have meromorphic analytic continuations to the whole s-plane and have no poles on the unitary axis $\mathrm{Re}(s) = 0$.

- The map

$$E^*(0) : I(0) \to \mathcal{A}(G), \ \Phi_0 \mapsto (g \mapsto E^*(g, 0, \Phi_s))$$

  gives a $(g_\infty, K_{G,\infty}) \times G(\mathbb{A}_f)$-intertwining map from the induced representation $I(0)$ at $s = 0$ to the space of automorphic forms on G with trivial central character.

- Let

$$\begin{aligned} \mathbf{G} &= (\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2)_0 \\ &= \left\{ (g_1, g_2, g_3) \in (\mathrm{GL}_2)^3 \,|\det(g_1) = \det(g_2) = \det(g_3) \right\}. \end{aligned}$$

  This group embeds diagonally in $G = \mathrm{GSp}_6$.

- For automorphic forms $f_i \in \pi_i, i = 1, 2, 3$, let $F = f_1 \otimes f_2 \otimes f_3$ be the corresponding function on $G(\mathbb{A})$. The global zeta integral is given by

$$Z(s, F, \Phi_s) = \int_{Z_G G(k) \backslash G(\mathbb{A})} E^*(g, s, \Phi_s) F(g) \, \mathrm{d}g.$$

- Suppose that the automorphic forms $f_i \in \pi_i$ have factorizable Whittaker functions $W_i^{\psi} = \otimes_v W_{i,v}^{\psi}$ and that the section Φ_s is factorizable. Let S be a finite set of places of k, including all archimedean places, such that, for $v \notin S$, the local data satisfies properties (i), (ii), and (iii) in the main text. Then

$$Z(s, F, \Phi_s) = L^S\left(s + \frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3\right) \cdot \prod_{v \in S} Z_v\left(s, W_v^{\psi}, \Phi_{s,v}\right),$$

for local zeta integrals $Z_v\left(s, W_v^{\psi}, \Psi_{s,v}\right)$, where $W_v^{\psi} = W_{1,v}^{\psi} W_{2,v}^{\psi} W_{3,v}^{\psi}$. $L^S(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ is the triple product L-functions with the factors for $v \in S$ omitted.

- The local zeta integrals are defined by

$$Z_v\left(s, W_v^{\psi}, \Phi_{s,v}\right) = \int_{Z_G(k_v) M(k_v) \backslash G(k_v)} \Phi_{s,v}(\delta g) W_v^{\psi}(g) \, \mathrm{d}g$$

where $\delta \in G(k)$ is a representative for the open orbit of G in $P \backslash G$.

2 Local zeta integrals.

- The key result is the existance of a local Euler factor. Let $\pi_{i,v},\ i = 1, 2, 3$, be a triple of admissible irreducible representations of $\mathrm{GL}(2, k_v)$ that arise as local components at v of cuspidal automorphic representations π_i. The quotient

$$\tilde{Z}_v\left(s, W_v^{\psi}, \Psi_{s,v}\right) = Z_v\left(s, W_v^{\psi}, \Phi_{s,v}\right) \cdot L\left(s + \frac{1}{2}, \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}\right)^{-1}$$

is entire as a function of s.

- Consequently, we have the identity

$$L\left(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3\right)\cdot\prod_{v\in S}Z_v^*\left(0,W_v^\psi,\Phi_{s,v}\right)$$
$$=\int_{Z_G(\mathbb{A})G(k)\backslash G(\mathbb{A})}E^*\left(g,0,\Psi_s\right)F\left(g\right)\,\mathrm{d}g,\quad (1)$$

where

$$Z_v^*\left(s,W_v^\psi,\Phi_{s,v}\right)=\begin{cases}\tilde{Z}_v\left(s,W_v^\psi,\Phi_{s,v}\right) & \text{if } v\in S_f,\\ Z_v\left(s,W_v^\psi,\Phi_{s,v}\right) & \text{if } v\in S_\infty.\end{cases}$$

