Portfolio

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Contents

1 Introduction

In this portfolio, I have included seven portfolio theorems, using a variety of proof methods and covering a variety of mathematical concepts.

The proof of Portfolio Theorem 1 is a *proof by contrapositive* using the definitions of prime and composite integers.

Portfolio Theorem 2 is a *direct proof* that uses the "divides" relation.

Portfolio Theorem 3 and Portfolio Theorem 5 are both *proofs by mathematical induction*, but the proof of Portfolio Theorem 3 is algebraic, while the proof of Portfolio Theorem 5 is a geometric argument.

I included two different proofs for Portfolio Theorem 4, one of which is a non-constructive *proof by cases*, the other of which is a *constructive direct proof*, but relies on Lemma 3, which was proven using a *proof by contradiction*.

For Portfolio Theorem 6, I prove that a function is a bijection. For Portfolio Theorem 7, I prove that a function is *not* an injection.

2 Conjectures and Proofs

2.1 A Theorem Concerning Primes

2.1.1 A Definition

Definition 1. Saying that an integer, n, is *prime* means that n has exactly two distinct positive factors. An integer with *more* than two distinct positive factors is *composite.*

For example, 7 is prime because its positive factors are 1 and 7, while 12 is composite because its positive factors are 1, 2, 3, 4, 6, and 12. (Note that 1 is neither prime nor composite because it has only one positive factor, namely 1.)

Another thing to be aware of about this definition is that if n is composite it must have at least two factors that are greater than 1.

Conjecture 1. *If n and b are positive integers and $1 + b + b^2 + \cdots + b^{n-2} + b^{n-1}$ is prime, then n is prime.*

2.1.2 Equivalent Expressions and the Negation

I began by rewriting the proposition so that the quantifiers are explicit. The quantifiers are in **bold** and logical connective are *emphasized* to better see the logical structure.

> **For all** positive integers b and **for all** positive integers n, *if* $1 + b + b^2 + \cdots + b^{n-2} + b^{n-1}$ is prime, *then* n is prime.

The variables b and n are universally quantified and both are taken from the set of positive integers, which I will denote by $\mathbb{Z}^+$. So, now I can see that this conjecture has the form

$$(\forall b \in \mathbb{Z}^+)(\forall n \in \mathbb{Z}^+)\big(S(b,n) \to P(n)\big) \tag{1}$$

where $S(b,n))$ is the predicate "$1 + b + b^2 + \cdots + b^{n-2} + b^{n-1}$ is prime," and $P(n)$ is the predicate "n is prime." This form makes it easier to find the negation and contrapositive.

When we find the negation, the quantifiers are changed, and the negation of an implication, $P \to Q$ is $P \wedge \neg Q$, so the negation of Statement (1) is

$$(\exists b \in \mathbb{Z}^+)(\exists n \in \mathbb{Z}^+)\big(S(b,n) \wedge \neg P(n)\big) \tag{2}$$

which, in plain language, is

> there are positive integers b and n such that $1+b+b^2+\cdots+b^{n-2}+b^{n-1}$ is prime and n is not prime.

It will probably be easier to start with knowing whether a value for n is prime or not and find out about the value for $1 + b + b^2 + \cdots + b^{n-2} + b^{n-1}$. Since Statement (2) is a conjunction, I can write the two parts in either order, so I will rewrite it as

Conjecture 1a (Negation)**.** *There are positive integers b and n such that n is not prime and $1 + b + b^2 + \cdots + b^{n-2} + b^{n-1}$ is prime.*

Since a counter-example is an example that proves the negation of a statement, a counter-example for Conjecture 1 would consist of a positive integer b and a positive integer n so that n is not prime and $1 + b + b^2 + \cdots + b^{n-2} + b^{n-1}$ is prime.

On the other hand, if I want to start with n and try to prove that Conjecture 1 is true, I can replace Conjecture 1 with its contrapositive, which will be logically equivalent. The contrapositive of Conjecture 1 will have the form

$$(\forall b \in \mathbb{Z}^+)(\forall n \in \mathbb{Z}^+)\big(\neg P(n) \to \neg S(b,n)\big) \tag{3}$$

which I will now write as Conjecture 1b

Conjecture 1b (Contrapositive)**.** *For all positive integers b and all positive integers n, if n is not prime, then $1 + b + b^2 + \cdots + b^{n-2} + b^{n-1}$ is not prime.*

I experimented to see if I could find a counter-example, or see a way to prove the conjecture. Since I would need n to be composite for a counter-example, I primarily looked at those cases, but looked at others, too, to look for patterns. I looked at several examples where I kept the same value of b and changed n, and I looked at other examples where I kept n the same and changed b.

