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Problem 1 (a)

Solution
$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_0^1\int_{-y}^{y} xydxdy = 0$

$\int_{-\infty}^{\infty} f(x,y)dy = \int_{|x|}^{1} 1dy = 1-|x|$
The marginal PDF of X is $f_X(x) = \{1-|x|$, if $-1 < x < 1$; 0 else.

$\int_{-\infty}^{\infty} f(x,y)dx = \int_{-y}^{y} 1dy = 2y$
The marginal PDF of Y is $f_Y(y) = \{2y$, if $0 < y < 1$; 0 else.

$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_{-1}^{1} x(1-|x|)dx = 0$
$E[Y] = \int_{-\infty}^{\infty} xf_Y(y)dy = \int_0^1 y(2y)dy = 2/3$

Then we have $E[XY] = 0 = E[X]E[Y] = 0 \times 2/3$

Problem 1 (b)

Solution
Give $A = \{x|0.1 < x < 0.2\}$, $B = \{y|0.5 < y < 0.6\}$.

$P(U \in A), V \in B) = \int_{0.1}^{0.2}\int_{0.5}^{0.6} f(x,y)dxdy = \int_{0.1}^{0.2}\int_{0.5}^{0.6} 1dxdy = 0.1 \times 0.1 = 0.01$
$P(U \in A) = \int_{0.1}^{0.2} f_X(x)dx = \int_{0.1}^{0.2} 1-|x|dx = \int_{0.1}^{0.2} 1-xdx = 0.2-0.02-0.1+0.005 = 0.085$
$P(V \in B) = \int_{0.5}^{0.6} f_Y(y)dy = \int_{0.5}^{0.6} 2ydx = 0.36-0.25 = 0.14$

Then $P(U \in A), V \in B) = 0.01 \neq P(U \in A)P(V \in B) = 0.14 \times 0.085 = 0.0119$.
Proven X and Y are not independent.

Problem 2 (a)

Solution
Define the joint PDF of X and Y is $f_{XY}(x,y)$.
We know $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx = 1$ and $f_{XY}(x,y)$ is uniform over the triangle with vertices at (0, 0), (0, 1), and (1, 0).
$\because \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx = \int_0^1\int_0^{1-x} f_{XY}(x,y)dydx = f_{XY}(x,y)/2 = 1$
$\therefore f_{XY}(x,y) = 2$, when $0 < y < 1-x$, $0 < x < 1$
Then we have $f_{XY}(x,y) = \{2$, if $0 < y < 1-x$, $0 < x < 1$; 0, else.

Define the marginal PDF of Y is $f_Y(y)$.
$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^{1-y} 2dx = 2-2y$

Define the conditional PDF of X given $Y = y$ with $y \in (0,1)$ is $f_{X|Y}(x|y)$.
$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{2-2y} = \frac{1}{1-y}$, if $0 < x < 1$
Then the conditional PDF of X given $Y = y$ with $y \in (0,1)$ is $f_{X|Y}(x|y) = \{\frac{1}{1-y}$, if $0 < x < 1$; 0, else.

Problem 2 (b)

Solution
$E[X|Y=y](y \in (0,1)) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \int_0^{1-y} x\frac{1}{1-y}dx = \frac{1-y}{2}$

By Law of Iterated Expectation,
$E[X] = E[E[X|Y=y]] = \int_0^1 f_Y(y)E[X|Y=y]dy = \int_0^1 (2-2y)\frac{(1-y)}{2}dy = \int_0^1 (1-y)^2 dy = 1/3$

Problem 3 (a)

Solution
$X \sim U(-1,3)$
Then the MGF of X is $M_X(t) = E[e^{tX}] = \frac{e^{tb}-e^{ta}}{t(b-a)} = \frac{e^{3t}-e^{-t}}{4t}$

$E[X] = \frac{d}{dt}M_X(t)|_{t=0} = \frac{d}{dt}\frac{e^{3t}-e^{-t}}{4t}|_{t=0} = \lim_{t\to 0}\frac{3te^{4t}+t+1-e^{4t}}{4t^2e^t} = \lim_{t\to 0}\frac{12te^{4t}-e^{4t}+1}{8te^t+4t^2e^t} = \lim_{t\to 0}\frac{48te^{4t}+8e^{4t}}{8e^t+16te^t+4t^2e^t} = \lim_{t\to 0}\frac{2e^{3t}+12te3t}{2+4t+t^2} = \frac{2+0}{2+0+0} = 1$
$E[X^2] = \frac{d^2}{dt^2}M_X(t)|_{t=0} = \frac{d}{dt}\frac{3te^{4t}+t+1-e^{4t}}{4t^2e^t}|_{t=0} = \lim_{t\to 0}\frac{9t^2e^{4t}-6te^{4t}-2t-t^2-2+2e^{4t}}{4t^3e^t} = \lim_{t\to 0}\frac{36t^2e^{4t}-6te^{4t}-2t-2+2e^{4t}}{4t^3e^t+12t^2e^t} = \lim_{t\to 0}\frac{72t^2e^{4t}+24te^{4t}-1+e^{4t}}{2t^3e^t+12t^2e^t+12te^t} = \lim_{t\to 0}\frac{288t^2e^{4t}+240te^{4t}+28e^{4t}}{2t^3e^t+18t^2e^t+36te^t+12e^t} = \lim_{t\to 0}\frac{14e^{3t}+120te^{3t}+144t^2e3t}{6+18t+9t^2+t^3} = \frac{14+0+0}{6++0+0+0} = 7/3$

