Digital Signal Processing SS 2024 – Exercise 1

Digital Signal Processing Tutorial

Group 23

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Exercise 1

We have three complex numbers given:

$$c_1 = -5 + 3j \qquad c_2 = \frac{\sqrt{2}}{2} e^{-\frac{3\pi j}{4}} \qquad c_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} j$$

a) We do the calculations by hand. First, let's use Euler's formula to simplify c_2:

$$c_2 = \frac{\sqrt{2}}{2} e^{-\frac{3\pi j}{4}} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{3\pi}{4}\right) + j \sin\left(\frac{3\pi}{4}\right)\right) = \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j\right) = -\frac{1}{2} + \frac{1}{2} j$$

With this, let us now compute the following numbers:

$$c_4 = c_1 + c_2 \qquad = -5 - \frac{1}{2} + 3j + \frac{1}{2} j = -\frac{11}{2} + \frac{7}{2} j$$

$$c_5 = c_1 \cdot c_2 \qquad = (-5 + 3j) \cdot \left(-\frac{1}{2} + \frac{1}{2} j\right) = \frac{5}{2} - \frac{5}{2} j - \frac{3}{2} j - \frac{3}{2} = 1 - 4j$$

$$c_6 = |c_3|^2 \qquad = \left(\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right)^2 = \left(\sqrt{\frac{1}{2} + \frac{1}{2}}\right)^2 = \left(\sqrt{1}\right)^2 = 1$$

$$c_7 = \arg(c_3) \qquad = \operatorname{atan2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \arctan\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = \arctan(1) = \frac{\pi}{4}$$

$$c_8 = \frac{c_1}{c_2} \qquad = \frac{-5 + 3j}{\frac{-1+1j}{2}} = \frac{-10 + 6j}{-1 + 1j} = \frac{(-1 - 1j)(-10 + 6j)}{(-1 - 1j)(-1 + 1j)} = \frac{10 - 6j + 10j + 6}{1 - 1j + 1j + 1} = 8 - 2j$$

$$c_9 = c_1 \cdot c_1^* \qquad = (-5 + 3j)(-5 - 3j) = 25 + 15j - 15j + 9 = 34$$

b) We check the result using MATLAB.

This is the code we use:

```
c1 = -5 + 3j
c2 = sqrt(2)/2 * exp((3j*pi)/4)
c3 = 1/sqrt(2) + 1j/sqrt(2)

c4 = c1 + c2
c5 = c1 * c2
```

```
c6 = abs(c3)^2
c7 = angle(c3)
c8 = c1 / c2
c9 = c1 * c1'
```

These are the results:

```
c1 = -5.0000 + 3.0000i
c2 = -0.5000 + 0.5000i
c3 = 0.7071 + 0.7071i
c4 = -5.5000 + 3.5000i
c5 = 1.0000 - 4.0000i
c6 = 1.0000
c7 = 0.7854
c8 = 8.0000 + 2.0000i
c9 = 34
```

Checks out. (Question: what is the best way to get this output from MATLAB to latex? The way I did it was copying it from the console, but that was rather cumbersome since I needed to remove the blank lines.)

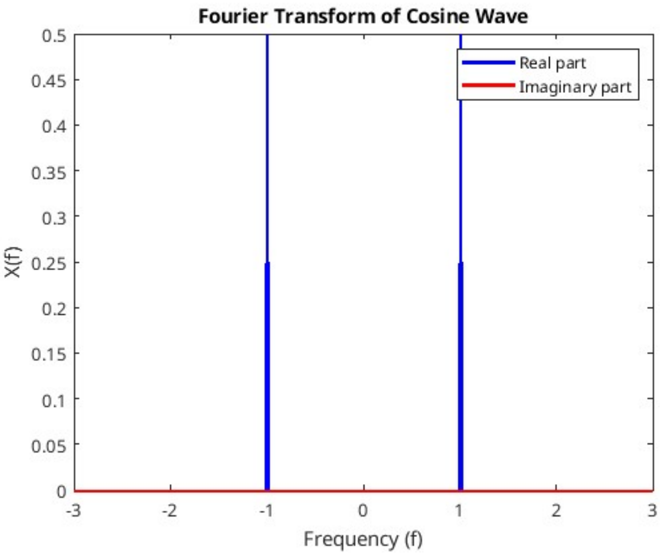
Exercise 2

We need to prooof the following relation (Fourier transform pair for the cosine wave)

