Problem 1

a) For ease of notation, let $S := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$

$T := \{ x \mid x \in S_n, \text{ for infinitely many } n \}$.

To show that $S = T$, we need to show $S \subseteq T$, $T \subseteq S$

- $T \subseteq S$

Let x be an element in T. Then for any positive integer k, there exist a positive integer $M_k \geq k$ s.t. $x \in S_{M_k}$. This implies that $x \in \bigcup_{n=k}^{\infty} S_n$ for all k. Hence, $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$

- $S \subseteq T$

Prove this by contradiction. Note that $T^c = \{ x \mid x \in S_n, \text{ for only finitely many } n \}$

Let y be an element in S.

Suppose $y \in T^c$. Then, y can only be in finitely many S_n.

Therefore, there must exist an integer M such that $y \notin \bigcup_{n=k}^{\infty} S_n$ for all $k \geq M$.

This implies that $y \notin \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$ (which leads to contradiction.

b) Recall that $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \{ x : x \in A_n, \text{ for all but finitely many } n \}$

$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n : \{ x : x \in A_n, \text{ for infinitely many } n \}$

$$\begin{aligned}
\cap_{n=1}^{\infty} A_n &= A_1 \cap A_2 \cap A_3 \cap A_4 \cdots A_M \cdots \\
&= (B-C) \cap (B-C) \cap (C-B) \cap (B-C) \cdots \\
&= (B-C) \cap (C-B) \\
&= \phi
\end{aligned}$$

$$\begin{aligned}
\cup_{n=1}^{\infty} A_n &= A_1 \cup A_2 \cup A_3 \cup A_4 \cdots A_m \cdots \\
&= (B-C) \cup (B-C) \cup (C-B) \cup (B-C) \cdots \\
&= (B-C) \cup (C-B) \\
&= (B \cup C) \cap (B^c \cup C^c)
\end{aligned}$$

By the definition of A_n, we know

- For any x, $x \in \underbrace{(B-C)\cup(C-B)}_{(B\cup C)\cap(B^c\cap C^c)}$ iff x is in infinitely many A_n's.
- For any x, $x \in \underbrace{(B-C)\cap(C-B)}_{\emptyset}$ iff x is in every A_n.

Therefore we can conclude that:

$$\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \emptyset$$

$$\bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_n = (B\cup C)\cap(B^c \cup C^c)$$

Problem 1

c) Prove by contradiction:

Suppose there are countably infinite real numbers in $(0,1)$.

We denote these numbers as $x_1, x_2, x_3 \ldots\ldots$

For each x_i, we express the number in decimal expansion, i.e.

$$x_i = 0.\, a_i^{(1)} a_i^{(2)} a_i^{(3)} a_i^{(4)} \ldots\ldots$$

Now we construct a real number $y = 0.b^{(1)} b^{(2)} b^{(3)} b^{(4)} \ldots$,

where $b^{(k)} = \begin{cases} 1, & \text{if } a^{(k)} = 2 \\ 2, & \text{otherwise} \end{cases} \Rightarrow y \in (0,1)$

Therefore, we know $y \neq x_i$ for all $i \Rightarrow$ Contradiction!

Hence, there are uncountably infinite real numbers in $(0,1)$

Problem 2

a) We first prove that $P(\bigcup_{n=1}^{N} A_n) \leq \sum_{n=1}^{N} P(A_n)$, for all $N \in \mathbb{N}$ by induction

For $N=1$: $P(A_1) \leq P(A_1)$

For $N=2$: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - \underbrace{P(A_1 \cap A_2)}_{\geq 0, \text{ by probability axiom } \#1} \leq \sum_{n=1}^{2} P(A_n)$ — $(*)$

Suppose for $N=k$, the following property holds:

$$P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n) \quad \text{———} \quad (**)$$

For $N=k+1$:
$$\begin{aligned} P(\bigcup_{n=1}^{k+1} A_n) &= P((\bigcup_{n=1}^{k} A_n) \cup A_{k+1}) \\ &\leq P(\bigcup_{n=1}^{k} A_n) + P(A_{k+1}) \quad \cdots\cdots \text{ by } (*) \\ &\leq \sum_{n=1}^{k+1} P(A_n) \quad \cdots\cdots \quad (**) \end{aligned}$$

Therefore, by induction, we know $P(\bigcup_{n=1}^{N} A_n) \leq \sum_{n=1}^{N} P(A_n)$ holds for all $N \in \mathbb{N}$

P.S.: To handle the countably infinite case, i.e. $\underbrace{P(\bigcup_{n=1}^{\infty} A_n)}_{LHS} \leq \underbrace{\sum_{n=1}^{\infty} P(A_n)}_{RHS}$, we need to discuss two scenarios separately:

① If the RHS is unbounded, then the inequality automatically holds

② If the RHS is bounded, then by the Monotone Convergence Theorem, we know $\lim_{N\to\infty} \sum_{n=1}^{N} P(A_n)$ exists

Since $P(\bigcup_{n=1}^{N} A_n) \leq \sum_{n=1}^{N} P(A_n)$ for all $N \in \mathbb{N}$, by the inequality rule of limits, the union bound indeed holds in the countably infinite case.

(If two convergent sequences $\{a_n\}, \{b_n\}$ satisfy $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$)

b) $\Omega = \{1,2,3,4,5\} \Rightarrow P(\{1,2,3,4,5\}) = 1 \quad \cdots\cdots (a)$

We know: $\begin{cases} P(\{1,5\}) = 0.5 & \text{——— (b)} \\ P(\{1,2,4\}) = 0.4 & \text{—— (c)} \\ P(\{3\}) = 0.3 & \text{——— (d)} \end{cases}$

By (a), (c), (d), we know $P(\{5\}) = 0.3$

Moreover, by (b), we know $P(\{1\}) = 0.2$

Then, $P(\{2,4\}) = 0.2$

Therefore, the possible valid probability assignments are those that satisfy:

$$\begin{cases} P(\{1\}) = 0.2 \\ P(\{3\}) = 0.3 \\ P(\{5\}) = 0.3 \\ P(\{2,4\}) = 0.2, \ P(\{2\}) \geq 0, \ P(\{4\}) \geq 0 \end{cases}$$

Moreover $P(\{2,3,5\}) = P(\{2\}) + P(\{3\}) + P(\{5\}) \geq \underline{0} + 0.3 + 0.3 = 0.6$

by the fact that

$0.2 \geq P(\{2\}) \geq 0$

Hence, the minimum possible value of $P(\{2,3,5\})$ is 0.6.

Problem 3

a) For ease of notation, let $B_k := \bigcup_{n=k}^{\infty} A_n$. $\{B_k\}$ is a decreasing sequence of event

Then, we have

$$0 \leq P(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n) = P(\bigcap_{k=1}^{\infty} B_k) \cdots \text{by the definition of } B_k$$

↑ by Probability Axiom 1

$$= P(\lim_{k\to\infty} B_k) \cdots \text{Since } B_k \text{ is a decreasing sequence}$$

$$= \lim_{k\to\infty} P(B_k) \cdots \text{by the continuity of probability}$$

$$= \lim_{k\to\infty} P(\bigcup_{n=k}^{\infty} A_n) \cdots \text{by the definition of } B_k$$

$$\leq \lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n) \cdots \text{by the union bound shown in Problem 2 (a)}$$

$$= 0 \cdots \text{by the condition that } \sum_{n=1}^{\infty} P(A_n) < \infty$$

Remark:

$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow$ The sum $\sum_{n=1}^{\infty} P(A_n)$ converge as $N \to \infty$

$\Rightarrow$ The residual $\sum_{n=k}^{\infty} P(A_n)$ converge to 0 as $k \to \infty$

b) Consider a sample space Ω when each outcome w is the experimental result of a countably infinite sequence of coin tosses.

