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Problem 5 (a)

Solution

$\lambda > 0$

We know the CDF of X is $F_X(t) = \{1 - e^{-\lambda t}$, if $t \geq 0$; 0, otherwise.

Then $Y = aX + b$.

The CDF of Y is $F_Y(t) = P(Y \leq t) = P(aX + b \leq t) = P(X \leq \frac{t-b}{a}) = F_X(\frac{t-b}{a}) = \{1 - e^{-\lambda \frac{t-b}{a}}$, if $\frac{t-b}{a} \geq 0$; 0 otherwise.

$\frac{d}{dt}(1 - e^{-\lambda \frac{t-b}{a}}) = \frac{\lambda}{a} e^{-\lambda \frac{t-b}{a}}$.

The PDF of Y is $f_Y(t) = \{\frac{\lambda}{a} e^{-\lambda \frac{t-b}{a}}$, if $\frac{t-b}{a} \geq 0$; 0 otherwise.

Define $\lambda^* = \frac{\lambda}{a}$.

λ^* should greater than 0 and t have interval $[0, \infty)$.

Hence $a > 0$ and $b = 0$.

When $a > 0$ and $b = 0$, Y also is an exponential random variable:

Problem 5 (b)

Solution

The PDF of standard normal variables is $P(X = x) = \frac{1}{2\pi} \exp(\frac{-x^2}{2})$, $\forall x \in \mathbb{R}$.

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} [e^{-x^2/2}]_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}} (\lim_{t\to\infty} [e^{-x^2/2}]_{-t}^{0} + \lim_{t\to\infty} [e^{-x^2/2}]_{0}^{t}) = \frac{-1}{\sqrt{2\pi}} \times 0 = 0$$

$$\begin{aligned} Var[X] &= \int_{-\infty}^{\infty} (x - E[X])^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = Var[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x)^2 e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} ([-xe^{-x^2/2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx) = \frac{1}{\sqrt{2\pi}} (0 + \int_{-\infty}^{\infty} e^{-x^2/2} dx) = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} = 1 \end{aligned}$$

Verified that a standard normal random variable X satisfies that $Var[X] = 1$.