Name: 陳品劭 ID: 109550206 Self link

Problem 1 (a)

Solution

We know $p_X(k) = \frac{e^{-\lambda T}(\lambda T)^k}{k!}$ and $p_X(k) \geq 0$, for any k.
Give an integer a.
$p_X(a) = \frac{e^{-\lambda T}(\lambda T)^a}{a!}$
$p_X(a-1) = \frac{e^{-\lambda T}(\lambda T)^{a-1}}{(a-1)!}$
$\rightarrow p_X(a) = p_X(a-1) \cdot \frac{\lambda T}{a}$

When $k = a$, and $0 \leq a \leq k^* \leq \lambda T$.
Since $0 \leq a \leq k^* \leq \lambda T$, $\frac{\lambda T}{a} \geq 1$.
However $p_X(a) \geq p_X(a-1)$, when $0 \leq a \leq k^* \leq \lambda T$.
Proven the PMF of X is monotonically non-decreasing with k in the range from 0 to k^*. $\ldots(1)$

When $k = a$, and $0 \leq k^* \leq \lambda T < a$.
Since $0 \leq k^* \leq \lambda T < a$, $\frac{\lambda T}{a} < 1$.
However $p_X(a-1) > p_X(a)$, when $0 \leq k^* \leq \lambda T < a$.
Proven the PMF of X is monotonically decreasing with k for $k \geq k^*$. $\ldots(2)$

By (1) and (2), we can have $p_X(k^*)$ is the largest number of the PMF of X.
Proven that $k^* = \arg\max_{k \in \mathbb{N} \cup \{0\}} p_X(k)$.

Problem 1 (b)

Solution

The PMF of X_i is $P(X_i = k) = (1-p)^{k-1}p$, $k = 1, 2, 3, \ldots$.
$P(X_i > k) = (1-p)^k$
$X > k$ means all $X_i > k$ and $X_1, ..., X_n$ are independent, so $P(X > k) = \prod_{i=1}^{n} P(X_i > k) = (1-p)^{kn}$.
$P(X \leq k) = 1 - (1-p)^{kn}$
$P(X = k) = P(X \leq k) - P(X \leq k-1) = (1-(1-p)^{kn}) - (1-(1-p)^{(k-1)n}) = (1-p)^{n(k-1)}(1-(1-p)^n)$
Define $P^* = (1-p)^n$.
Then $P(X = k) = (1-p)^{n(k-1)}(1-(1-p)^n) = (P^*)^{k-1}(1-P^*)$
Define $P = (1-P^*)$.
Then $P(X = k) = (P^*)^{k-1}(1-P^*) = P(1-P)^{k-1}$.
The PMF of X is $P(X = k) = P(1-P)^{k-1}$, where $P = 1-(1-p)^n$, $k = 1, 2, 3, \ldots$.

So, it follow that the minimum of n independent Geometric random variables has geometric distribution.
Then the PMF of X is also a Geometric random variables.

Problem 2 (a)

Solution

There are r indistinguishable balls which are need to distributing to n different cells.
Define the number of balls in the ith cells is X_i.
Then we have $\sum_{i=1}^{n} X_i = r$, and the number of arrangements is C_r^{n+r-1}.
When the number of balls in the 1st cell is k, means $k + X_2 + ...X_n = r$.
Then we have $X_2 + ...X_n = r - k = \sum_{i=2}^{n} X_i = r - k$, and the number of arrangements is $C_{r-k}^{n+r-k-2}$.

However the probability of the event that $X = k$ is $\frac{C_{r-k}^{n+r-k-2}}{C_r^{n+r-1}}$.

