Büchi Automata

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Regular Expressions and ω-Regular Expressions

Non-Deterministic Büchi Automata

Characterizing Büchi Recognisable Languages

Logic of Sequences and Büchi Automata

Regular Expressions and ω-Regular Expressions

Regular Expressions over Σ

Definition

A regular expression is a sequence of characters that define a search pattern.

$$\alpha := \phi \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 \cdot \alpha_2 \mid \alpha^*$$

Where A is an alphabet and ϕ is the empty string.

Some semantics used throughout the presentation:

- $\alpha_1 + \alpha_2$ is the union of α_1 and α_2
- $\alpha_1 \cdot \alpha_2$ is the concatenation of α_1 and α_2
- α^* is the Kleene closure of α
- $\mathcal{L}(\phi) = \phi$, $\mathcal{L}(A) = \{A\}$ and $\mathcal{L}(\epsilon) = \{\epsilon\}$

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ω-Regular Expression

Definition

ω-regular expression = regex + ω-operator α^ω

Where we choose any string generated by α and repeat it infinitely many times, denoted by ω-operator.

$$\text{for } \mathcal{L} \subseteq \Sigma^* :$$

$$\mathcal{L}^\omega = \{w_1 w_2 w_3 \cdots : w_i \in \mathcal{L}\ \forall i \geq 1\}$$

$$\mathcal{L}(\omega) \subseteq \Sigma^\omega \text{ if } \epsilon \notin \mathcal{L}$$

- Kleene Star: 'finite repetition'
- ω-operator: 'infinite repetition'

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Syntax and semantics of ω-regular expressions

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Syntax of ω-regular expressions over alphabet Σ:

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Example

The language of $(A^*B)^\omega$ = set of all infinite words containing infinitely many B's

ω Regular Language

ω Regular Language

A language $\mathcal{L} \subseteq \Sigma^\omega$ is called ω-regular iff there exists an ω regular expression such that $\mathcal{L} = \mathcal{L}_\omega(\gamma)$

ω - regex example

- Alphabet $\Sigma = \{a, b\}$ such that the set of all infinite words contain finitely many a's.

$$(a + b)^* b^\omega$$

- Set of all infinite words where each a is followed immediately by letter b.

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Non-Deterministic Büchi Automata

Büchi Automata (NBA)

A Büchi automaton is a method of defining a set of ω words over a finite alphabet Σ

Definition

A (non-deterministic) Büchi automaton is a tuple $A = (Q, \Sigma, \delta, Q_0, F)$ where:

- Q is a finite set of states
- Σ is a finite alphabet
- $Q_0 \subseteq Q$ set of initial states
- $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation.
- F is the set of final states.

Büchi Automata (NBA)

Acceptance

Let $A_0 A_1 A_2 \cdots \in \Sigma^\omega$ be a run of a word, and π be the sequence of states visited:

$$\pi = q_0 q_1 q_2 \cdots \text{ where } q_0 \in Q_0$$

$$q_{i+1} \in \delta(q_i, A_i) \text{ for } i \geq 0$$

We say that run π is accepted if there exist infinitely many i's such that $q_i \in F$.

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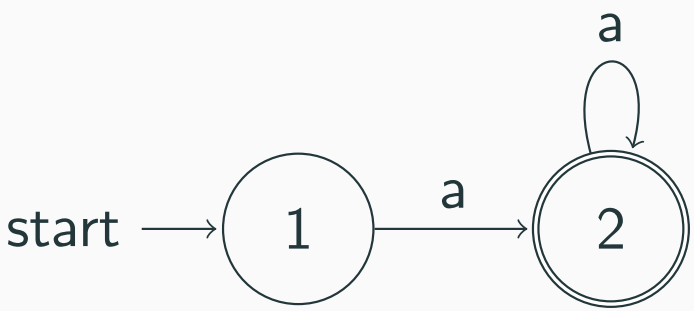

