L-functions for GL(2)

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The case of modular forms. Before discussing L-functions for GL(2), let us first recall L-functions for cusp forms, so that we can draw analogies between the two theories. Let f be a cusp form of level 1 and weight k with Fourier expansion

$$f(z) = \sum a_n e^{2\pi i n z}.$$

There is an associated L-function

$$L(s, f) = \sum a_n n^{-s}$$

which can be recognized as a Mellin transform of f

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f) = \int_0^\infty f(iy)y^s \, \mathrm{d}^*y.$$

This is seen by substituting the Fourier expansion for f,

$$\Lambda(s, f) = \int_0^\infty f(iy)y^s \, d^*y = \int_0^\infty \sum a_n e^{-2\pi n y} y^s \, \mathrm{d}^*s.$$

Swapping the integral and sum and making the substitution $2\pi n y \mapsto y$, we get

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s) \sum a_n n^{-s}$$

Convergence of the integral and the validity of these operations follows from the "moderate growth" properties of f. In case f is a normalized Hecke eigenform, we also have an Euler product

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s) \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}.$$

The functional equation for $\Lambda(s, f)$ follows from the functional equation $f(iy) = (-1)^{k/2} y^{-k} f(i/y)$ in the following way

$$\Lambda(s, f) = \int_0^\infty f(iy) y^s \, \mathrm{d}^* s = (-1)^{k/2} \int_0^\infty f(i/y) y^{s-k} \, \mathrm{d}^* s.$$

The substitution $y \mapsto y^{-1}$ gives

$$\Lambda(s, f) = (-1)^{k/2} \Lambda(k - s, f).$$

In fact, the functional equation for f is *equivalent* to the functional equation for Λ, in a sense which is made precise by the *converse theorem*:

Let a_n be a sequence of complex numbers such that $|a_n| = O(n^k)$ for some real number k. Let $f(z) = \sum a_n e^{2\pi i n z}$. If $\Lambda(s, f)$ has analytic continuation to all s, is bounded in every vertical strip $\sigma_1 \le \mathrm{re}(s) \le \sigma_2$, and satisfies $\Lambda(s, f) = (-1)^{k/2} \Lambda(k - s, f)$, then f is a cusp form of level 1 and weight k. If the Euler product is valid, f is a normalized Hecke eigenform.

All of the above result can be generalized to the case of "twisting" by a multiplicative character χ.

Local Whittaker models and local multiplicity one. The analogue of Fourier expansion in the theory of automorphic forms comes from Whittaker models. Let F be a non-Archimedean local field, ψ a nontrivial additive character of F, and let (π, V) be an irreducible admissible reprensetation of $\mathrm{GL}(2, F)$. There exists at most one space $\mathcal{W}$ of functions in $\mathrm{GL}(2, F)$ such that if $W \in \mathcal{W}$, then

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi(x) W(g) \;\; x \in F, g \in \mathrm{GL}(2, F)$$

and such that W is closed under right translation by elements of $\mathrm{GL}(2, F)$, and the resulting reprensetation of $\mathrm{GL}(2, F)$ is isomorphic of π. Such a space of function $\mathcal{W}$ is called a *Whittaker model* for (π, V). There is a similar definition for Archimedean local fields, which I will omit. The local multiplicity one theorem holds in the Archimedean case as well.

Whittaker models. Let F be a global field and A its adele ring. We fix ψ a nontrivial character of A/F. ψ decomposes into local characters ψ_v (c.f. Tate's thesis, section 4.1.). Let π be an irreducible admissible representation of $\mathrm{GL}(2, A)$. A *Whittaker model* of π with respect to the nontrivial character ψ is a space of functions of $\mathrm{GL}(2, A)$ which are

(*i*) smooth,
(*ii*) K-finite,
(*iii*) of moderate growth,
(*iv*) and satisfying

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)W(g)$$

for all $x \in A$.

Multiplicity one. (π, V) has a Whittaker model $\mathcal{W}$ with respect to ψ if and only if each (π_v, V_v) has a Whittaker model W_v with respect to the character ψ_v of F_v. If this is the case, the $\mathcal{W}$ is unique and consists of all finite linear combinations of functions of the form $W(g) = \prod_v W_v(g_v)$ where $W_v \in \mathcal{W}_v$, and $W = W_v^\circ$ for almost all v, where W_v° spherical elements of $\mathcal{W}_v$, normalized so $W_v^\circ(k_v) = 1$ for $k_v \in \mathrm{GL}(2, \mathfrak{o}_v)$.

Existence of Whittaker models for automorphic cuspidcal representations. Now suppose that π is an automorphic cuspidal representation. If $\phi \in V$ and $g \in \mathrm{GL}(2, A)$, let

$$W_\phi(g) = \int_{A/F} \phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi(-x)\,\mathrm{d}x.$$

Then the space $\mathcal{W}$ of functions W_ϕ is a Whittaker model for π. We have the "Fourier expansion"