2.1.3 Experimentation

1. $b = 6$, $n = 1$ (1 is not prime): $6^{1-1} = 6^0 = 1$, 1 is not prime.

2. $b = 2$, $n = 2$ (2 is prime): $1 + 2 = 3$, 3 is prime.

3. $b = 2$, $n = 3$ (3 is prime): $1 + 2 + 2^2 = 1 + 2 + 4 = 7$, 7 is prime.

4. $b = 2$, $n = 4$ ($4 = 2 \cdot 2$ is not prime): $1 + 2 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$, $15 = 3 \cdot 5$ is not prime.

5. $b = 2$, $n = 5$ (5 is prime): $1 + 2 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$, 31 is prime.

6. $b = 2$, $n = 6$ ($6 = 2 \cdot 3$ is not prime): $1+2+2^2+2^3+2^4+2^5 = 1+2+4+8+16+32 = 63$, $63 = 3 \cdot 3 \cdot 7$ is not prime.

7. $b = 3$, $n = 1$ (1 is not prime): $3^{1-1} = 3^0 = 1$, 1 is not prime.

8. $b = 3$, $n = 4$ (4 is not prime): $1 + 3 + 3^2 + 3^3 = 1 + 3 + 9 + 27 = 40$, 40 is not prime.

9. $b = 3$, $n = 5$ (5 is prime): $1 + 3 + 3^2 + 3^3 + 3^4 = 1 + 3 + 9 + 27 + 81 = 121$, $121 = 11 \cdot 11$ is not prime, (but the conjecture says that *if* 121 is *not* prime then n is not prime, so this isn't a counter-example.)

I noticed that $n = 1$ is not prime, and $b^{1-1} = b^0 = 1$ (and again, 1 is not prime), so the conjecture is true in the case where $n = 1$.

I realized that any positive integer greater than 1 must either be prime or composite, and if $n > 1$ is composite, then it must have a positive integer factor k so that $k > 1$ and $k \neq n$. That means that we can find an integer j so that $n = kj$ where $j > 1$ and $j \neq n$.

Next I tried to see if I could figure out how being able to factor n this way applied to my experiments where n is not prime.

Let's look at $n = 6 = 2 \cdot 3$, $b = 2$. $63 = 7 \cdot 9 = 7 \cdot 3 \cdot 3$.

I noticed that when $b = 2$ and $n = 2 \cdot 3$, the value I got for the sum was $63 = 3 \cdot 3 \cdot 7$. But when $b = 2$ and $n = 2$ the sum was 3, and when $b = 2$ and $n = 3$ the sum was 7 and both of these sums are factors of 63.

After a couple more examples, I started to see a pattern: the sum would have n things added up, so if n was composite, I could break it into smaller groups of the same size. For example, I can break a sum of six things into three groups of two or two groups of three:

$$\begin{aligned}
(1+2+2^2+2^3+2^4+2^5) &= (1+2)+(2^2+2^3)+(2^4+2^5) \qquad (4)\\
&= (1+2)+2^2(1+2)+2^4(1+2)\\
&= (1+2^2+2^4)(1+2)\\
&= (21)(3)
\end{aligned}$$

or

$$\begin{aligned}
(1+2+2^2+2^3+2^4+2^5) &= (1+2+2^2)+(2^3+2^4+2^5) \qquad (5)\\
&= (1+2+2^2)+2^3(1+2+2^2)\\
&= (1+2^3)(1+2+2^2)\\
&= (9)(7)
\end{aligned}$$

Then I was ready to write the proof, which meant that the conjecture would be a theorem!

2.1.4 Theorem and Proof

Portfolio Theorem 1. *If n and b are positive integers and $1+b+b^2+\cdots+b^{n-2}+b^{n-1}$ is prime, then n is prime.*

Proof. We will prove this theorem using the contrapositive. That is, we will prove that if n and b are positive integers and n is not prime , then $1+b+b^2+\cdots+b^{n-2}+b^{n-1}$ is not prime. Thus, we will assume that n and b are positive integers and that n is not prime, and we will show that the sum $1+b+b^2+\cdots+b^{n-2}+b^{n-1}$ is not prime.

