Problem 1

(a) Recall the PMF of a Poisson random variable with parameters λ and T

$$P_X(k) = \begin{cases} \dfrac{e^{-\lambda T} \cdot (\lambda T)^k}{k!}, & k = 0, 1, 2, \cdots \\ 0, & \text{otherwise} \end{cases}$$

Note that for any $k \in \mathbb{N} \cup \{0\}$, we have

$$\frac{P_X(k+1)}{P_X(k)} = \frac{\dfrac{e^{-\lambda T} \cdot (\lambda T)^{k+1}}{(k+1)!}}{\dfrac{e^{-\lambda T} \cdot (\lambda T)^k}{k!}} = \frac{\lambda T}{(k+1)}$$

Then, we know

$$\begin{cases} \dfrac{P_X(k+1)}{P_X(k)} \geq 1, & \text{for all integers } k \leq \lfloor \lambda T \rfloor - 1 \\ \dfrac{P_X(k+1)}{P_X(k)} < 1, & \text{for all integers } k > \lfloor \lambda T \rfloor - 1 \end{cases}$$

Therefore, we know the maximum of PMF is located at $\lfloor \lambda T \rfloor \equiv k^*$

Hence, $k^* = \underset{k \in \mathbb{N} \cup \{0\}}{\arg\max}\ P_X(k)$.

$\square$

(b). $X_1, X_2, \cdots, X_n$ are n independent random variables and $X_i \sim \text{Geometric}(p)$ for all i.

$X := \min(X_1, \cdots, X_n)$, and X is a positive discrete random variable.

To find the PMF of X, let's consider $P(X>k)$ for a positive integer k:

$$P(X>k) = P(\min(X_1,\cdots,X_n)>k)$$

$$= P(X_1>k \text{ and } X_2>k, \cdots, \text{and } X_n>k)$$

by the independence of $X_1, \cdots, X_k$

$$= P(X_1>k) \cdot P(X_2>k) \cdots P(X_n>k)$$

$$= \underbrace{(1-p)^k \cdot (1-p)^k \cdot \cdots (1-p)^k}_{n \text{ terms}}$$

$$= ((1-p)^n)^k. \quad \text{——— } (*)$$

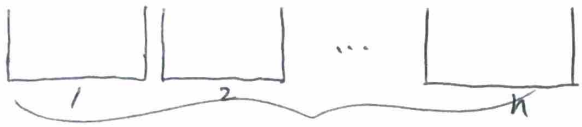
By $(*)$, we know $X \sim \text{Geometric}(1-(1-p)^n)$

Therefore, the PMF of X is $$P_X(k) = \begin{cases} ((1-p)^n)^{k-1} \cdot (1-(1-p)^n), & k=1,2,\cdots \\ 0, & \text{otherwise} \end{cases}$$

$\square$

Problem 2


r indistinguishable balls



n cells

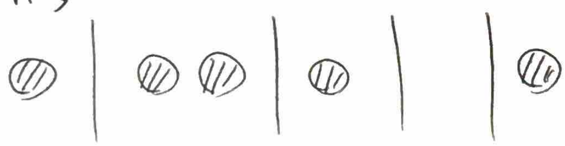
(a) To find q_k, we need to figure out two things:

Define $N(r,n) :=$ "total number of distinguishable arrangements with n cells and r indistinguishable balls"

Then, we know $q_k = \dfrac{N(r-k, n-1)}{N(r,n)}$, for any $k = 0, 1, 2, \cdots, r$

We know finding $N(r,n)$ can be reformulated as "finding # of ordered arrangements with r distinguishables and $n-1$ indistinguishable spacings":

e.g.: $r=5$, $n=5$:



Therefore, $N(r,n) = C^{n+r-1}_{r}$

Similarly, we have $N(r-k, n-1) = C^{(n-1)+(r-k)-1}_{r-k}$

Hence, $q_k = \begin{cases} \dfrac{C^{n+r-k-2}_{r-k}}{C^{n+r-1}_{r}}, & \text{if } k = 0, 1, 2, \cdots, r \\ 0, & \text{otherwise} \end{cases}$

