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**Problem 1 (a)**

**Solution**

$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp(-\frac{\frac{(z_1-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)})$

$f_{Z_1(z_1)} = \int_{-\infty}^{\infty} f_{Z_1,Z_2}(z_1,z_2)dz_2 = \frac{1}{\sqrt{2\pi}\sigma_1}\exp(-\frac{(z_1-\mu_1)^2}{2\sigma_1^2})$

$f_{Z_2|Z_1}(z_2|z_1) = \frac{f_{Z_1,Z_2}(z_1,z_2)}{f_{Z_1(z_1)}}$

$= \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp(-\frac{\frac{(z_1-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)})}{\frac{1}{\sqrt{2\pi}\sigma_1}\exp(-\frac{(z_1-\mu_1)^2}{2\sigma_1^2})}$

$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\cdot\sqrt{2\pi}\sigma_1\cdot\exp(-\frac{\frac{(z_1-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)})\exp(\frac{(z_1-\mu_1)^2}{2\sigma_1^2})$

$= \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}\cdot\exp(-\frac{\frac{(z_1-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}+\frac{(z_1-\mu_1)^2}{2\sigma_1^2})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}-\frac{\frac{(z_1-\mu_1)^2}{\sigma_1^2}}{2(1-\rho^2)}+\frac{(z_1-\mu_1)^2}{2\sigma_1^2})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}-\frac{(z_1-\mu_1)^2}{2\sigma_1^2(1-\rho^2)}+\frac{(z_1-\mu_1)^2(1-\rho^2)}{2\sigma_1^2(1-\rho^2)})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}+\frac{((1-\rho^2)-1)(z_1-\mu_1)^2}{2\sigma_1^2(1-\rho^2)})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}-\frac{\rho^2(z_1-\mu_1)^2}{2\sigma_1^2(1-\rho^2)})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{\frac{\rho^2(z_1-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)})$

Define $X = \frac{(z_1-\mu_1)}{\sigma_1}$ and $Y = \frac{(z_2-\mu_2)}{\sigma_2}$

Then,

$f_{Z_2|Z_1}(z_2|z_1) = \frac{f_{Z_1,Z_2}(z_1,z_2)}{f_{Z_1(z_1)}}$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{(X\rho)^2-2XY\rho+Y^2}{2(1-\rho^2)})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{(X\rho-Y)^2}{2(1-\rho^2)})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{(\frac{(z_1-\mu_1)}{\sigma_1}\rho-\frac{(z_2-\mu_2)}{\sigma_2})^2}{2(1-\rho^2)})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{(\frac{(\rho\sigma_2(z_1-\mu_1)}{\sigma_1}-(z_2-\mu_2))^2}{2\sigma_2^2(1-\rho^2)})$

$= \frac{1}{\sqrt{2\pi(\sigma_2^2(1-\rho^2))}}\cdot\exp(-\frac{(z_2-(\mu_2+\frac{\rho\sigma_2(z_1-\mu_1)}{\sigma_1}))^2}{2\sigma_2^2(1-\rho^2)})$

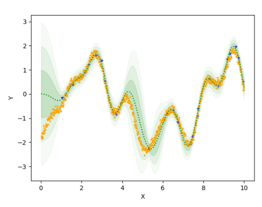
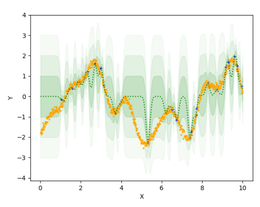
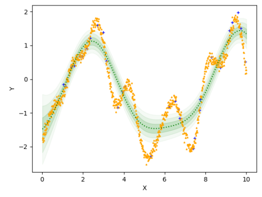
Proven that conditioned on that $Z_1 = z_1$, the conditional distribution of $Z_2$ is normal with mean $\mu_2 + \frac{\rho\sigma_2(z_1-u_1)}{\sigma_1}$ and variance $(1-\rho^2)\sigma_2^2$.

# HW4 PartII

**[Self Link](https://hackmd.io/@pinchen/ProbabilityHW4PartII)** (https://hackmd.io/@pinchen/ProbabilityHW4PartII)

## Problem 1

### (b)

|  |  |  |
| --- | --- | --- |
| Fig. 1 $\sigma_f = 1$, $\sigma = 0.1$, $\ell = 0.5$ | Fig. 2 $\sigma_f = 1$, $\sigma = 0.1$, $\ell = 0.1$ | Fig. 3 $\sigma_f = 1$, $\sigma = 0.1$, $\ell = 2.0$ |

根據上三圖的分布狀況可以發現，當$\ell$越小其綠色區間越小(標準差越小)且虛線越平滑(期望值)，反之越大、越多轉折。應該是因為當$\ell$越大數據間的差異被縮小，反之差異被放大，且我們可以發現$\ell$為0.1或2.0都比0.5偏離真實數據，要預測數據分布可能需要選擇適當的$\ell$。

## Problem 2

### (a)

```
n = 10^3:
[0.5457421795388181, 0.6390370690555895, 0.5494161446896537, 0.5735691859890404, 0
n = 10^5:
[0.6069972165150564, 0.606546467920867, 0.6081740914680214, 0.6060589053706937, 0.
```

可以明顯感覺到後者的數據更為集中在0.606左右，偏差不大，而前者隨然也在這附近，但互相的差距較為寬。展現出測試越多次其會更為收斂至某一個數，且其值可能為真實數值。

### (b)

```
n = 10^1:
0.4
n = 10^3:
0.588
n = 10^5:
0.55184
n = 10^7:
0.5550588
```