First, we note that if $n=1$, then the sum is just 1, which is not prime, so the theorem holds in this case.

Next, in the case where $n>1$ and n is not prime, we know that n is composite, so we will be able to find integers $k>1$ and $j>1$ so that $n=kj$. We can think of this as grouping n objects into k groups of size j. So, because the sum $1+b+b^2+\cdots+b^{n-2}+b^{n-1}$ is a sum of n terms, we can write it as

$$\begin{aligned}
1+b+b^2+\cdots+b^{n-2}+b^{n-1} &= (1+b+b^2+\cdots+b^{n-2}+b^{j-1}) \qquad (6)\\
&\quad + (b^j+b^{j+1}+\cdots+b^{j+(j-2)}+b^{2j-1})\\
&\quad\quad +\cdots+(b^{(k-1)j}+b^{(k-1)j+1}+\cdots+b^{(k-1)j+(j-2)}+b^{kj-1})\\
&= (1+b^j+b^{2j}+\cdot+b^{(k-1)j})\\
&\quad \times (1+b+b^2+\cdots+b^{n-2}+b^{j-1}). \qquad (7)
\end{aligned}$$

Since $b>0$, $j>1$, and $k>1$, each of the two sums in equation (7) will be greater than 1, and so the sum $1+b+b^2+\cdots+b^{n-2}+b^{n-1}$ must be composite, and therefore not prime.

Thus, we have shown that if n and b are positive integers and n is not prime, then the sum $1+b+b^2+\cdots+b^{n-2}+b^{n-1}$ is not prime.

□

2.2 Divisibility by Six

A commonly taught “rule” for testing for divisibility by six is to check that a number (an integer) is even and divisible 3. This always works, which means that it is a theorem. Using the fact that an integer is even exactly when it is divisible by 2, we the following theorem.

Portfolio Theorem 2. *Let M be any integer. If M is divisible by 2 and M is divisible by 3 then M is divisible by 6.*

Proof. Stating that M is divisible by 3 means that we can express M as $M = 3k$ where k is an integer. Similarly, Stating that M is divisible by 2 means that we can express M as $M = 2j$ where j is an integer.

We wish to show that if we assume that M is divisible by both 2 and 3, then we will be able to find an integer, say q, so that $M = 6q$.

Because we are given that M is divisible by 3, there is an integer, k so that $M = 3k$. Furthermore, there is an integer j so that $M = 2j$ because we are given that M is divisible by 2. Thus, substituting, we see that

$$3k = 2j. \tag{8}$$

Now, we know that $3 = 2 + 1$, so substituting this into equation (8) and using the distributive property, we have

$$\begin{aligned} 3k &= 2j \\ (2+1)k &= 2j \\ 2k + k &= 2j \\ &\text{so} \\ k &= 2j - 2k \\ &= 2(j-k). \end{aligned} \tag{9}$$

Now, substituting equation (9) into the equation $M = 3k$, we see that

$$\begin{aligned} M &= 3\big(2(j-k)\big) \\ &= 6(j-k). \end{aligned} \tag{10}$$

Thus, given that the integer M is divisible by 2 and 3, we see from equation (10) that M is divisible by 6. □

2.3 A Lemma and a Theorem

Lemma 2. *For any real numbers a, b, and c, if $a \geq b$ and $c \geq 0$ then $ca \geq cb$.*

Proof. Let a, b, and c be real numbers with $a \geq b$ and $c \geq 0$. We will show that $ca \geq cb$ using a direct proof.

We know that $a \geq b$ if and only if $a - b \geq 0$. Furthermore, we know that the product of any two non-negative real numbers will be non-negative. Thus, we know that

$$c(a - b) = ca - cb \geq 0$$

and so

$$ca \geq cb$$

which is what we wanted to show. □

Portfolio Theorem 3. *For any non-negative real numbers a and b, if $a \geq b$, then for all natural numbers n, $a^n \geq b^n$.*

Proof. Let a and b be non-negative real numbers and assume that $a \geq b$. We will prove that for any natural number n, $a^n \geq b^n$ using induction.

Basis Step

For our basis step, we know that $a > b$, so

$$a^1 = a \geq b = b^1.$$

So $a^1 \geq b^1$.