$\square$

(b) For any $k=0,1,\cdots,r$,

$$q_k = \frac{C^{n+r-k-2}_{r-k}}{C^{n+r-1}_{r}}$$

$$= \frac{\dfrac{(n+r-k-2)!}{(r-k)!\,(n-2)!}}{\dfrac{(n+r-1)!}{r!\,(n-1)!}}$$

$$= \frac{(n-1)\,r(r-1)\cdots(r-k+1)}{(n+r-1)\cdots\cdots(n+r-k-1)}$$

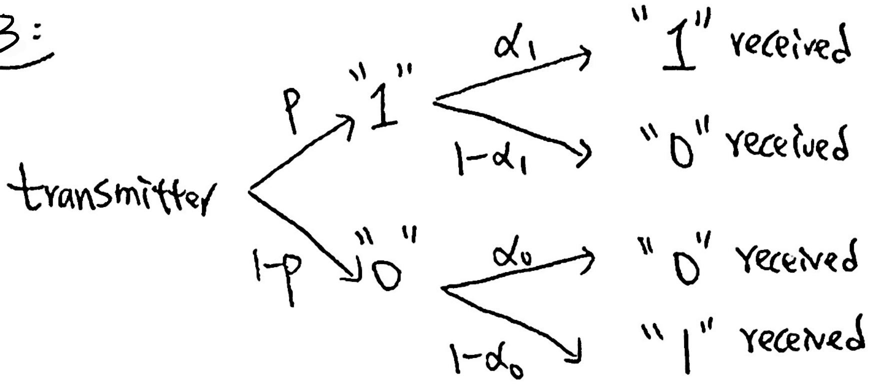
divide both the numerator and the denominator by n^{k+1}

$$= \frac{\left(1-\frac{1}{n}\right)\left(\frac{r}{n}\right)\cdot\left(\frac{r-1}{n}\right)\cdots\left(\frac{r-k+1}{n}\right)}{\left(1+\frac{r-1}{n}\right)\left(1+\frac{r-2}{n}\right)\cdots\left(1+\frac{r-k-1}{n}\right)} \xrightarrow[\frac{r}{n}\to\lambda]{\text{as } r\to\infty,\ n\to\infty} \frac{1\cdot\lambda^k}{(1+\lambda)^{k+1}}$$

Hence, in the limit, X is a Geometric random variable with success probability $\frac{1}{1+\lambda}$.

$\square$

Problem 3:



(a). Define X = # of 1's transmitted in the interval

V = # of transmitted bits " "

# of 0's transmitted

$$P(X=k) = \sum_{n=0}^{\infty} P(X=k \mid V=k+n)\cdot P(V=k+n)$$

$$= \sum_{n=0}^{\infty} \left(C_k^{k+n} \cdot p^k \cdot (1-p)^n \right) \cdot \frac{e^{-\lambda T} \cdot (\lambda T)^{k+n}}{(k+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{(k+n)!}}{k!\, n!}\, p^k (1-p)^n \, \frac{e^{-\lambda T} (\lambda T)^{k+n}}{\cancel{(k+n)!}}$$

$$= \frac{e^{-\lambda p T} \cdot p^k \cdot (\lambda T)^k}{k!} \sum_{n=0}^{\infty} \frac{e^{-\lambda (1-p) T} \cdot (1-p)^n \cdot (\lambda T)^n}{n!}$$

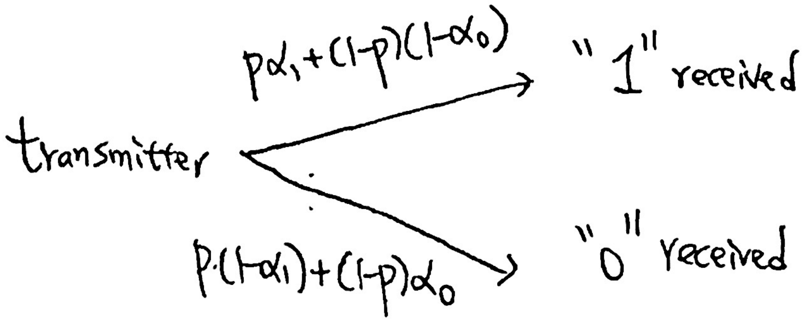
PMF of a Poisson($\lambda(1-p)$, T)

This sum is 1.

$$= \frac{e^{-(\lambda p)T} \cdot ((\lambda p) T)^k}{k!}$$

Remark: The above procedure is usually called "Bernoulli splitting of a Poisson" r.v.

(b). The communication chahell can be cast as follows:



By reusing the results of (a), we know Y is also Poisson, with an average rate $\lambda\cdot[p\alpha_1+(1-p)(1-\alpha_0)]$.