Proven $q_k = \frac{C_{r-k}^{n+r-k-2}}{C_r^{n+r-1}}$, for any $k \in \mathbb{N} \cup \{0\}$

Problem 2 (b)

Solution

$q_k = \frac{C_{r-k}^{n+r-k-2}}{C_r^{n+r-1}} = \frac{(n+r-k-2)!}{(r-k)!(n-2)!} \cdot \frac{r!(n-1)!}{(n+r-1)!} = (n-1)\frac{r\times(r-1)\times\cdots\times(r-k+1)}{(n+r-1)\times\cdots\times(n+r-k-1)}$

$\because \lim\limits_{n\to\infty,r\to\infty} \frac{n+r-i-1}{r-i} = \lim\limits_{n\to\infty,r\to\infty} (\frac{n}{r-i} + \frac{r-i}{r-i} - \frac{1}{r-i}) = \lim\limits_{n\to\infty,r\to\infty} \frac{n}{r-i} + \lim\limits_{n\to\infty,r\to\infty} \frac{r-i}{r-i} - \lim\limits_{n\to\infty,r\to\infty} \frac{1}{r-i}$

$= \lim\limits_{n\to\infty,r\to\infty} \frac{n}{r-i} + 1 - 0 = \frac{1}{\lambda} + 1$, where $i = 0, 1, \ldots, k-1$.

$\therefore \lim\limits_{n\to\infty,r\to\infty} q_k = \lim\limits_{n\to\infty,r\to\infty} (\frac{(n-1)}{(n+r-k-1)}) \cdot (\frac{1}{1/\lambda+1})^k$

$\because \lim\limits_{n\to\infty,r\to\infty} \frac{n+r-k-1}{n-1} = \lim\limits_{n\to\infty,r\to\infty} (\frac{n-1}{n-1} + \frac{r}{n-1} - \frac{k}{n-1}) = \lim\limits_{n\to\infty,r\to\infty} \frac{n-1}{n-1} + \lim\limits_{n\to\infty,r\to\infty} \frac{r}{n-1} - \lim\limits_{n\to\infty,r\to\infty} \frac{k}{n-1}$

$= \lim\limits_{n\to\infty,r\to\infty} 1 + \frac{r}{n-1} - 0 = \lambda + 1$.

$\therefore \lim\limits_{n\to\infty,r\to\infty} q_k = \lim\limits_{n\to\infty,r\to\infty} (\frac{(n-1)}{(n+r-k-1)}) \cdot (\frac{1}{1/\lambda+1})^k = \frac{1}{1+\lambda}(\frac{\lambda}{\lambda+1})^k = \frac{\lambda^k}{(1+\lambda)^{k+1}}$

Proven that if the average number of particles per cell r/n tends to λ as $n \to \infty$ and $r \to \infty$, then $q_k \to \frac{\lambda^k}{(1+\lambda)^{k+1}}$ as $n \to \infty$ and $r \to \infty$.

Define $P = \frac{1}{1+\lambda}$, then $q_k \to \frac{\lambda^k}{(1+\lambda)^{k+1}} = \frac{1}{1+\lambda}(\frac{\lambda}{\lambda+1})^k = P(1-P)^k$, $k = 0, 1, 2, \ldots$.

Then the kind of random variable of X is Geometric random variable in the limit.

Problem 3 (a)

Solution

Define V = total transmitted bits in the given interval.

By the total probability theorem, $P(X = k) = \sum\limits_{n=0}^{\infty} P(X = k|V = k+n) \cdot P(V = k+n)$.

$P(X = k|V = k+n) = C_k^{n+k} p^k (1-p)^n$.

$P(V = k+n) = \frac{e^{-\lambda T}(\lambda T)^{k+n}}{(k+n)!}$.

$$
\begin{aligned}
P(X = k) &= \sum_{n=0}^{\infty} C_k^{n+k} p^k (1-p)^n \cdot \frac{e^{-\lambda T}(\lambda T)^{k+n}}{(k+n)!} \\
&= \sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!} p^k (1-p)^n \cdot \frac{e^{-\lambda T}(\lambda T)^{k+n}}{(k+n)!} \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^{k+n} p^k (1-p)^n}{n!k!} \\
&= \sum_{n=0}^{\infty} \frac{e^{-(1-p)\lambda T}(\lambda T)^n (1-p)^n}{n!} \cdot \frac{e^{-p\lambda T}(\lambda T)^k p^k}{k!} \\
&= \frac{e^{-p\lambda T}(\lambda T)^k p^k}{k!} \cdot \sum_{n=0}^{\infty} \frac{e^{-(1-p)\lambda T}(\lambda T)^n (1-p)^n}{n!} \\
&= \frac{e^{-p\lambda T}(p\lambda T)^k}{k!} \cdot \sum_{n=0}^{\infty} \frac{e^{-(1-p)\lambda T}((1-p)\lambda T)^n}{n!} \\
&= \frac{e^{-p\lambda T}(p\lambda T)^k}{k!} \cdot 1 \text{ (by Taylor's expansion)} \\
&= \frac{e^{-p\lambda T}(p\lambda T)^k}{k!}
\end{aligned}
$$

