Character sheaves and Hecke algebras

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joint work with K. Vilonen, partly with M. Grinberg

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Here

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Our goal Describe the character sheaves in $\mathrm{Char}_K(\mathfrak{g}_1)$ explicitly.

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- The bijection

$$\mathrm{SPerv}_K(\mathcal{N}_{-1}) \xrightarrow{\mathfrak{F}} \mathrm{Char}_K(\mathfrak{g}_1)$$

  can be viewed as an analogue of Springer correspondence/Lusztig's generalized Springer correspondence for $\mathbb{Z}/m\mathbb{Z}$-graded Lie algebras.

Ungraded case/Springer correspondence

Recall Grothendieck-Springer resolution

$$\begin{tikzcd} \widetilde{\mathcal{N}} = G \times^B \mathfrak{n} \arrow[r] \arrow[d, "\pi"'] & \widetilde{\mathfrak{g}} = G \times^B \mathfrak{b} \arrow[d, "\check{\pi}"] \\ \mathcal{N} \arrow[r] & \mathfrak{g} \end{tikzcd}$$

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$$\begin{array}{ccc}
\mathfrak{F}(\pi_* \mathbb{C}_{\widetilde{\mathcal{N}}}) & \cong & \check{\pi}_* \mathbb{C}_{\widetilde{\mathfrak{g}}}[-] \\
\Vert & & \Vert \\
\mathfrak{F}(\oplus \mathrm{IC}(\mathcal{O}, \mathcal{L}) \otimes V_{\mathcal{O},\mathcal{L}}) & \cong & \bigoplus_{\rho \in \hat{W}_G} \mathrm{IC}(\mathfrak{g}^{rs}, \mathcal{M}_\rho) \otimes V_\rho
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$\Rightarrow \mathfrak{F}\,\mathrm{IC}(\mathcal{O}, \mathcal{L}) \cong \mathrm{IC}(\mathfrak{g}^{rs}, \mathcal{M}_\rho)$ (Springer correspondence)

$\mathbb{Z}/m\mathbb{Z}$-graded Lie algebras

Remark: In our context, an analogue of Grothendieck-Springer resolution does not seem to exist.
Cohomology of Hessenberg varieties (of regular semisimple elements) occuring in our situation is more complicated.

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Definition: A character sheaf is called *cuspidal* if it does not arise from parabolic induction of character sheaves in $\mathrm{Char}_{L_K}\,\mathfrak{l}_1$ for θ-stable Levi subgroups L contained in proper θ-stable parabolic subgroups.

Cuspidal character sheaves

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They are associated to irreducible representations of Hecke algebras of complex reflection groups at roots of unity.

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- We can view the nearby cycle construction as a replacement of the Grothendieck-Springer resolution.

The adjoint quotient

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Let $\mathfrak{a} \subset \mathfrak{g}_1$ be a Cartan subspace. By Vinberg, we have

$$f : \mathfrak{g}_1 \to \mathfrak{g}_1/\!\!/K \cong \mathfrak{a}/W$$

where $W \cong N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. It is in general a complex reflection group.

Nearby cycle construction

Let $\bar{a} \in \mathfrak{a}^{rs}/W$. Then $X_{\bar{a}} = f^{-1}(a)$ is a regular semisimple K-orbit. We have

$$\pi_1^K(X_{\bar{a}}) \cong Z_K(\mathfrak{a})/Z_K(\mathfrak{a})^0 := I.$$

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Theorem (Grinberg) We have

$$\mathfrak{F}P_\chi \cong \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_\chi)$$

where $\mathcal{M}_\chi$ is given by a representation M_χ of $\pi_1^K(\mathfrak{g}_1^{rs})$.

The equivariant fundamental group $\pi_1^K(\mathfrak{g}_1^{rs})$

We have

$$\begin{tikzcd}
1 \arrow[r] & I \arrow[r] \arrow[d, equal] & \pi_1^K(\mathfrak{g}_1^{rs}) \arrow[r] \arrow[d] & B_W \arrow[r] \arrow[d] & 1 \\
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The top sequence splits via a Kostant slice. We write

$$\pi_1^K(\mathfrak{g}_1^{rs}) := \widetilde{B}_W \cong B_W \ltimes I.$$

The local system $\mathcal{M}_\chi$

Theorem (Grinberg-Vilonen-X.) We have

$$M_\chi \cong \mathbb{C}[\widetilde{B}_W] \otimes_{\mathbb{C}[\widetilde{B}^0_{W,\chi}]} \left(\mathcal{H}_{W^0_\chi} \otimes \mathbb{C}_\chi\right)$$

where $W^0_\chi \subset \mathrm{Stab}_W\,\chi$ is a complex reflection subgroup and $\mathcal{H}_{W^0_\chi}$ is a cyclotomic Hecke algebra associated to W^0_χ.

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- The full support constituents of $\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_\chi)$ are full support cuspidal character sheaves (conjecturally all).
- The above theorem holds in the setting of stable polar representations.

The Hecke algebra $\mathcal{H}_{W^0_\chi}$

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Example: when $m = 2$ and (G, K) split pair, W_χ^0 is a Coxeter group and the Hecke relations for $\mathcal{H}_{W_\chi^0}$ are $(x - 1)^2 = 0$. (Endoscopic interpretation as pointed out by C.C. Tsai.)

Connection with Lusztig-Yun's work

$$\begin{tikzcd}[column sep=huge, row sep=large]
\left\{\begin{array}{c}\textit{cuspidal}\text{ character}\\ \text{sheaves}\end{array}\right\} \arrow[r, leftrightarrow, "\text{Nearby}", "\text{cycle}"'] \arrow[d, leftrightarrow, "\text{Fourier}"', "\text{transform}"] & \left\{\begin{array}{c}\text{simple modules of}\\ \text{cyclotomic Hecke}\\ \text{algebras (at roots of unity)}\end{array}\right\} \\
\left\{\begin{array}{c}\textit{cuspidal}\text{ nilpotent}\\ \text{orbital sheaves}\end{array}\right\} \arrow[d, "\begin{array}{c}\text{Lusztig-Yun}\\ \text{W. Liu}\end{array}"'] & \left\{\begin{array}{c}\textit{full support}\text{ simple modules of}\\ \text{rational DAHAs}\\ \text{of complex reflection groups}\end{array}\right\} \arrow[u, "\text{GGOR}", "\text{KZ functor}"'] \\
\left\{\begin{array}{c}\textit{f.d.}\text{ simple modules of}\\ \text{trignometric DAHAs}\\ \text{of Coxeter groups}\end{array}\right\} \arrow[r, "\text{Etingof}", "\text{reduction}"'] & \left\{\begin{array}{c}\textit{f.d.}\text{ simple modules}\\ \text{of rational DAHAs}\\ \text{of Coxeter groups}\end{array}\right\} \arrow[u, leftrightarrow, "\text{Koszul duality}"']
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Moreover, in classical types, the arrow *Koszul duality* coincides with the Koszul duality for blocks of cyclotomic rational DAHAs (type $G_{m,1,r}$) conjectured by Chuang-Miyachi, established by Rouquier-Shan-Varagnolo-Vasserot.

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Remark: The diagram also suggests Koszul duality between blocks of rational DAHAs of exceptional types.