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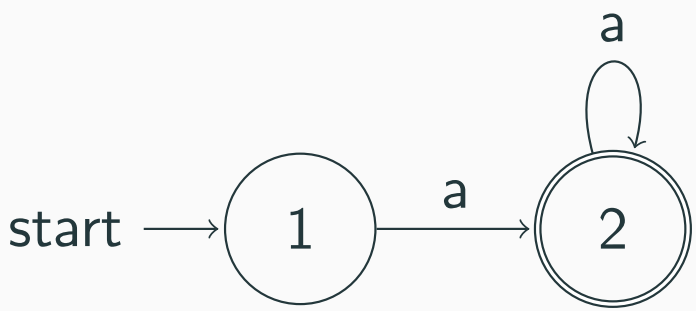

This automaton accepts the language of all ω-words over $\{a\}$ that contain infinitely many a's.

$$\mathcal{L}_\omega(A_1) = \{a\}^\omega$$

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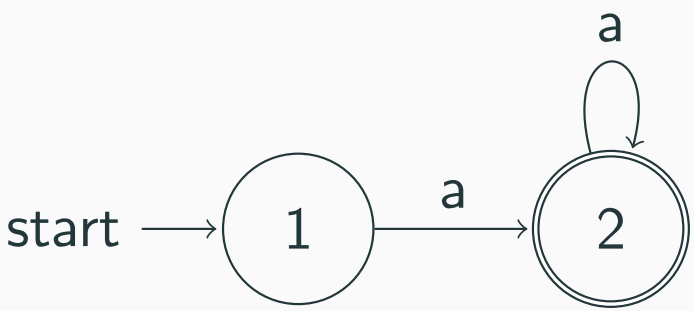

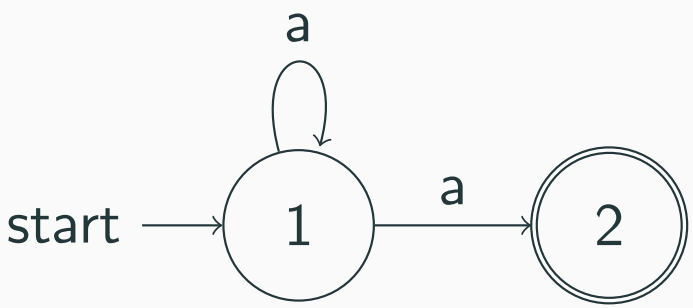

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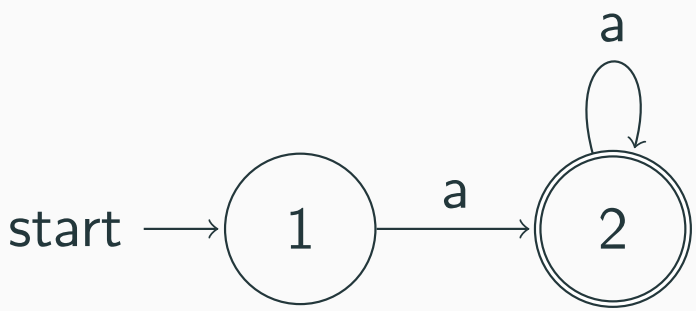

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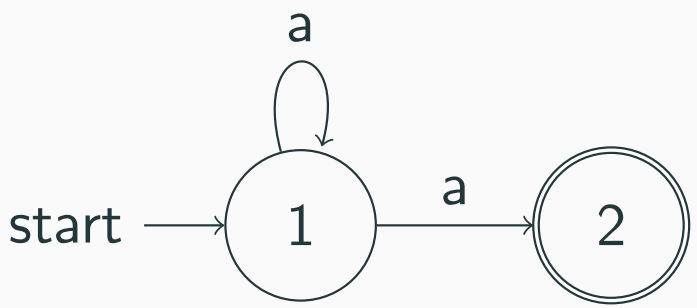