$$x(t) = \hat{X}\cos(2\pi f_0 t) \leftrightarrow X(f) = \frac{\hat{X}}{2}\delta(f - f_0) + \frac{\hat{X}}{2}\delta(f + f_0)$$

1. We use Eulers formula to express the cosine in the time domain as sum of complex exponents.
   - For this we use the formula for the cosine wave from (DSP_2.pdf, page 5)
     $s(t) = \hat{s}\cos(2\pi f_0 t + \varphi_0) = \frac{\hat{s}}{2}\left(e^{j(\omega_0 t + \varphi_0)} + e^{-j(\omega_0 t + \varphi_0)}\right)$
   - And we get:
     $x(t) = \hat{X}\cos(2\pi f_0^t t) = \frac{\hat{X}}{2}e^{j2\pi f_0 t} + \frac{\hat{X}}{2}e^{-j2\pi f_0 t}$
2. We express the Fourier Transform using (DSP_2.pdf, page 38).
   - The Fourier Transform of a complex exponential function is a delta function.
     $x(t) = \hat{X}e^{j2\pi f_0 t} \quad \circ\!-\!\bullet \quad X(f) = \hat{X}\delta(f - f_0)$
3. From 1. and 2. we get:
   $\Longrightarrow X(f) = \frac{\hat{X}}{2}\delta(f - f_0) + \frac{\hat{X}}{2}\delta(f + f_0) \qquad$ q.e.d

Add a diagram of $X(f)$ in the report (draw the real and imaginary part of $X(f)$ in the same diagram).



Exercise 3

Given are two sines according to the following formula:

$x_i(t) = \sin(2\pi f_i t)$, with $i \in \{1,2\}$ with $f_1 = 1$ Hz and $f_2 = 3$ Hz.

All two sines are time delayed by $\tau = 0.1$ s to yield

$y_i(t) = \sin(2\pi f_i(t - 0.1))$

This corresponds to a phase shift. Thus, the delayed sines may also be written as

$y_i(t) = \sin(2\pi f_i t + \phi_i)$

a) We have to calculate the phase shifts ϕ_i for each sine and verify that this corresponds to the "Shift Theorem"

1. Calculate the phase shifts ϕ_i

   - We know that:

     $\sin(2\pi f_i(t-\tau)) = \sin(2\pi f_i t + \phi_i)$
   - So:

     $2\pi f_i(t-\tau) = 2\pi f_i t + \phi_i$

     $\implies \phi_i = 2\pi f_i t - 2\pi f_i t\tau - 2\pi f_i t = 2\pi f_i t\tau$
   - So we get:

     $\phi_1 = -2\pi \cdot 1 \cdot 0.1 = -0.2\pi$

     $\phi_2 = -2\pi \cdot 3 \cdot 0.1 = -0.6\pi$

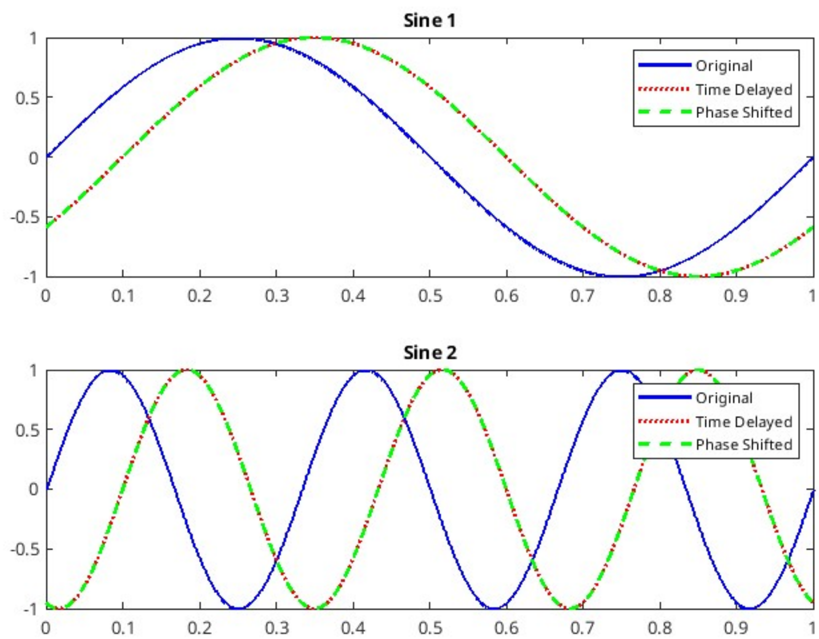
2. Verify that this corresponds to the "Shift Theorem" of the Fourier Transform

   - We use the formula for the "Shift Theorem" (DSP_2.pdf, page 41)

     $x(t-T) \circ\!-\!\bullet\ X(f)e^{-j2\pi fT}$
   - So, since $\phi_i = 2\pi f_i t\tau$, we get

     $x(t-\tau) \circ\!-\!\bullet\ X(f)e^{-j2\pi f\tau} = X(f)e^{-j\phi_i}$ q.e.d

b) For both sines in separated plots: Plot the original signal, the time delayed signal and the phase shifted signal. Since the latter two are identical, show this by plotting the first with a solid line and the overlaid one in a different colour with a dashed line. Plot each signal from 0 to 1s. In Matlab use the following time-vector: $t = 0 : 0.001 : 1$.



Exercise 4

We need to find out whether

a) $y(t) = (x(t))^2$ b) $y(t) = x(t) \cdot \sin(\Omega_0 t)$

are linear and time invariant.

a) Let $y(t) = (x(t))^2$.

1) **Linearity.** It is easy to see that the system is not linear:

$$(\alpha x(t))^2 = \alpha^2 (x(t))^2 = \alpha^2 y(t) \neq \alpha y(t).$$

2) **Time invariance.** Let $y_1(t) = (x(t-\tau))^2$ be the system in which input is delayed, and let $y_2(t) = y(t-\tau)$ be the system in which the output is delayed. Since $y_1 = (x(t-\tau))^2 = y(t-\tau) = y_2$, we can see that the system is time invariant.

b) Let $y(t) = x(t) \cdot \sin(\Omega_0 t)$.

1) **Linearity.** Let $y_1(t) = (\alpha x_1(t) + \beta x_2(t)) \sin(\Omega_0 t)$, and let $y_2(t) = \alpha x_1(t) \sin(\Omega_0 t) + \beta x_2(t) \sin(\Omega_0 t)$. Then we can see that

$$y_1(t) = (\alpha x_1(t) + \beta x_2(t)) \sin(\Omega_0 t) = \alpha x_1(t) \sin(\Omega_0 t) + \beta x_2(t) \sin(\Omega_0 t) = y_2(t).$$

Therefore, the system is linear.

2) **Time invariance.** Let $y_1(t) = x(t-\tau) \sin(\Omega_0 t)$ be the system in which the input is delayed, and let $y_2(t) = y(t-\tau)$ be the system in which the output is delayed. Since $y_1 = x(t-\tau) \sin(\Omega_0 t) \neq x(t-\tau) \sin(\Omega_0 (t-\tau)) = y(t-\tau) = y_2$, we can see that the system in not time invariant.