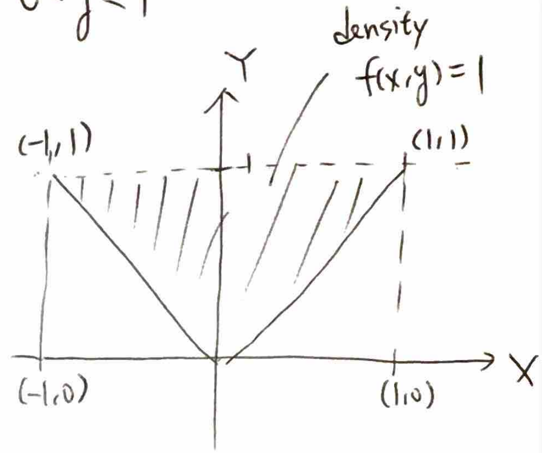
Problem 1

$$f(x,y) = \begin{cases} 1, & \text{if } |x|<y,\ 0<y<1 \\ 0, & \text{else} \end{cases}$$



(a)

To find $E[X], E[Y]$, we start by deriving the marginal PDFs of X and Y:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy \quad \begin{cases} ① \text{ if } x \in (-1,1): & f_X(x) = \int_{|x|}^{1} 1 \cdot dy = 1-|x| \\ ② \text{ Otherwise}: & f_X(x) = 0 \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dy \quad \begin{cases} ① \text{ if } y \in (0,1): & f_Y(y) = \int_{-y}^{y} 1 \cdot dx = 2y \\ ② \text{ Otherwise}: & f_Y(y) = 0. \end{cases}$$

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_{-1}^{1} x \cdot (1-|x|)\,dx = 0$$

$$E[Y] = \int_{-\infty}^{+\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot 2y\,dy = \frac{2}{3}y^3\Big|_0^1 = \frac{2}{3}$$

Moreover, we know $E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \cdot y \cdot f(x,y)\,dx\,dy$

$$= \int_0^1 \left(\int_{-y}^{y} xy \cdot 1 \cdot dx \right) dy = \int_0^1 \left(\frac{1}{2}x^2 y \Big|_{-y}^{y} \right) dy = 0.$$

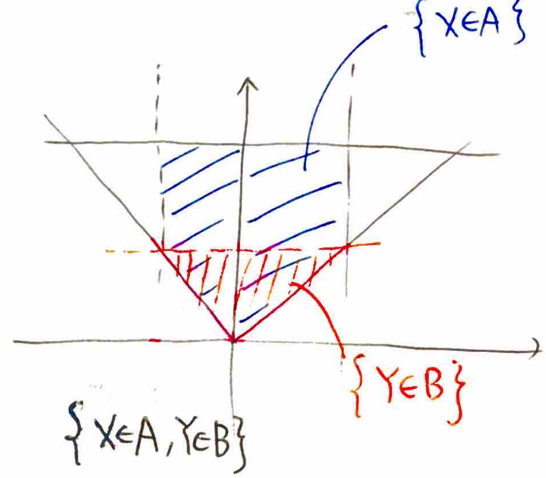
Hence, $E[XY] = E[X] \cdot E[Y]$.

$\square$

(Cont.).

(b). Let's show that X and Y are <u>NOT</u> independent :

Construct two sets A an B as $A = [-\frac{1}{2}, \frac{1}{2}]$, $B = [0, \frac{1}{2}]$.



$P(X \in A) = \frac{6}{8}$

$P(Y \in B) = \frac{1}{4}$

$P(X \in A, Y \in B) = \frac{1}{4}$.

Hence, we know $P(X \in A, Y \in B) \neq P(X \in A) \cdot P(Y \in B)$.

This implies that X and Y are not independent.

