Notes on "The Theta Correspondence for Similitudes" By Brooks Roberts

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February 4, 2025

Fixing notation.

- k a nonarchimedean local field.
- X a finite dimensional nondegenerate symmetric bilinear space over k.
- Y a finite dimensional nondegenerate symplectice bilinear space over k.
- p the projection $\mathrm{Mp}\left(X \otimes_k Y\right) \to \mathrm{Sp}\left(X \times_k Y\right)$.
- r the smooth Weil representation of $\mathrm{Mp}\left(X \times_k Y\right)$ (corresponding to some fixed nontrivial additive character of k).

The theta correspondence.
The restriction of r to $p^{-1}\left(\mathrm{O}\left(X\right)\right) p^{-1}\left(\mathrm{Sp}\left(Y\right)\right)$ definites a correspondence between the smooth admissible duals of $p^{1}\left(\mathrm{O}\left(X\right)\right)$ and $p^{-1}\left(\mathrm{Sp}\left(Y\right)\right)$. When the residue characteristic of k is odd, this correspondence satisfies strong Howe duality.

Biblographic notes for ease of reference.
The Weil representation was first constructed in Weil (1964) "Sur certain groupes d'opérateurs unitaires." The theta correspondence was introduced in Howe (1979) "θ-series and invariant theory." Howe duality was proved for archimedean local fields in Howe (1989) "Transcending classical invariant theory" and for p-adic fields with p odd in Waldspurger (1990) "Démonstration d'une conjecture de dualité de Howe danse le case p-adique, $p \neq 2$." A classic reference (in French) is MVW (1987) "Correspondance de Howe sur un corps p-adique." A useful set of notes is Kulda (1996) "Notes on the Local Theta

Correspondence" and Gan's AWS notes. A very comprehensive modern textbook is GKT "The Local Theta Correspondence."

Summary of the paper.

The two main approaches to a theta correspondence for similitudes are "essentially the same" and a version of strong Howe duality holds for both constructions. The two main constructions are:

- Extend the restriction of r to $p^1(\mathrm{O}(X))\,p^{-1}(\mathrm{Sp}(Y))$ to a representation ω of a larger group involving similitudes.
- Induce the restriction of r to $p^{-1}(\mathrm{Sp}(Y))$ to obtain a representation Ω that involves similitues.

More notation.

- $H = \mathrm{GO}(X)$.
- If $\dim_k X$ is even, let $G' = \mathrm{GSp}(Y)$.
- If $\dim_k X$ is odd, let G' be a certain two-fold cover of $\mathrm{GSp}(Y)$.
- For $g \in G'$, let $\lambda(g)$ be the similitude factor of the projection of g to $\mathrm{GSp}(Y)$. Let $\lambda(h)$ be the similitude factor of $h \in H$.
- Let G be the subgroup of $g \in G'$ such that $\lambda(g) \in \lambda(H)$ (recall that the map λ is surjective on $\mathrm{GSp}(Y)$ but not necessarily on $\mathrm{GO}(X)$).
- Let G_1 and H_1 be the subgroups of $g \in G$ and $h \in H$ such that $\lambda(g) = 1$ and $\lambda(h) = 1$, respectively (so H_1 is just $\mathrm{O}(X)$ and G_1 is $\mathrm{Sp}(Y)$ in case $\dim_k X$ is even.
- ω is a representation of the group

$$R = \{(g,h) \in G' \times H : \lambda(g) = \lambda(h)\}$$

  and Ω is a representation of $G' \times H$.

Detailed summary.

- Section 1 defines Howe duality, multiplication preservation, and strong Howe duality. It also shows that Howe duality and multiplication preservation are equivalent to strong Howe duality.

- Sections 2 and 3 construct and relate ω and Ω. Ω is obtained from ω via compact induction:

$$\Omega \cong \text{c-Ind}_R^{G' \times H} \omega.$$

- Section 4 studies the correspondence defined by ω. In particular, it proves that the analogues of Howe duality and multiplicity preservation hold.

- Sections 5 and 6 consider the consequences of section 4 for Ω. It gives a condition for Howe duality called theta dichotomy. It shows that Howe duality does not hold for Ω in the stable range. When $\dim_k X \leq \dim_k Y$, strong Howe duality for Ω is expected to hold.

Some more notation.

- Let J be a group of td-type. This means that J is a topological group and every neighborhood of the identity element of J contains a compact open subgroup. For such groups, Schur's lemma holds.

- Let $\mathrm{Irr}(J)$ be the set of equivalence classes of smooth admissible irreducible representations of J.

- If $\pi \in \mathrm{Irr}(J)$ then $\pi^\vee \in \mathrm{Irr}(J)$ denotes its contragredient representation.

- A character of J is a continuous homomorphism from J to $\mathbb{C}^\times$.

- If L is a closed normal subgroup of J, $\pi \in \mathrm{Irr}(L)$ and $g \in J$, then $g\pi \in \mathrm{Irr}(L)$ is the representation with the same space as π and action defined by $(g\pi)(h) = \pi(g^{-1}hg)$, and J_π is the subgroup of $g \in J$ such that $g\pi \cong \pi$.

- Let $(\ ,\ )_k$ denote the Hilbert symbol of k.

1 Howe duality and multiplicity preservation

Existence and uniquenss of the big theta lift.

Let A and B be groups of td-type, with countable bases. Let $(\rho, \mathcal{U})$ be a smooth representation of $A \times B$. Let $\pi \in \mathrm{Irr}(A)$. Define

$$\mathcal{U}(\pi) = \mathcal{U} / \bigcap_{t \in \mathrm{Hom}_A(p,\pi)} \ker(t).$$

Via ρ, $A \times B$ acts on $\mathcal{U}(\pi)$. Call this representation $\rho(\pi)$. By [MVW] there exists a smooth representation $\Theta(\pi)$ of B, unique up to isomorphism, such that

$$\rho(\pi) \cong \pi \otimes_{\mathbb{C}} \Theta(\pi)$$

as representations of $A \times B$. Analogous remarks apply for elements of $\mathrm{Irr}(B)$.

Strong Howe duality.

Let $\mathcal{R}(A)$ be the set of equivalence classes of $\pi \in \mathrm{Irr}(A)$ such that $\mathcal{U}(\pi) \neq 0$ and define $\mathcal{R}(B)$ similarly. We say that strong Howe duality holds for ρ if for every $\pi \in \mathcal{R}(A)$ the representation $\Theta(\pi)$ has a unique nonzero irreducible quotient $\theta(\pi) \in \mathcal{R}(B)$, and for every $\tau \in \mathcal{R}(B)$ the representation $\Theta(\tau)$ has a unique nonzero irreducible quotient $\theta(\tau) \in \mathcal{R}(A)$.

Howe duality. We say that Howe duality holds for ρ if the set

$$\mathcal{R}(A \times B) = \{(\pi, \tau) \in \mathcal{R}(A) \times \mathcal{R}(B) : \mathrm{Hom}_{A \times B}(\rho, \pi \otimes_{\mathbb{C}} \tau) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(A)$ and $\mathcal{R}(B)$. Equivalently, Howe duality holds for ρ if and only if (1) every $\pi \in \mathcal{R}(A)$ occurs as the first entry of an element of $\mathcal{R}(A \times B)$ and every $\tau \in \mathcal{R}(B)$ occurs as the second entry of an element of $\mathcal{R}(A \times B)$; and (2) for all $\pi \in \mathrm{Irr}(A)$ and $\tau_1, \tau_2 \in \mathrm{Irr}(B)$,

$$\mathrm{Hom}_{A \times B}(\rho, \pi \otimes_{\mathbb{C}} \tau_1) \neq 0,\ \mathrm{Hom}_{A \times B}(\rho, \pi \otimes_{\mathbb{C}} \tau_2) \neq 0 \implies \tau_1 \equiv \tau_2;$$

for all $\pi_1, \pi_2 \in \mathrm{Irr}(A)$ and $\tau \in \mathrm{Irr}(B)$,

$$\mathrm{Hom}_{A \times B}(\rho, \pi_1 \otimes_{\mathbb{C}} \tau) \neq 0,\ \mathrm{Hom}_{A \times B}(\rho, \pi_2 \otimes_{\mathbb{C}} \tau) \neq 0 \implies \pi_1 \cong \pi_2.$$

Multiplicity preservation. We show that multiplicity preservation holds for ρ if for all $\pi \in \mathrm{Irr}(A)$ and $\tau \in \mathrm{Irr}(B)$,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{A \times B}(\rho, \pi \otimes_C \tau) \leq 1.$$

Proposition 1.1.

