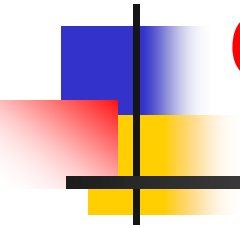

Chapter 2

Boolean Algebra and Logic Gates

J.J. Shann

Chapter Overview

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2-2 Basic Definitions

2-3 Axiomatic Definition of Boolean Algebra

2-4 Basic Theorems and Properties of Boolean Algebra

2-5 Boolean Functions

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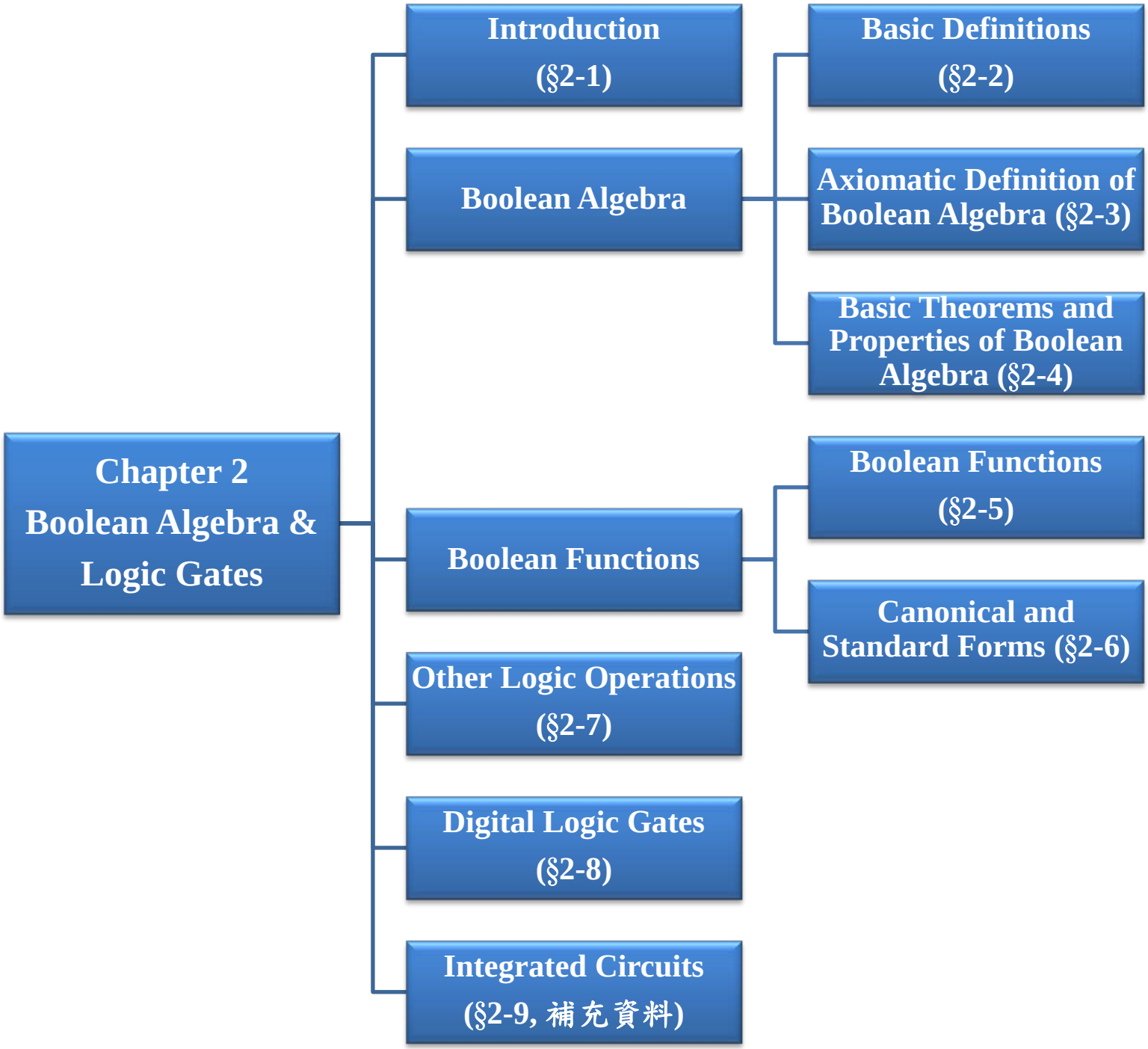
補充資料：Technology Parameters


Chapter 2
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2-1 Introduction

- Important factor of digital circuit design:
  - the cost of the circuit

  $\Rightarrow$ Find simpler and cheaper, but equivalent, realizations of a circuit
- **Boolean algebra**:
  - Mathematical methods that simplify circuits rely primarily on Boolean algebra.

2-2 Basic Definitions

- A deductive mathematical system may be defined with
  - a set of elements: *S*
  - a set of operators: binary/unary operator
  - a number of unproven axioms or postulates:
    - Form the basic assumptions from which it is possible to deduce the rules, theorems, and properties of the system

Common Postulates

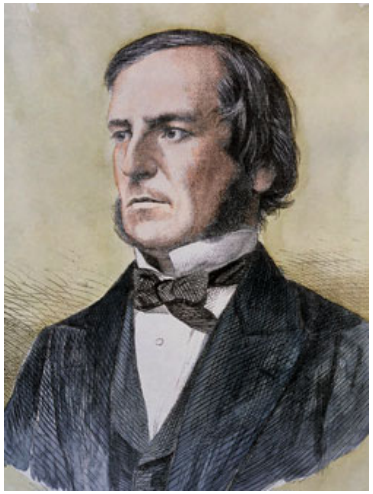
- elements: $\in S$
- operators
- axioms or postulates

- For algebraic structures:
  1. Closure
     - E.g.: the set of natural numbers $N = \{1, 2, 3, 4, \ldots\}$ is closed w.r.t. "+" (arithmetic addition), but not to "–" (arithmetic subtraction)
  2. Associative law: $(x * y) * z = x * (y * z)$
  3. Commutative law: $x * y = y * x$
  4. Identity element: e, $e * x = x * e = x$
     - E.g.: 0 is an identity element w.r.t. "+" on the set of integers $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
  5. Inverse: $x * y = e$
     - E.g.: In the set of integers, I, and the operator +, with $e = 0$, the inverse of an element a is $(-a)$.
  6. Distributive law: $x * (y \bullet z) = (x * y) \bullet (x * z)$

2-3 Axiomatic Definition of Boolean Algebra

- 1854 George Boole: Boolean algebra
- 1904 E. V. Huntington: postulates
- 1938 C. E. Shannon: 2-valued Boolean algebra
  (switching algebra, binary logic)

https://en.wikipedia.org/wiki/Edward_Vermilye_Huntington



George Boole

(1815~1864)

https://en.wikipedia.org/wiki/George_Boole

C.E. Shannon

(1916~2001)

https://en.wikipedia.org/wiki/Claude_Shannon

Boolean Algebra

- Boolean algebra: an algebra structure
  - a set of elements: $\boldsymbol{B}$
  - a set of operators: 2 binary operators: +, •
  - (Huntington) postulates

Huntington Postulates

- (Huntington) postulates:
  1. (a) Closure w.r.t the operator +.
     (b) Closure w.r.t the operator •.
  2. (a) An identity element w.r.t. +: 0, $x + 0 = 0 + x = x$
     (b) An identity element w.r.t. •: 1, $x \cdot 1 = 1 \cdot x = x$
  3. (a) Commutative w.r.t. +: $x + y = y + x$
     (b) Commutative w.r.t. • : $x \cdot y = y \cdot x$
  4. (a) • is distributive over +: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
     (b) + is distributive over • : $x + (y \cdot z) = (x + y) \cdot (x + z)$
  5. Complement: x', (a) $x + x' = 1$ (b) $x \cdot x' = 0$
  6. There exists at least two elements $x, y \in \boldsymbol{B}$ s.t. $x \neq y$.
  - ❖ Associative law: can be derived from the other postulates

Two-Valued Boolean Algebra

- Set of elements: $\boldsymbol{B} = \{0, 1\}$
- Set of operators: 2 binary operators: +, •

x y	$x \cdot y$
0 0	0
0 1	0
1 0	0
1 1	1

AND

x y	$x+y$
0 0	0
0 1	1
1 0	1
1 1	1

OR

x	x'
0	1
1	0

NOT

(Postulate 5)

- (Huntington) postulates:

$x\ y$	$x \cdot y$
0 0	0
0 1	0
1 0	0
1 1	1

$x\ y$	$x+y$
0 0	0
0 1	1
1 0	1
1 1	1

x	x'
0	1
1	0

- (Huntington) postulates:

1. Closure
2. Identity elements: two; 0 for +, 1 for •

   $x + 0 = 0 + x = x$, $x \cdot 1 = 1 \cdot x = x$
3. Commutative laws: $x + y = y + x$, $x \cdot y = y \cdot x$
4. (a) • is distributive over +: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

   (b) + is distributive over • : $x + (y \cdot z) = (x + y) \cdot (x + z)$
5. Complement:

   (a) $x + x' = 1$: $0 + 0' = 0 + 1 = 1$, $1 + 1' = 1 + 0 = 1$

   (b) $x \cdot x' = 0$: $0 \cdot 0' = 0 \cdot 1 = 0$, $1 \cdot 1' = 1 \cdot 0 = 0$
6. $\boldsymbol{B} = \{0,1\}$, $0 \neq 1$.

Proof by Truth Table

*** Prove 4(b)!**

<Proof> 4(a) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

$x\ y\ z$	$y + z$	$x \cdot (y + z)$	$x \cdot y$	$x \cdot z$	$(x \cdot y) + (x \cdot z)$
0 0 0	0	0	0	0	0
0 0 1	1	0	0	0	0
0 1 0	1	0	0	0	0
0 1 1	1	0	0	0	0
1 0 0	0	0	0	0	0
1 0 1	1	1	0	1	1
1 1 0	1	1	1	0	1
1 1 1	1	1	1	1	1

2-4 Basic Theorems and Properties of Boolean Algebra

- Duality:
  - Every algebraic expression deducible from the postulates of Boolean algebra (or truth table) remains valid

    if the operators and identity elements are interchanged

    (OR $\leftrightarrow$ AND, 0 $\leftrightarrow$ 1)

    * Positive vs. Negative Logic (p.2-74~2-76)

  - E.g.:

    4.(a) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

    (b) $x + (y \cdot z) = (x + y) \cdot (x + z)$] dual

    5.(a) $x + x' = 1$

    (b) $x \cdot x' = 0$] dual

A. Basic Theorems

- Postulates and theorems of Boolean algebra:
  - Postulates: basic axioms, need no proof
  - Theorems: must be proven
    i. From the postulates and proven theorems
    ii. From truth table

dual

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

Theorem 1

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

- 1(a): $x + x = x$

$$x + x = (x + x) \bullet 1$$ by postulate: 2(b)

$$= (x + x)\,(x + x')$$ 5(a)

$$= x + xx'$$ 4(b)

$$= x + 0$$ 5(b)

$$= x$$ 2(a)

x	$x + x$
0	$0 + 0 = 0$
1	$1 + 1 = 1$

- 1(b): $x \bullet x = x$

$$x \bullet x = x\,x + 0$$ by postulate: 2(a)

$$= x\,x + x\,x'$$ 5(b)

$$= x\,(x + x')$$ 4(a)

$$= x \bullet 1$$ 5(a)

$$= x$$ 2(b)

x	$x \cdot x$
0	$0 \cdot 0 = 0$
1	$1 \cdot 1 = 1$

Theorem 2

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

- 2(a): $x + 1 = 1$

$x + 1 = 1 \bullet (x + 1)$ by postulate: 2(b), 3(b)

$= (x + x')\,(x + 1)$ 5(a)

$= x + x' \bullet 1$ 4(b)

$= x + x'$ 2(b)

$= 1$ 5(a)

x	$x + 1$
0	0 + 1 = 1
1	1 + 1 = 1

- 2(b): $x \bullet 0 = 0$

$x + 1 = 1$ is proven

by duality principle (OR ↔ AND, 0 ↔ 1)

$x \bullet 0 = 0$

Theorem 3: Involution

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

- $(x')' = x$
  - Postulate 5 defines the complement of x:

    $x + x' = 1$ and $x\,x' = 0$
  - The complement of x' is x and is also $(x')'$.