Then we have that $P(X = k)$ is Poisson PMF with average rate $p\lambda$.

Proven that X has a Poisson PMF with average rate $p\lambda$.

Problem 3 (b)

Solution

Define a random variable Z to be the number of 0's transmitted in that interval.

And $P(Z=n) = \frac{e^{-(1-p)\lambda T}((1-p)\lambda T)^n}{n!}$

Then $P(Y=m) = \sum_{j=0}^{m}\sum_{k=j}^{\infty}\sum_{n=m-j}^{\infty} C_j^k a_1^k(1-a_1)^{k-j}P(X=k)\cdot C_{m-j}^m a_0^{m-j}(1-a_0)^{n-m+j}P(Z=n)$

$= \sum_{j=0}^{m}\sum_{k=j}^{\infty}\sum_{n=m-j}^{\infty} C_j^k a_1^k(1-a_1)^{k-j}\frac{e^{-p\lambda T}(p\lambda T)^k}{k!}\cdot C_{m-j}^m a_0^{m-j}(1-a_0)^{n-m+j}\frac{e^{-(1-p)\lambda T}((1-p)\lambda T)^n}{n!}$

$= \sum_{j=0}^{m}\sum_{k=j}^{\infty}\sum_{n=m-j}^{\infty} \frac{e^{-a_1p\lambda T}(a_1p\lambda T)^j}{j!}\cdot\frac{e^{-(1-a_1)p\lambda T}((1-a_1)p\lambda T)^{k-j}}{(k-j)!}\cdot\frac{e^{-(1-a_0)(1-p)\lambda T}((1-a_0)(1-p)\lambda T)^{m-j}}{(m-j)!}\cdot\frac{e^{-a_0(1-p)\lambda T}(a_0(1-p)\lambda T)^{n-m-j}}{(n-m-j)!}$

$= \sum_{j=0}^{m}\frac{e^{-a_1p\lambda T}(a_1p\lambda T)^j}{j!}\cdot\frac{e^{-(1-a_0)(1-p)\lambda T}((1-a_0)(1-p)\lambda T)^{m-j}}{(m-j)!}\cdot\sum_{k=j}^{\infty}\frac{e^{-(1-a_1)p\lambda T}((1-a_1)p\lambda T)^{k-j}}{(k-j)!}\cdot\sum_{n=m-j}^{\infty}\frac{e^{-a_0(1-p)\lambda T}(a_0(1-p)\lambda T)^{n-m-j}}{(n-m-j)!}$

$= \sum_{j=0}^{m}\frac{e^{-a_1p\lambda T}(a_1p\lambda T)^j}{j!}\cdot\frac{e^{-(1-a_0)(1-p)\lambda T}((1-a_0)(1-p)\lambda T)^{m-j}}{(m-j)!}\cdot 1\cdot 1$ (by Taylor's expansion)

$= \sum_{j=0}^{m}\frac{e^{-a_1p\lambda T}(a_1p\lambda T)^j}{j!}\cdot\frac{e^{-(1-a_0)(1-p)\lambda T}((1-a_0)(1-p)\lambda T)^{m-j}}{(m-j)!}\cdot\frac{m!}{m!}$

$= \frac{e^{-(a_1p+(1-a_0)(1-p))\lambda T}}{m!}\sum_{j=0}^{m} C_j^m\cdot(a_1p\lambda T)^j\cdot((1-a_0)(1-p)\lambda T)^{m-j}$

$= \frac{e^{-(a_1p+(1-a_0)(1-p))\lambda T}}{m!}((a_1p+(1-a_0)(1-p))\lambda T)^m$

Define $\lambda^* = (a_1p+(1-a_0)(1-p))\lambda$

Then $P(Y=m) = \frac{e^{-\lambda^*T}(\lambda^*T)^m}{m!}$.

