Digital Signal Processing SS 2024 – Exercise 5

Digital Signal Processing Tutorial

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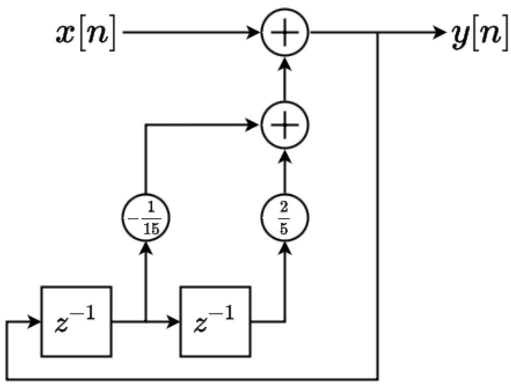
Exercise 1

We are given the difference equation

$$y[n] = x[n] - \frac{1}{15}y[n-1] + \frac{2}{5}y[n-2].$$

a) We sketch the corresponding block diagram:



b) The system is an IIR Filter. This is because in the difference equation we have non-zero a coefficients, i.e., it is a recursive filter.

c) To find the transfer function, we first need to transform the system to the z-domain. For the transformation we use the linearity of the z-transformation. We assume that the signal is causal, i.e., $x[n] = y[n] = 0$ for $n < 0$.

$$y[n] = x[n] - \frac{1}{15}y[n-1] + \frac{2}{5}y[n-2] \tag{1}$$

$$\circ\!\!-\!\!\bullet \tag{2}$$

$$Y(z) = X(z) - \frac{1}{15}z^{-1}Y(z) + \frac{2}{5}z^{z-2}Y(z) \quad \Longleftrightarrow \tag{3}$$

$$Y(z)\left(1 + \frac{1}{15}z^{-1} - \frac{2}{5}z^{-2}\right) = X(z) \quad \Longleftrightarrow \tag{4}$$

$$Y(z) = \left(1 + \frac{1}{15}z^{-1} - \frac{2}{5}z^{-2}\right)^{-1} X(z) \tag{5}$$

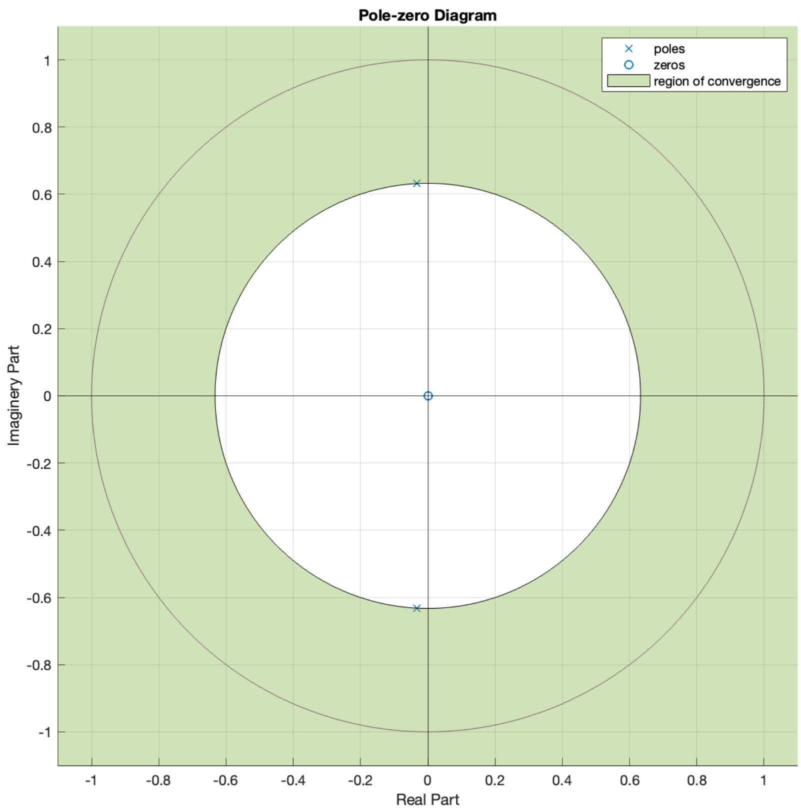
We can see that the transfer function $H(z) = \frac{Y(z)}{X(z)}$ is given by

$$
\begin{aligned}
H(z) &= \left(1 + \frac{1}{15}z^{-1} - \frac{2}{5}z^{-2}\right)^{-1} && (6)\\
&= \frac{1}{1 + \frac{1}{15}z^{-1} - \frac{2}{5}z^{-2}} && (7)\\
&= \frac{1}{1 + \frac{1}{15}z^{-1} - \frac{2}{5}z^{-2}}\frac{z^2}{z^2} && (8)\\
&= \frac{z^2}{z^2 + \frac{1}{15}z - \frac{2}{5}} && (9)\\
& && (10)
\end{aligned}
$$

d) The roots of the numerator are clearly 0. The roots of the denominator can be determined from $z^2 + \frac{1}{15}z - \frac{2}{5} = 0$:

$$
\begin{aligned}
z_{1,2} &= -\frac{1}{30} \pm \sqrt{\frac{1}{1125} - \frac{8}{20}} && (11)\\
&= -\frac{1}{30} \pm j\sqrt{\frac{8980}{22500}} && (12)\\
&= -\frac{1}{30} \pm j\frac{\sqrt{8980}}{150} && (13)\\
&\approx -0.0333 \pm j0.0574 && (14)
\end{aligned}
$$

You can see a sketch of the pole-zero map:



e) The system is stable. This can be seen by the region of convergence including the unit circle.

