

Name: 陳品劭 ID: 109550206 [Self link](#)**Problem 1 (a)**Let S_1, S_2, \dots be an infinite sequence of sets. Prove that

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = \{x | x \in S_n, \text{ for infinitely many } n\}.$$

Solution

Proof.

Let $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = A$ and $\{x | x \in S_n, \text{ for infinitely many } n\} = B$.1. $A \subseteq B$ Suppose $x \in A$.→ $x \in \bigcup_{n=k}^{\infty} S_n$, for any positive integer k .→ We can find infinitely integer n^* s.t. $n^* \geq k$ and $x \in S_{n^*}$ for any positive integer k .→ $x \in S_n$, for infinitely many n .→ $x \in B$.→ $A \subseteq B$.2. $B \subseteq A$ Suppose $y \in B$.→ Give any positive integer k , there exists some positive integer n^* s.t. $n^* \geq k$ and $y \in S_{n^*}$.→ $y \in \bigcup_{n=k}^{\infty} S_n$, for all positive integer k .→ $y \in A$.→ $B \subseteq A$.Since 1. and 2. we have $A = B$.Proven $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = \{x | x \in S_n, \text{ for infinitely many } n\}$.**Problem 1 (b)**Let Ω be the universal set and B, C be two sets that satisfy $B \subseteq \Omega$ and $C \subseteq \Omega$. Let $\{F_k\}_{k=1}^{\infty}$ denote the Fibonacci sequence, i.e., $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$, for $k \geq 2$. Define a countably infinite sequence of sets A_1, A_2, A_3, \dots as

$$A_n = \{B - C, \text{ if } n \text{ is in the Fibonacci sequence } \{F_k\}, C - B, \text{ otherwise.}\}$$

What are $\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} A_n$, $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$, and $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$? Please clearly explain your answer.**Solution**We know A_n can only be $B - C$ or $C - B$ by definition. Since 1 is in $\{F_k\}$ and 4 is not in $\{F_k\}$, then we have $A_1 = B - C, A_4 = C - B$.

→

$$1. \bigcap_{n=1}^{\infty} A_n = (B - C) \cap (C - B) = \phi.$$

$$2. \bigcup_{n=1}^{\infty} A_n = (B - C) \cup (C - B).$$

We know $F_k \rightarrow \infty$ as $k \rightarrow \infty$.→ Give any positive integer k , there exists an integer $n^* > k$ and in the Fibonacci sequence. And $n^* + 1$ will be not in the Fibonacci sequence.→ We have $A_{n^*} = B - C$ and $A_{n^*+1} = C - B$, and $n^* > k$ for any positive integer k .

→

3. $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} (\phi) = \phi.$
 4. $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} ((B - C) \cup (C - B)) = (B - C) \cup (C - B).$

Problem 1 (c)

Show that there are uncountably infinite many real numbers in the interval $(0, 1)$.

Solution

Assume that there are countably infinite real numbers in $(0, 1)$.

Define $x_1, x_2, x_3, \dots, x_i, \dots$ are the infinite real numbers in $(0, 1)$, and each real number x_i between 0 and 1 in decimal expansion.

And give a sequence as the following:

$$x_1 = 0.127368\dots$$

$$x_2 = 0.212562\dots$$

$$x_3 = 0.137611\dots$$

...

Then we let a number y whose i th decimal place is $x = \{ 1 \text{ if } x_i \text{'s } i\text{th decimal place} \neq 1, 2 \text{ if } x_i \text{'s } i\text{th decimal place} = 1.$

However for this case we will get $y = 0.221\dots$ which is $\neq x_i$, for any i .

That is contradiction.

Proven there are uncountably infinite many real numbers in the interval $(0, 1)$.

Problem 2 (a)

Let A_1, A_2, \dots, A_N be a sequence of events of an experiment. Prove that the following inequality holds for any $N \in \mathbb{N}$:

$$P\left(\bigcup_{n=1}^N A_n\right) \leq \sum_{n=1}^N P(A_n).$$

Solution

When $N = 1$, $P\left(\bigcup_{n=1}^1 A_n\right) = P(A_1) = \sum_{n=1}^1 P(A_n).$

$\rightarrow P\left(\bigcup_{n=1}^N A_n\right) \leq \sum_{n=1}^N P(A_n)$ is true when $N = 1$.

