JOHANNES KEPLER
UNIVERSITY LINZ

# CLOOB: Modern Hopfield Networks with InfoLOOB Outperform CLIP

Seminar in AI
Pascal Pilz
Institute for Machine Learning

# Motivation and Setup

# Contrastive Learning and Zero-shot Transfer Learning

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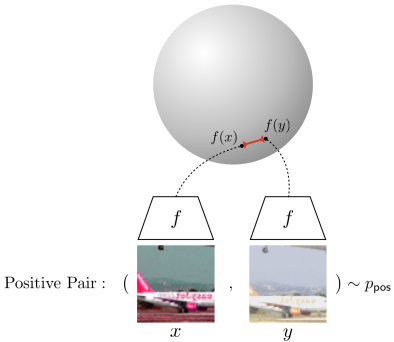
## Contrastive Learning:

- self-supervised technique



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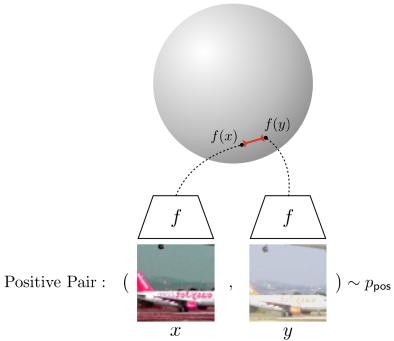


## Zero-shot Transfer Learning:

- model needs to generalize well

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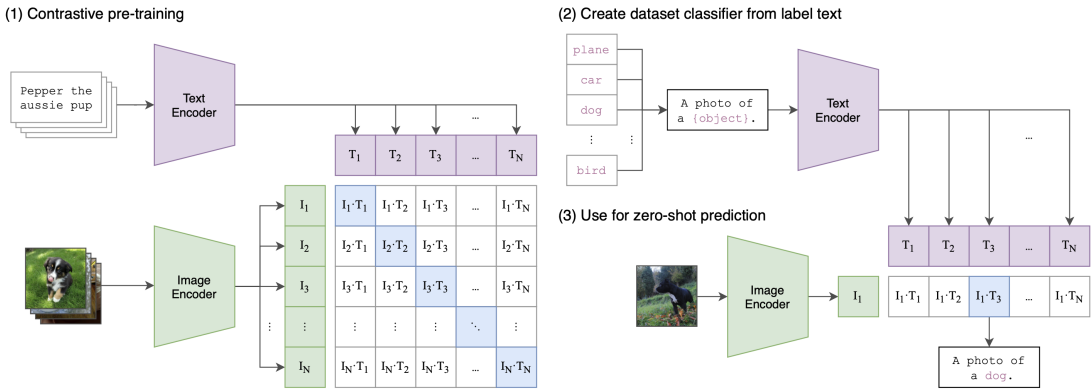

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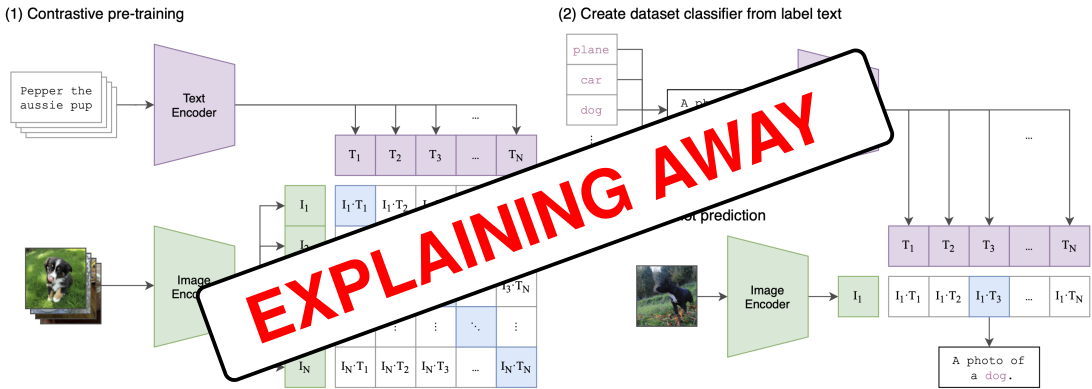