We make the abbreviation

$$Z^*\left(F,\Phi\right)=\prod_{v\in S}Z_v^*\left(0,W_v^\psi,\Phi_{s,v}\right).$$

3 The Weil representation for similitudes.

- Let B be a quaternion algebra over k, and let $V=B$ be a 4 dimensinal quadratic space over k where the quadratic form is given by $Q\left(x\right)=\alpha v\left(x\right)$, where v is the reduced norm on B and $\alpha\in k^\times$
- Let $H=\mathrm{GO}\left(V\right)$ and $H_1=\mathrm{O}\left(V\right)$. Let $G=\mathrm{GSp}_6$ and let $G_1=\mathrm{Sp}_6$.
- Let
$$R=\left\{(h,g)\in H\times G\mid v\left(h\right)=v\left(g\right)\right\}.$$
There exists an extension of the standard Weil representation $\omega=\omega_\psi$ of $H_1\left(\mathbb{A}\right)\times G_1\left(\mathbb{A}\right)$ on the Schwartz space $S\left(V\left(\mathbb{A}\right)^3\right)$ to $R\left(\mathbb{A}\right)$.
- Let
$$G\left(\mathbb{A}\right)^+:=\left\{g\in G\left(\mathbb{A}\right)\mid v\left(g\right)\in v\left(H\left(\mathbb{A}\right)\right)\right\}$$
$$=\left\{g\in G\left(\mathbb{A}\right)\mid v\left(g\right)_v>0,\ \forall v\in\Sigma_\infty\left(V\right)\right\}.$$
- For $g\in G\left(\mathbb{A}\right)^+$, and $\varphi\in S\left(V\left(\mathbb{A}\right)^3\right)$, and for V anisotropic over k, the theta integral is defined by
$$I\left(g,\varphi\right)=\int_{H_1(k)\backslash H_1(\mathbb{A})}\theta\left(h_1h,g;\varphi\right)\,\mathrm{d}h_1,$$
where $h\in H\left(\mathbb{A}\right)$ with $v\left(h\right)=v\left(g\right)$. An alternate definition has to made if V is not anisotropic over k. $I\left(g,\varphi\right)$ has a unique extension to a left $G\left(k\right)$-invariant function on $G\left(\mathbb{A}\right)$.

4 The Siegel-Weil formula for $(\mathbf{GO}(V), \mathbf{GSp}_6)$.

5 Proof of Jacquet's conjecture.

- Applying the Siegel-Weil formula for similitudes to the basic identity (1), we obtain
$$\begin{aligned} &L\left(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3\right) \cdot Z^*(F, \Phi) \\ &= \int_{Z_G(\mathbb{A})G(k)\backslash G(\mathbb{A})} E^*(g, 0, \Psi_s) F(g)\, \mathrm{d}g \\ &= 2\zeta_k(2)^2 \sum_V \int_{Z_G(\mathbb{A})G(k)\backslash G(\mathbb{A})} I\left(g, \varphi^V\right) F(g)\, \mathrm{d}g. \end{aligned}$$

- Next, we want to apply the seesaw identity. We set
$$H = \mathrm{GO}(V)$$
$$\mathbf{H} = \left\{(h_1, h_2, h_3) \in H^3 | v(h_1) = v(h_2) = v(h_3)\right\}$$
For F a cuspidal automorphic form on $\mathbf{G}(\mathbb{A})$ and for $h \in \mathbf{H}(\mathbb{A})$, let
$$I(h, \varphi; F) = \int_{\mathbf{G}_1(k)\backslash \mathbf{G}_1(\mathbb{A})} \theta(h, \mathbf{g}_1\mathbf{g}; \varphi) F(\mathbf{g}_1\mathbf{g})\, \mathrm{d}\mathbf{g}_1.$$
The seesaw identity is
$$\int_{Z_G(\mathbb{A})\mathbf{G}(k)\backslash \mathbf{G}(\mathbb{A})} I(g, \varphi) F(g)\, \mathrm{d}g = \int_{Z_H(\mathbb{A})H(k)\backslash H(\mathbb{A})} I(h, \varphi; F)\, \mathrm{d}h.$$

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