Inductive Step

For our inductive step, we will show that if k is any natural number and $a^k \geq b^k$, then $a^{k+1} \geq b^{k+1}$. Thus, we let k be an arbitrary natural number and we assume that $a^k \geq b^k$. We first note that $a^{k+1} = a \cdot a^k$ and $b^{k+1} = b \cdot b^k$. Now, because $a \geq 0$, we know from Lemma 2 that

$$a \cdot a^k \geq a \cdot b^k. \tag{11}$$

Furthermore, because $b^k \geq 0$ and $a \geq b$, applying Lemma 2 again, we have

$$a \cdot b^k \geq b \cdot b^k. \tag{12}$$

Thus, combining equations (11) and (12), we have

$$\begin{aligned} a^{k+1} &= a \cdot a^k \\ &\geq a \cdot b^k \\ &\geq b \cdot b^k \\ &= b^{k+1}. \end{aligned}$$

So, we have shown that if $a \geq b$, then for any natural number k, if $a^k \geq b^k$, then $a^{k+1} \geq b^{k+1}$, which concludes our inductive step.

Since we have shown the basis step and the inductive step, we have shown that for any non-negative real numbers a and b, if $a \geq b$, then for all natural numbers n, $a^n \geq b^n$. □

2.4 Two Proofs of a Conjecture

Here we present a conjecture together with both a constructive and a non-constructive proof of the conjecture.

Portfolio Theorem 4. *The set of irrational real numbers is not closed under exponentiation. That is, there exist irrational numbers a and b so that a^b is rational.*

2.4.1 A Lemma

Before we begin our proofs of Portfolio Theorem 4, we will prove a lemma that will be needed in the constructive proof. (It is not needed in the non-constructive proof.)

Lemma 3. *The logarithm base* 2 *of* 9*,* $\log_2(9)$*, is irrational.*

Proof of Lemma 3. We will prove that the logarithm base 2 of 9 is irrational by means of a proof by contradiction. Thus, we will assume that $\log_2(9)$ is rational, and we will show that this gives rise to a contradiction.

If $\log_2(9)$ is rational, there must exist (because of what it means to be a rational number) integers p and q, with $q \neq 0$, so that $\log_2(9) = p/q$. This means (from properties of logarithms) that

$$2^{p/q} = 9. \tag{13}$$

Furthermore, because $9 > 2$, we must have that $p/q > 1 > 0$, and we can assume that both p and q are positive integers, since otherwise they would both have to be negative and we would use instead the integers $-p$ and $-q$.

Raising each side of equation (13) to the q power, we obtain

$$2^p = 9^q. \tag{14}$$

Because p and q are positive integers, both sides of equation (14) are integers. Furthermore, the integer on the left, 2^p, must be even, and the number on the right, 9^q, is odd. However, no integer can be both even and odd, and so we have arrived at a contradiction. Therefore, we have proven that $\log_2(9)$ is irrational. □

2.4.2 A Non-Constructive Proof of Portfolio Theorem 4

Non-Constructive Proof of Portfolio Theorem 4. We will show that there must be irrational numbers a and b such that a^b is rational by considering two possibilities for the real number $\left(\sqrt{3}\right)^{\sqrt{2}}$, namely that this number must either be irrational or rational.

Case 1 ($\left(\sqrt{3}\right)^{\sqrt{2}}$ rational) We have shown previously that $\sqrt{2}$ and $\left(\sqrt{3}\right)$ are irrational. If $\left(\sqrt{3}\right)^{\sqrt{2}}$ is rational, then we can set $a = \left(\sqrt{3}\right)$ and $b = \sqrt{2}$, and have that

$$a^b = \left(\sqrt{3}\right)^{\sqrt{2}}$$

so that a and b are irrational and a^b is rational, and so our theorem is true.

Case 2 ($\left(\sqrt{3}\right)^{\sqrt{2}}$ irrational) If $\left(\sqrt{3}\right)^{\sqrt{2}}$ is irrational, then we can set $a = \left(\sqrt{3}\right)^{\sqrt{2}}$ and $b = \sqrt{2}$, and we have

$$\begin{aligned} a^b &= \left(\left(\sqrt{3}\right)^{\sqrt{2}}\right)^{\sqrt{2}} \\ &= \left(\sqrt{3}\right)^{\sqrt{2}\cdot\sqrt{2}} \\ &= \left(\sqrt{3}\right)^{2} \\ &= 3 \end{aligned}$$

and 3 is rational, so we have found a and b irrational with a^b rational, so our theorem is true.