# Explaining Away — Co-occurrences and Covariance structure

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# Modern Hopfield Networks

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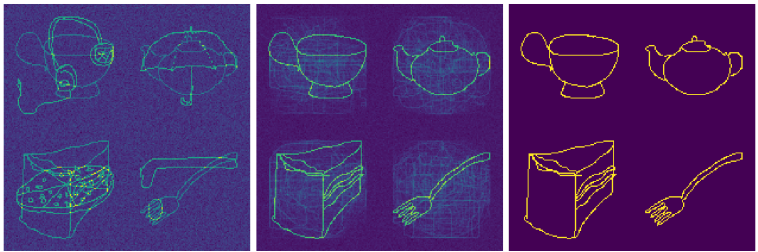

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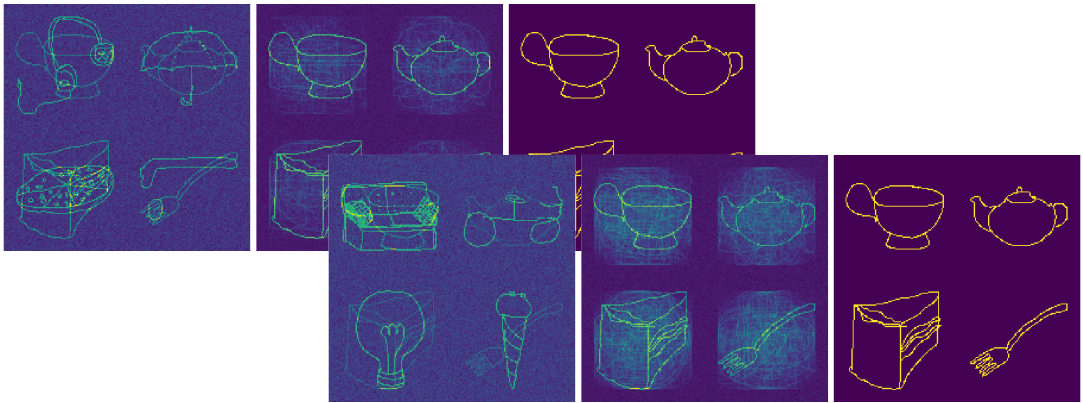

# CLOOB Architecture

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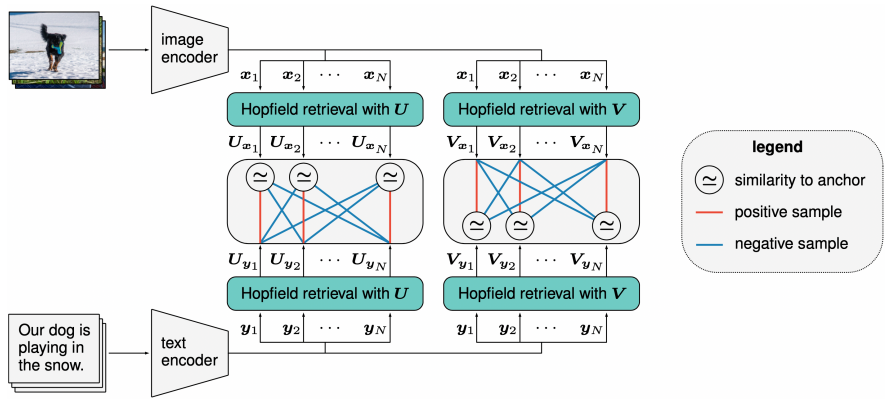

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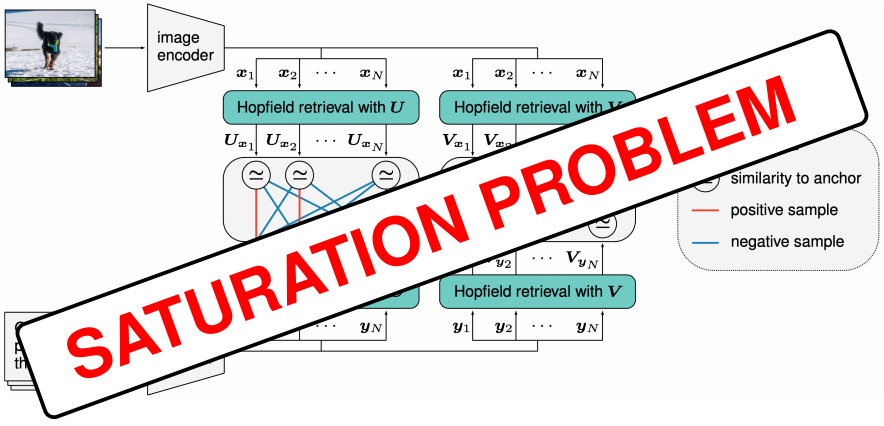