    Since the complement is unique $\Rightarrow (x')' = x$.

x	x'	$(x')'$
0	1	0
1	0	1

Theorem 5: DeMorgan

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

- 5(a): $(x + y)' = x'\,y'$

x y	$x + y$	$(x + y)'$	x'	y'	$x'y'$
0 0	0	1	1	1	1
0 1	1	0	1	0	0
1 0	1	0	0	1	0
1 1	1	0	0	0	0

- 5(b): $(x\,y)' = x' + y'$

* Prove Theorem 5 by using Postulate 5!

- Two variables: $(x + y)' = x'\,y'$, $(x\,y)' = x' + y'$
- Three or more variables:

$(A + B + C)'$	$= (A + X)'$	let $B + C = X$
	$= A'\,X'$	by DeMorgan's
	$= A'\,(B + C)'$	substitute $B + C = X$
	$= A'\,(B'\,C')$	by DeMorgan's
	$= A'\,B'\,C'$	associative

- Generalized form:

$(A + B + C + \ldots + G)' = A'\,B'\,C' \ldots G'$

$(A\,B\,C \ldots G)' = A' + B' + C' + \ldots + G'$

$\Rightarrow$ AND $\leftrightarrow$ OR and complement each literal

Theorem 6: Absorption

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
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Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
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Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

- 6(a): $x + xy = x$

i. $x + xy = x \cdot 1 + xy$ by postulate: 2(b)

$= x\,(1 + y)$ 4(a)

$= x\,(y + 1)$ 3(a)

$= x \cdot 1$ (Theorem) 2(a)

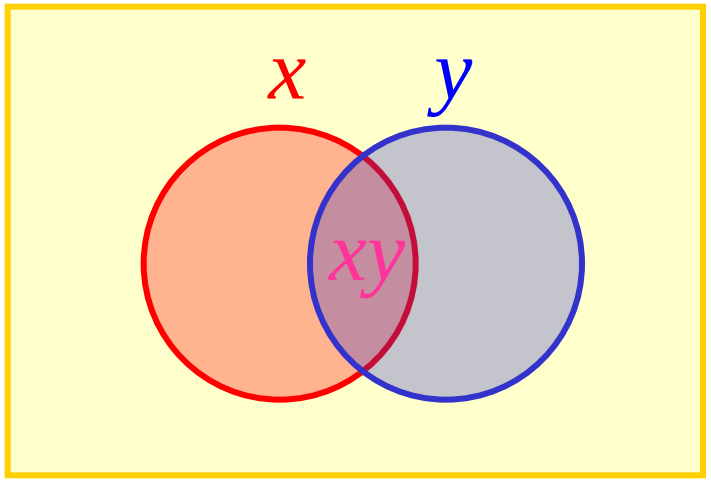
$= x$ 2(b)

ii. Truth table:

x	y	xy	$x + xy$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

=

- 6(a): $x + xy = x$
  * Venn diagram:



Theorem * : Consensus

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
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Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

- $xy + x'z + yz = xy + x'z$

  $(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

$xy + x'z + yz$

$= xy + x'z + (x + x')\,yz$ … P2(b), P3(b), P5(a)

$= xy + x'z + xyz + x'yz$ … P4(a)

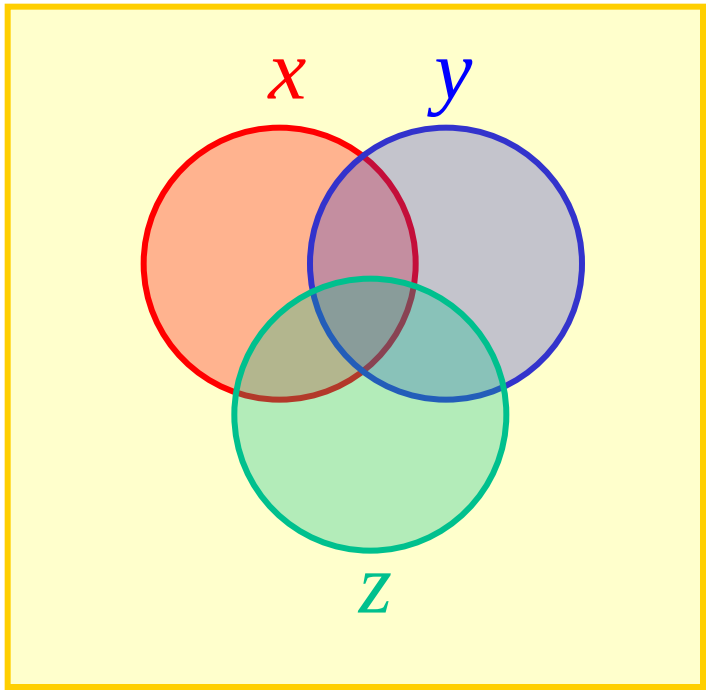
$= xy\,(1 + z) + x'z\,(1 + y)$ … P3(a), P4(a)

$= xy + x'z$ … T2(a), P2(b)

- can be used to eliminate redundant terms from Boolean expressions.
- can be used to generate redundant terms for further simplification.

* Venn diagram

- $xy + x'z + yz = xy + x'z$
  * Venn diagram:



B. Operator Precedence

- Operator precedence for evaluating Boolean expression:
  - parenthesis
  - NOT
  - AND
  - OR

- Example:

$$(x + y)' + x'y$$

2-5 Boolean Functions

- Boolean algebra:
  - is an algebra dealing w/ binary variables and logic ops
    - binary variables: are designated by letters of the alphabet
    - logic ops: AND, OR, NOT
- Boolean expression:
  - an algebraic expression formed by using

    binary variables,

    the constants 0 and 1,

    the logic op symbols, and

    parentheses.
  - E.g.: $X \cdot (\overline{Y} + Z) + \mathrm{Z} \cdot 1$

Boolean Function

- Boolean function:
  - can be described by
    - a Boolean equation,
    - a truth table, or
    - a logic ckt diagram

Boolean Equation

- Boolean equation:
  - consists of a binary variable identifying the function followed by an equal sign and a Boolean expression.
  - expresses the logical relationship b/t binary variables
  - can be expressed in a variety of ways
    * Obtain a simpler expression for the same function.
  - E.g.:

$$F(W,X,Y,Z) = X\overline{Y}\overline{Z} + \overline{Y}Z + \overline{W}XYZ + WXY + \overline{W}XY\overline{Z}$$

$$= X + \overline{Y}Z$$

terms

Truth Table

- Truth table for a function: is unique
  - a list of all combinations of 1's and 0's that can be assigned to the binary variables and a list that shows the value of the function for each binary combination.
  - E.g.:

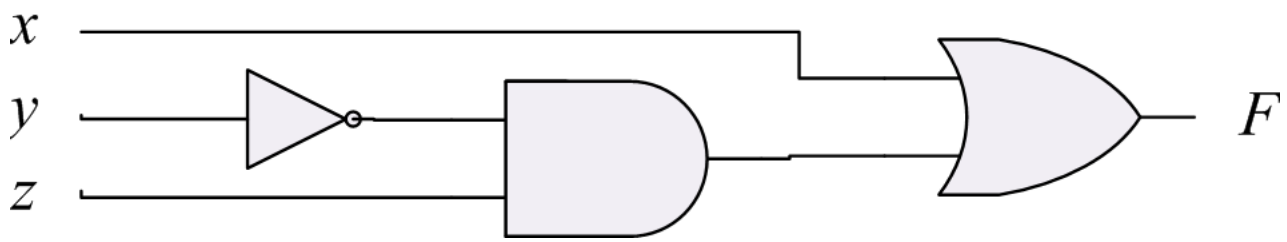
$$F(X,Y,Z) = X + \overline{Y}Z$$

X Y Z	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Logic Circuit Diagram

- Logic circuit diagram:
  - An algebraic expression for a Boolean function

    $\Rightarrow$ A ckt diagram composed of logic gates
  - Circuit gates are interconnected by wires that carry logic signals.
  - E.g.: $F(x, y, z) = x + \bar{y}z$



* Combinational logic circuits

Example

- Present a set of requirements under which an insurance policy will be issued:

The applicant must be

1. a married female 25 years old or over, or
2. a female under 25, or
3. a married male under 25 who has not been involved in a car accident, or
4. a married male who has been involved in a car accident, or
5. a married male 25 years or over who has not been involved in a car accident.