Suppose $P\left(\bigcup_{n=1}^N A_n\right) \leq \sum_{n=1}^N P(A_n)$ is true when $N = k$.

When $N = k + 1$,

$$\begin{aligned} P\left(\bigcup_{n=1}^{k+1} A_n\right) &= P\left(\bigcup_{n=1}^k A_n \cup A_{k+1}\right) \\ &= P\left(\bigcup_{n=1}^k A_n\right) + P(A_{k+1}) - P\left(\left(\bigcup_{n=1}^k A_n\right) \cap A_{k+1}\right) \\ &\leq P\left(\bigcup_{n=1}^k A_n\right) + P(A_{k+1}) \\ &\leq \sum_{n=1}^k P(A_n) + P(A_{k+1}) \\ &= \sum_{n=1}^{k+1} P(A_n) \end{aligned}$$

$\rightarrow P\left(\bigcup_{n=1}^{k+1} A_n\right) \leq \sum_{n=1}^{k+1} P(A_n)$ if $P\left(\bigcup_{n=1}^k A_n\right) \leq \sum_{n=1}^k P(A_n)$ is true, for any integer $k > 1$.

Proven $P\left(\bigcup_{n=1}^N A_n\right) \leq \sum_{n=1}^N P(A_n)$ for any $N \in \mathbb{N}$ by induction.

Problem 2 (b)

Consider an experiment with a sample space $\Omega = \{1, 2, 3, 4, 5\}$. Suppose we know $P(\{1, 5\}) = 0.5$, $P(\{1, 2, 4\}) = 0.4$, and $P(\{3\}) = 0.3$. Please write down all possible valid probability assignments. Moreover, among all the possible valid probability assignments, what is the minimum possible value of $P(\{2, 3, 5\})$? Please explain your answer.

Solution

By axioms 1, 3, we have following:

$$P(\{5\}) = P(\Omega) - P(\{1, 2, 3, 4\}) = 1 - (P(\{1, 2, 4\}) + P(\{3\})) = 1 - (0.4 + 0.3) = 0.3$$

$$P(\{1\}) = P(\{1, 5\}) - P(\{5\}) = 0.5 - 0.3 = 0.2$$

$$P(\{2\}) + P(\{4\}) = P(\{1, 2, 4\}) - P(\{1\}) = 0.4 - 0.2 = 0.2$$

Then the all possible valid probability assignments are the assignments which is satisfied $P(\{1\}) = 0.2$, $P(\{3\}) = 0.3$, $P(\{5\}) = 0.3$, and $P(\{2\}) + P(\{4\}) = 0.2$.

The minimum possible value of $P(\{2, 3, 5\})$ is 0.6 if $P(\{2\}) = 0$ and $P(\{4\}) = 0.2$.

Problem 3 (a)

Let A_1, A_2, A_3, \dots be a countably infinite sequence of events. Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$.

Solution

By Boole's inequality, we know $P(\bigcup_{n=k}^{\infty} A_n) \leq \sum_{n=k}^{\infty} P(A_n)$.

The assumption, $\sum_{n=1}^{\infty} P(A_n) < \infty$, means the series $\sum_{n=1}^{\infty} P(A_n)$ converges,

$$\lim_{k \rightarrow \infty} \sum_{n=k}^{\infty} P(A_n) = 0.$$

$$\rightarrow \lim_{k \rightarrow \infty} P(\bigcup_{n=k}^{\infty} A_n) = 0$$

$$\rightarrow P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$$

Proven if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$.

Problem 3 (b)

Consider a countably infinite sequence of coin tosses. The probability of having a head at the k -th toss is p_k , with $p_k = 100 \cdot k^{-N}$. We use I to denote the event of observing an infinite number of heads. Show that $P(I) = 0$ if $N > 1$. Please clearly explain your answer.

Solution

$$\lim_{k \rightarrow \infty} p_k = \lim_{k \rightarrow \infty} 100 \cdot k^{-N} = 0, \text{ when } N > 1.$$

$$\rightarrow \sum_{n=1}^{\infty} p_k \text{ converges.}$$

$$\rightarrow \sum_{n=1}^{\infty} p_k < \infty \dots (1)$$

By the definition, $I = \limsup_{k \rightarrow \infty} p_k$.

By Borel-Cantelli Lemma (result in (a)), $\limsup_{k \rightarrow \infty} p_k = 0$, when $N > 1$ since (1).