$\square$

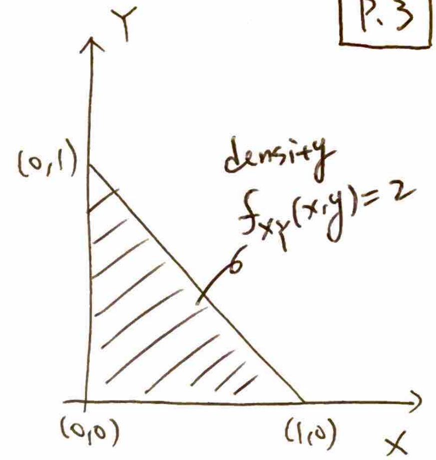
<u>Final Remark</u> :

This problem serves as an example showing that "$E[XY] = E[X]E[Y]$" does not imply that "X, Y are independent".

Problem 2

(a) To begin with, the joint PDF of X, Y is

$$f_{XY}(x,y) = \begin{cases} 2, & \text{if } x \geq 0, y \geq 0, x+y \leq 1 \\ 0, & \text{else} \end{cases}$$



$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Note that for any $y \in (0,1)$, we have

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_0^{1-y} 2 \cdot dx = 2 \cdot (1-y)$$

Therefore, we know for any $y \in (0,1)$,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{2}{2(1-y)}, & \text{if } x \geq 0, x+y \leq 1 \\ 0, & \text{else} \end{cases}$$

(b) For any $y \in (0,1)$, we have

$$E[X|Y=y] = \int_{-\infty}^{+\infty} x \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_0^{1-y} x \cdot \frac{2}{2 \cdot (1-y)}\,dx$$

$$= \frac{1}{2(1-y)} \cdot x^2 \Big|_0^{1-y} = \underline{\frac{1-y}{2}} \#$$

(Conti).

By Law of Iterated Expectation, we have

$$
\begin{aligned}
E[X] &= E[\,E[X|Y]\,] \\
&= \int_0^1 2\cdot(1-y)\left(\frac{1-y}{2}\right) dy \\
&= \int_0^1 (1-y)^2\, dy \\
&= \frac{1}{3}(y-1)^3 \Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

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Problem 3

(a) $X \sim \text{Unif}(-1,3)$.

The PDF of X is $f_X(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in (-1,3) \\ 0, & \text{otherwise} \end{cases}$

Consider the following two cases:

1. $t \neq 0$:

$$M_X(t) = E[e^{tX}] = \int_{-1}^{3} \frac{1}{4} e^{tx}\, dx = \frac{1}{4t} e^{tx} \Big|_{-1}^{3} = \frac{1}{4t}\left(e^{3t} - e^{-t}\right).$$

2. $t = 0$:

$$M_X(t) = \int_{-1}^{3} \frac{1}{4} \cdot e^{0}\, dx = 1.$$

Therefore, we have $M_X(t) = \begin{cases} \frac{1}{4t}\left(e^{3t} - e^{-t}\right), & \text{if } t \neq 0 \\ 1, & \text{if } t = 0 \end{cases}$

Now we are ready to find $E[X]$ and $Var[X]$:

$$E[X] = \frac{d M_X(t)}{dt}\Big|_{t=0} = \lim_{h \to 0} \frac{M_X(h) - M_X(0)}{h} = \lim_{h \to 0} \frac{\frac{1}{4h}\left(e^{3h} - e^{-h}\right) - 1}{h}$$

$$= \lim_{h \to 0} \frac{e^{3h} - e^{-h} - 4h}{4h^2}$$

L'Hôpital's rule

$$= \lim_{h \to 0} \frac{3e^{3h} + e^{-h} - 4}{8h}$$

$$= \lim_{h \to 0} \frac{9e^{3h} - e^{-h}}{8} = 1$$

(Cont.).

$$E[X^2] = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = \lim_{h \to 0} \frac{\left.\frac{dM_X(t)}{dt}\right|_{t=h} - \left.\frac{dM_X(t)}{dt}\right|_{t=0}}{h}$$

Remark: For $t \neq 0$

$$\frac{dM_X(t)}{dt} = -\frac{1}{4t^2}(e^{3t} - e^{-t}) + \frac{1}{4t}(3e^{3t} + e^{-t})$$