<Ans.>

- Define input variables: 4; w, x, y, z
  - $w = 1$ if applicant has been involved in a car accident
  - $x = 1$ if applicant is married
  - $y = 1$ if applicant is a male
  - $z = 1$ if applicant is under 25
- Find a Boolean expression which assumes the value 1 whenever the policy should be issued:

$x\,y'\,z'$	1. a married female 25 years old or over
$y'\,z$	2. a female under 25
$w'\,x\,y\,z$	3. a married male under 25 who has not been involved in a car accident
$w\,x\,y$	4. a married male who has been involved in a car accident
$w'\,x\,y\,z'$	5. a married male 25 years or over who has not been involved in a car accident

$\Rightarrow$ $F = x\,y'\,z' + y'\,z + w'\,x\,y\,z + w\,x\,y + w'\,x\,y\,z'$

- Simplify the expression and suggest a simpler set of requirements:

$$F = x\,y'\,z' + y'\,z + \underbrace{w'\,x\,y\,z + w\,x\,y + w'\,x\,y\,z'}_{x\,y}$$

$$= x + y'\,z$$

Insurance policy

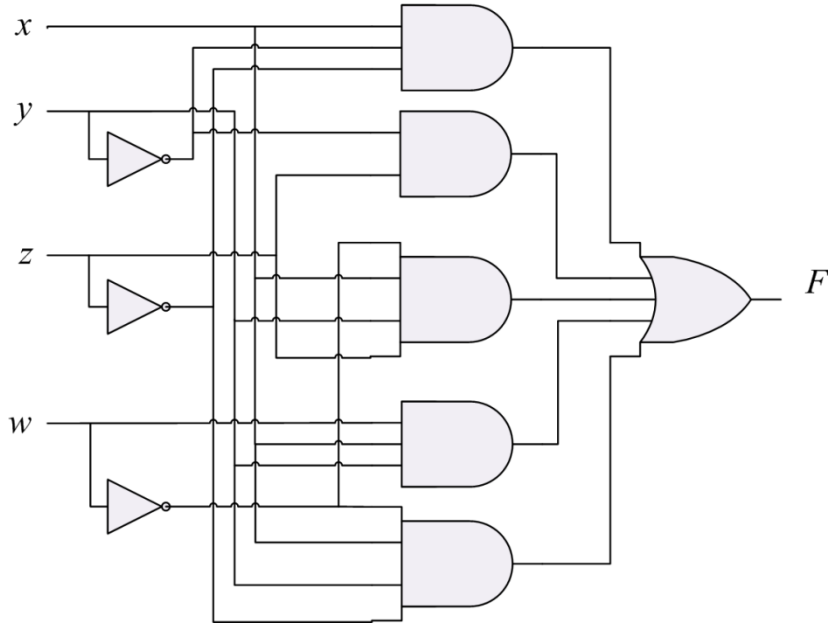
$w = 1$ if applicant has been involved in a car accident
$x = 1$ if applicant is married
$y = 1$ if applicant is a male
$z = 1$ if applicant is under 25

The applicant must be

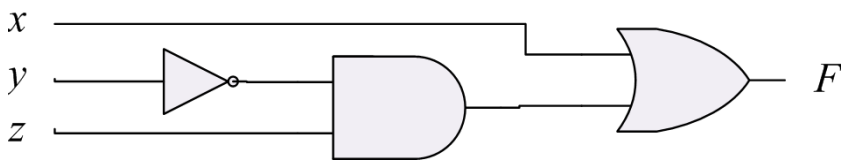
1. married or
2. a female under 25.

- Draw the logic circuit diagram:

$$F = x\,y'\,z' + y'\,z + w'\,x\,y\,z + w\,x\,y + w'\,x\,y\,z'$$



$$= x + y'\,z$$



Simplification of Boolean Functions

- Boolean algebra is a useful tool for simplifying digital ckts.
  - E.g.:

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

$$F_2 = x'y'z + x'yz + xy'$$

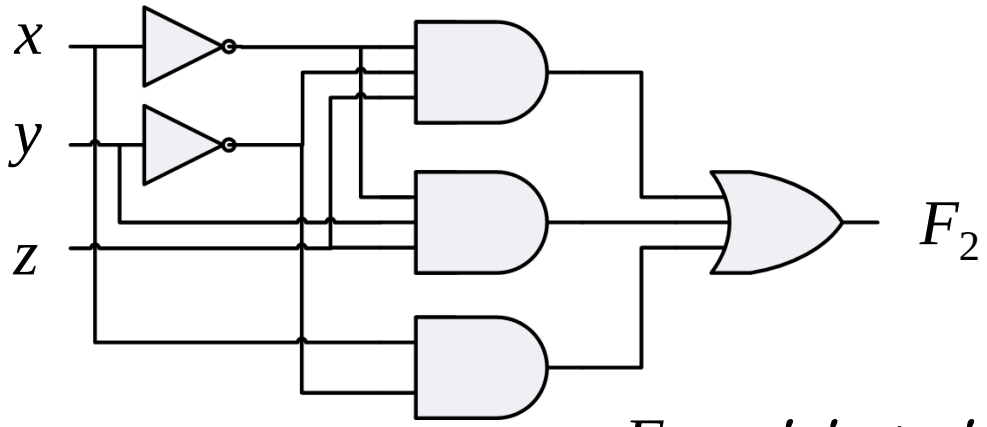
$$= x'z\,(y' + y) + xy' \quad \ldots \text{ Postulate 4(a)}$$

$$= x'z \cdot 1 + xy' \quad \ldots \text{ Postulate 5(a)}$$

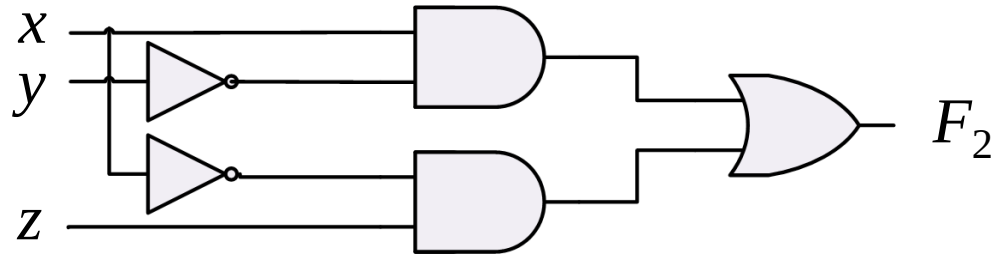
$$= x'z + xy' \quad \ldots \text{ Postulate 2(b)}$$

$F_2 = x'y'z + x'yz + xy'$ ..... (a)

$= x'z + xy'$ ..... (b)



$F_2 = x'y'z + x'yz + xy'$



$F_2 = x'z + xy'$

x y z	F_2
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	1
1 0 1	1
1 1 0	0
1 1 1	0

A. Algebraic Manipulation

- Implementation of Boolean function w/ gates:
  - each term is implemented w/ a gate
  - each literal designates an input to a gate
    - *literal*: a primed or unprimed variable
  - E.g.: $F_2 = x'y'z + x'yz + xy'$ … 3 terms & 8 literal
    $= x'z + xy'$ … 2 terms & 4 literals
- Criterion of equipment minimization:
  - Minimize the # of "literals"
  - Minimize the # of "terms"
- Algebraic manipulation: literal minimization
  - Method: cut-and-try (hard)
    - employ the postulates, basic theorem, ...

Example 2-1

- Simplify the following Boolean functions to a minimum # of literals:

1. $x(x' + y)$

   $= xx' + xy$

   $= 0 + xy$

   $= xy$ $\quad 3 \rightarrow 2$

2. $x + x'y$

   $= (x + x')(x + y)$

   $= 1\ (x + y)$

   $= x + y$ $\quad 3 \rightarrow 2$ (or by duality from 1)

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Theorem 4, associative	$x + (y + z) = (x + y) + z$	$x\,(y\,z) = (x\,y)\,z$
Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

3. $(x + y)(x + y')$

$= xx + xy + xy' + yy'$

$= x + xy + xy' + 0$

$= x(1 + y + y')$

$= x$ $\quad 4 \rightarrow 1$

Postulate/Theorem	(a)	(b)
Postulate 2, identity element	$x + 0 = x$	$x \cdot 1 = x$
Postulate 5, complement	$x + x' = 1$	$x \cdot x' = 0$
Postulate 3, commutative	$x + y = y + x$	$x\,y = y\,x$
Postulate 4, distributive	$x\,(y + z) = xy + xz$	$x + (y\,z) = (x + y)\,(x + z)$
Theorem 1	$x + x = x$	$x \cdot x = x$
Theorem 2	$x + 1 = 1$	$x \cdot 0 = 0$
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Theorem 5, DeMorgan	$(x + y)' = x'\,y'$	$(xy)' = x' + y'$
Theorem 6, absorption	$x + xy = x$	$x\,(x + y) = x$
Theorem *, consensus	$xy + x'z + yz = xy + x'z$	$(x + y)\,(x' + z)\,(y + z) = (x + y)\,(x' + z)$

4. $xy + x'z + yz$

$= xy + x'z + yz(x+x')$

$= xy + x'z + xyz + x'yz$

$= xy(1 + z) + x'z(1 + y)$

$= xy + x'z$ $\quad 6 \rightarrow 4$

* 4 & 5

$\rightarrow$ Consensus theorem

5. $(x + y)(x' + z)(y + z)$

$= (x + y)(x' + z)$ $\quad 6 \rightarrow 4$ (duality from 4)

B. Complement of a Function

- Complement of a function F:
  - E.g.: p.2-30 (Insurance Policy)

    F = 1 whenever the insurance policy should be issued

    $\Rightarrow$ G = F′ = 1 whenever the insurance policy should ***not*** be issued

  i. interchange of 1's to 0's and 0's to 1's for the values of F in the truth table

  ii. can be derived algebraically by applying DeMorgan's theorem $\Rightarrow$ interchange AND and OR ops and complement each variable and constant

$$\overline{X+Y} = \overline{X} \cdot \overline{Y} \qquad \overline{X \cdot Y} = \overline{X} + \overline{Y}$$

  iii. take the dual (OR $\leftrightarrow$ AND, 0 $\leftrightarrow$ 1) of the function eq & complement each literal

Example 2.2

$$\overline{X+Y}=\overline{X}\cdot\overline{Y} \qquad \overline{X\cdot Y}=\overline{X}+\overline{Y}$$

- Complementing functions by applying DeMorgan's theorem:
  - E.g.s:

$$F_1=\overline{X}Y\overline{Z}+\overline{X}\overline{Y}Z \qquad \overline{F}_1=(\overline{X}Y\overline{Z}+\overline{X}\overline{Y}Z)'$$
$$=(\overline{X}Y\overline{Z})'\cdot(\overline{X}\overline{Y}Z)'$$
$$=(X+\overline{Y}+Z)\cdot(X+Y+\overline{Z})$$

$$F_2=X(\overline{Y}\ \overline{Z}+YZ) \qquad \overline{F}_2=(X(\overline{Y}\ \overline{Z}+YZ))'$$
$$=\overline{X}+(\overline{Y}\ \overline{Z}+YZ)'$$
$$=\overline{X}+(\overline{Y}\ \overline{Z})'\cdot(YZ)'$$
$$=\overline{X}+(Y+Z)\cdot(\overline{Y}+\overline{Z})$$