# InfoNCE, InfoLOOB, and CLOOB

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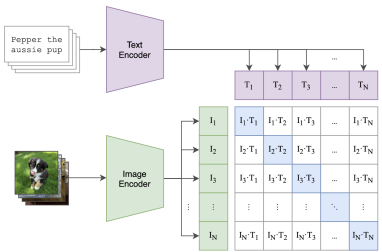

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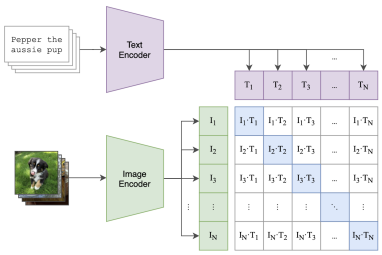

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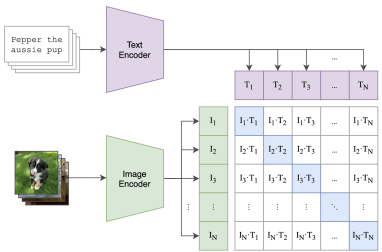
- $\|\mathbf{x}_i\| = \|\mathbf{y}_j\| = 1 \Rightarrow \cos(\mathbf{x}_i, \mathbf{y}_j) = \mathbf{x}_i^T\mathbf{y}_j$



# InfoLOOB (Leave One Out Bound) — CLOOB's Objective

$$\mathrm{L}_{\mathrm{InfoLOOB}} = -\frac{1}{N} \ln \sum_{i=1}^{N} \frac{\exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_i)}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_j)} - \frac{1}{N} \sum_{i=1}^{N} \ln \frac{\exp(\tau^{-1} \mathbf{x}_i^T \mathbf{y}_i)}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{x}_j^T \mathbf{y}_i)}$$

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# W_i[d_i, d_e] – image projection
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# Hopfield retrieval H with batch stored
# H(beta, A, B) = B.T @ softmax(beta * A @ B.T)
U_x = H(beta, x, x).T #[n, d_e]
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# normalize retrievals
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# loss: info_loob(tau, anchors, samples)
# samples contain pos. and neg. embeddings
loss_i = info_loob(tau, U_x, U_y)
loss_t = info_loob(tau, V_y, V_x)
loss = (loss_i + loss_t) * tau
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- Ablation studies
  - Modern Hopfield networks
  - InfoLOOB

# CLIP and CLOOB: Results

CC — mean accuracy over 5 runs

| Dataset | CLIP RN-50 | CLOOB RN-50 | CLIP* RN-50 | CLOOB* RN-50 |
|---|---|---|---|---|
| Birdsnap | $2.26 \pm 0.20$ | $\mathbf{3.06 \pm 0.30}$ | $2.8 \pm 0.16$ | $\mathbf{3.24 \pm 0.31}$ |
| Country211 | $0.67 \pm 0.11$ | $0.67 \pm 0.05$ | $0.7 \pm 0.04$ | $0.73 \pm 0.05$ |
| Flowers102 | $12.56 \pm 0.38$ | $13.45 \pm 1.19$ | $13.32 \pm 0.43$ | $14.36 \pm 1.17$ |
| GTSRB | $7.66 \pm 1.07$ | $6.38 \pm 2.11$ | $8.96 \pm 1.70$ | $7.03 \pm 1.22$ |
| UCF101 | $20.98 \pm 1.55$ | $22.26 \pm 0.72$ | $21.63 \pm 0.65$ | $\mathbf{23.03 \pm 0.85}$ |
| Stanford Cars | $0.91 \pm 0.10$ | $\mathbf{1.23 \pm 0.10}$ | $0.99 \pm 0.16$ | $\mathbf{1.41 \pm 0.32}$ |
| ImageNet | $20.33 \pm 0.28$ | $\mathbf{23.97 \pm 0.15}$ | $21.3 \pm 0.42$ | $\mathbf{25.67 \pm 0.22}$ |
| ImageNet V2 | $20.24 \pm 0.50$ | $\mathbf{23.59 \pm 0.15}$ | $21.24 \pm 0.22$ | $\mathbf{25.49 \pm 0.11}$ |

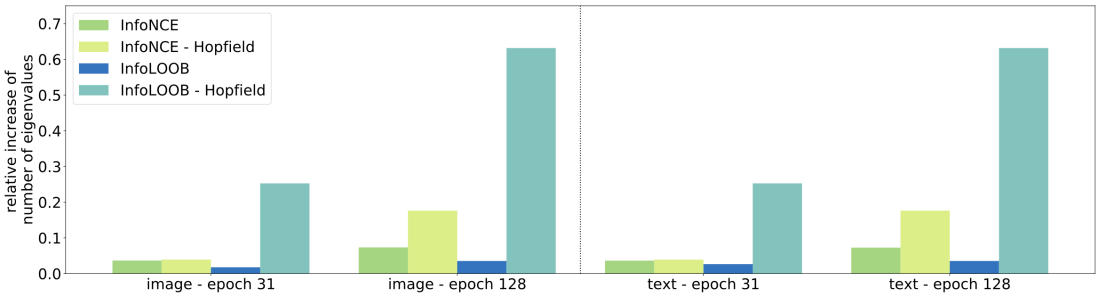
Bold: statistically significant

YFCC — one run

| | RN-50 | | RN-101 | | RN-50x4 | |
|---|---|---|---|---|---|---|
| Dataset | CLIP | CLOOB | CLIP | CLOOB | CLIP | CLOOB |
| Birdsnap | 21.8 | **28.9** | 22.6 | **30.3** | 20.8 | **32.0** |
| Country211 | 6.9 | **7.9** | 7.8 | **8.5** | 8.1 | **9.3** |
| Flowers102 | 48.0 | **55.1** | 48.0 | **55.3** | 50.1 | **54.3** |
| GTSRB | 7.9 | **8.1** | 7.4 | **11.6** | 9.4 | **11.8** |
| UCF101 | **27.2** | 25.3 | 28.6 | **28.8** | 31.0 | **31.9** |
| Stanford Cars | 3.7 | **4.1** | 3.8 | **5.5** | 3.5 | **6.1** |
| ImageNet | 34.6 | **35.7** | 35.3 | **37.1** | 37.7 | **39.0** |
| ImageNet V2 | 33.4 | **34.6** | 34.1 | **35.6** | 35.9 | **37.3** |

Bold: higher value

# Results Ablation Study



# Conclusion

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Paper fulfills all NeurIPS check marks

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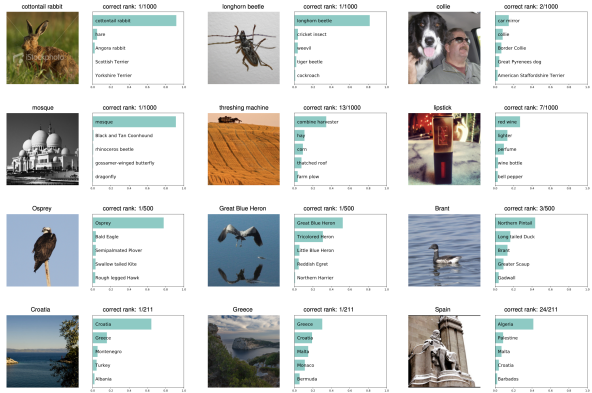
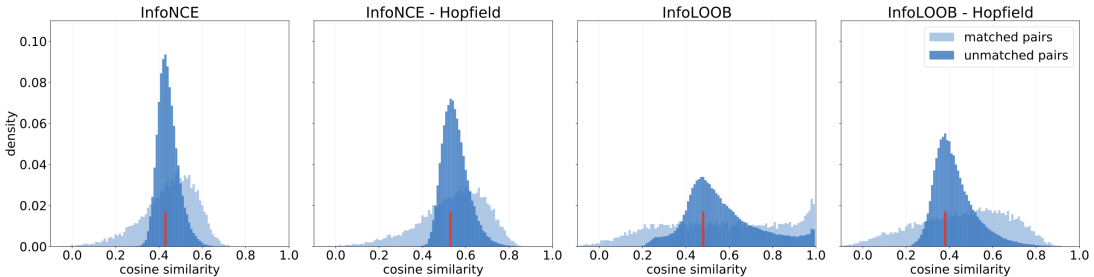