The PMF of Y is $P(Y=m) = \frac{e^{-\lambda^*T}(\lambda^*T)^m}{m!}$, where $\lambda^* = (a_1p+(1-a_0)(1-p))\lambda$, $m=0,1,2,\ldots$

Problem 4 (a)

Solution

$P(X=k) = p(1-p)k-1$

$E[X] = \sum_{k=1}^{\infty}p(1-p)^{k-1}k = \sum_{k=0}^{\infty}p(1-p)^k(k+1) = p+\sum_{k=1}^{\infty}p(1-p)^k(k+1)$

$(1-p)E[X] = \sum_{k=1}^{\infty}k(1-p)^kp$

$E[X]-(1-p)E[X] = p+\sum_{k=1}^{\infty}p(1-p)^k(k+1)-\sum_{k=1}^{\infty}k(1-p)^kp = p+\sum_{i=1}^{\infty}(1-p)^kp = p+(1-p)\sum_{i=1}^{\infty}(1-p)^{k-1}p =$
$p+(1-p)\cdot 1 = 1$

$E[X]-(1-p)E[X] = E[X]-E[X]-pE[X] = 1$

$E[X] = \frac{1}{p}$

$E[X^2] = \sum_{k=1}^{\infty}p(1-p)^{k-1}k^2 = \sum_{k=0}^{\infty}p(1-p)^k(k+1)^2 = p+\sum_{k=1}^{\infty}p(1-p)^k(k+1)^2$

$(1-p)E[X^2] = \sum_{k=1}^{\infty}p(1-p)^kk^2$

$E[X^2]-(1-p)E[X^2] = p+\sum_{k=1}^{\infty}p(1-p)^k(k+1)^2-\sum_{k=1}^{\infty}p(1-p)^kk^2 = p+\sum_{k=1}^{\infty}p(1-p)^k(2k+1)$

$= p+2(1-p)E[X]+(1-p)\cdot 1 = \frac{2-2p}{p}+1 = \frac{2-p}{p}$

$Var[X] = E[X^2]-E[X]^2 = \frac{2-p}{p}-(\frac{1}{p})^2 = \frac{1-p}{p^2}$

Proven $E[X] = \frac{1}{p}$ and $Var[X] = \frac{1-p}{p^2}$.

Problem 4 (b)

Solution

Suppose $E[X^m] = E[Y^m]$, for all $m \in \{1, 2, \ldots, n-1\}$.

$E[X^m] = \sum_{i=1}^{n} a_i^m P(X = a_i)$, for all $m \in \{1, 2, \ldots, n-1\}$.

$E[Y^m] = \sum_{i=1}^{n} a_i^m P(Y = a_i)$, for all $m \in \{1, 2, \ldots, n-1\}$.

Since the set is finite and a_i are real number ($i = 1, 2, \ldots, n$), we can arbitrary change the order of the set.

$$A = \begin{bmatrix} E[X^m] \\ \vdots \\ E[X] \\ 1 \end{bmatrix} = \begin{bmatrix} a_1^m & \ldots & a_n^m \\ \vdots & & \vdots \\ a_1^0 & \ldots & a_n^0 \end{bmatrix} \begin{bmatrix} P(X = a_1) \\ \vdots \\ P(X = a_n) \end{bmatrix} = \begin{bmatrix} a_1^m & \ldots & a_n^m \\ \vdots & & \vdots \\ a_1^0 & \ldots & a_n^0 \end{bmatrix} \begin{bmatrix} P(Y = a_1) \\ \vdots \\ P(Y = a_n) \end{bmatrix} = \begin{bmatrix} E[Y^m] \\ \vdots \\ E[Y] \\ 1 \end{bmatrix}$$

Define M= $\begin{bmatrix} a_1^m & \ldots & a_n^m \\ \vdots & & \vdots \\ a_1^0 & \ldots & a_n^0 \end{bmatrix}$.