Hence, the PMF of Y =

$$\begin{cases} \dfrac{e^{-\lambda[p\alpha_1+(1-p)(1-\alpha_0)]T}\cdot\left(\lambda[p\alpha_1+(1-p)(1-\alpha_0)]T\right)^k}{k!}, & k\geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 4

(a) Since $X \sim \text{Geometric}(p)$, then $P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p, & k = 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$

Method 1:

$$E[X] = \sum_{k=1}^{\infty} k \cdot P_X(k)$$

$$= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$$

$$= p \cdot \left[\sum_{k=1}^{\infty} (1-p)^{k-1} + \sum_{k=2}^{\infty} (1-p)^{k-1} + \sum_{k=3}^{\infty} (1-p)^{k-1} + \dots + \sum_{k=n}^{\infty} (1-p)^{k-1} + \dots \right]$$

$$= p \cdot \sum_{n=1}^{\infty} \underbrace{\sum_{k=n}^{\infty} (1-p)^{k-1}}_{= \frac{(1-p)^{n-1}}{1-(1-p)}}$$

$$= \cancel{p} \cdot \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}}{\cancel{p}}$$

$$= \sum_{n=1}^{\infty} (1-p)^{n-1}$$

$$= \frac{1}{p}$$

Method 2:

$$E[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$$

can be viewed as $\frac{d}{dp}\left[-(1-p)^{k-1}\right]$

$$= -p \cdot \sum_{k=1}^{\infty} \frac{d}{dp}\left[(1-p)^{k-1}\right]$$

$$= -p \cdot \frac{d}{dp}\left[\underbrace{\sum_{k=1}^{\infty} (1-p)^{k-1}}_{= \frac{1}{p}}\right] \quad \dots \text{by the linearity of diffrentiation.}$$

$$= -p \cdot \frac{-1}{p^2} \quad \dots \quad \frac{d}{dp}\left[\frac{1}{p}\right] = \frac{-1}{p^2}$$

$$= \frac{1}{p}.$$

Similarly

$$E[X^2] = \sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1} \cdot p$$

$$= \sum_{k=1}^{\infty} \left((k+1) \cdot k - k \right) \cdot (1-p)^{k-1} \cdot p$$

$$= \sum_{k=1}^{\infty} (k+1) \cdot k \cdot (1-p)^{k-1} \cdot p - \underbrace{\sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p}_{= \frac{1}{p} \text{ by the result in the last page}}$$

$$= p \cdot \sum_{k=1}^{\infty} \frac{d}{dp} \left[(1-p)^{k+1} \right] - \frac{1}{p}$$

$$= p \cdot \frac{d^2}{dp^2} \Big[\underbrace{\sum_{k=1}^{\infty} (1-p)^{k+1}}_{= \frac{1}{p}} \Big] - \frac{1}{p}$$

$$= p \cdot \frac{2}{p^3} - \frac{1}{p} \qquad \text{...... since } \frac{d^2}{dp^2}\left[\frac{1}{p}\right] = \frac{2}{p^3}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

Therefore, $Var[X] = E[X^2] - (E[X])^2$

$$= \left(\frac{2}{p^2} - \frac{1}{p} \right) - \left(\frac{1}{p} \right)^2$$

$$= \frac{1}{p^2} - \frac{1}{p}$$

$$= \frac{(1-p)}{p^2}$$

$\square$

(b). Define $P(X=a_i)=p_i$ and $P(Y=a_i)=q_i$, for all $i=1,2,\cdots,n$

Suppose $E[X^m]=E[Y^m]=C_m$, for $m=1,\cdots,n-1$

By the definition of moments, we know that for X we have the following system of linear equations:

$$
\begin{aligned}
1\cdot p_1 + 1\cdot p_2 + \cdots + 1\cdot p_n &= 1\\
a_1\cdot p_1 + a_2\cdot p_2 + \cdots + a_n\cdot p_n &= E[X^1] = C_1\\
a_1^2\cdot p_1 + a_2^2\cdot p_2 + \cdots + a_n^2\cdot p_n &= E[X^2] = C_2\\
&\vdots\\
a_1^m\cdot p_1 + a_2^m\cdot p_2 + \cdots + a_n^m p_n &= E[X^m] = C_m\\
&\vdots\\
a_1^{n-1}\cdot p_1 + a_2^{n-1}\cdot p_2 + \cdots + a_n^{n-1} p_n &= E[X^{n-1}] = C_{n-1}
\end{aligned}
$$