Example 2.3

- Complementing functions by using duals:
  - E.g.s: (dual: OR $\leftrightarrow$ AND, 0 $\leftrightarrow$ 1)

$$F_1 = \overline{X}Y\overline{Z} + \overline{X}\,\overline{Y}Z$$

$$\text{dual of } F_1 \Rightarrow (\overline{X} + Y + \overline{Z}) \cdot (\overline{X} + \overline{Y} + Z)$$

$$\text{complement each literal} \Rightarrow (X + \overline{Y} + Z) \cdot (X + Y + \overline{Z})$$

$$= \overline{F}_1$$

$$F_2 = X(\overline{Y}\ \overline{Z} + YZ)$$

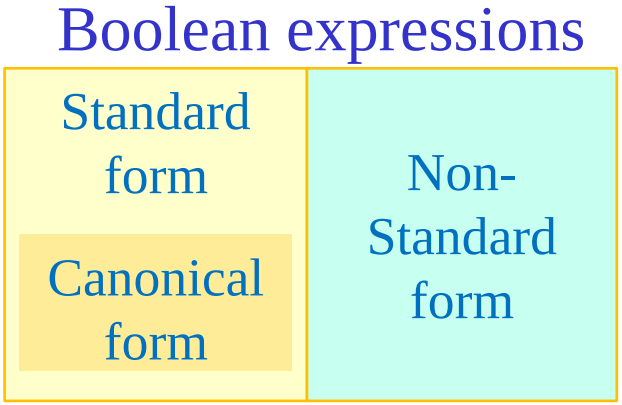
$$\text{dual of } F_2 \Rightarrow X + (\overline{Y} + \overline{Z}) \cdot (Y + Z)$$

$$\text{complement each literal} \Rightarrow \overline{X} + (Y + Z) \cdot (\overline{Y} + \overline{Z})$$

$$= \overline{F}_2$$

2-6 Canonical & Standard Forms

- Boolean expressions:
  - an expression formed w/:
    - binary variables
    - constants 0 and 1
    - binary operators OR and AND
    - unary operator NOT
    - parentheses
    - equal sign
  - Special forms:
    - Canonical forms
    - Standard forms



A. Canonical Forms

- Canonical forms:
  - special forms of Boolean expression being expressed directly from truth tables
  - Two types:
    i. *sum-of-minterms form*

       (*min*term) + … + (*min*term)

    ii. *product-of-maxterms form*

       (*max*term) • … • (*max*term)

$x\ y\ z$	F_2
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	1
1 0 1	1
1 1 0	0
1 1 1	1

minterms

- *minterm*: standard product, m_j
  - an AND term of the n variables, w/ each variable being primed if the corresponding bit is 0 and unprimed if a 1.
  - n variables $\rightarrow 2^n$ minterms
  - E.g.: minterms for 3 variables

$x\ y\ z$	minterm		m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
0 0 0	$x'y'z'$	(m_0)	1	0	0	0	0	0	0	0
0 0 1	$x'y'z$	(m_1)	0	1	0	0	0	0	0	0
0 1 0	$x'yz'$	(m_2)	0	0	1	0	0	0	0	0
0 1 1	$x'yz$	(m_3)	0	0	0	1	0	0	0	0
1 0 0	$xy'z'$	(m_4)	0	0	0	0	1	0	0	0
1 0 1	$xy'z$	(m_5)	0	0	0	0	0	1	0	0
1 1 0	xyz'	(m_6)	0	0	0	0	0	0	1	0
1 1 1	xyz	(m_7)	0	0	0	0	0	0	0	1

Maxterms

- *Maxterm*: standard sum, M_j (= m_j')
  - an OR term of the n variables, w/ each variable being unprimed if the corresponding bit is 0 and primed if a 1.
  - n variables $\rightarrow 2^n$ Maxterms
  - E.g.: Maxterms for 3 variables

$x\ y\ z$	Maxterm		M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0 0 0	$x + y + z$	(M_0)	0	1	1	1	1	1	1	1
0 0 1	$x + y + z'$	(M_1)	1	0	1	1	1	1	1	1
0 1 0	$x + y' + z$	(M_2)	1	1	0	1	1	1	1	1
0 1 1	$x + y' + z'$	(M_3)	1	1	1	0	1	1	1	1
1 0 0	$x' + y + z$	(M_4)	1	1	1	1	0	1	1	1
1 0 1	$x' + y + z'$	(M_5)	1	1	1	1	1	0	1	1
1 1 0	$x' + y' + z$	(M_6)	1	1	1	1	1	1	0	1
1 1 1	$x' + y' + z'$	(M_7)	1	1	1	1	1	1	1	0

minterms vs. Maxterms

- E.g.: for 3 binary variables **(* $M_j = m_j'$)**

x y z	**minterm**		**Maxterm**	
000	$x'y'z'$	(m_0)	$x + y + z$	(M_0)
001	$x'y'z$	(m_1)	$x + y + z'$	(M_1)
010	$x'yz'$	(m_2)	$x + y' + z$	(M_2)
011	$x'yz$	(m_3)	$x + y' + z'$	(M_3)
100	$xy'z'$	(m_4)	$x' + y + z$	(M_4)
101	$xy'z$	(m_5)	$x' + y + z'$	(M_5)
110	xyz'	(m_6)	$x' + y' + z$	(M_6)
111	xyz	(m_7)	$x' + y' + z'$	(M_7)

– E.g.:

$$m_3 = \bar{x}yz$$

$$\overline{m_3} = \overline{\bar{x}yz}$$
$$= x + \bar{y} + \bar{z}$$
$$= M_3$$

Canonical Forms

- Canonical forms:
  - A Boolean function can be expressed algebraically from a given truth table by

i. **Sum-of-minterms form**:

(*min*term) + … + (*min*term)

- form a minterm for each combination of the variables that produces a 1 in the function, and then take the OR of all those terms

ii. **Product-of-Maxterms form**:

(*max*term) • … • (*max*term)

- form a maxterm for each combination of the variables that produces a 0 in the function, and then take the AND of all those terms

Example

- Given the truth table of two functions, f_1 and f_2:

x y z	f_1	f_2
0 0 0	0	0
0 0 1	1	0
0 1 0	0	0
0 1 1	0	1
1 0 0	1	0
1 0 1	0	1
1 1 0	0	1
1 1 1	1	1

  - Sum of minterms:

$$f_1 = x'y'z + xy'z' + xyz = m_1 + m_4 + m_7 = \sum m(1, 4, 7)$$

$$f_2 = x'yz + xy'z + xyz' + xyz = m_3 + m_5 + m_6 + m_7$$

$$= \sum m(3, 5, 6, 7)$$

Example (cont'd)

$f_1 = x'y'z + xy'z' + xyz$
$= m_1 + m_4 + m_7 = \sum m(1,4,7)$
$f_2 = x'yz + xy'z + xyz' + xyz$
$= m_3 + m_5 + m_6 + m_7 = \sum m(3,5,6,7)$

x y z	f_1	f_2
0 0 0	0	0
0 0 1	1	0
0 1 0	0	0
0 1 1	0	1
1 0 0	1	0
1 0 1	0	1
1 1 0	0	1
1 1 1	1	1

- Product of maxterms:

$$f_1' = m_0 + m_2 + m_3 + m_5 + m_6 = x'y'z' + x'yz' + x'yz + xy'z + xyz'$$

$$f_1 = (f_1')' = (x+y+z)(x+y'+z)(x+y'+z')(x'+y+z')(x'+y'+z)$$
$$= M_0 M_2 M_3 M_5 M_6 = \Pi M(0, 2, 3, 5, 6)$$

$$f_2 = (x+y+z)(x+y+z')(x+y'+z)(x'+y+z)$$
$$= M_0 M_1 M_2 M_4 = \Pi M(0, 1, 2, 4)$$

Conversion to Sum of Minterms Form

- Express a Boolean function in its sum-of-minterms form:
  i. Expand the expression into a sum of AND terms by using the distributive law, $x\,(y + z) = xy + xz$.

     Any missing variable x in each AND term is ANDed with $(x + x')$.

     – E.g.: $F(A, B, C) = \boldsymbol{A + B'C}$

– E.g.: $F(A, B, C) = A + B'C$

$$A = A\,(B+B')\,(C+C')$$
$$= (AB + AB')\,(C+C')$$
$$= AB\,(C+C') + AB'(C+C')$$
$$= ABC + ABC' + AB'C + AB'C'$$

$$B'C = (A+A')\,B'C$$
$$= AB'C + A'B'C$$

$$F(A, B, C) = A'B'C + AB'C' + AB'C + ABC' + ABC$$
$$= m1 + m4 + m5 + m6 + m7$$
$$= \Sigma m(1, 4, 5, 6, 7)$$

ii. Obtain the truth table of the function directly from the algebraic expression and then read the minterms from the truth table.

– E.g.: $F(A, B, C) = \boldsymbol{A + B'C}$

A	*B*	*C*	*F*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$$F(A, B, C)$$
$$= A'B'C + AB'C' + AB'C$$
$$+ ABC' + ABC$$
$$= \Sigma m(1, 4, 5, 6, 7)$$

Conversion to Product-of-Maxterms Form

- Express a Boolean function in its product-of-maxterms form:
  i. Convert the function into OR terms by using the distributive law, $x + yz = (x + y)(x + z)$.

     Any missing variable x in each OR term is ORed with xx'.

     – E.g.: $F(x,y,z) = xy + x'z$

– E.g.: $F(x,y,z) = xy + x'z$

$$F = xy + x'z$$
$$= (xy + x')(xy + z)$$
$$= (x+x')(y+x')(x+z)(y+z)$$
$$= (x'+y)(x+z)(y+z)$$

$$x'+y = x' + y + zz'$$
$$= (x'+y+z)(x'+y+z')$$

$$x+z = x + z + yy'$$
$$= (x+y+z)(x+y'+z)$$

$$y+z = y + z + xx'$$
$$= (x+y+z)(x'+y+z)$$

$$F = (x+y+z)(x+y'+z)(x'+y+z)(x'+y+z')$$
$$= M_0 M_2 M_4 M_5$$
$$= \Pi M(0, 2, 4, 5)$$

ii. Use the truth table and the canonical conversion procedure.

– E.g.: $F = xy + x'z$

x y z	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	0
1 1 0	1
1 1 1	1

$$F(x, y, z)$$
$$= (x+y+z)(x+y'+z)$$
$$(x'+y+z)(x'+y+z')$$
$$= \Pi M(0, 2, 4, 5)$$

Conversion b/t Canonical Forms

- To convert from one canonical form to another:
  - Interchange the symbols Σ and Π &
    list those numbers missing from the original form
  - E.g.: $F = xy + x'z$

x	*y*	*z*	*F*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

$$F(x, y, z)$$
$$= \Sigma m(1, 3, 6, 7)$$
$$= \Pi M(0, 2, 4, 5)$$

B. Standard Forms

- Canonical forms: special cases of standard forms
  - sum of ***minterms*** & product of ***maxterms***
  - the basic forms that one obtains from reading a function from the truth table.
  - Each terms must contain all the variables
    - Seldom w/ the least # of literals (not minimized)
- Standard forms:
  - Sum of ***products*** & product of ***sums***

    (***AND*** terms) (***OR*** terms)
  - The terms may contain any number of literals

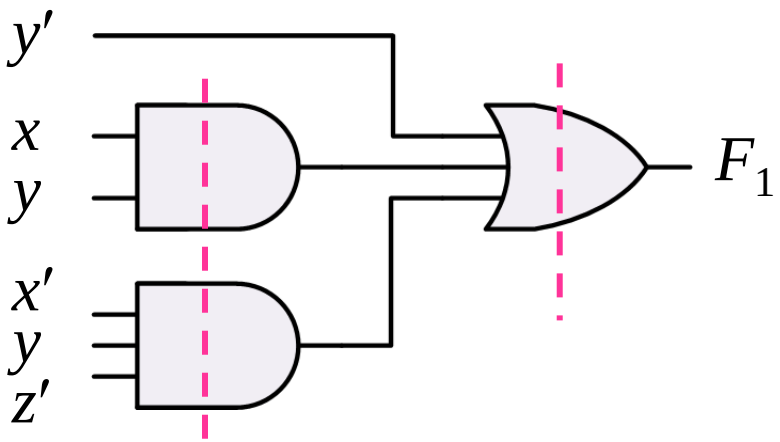
Sum of Products (SOP)

- SOP: (*AND term*) + … + (*AND term*)
  - is a Boolean expression containing ***AND*** terms (***product*** terms) of one or more literals each.