For $t = 0$:

$$\frac{dM_X(t)}{dt} = 1$$

$$= \lim_{h \to 0} \frac{-\frac{1}{4h^2}(e^{3h} - e^{-h}) + \frac{1}{4h}(3e^{3h} + e^{-h}) - 1}{h}$$

$$= \lim_{h \to 0} \frac{-(e^{3h} - e^{-h}) + h(3e^{3h} + e^{-h}) - 4h^2}{4h^3}$$

$$= \lim_{h \to 0} \frac{\cancel{-(3e^{3h} + e^{-h})} \cancel{+ (3e^{3h} + e^{-h})} + h \cdot (9e^{3h} - e^{-h}) - 8h}{12h^2}$$

$$= \lim_{h \to 0} \frac{(9e^{3h} - e^{-h}) + h(27e^{3h} + e^{-h}) - 8}{24h}$$

$$= \lim_{h \to 0} \frac{(27e^{3h} + e^{-h}) + (27e^{3h} + e^{-h}) + h \cdot (81e^{3h} - e^{-h})}{24}$$

$$= \frac{7}{3}$$

Therefore, $\mathrm{Var}[X] = E[X^2] - (E[X])^2$

$$= \frac{7}{3} - 1^2$$

$$= \frac{4}{3}$$

□.

(b) $$P_Y(k) = \begin{cases} \dfrac{6}{\pi^2 k^2}, & \text{if } k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} M_Y(t) &= E[e^{tY}] \\ &= \sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} \cdot e^{tk} \\ &\geq \sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} \cdot (1+tk) \\ &= \sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} + \frac{6t}{\pi^2 k} \end{aligned}$$

Remark:

$e^x \geq 1+x$, for all $x \geq 0$

Since $\sum_{k=1}^{\infty} \frac{6t}{\pi^2 k} = \infty$ for all $t > 0$, we know $M_Y(t)$ does not exist for any $t > 0$.

Hence, the MGF of Y does not exist.

$\square$

Problem 4

(a). Binomial (n,p) has an MGF as $(1-p+pe^t)^n$

If $M_X(t) = \left(\frac{1}{3}e^t + \frac{2}{3}\right)^5$, then $X \sim$ Binomial$\left(n=5,\ p=\frac{1}{3}\right)$.

Hence, the PMF of X is $P_X(k) = \begin{cases} C_k^5 \cdot \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{5-k}, & k=0,1,2,3,4,5 \\ 0, & \text{otherwise} \end{cases}$

(b). Poisson (λT) has an MGF as $e^{\lambda T(e^t-1)}$

If $M_X(t) = \exp\left(5\cdot(e^t-1)\right)$, then $X \sim$ Poisson(5).

Hence, the PMF of X is $P_X(k) = \begin{cases} \dfrac{e^{-5}\cdot 5^k}{k!}, & k=0,1,2,\cdots \\ 0, & \text{otherwise} \end{cases}$

Problem 5

$$X_1 = \sigma_1 Z + \mu_1 \quad , \quad Z \sim N(0,1)$$

$$X_2 = \sigma_2\left(\rho Z + \sqrt{1-\rho^2}\,W\right) + \mu_2 \quad , \quad W \sim N(0,1)$$

Z, W are independent.

For simplicity, define $X_1^* = X_1 - \mu_1 = \sigma_1 Z$

$$X_2^* = X_2 - \mu_2 = \sigma_2\left(\rho Z + \sqrt{1-\rho^2}\,W\right)$$

We can further express X_1^*, X_2^* in matrix form:

$$\begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\cdot\sqrt{1-\rho^2} \end{bmatrix}}_{= A} \begin{bmatrix} Z \\ W \end{bmatrix}$$