Proven $P(I) = 0$ if $N > 1$.

Problem 4

Suppose we are given a special pair of moon blocks with unknown characteristics. Let $\theta_Y, \theta_L, \theta_N$ denote the unknown probabilities of getting a "Yes" (Y), "Laughing" (L), and "No" (N) at each toss, respectively. Moreover, suppose that the tuple of the unknown parameters $(\theta_Y, \theta_L, \theta_N)$ can only be one of the following three possibilities: $(\theta_Y, \theta_L, \theta_N) \in \{(0.1, 0.3, 0.6), (0.3, 0.6, 0.1), (0.6, 0.3, 0.1)\}$. In order to infer the values $(\theta_Y, \theta_L, \theta_N)$, we experiment with the moon blocks and consider Bayesian inference as follows: Define events $A_1 = \{\theta_Y = 0.1, \theta_L = 0.3, \theta_N = 0.6\}$, $A_2 = \{\theta_Y = 0.3, \theta_L = 0.6, \theta_N = 0.1\}$, $A_3 =$

$\{\theta_Y = 0.6, \theta_L = 0.3, \theta_N = 0.1\}$. Since initially we have no further information about $(\theta_Y, \theta_L, \theta_N)$, we simply consider the prior probability assignment to be $P(A_1) = P(A_2) = P(A_3) = 1/3$.

Question (a)

Suppose we toss the pair of moon blocks once and observe a "Y" (for ease of notation, we define the event $B = \{\text{the first toss is a Y}\}$). What is the posterior probability $P(A_1|B)$? How about $P(A_2|B)$ and $P(A_3|B)$?

Solution

$$\begin{aligned} P(A_1|B) &= \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)} = \frac{1/3 \times 0.1}{1/3 \times 0.1 + 1/3 \times 0.3 + 1/3 \times 0.6} = \frac{0.1}{0.1+0.3+0.6} = 0.1 \\ P(A_2|B) &= \frac{P(A_2)P(B|A_2)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)} = \frac{1/3 \times 0.3}{1/3 \times 0.1 + 1/3 \times 0.3 + 1/3 \times 0.6} = \frac{0.3}{0.1+0.3+0.6} = 0.3 \\ P(A_3|B) &= \frac{P(A_3)P(B|A_3)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+P(A_3)P(B|A_3)} = \frac{1/3 \times 0.6}{1/3 \times 0.1 + 1/3 \times 0.3 + 1/3 \times 0.6} = \frac{0.6}{0.1+0.3+0.6} = 0.6 \end{aligned}$$

Question (b)

Suppose we toss the pair of moon blocks for 12 times and observe $YLYNLYLLYLLL$ (for ease of notation, we define the event $C = \{YLYNLYLLYLLL\}$). Moreover, all the tosses are assumed to be independent. What is the posterior probability $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ ?

Solution

$$P(C) = \theta_Y \theta_L \theta_Y \theta_N \theta_L \theta_Y \theta_L \theta_L \theta_Y \theta_L \theta_L \theta_L$$

$$P(C|A_1) = \frac{P(C) \cap P(A_1)}{P(A_1)} = \frac{(0.1)^4 (0.3)^7 (0.6) P(A_1)}{P(A_1)}$$

$$P(C|A_2) = \frac{P(C) \cap P(A_2)}{P(A_2)} = \frac{(0.3)^4 (0.6)^7 (0.1) P(A_2)}{P(A_2)}$$

$$P(C|A_3) = \frac{P(C) \cap P(A_3)}{P(A_3)} = \frac{(0.6)^4 (0.3)^7 (0.1) P(A_3)}{P(A_3)}$$

$$\begin{aligned} P(A_1|C) &= \frac{P(A_1)P(C|A_1)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\ &= \frac{(0.1)^4 (0.3)^7 (0.6) P(A_1)}{(0.1)^4 (0.3)^7 (0.6) P(A_1) + (0.3)^4 (0.6)^7 (0.1) P(A_2) + (0.6)^4 (0.3)^7 (0.1) P(A_3)} \\ &= \frac{6}{6+3^4 \times 2^7 + 6^4} = \frac{1}{1+3^3 \times 2^6 + 6^3} = \frac{1}{6+1728+216} = \frac{1}{1945} \approx 0.00051 \dots \end{aligned}$$