Strong Howe duality holds for ρ if and only if Howe duality and multiplicity preservation hold for ρ. If strong Howe duality holds for ρ, then the map $\theta : \mathcal{R}(A) \to \mathcal{R}(B)$ is the bijection given by Howe duality.

Proposition 1.2. (used in section 6)

Assume that A is contained in a group A' of td-type with countable basis as a closed normal subgroup of index two. Let a be a representative for the nontrivial coset of A'/A. Let $\rho' = \operatorname{Ind}_{A\times B}^{A'\times B}\rho$. All of the above definitions apply with ρ' in place of ρ. If Howe duality holds for ρ, then Howe duality holds for ρ' if and only if $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A) = \varnothing$. If strong Howe duality holds for ρ, then strong Howe duality holds for ρ' if and only if $\mathcal{R}(A) \cap a \cdot \mathcal{R}(A)$.

2 The groups

Polarization of symplectic space. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be a finite dimensional nondegenerate symplectic vector space over k. Assume $\mathbb{W} \neq 0$. There exists a basis $\mathbb{W}$ relative to which the symplectic form has a the matrix representation

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and $2n = \dim(\mathbb{W})$. Take $\mathbb{U}$ to be the subspace generated by the first n vectors in this basis (the symplectic form restricted to $\mathbb{U}$ is 0). We see that $\mathbb{U}^*$ can be identified with the subspace generated by the remaining n vectors. Hence, $\mathbb{W} \cong \mathbb{U} \oplus \mathbb{U}^*$. We write elements of $\mathrm{GL}(\mathbb{W})$ with repsect to this basis.

Definition and structure of GSp $(\mathbb{W})$**.** Let $\mathrm{GSp}(\mathbb{W})$ be the subgroup of $\mathrm{GL}(\mathbb{W})$ such that there exists $\lambda \in k^\times$ such that $\langle\!\langle gw, gw' \rangle\!\rangle = \lambda \langle\!\langle w, w' \rangle\!\rangle$ for all $w, w' \in \mathbb{W}$. If $g \in \mathrm{GSp}(\mathbb{W})$, then such a λ is unique, and it will be denoted $\lambda(g)$. Let $\mathrm{Sp}(\mathbb{W})$ be the subgroup of $g \in \mathrm{GSp}(\mathbb{W})$ such that $\lambda(g) = 1$. λ is a homomorphism and the map

$$y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$$

is a splitting for the short exact sequence

$$1 \to \mathrm{Sp}(\mathbb{W}) \to \mathrm{GSp}(\mathbb{W}) \to k^\times \to 1.$$

Hence, $\mathrm{GSp}(\mathbb{W}) \cong k^{\times} \ltimes \mathrm{Sp}(\mathbb{W})$ and $k^{\times}$ acts on $\mathrm{Sp}(\mathbb{W})$ by conjugation.

The metaplectic similitude group.

We denote by $\mathrm{GMp}(\mathbb{W})$ the metaplectic cover of $\mathrm{GSp}(\mathbb{W})$, constructed in more detail in the text. There is also a two-fold cover of $\mathrm{GSp}(\mathbb{W})$ denoted $\widehat{\mathrm{GSp}(\mathbb{W})}$ and an inclusion of $\widehat{\mathrm{GSp}(\mathbb{W})}$ in $\mathrm{GMp}(\mathbb{W})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GMp}(\mathbb{W}) & \longrightarrow & \mathrm{GSp}(\mathbb{W}) \\ \uparrow & & \uparrow \\ \widehat{\mathrm{GSp}(\mathbb{W})} & \longrightarrow & \mathrm{GSp}(\mathbb{W}). \end{array}$$

GO(X) and GSp(Y).

