Problem 1

$X_1, X_2, \cdots, X_N$ are non-negative independent random variables.

Moreover, it is assumed that the PDFs of X_i's are uniformly bounded by 1.

(a).
$$E[e^{-tX_i}] = \int_0^{+\infty} f_{X_i}(x) \cdot \underset{\substack{\uparrow \\ \text{positive}}}{e^{-tx}}\, dx \le \int_0^{+\infty} 1 \cdot e^{-tx}\, dx = -\frac{1}{t} e^{-tx} \Big|_0^{\infty}$$

$$= \frac{1}{t}, \text{ for all } t > 0.$$

(Note that $E[e^{-tX_i}]$ does not exist for any $t \le 0$)

(b).

For any $t > 0$, we have

$$P\left(\sum_{i=1}^{N} X_i \le \varepsilon N\right) \overset{\text{Chernoff technique}}{=} P\left(e^{t\sum_{i=1}^{N} X_i} \le e^{t\varepsilon N}\right)$$

$$= P\left(e^{-t\sum_{i=1}^{N} X_i} \ge e^{-t\varepsilon N}\right)$$

$$\overset{\text{Markov inequality}}{\le} \frac{E[e^{-t\sum_{i=1}^{N} X_i}]}{e^{-t\varepsilon N}}$$

$$\overset{\text{independence of } X_1, X_2, \cdots, X_N}{=} \frac{\prod_{i=1}^{N} E[e^{-tX_i}]}{e^{-t\varepsilon N}}$$

$$\le \left(\frac{1}{t} e^{t\varepsilon}\right)^N \quad \text{———} \quad (*)$$

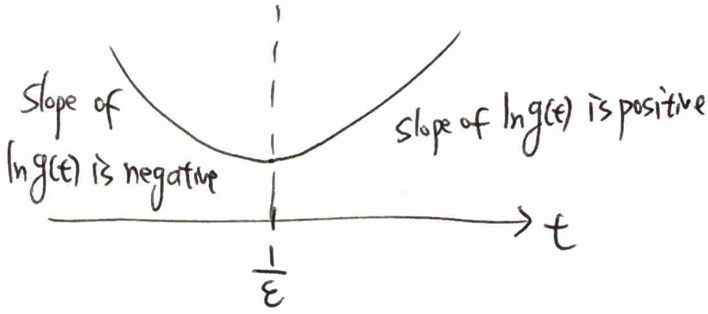
(Cont.).

Finally, we shall minimize (*) over $t>0$:

Define $g(t) = \left(\frac{1}{t} e^{t\varepsilon}\right)^N$.

Then, $\ln g(t) = N\cdot\left(-\ln t + t\varepsilon\right)$.

$\frac{d(\ln g(t))}{dt} = N\cdot\left(-\frac{1}{t}+\varepsilon\right) \Rightarrow$



Therefore, we know the minimizer of $\ln g(t)$ and $g(t)$ is $t=\frac{1}{\varepsilon}$,

Hence, we conclude that

$$P\left(\sum_{i=1}^{N} X_i \le \varepsilon N\right) \le \left(\frac{1}{\frac{1}{\varepsilon}} e^{\frac{1}{\varepsilon}\cdot\varepsilon}\right)^N = (e\varepsilon)^N$$

Problem 2 Define $A=\{\omega: X_n(\omega) \text{ does not converge to } a\}$ and $B=\{\omega: Y_n(\omega) \text{ does not converge to } b\}$.

We know $X_n \xrightarrow{a.s} a$ and $Y_n \xrightarrow{a.s.} b$

Therefore, $P(A)=0$ and $P(B)=0$.

For any $\omega' \in (A \cup B)^c$, $X_n(\omega') \cdot Y_n(\omega')$ must converge to $a \cdot b$.

Then, we have

$$P(\{\omega: X_n(\omega) \cdot Y_n(\omega) \text{ converge to } a \cdot b\})$$

$$\geq P((A \cup B)^c)$$

$$= 1 - P(A \cup B)$$

$$\geq 1 - (P(A) + P(B))$$

$$= 1$$

$\square$

Problem 3

$X_1, X_2, \cdots,$ converges to a number c in the mean square if $\lim_{n\to\infty} E[(X_n - c)^2] = 0$

(a). For any $\varepsilon > 0$:

$$0 \leq P(|X_n - c| \geq \varepsilon) = P(|X_n - c|^2 \geq \varepsilon^2)$$

by Markov's inequality

$$\leq \frac{E[|X_n - c|^2]}{\varepsilon^2}$$

By letting $n \to \infty$, we have

by convergence in the mean square

$$0 \leq \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{E[|X_n - c|^2]}{\varepsilon^2} = 0$$

Hence, $\lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0$ and therefore $X_n \xrightarrow{P} c$.

□.

(b). Define $X_n = \begin{cases} 0, & \text{with probability } 1 - \frac{1}{n} \\ \sqrt{n}, & \text{with probability } \frac{1}{n} \end{cases}$

For any $\varepsilon > 0$:

$$P(|X_n - 0| \geq \varepsilon) \leq \frac{1}{n}, \quad \text{for any } n > \varepsilon^2$$

This implies that $\lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) = 0$, i.e. $X_n \xrightarrow{P} 0$

However, $E[(X_n - 0)^2] = \frac{1}{n} \cdot (\sqrt{n})^2 + (1 - \frac{1}{n}) \cdot 0^2 = 1.$ for all n.

Therefore, X_n does not converge to 0 in the mean square.

□