Chip firing games: applications of the Smith Normal Forms of Combinatorial matrices

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1 Introduction

The Smith Normal Forms (SNF) of an incidence matrix is a powerful invariant that may help distinguish the underlying incidence structure. Of special interest is the SNF of the Laplacian matrix of a graph. This comes up in the context of the popular chip firing game on graphs (cf. [2]). We will now briefly describe this game and its connections to the Laplacian matrix.

The chip firing game on graphs was introduced by Bjorner, Lovasz, and Shor in [3]. Fix a graph $\mathcal{G}$ on vertex set $\{1, 2 \ldots, n\}$, and start by putting chips a_i at the ith vertex. So $a = (a_1, \ldots a_1 \ldots a_n) \in \mathbb{Z}_+^n$, and let $N = \sum a_i$. Each step of the game involves *firing* a vertex v, that is, one chip from v goes to each of its adjacent vertices. A vertex v can be *fired* if the number of chips currently held at v is atleast the degree of v.

A *position* of size N on $\mathcal{G}$ is a distribution of N chips to vertices of $\mathcal{G}$, that is a vector $b = (b_1, \ldots b_i, \ldots b_n)$ such that $b \in \mathbb{Z}_+^n$ and $\sum b_i = N$. A *legal* game is any sequence of *positions*, such that each *position* is obtained from the previous one by *firing* at a vertex of $\mathcal{G}$. The following was proved in [3].

Theorem 1. *(Theorem 2.1 of [3]) Given a connected graph, with an initial distribution of chips, either every legal game can be continued indefinitely, or every legal game terminated after the same number of steps and the same final position.*

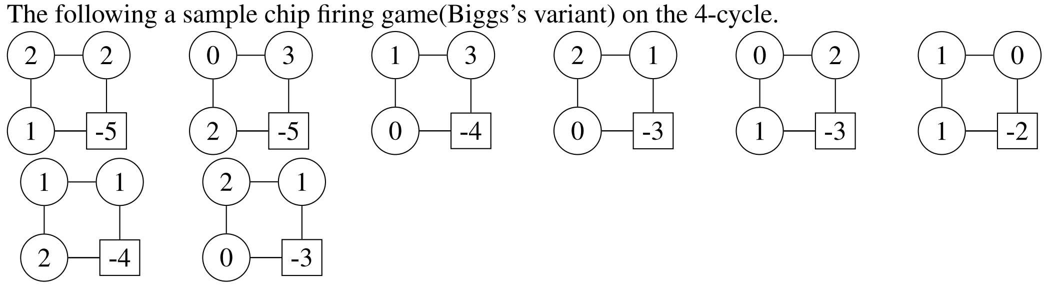
Note that if the game terminates, irrespective of the order in which we fire vertices, we end up at the same final *position*.

In [2], Biggs introduced a variant of this game. In this we fix a vertex q that can have a negative number of chips, in fact we need q to always be in "debt". A *configuration* on G is a distribution of $s(j)$ chips at every vertex $j \neq q$ and $s(q) = -\sum s(j)$ chips at q. Just as in the previous game, a vertex $v \neq q$ can be *fired* if the number of chips currently held at v is atleast the degree of v. The vertex q can be *fired* if no other vertex can be fired.

Each step of the game involves *firing* a vertex v, that is, one chip from v goes to each of its adjacent vertices. A vertex $v \neq q$ can be *fired* if the number of chips currently held at v is atleast the degree of v.

A *legal* game is any sequence of *configuration*, such that each *configuration* is obtained from the previous one by *firing* at a vertex of $\mathcal{G}$. A *configuration* is said to be recurrent if it is the initial and terminal *configuration* of a *legal* game. A *configuration* is said to be stable if no vertex but q can be *fired*. A *configuration* is said to be *critical* if it is both recurrent and stable.

The following a sample chip firing game(Biggs's variant) on the 4-cycle.



Let s and s' be the initial and terminal positions of a *legal* game. If the ith vertex is *fired* x_i times, we can see that $s' - s = Lx$, where L is the Laplacian of $\mathcal{G}$. By the *critical* group of $\mathcal{G}$, we mean the torsion part of the cokernel of the Laplacian matrix. We denote this abelian group by $K(\mathcal{G})$. By Kirchoff's Matrix-Tree Theorem, $|K(\mathcal{G})|$ is equal to the number of spanning trees in $\mathcal{G}$.

The following was proved by Biggs in [2].

Theorem 2 (Biggs 1997)**.** *Any starting configuration of a graph $\mathcal{G}$ leads to a unique critical configuration. The set of critical configuration has a natural group operation that is isomorphic to the critical group $K(\mathcal{G})$.*

The structure of the critical group of a graph follows from finding the SNF of the Laplacian matrix. For a nice survey on critical groups of graphs, we refer to [13, §3]. Some papers with computations of critical groups of families of graphs include [14], [10], [9], [5], [1], [8], [6], [4], [12], and [11]. In [9], Lorenzini examined the proportion of graphs with cyclic critical groups among graphs with critical groups of particular order.

2 Potential research plan.

Literature indicates that the class of Strongly regular graphs have been amenable to the computation of the critical groups. There have been many papers computing the critical groups of various classes of Strongly Regular graphs. Quite a few of these have been co-authored by undergraduate researchers. The students participating in this program will be encouraged to pick a class of strongly regular graphs and perform some computer experiments to conjecture the structure of the associated critical group. Upon doing so, taking ideas from literature, they can try to prove their conjectures.

Another approach would be to possibly extend these computations to graphs which are not strongly regular. In [7], the critical group of the Kneser graph $K(n, 2)$ was computed. The authors used the strong regularity of $K(n, 2)$ and the representation theory of the symmetric group. We could look at possibly computing the same for $K(n, 3)$. While $K(n, 3)$ is not strongly regular, we could make use of the fact that it is distance regular and that S_n acts as a group of automorphisms.

3 Prerequisites

Strong back ground in linear algebra is a must. We may be using some techniques from ring theory and representation theory of groups, but students will be able to learn these concepts on the go.

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