Since we have shown that whether $\left(\sqrt{3}\right)^{\sqrt{2}}$ is rational or irrational, there exist irrational numbers a and b with a^b rational, we have proven our theorem. □

2.4.3 A Constructive Proof of Portfolio Theorem 4

Constructive Proof of Portfolio Theorem 4. We will show that there exist irrational numbers a and b with the property that a^b is rational by exhibiting such numbers. Let $a = \sqrt{2}$ and let $b = \log_2(9)$. We have shown previously that $\sqrt{2}$ is irrational, and by Lemma 3 we know that $\log_2(9)$ is irrational. So, using properties of logarithms and of exponents, we have

$$\begin{aligned} a^b &= \left(\sqrt{2}\right)^{\log_2(9)} \\ &= \left(2^{1/2}\right)^{\log_2\left(3^2\right)} \\ &= \left(2^{1/2}\right)^{2\log_2(3)} \\ &= (2)^{\log_2(3)} \\ &= 3. \end{aligned}$$

Because 3 is rational, we have shown that there exist irrational numbers a and b such that a^b is rational, as desired. □

2.5 A Non-Algebraic Induction Proof

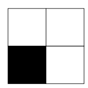
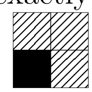
Definition 2. An L-triomino is a geometric figure consisting of three congruent squares arranged into an L,  . We say that a region R can be tiled by L-triominoes if the region can be completely covered by L-triominoes without overlapping.

Portfolio Theorem 5. *For any natural number n, any $2^n \times 2^n$ chessboard composed of squares of a particular size with one square removed can be completely tiled using L-triominoes consisting of squares of the same given size.*

Proof. We will show that for any natural number n, any chessboard of size $2^n \times 2^n$ with one square deleted can always be tiled using L-triominoes using mathematical induction.

Basis Step

For our basis step, we need to show that any $2^1 \times 2^1 = 2 \times 2$ chess board with one square deleted can be tiled using L-triominoes.

We first note that we can always rotate the 2×2 chessboard so that the deleted square is in the lower left corner: . Here, it is obvious that we can exactly tile the three remaining squares of the 2×2 chessboard with one L-triomino, .

Inductive Step

For our inductive step, we will fix k to be some positive integer and show that if every $2^k \times 2^k$ chessboard with one square deleted can be tiled using L-triominoes, then every $2^{k+1} \times 2^{k+1}$ chessboard with one square deleted can be tiled using L-triominoes.

Given any $2^{k+1} \times 2^{k+1}$ chessboard with one square deleted, we first partition the chessboard into four regions of size $2^k \times 2^k$, and then rotate the chessboard so that the missing square is in the lower left $2^k \times 2^k$ region, as shown in Figure 1.

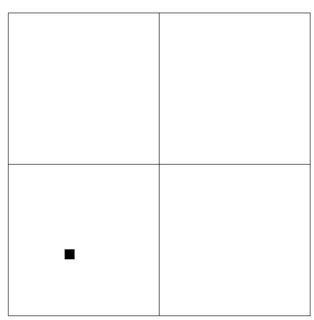


Figure 1: A $2^{k+1} \times 2^{k+1}$ chessboard partitioned into four $2^k \times 2^k$ regions.

It is clear that the lower left region is precisely a $2^k \times 2^k$ chessboard with one square deleted and thus, by our inductive hypothesis, can be tiled with L-triominoes.

Next, we place one L-triomino adjacent to the lower left region so that it occupies one square in each of the remaining three $2^k \times 2^k$ regions as shown in Figure 2.

Because the L-triomino covers exactly one tile in each of the three $2^k \times 2^k$ regions, the remaining squares form three $2^k \times 2^k$ chessboards each with one square deleted, and

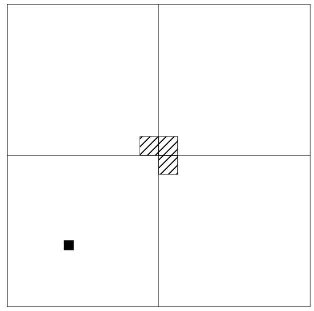


Figure 2: A $2^{k+1} \times 2^{k+1}$ chessboard with one L-triomino.

thus they can, by our inductive hypothesis, be tiled with L-triominoes as well, and thus the entire $2^{k+1} \times 2^{k+1}$ chessboard can be tiled with L-triominoes. This completes the inductive step.

