

Rankin-Selberg Method

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Let $f(z), g(z)$ be modular forms and let $A(n), B(n)$ be their Fourier coefficients. Define the Eisenstein series for $SL(2, \mathbb{Z})$ by

$$E(z, s) = \pi^{-s} \Gamma(s) \zeta(2s) \sum_{\Gamma_\infty \backslash \Gamma(1)} \text{im}(\gamma(z))^s$$

Also, define

$$\Lambda(s) = 4^{-s-k+1} \pi^{-2s-k+1} \Gamma(s) \Gamma(s+k-1) \zeta(2s) \sum_{n=1}^{\infty} A(n) B(n) n^{-s-k+1}$$

We show (*)

$$\Lambda(s) = \int_{\Gamma(1) \backslash \mathcal{H}} f(z) \bar{g}(z) E(z, s) y^k \frac{dx dy}{y^2}$$

First note that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ we have

$$f(\gamma(z)) g(\gamma(z)) \text{im}(\gamma(z))^k = (cz+d)^k (\bar{c}z+d)^k f(z) g(z) \frac{y^k}{|cz+d|^{2k}} = f(z) g(z) y^k$$

hence $f(z) \bar{g}(z) y^k$ is invariant under fractional linear transformations. Also, $E(z, s)$ is automorphic in z . Finally, it is easy to check that the differential $\frac{dx dy}{y^2}$ is invariant under linear fractional transformations by checking it for the generators S and T of $\Gamma(1)$. Hence, the integral is well-defined.

The right hand side of (*) is equal to

$$\pi^{-s} \Gamma(s) \zeta(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \int_{\Gamma(1) \backslash \mathcal{H}} f(\gamma(z)) \bar{g}(\gamma(z)) \text{im}(\gamma(z))^{s+k} \frac{dx dy}{y^2}$$

$$= \pi^{-s} \Gamma(s) \zeta(2s) \int_{\Gamma_\infty \backslash \mathcal{H}} f(z) \bar{g}(z) y^{s+k} \frac{dx \, dy}{y^2}$$

Now,

$$\begin{aligned} & \int_{\Gamma_\infty \backslash \mathcal{H}} f(z) \bar{g}(z) y^{s+k} \frac{dx \, dy}{y^2} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A(n) \bar{B}(m) \int_{\Gamma_\infty \backslash \mathcal{H}} e^{2\pi i(n-m)x} e^{-2\pi(n+m)y} y^{s+k} \frac{dx \, dy}{y^2} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A(n) \bar{B}(m) \int_0^\infty \int_0^1 e^{2\pi i(n-m)x} e^{-2\pi(n+m)y} y^{s+k-1} \, dx \, \frac{dy}{y} \\ &= \sum_{n=1}^{\infty} A(n) \bar{B}(n) \int_0^\infty e^{-4\pi n y} y^{s+k-1} \frac{dy}{y} \end{aligned}$$

Making the substitution $y = \frac{t}{4\pi n}$ this is

$$= (4\pi)^{-s-k+1} \Gamma(s+k-1) \sum_{n=1}^{\infty} A(n) \bar{B}(n) n^{-s-k+1}$$

Finally, $B(n) = \bar{B}(n)$ because the $B(n)$ are eigenvalues of the Hecke operators, which are self adjoint. This proves (*).