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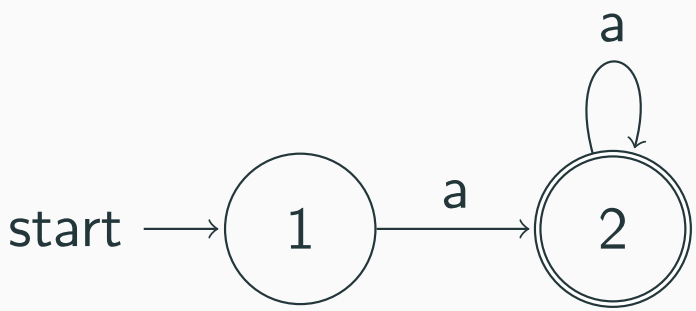

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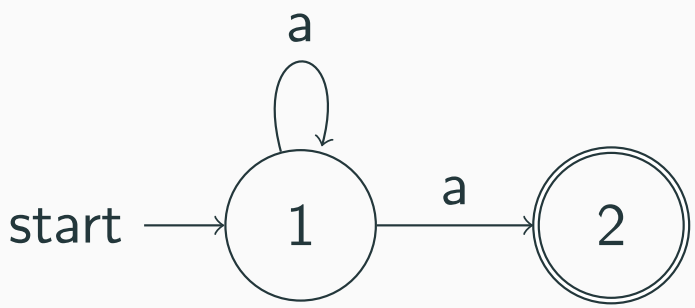

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In NBA "perspective", both languages are different.

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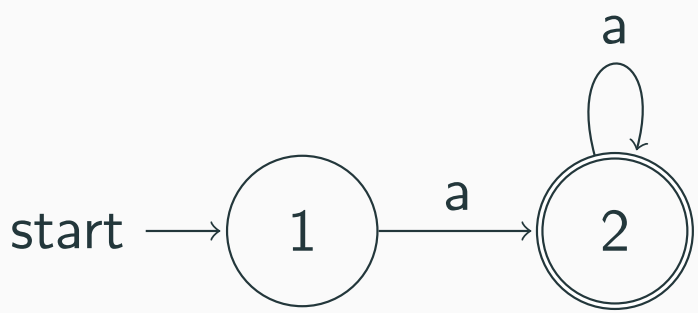

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$\mathcal{L}(A_1) = \{a^{n+1} \mid n \in N\}$

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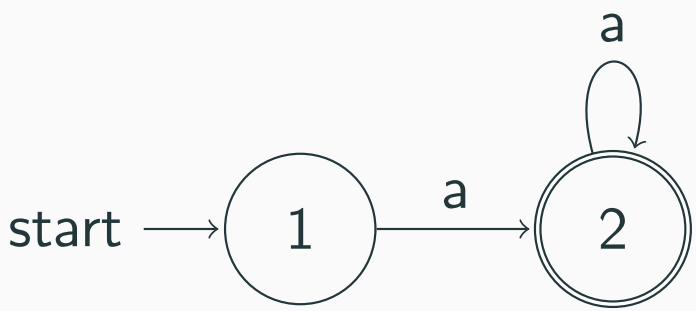

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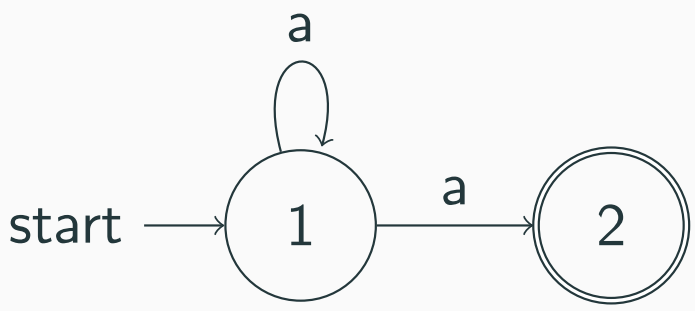