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right).$$

The global zeta integral. Z is analogous to Λ from the case of modular forms. It is the "completed global L-function." For $\phi \in V$,

$$\phi\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}$$

is rapidly decreasing as $|y| \to \infty$. That is, for any $N > 0$ there exists a constant C_N such that

$$\phi\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} < C_N|y|^{-N}.$$

Hence,

$$Z(s,\phi) = \int_{A^\times/F^\times} \phi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s-1/2} \, \mathrm{d}^* y$$

is absolutely convergent for all values of s. By the Fourier expansion, we have

$$Z(s,\phi) = \int_{A^\times/F^\times} \sum_{\alpha \in F^\times} W_\phi \begin{pmatrix} \alpha y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s-1/2} \, \mathrm{d}^* y = \int_{A^\times} W_\phi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s-1/2} \, \mathrm{d}^* y$$

provided this integral is absolutely convergent. Absolute converges occurs when $\mathrm{re}(s) > 3/2$. We may write $W(g) = \prod_v W_v(g_v)$ and suppose that the vector ϕ corresponds to a pure tensor $\otimes_v \phi_v$. Z decomposes as an Euler product

$$Z(s,\phi) = \prod_v Z_v(s, W_v),$$

where

$$Z_v(s, W_v) = \int_{F_v^\times} W_v \begin{pmatrix} y_v & 0 \\ 0 & 1 \end{pmatrix} |y_v|_v^{s-1/2} \, \mathrm{d}^* y_v.$$

Absolute convergence of the local factors occurs when $\mathrm{re}(s) > 1/2$.

The local L-functions. Let F be a non-Archimedean local field and $\mathfrak{o}$ its ring of integers. We say that a mutliplicative character of F is unramified it is trivial on the group of units of F. If χ_1 and χ_2 are unramified multiplicative characters of F, their principal series representation of $\mathrm{GL}(2,F)$, denoted $\pi(\chi_1,\chi_2)$, is spherical, i.e., has a $\mathrm{GL}(2,\mathfrak{o})$-fixed vector. Let $\alpha_1 = \chi(\varpi)$ and $\alpha_2 = \chi_2(\varpi)$ where ϖ is a uniformizer of F. We call α_1 and α_1 the *Satake parameters* of $\pi(\chi_1,\chi_2)$. If $\pi(\chi_1,\chi_2)$ is unitary, α_1 and α_2 lie on the unit circle. Let q be the cardinality of the residue fied $\mathfrak{o}/(\varpi)$. We call

$$L(s,\pi) = (1-\alpha_1 q^{-s})^{-1}(1-\alpha_2 q^{-s})^{-1}$$

the local L-function of π.

Equality of Z_v and L_v. Write $\pi = \otimes_v \pi_v$ is a tensor product of local reprensetations (π_v, V_v). For a place v, F_v denotes its corresponding local field and $\mathfrak{o}_v$ its ring of integers for non-Archimedean v. We say that a place v is unramified if v is non-Archimedean, π_v is a spherical principal series, the conductor of the additive character ψ_v is $\mathfrak{o}_v$, the vector ϕ_v is the spherical vector in the representation, and Whittaker function W_v is normalized so that $W_v(1) = 1$. These conditions are true for almost all v.

If v is unramified, then for s sufficiently large,

$$Z_v(s, W_v) = L_v(s, \pi_v).$$

The (partial) global L-function. We assume the central chracter ω of π is unitary. Let S be a finite set of places such that if $v \notin S$, π_v is spherical. Let

$$L_S(s, \pi) = \prod_{v \notin S} L_v(s, \pi_v).$$

Twisting. All of the previous results hold in the more general case of twisting by a unitary Hecke character ξ. For instance, we define

$$Z(s, \psi, \xi) = \int_{A^\times/F^\times} \phi \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \xi(y)|y|^{s-1/2} \, \mathrm{d}^\times y$$

and all the other definitions are modified in a similar way.

The global functional equation. Let

$$w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Because ϕ is automorphic,

$$\begin{aligned} Z(s, \phi, \xi) &= \int_{A^\times/F^\times} \phi\left(w_1 \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|^{s-1/2} \xi(y) \, \mathrm{d}^* y \\ &= \int_{A^\times/F^\times} \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} w_1 \right) |y|^{s-1/2} \xi(y) \, \mathrm{d}^* y. \end{aligned}$$

Substituting y^{-1} for y, the above is equal to

$$\int_{A^\times/F^\times} (\pi(w_1)\phi) \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{-s+1/2} (\xi\omega)^{-1}(y) \, \mathrm{d}^* y.$$

Hence,

$$Z(s, \phi, \xi) = Z(1-s, \pi(w_1)\phi, \xi^{-1}\omega^{-1})$$

Local funcitonal equation. The local zeta integral $Z_v(s, W_v)$ has meromorphic continuation to all s. There exists a meromorphic function $\gamma_v(s, \pi_v, \xi_v, \psi_v)$ such that

$$Z_v(1-s, \pi_v(w_1)W_v, \xi_v^{-1}\omega_v^{-1}) = \gamma_v(s, \pi_v, \xi_v, \psi_v) Z_v(s, W_v, \xi_v).$$

Functional equation for the partial global L-function. Let ξ be a Hecke character of F. Let S be a finite set of places of F containing all the Archimedean ones such that if $v \notin S$, then π_v is spherical and ξ_v is nonramified and the additive character ψ_v has conductor $\mathfrak{o}_v$. Then

$$L_S(s, \pi, \xi) = \left\{ \prod_{v \in S} \gamma_v(s, \pi_v, \xi_v, \psi_v) \right\} L_S(1 - s, \hat{\pi}, \xi^{-1})$$

where $\hat{\pi}$ is the contragradient representation

A "local converse theoem." Now let F be a non-Archimedean local field. Let π_1 and π_2 be irreducible admissible representations of $\mathrm{GL}(2, F)$. Suppose that π_1 and π_2 have the same central quasicharacter ω and that $\gamma(s, \pi_1, \xi, \psi) = \gamma(s, \pi_2, \xi, \psi)$ for all characters ξ of $F^\times$. Then $\pi_1 \cong \pi_2$.