(Fall 2021) 1179 Probability **(Due: 2021/10/07, 9pm)**

Homework 1: Probability Axioms, Set Operations, and Conditioning

Submission Guidelines: Please compress all your write-ups (photos/scanned copies are acceptable; please make sure that the electronic files are of good quality and reader-friendly) into one .zip file and submit the compressed file via E3.

Problem 1 (Set Operations) (10+10+10=30 points)

(a) Let $S_1, S_2, \cdots$ be an infinite sequence of sets. Prove that

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = \{x | x \in S_n, \text{for infinitely many } n\}.$$

(Hint: To prove that $S = T$, we need to show $S \subseteq T$ and $T \subseteq S$)

(b) Let Ω be the universal set and B, C be two sets that satisfy $B \subseteq \Omega$ and $C \subseteq \Omega$. Let $\{F_k\}_{k=1}^{\infty}$ denote the Fibonacci sequence, i.e., $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$, for $k \geq 2$. Define a countably infinite sequence of sets $A_1, A_2, A_3, \cdots$ as

$$A_n = \begin{cases} B - C, & \text{if } n \text{ is in the Fibonacci sequence } \{F_k\}, \\ C - B, & \text{otherwise.} \end{cases}$$

What are $\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} A_n$, $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$, and $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$? Please clearly explain your answer.

(c) Show that there are *uncountably* infinite many real numbers in the interval $(0, 1)$. (Hint: Prove this by contradiction. Specifically, (i) assume that there are countably infinite real numbers in $(0, 1)$ and denote them as $x_1, x_2, x_3, \cdots, x_i, \cdots$; (ii) express each real number x_i between 0 and 1 in decimal expansion; (iii) construct a number y whose digits are either 1 or 2. Can you find a way to choose 1 or 2 such that y is different from all the x_is?)

Problem 2 (Probability Axioms) (10+12=22 points)

(a) Let $A_1, A_2, \cdots, A_N$ be a sequence of events of an experiment. Prove that the following inequality holds for any $N \in \mathbb{N}$:

$$P\Big(\bigcup_{n=1}^{N} A_n\Big) \leq \sum_{n=1}^{N} P(A_n).$$

This property is called *Boole's inequality* or the *union bound*. (Hint: Prove this by induction) (P.S.: By the way, if you are interested, you could also try figuring out whether the same property holds under the countable union of A_ns?)

(b) Consider an experiment with a sample space $\Omega = \{1, 2, 3, 4, 5\}$. Suppose we know $P(\{1, 5\}) = 0.5$, $P(\{1, 2, 4\}) = 0.4$, and $P(\{3\}) = 0.3$. Please write down all possible valid probability assignments. Moreover, among all the possible valid probability assignments, what is the minimum possible value of $P(\{2, 3, 5\})$? Please explain your answer.

Problem 3 (Continuity of Probability Functions) (12+12=24 points)

(a) Let $A_1, A_2, A_3, \cdots$ be a countably infinite sequence of events. Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$. This property is known as the *Borel-Cantelli Lemma*. (Hint: Consider the continuity of probability function for $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$ and then apply the union bound)

(b) Consider a countably infinite sequence of coin tosses. The probability of having a head at the k-th toss is p_k, with $p_k = 100 \cdot k^{-N}$ (Note: different tosses are NOT necessarily independent). We use I to denote the event of observing an infinite number of heads. Show that $P(I) = 0$ if $N > 1$. Please clearly explain your answer. (Hint: Leverage the result in (a))

Problem 4 (Inference via Bayes' Rule) (10+10+10=30 points)

Suppose we are given a special pair of moon blocks with unknown characteristics. Let $\theta_Y, \theta_L, \theta_N$ denote the unknown probabilities of getting a "Yes" (Y), "Laughing" (L), and "No" (N) at each toss, respectively. Moreover, suppose that the tuple of the unknown parameters $(\theta_Y, \theta_L, \theta_N)$ can only be one of the following three possibilities: $(\theta_Y, \theta_L, \theta_N) \in \{(0.1, 0.3, 0.6), (0.3, 0.6, 0.1), (0.6, 0.3, 0.1)\}$. In order to infer the values $(\theta_Y, \theta_L, \theta_N)$, we experiment with the moon blocks and consider Bayesian inference as follows: Define events $A_1 = \{\theta_Y = 0.1, \theta_L = 0.3, \theta_N = 0.6\}$, $A_2 = \{\theta_Y = 0.3, \theta_L = 0.6, \theta_N = 0.1\}$, $A_3 = \{\theta_Y = 0.6, \theta_L = 0.3, \theta_N = 0.1\}$. Since initially we have no further information about $(\theta_Y, \theta_L, \theta_N)$, we simply consider the prior probability assignment to be $P(A_1) = P(A_2) = P(A_3) = 1/3$.

(a) Suppose we toss the pair of moon blocks once and observe a "Y" (for ease of notation, we define the event $B = \{\text{the first toss is a Y}\}$). What is the posterior probability $P(A_1|B)$? How about $P(A_2|B)$ and $P(A_3|B)$? (Hint: use the Bayes' rule)

(b) Suppose we toss the pair of moon blocks for 12 times and observe YLYNLYLLYLLL (for ease of notation, we define the event $C = \{\text{YLYNLYLLYLLL}\}$). Moreover, all the tosses are assumed to be independent. What is the posterior probability $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ?

(c) Given the same setting as (b), suppose we instead choose to use a different prior probability assignment $P(A_1) = 3/5, P(A_2) = 1/5, P(A_3) = 1/5$. What is the posterior probabilities $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ?