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Problem 1 (a)
Let $S_1, S_2, ...$ be an infinite sequence of sets. Prove that
$\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n = \{x|x \in S_n, \text{ for infinitely many n}\}$.

Solution
Proof.
Let $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n = A$ and $\{x|x \in S_n, \text{ for infinitely many n}\} = B$.

1. $A \subseteq B$
Suppose $x \in A$.
$\rightarrow$ $x \in \bigcup_{n=k}^{\infty} S_n$, for any positive integer k.
$\rightarrow$ We can find infinitely integer n^* s.t. $n^* \geq k$ and $x \in S_{n^*}$ for any positive integer k.
$\rightarrow$ $x \in S_n$, for infinitely many n.
$\rightarrow$ $x \in B$.
$\rightarrow$ $A \subseteq B$.

2. $B \subseteq A$
Suppose $y \in B$.
$\rightarrow$ Give any positive integer k, there exists some positive integer n^* s.t. $n^* \geq k$ and $y \in S_{n^*}$.
$\rightarrow$ $y \in \bigcup_{n=k}^{\infty} S_n$, for all positive integer k.
$\rightarrow$ $y \in A$.
$\rightarrow$ $B \subseteq A$.

Since 1. and 2. we have $A = B$.

Proven $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n = \{x|x \in S_n, \text{ for infinitely many n}\}$.

Problem 1 (b)
Let Ω be the universal set and B, C be two sets that satisfy $B \subseteq \Omega$ and $C \subseteq \Omega$. Let $\{F_k\}_{k=1}^{\infty}$ denote the Fibonacci sequence, i.e., $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$, for $k \geq 2$. Define a countably infinite sequence of sets $A_1, A_2, A_3, ...$ as
$A_n = \{B - C$, if n is in the Fibonacci sequence $\{F_k\}$, $C - B$, otherwise.$\}$
What are $\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} A_n$, $\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n, and \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n$? Please clearly explain your answer.

Solution
We know A_n can only be $B - C$ or $C - B$ by definition. Since 1 is in $\{F_k\}$ and 4 is not in $\{F_k\}$, then we have $A_1 = B - C, A_4 = C - B$.
$\rightarrow$
1. $\bigcap_{n=1}^{\infty} A_n = (B - C) \cap (C - B) = \phi$.
2. $\bigcup_{n=1}^{\infty} A_n = (B - C) \cup (C - B)$.

We know $F_k \rightarrow \infty$ as $k \rightarrow \infty$.
$\rightarrow$ Give any positive integer k, there exits an integer $n^* > k$ and in the Fibonacci sequence. And $n^* + 1$ will be not in the Fibonacci sequence.
$\rightarrow$ We have $A_{n^*} = B - C$ and $A_{n^*+1} = C - B$, and $n^* > k$ for any positive integer k.
$\rightarrow$

3. $\bigcup\limits_{k=1}^{\infty}\bigcap\limits_{n=k}^{\infty} A_n = \bigcup\limits_{k=1}^{\infty}(\phi) = \phi.$
4. $\bigcap\limits_{k=1}^{\infty}\bigcup\limits_{n=k}^{\infty} A_n = \bigcap\limits_{k=1}^{\infty}((B-C)\cup(C-B)) = (B-C)\cup(C-B).$

Problem 1 (c)
Show that there are uncountably infinite many real numbers in the interval $(0,1)$.

Solution
Assume that there are countably infinite real numbers in $(0,1)$.
Define $x_1, x_2, x_3, ..., x_i, ...$ are the infinite real numbers in $(0,1)$, and each real number x_i between 0 and 1 in decimal expansion.
And give a sequence as the following:
$x_1 = 0.127368\ldots$
$x_2 = 0.212562\ldots$
$x_2 = 0.137611\ldots$
$\ldots$
Then we let a number y whose ith decimal place is $x = \{$ 1 if x_i's ith decimal place $\neq 1$, 2 if x_i 's ith decimal place $= 1$.
However for this case we will get $y = 0.221...$ which is $\neq x_i$, for any i.
That is contradiction.