    There ***AND*** terms are ***OR***ing together.

    → 2-level gating structure

    (if the complements of the input variables are available)
  - E.g.: $F_1 = y' + xy + x'yz'$



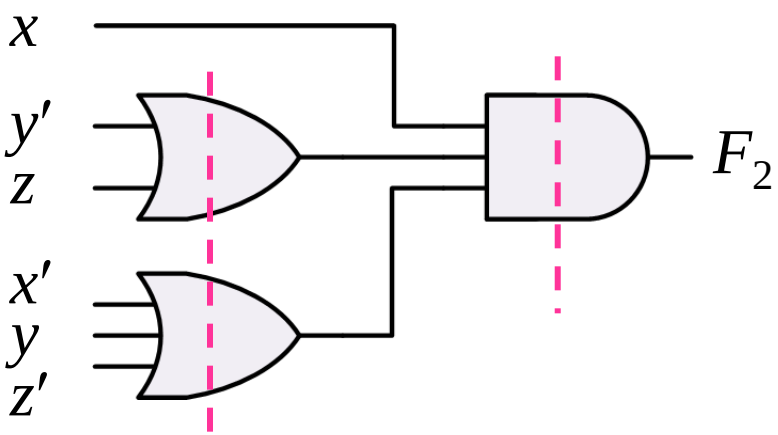
Product of Sums (POS)

- POS: (*OR term*) • … • (*OR term*)
  - is a Boolean expression containing ***OR*** terms (***sum*** terms) of one or more literals each.

    These ***OR*** terms are ***AND***ing together.

    → 2-level gating structure

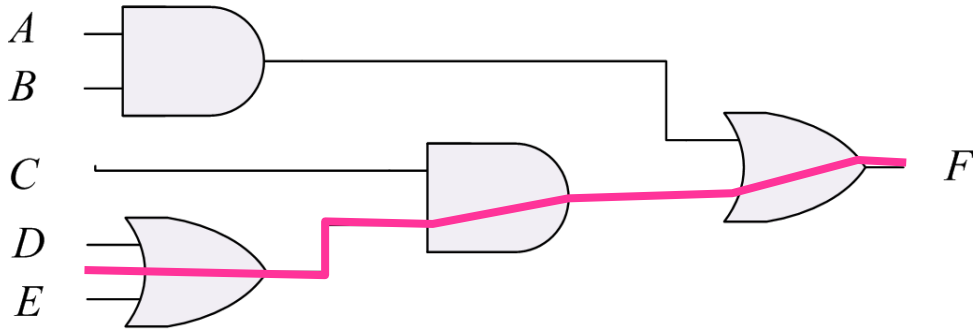
    (if the complements of the input variables are available)
  - E.g.: $F_2 = x(y'+z)(x'+y+z')$



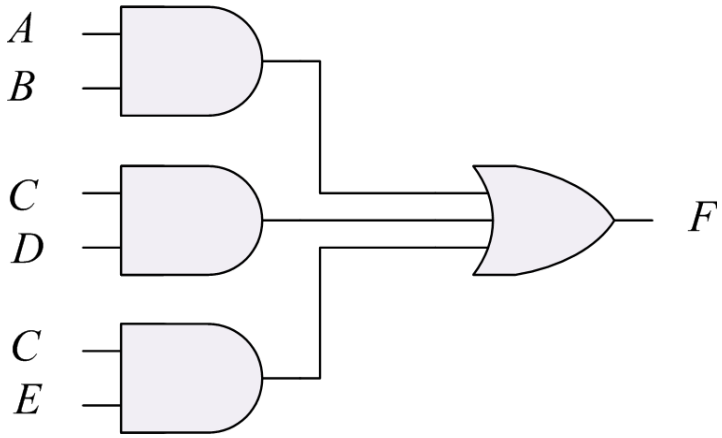
C. Nonstandard Form

- Nonstandard form:
  - E.g.: $F = AB + C\,(D + E)$ → 3-level
- Nonstandard form → Standard form:
  - By using the distributive laws
  - E.g.: $F = AB + C\,(D + E)$
    $= AB + CD + DE$ → 2-level

Standard form
vs
Nonstandard form



$AB + C\,(D + E)$



$AB + CD + CE$

2-7 Other Logic Operations

- Basic logic operations: AND, OR, NOT
- All possible functions of *n* binary variables:

  n binary variables $\rightarrow 2^n$ distinct minterms

  $\rightarrow 2^{2^n}$ possible functions

- E.g.: $n = 2 \rightarrow 4$ minterms $\rightarrow 16$ possible functions

x *y*	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0 0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0 1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1 0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1 1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

x y	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0 0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0 1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1 0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1 1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

$F_0 = 0$	Null	$F_{15} = 1$	Identity
$F_1 = xy$	AND, $x \cdot y$	$F_{14} = (xy)'$	NAND, $x \uparrow y$
$F_2 = xy'$	Inhibition, x/y	$F_{13} = x' + y$	Implication, $x \supset y$
$F_3 = x$	Transfer	$F_{12} = x'$	Complement, x'
$F_4 = x'y$	Inhibition, y/x	$F_{11} = x + y'$	Implication, $x \subset y$
$F_5 = y$	Transfer	$F_{10} = y'$	Complement, y'
$F_6 = xy' + x'y$	Exclusive-OR, $x \oplus y$	$F_9 = xy + x'y'$	Eauivalence, $(x \oplus y)'$
$F_7 = x + y$	OR, $x+y$	$F_8 = (x + y)'$	NOR, $x \downarrow y$

$F_0 = 0$	Null	$F_{15} = 1$	Identity
$F_1 = xy$	AND, $x \cdot y$	$F_{14} = (xy)'$	NAND, $x \uparrow y$
$F_2 = xy'$	Inhibition, x/y	$F_{13} = x' + y$	Implication, $x \supset y$
$F_3 = x$	Transfer	$F_{12} = x'$	Complement, x'
$F_4 = x'y$	Inhibition, y/x	$F_{11} = x + y'$	Implication, $x \subset y$
$F_5 = y$	Transfer	$F_{10} = y'$	Complement, y'
$F_6 = xy' + x'y$	Exclusive-OR, $x \oplus y$	$F_9 = xy + x'y'$	Eauivalence, $(x \oplus y)'$
$F_7 = x + y$	OR, $x+y$	$F_8 = (x + y)'$	NOR, $x \downarrow y$

- The 16 functions can be subdivided into 3 categories:
  1. 2 functions that produce a constant 0 or 1.
  2. 4 functions w/ unary operations complement (NOT) and transfer.
  3. 10 functions w/ binary operators that define eight different operations AND, OR, NAND, NOR, exclusive-OR, equivalence, inhibition, and implication.

2-8 Digital Logic Gates

- Boolean expression:
  - AND, OR and NOT operations
  - It is easier to implement a Boolean function in these types of gates.
- Factors in considering the construction of other types of logic gates:
  - the feasibility and economy of implementing the gate w/ electronic components
  - the possibility of extending the gate to more than two inputs
  - the basic properties of the binary operator
    - E.g.: commutativity, associativity, …
  - the ability of the gate to implement Boolean functions alone or in conjunction w/ other gates

*** Why?**

*** Why?**

$F_0 = 0$	Null	$F_{15} = 1$	Identity
$F_1 = xy$	AND, x·y	$F_{14} = (xy)'$	NAND, x↑y
$F_2 = xy'$	Inhibition, x/y	$F_{13} = x'+y$	Implication, $x \supset y$
$F_3 = x$	Transfer	$F_{12} = x'$	Complement, x'
$F_4 = x'y$	Inhibition, y/x	$F_{11} = x+y'$	Implication, $x \subset y$
$F_5 = y$	Transfer	$F_{10} = y'$	Complement, y'
$F_6 = xy' + x'y$	Exclusive-OR, $x \oplus y$	$F_9 = xy + x'y'$	Eauivalence, $(x \oplus y)'$
$F_7 = x + y$	OR, $x+y$	$F_8 = (x + y)'$	NOR, $x \downarrow y$

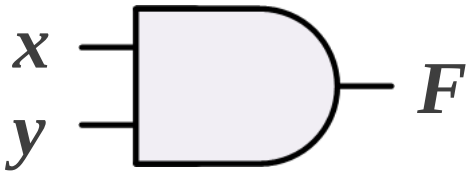
- Consider the 16 functions
  - two are equal to a constant 0, 1
  - four are repeated twice

    Transfer, Complement, Inhibition, Implication
  - inhibition and implication are not commutative or associative (×)
  - complement, transfer, AND, OR, NAND, NOR, XOR, and equivalence (XNOR) are used as standard gates
    - complement: inverter, NOT
    - transfer: buffer (drive strength)

    * NAND and NOR gates are far more popular than the AND and OR gates.

