(Fall 2021) 1179 Probability **Due: 2021/12/17 (Friday), 1pm**

Homework 3: Joint Distributions, Bivariate Normal, and MGF

Submission Guidelines: Please combine all your write-ups (photos/scanned copies are acceptable; please make sure that the electronic files are of good quality and reader-friendly) into one single .pdf file and submit the file via E3.

Problem 1 (Independence and Expected Value of Two Random Variables) (12+12=24 points)

Let the joint PDF of the two random variables X and Y be given by

$$f(x,y) = \begin{cases} 1, & \text{if } |x| < y, 0 < y < 1 \\ 0, & \text{else} \end{cases}$$

(a) Show that $E[XY] = E[X]E[Y]$. (Hint: To find $E[X]$ and $E[Y]$, you need to first obtain the marginal PDF of X and Y.)

(b) Show that X and Y are NOT independent. (Hint: Construct an example of two sets A, B such that $P(U \in A, V \in B) \neq P(U \in A)P(V \in B)$)

Problem 2 (Joint and Conditional Distributions) (10+14=24 points)

Let X, Y be two random variables that have a joint PDF which is uniform over the triangle with vertices at $(0,0)$, $(0,1)$, and $(1,0)$.

(a) Find the conditional PDF of X given $Y = y$ with $y \in (0,1)$.

(b) Find $E[X|Y = y]$ $(y \in (0,1))$. Then, please use this result and Law of Iterated Expectation to find $E[X]$.

Problem 3 (Moment Generating Functions) (14+12=26 points)

(a) Let X be a continuous uniform random variable between -1 and 3. Find the MGF of X (denoted by $M_X(t)$) and use the derived $M_X(t)$ to find $E[X]$ and $\text{Var}[X]$. (Hint: When evaluating the first-order and second-order derivatives of $M_X(t)$, you may need to leverage the L'Hôpital's rule)

(b) Let Y be a discrete random variable with PMF

$$p_Y(k) = \begin{cases} \frac{6}{\pi^2 k^2}, & \text{if } k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Show that the MGF of Y (denoted by $M_Y(t)$) does NOT exist, i.e., there exists no interval of the form $(-\delta, \delta)$ (with $\delta > 0$) such that $M_Y(t)$ exists. (Hint: Show that $M_Y(t)$ is not finite on $t \in (0, \infty)$)

Problem 4 (Use MGFs to Find Distributions) (6+6=12 points)

In each of the following cases, $M_X(t)$, the moment generating functions of X, is given. Please determine the distribution of X. (Hint:You could use the MGF table in the slides or the one in the textbook. Please clearly write down the PMF or PDF of X)

(a) $M_X(t) = \left(\frac{1}{3}e^t + \frac{2}{3}\right)^5$.

(b) $M_X(t) = \exp\left[5(e^t - 1)\right]$.

Problem 5 (Bivariate Normal) (14 points)

Let Z and W be two independent standard normal random variables. Let X_1 and X_2 be defined as

$$X_1 = \sigma_1 Z + \mu_1,$$
$$X_2 = \sigma_2(\rho Z + \sqrt{1-\rho^2}W) + \mu_2,$$

where $\sigma_1, \sigma_2 > 0$, μ_1, μ_2 are finite real numbers, and $\rho \in (-1, 1)$. Show that the joint PDF of X_1, X_2 is bivariate normal, i.e., for all $x_1, x_2 \in \mathbb{R}$

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}{2(1-\rho^2)}\right].$$

(Hint: Leverage the theorem of linear transformation of two random variables in Lecture 22)

Problem 6 (Beyond Cauchy-Schwarz Inequality) (12 points)

Let p and q be two positive real numbers that satisfy $1/p + 1/q = 1$. Let X and Y be two random variables with $E[|X|^p] < \infty$, $E[|Y|^q] < \infty$, and $E[|XY|] < \infty$. Show that

$$E[|XY|] \le E[|X|^p]^{\frac{1}{p}} E[|Y|^q]^{\frac{1}{q}}$$

The above is known as the Hölder's inequality. When $p = q = 2$, Hölder's inequality exactly recovers the Cauchy-Schwarz inequality. (Hint: Prove this by Young's inequality. That is, given positive real numbers p, q that satisfy $1/p + 1/q = 1$ and any two positive real numbers a, b, we always have $ab \le \frac{a^p}{p} + \frac{b^q}{q}$)