CLOOB: Modern Hopfield Networks with InfoLOOB Outperform CLIP

Seminar in AI
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JOHANNES KEPLER UNIVERSITY LINZ

Motivation and Setup

Contrastive Learning and Zero-shot Transfer Learning

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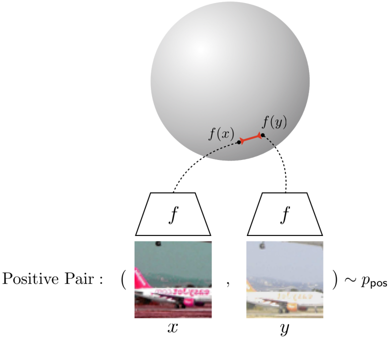
Contrastive Learning:

- self-supervised technique



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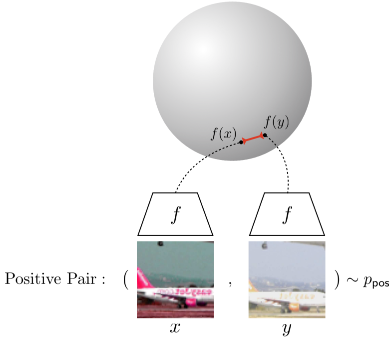
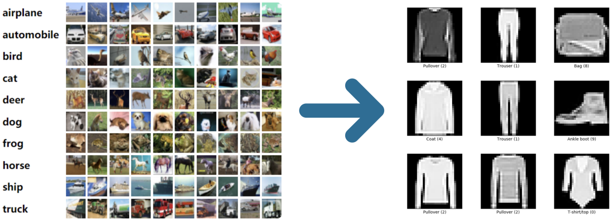


Zero-shot Transfer Learning:

- model needs to generalize well



CLIP (Contrastive Language-Image Pretraining) by OpenAI

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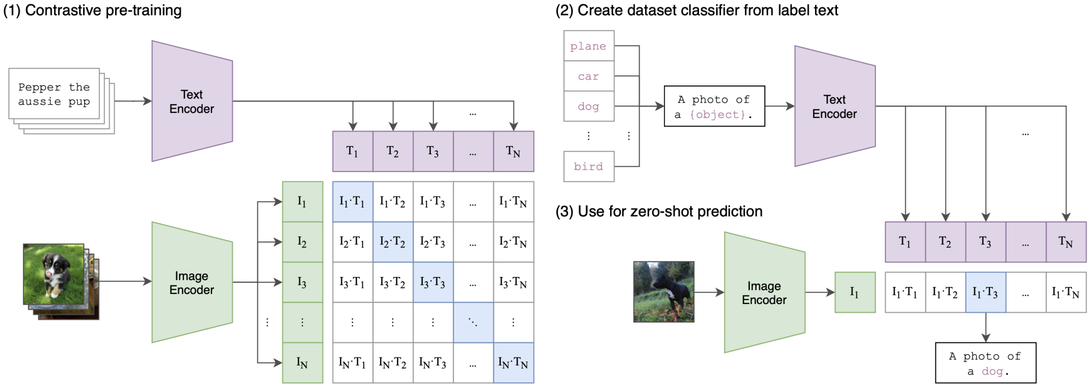

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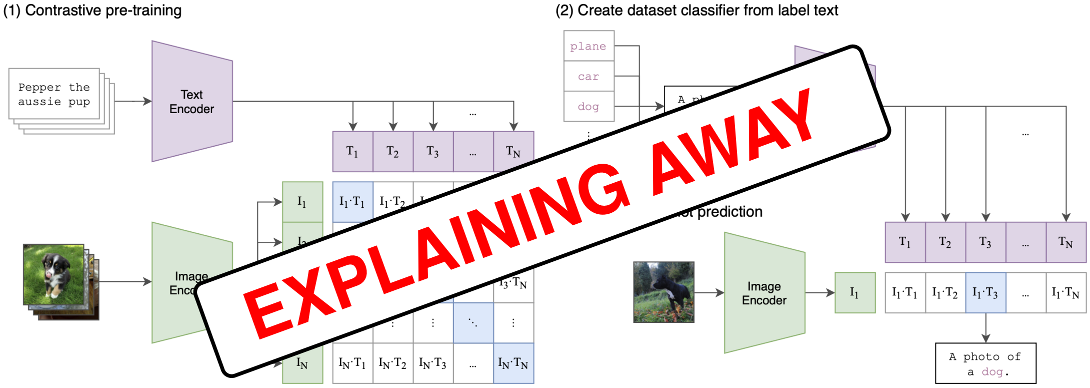