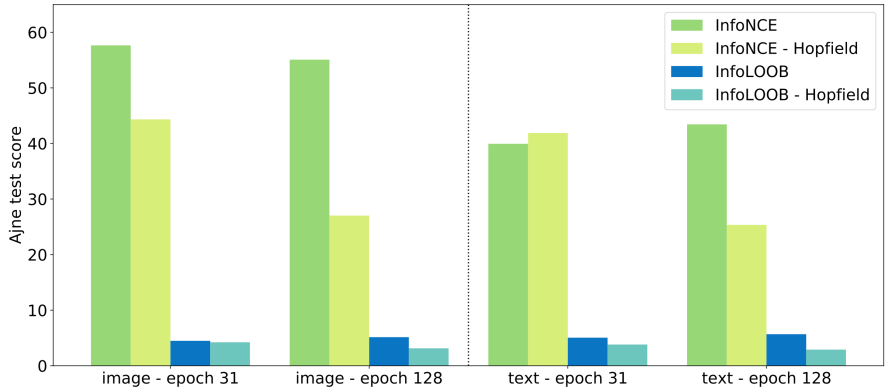
# Additional Material

# Distribution of Cosine Similarity




Distribution of cosine similarity of matched pairs and of the 10 unmatched pairs that have the highest similarity score with anchor

# Ajne Test


InfoNCE
InfoNCE - Hopfield
InfoLOOB
InfoLOOB - Hopfield
Ajne test score
0
10
20
30
40
50
60
image - epoch 31
image - epoch 128
text - epoch 31
text - epoch 128

# InfoLOOB – Further details

| | $\mathbf{u}_i$ - stored image emb. | $\mathbf{v}_i$ - stored text emb. |
|---|---|---|
| $\mathbf{x}_i$ - image query emb. | $\mathbf{U}_{\mathbf{x}_i}$ - image-retr. image emb. | $\mathbf{V}_{\mathbf{x}_i}$ - image-retr. text emb. |
| $\mathbf{y}_i$ - text query emb. | $\mathbf{U}_{\mathbf{y}_i}$ - text-retr. image emb. | $\mathbf{V}_{\mathbf{y}_i}$ - text-retr. text emb. |

$$L_{\text{InfoLOOB}} = -\frac{1}{N} \ln \sum_{i=1}^{N} \frac{\exp(\tau^{-1} \mathbf{U}_{\mathbf{x}_i}^T \mathbf{U}_{\mathbf{y}_i})}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{U}_{\mathbf{x}_i}^T \mathbf{U}_{\mathbf{y}_j})} - \frac{1}{N} \sum_{i=1}^{N} \ln \frac{\exp(\tau^{-1} \mathbf{V}_{\mathbf{x}_i}^T \mathbf{V}_{\mathbf{y}_i})}{\sum_{j \neq i}^{N} \exp(\tau^{-1} \mathbf{V}_{\mathbf{x}_j}^T \mathbf{V}_{\mathbf{y}_i})}$$

# Gradients of InfoLOOB and InfoNCE

$$\frac{\partial}{\partial \mathbf{y}} \mathsf{L}_{\text{InfoNCE}}(\mathbf{y}) = \frac{\partial}{\partial \mathbf{y}} - \ln \frac{\exp(\tau^{-1} \mathbf{x}_1^T \mathbf{y}_1)}{\sum_{j=1}^{N} \exp(\tau^{-1} \mathbf{x}_j^T \mathbf{y})} = -\tau^{-1} \mathbf{y}^T \mathbf{x}_1 + \tau^{-1} \operatorname{lse}(\tau^{-1}, \mathbf{X}^T \mathbf{y})$$

$$\frac{\partial}{\partial \mathbf{y}} \mathsf{L}_{\text{InfoLOOB}}(\mathbf{y}) = \frac{\partial}{\partial \mathbf{y}} - \ln \frac{\exp(\tau^{-1} \mathbf{x}_1^T \mathbf{y}_1)}{\sum_{j \neq 1}^{N} \exp(\tau^{-1} \mathbf{x}_j^T \mathbf{y})} = -\tau^{-1} \mathbf{y}^T \mathbf{x}_1 + \tau^{-1} \operatorname{lse}(\tau^{-1}, \tilde{\mathbf{X}}^T \mathbf{y})$$

$$\frac{\partial}{\partial \mathbf{y}} \mathsf{L}_{\text{InfoNCE}}(\mathbf{y}) = -\tau^{-1}(1 - p_1)(\mathbf{x}_1 - \tilde{\mathbf{X}} \operatorname{softmax}(\tau^{-1} \tilde{\mathbf{X}}^T \mathbf{y})) = (1 - p_1) \frac{\partial}{\partial \mathbf{y}} \mathsf{L}_{\text{InfoLOOB}}(\mathbf{y})$$

$$\operatorname{lse}(\beta, \boldsymbol{\alpha}) = \beta^{-1} \log \left( \sum_{i=1}^{N} \exp(\beta \alpha_i) \right)$$

$$\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N), \quad \tilde{\mathbf{X}} = (\mathbf{x}_2, \ldots, \mathbf{x}_N)$$