Problem 5: (a) $Y = aX + b$, is a linear transformation of X, and $X \sim Exp(\lambda)$

CDF:

$$F_Y(t) = P(Y \le t) = P(aX + b \le t)$$

Note that the CDF of X is

$$F_X(t) = \begin{cases} 1 - e^{-\lambda t}, & \text{if } t \ge 0 \\ 0, & \text{else.} \end{cases}$$

To find $F_Y(t)$, we need to discuss two cases, namely $a > 0$ and $a < 0$.

① $a > 0$:

$$F_Y(t) = P(aX + b \le t) = P\left(X \le \frac{t-b}{a}\right) = \begin{cases} 1 - e^{-\lambda\left(\frac{t-b}{a}\right)}, & \text{if } t \ge b \\ 0, & \text{else} \end{cases}$$

② $a < 0$:

$$F_Y(t) = P(aX + b \le t) = P\left(X \ge \frac{t-b}{a}\right) = \begin{cases} e^{-\lambda\left(\frac{t-b}{a}\right)}, & \text{if } t \le b \\ 1, & \text{else} \end{cases}$$

since $a < 0$

From the above discussion, we know for Y to be an exponential r.v.,

we need to have $a > 0$ and $b = 0$

(Conti).

PDF:

As it is assumed that the PDF of Y is continuous, we have $f_Y(t) = F_Y'(t)$.

Again, we consider the following two cases:

① $a>0$:

$$f_Y(t) = F_Y'(t) = \begin{cases} \frac{\lambda}{a} e^{-\lambda\left(\frac{t-b}{a}\right)}, & \text{if } t>b \\ 0, & \text{else} \end{cases}$$

② $a<0$:

$$f_Y(t) = F_Y'(t) = \begin{cases} -\frac{\lambda}{a} e^{-\lambda\left(\frac{t-b}{a}\right)}, & \text{if } t<b \\ 0, & \text{else} \end{cases}$$

(b) $X \sim N(0,1) \Rightarrow$ the PDF of X: $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, $\forall x \in \mathbb{R}$.

Since $Var[X] = E[X^2] - (E[X])^2$ and $E[X] = 0$,

all we need is to show that $E[X^2] = 1$.

Recall: Integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\frac{d\left(\frac{-1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right)}{dx} = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x \cdot \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \underbrace{x \cdot -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Big|_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{+\infty} -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}_{\text{PDF of } X} dx = 1$$

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