Let $(X, (\ ,\ ))$ be a nondegenearte symmetric bilinear space over k of dimension m, and let $(Y, \langle\ ,\ \rangle)$ be a nondegenerate symplectic bilinear space over k of dimension $2n$. For the remainder we will assume that

$$(\mathbb{W}, \langle\!\langle\ ,\ \rangle\!\rangle) = (X, (\ ,\ )) \otimes_k (Y, \langle\ ,\ \rangle),$$

and there is a complete polarization $Y = U \oplus U^{*}$ such that $\mathbb{U} = X \otimes_k U$ and $\mathbb{U}^{*} = X \otimes_k U^{*}$. Let $\mathrm{GO}(X)$ be the similitude group of X defined in a similar way as for symplectic spaces. There are inclusions

$$\mathrm{GO}(X) \to \mathrm{GSp}(\mathbb{W}),\ \mathrm{GSp}(Y) \to \mathrm{GSp}(\mathbb{W}).$$

The elements of $p^{-1}(\mathrm{GO}(X))$ and $p^{-1}(\mathrm{GSp}(Y))$ in general do not commute. We have the following results on the structure of these groups.

Proposition 2.1.

If m is even, then $p^{-1}(\mathrm{GSp}(Y))$ is trivial as an extension of $\mathrm{GSp}(Y)$ by $\mathbb{C}^1$. If m is odd, then $p^{-1}(\mathrm{GSp}(Y))$ is the metaplectic cover of $\mathrm{GSp}(Y)$.

Proposition 2.3. Assume that the residual characteristic of k is odd. If m is odd, $m = 2$, or $m = 4$ and X is anisotropic, then $p^{-1}(\mathrm{GO}(X))$ is trivial as an extension of $\mathrm{GO}(X)$ by $\mathbb{C}^1$. If m is even, $m \geq 4$ and X is not four dimensional and anisotropic, then $p^{-1}(\mathrm{GO}(X))$ is trivial if and only if the character of $\mathrm{GSO}(X)$ defined by $h \mapsto (-1, \lambda(h))_k^n$ is trivial.

3 The representations

Proposition 3.1. Consider the subgroup of $\mathrm{GSp}(\mathbb{W})$ generated by the elements

$$\begin{pmatrix} \lambda(h)^{-1} & 0 \\ 0 & 1 \end{pmatrix} (h \otimes 1)$$

for $h \in \mathrm{GO}(X)$. This subgroup is isomorphic to $\mathrm{GO}(X)$. Its preimage under p is trivial as an extension of $\mathrm{GO}(X)$ by $\mathbb{C}^1$.

Lemma 3.2. (Fundamental identity).

Let L be a splitting of the preimage of the subgroup from Proposition 3.1. If $h \in \mathrm{GO}(X)$ and $g \in p^{-1}(\mathrm{Sp}(Y))$ then

$$L(h)\, g L(h)^{-1} = g^{\lambda(h)^{-1}}.$$

Proposition 3.3. (Shimizu-Harris-Kudla).

Define an action of H on G_1 by $h \cdot g = g^{\lambda(h)^{-1}}$ and form the semidirect product $G_1 \rtimes H$. The map $G_1 \rtimes H \to \mathrm{Mp}(\mathbb{W})$ defined by $(g,h) \mapsto gL(h)$ is a homomorphism. Let

$$R = \{(g,h) \in G' \times H : \lambda(g) = \lambda(h)\}.$$

The map $R \to G_1 \rtimes H$ defined by

$$(g,h) \mapsto \left(gd(\lambda(g))^{-1}, h\right)$$

is a homomorphism. Thus, the composition ω

$$R \to G_1 \rtimes H \to \mathrm{Mp}(\mathbb{W}) \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{S})$$

is a homomorphism. This representation is smooth.

ω is called the **extended Weil representation associated to X and Y**. We have

$$\omega(g,h) = r(gh).$$

Proposition 3.4. Let $\Omega = \text{c-Ind}_{G_1}^{G'} r$. For each $h \in H$, define an operator $\Omega(h)$ on the space $\mathcal{T}$ of Ω by

$$(\Omega(h) f)(g) = r(L(h)) \cdot f\left(d(\lambda(h))^{-1} g\right).$$

Then the map

$$\Omega : G' \times H \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{T})$$

defined by $(g,h) \mapsto \Omega(g)\Omega(h)$ is a homomorphism. This representation is smooth. We call Ω the **induced Weil representation**.

Proposition 3.5. We have

$$\Omega \cong \text{c-Ind}_R^{G' \times H} \omega.$$

The group G and the representation Ω^+. Let G be the group of $g \in G'$ such that $\lambda(g) \in \lambda(H)$. A smooth representation $(\Omega^+, \mathcal{T}^+)$ of $\Omega|_G$ is defined, and turns out to be isomorphic to $\text{c-Ind}_R^{G \times H} \omega$.