For example:

$w = HTTH \cdots$

$w' = THHT \cdots$

$w'' = HTTT \cdots$

Define the following events: For each $n \in \mathbb{N}$, define

$$A_i = \{ w \in \Omega : \text{the } i\text{-th toss of } w \text{ is a head} \}$$

$$I = \{ w \in \Omega : \text{there are infinite number of heads in } w \}$$

Then, we know $I = \bigcap_{k=1}^{\infty} \bigcup_{i=k} A_i$

Moreover, we already know $P(A_i) = p_i = 100 \cdot i^{-N}$, for any $i \in \mathbb{N}$

$\sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} 100 \cdot \frac{1}{i^N}$, $\quad \sum_{i=1}^{\infty} 100 \frac{1}{i^N} < \infty$ if $N > 1$ $\cdots\cdots (*)$

By the Borel - Cantelli Lemma in Problem 3 (a), since $\sum_{i=1}^{\infty} P(A_i) < \infty$

then $P(\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{\infty} A_i) = 0$

Hence, we have $P(I) = 0$ if $N > 1$, by $(*)$

Problem 4

a) By Bayes' rule

$$P(A_1 \mid B) = \frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)}$$

$$= \frac{\frac{1}{3}\cdot 0.1}{\frac{1}{3}\cdot 0.1 + \frac{1}{3}\cdot 0.3 + \frac{1}{3}\cdot 0.6}$$

$$= \underline{\frac{1}{10}}$$

Similarly, $P(A_2|B) = \dfrac{\frac{1}{3}\cdot 0.3}{\frac{1}{3}\cdot 0.1 + \frac{1}{3}\cdot 0.3 + \frac{1}{3}\cdot 0.6}$

$$= \underline{\frac{3}{10}}$$

$$P(A_3 \mid B) = \frac{\frac{1}{3}\cdot 0.6}{\frac{1}{3}\cdot 0.1 + \frac{1}{3}\cdot 0.3 + \frac{1}{3}\cdot 0.6}$$

$$= \underline{\frac{3}{5}}$$

b) $$P(A_1 \mid C) = \frac{P(A_1)\cdot P(C \mid A_1)}{P(A_1)\cdot P(C \mid A_1) + P(A_2)\cdot P(C \mid A_2) + P(A_3)\cdot P(C \mid A_3)}$$

4.374 e^{-9}
7.558 e^{-6}
9.447 e^{-7}

$$= \frac{\frac{1}{3}\cdot(0.1)^4\cdot(0.3)^7\cdot(0.6)^1}{\frac{1}{3}\cdot(0.1)^4\cdot(0.3)^7\cdot(0.6) + \frac{1}{3}\cdot(0.3)^4\cdot(0.6)^7\cdot(0.1) + \frac{1}{3}\cdot(0.6)^4(0.3)^7\cdot(0.1)}$$

$$\cong \underline{0.000514}$$

Similarly, $P(A_2 \mid C) \cong \underline{0.8884}$

$P(A_3 \mid C) \cong \underline{0.111}$

Therefore, the most probable value for θ is $(\theta_Y, \theta_L, \theta_N)$

$= \underline{(0.3, 0.6, 0.1)}$

c) Similar to (b)

$$P(A_1 \mid C) = \frac{P(A_1)\cdot P(C \mid A_1)}{P(A_1)\cdot P(C \mid A_1) + P(A_2)\cdot P(C \mid A_2) + P(A_3)\cdot P(C \mid A_3)}$$

7.87 e^{-9}
4.53 e^{-6}
5.66 e^{-7}

$$= \frac{\frac{3}{5}\cdot(0.1)^4\cdot(0.3)^7\cdot 0.6}{\frac{3}{5}\cdot(0.4)^4\cdot(0.3)^7\cdot 0.6 + \frac{1}{5}\cdot(0.3)^4(0.6)^7\cdot(0.1) + \frac{1}{5}\cdot(0.6)^4\cdot(0.3)^7\cdot 0.1}$$

$$\simeq \underline{0.00154}$$

Similarly, $P(A_2 \mid C) \cong \underline{0.887}$

$P(A_3 \mid C) \cong \underline{0.110}$

Therefore, the most probable value for θ is $(\theta_Y, \theta_L, \theta_N)$

$= \underline{(0.3, 0.6, 0.1)}$