Explaining Away — Co-occurrences and Covariance structure

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Modern Hopfield Networks

Excursus: Modern Hopfield Networks

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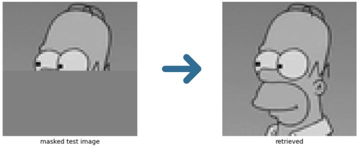

Amplifying Co-occurrences and Covariance Structures

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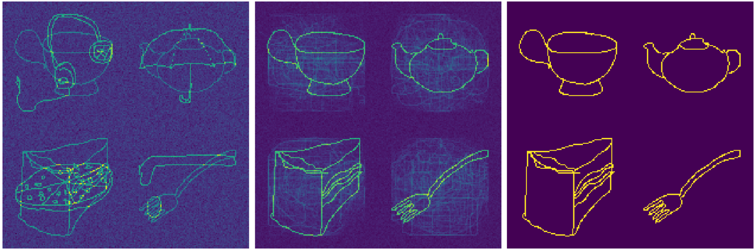

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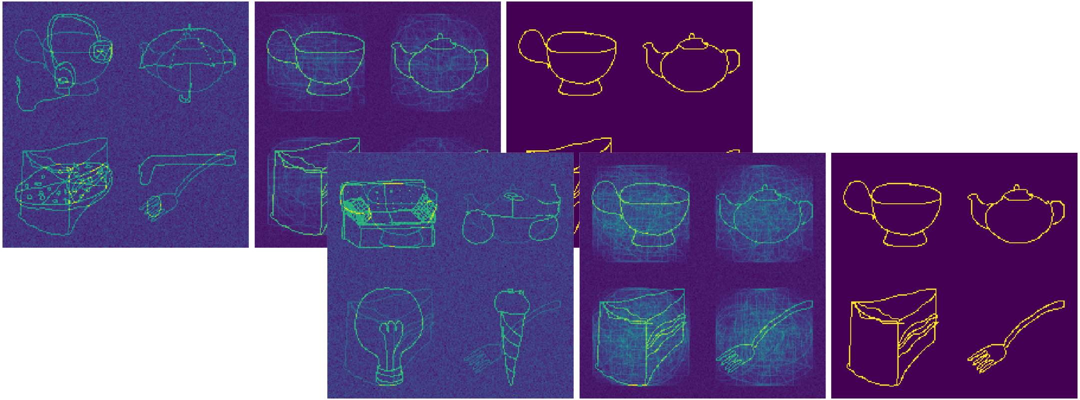

CLOOB Architecture

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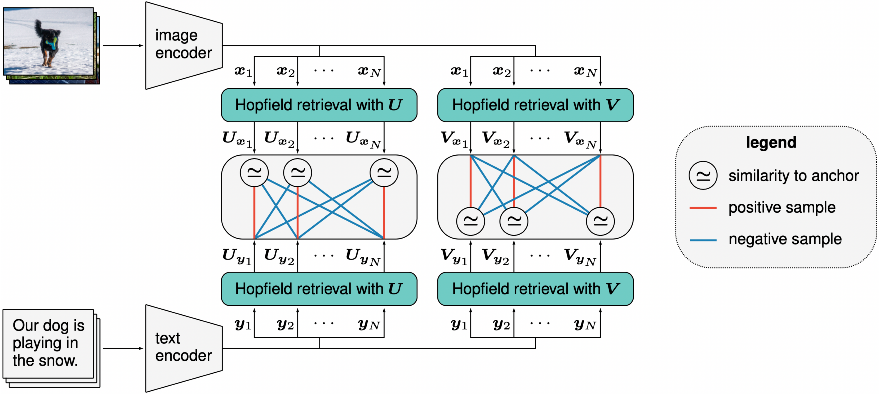

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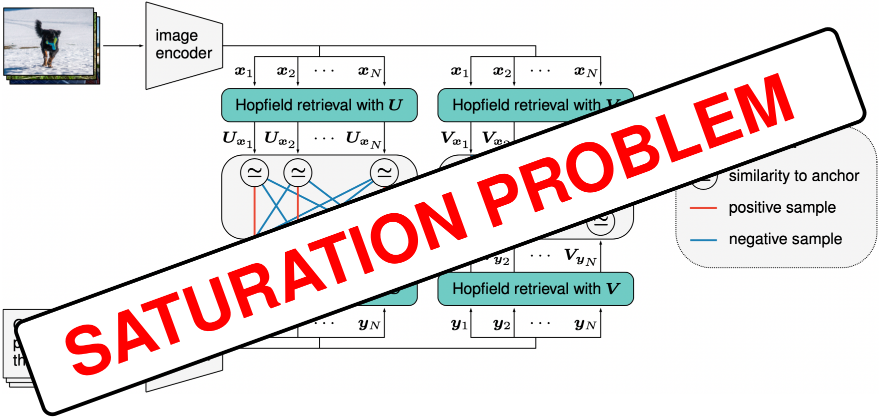