The above can be written in matrix form as

$$
\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & & \ddots & \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}}_{= V}
\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}
=
\begin{bmatrix} 1 \\ C_1 \\ \vdots \\ C_{n-1} \end{bmatrix}.
$$

Remark:
V is usually called "Vandermonde matrix"!
Moreover, it is known that

$$\det(V)=\prod_{1\le i<j\le n}(a_j-a_i)$$

Since a_i's are distinct, we know $\det(V)>0$.
Hence, V is invertible.

Therefore, we have

$$
\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}
= V^{-1}
\begin{bmatrix} 1 \\ C_1 \\ \vdots \\ C_{n-1} \end{bmatrix}.
$$

(Cont.)

Similarly, for Y we have a similar system of linear equations

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} 1 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

Therefore, we have $\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = V^{-1} \begin{bmatrix} 1 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$.

To conclude, we know that X and Y are identically distributed.

$\square$

(c). $Z_n = (-1)^n \sqrt{n}$, for $n = 1, 2, \cdots$

Z is a discrete random variable with PMF

$$P_Z(z) = \begin{cases} \frac{6}{(\pi n)^2}, & \text{if } z = Z_n \\ 0, & \text{otherwise} \end{cases}$$

- To find $Var[Z]$, we need to obtain $E[Z^2]$ and $E[Z]$.

However, $E[|Z|^2] = E[Z^2] = \sum_{n=1}^{\infty} ((-1)^n \cdot \sqrt{n})^2 \cdot \frac{6}{(\pi n)^2}$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{6}{\pi^2}$$

$$= \infty \quad \left(\text{since } \sum_{n=1}^{\infty} \frac{1}{n} = \infty \right)$$

Therefore, we know the 2nd moment of Z does not exist.

This implies that $Var[Z]$ does not exist.

Dirichlet eta function evaluated at $\frac{1}{2}$

- $\sum_{n=1}^{\infty} Z_n^3 \cdot P_Z(z_n) = \sum_{n=1}^{\infty} (-1)^n (\sqrt{n})^3 \cdot \frac{6}{(\pi n)^2} = \frac{-6}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{\sqrt{n}}$

$$\approx \frac{-6}{\pi^2} \cdot (0.6049)$$

- $E[|Z|^3] = \sum_{n=1}^{\infty} |Z_n|^3 \cdot P_Z(Z_n) = \sum_{n=1}^{\infty} \frac{6}{\pi^2} \cdot \sqrt{n} = \infty$

Therefore, the 3rd moment of Z does not exist.

(Cont.)

- To figure out the existence of $E[Z^{10}]$, we can show the following property:

<u>Property:</u> If $E[|Z|^{\alpha}]=\infty$ for some $\alpha \in \mathbb{R}$, then we have $E[|Z|^{\beta}]=\infty$, for all $\beta > \alpha$.

<u>Pf of property:</u>

$$\begin{aligned}
E[|Z|^{\beta}] &= \sum_{n=1}^{\infty} |z_n|^{\beta} p_Z(z_n) \\
&= \sum_{z_n: |z_n| \geq 1} |z_n|^{\beta} \cdot p_Z(z_n) + \sum_{z_n: |z_n| < 1} |z_n|^{\beta} \cdot p_Z(z_n) \\
&\geq \sum_{z_n: |z_n| \geq 1} |z_n|^{\alpha} \cdot p_Z(z_n) + \sum_{z_n: |z_n| < 1} |z_n|^{\beta} \cdot p_Z(z_n) \\
&= \underbrace{\sum_{n=1}^{\infty} |z_n|^{\alpha} \cdot p_Z(z_n)}_{= \infty} - \underbrace{\sum_{z_n: |z_n| \leq 1} |z_n|^{\alpha} \cdot p_Z(z_n)}_{\text{between 0 and 1}} + \underbrace{\sum_{z_n: |z_n| < 1} |z_n|^{\beta} \cdot p_Z(z_n)}_{\text{between 0 and 1}} \\
&= \infty
\end{aligned}$$

Therefore, based on the above property, we know that $E[|Z|^3]=\infty$ implies that $E[|Z|^{10}]=\infty$.

Hence, the 10th-moment of Z does not exist.

$\square$