AND, OR

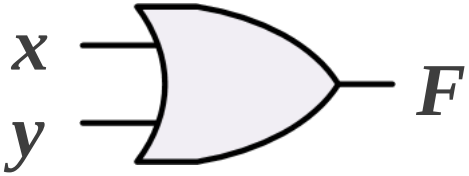
AND



$F = xy$

x	y	F
0	0	0
0	1	0
1	0	0
1	1	1

OR

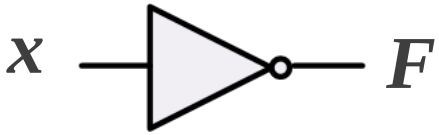


$F = x + y$

x	y	F
0	0	0
0	1	1
1	0	1
1	1	1

Inverter, Buffer

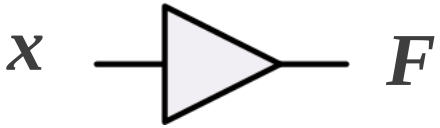
Inverter



$F = x'$

x	F
0	1
1	0

Buffer



$F = x$

x	F
0	1
1	0

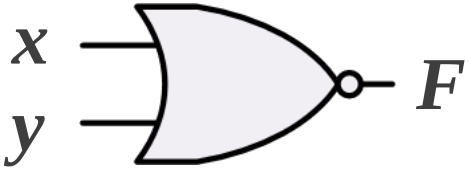
NAND, NOR

NAND

$F = (xy)'$

x	y	F
0	0	1
0	1	1
1	0	1
1	1	0

NOR

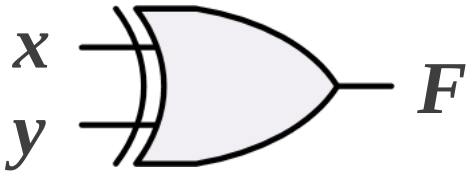


$F = (x + y)'$

x	y	F
0	0	1
0	1	0
1	0	0
1	1	0

XOR, NXOR

XOR

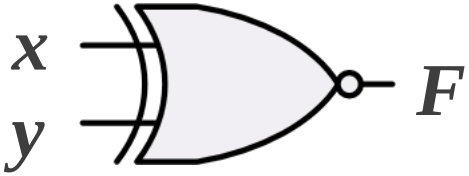


$F = xy' + x'y$

$= x \oplus y$

x	y	F
0	0	0
0	1	1
1	0	1
1	1	0

XNOR



$F = xy + x'y'$

$= (x \oplus y)'$

x	y	F
0	0	1
0	1	0
1	0	0
1	1	1

Extension to Multiple Inputs

- Extension to multiple inputs
  - A gate can be extended to multiple inputs if its binary operation is commutative and associative.
- AND, OR: commutative and associative:

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

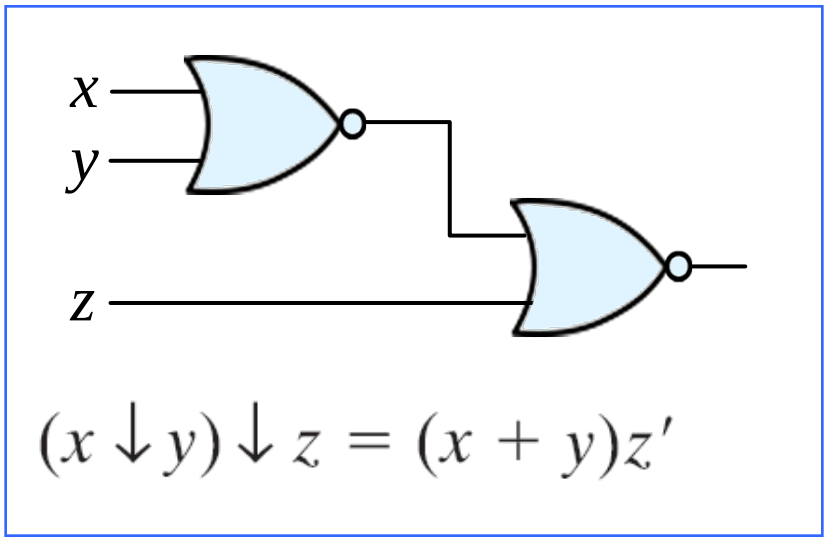
$$x \bullet y = y \bullet x$$

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

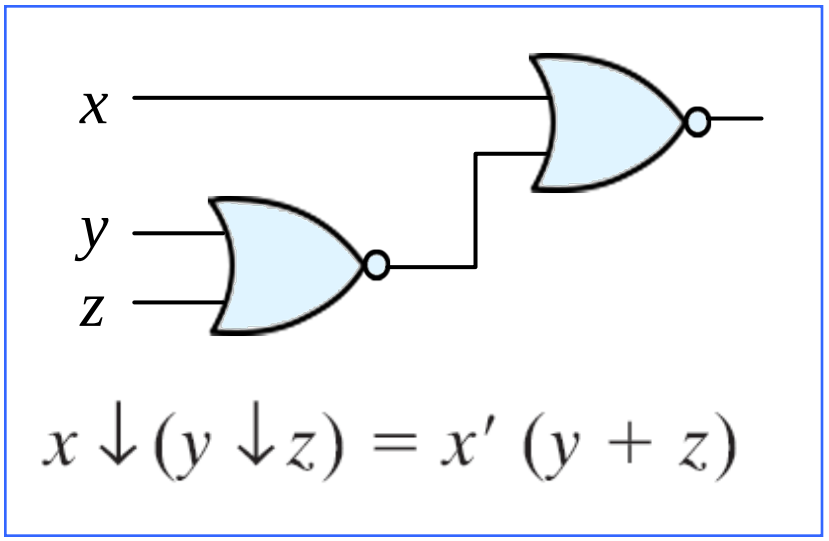
- NAND ($\uparrow$) , NOR ($\downarrow$):
  - commutative but not associative $\Rightarrow$ not extendable
  - E.g.: $(x \downarrow y) \downarrow z \neq x \downarrow (y \downarrow z)$

$$(x \downarrow y) \downarrow z = [(x + y)' + z]' = (x + y)\, z' = xz' + yz'$$

$$x \downarrow (y \downarrow z) = [x + (y + z)' \,]' = x'\,(y + z) = x'y + x'z$$



$\neq$

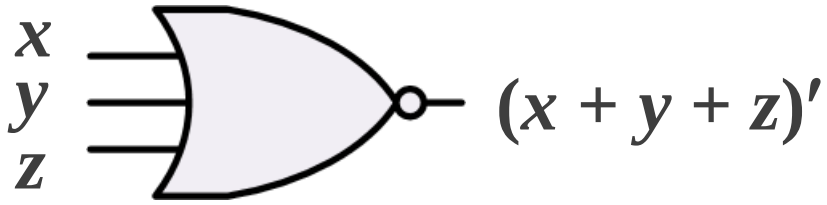


$\Sigma m(2, 4, 6)$ $\Sigma m(1, 2, 3)$

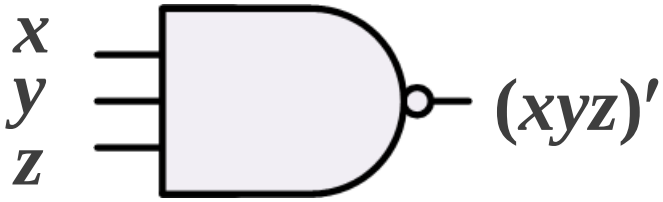
- Modification of the definition:
  - The multiple NOR (or NAND) gate is a complement OR (or AND) gate.

$$x \downarrow y \downarrow z = (x + y + z)'$$

$$x \uparrow y \uparrow z = (x \cdot y \cdot z)'$$

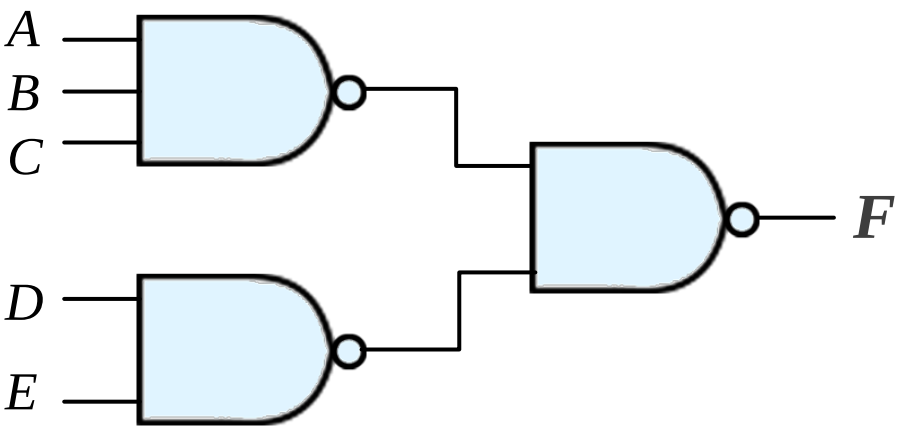


3-input NOR gate

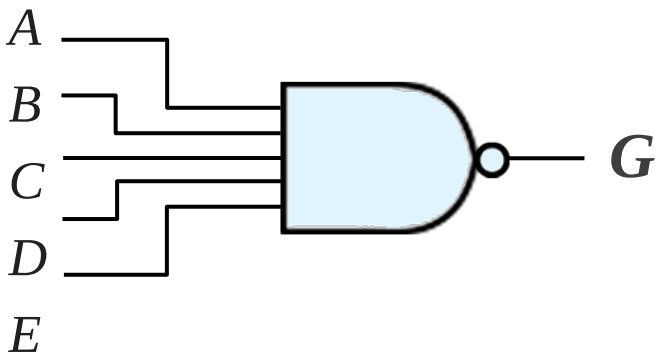


3-input NAND gate

- In writing cascaded NOR and NAND operations:
  - Use the correct parentheses to signify the proper sequence of the gates.
  - E.g.: Cascaded NAND gates



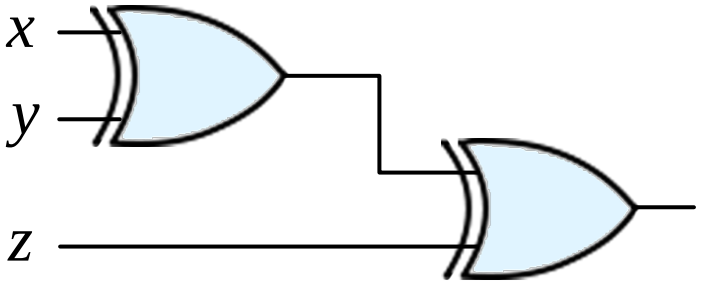
$\neq$



$$F = [(ABC)' \cdot (DE)']' = ABC + DE$$

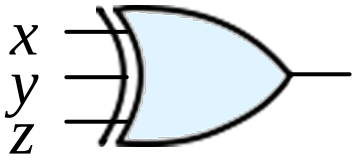
$$G = (ABCDE)'$$
$$= A' + B' + C' + D' + E'$$

- Exclusive-OR (⊕) and Equivalence:
  - are commutative and associative
  - Modification of the definition of multiple-input XOR:
    - an odd function: = 1, if the input variables have an odd # of 1's



$F = x \oplus y \oplus z$

Using 2-input gates



$F = x \oplus y \oplus z$

Using 3-input gate

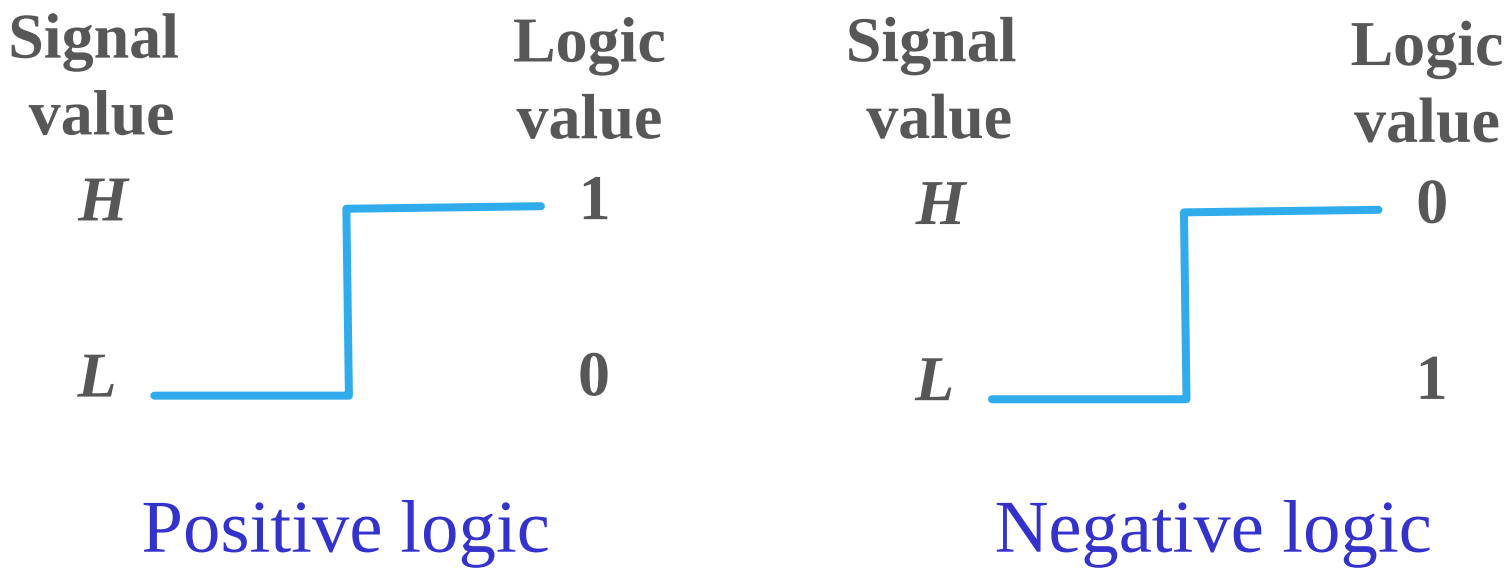
x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

(c) Truth table

  - Equivalence: an even function

Positive & Negative Logic

- Logic-value assignment: polarity assignment
  - Positive logic
  - Negative logic



- Demonstration of positive & negative logic:

x	*y*	*z*
L	*L*	*L*
L	*H*	*L*
H	*L*	*L*
H	*H*	*H*

Truth table with *H* and *L*

H = 1
L = 0

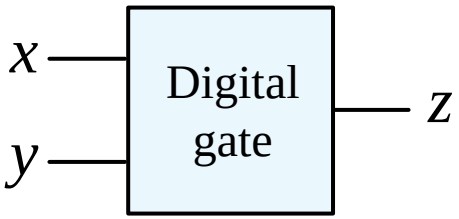
x	*y*	*z*
0	0	0
0	1	0
1	0	0
1	1	1

Truth table for positive logic

H = 0
L= 1

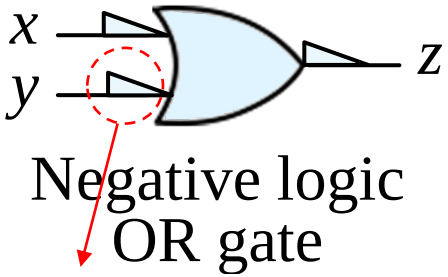
x	*y*	*z*
1	1	1
1	0	1
0	1	1
1	0	0

Truth table for negative logic



Gate block diagram

Positive logic AND gate



Negative logic OR gate

* Polarity indicator

- Conversion b/t positive and negative logic:
  - Change 1's to 0's and 0's to 1's in both inputs and output of a gate (truth table)

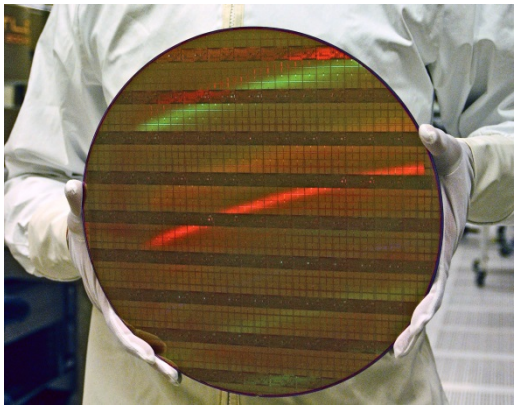
    $\Rightarrow$ take the dual of a function

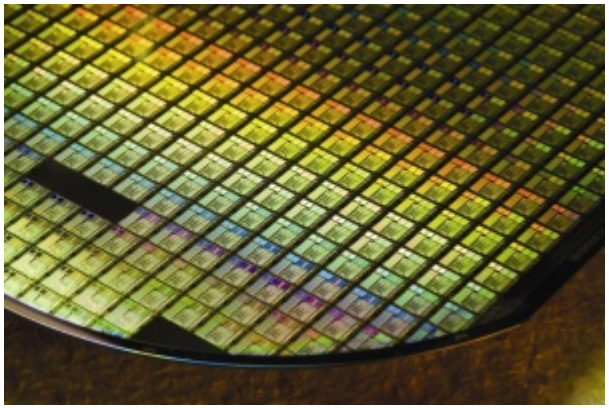
    (AND $\rightarrow$ OR, OR $\rightarrow$ AND, 0 $\rightarrow$ 1, 1 $\rightarrow$ 0)

*** Duality principle**

2-9 Integrated Circuits (ICs)

- Digital ckts are constructed w/ integrated ckts.
- Integrated circuit (IC):
  - is a silicon semiconductor crystal (chip) containing the electronic components for the digital gates and storage elements.
  - The various components are interconnected on the chip.
  - The chip is mounted in a ceramic or plastic container, and connections are welded from the chip to the external pins.





A. Levels of Integration

- Levels of integration: ckt complexity

  (# of logic gates in a single silicon chip)
  - SSI: small-scale integration, <10 gates
  - MSI: medium-scale integration, 10 ~ 100 (Ch 4~7)
    - performs specific elementary digital functions
    - e.g.: addition of 4 bits
  - LSI: large-scale integration: 100 ~ x000
    - eg.: small processors, small memories, & programmable modules
  - VLSI: very large-scale integration, x000 ~
    - e.g.: complex microprocessors, digital signal processing chips

B. Digital Logic Families

- Digital logic families: ckt technology

  TTL: transistor-transistor logic

  ECL: emitter-coupled logic (speed)

  MOS: metal-oxide semiconductor (density)

  CMOS: complementary MOS (power)

  – The basic ckt in each family is a **NAND**, **NOR**, or **inverter** gate. (Ch3)

Technology Parameters（補充資料）

- The most important parameters of digital logic families:
  - ***Fan-in***: # of inputs available on a gate
  - ***Fan-out***: # of standard loads driven by a gate output
    - *max fan-out* : the fan-out that the output can drive w/o impairing gate performance
  - ***Noise margin***: the max external noise voltage superimposed on a normal input value that will not cause an undesirable change in the ckt output
  - ***Cost*** for a gate: is usually based on the area occupied by the layout cell ($\propto$ the size of the transistors & the wiring in the gate layout)

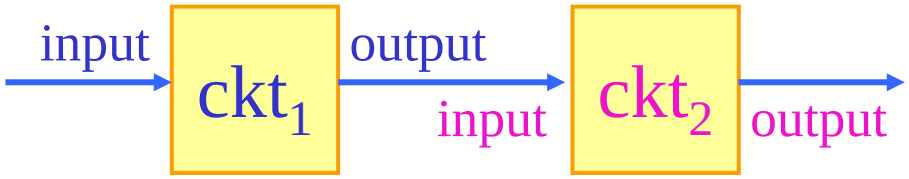
- ***Propagation delay***: the time required for a change in value of a signal to propagate from input to output
  - The operating speed of a ckt is inversely related to the longest propagation delays through the gates of the ckt.
- ***Power dissipation***: the power drawn from the power supply and consumed by the gate
  - must be considered in relation to the operating temperature and cooling

Reference:

M. Morris Mano & Charles R. Kime, *Logic and Computer Design Fundamentals*, 3rd Edition , 2004, Pearson Prentice Hall. (§3-2)

Noise Margin

- Noise margin: the max external noise voltage superimposed on a normal input value that will not cause an undesirable change in the ckt output



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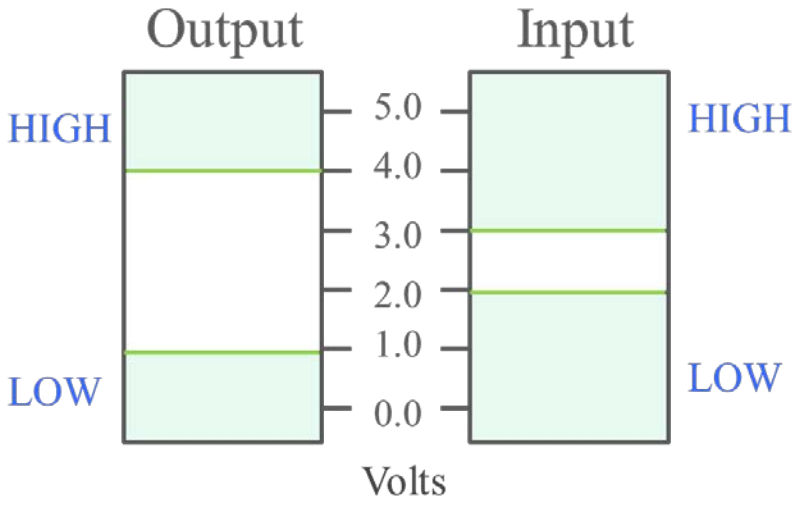
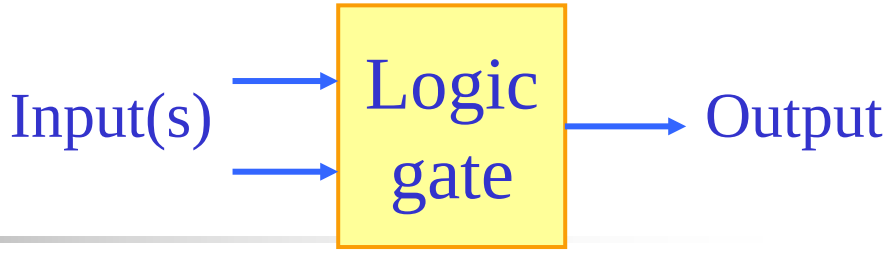
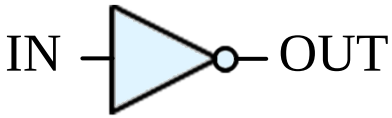