InfoNCE, InfoLOOB, and CLOOB

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$$\mathrm{L}_{\mathrm{InfoNCE}} = -\frac{1}{N} \ln \sum_{i=1}^{N} \frac{\exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_i)}{\sum_{j=1}^{N} \exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_j)} - \frac{1}{N} \sum_{i=1}^{N} \ln \frac{\exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_i)}{\sum_{j=1}^{N} \exp(\tau^{-1} \mathbf{x}_j^T \mathbf{y}_i)}$$

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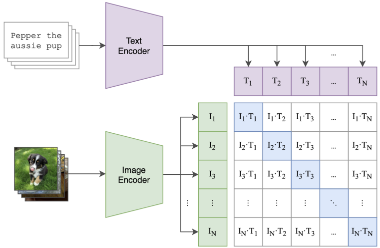

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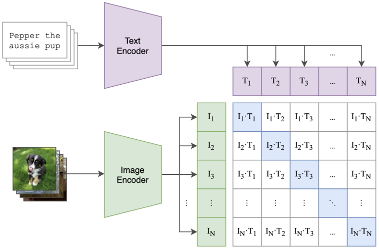

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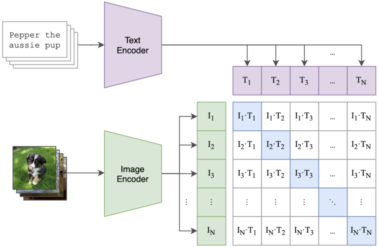
- $\|\mathbf{x}_i\| = \|\mathbf{y}_j\| = 1 \;\Rightarrow\; \cos(\mathbf{x}_i, \mathbf{y}_j) = \mathbf{x}_i^T \mathbf{y}_j$



InfoLOOB (Leave One Out Bound) — CLOOB's Objective

$$\mathrm{L}_{\mathrm{InfoLOOB}} = -\frac{1}{N} \ln \sum_{i=1}^{N} \frac{\exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_i)}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_j)} - \frac{1}{N} \sum_{i=1}^{N} \ln \frac{\exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_i)}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{x}_j^T \mathbf{y}_i)}$$

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# text_encoder – Text Transformer