Proven there are uncountably infinite many real numbers in the interval $(0,1)$.

Problem 2 (a)
Let $A_1, A_2, ..., A_N$ be a sequence of events of an experiment. Prove that the following inequality holds for any $N \in \mathbb{N}$:
$P(\bigcup\limits_{n=1}^{N} A_n) \leq \sum\limits_{n=1}^{N} P(A_n).$

Solution
When $N = 1$, $P(\bigcup\limits_{n=1}^{1} A_n) = P(A_1) = \sum\limits_{n=1}^{1} P(A_n).$
$\rightarrow P(\bigcup\limits_{n=1}^{N} A_n) \leq \sum\limits_{n=1}^{N} P(A_n)$ is true when $N = 1$.
Suppose $P(\bigcup\limits_{n=1}^{N} A_n) \leq \sum\limits_{n=1}^{N} P(A_n)$ is true when $N = k$.
When $N = k+1$,

$$\begin{aligned}
P(\bigcup_{n=1}^{k+1} A_n) &= P(\bigcup_{n=1}^{k} A_n \cup A_{k+1}) \\
&= P(\bigcup_{n=1}^{k} A_n) + P(A_{k+1}) - P((\bigcup_{n=1}^{k} A_n) \cap A_{k+1}) \\
&\leq P(\bigcup_{n=1}^{k} A_n) + P(A_{k+1}) \\
&\leq \sum_{n=1}^{k} P(A_n) + P(A_{k+1}) \\
&= \sum_{n=1}^{k+1} P(A_n)
\end{aligned}$$

$\rightarrow P(\bigcup\limits_{n=1}^{k+1} A_n) \leq \sum\limits_{n=1}^{k+1} P(A_n)$ if $P(\bigcup\limits_{n=1}^{k} A_n) \leq \sum\limits_{n=1}^{k} P(A_n)$ is true, for any integer $k > 1$.
Proven $P(\bigcup\limits_{n=1}^{N} A_n) \leq \sum\limits_{n=1}^{N} P(A_n)$ for any $N \in \mathbb{N}$ by induction.

Problem 2 (b)
Consider an experiment with a sample space $\Omega = \{1, 2, 3, 4, 5\}$. Suppose we know $P(\{1, 5\}) = 0.5$, $P(\{1, 2, 4\}) = 0.4$, and $P(\{3\}) = 0.3$. Please write down all possible valid probability assignments. Moreover, among all the possible valid probability assignments, what is the minimum possible value of P({2, 3, 5})? Please explain your answer.

Solution
By axioms 1, 3, we have following:
P({5}) = P(Ω) - P({1, 2, 3, 4}) = 1 - (P({1, 2, 4}) + P({3})) = 1 - (0.4 + 0.3) = 0.5
P({1}) = P({1, 5}) - P({5}) = 0.5 - 0.3 = 0.2
P({2}) + P({4}) = P({1, 2, 4}) - P({1}) = 0.4 - 0.2 = 0.2
Then the all possible valid probability assignments are the assignments which is satisfied P({1}) = 0.2, P({3})) = 0.3, P({5}) = 0.3, and P({2}) + P({4}) = 0.2.
The minimum possible value of P({2, 3, 5}) is 0.6 if P({2}) = 0 and P({4}) = 0.2.

Problem 3 (a)
Let $A_1, A_2, A_3, \dots$ be a countably infinite sequence of events. Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=k}^{\infty} A_n) = 0$.

Solution
By Boole's inequality, we know $P(\bigcup\limits_{n=k}^{\infty} A_n) \leq \sum\limits_{n=k}^{\infty} P(A_n)$.
The assumption, $\sum_{n=1}^{\infty} P(A_n) < \infty$, means the series $\sum_{n=1}^{\infty} P(A_n)$ converges,
$\lim\limits_{k\to\infty} \sum\limits_{n=k}^{\infty} P(A_n) = 0$.
$\to \lim\limits_{k\to\infty} P(\bigcup\limits_{n=k}^{\infty} A_n) = 0$
$\to P(\bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=k}^{\infty} A_n) = 0$
Proven if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=k}^{\infty} A_n) = 0$.