Since we have proven the basis step and the inductive step, we have proven our proposition, that for any natural number n, any $2^n \times 2^n$ chessboard with one square removed can be completely tiled using L-triominoes. $\square$

2.6 A Proof of Bijection

Portfolio Theorem 6. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 2; & x = 3 \\ \frac{4x+3}{2x-6}; & x \neq 3 \end{cases} \tag{15}$$

is a bijection.

Proof. We will prove that the function, f, defined by equation (15) is a bijection by proving that f is an injection and that f is a surjection.

Injectivity To prove that f is an injection, we need to show that for any x_1 and x_2 that are elements of the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. We will do this by using the contrapositive. That is, we will let x_1 and x_2 be arbitrary elements of the domain of f and show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. In this case, this means that we assume x_1 and x_2 are arbitrary real numbers and that $f(x_1) = f(x_2)$ and we will show that $x_1 = x_2$.

Because f is piecewise defined, we need to consider that we could have $x_1 = 3$ or $x_1 \neq 3$, and similarly that we could have $x_2 = 3$ or $x_2 \neq 3$. If $x_1 = 3$ and $x_2 = 3$, then there is nothing to show, since $x_1 = 3 = x_2$. Thus we have to consider the following cases: $x_1 = 3$ and $x_2 \neq 3$, $x_1 \neq 3$ and $x_2 = 3$, and $x_1 \neq 3$ and $x_2 \neq 3$. It is easy to see that by simply relabeling x_1 and x_2, the first two cases are really the same. That leaves us with two cases to consider.

Case 1 ($x_1 = 3$ and $x_2 \neq 3$) This is a proof by contradiction, and thus we will assume that $x_1 = 3$, $x_2 \neq 3$, and $f(x_1) = f(x_2)$, and show that a contradiction arises. Since $x_1 = 3$ and $x_2 \neq 3$, we know that $f(x_1) = f(3) = 2$ and $f(x_2) = \frac{4x_2+3}{2x_2-6}$. Because we assume that $f(x_1) = f(x_2)$, we thus have

$$\frac{4x_2 + 3}{2x_2 - 6} = 2. \tag{16}$$

Multiplying each side of equation (16) by $2x_2 - 6$, we obtain

$$\begin{aligned} 4x_2 + 3 &= 2(2x_2 - 6) \\ &= 4x_2 - 12. \end{aligned} \tag{17}$$

By subtracting $4x_2$ from both sides of equation (17) we get $3 = -12$), which is a contradiction. Thus, the $x_1 = 3$, $x_2 \neq 3$, and $f(x_1) = f(x_2)$ cannot occur.

Case 2 ($x_1 \neq 3$ and $x_2 \neq 3$) It now remains to show that if x_1 and x_2 are any real numbers with $x_1 \neq 3$, $x_2 \neq 3$, and $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Because $x_1 \neq 3$, $x_2 \neq 3$, and $f(x_1) = f(x_2)$, we have that

$$\frac{4x_1 + 3}{2x_1 - 6} = \frac{4x_2 + 3}{2x_2 - 6}. \tag{18}$$

Multiplying both sides of equation (18) by $(2x_1 - 6)(2x_2 - 6)$ and expanding, we get

$$(4x_1 + 3)(2x_2 - 6) = (4x_2 + 3)(2x_1 - 6)$$
$$\text{and so}$$
$$8x_1x_2 - 24x_1 + 6x_2 - 18 = 8x_1x_2 + 6x_1 - 24x_2 - 18. \tag{19}$$

Subtracting the left hand side of equation (19) from both sides, we see that

$$\begin{aligned} 0 &= 30x_1 - 30x_2 \\ &= 30(x_1 - x_2). \end{aligned} \tag{20}$$

From equation (20), since $30 \neq 0$, we see that $x_1 - x_2 = 0$, and thus $x_1 = x_2$, as desired.

Thus, we have shown that the function f is injective.

Surjectivity We will now show that f is surjective by showing that for any b in the codomain of f, there is an element x in the domain of f so that $f(x) = b$. Thus, we let b be an arbitrary element of the codomain of f, which is to say that b is any real number.

We will consider two cases for the value of b: either $b = 2$ or $b \neq 2$.

Case 1 ($b = 2$) If $b = 2$, then we see that if $x = 3$, then $f(x) = 2 = b$ as desired.