Since $a_1, \ldots, a_n$ are n distinct real number, $\det(M) \neq 0$.

Then we have M^{-1}.

$$M^{-1}M \begin{bmatrix} P(X = a_1) \\ \vdots \\ P(X = a_n) \end{bmatrix} = M^{-1}M \begin{bmatrix} P(Y = a_1) \\ \vdots \\ P(Y = a_n) \end{bmatrix} \to \begin{bmatrix} P(X = a_1) \\ \vdots \\ P(X = a_n) \end{bmatrix} = \begin{bmatrix} P(Y = a_1) \\ \vdots \\ P(Y = a_n) \end{bmatrix}$$

Proven X and Y are identically distributed, for all $t \in \{A_1, \ldots, a_n\}$, if $E[X^m] = E[Y^m]$, for all $m \in \{1, 2, \ldots, n-1\}$.

Problem 4 (c)

Solution

$E[|Z|] = \sum_{n=1}^{\infty} |z_n| P(Z = z_n) = \sum_{n=1}^{\infty} \sqrt{n} \frac{6}{(\pi n)^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$

Since $E[|Z|] < \infty$, $E[Z]$ exists.

$E[|Z^2|] = \sum_{n=1}^{\infty} |z_n^2| P(Z = z_n) = \sum_{n=1}^{\infty} z_n^2 P(Z = z_n) = E[Z^2] = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$

$Var[Z] = E[Z^2] - E[Z]^2 = \infty - E[Z]^2 = \infty$

$\sum_{n=1}^{\infty} z_n^3 \cdot p_Z(z_n) = \sum_{n=1}^{\infty} (-1)^n \frac{6}{(\pi)^2 \sqrt{n}} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Since $\lim_{n \to \infty} \frac{(-1)^n}{\sqrt{n}} = 0$, and $\frac{d}{dx}(\frac{(-1)^n}{\sqrt{n}}) < 0$, $\sum_{n=1}^{\infty} z_n^3 \cdot p_Z(z_n)$ converges.

Since $E[|Z|^3] = \sum_{n=1}^{\infty} \frac{6}{(\pi)^2 \sqrt{n}} = \frac{6}{(\pi)^2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{6}{(\pi)^2} \times \infty = \infty$ and $E[Z^3]$ converges, $E[Z^3]$ DNE.

$E[|Z^{10}|] = \sum_{n=1}^{\infty} |z_n^{10}| p_Z(z_n) = \sum_{n=1}^{\infty} z_n^{10} p_Z(z_n) = E[Z^{10}] = \sum_{n=1}^{\infty} n^5 \frac{6}{(\pi)^2 \sqrt{n}} = \infty$.

Since $E[|Z^{10}|] = E[Z^{10}] = \infty$, $E[Z^{10}]$ exists.

Ans: $Var[Z] = \infty$, $\sum_{n=1}^{\infty} z_n^3 \cdot p_Z(z_n)$ converges, $E[Z^3]$ DNE, $\sum_{n=1}^{\infty} z_n^{10} p_Z(z_n)$ exists.

Problem 5 (a)

Solution
5(a)'s solution.

Problem 5 (b)

Solution
The PDF of standard normal variables is $P(X=x) = \frac{1}{2\pi}\exp(\frac{-x^2}{2})$, $\forall x \in \mathbb{R}$.

$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{1}{\sqrt{2\pi}}[e^{-x^2/2}]_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}(\lim_{t\to\infty}[e^{-x^2/2}]_{-t}^{0} + \lim_{t\to\infty}[e^{-x^2/2}]_{0}^{t}) = \frac{-1}{\sqrt{2\pi}} \times 0 = 0$

$$\begin{aligned} Var[X] &= \int_{-\infty}^{\infty} (x - E[X])^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = Var[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x)^2 e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}}([-xe^{-x^2/2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx) = \frac{1}{\sqrt{2\pi}}(0 + \int_{-\infty}^{\infty} e^{-x^2/2} dx) = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} = 1 \end{aligned}$$

Verified that a standard normal random variable X satisfies that $Var[X] = 1$.