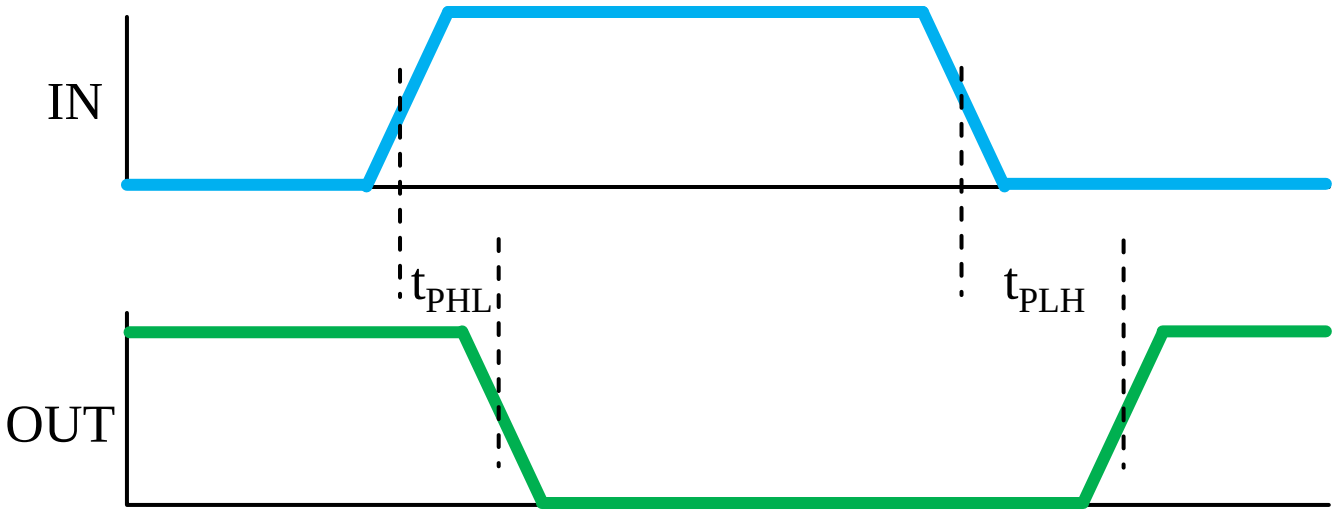
Propagation Delay



- 3 propagation delay parameters:
  - high-to-low propagation time t_{PHL}
  - low-to-high propagation time t_{PLH}
  - propagation delay t_{pd} : max{t_{PHL}, t_{PLH})

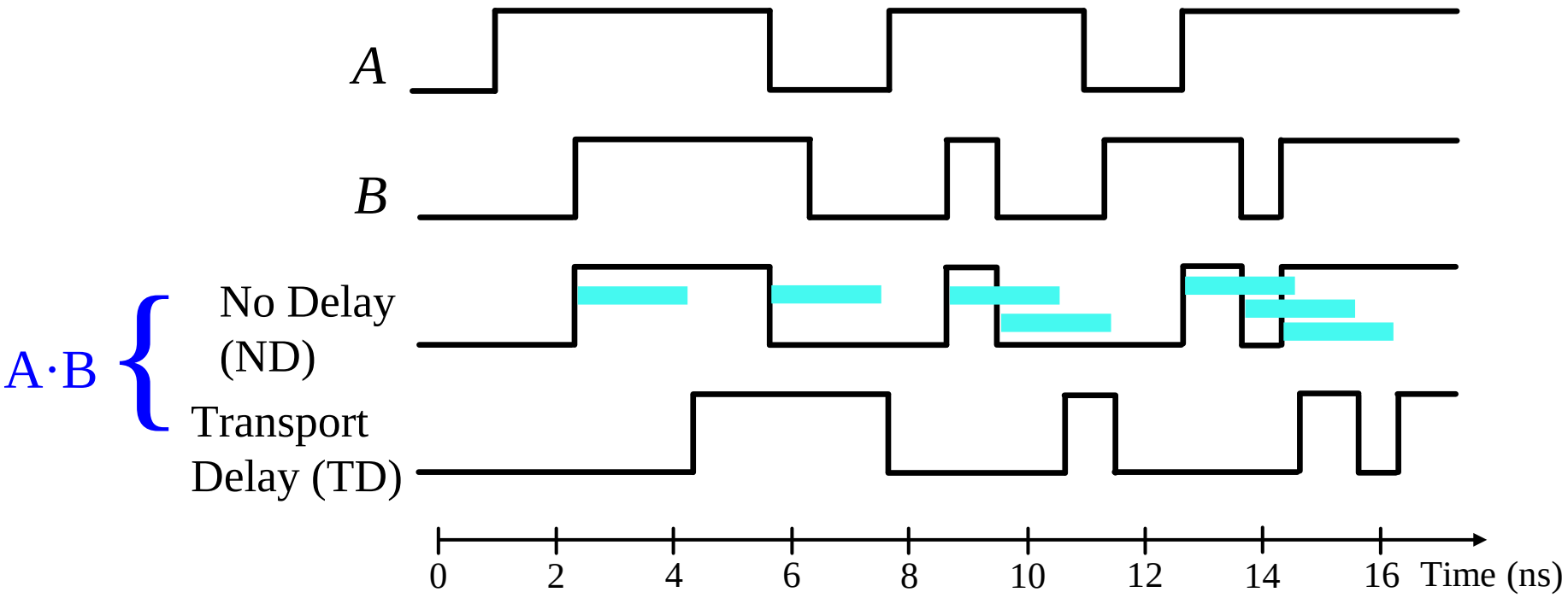
- E.g.:





t_{pd} = max (t_{PHL}, t_{PLH})

- E.g.: behavior of propagation delay of an AND gate

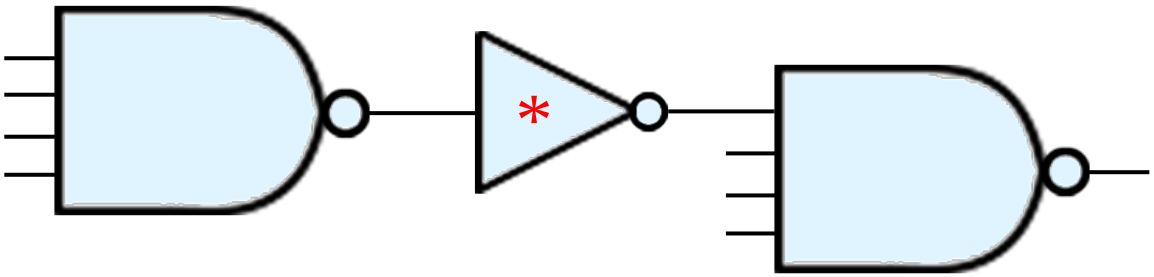


: propagation delay = 2 ns

Fan-in

- For high-speed technologies, fan-in is often restricted on gate primitives to ≤ 4 or 5.

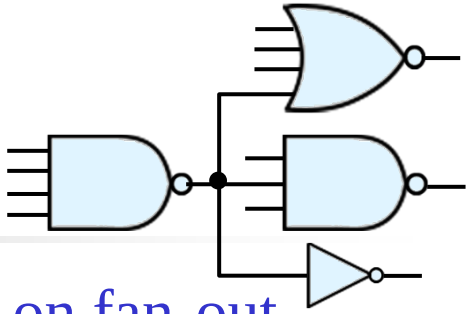
- E.g.: Implementation of a 7-input NAND gate using NAND gates w/ 4 or fewer inputs



* NAND is not associative!

Fan-Out

- *Fan-out*: # of standard loads driven by a gate output
- *Max fan-out* : the fan-out that the output can drive w/o impairing gate performance
- For CMOS gates:
  - The load on the output of a gate determines the time required for the output of the gate to change from L to H and from H to L. $\rightarrow$ *transition time*



- Example: Calculation of gate delay based on fan-out

A 4-input NAND gate output is attached to the inputs of the following gates w/ the given # of standard loads representing their inputs:

4-input NOR gate-- 0.80 standard load

3-input NAND gate-- 1.00 standard load

Inverter-- 1.00 standard load

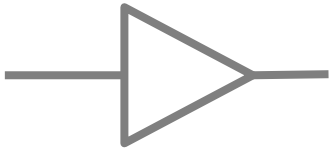
The formula for the delay of the 4-input NAND gate is

$$t_{pd} = 0.07 + 0.021 \times SL \; ns$$

SL: the sum of the standard loads driven by the gate

<Ans.> Ignoring the wiring delay

$$t_{pd} = 0.07 + 0.021 \times (0.80 + 1.00 + 1.00) = 0.129 \; ns$$



- Example:

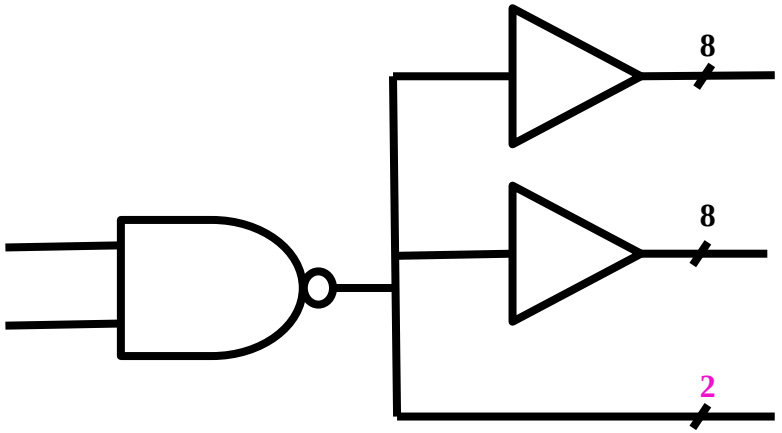
An integrated circuit logic family has

NAND gates with a fan-out of 4 standard loads &

buffers with a fan-out of 8 standard loads.

Sketch a schematic showing how the output signal of a single NAND gate can be applied to 18 other gate inputs. Assume that each input is one standard load.

<Ans.>



- Fan-in & Fan-out:
  - must be dealt w/ in the technology mapping step of the design process.
  - Gate w/ fan-ins larger than available ⇒ Multiple gates
  - Gate w/ fan-outs either exceed its max allowable fan-out or have too high a delay ⇒ Multiple gates or added buffers at its output

C. Computer-Aided Design (CAD) of VLSI Circuits

- CAD tools:
  - software programs that support computer-based representation and aid in the development of digital hardware by automating the design process.
  - Schematic capture tools:
    - support the drawing of blocks and interconnections at all levels of the hierarchy.
    - At the level of primitives and functional blocks, libraries of graphics symbols are provided.
  - Logic simulator:
    - verify the behavior and the timing of the hierarchical blocks and the entire ckt
  - Logic synthesizer:
    - optimize design being generated automatically from HDL (hardware description language) specifications in physical area or delay

* Chapter Summary

- Primitive logic ops: AND, OR, NOT
- Boolean algebra
- Canonical form: minterm & maxterm standard forms
- Standard forms: SoP and PoS $\Rightarrow$ 2-level gate ckts
- Other logic gates:
  - NAND, NOR
  - XOR, XNOR
- Integrated circuits
  - Technology parameters

Problems & Homework (6th ed)

Sections	Exercises	Homework
§2-3	2.1	2.1(a)
§2-4	2.2~2.4, 2.24	2.3, 2.4
§2-5	2.5~2.9, 2-13	2.6
§2-6	2.10, 2.11, 2.15~2.23, 2.27, 2.29~2.31	2.17(a)*, 2.20, 2.22*, 2.30
§2-7	2.12, 2.14, 2.25	2.14, 2.25
§2-8	2.26, 2.28, 2.32, 2.33	2.28, 2.32*