$Var[X] = E[X^2] - E[X]^2 = 7/3 - 1 = 4/3$

Problem 3 (b)

Solution
$M_Y(t) = E[e^{tY}] = \sum_{k=0}^{\infty} e^{tk}\frac{6}{\pi^2k^2} = \frac{6}{\pi^2}\sum_{k=0}^{\infty}\frac{e^{tk}}{k^2} = \infty$, for $t \in (0,\infty)$ (Since $\lim_{k\to\infty}\frac{e^{tk}}{k^2} = \infty$)

Since $M_Y(t)$ is not finite on $t \in (0,\infty)$, $M_Y(t)$ does not exist, i.e., there exists no interval of the form (δ, δ) (with $\delta > 0$) such that $M_Y(t)$ exists.

Proven that the MGF of Y does not exist, i.e., there exists no interval of the form (δ, δ) (with $\delta > 0$) such that $M_Y(t)$ exists.

Problem 4 (a)

Solution
By the MGF table,
$X \sim B(n,p) \to M_X(t) = (1-p+pe^t)^n$

$M_X(t) = (1/3e^t + 2/3)^5$
$\to n = 5, p = 1/3$
$P(X=k) = \{C_k^5(1/3)^k(2/3)^{5-k}$, if $k = 0,1,2,3,4,5$; 0, else.

X is a Binomial Random Variables.

Problem 4 (b)

Solution
By the MGF table,
$X \sim Pois(\lambda) \to M_X(t) = e^{\lambda(e^t-1)}$

$M_X(t) = e^{5(e^t-1)}$
$\to \lambda = 5$

$P(X=n)=\{\frac{e^{-5}(5)^n}{n!}$, if $n=0,1,2\ldots$; 0, else.

X is a Poisson Random Variables.

Problem 5

Solution
Define A=$\begin{bmatrix}\sigma_1 & 0\\ \sigma_2 & \sigma_2\sqrt{1-p^2}\end{bmatrix}$.
$\begin{bmatrix}X_1\\X_2\end{bmatrix}$=A $\begin{bmatrix}Z\\W\end{bmatrix}$.
$det(A)=\sigma_1\sigma_2\sqrt{1-p^2}\neq 0$

By the theorem of linear transformation of two random variables,
$f_{X_1,X_2}(x_1,x_2)=\frac{1}{|det(A)|}f_{Z,W}(A^{-1}[x_1,x_2]^T)$

Sine Z and W are two independent standard normal random variables,
$f_{Z,W}(A^{-1}[x_1,x_2]^T)=f_{Z,W}(z,w)=f_Z(z)f_W(w)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\frac{1}{\sqrt{2\pi}}e^{-w^2/2}$
$f_{X_1,X_2}(x_1,x_2)=\frac{1}{\sigma_1\sigma_2\sqrt{1-p^2}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\frac{1}{\sqrt{2\pi}}e^{-w^2/2}$
$Z=\frac{X_1-u_1}{\sigma_1},W=\frac{\frac{X_2-u_2}{\sigma_2}-\rho\frac{X_1-u_1}{\sigma_1}}{\sqrt{1-p^2}}$

$f_{X_1,X_2}(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}}e^{-1/2(z^2+w^2)}=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}}e^{-\frac{(\frac{(x_1-u_1)^2}{\sigma_1^2}-2\rho\frac{(x_1-u_1)(x_2-u_2)}{\sigma_1\sigma_2}+\frac{(x_2-u_2)^2}{\sigma_2^2})}{2(1-\rho^2)}}$

Proven that the joint PDF of $X1$, $X2$ is bivariate normal, i.e., for all $x1,x2\in\mathbb{R}$

Problem 6

Solution
Let $a=\frac{|X|}{E[|X|^p]^{1/p}}$, $b=\frac{|Y|}{E[|Y|^q]^{1/q}}$
By Young's inequality,
$\frac{|X|}{E[|X|^p]^{1/p}}\frac{|Y|}{E[|Y|^q]^{1/q}}\leq 1/p\frac{|X|^p}{E[|X|^p]}+1/q\frac{|Y|^q}{E[|Y|^q]}$

Take expected value,
$\frac{E[|XY|]}{E[|X|^p]^{1/p}E[|Y|^q]^{1/q}}\leq 1/p\frac{E[|X|^p]}{E[|X|^p]}+1/q\frac{E[|Y|^q]}{E[|Y|^q]}=1/p+1/q=1$
$\rightarrow\frac{E[|XY|]}{E[|X|^p]^{1/p}E[|Y|^q]^{1/q}}\leq 1$
$\rightarrow E[|XY|]\leq E[|X|^p]^{1/p}E[|Y|^q]^{1/q}$

Proven $E[|XY|]\leq E[|X|^p]^{1/p}E[|Y|^q]^{1/q}$.