# I[n, h, w, c] – minibatch of images
# T[n, l] – minibatch of texts

# W_i[d_i, d_e] – image projection
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# Hopfield retrieval H with batch stored
# H(beta, A, B) = B.T @ softmax(beta * A @ B.T)
U_x = H(beta, x, x).T #[n, d_e]
U_y = H(beta, y, x).T #[n, d_e]
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# normalize retrievals
U_x = l2_normalize(U_x) #[n, d_e]
U_y = l2_normalize(U_y) #[n, d_e]
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# loss: info_loob(tau, anchors, samples)
# samples contain pos. and neg. embeddings
loss_i = info_loob(tau, U_x, U_y)
loss_t = info_loob(tau, V_y, V_x)
loss = (loss_i + loss_t) * tau
```

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- Ablation studies
  - Modern Hopfield networks
  - InfoLOOB

CLIP and CLOOB: Results

CC — mean accuracy over 5 runs

Dataset	CLIP RN-50	CLOOB RN-50	CLIP* RN-50	CLOOB* RN-50
Birdsnap	2.26 ± 0.20	$\mathbf{3.06 \pm 0.30}$	2.8 ± 0.16	$\mathbf{3.24 \pm 0.31}$
Country211	0.67 ± 0.11	0.67 ± 0.05	0.7 ± 0.04	0.73 ± 0.05
Flowers102	12.56 ± 0.38	13.45 ± 1.19	13.32 ± 0.43	14.36 ± 1.17
GTSRB	7.66 ± 1.07	6.38 ± 2.11	8.96 ± 1.70	7.03 ± 1.22
UCF101	20.98 ± 1.55	22.26 ± 0.72	21.63 ± 0.65	$\mathbf{23.03 \pm 0.85}$
Stanford Cars	0.91 ± 0.10	$\mathbf{1.23 \pm 0.10}$	0.99 ± 0.16	$\mathbf{1.41 \pm 0.32}$
ImageNet	20.33 ± 0.28	$\mathbf{23.97 \pm 0.15}$	21.3 ± 0.42	$\mathbf{25.67 \pm 0.22}$
ImageNet V2	20.24 ± 0.50	$\mathbf{23.59 \pm 0.15}$	21.24 ± 0.22	$\mathbf{25.49 \pm 0.11}$

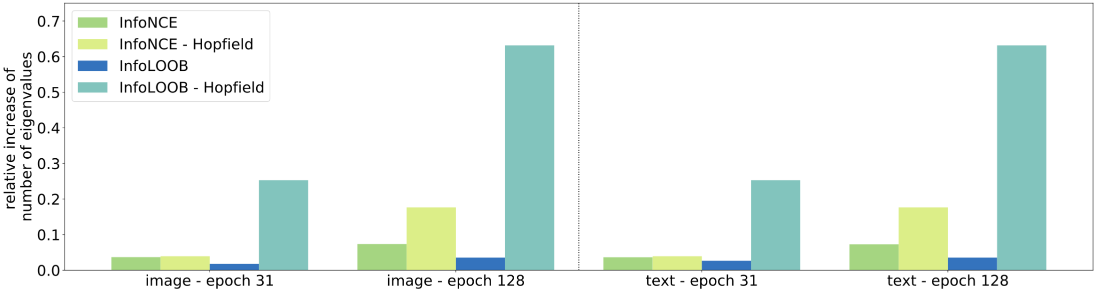
Bold: statistically significant

YFCC — one run

	RN-50		RN-101		RN-50x4	
Dataset	CLIP	CLOOB	CLIP	CLOOB	CLIP	CLOOB
Birdsnap	21.8	**28.9**	22.6	**30.3**	20.8	**32.0**
Country211	6.9	**7.9**	7.8	**8.5**	8.1	**9.3**
Flowers102	48.0	**55.1**	48.0	**55.3**	50.1	**54.3**
GTSRB	7.9	**8.1**	7.4	**11.6**	9.4	**11.8**
UCF101	**27.2**	25.3	28.6	**28.8**	31.0	**31.9**
Stanford Cars	3.7	**4.1**	3.8	**5.5**	3.5	**6.1**
ImageNet	34.6	**35.7**	35.3	**37.1**	37.7	**39.0**
ImageNet V2	33.4	**34.6**	34.1	**35.6**	35.9	**37.3**

Bold: higher value

Results Ablation Study



Conclusion

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Paper fulfills all NeurIPS check marks

Summary

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➔ Modern Hopfield networks

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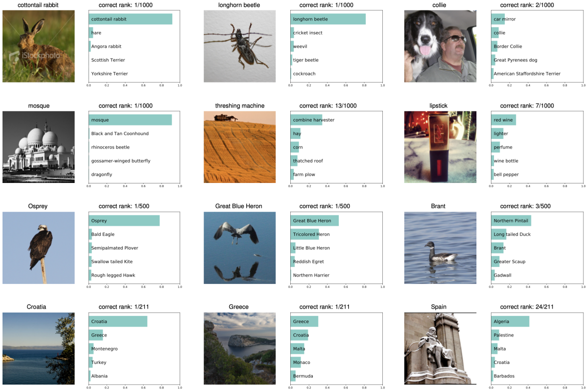
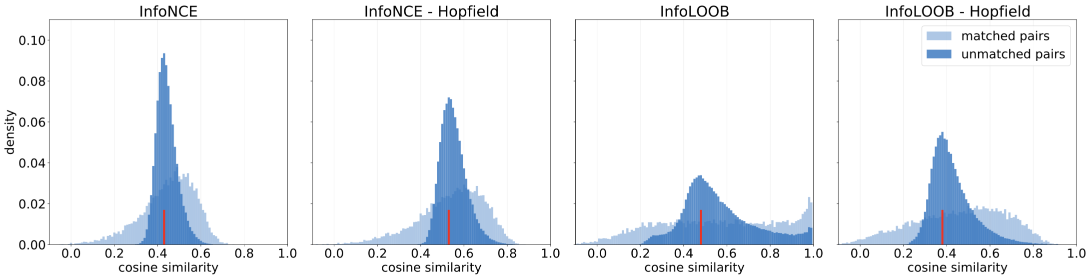


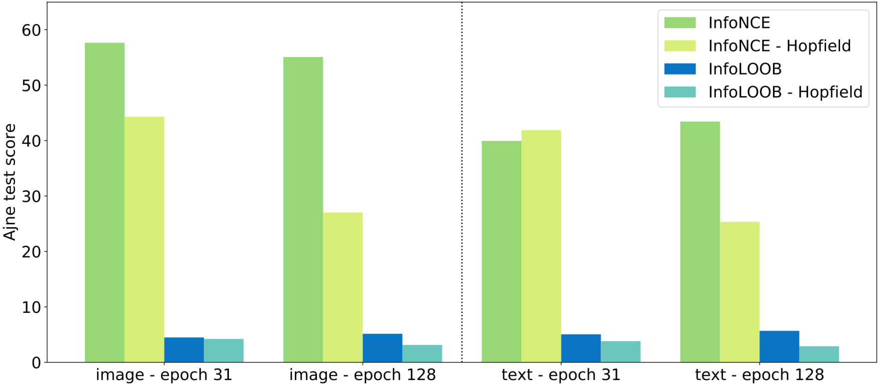