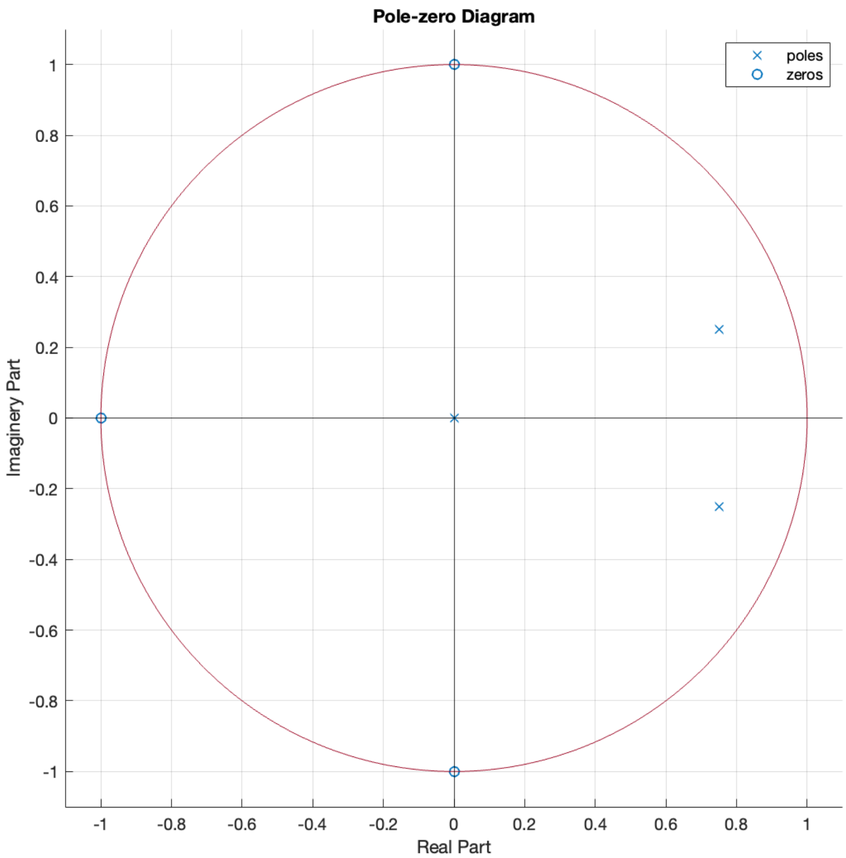
Exercise 2

We are given a set of poles and zeros:

- Zeros: $N_1 = -1,\ N_2 = j,\ N_3 = -j$
- Poles: $P_1 = 0,\ P_2 = 0.75 + j0.25,\ P_3 = 0.27 - 0.25j$

a) This filter has only real coefficients, since when we express the transfer function in the pole-zero representation then we can see that the numerator and denominator both contain a complex number and their corresponding conjugate, meaning that when we multiply it out we will be left with only real coefficients.

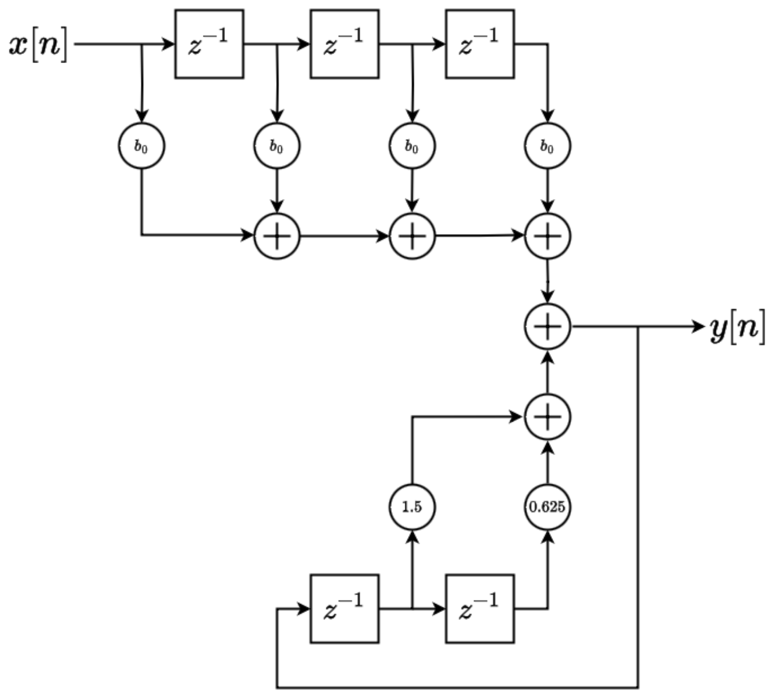
b) You can see a sketch of the pole-zero map:



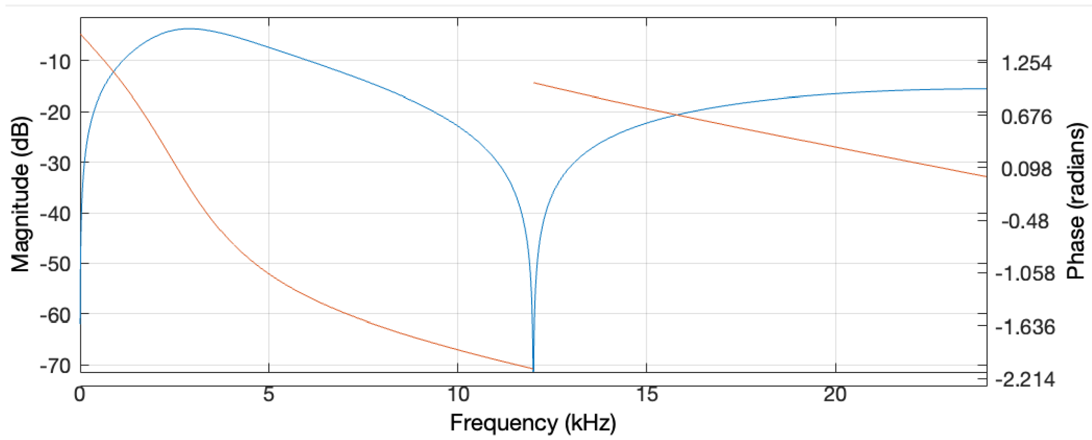
c) The transfer function can be obtained by writing it first in pole-zero representation and then multiplying out the clauses :

$$
\begin{aligned}
H(z) &= b_0 z^{M-N} \frac{(z-N_1)(z-N_2)(z-N_3)}{(z-P_1)(z-P_2)(z-P_3)} && (15)\\
&= b_0 \frac{(z+1)(z+j)(z-j)}{z(0.75+j0.25)(0.27-0.25j)} && (16)\\
&= b_0 \frac{z^3+z^2+z+1}{z^3-1.5z^2+0.625z} && (17)\\
&= \frac{b_0z^3+b_0z^2+b_0z+b_0}{z^3-1.5z^2+0.625z} && (18)\\
&= \frac{b_0+b_0z^{-1}+b_0z^{-2}+b_0z^{-3}}{1-1.5z^{-1}+0.625z^{-2}} && (19)\\
& && (20)
\end{aligned}
$$

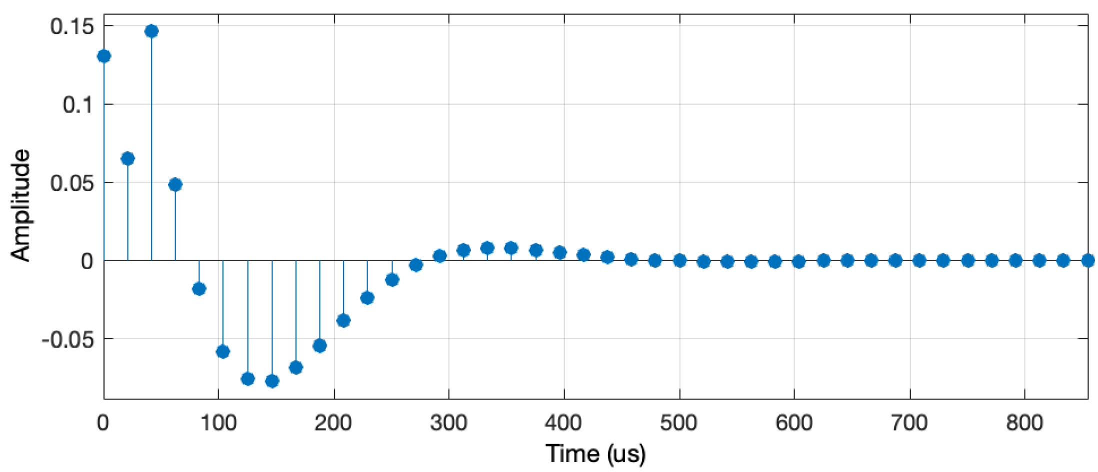
d) The block diagram of a direct-form-I implementation of the filter can be seen:



e) The magnitude and phase response can be seen:



f) The impulse response for $0 \leq n \leq 50$ can be seen:



Exercise 3

Exercise 4