If $b \neq 2$, $2b-4 \neq 0$ and so $x = \frac{6b+3}{2b-4}$ is a real number, and thus is in the domain of f. Furthermore,

$$\begin{aligned} f(x) &= f\left(\frac{6b+3}{2b-4}\right) \\ &= \frac{4\left(\frac{6b+3}{2b-4}\right)+3}{2\left(\frac{6b+3}{2b-4}\right)-6} \\ &= \frac{4\left(6b+3\right)+3\left(2b-4\right)}{2\left(6b+3\right)-6\left(2b-4\right)} \\ &= \frac{24b+12+6b-12}{12b+6-12b+24} \\ &= \frac{30b}{30} \\ &= b. \end{aligned}$$

We have thus shown that for any element b of the codomain of f, there is an element x of the domain of f so that $f(x) = b$, which completes our proof that f is surjective.

Since we have shown that f is injective and surjective, this completes our proof that f is a bijection. $\square$

2.7 Checking and Disproving Injectivity

We start by defining the set $D = \mathbb{R} - \{3\}$, and looking at the function $f : D \to \mathbb{R}$ defined by

$$f : x \mapsto \frac{2x^3}{x-3}.$$

We will investigate the question of whether f is injective or not.

We will check by letting x and z be arbitrary elements of the domain, $D = \mathbb{R} - \{3\}$, of f. If we can show that the *only* way for this to happen is for $x = z$, then we know that f is injective. On the other hand, if we can find $x \neq z$ so that $f(x) = f(z)$, then we know that f is *not* injective.

We assume that $f(x) = f(z)$. So,

$$\frac{2x^3}{x-3} = \frac{2z^3}{z-3}. \tag{21}$$

We can multiply equation (21) by $(x-3)(z-3)$ and simplify to get

$$\frac{\cancel{2}x^3}{\cancel{(x-3)}}\cancel{(x-3)}(z-3) = \frac{\cancel{2}z^3}{\cancel{(z-3)}}(x-3)\cancel{(z-3)}$$

or

$$x^3(z-3) = z^3(x-3). \tag{22}$$

Next we subtract $z^3(x-3)$ to obtain

$$x^3(z-3) - z^3(x-3) = 0. \tag{23}$$

By expanding, collecting like terms, and factoring equation (23), we have

$$\begin{aligned}
0 &= x^3(z-3) - z^3(x-3) \\
&= x^3 z - 3x^3 - x\,z^3 + 3z^3 \\
&= x^3 z - x\,z^3 - 3x^3 + 3z^3 \\
&= x^3 z - x\,z^3 - 3(x^3 - z^3) \\
&= xz(x^2 - z^2) - 3(x^3 - z^3) \\
&= xz(x+z)(x-z) - 3(x^2 + x\,z + z^2)(x-z) \\
&= \big(xz(x+z) - 3(x^2 + x\,z + z^2)\big)\,(x-z).
\end{aligned} \tag{24}$$

From equation (24), we can see that if $f(x) = f(z)$, then either

$$x - z = 0 \text{ and thus } x = z$$
$$\text{or}$$
$$xz(x+z) - 3(x^2 + x\,z + z^2) = 0. \tag{25}$$

If we can show that for all $x, z \in D$, $xz(x+z) - 3(x^2 + x\,z + z^2) \neq 0$, then we will have shown that f is injective.

Otherwise, if we can find $x \neq z$ so that $xz(x+z) - 3(x^2 + x\,z + z^2) = 0$, we have a likely candidate to show that f is *not* injective.

Now, if $x = 0$, then $f(x) = f(0) = \frac{2 \cdot 0^3}{0-3} = 0$, and thus if $f(x) = f(z)$, then $f(z) = \frac{2z^3}{z-3} = 0$. Multiplying both sides by $z - 3$ gives us $z = 0$. So, of $f(0) = f(z)$, then $z = 0$.

Having dealt with the case $x = 0$ separately, we can now assume that $x \neq 0$ and, by symmetry, $z \neq 0$.