Problem 3 (b)
Consider a countably infinite sequence of coin tosses. The probability of having a head at the $k-$th toss is p_k, with $p_k = 100 \cdot k^{-N}$. We use I to denote the event of observing an infinite number of heads. Show that $P(I) = 0$ if $N > 1$. Please clearly explain your answer.

Solution
$\lim\limits_{k\to\infty} p_k = \lim\limits_{k\to\infty} 100 \cdot k^{-N} = 0$, when $N > 1$.
$\to \sum_{n=1}^{\infty} p_k$ converges.
$\to \sum_{n=1}^{\infty} p_k < \infty \dots (1)$
By the definition, $I = \limsup\limits_{k\to\infty} p_k$.
By Borel-Cantelli Lemma (result in (a)), $\limsup\limits_{k\to\infty} p_k = 0$, when $N > 1$ since (1).
Proven $P(I) = 0$ if $N > 1$.

Problem 4
Suppose we are given a special pair of moon blocks with unknown characteristics. Let $\theta_Y, \theta_L, \theta_N$ denote the unknown probabilities of getting a "Yes" (Y), "Laughing" (L), and "No" (N) at each toss, respectively. Moreover, suppose that the tuple of the unknown parameters $(\theta_Y, \theta_L, \theta_N)$ can only be one of the following three possibilities: $(\theta_Y, \theta_L, \theta_N) \in \{(0.1, 0.3, 0.6), (0.3, 0.6, 0.1), (0.6, 0.3, 0.1)\}$. In order to infer the values $(\theta_Y, \theta_L, \theta_N)$, we experiment with the moon blocks and consider Bayesian inference as follows: Define events $A_1 = \{\theta_Y = 0.1, \theta_L = 0.3, \theta_N = 0.6\}$, $A_2 = \{\theta_Y = 0.3, \theta_L = 0.6, \theta_N = 0.1\}$, $A_3 =$

$\{\theta_Y = 0.6, \theta_L = 0.3, \theta_N = 0.1\}$. Since initially we have no further information about $(\theta_Y, \theta_L, \theta_N)$, we simply consider the prior probability assignment to be $P(A_1) = P(A_2) = P(A_3) = 1/3$.

Question (a)
Suppose we toss the pair of moon blocks once and observe a "Y" (for ease of notation, we define the event B = {the first toss is a Y}). What is the posterior probability $P(A_1|B)$? How about $P(A_2|B)$ and $P(A_3|B)$?

Solution

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)} = \frac{1/3\times0.1}{1/3\times0.1+1/3\times0.3+1/3\times0.6} = \frac{0.1}{0.1+0.3+0.6} = 0.1$$
$$P(A_2|B) = \frac{P(A_2)P(B|A_2)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)} = \frac{1/3\times0.3}{1/3\times0.1+1/3\times0.3+1/3\times0.6} = \frac{0.3}{0.1+0.3+0.6} = 0.3$$
$$P(A_3|B) = \frac{P(A_3)P(B|A_3)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)} = \frac{1/3\times0.6}{1/3\times0.1+1/3\times0.3+1/3\times0.6} = \frac{0.6}{0.1+0.3+0.6} = 0.6$$

Question (b)
Suppose we toss the pair of moon blocks for 12 times and observe $YLYNLYLLYLLL$ (for ease of notation, we define the event C = $\{YLYNLYLLYLLL\}$). Moreover, all the tosses are assumed to be independent. What is the posterior probability $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ?

