Notes on Local Theta Correspondence (Gan, Kudla Takeda) - Chapter 1 - Heisenberg Representation

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$(W, \langle \cdot, \cdot \rangle)$ is a nondegenerate symplectic space. F is a non-archimedean local field or a finite field of characteristic differnt from 2. The Heisenberg group associated to W is defined is the set

$$H(W) := W \times F$$

with multiplication defined by

$$(w, t) \cdot (w', t') = \left(w + w', t + t' + \frac{1}{2} \langle w, w' \rangle \right).$$

$H(W)$ is given the product topology. We verify that $H(W)$ is a topological group.

The multiplication law is associative

$$((w_1, t_1)(w_2, t_2))(w_3, t_3)$$

$$= \left(w_1 + w_2, t_1 + t_2 + \frac{1}{2} \langle w_1, w_2 \rangle \right) (w_3, t_3)$$

$$= \left(w_1 + w_2 + w_3, t_1 + t_2 + t_3 + \frac{1}{2} \langle w_1, w_2 \rangle + \frac{1}{2} \langle w_1 + w_2, w_3 \rangle \right).$$

$$= \left(w_1 + w_2 + w_3, t_1 + t_2 + t_3 + \frac{1}{2} \langle w_1, w_2 \rangle + \frac{1}{2} \langle w_1, w_3 \rangle + \frac{1}{2} \langle w_2, w_3 \rangle \right)$$

$$= \left(w_1 + w_2 + w_3, t_1 + t_2 + t_3 + \frac{1}{2}\langle w_1, w_2 + w_3\rangle + \frac{1}{2}\langle w_2, w_3\rangle\right)$$

$$(w_1, t_1)\left(w_2 + w_3, t_2 + t_3 + \frac{1}{2}\langle w_2, w_3\rangle\right)$$

$$(w_1, t_1)\left((w_2, t_2)(w_3, t_3)\right).$$

The idetity element is $(0, 0)$;

$$(0, 0)(w, t) = \left(0 + w, 0 + t + \frac{1}{2}\langle 0, t\rangle\right) = (w, t).$$

The inverse of (w, t) is $(-w, -t)$;

$$(w, t)(-w, -t) = \left(w - w, t - t + \frac{1}{2}\langle w, -w\rangle\right) = (0, 0).$$

It is clear that the operations of multiplication and inversion are continuous, making $H(W)$ into a topological group.

We show the center of the Heisenberg group is

$$Z_{H(W)} = \{(0, t) : t \in F\} \cong F.$$

$Z_{H(W)}$ is a subgroup isomorphic to F since

$$(0, t)(0, t') = (0, t + t').$$

$Z_{H(W)}$ is in the center of $H(W)$ because

$$(0, t)(w', t') = (w', t + t') = (w', t')(0, t).$$

Suppose that (w, t) is an element of $H(W)$ but $w \neq 0$. Since $\langle\cdot,\cdot\rangle$ is nondegenerate, there exists $w' \in W$ such that $\langle w, w'\rangle \neq 0$. We have

$$(w, t)(w', -t) = \left(w + w', \frac{1}{2}\langle w, w'\rangle\right)$$

and

$$(w', -t)\,(w, t) = \left(w + w', \frac{1}{2}\langle w', w\rangle\right) = \left(w + w', -\frac{1}{2}\langle w, w'\rangle\right).$$

Hence, (w, t) is not in the center.
We verify the identities

$$(w + w', 0) = \left(w + w', -\frac{1}{2}\langle w, w'\rangle + \frac{1}{2}\langle w, w'\rangle\right) = \left(w, -\frac{1}{2}\langle w, w'\rangle\right)(w', 0)$$

and

$$(w, 0)\,(w', 0) = \left(w + w', \frac{1}{2}\langle w, w'\rangle\right) = \left(w + w', -\frac{1}{2}\langle w', w\rangle\right)$$

$$= \left(w + w', -\langle w', w\rangle + \frac{1}{2}\langle w', w\rangle\right) = \left(w + w', \langle w, w'\rangle + \frac{1}{2}\langle w', w\rangle\right)$$

$$= (w', \langle w, w'\rangle)\,(w, 0)$$

for all $w, w' \in W$ and $t \in F$.