Since $x \neq 0$, we can define $a = {}^{z}\!/\!{}_{x}$, so that $z = a\,x$, and substitute $z = a\,x$ in equation (25). Expanding and then factoring a common factor of x^2, we get

$$\begin{aligned}
0 &= xz(x+z) - 3(x^2 + x\,z + z^2) \\
&= x(a\,x)(x + a\,x) - 3x^2 - 3x\,(a\,x) - 3(a\,x)^2 \\
&= x(a\,x)(x + a\,x) - 3(x^2 + x\,(a\,x) + (a\,x)^2) \\
&= a\,x^3 + a^2 x^3 - 3(x^2 + a\,x^2 + a^2 x^2) \\
&= (a^2 + a)x^3 - 3(a^2 + a + 1)x^2 \\
&= x^2\left((a^2 + a)x - 3(a^2 + a + 1)\right).
\end{aligned} \tag{26}$$

We have already said that $x \neq 0$, so equation (26) implies that

$$(a^2 + a)x - 3(a^2 + a + 1) = 0.$$

and thus

$$(a^2 + a)x = 3(a^2 + a + 1). \tag{27}$$

Recalling that $a = {}^{z}\!/\!{}_{x}$ and we want to find $x \neq z$ so that $f(x) = f(z)$, any value of a other than $a = 1$ is reasonable. In particular, as long as we choose a so that $a \neq 0$ and $a \neq -1$ we can divide by $a^2 + a$ in equation (27) getting

$$x = \frac{3(a^2 + a + 1)}{a^2 + a}. \tag{28}$$

Now let's pick a value of a. If we try $a = -2$, we get

$$\begin{aligned} x &= \frac{3\big((-2)^2 + (-2) + 1\big)}{(-2)^2 + (-2)} \\ &= \frac{3(4 - 2 + 1)}{4 - 2} \\ &= \frac{3(3)}{2} \\ &= \frac{9}{2}, \end{aligned} \tag{29}$$

and $z = a\,x = (-2)({}^{9}\!/\!{}_{2}) = -9$.

Clearly $-9 \neq {}^{9}\!/\!{}_{2}$, so if $f(-9) = f({}^{9}\!/\!{}_{2})$, then we will have shown that f is not injective. Checking, we see that

$$\begin{aligned} f(-9) &= \frac{2 \cdot (-9)^3}{-9 - 3} \\ &= \frac{-2 \cdot 9^3}{-12} \\ &= \frac{9^3}{6} \\ &= \frac{243}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} f\left({}^{9}\!/\!{}_{2}\right) &= \frac{2 \cdot \left({}^{9}\!/\!{}_{2}\right)^3}{\left({}^{9}\!/\!{}_{2}\right) - 3} \\ &= \frac{9^3/2^2}{(9 - 6)/2} \\ &= \frac{9^3/2}{3} \\ &= \frac{9^3}{6} \\ &= \frac{243}{2}, \end{aligned}$$

and thus we have that $-9 \neq {}^{9}\!/\!{}_{2}$ and $f\left(-9\right) = f\left({}^{9}\!/\!{}_{2}\right)$, so f is not injective. We are now ready to state a conjecture and proof.

Portfolio Theorem 7. *Let $D = \mathbb{R} - \{3\}$, and let the function $f\colon D \to \mathbb{R}$ be defined by*

$$f\colon x \mapsto \frac{2x^3}{x - 3}.$$

The function f is not injective.

Proof. We will show that the function $f(x) = \frac{2x^3}{x-3}$ is not an injection by showing two distinct elements of the domain that map to the same element of the codomain. That is, we will find $x_1, x_2 \in D$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Let $x_1 = -9$ and let $x_2 = {}^9\!/_2$. Since $x_1 \neq 3$ and $x_2 \neq 3$, we have that $x_1 \in D$ and $x_2 \in D$. It is also clear that $x_1 \neq x_2$. Thus, it remains to show that $f(x_1) = f(x_2)$.

By direct calculation, we see that

$$\begin{aligned} f(x_1) &= f(-9) \\ &= \frac{2 \cdot (-9)^3}{-9-3} \\ &= \frac{-2 \cdot 9^3}{-12} \\ &= \frac{9^3}{6} \\ &= \frac{243}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} f(x_2) &= f\left({}^9\!/_2\right) \\ &= \frac{2 \cdot \left({}^9\!/_2\right)^3}{\left({}^9\!/_2\right) - 3} \\ &= \frac{9^3/2^2}{(9-6)/2} \\ &= \frac{9^3/2}{3} \\ &= \frac{9^3}{6} \\ &= \frac{243}{2}, \end{aligned}$$

and thus $f(x_1) = f(x_2)$.

Since we have shown that $-9 \neq {}^9\!/_2$ and $f(-9) = f\left({}^9\!/_2\right)$, we have shown that f is not injective. $\square$