Solution

$$P(C) = \theta_Y\theta_L\theta_Y\theta_N\theta_L\theta_Y\theta_L\theta_L\theta_Y\theta_L\theta_L\theta_L$$

$$P(C|A_1) = \frac{P(C)\cap P(A_1)}{P(A_1)} = \frac{(0.1)^4(0.3)^7(0.6)P(A_1)}{P(A_1)}$$
$$P(C|A_2) = \frac{P(C)\cap P(A_2)}{P(A_2)} = \frac{(0.3)^4(0.6)^7(0.1)P(A_2)}{P(A_2)}$$
$$P(C|A_3) = \frac{P(C)\cap P(A_3)}{P(A_3)} = \frac{(0.6)^4(0.3)^7(0.1)P(A_3)}{P(A_3)}$$

$$\begin{aligned}
P(A_1|C) &= \frac{P(A_1)P(C|A_1)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \frac{(0.1)^4(0.3)^7(0.6)P(A_1)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \frac{6}{6+3^4\times2^7+6^4} = \frac{1}{1+3^3\times2^6+6^3} = \frac{6}{6+1728+216} = \frac{1}{1945} \approx 0.00051\ldots \\
P(A_2|C) &= \frac{P(A_2)P(C|A_2)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \frac{(0.3)^4(0.6)^7(0.1)P(A_2)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \frac{3^4\times2^7}{6+3^4\times2^7+6^4} = \frac{3^3\times2^6}{1+3^3\times2^6+6^3} = \frac{1728}{1+1728+216} = \frac{1728}{1945} \approx 0.88843\ldots \\
P(A_3|C) &= \frac{P(A_3)P(C|A_3)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \frac{(0.6)^4(0.3)^7(0.1)P(A_3)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \frac{6^4}{6+3^4\times2^7+6^4} = \frac{6^3}{1+3^3\times2^6+6^3} = \frac{216}{1+1728+216} = \frac{216}{1945} \approx 0.11105\ldots
\end{aligned}$$

The most probable value for θ is $A_2 = \{\theta_Y = 0.3, \theta_L = 0.6, \theta_N = 0.1\}$.

Question (c)
Given the same setting as (b), suppose we instead choose to use a different prior probability assignment $P(A_1) = 3/5$, $P(A_2) = 1/5$, $P(A_3) = 1/5$. What is the posterior probabilities $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ?

Solution

$$P(C) = \theta_Y\theta_L\theta_Y\theta_N\theta_L\theta_Y\theta_L\theta_L\theta_Y\theta_L\theta_L\theta_L$$

$$P(C|A_1) = \frac{P(C)\cap P(A_1)}{P(A_1)} = \frac{(0.1)^4(0.3)^7(0.6)P(A_1)}{P(A_1)}$$
$$P(C|A_2) = \frac{P(C)\cap P(A_2)}{P(A_2)} = \frac{(0.3)^4(0.6)^7(0.1)P(A_2)}{P(A_2)}$$

$P(C|A_3) = \frac{P(C)\cap P(A_3)}{P(A_3)} = \frac{(0.6)^4(0.3)^7(0.1)P(A_3)}{P(A_3)}$

$$
\begin{aligned}
P(A_1|C) &= \tfrac{P(A_1)P(C|A_1)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \tfrac{(0.1)^4(0.3)^7(0.6)P(A_1)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \tfrac{6\times 3}{6\times 3+3^4\times 2^7+6^4} = \tfrac{1}{1+3^2\times 2^6+2^3\times 3^2} = \tfrac{1}{1+576+72} = \tfrac{1}{649} \approx 0.00154\ldots \\
P(A_2|C) &= \tfrac{P(A_2)P(C|A_2)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \tfrac{(0.3)^4(0.6)^7(0.1)P(A_2)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \tfrac{3^4\times 2^7}{6\times 3+3^4\times 2^7+6^4} = \tfrac{3^2\times 2^6}{1+3^2\times 2^6+2^3\times 3^2} = \tfrac{576}{1+576+72} = \tfrac{576}{649} \approx 0.88751\ldots \\
P(A_3|C) &= \tfrac{P(A_3)P(C|A_3)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \tfrac{(0.6)^4(0.3)^7(0.1)P(A_3)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \tfrac{6^4}{6\times 3+3^4\times 2^7+6^4} = \tfrac{2^3\times 3^2}{1+3^2\times 2^6+2^3\times 3^2} = \tfrac{72}{1+576+72} = \tfrac{72}{649} \approx 0.11093\ldots
\end{aligned}
$$

The most probable value for θ is $A_2 = \{\theta_Y = 0.3, \theta_L = 0.6, \theta_N = 0.1\}$.