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Problem 1 (a)

Solution
$E[e^{-tX_i}] = \int_0^\infty e^{-tX_i} f_{X_i}(x)dx \leq \int_0^\infty e^{-tX_i}(1)dx = \int_0^\infty e^{-tX_i}dx = 0 - \frac{-1}{t} = 1/t$

Proven $E[e^{-tX_i}] \leq 1/t$, for every i, for all $t > 0$.

Problem 1 (b)

Solution
$P(\sum_{i=1}^{N} \leq \epsilon N) = P(e^{t\sum_{i=1}^{N} X_i} \leq e^{t\epsilon N}) = P(e^{-t\sum_{i=1}^{N} X_i} \geq e^{-t\epsilon N})$
By Markov's Inequality,
$P(e^{-t\sum_{i=1}^{N} X_i} \geq e^{-t\epsilon N}) \leq e^{t\epsilon N} E[e^{-t\sum_{i=1}^{N} X_i}]$
$\rightarrow P(\sum_{i=1}^{N} \leq \epsilon N) \leq e^{t\epsilon N} E[e^{-t\sum_{i=1}^{N} X_i}] = e^{t\epsilon N} \prod_{i=1}^{N} E[e^{-tX_i}] \leq e^{t\epsilon N}(1/t)^N$ (by 1(a))
$\frac{d}{dt}(e^{t\epsilon}/t)^N = \frac{t\epsilon e^{t\epsilon} - e^{t\epsilon}}{t^2}$
$\frac{t\epsilon e^{t\epsilon} - e^{t\epsilon}}{t^2} = 0$
$\rightarrow t\epsilon - 1 = 0$
$\rightarrow t = 1/\epsilon$
$\rightarrow P(\sum_{i=1}^{N} \leq \epsilon N) \leq (e^{t\epsilon}/t)^N \leq (e\epsilon)^N$
Proven $P(\sum_{i=1}^{N} \leq \epsilon N) \leq (e\epsilon)^N$, for any $\epsilon > 0$.

Problem 2

Solution
Define $A = \{\omega : X_n(\omega)$ does not converges to $a\}$ and $B = \{\omega : Y_n(\omega)$ does not converges to $b\}$.
By definition of almost sure convergence, we conclude $P(A) = P(B) = 0$ and $P(A^c) = P(B^c) = 1$
$P(A^c \cap B^c) = 1 - P(A \cup B) \geq 1 - P(A) - P(B) = 1$

Define $Z_n = X_n \cdot Y_n$.
Define $C = \{\omega : Z_n(\omega)$ converges to $a \cdot b\}$.
Suppose $s \in A^c \cap B^c$.
$\because$
$\lim_{n\to\infty} X_n(s) = a$
$\lim_{n\to\infty} Y_n(s) = b$
$\therefore$
$\lim_{n\to\infty} Z_n(s) = \lim_{n\to\infty} X_n(s)Y_n(s) = \lim_{n\to\infty} X_n(s) \cdot \lim_{n\to\infty} Y_n(s) = a \cdot b$

Thus $s \in C$, $A^c \cap B^c \subset C$.

Then $P(C) \geq P(A^c \cap B^c) = 1$, which implies $P(C) = P(\{\omega : Z_n(\omega)$ converges to $a \cdot b\}) = 1$.
That means $Z_n = X_n \cdot Y_n$ converges to $a \cdot b$, almost surely.

Proven that $X_n \cdot Y_n$ converges to $a \cdot b$, almost surely.

Problem 3(a)

Solution
Define $Y = |X_n - c|$.
$P(|X_n - c| \geq \epsilon) = P(Y \geq \epsilon) = P(Y^2 \geq \epsilon^2)$

By Markov's Inequality,
$P(Y^2 \geq \epsilon^2) \leq \frac{E[Y^2]}{\epsilon^2}$
$\rightarrow \lim_{n\rightarrow\infty} P(|X_n - c| \geq \epsilon) = \lim_{n\rightarrow\infty} P(Y^2 \geq \epsilon^2) \leq \lim_{n\rightarrow\infty} \frac{E[Y^2]}{\epsilon^2} = \lim_{n\rightarrow\infty} \frac{E[(|X_n - c|)^2]}{\epsilon^2} = 0/\epsilon^2 = 0$

Proven that convergence in the mean square implies convergence in probability.

Problem 3(b)

Solution
Define $P(X_n = t) = \{1 - 1/n$, if $t = c$, $1/n$, if $t = n$, 0 else.
$\lim_{n\rightarrow\infty} P(|X_n - c| < \epsilon) = \lim_{n\rightarrow\infty} P(X_n = c) = \lim_{n\rightarrow\infty} 1 - 1/n = 1$
$\rightarrow \lim_{n\rightarrow\infty} P(|X_n - c| \geq \epsilon) = \lim_{n\rightarrow\infty} 1 - P(|X_n - c| < \epsilon) = 1 - 1 = 0$

Then we get it is convergence in probability.

$E[(X_n - c)^2] = E[X_n^2 - 2cX_n + c^2] = E[X_n^2] - 2cE[X_n] + c^2$
$\lim_{n\rightarrow\infty} E[X_n^2] = \lim_{n\rightarrow\infty} c^2 \cdot (1 - 1/n) + n^2 \cdot 1/n = c^2 + \infty = \infty$
$\lim_{n\rightarrow\infty} E[X_n] = \lim_{n\rightarrow\infty} c \cdot (1 - 1/n) + n \cdot 1/n = c + 1$
$\lim_{n\rightarrow\infty} E[(X_n - c)^2] = \lim_{n\rightarrow\infty} E[X_n^2] - 2cE[X_n] + c^2 = \infty - 2c(c + 1) + c^2 = \infty$

Then we get it is not convergence in the mean square.

Proven that convergence in probability does not imply convergence in the mean square.