(Fall 2021) 1179 Probability **Due: 2021/11/01 (Monday), 9pm**

Homework 2: Random Variables and Expected Value

Submission Guidelines: Please combine all your write-ups (photos/scanned copies are acceptable; please make sure that the electronic files are of good quality and reader-friendly) into one single .pdf file and submit the file via E3.

Problem 1 (Special Discrete Random Variables) (10+10=20 points)

(a) Consider a Poisson random variable X with parameters λ and T. Let $p_X(k)$ denote the PMF of X and let k^* be the largest integer that is less than or equal to λT. Show that $k^* = \arg\max_{k\in\mathbb{N}\cup\{0\}} p_X(k)$. (Hint: Show that the PMF of X is monotonically non-decreasing with k in the range from 0 to k^* and is monotonically decreasing with k for $k \geq k^*$.)

(b) Let $X_1, X_2, \cdots, X_n$ be n independent Geometric random variables with the same success probability $p \in (0, 1)$. Define $X = \min(X_1, \cdots, X_n)$. What is the PMF of X? What kind of random variable is X?

Problem 2 (PMFs and Combinatorics) (10+8=18 points)

Consider a random experiment of distributing r *indistinguishable* balls to n different cells (for simplicity, let us index the cells as $1, 2, \cdots, n$). Suppose that all distinguishable arrangements are equally likely to occur (For example, in the case of $r = 2$ and $n = 2$, then the three arrangements $(2, 0)$, $(1, 1)$, and $(0, 2)$ have equal probabilities). This setting is closely related to the Bose-Einstein statistics in quantum physics.

(a) Let X be the number of balls in the 1st cell (and X is clearly a random variable), and let q_k be the probability of the event that $X = k$. Show that for any $k \in \mathbb{N} \cup \{0\}$,

$$q_k = C^{n+r-k-2}_{r-k} / C^{n+r-1}_{r}.$$

Please clearly explain each step of your answer.

(b) Based on the result in (a), show that if the average number of particles per cell r/n tends to λ as $n \to \infty$ and $r \to \infty$, then we have $q_k \to \lambda^k/(1 + \lambda)^{k+1}$ as $n \to \infty$ and $r \to \infty$. What kind of random variable is X in the limit?

Problem 3 (Communication Over a Binary Channel and Poisson) (10+10=20 points)

At each time, the transmitter sends out either a '1' with probability p, or a '0' with probability $1-p$. Given that the transmitted bit is '1', the receiver successfully receives the '1' with probability α_1 or it receives an incorrect message of '0' with probability $1 - \alpha_1$. Similarly, given that the transmitted bit is '0', the receiver either receives a '0' with probability α_0, or a '1' with probability $1 - \alpha_0$, respectively.

(a) Suppose the number of transmissions within a given observation window T has a Poisson PMF with average rate λ. Define a random variable X to be the number of 1's *transmitted* in that time interval. Show that X has a Poisson PMF with average rate λp. (Hint: Define V = total transmitted bits in the given interval. Try to use the total probability theorem $P(X = k) = \sum_{n=0}^{\infty} P(X = k|V = k + n) \cdot P(V = k + n)$)

(b) By using the setting in (a), we suppose that the number of transmissions within a given observation window T has a Poisson PMF with average rate λ. Define a random variable Y to be the total number of 1's *received* in that time interval. What is the PMF of Y?

Problem 4 (Expected Value, Variance and Moments) (8+10+10=28 points)

(a) Suppose $X \sim \text{Geometric}(p)$. Show that $E[X] = 1/p$ and $\text{Var}[X] = (1 - p)/p^2$. (Hint: Write down the PMF and try to reuse the fact that the total probability is 1.)

(b) Let $n \geq 3$ be a positive integer. Let X and Y be two discrete random variables with the identical set of possible values $\{a_1, a_2, \cdots, a_n\}$, where $a_1, a_2, \cdots, a_n$ are n distinct real numbers. Show that if $E[X^m] = E[Y^m]$ for all $m \in \{1, 2, \cdots, n-1\}$, then X and Y are identically distributed, i.e., $P(X = t) = P(Y = t)$, for all $t \in \{a_1, \cdots, a_n\}$.

(c) Let $z_n = (-1)^n \sqrt{n}$, for $n = 1, 2, 3 \cdots$. Let Z be a discrete random variable with the set of possible values $\{z_n : n = 1, 2, 3 \cdots\}$. The PMF of Z is

$$p_Z(z_n) = P(Z = z_n) = \frac{6}{(\pi n)^2}, \ \forall n \in \mathbb{N}.$$

What is $\mathrm{Var}[Z]$? How about the value of $\sum_{n=1}^{\infty} z_n^3 \cdot p_Z(z_n)$? Does $E[Z^3]$ exist? Moreover, does $E[Z^{10}]$ exist? Please carefully justify each step of your answer.

Problem 5 (Continuous Random Variables) (10+10=20 points)

(a) Let X be an exponential random variable with parameter λ. Consider another random variable $Y = aX + b$, where a, b are real numbers and $a \neq 0$. Please write down the CDF and PDF of Y (Note: we assume that the PDF of Y is continuous everywhere, except at b). Under what condition is Y also an exponential random variable? (Hint: $a > 0$ and $a < 0$ may lead to different characteristics of CDF and PDF)

(b) Verify that a standard normal random variable X satisfies that $\mathrm{Var}[X] = 1$. (Hint: Use integration by parts)