Additional Material

Distribution of Cosine Similarity



Distribution of cosine similarity of matched pairs and of the 10 unmatched pairs that have the highest similarity score with anchor

Ajne Test



InfoNCE
InfoNCE - Hopfield
InfoLOOB
InfoLOOB - Hopfield
Ajne test score
0
10
20
30
40
50
60
image - epoch 31
image - epoch 128
text - epoch 31
text - epoch 128

InfoLOOB – Further details

	$\mathbf{u}_i$ - stored image emb.	$\mathbf{v}_i$ - stored text emb.
$\mathbf{x}_i$ - image query emb.	$\mathbf{U}_{\mathbf{x}_i}$ - image-retr. image emb.	$\mathbf{V}_{\mathbf{x}_i}$ - image-retr. text emb.
$\mathbf{y}_i$ - text query emb.	$\mathbf{U}_{\mathbf{y}_i}$ - text-retr. image emb.	$\mathbf{V}_{\mathbf{y}_i}$ - text-retr. text emb.

$$\mathrm{L}_{\mathrm{InfoLOOB}} = -\frac{1}{N} \ln \sum_{i=1}^{N} \frac{\exp(\tau^{-1} \mathbf{U}_{\mathbf{x}_i}^T \mathbf{U}_{\mathbf{y}_i})}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{U}_{\mathbf{x}_i}^T \mathbf{U}_{\mathbf{y}_j})} - \frac{1}{N} \sum_{i=1}^{N} \ln \frac{\exp(\tau^{-1} \mathbf{V}_{\mathbf{x}_i}^T \mathbf{V}_{\mathbf{y}_i})}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{V}_{\mathbf{x}_j}^T \mathbf{V}_{\mathbf{y}_i})}$$

Gradients of InfoLOOB and InfoNCE

$$\frac{\partial}{\partial \mathbf{y}} \mathrm{L}_{\mathrm{InfoNCE}}(\mathbf{y}) = \frac{\partial}{\partial \mathbf{y}} - \ln \frac{\exp(\tau^{-1} \mathbf{x}_1^T \mathbf{y}_1)}{\sum_{j=1}^{N} \exp(\tau^{-1} \mathbf{x}_j^T \mathbf{y})} = -\tau^{-1} \mathbf{y}^T \mathbf{x}_1 + \tau^{-1} \operatorname{lse}(\tau^{-1}, \mathbf{X}^T \mathbf{y})$$

$$\frac{\partial}{\partial \mathbf{y}} \mathrm{L}_{\mathrm{InfoLOOB}}(\mathbf{y}) = \frac{\partial}{\partial \mathbf{y}} - \ln \frac{\exp(\tau^{-1} \mathbf{x}_1^T \mathbf{y}_1)}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{x}_j^T \mathbf{y})} = -\tau^{-1} \mathbf{y}^T \mathbf{x}_1 + \tau^{-1} \operatorname{lse}(\tau^{-1}, \tilde{\mathbf{X}}^T \mathbf{y})$$

$$\frac{\partial}{\partial \mathbf{y}} \mathrm{L}_{\mathrm{InfoNCE}}(\mathbf{y}) = -\tau^{-1} (1 - p_1)(\mathbf{x}_1 - \tilde{\mathbf{X}} \operatorname{softmax}(\tau^{-1} \tilde{\mathbf{X}}^T \mathbf{y})) = (1 - p_1) \frac{\partial}{\partial \mathbf{y}} \mathrm{L}_{\mathrm{InfoLOOB}}(\mathbf{y})$$

$$\operatorname{lse}(\beta, \boldsymbol{\alpha}) = \beta^{-1} \log \left(\sum_{i=1}^{N} \exp(\beta \alpha_i) \right)$$

$$\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N), \quad \tilde{\mathbf{X}} = (\mathbf{x}_2, \ldots, \mathbf{x}_N)$$