By the property of linear transformation of 2 random variables, we have

$$f_{X_1^* X_2^*}(x_1^*, x_2^*) = \frac{1}{|\det(A)|}\cdot f_{ZW}\left(A^{-1}\cdot[x_1^*\ x_2^*]^T\right),$$

where $|\det(A)| = \sigma_1\sigma_2\cdot\sqrt{1-\rho^2}$

$$A^{-1} = \frac{1}{\sigma_1\sigma_2\cdot\sqrt{1-\rho^2}}\begin{bmatrix} \sigma_2\cdot\sqrt{1-\rho^2} & 0 \\ -\rho\cdot\sigma_2 & \sigma_1 \end{bmatrix}$$

$$f_{ZW}(z,w) = f_Z(z)\cdot f_W(w) = \frac{1}{2\pi}\exp\left(-\frac{z^2}{2} - \frac{w^2}{2}\right)$$

Therefore, we have

$$f_{X_1^* X_2^*}(x_1^*, x_2^*) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{\left(\frac{\cancel{\sigma_2\sqrt{1-\rho^2}}}{\sigma_1\cancel{\sigma_2\sqrt{1-\rho^2}}}x_1^*\right)^2}{2} - \frac{\left(\frac{-\rho\sigma_2 x_1^* + \sigma_1 x_2^*}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\right)^2}{2}\right)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{x_1^{*2}}{2\sigma_1^2} - \left(\frac{\rho^2 x_1^{*2}}{2\sigma_1^2(1-\rho^2)} - \frac{2\rho x_1^* \cdot x_2^*}{2\sigma_1\sigma_2(1-\rho^2)} + \frac{x_2^{*2}}{2\sigma_2^2(1-\rho^2)}\right)\right]$$

$$f_{X_1^* X_2^*}(x_1^*, x_2^*) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[- \frac{\frac{x_1^{*2}}{\sigma_1^2} - 2\rho\frac{x_1^* x_2^*}{\sigma_1\sigma_2} + \frac{x_2^{*2}}{\sigma_2^2}}{2(1-\rho^2)} \right]$$

Sine $X_1^* = X_1 - \mu_1$, $X_2^* = X_2 - \mu_2$, then we conclude that

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[- \frac{\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)} \right].$$

$\square$

Problem 6 Our goal is to show $E[|XY|] \le E[|X|^p]^{\frac{1}{p}} E[|Y|^q]^{\frac{1}{q}}$

- Young's inequality: Given positive real numbers p, q that satisfy $\frac{1}{p}+\frac{1}{q}=1$, we have $ab \le \frac{a^p}{p}+\frac{b^q}{q}$, for any $a, b > 0$.

Define two random variables $\tilde{X}, \tilde{Y}$ as follows: (Let Ω denote the sample space)

$$\tilde{X}(\omega) = \frac{X(\omega)}{E[|X|^p]^{\frac{1}{p}}}, \quad \text{for all } \omega \in \Omega$$

$$\tilde{Y}(\omega) = \frac{Y(\omega)}{E[|Y|^q]^{\frac{1}{q}}}, \quad \text{for all } \omega \in \Omega$$

By Young's inequality, we have

$$|\tilde{X}(\omega)\cdot\tilde{Y}(\omega)| = |\tilde{X}(\omega)|\cdot|\tilde{Y}(\omega)| \le \frac{|\tilde{X}(\omega)|^p}{p} + \frac{|\tilde{Y}(\omega)|^q}{q}, \quad \text{for all } \omega \in \Omega$$

$$= \frac{1}{p}\cdot\frac{|X(\omega)|^p}{E[|X|^p]} + \frac{1}{q}\cdot\frac{|Y(\omega)|^q}{E[|Y|^q]} \quad \text{——} (*)$$

Since $(*)$ holds for all $\omega \in \Omega$, we thereby have

$$E[|\tilde{X}\cdot\tilde{Y}|] \le \frac{1}{p}\cdot E\left[\frac{|X|^p}{E[|X|^p]}\right] + \frac{1}{q}\cdot E\left[\frac{|Y|^q}{E[|Y|^q]}\right]$$

$$= \frac{1}{p}+\frac{1}{q}$$

$$= 1.$$

This is equivalent to $E[|XY|] \le E[|X|^p]^{\frac{1}{p}} \cdot E[|Y|^q]^{\frac{1}{q}}$.

$\square$