(Fall 2021) 1179 Probability **Due: 2021/12/30 (Thursday), 9pm**

Homework 4, Part I: Concentration Inequalities and Law of Large Numbers

Submission Guidelines: Please combine all your write-ups (photos/scanned copies are acceptable; please make sure that the electronic files are of good quality and reader-friendly) into one single .pdf file and submit the file via E3.

Problem 1 (Techniques of Chernoff Bound) (12+12=24 points)

Let $X_1, \cdots, X_N$ be non-negative independent random variables with continuous distributions (but $X_1, \cdots, X_N$ are not necessarily identically distributed). Assume that the PDFs of X_i's are uniformly bounded by 1 (that is, for every X_i, we have $f_{X_i}(x) \leq 1$, for all $x \in \mathbb{R}$).

(a) Show that for every i, $E[\exp(-tX_i)] \leq \frac{1}{t}$, for all $t > 0$.

(b) By using (a), show that for any $\varepsilon > 0$, we have

$$P\Big(\sum_{i=1}^{N} X_i \leq \varepsilon N\Big) \leq (e\varepsilon)^N.$$

(Hint: For any $t > 0$, $P(\sum_{i=1}^{N} X_i \leq \varepsilon N) = P(e^{t\sum_{i=1}^{N} X_i} \leq e^{t\varepsilon N}) = P\big(e^{-t\sum_{i=1}^{N} X_i} \geq e^{-t\varepsilon N}))$

Problem 2 (Strong Law of Large Numbers) (12 points)

Consider two sequences of random variables $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ defined on the same sample space. Suppose that X_n converges to a and Y_n converges to b, almost surely. Show that $X_n \cdot Y_n$ converges to $a \cdot b$, almost surely. (Hint: Consider two events A, B defined as $A = \{\omega : X_n(\omega) \text{ does not converge to } a\}$ and $B = \{\omega : Y_n(\omega) \text{ does not converge to } b\}$)

Problem 3 (Convergence in Probability) (12+12=24 points)

A sequence of random variables $X_1, X_2, \cdots$ is said to converge to a number c **in the mean square**, if

$$\lim_{n\to\infty} E[(X_n - c)^2] = 0.$$

(a) Show that convergence in the mean square implies convergence in probability. (Hint: For every $\varepsilon > 0$, consider $P(|X_n - c| \geq \varepsilon)$ and use Markov's inequality)

(b) Please construct an example that shows that "convergence in probability" does not imply "convergence in the mean square."