$$\begin{aligned} P(A_2|C) &= \frac{P(A_2)P(C|A_2)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\ &= \frac{(0.3)^4 (0.6)^7 (0.1) P(A_2)}{(0.1)^4 (0.3)^7 (0.6) P(A_1) + (0.3)^4 (0.6)^7 (0.1) P(A_2) + (0.6)^4 (0.3)^7 (0.1) P(A_3)} \\ &= \frac{3^4 \times 2^7}{6+3^4 \times 2^7 + 6^4} = \frac{3^3 \times 2^6}{1+3^3 \times 2^6 + 6^3} = \frac{1728}{6+1728+216} = \frac{1728}{1945} \approx 0.88843 \dots \end{aligned}$$

$$\begin{aligned} P(A_3|C) &= \frac{P(A_3)P(C|A_3)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\ &= \frac{(0.6)^4 (0.3)^7 (0.1) P(A_3)}{(0.1)^4 (0.3)^7 (0.6) P(A_1) + (0.3)^4 (0.6)^7 (0.1) P(A_2) + (0.6)^4 (0.3)^7 (0.1) P(A_3)} \\ &= \frac{6^4}{6+3^4 \times 2^7 + 6^4} = \frac{6^3}{1+3^3 \times 2^6 + 6^3} = \frac{216}{6+1728+216} = \frac{216}{1945} \approx 0.11105 \dots \end{aligned}$$

The most probable value for θ is $A_2 = \{\theta_Y = 0.3, \theta_L = 0.6, \theta_N = 0.1\}$.

Question (c)

Given the same setting as (b), suppose we instead choose to use a different prior probability assignment $P(A_1) = 3/5$, $P(A_2) = 1/5$, $P(A_3) = 1/5$. What is the posterior probabilities $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ ?

Solution

$$P(C) = \theta_Y \theta_L \theta_Y \theta_N \theta_L \theta_Y \theta_L \theta_L \theta_Y \theta_L \theta_L \theta_L$$

$$P(C|A_1) = \frac{P(C) \cap P(A_1)}{P(A_1)} = \frac{(0.1)^4 (0.3)^7 (0.6) P(A_1)}{P(A_1)}$$

$$P(C|A_2) = \frac{P(C) \cap P(A_2)}{P(A_2)} = \frac{(0.3)^4 (0.6)^7 (0.1) P(A_2)}{P(A_2)}$$

$$\begin{aligned}
P(C|A_3) &= \frac{P(C) \cap P(A_3)}{P(A_3)} = \frac{(0.6)^4(0.3)^7(0.1)P(A_3)}{P(A_3)} \\
P(A_1|C) &= \frac{P(A_1)P(C|A_1)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \frac{(0.1)^4(0.3)^7(0.6)P(A_1)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \frac{6 \times 3}{6 \times 3 + 3^4 \times 2^7 + 6^4} = \frac{1}{1+3^2 \times 2^6 + 2^3 \times 3^2} = \frac{1}{1+576+72} = \frac{1}{649} \approx 0.00154 \dots \\
P(A_2|C) &= \frac{P(A_2)P(C|A_2)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \frac{(0.3)^4(0.6)^7(0.1)P(A_2)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \frac{3^4 \times 2^7}{6 \times 3 + 3^4 \times 2^7 + 6^4} = \frac{3^2 \times 2^6}{1+3^2 \times 2^6 + 2^3 \times 3^2} = \frac{576}{1+576+72} = \frac{576}{649} \approx 0.88751 \dots \\
P(A_3|C) &= \frac{P(A_3)P(C|A_3)}{P(A_1)P(C|A_1)+P(A_2)P(C|A_2)+P(A_3)P(C|A_3)} \\
&= \frac{(0.6)^4(0.3)^7(0.1)P(A_3)}{(0.1)^4(0.3)^7(0.6)P(A_1)+(0.3)^4(0.6)^7(0.1)P(A_2)+(0.6)^4(0.3)^7(0.1)P(A_3)} \\
&= \frac{6^4}{6 \times 3 + 3^4 \times 2^7 + 6^4} = \frac{2^3 \times 3^2}{1+3^2 \times 2^6 + 2^3 \times 3^2} = \frac{72}{1+576+72} = \frac{72}{649} \approx 0.11093 \dots
\end{aligned}$$

The most probable value for θ is $A_2 = \{\theta_Y = 0.3, \theta_L = 0.6, \theta_N = 0.1\}$.