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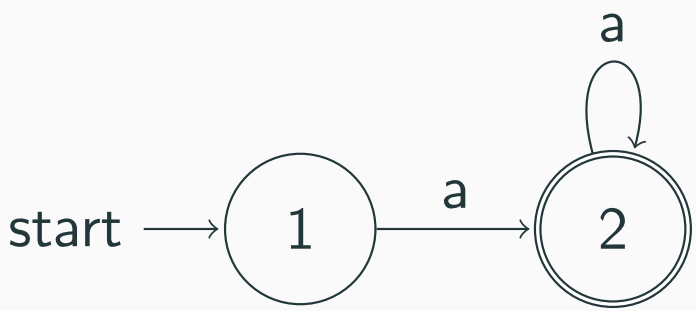

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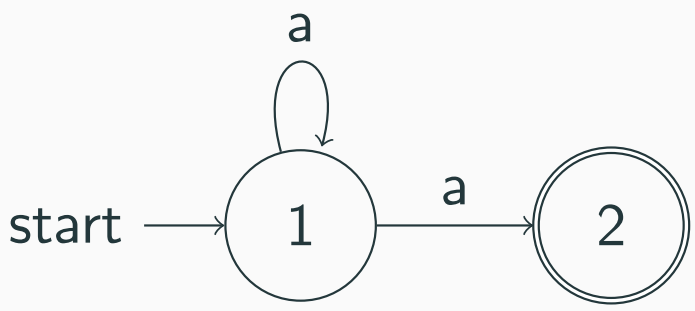

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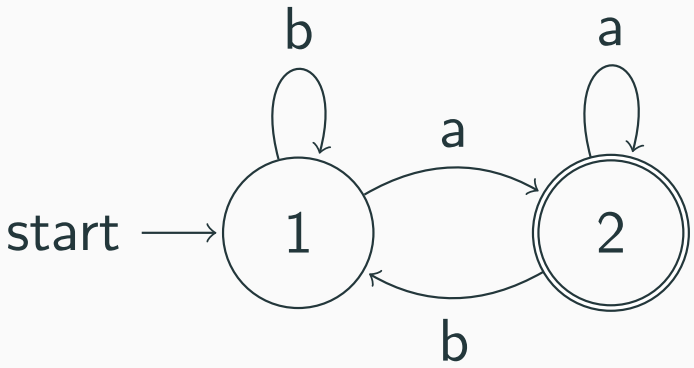
We can see that, the languages accepted by the automata A_1 and A_2 are same in NFA "perspective".
Thus both automaton are equivalent in NFA "perspective", but different in NBA "perspective".

Example of NBA over $\Sigma = \{a, b\}$



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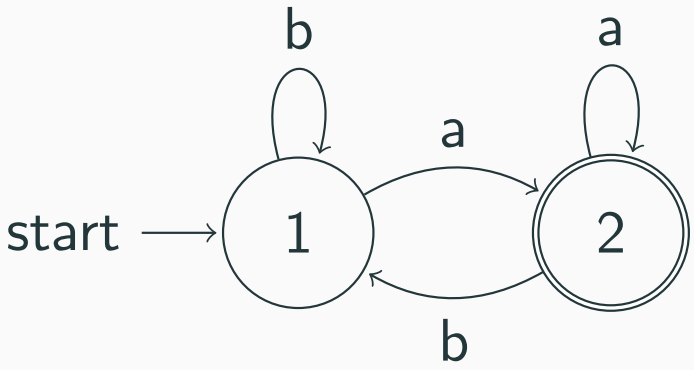

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Characterizing Büchi Recognisable Languages

Converting NBA to ω-regular expression

Theorem

A language is Buchi recognisable iff it is ω regular.

Proof.

($\implies$) Converting NBA to ω regular language.

Let A be an NBA $(Q, \Sigma, \delta, Q_0, F)$ and $q, p \in Q$.

Let $A_{q,p}$ be the NFA $(Q, \Sigma, \delta, q, \{p\})$. Then:

$$\mathcal{L}\omega(A) = \bigcup_{q \in Q_0} \bigcup_{p \in F} \mathcal{L}(A_{q,p})(\mathcal{L}(A_{p,p}) \backslash \{\epsilon\})^{\omega}$$

is ω regular as $\mathcal{L}(A_{q,p})$ and $\mathcal{L}(A_{p,p}) \backslash \{\epsilon\}$ are regular by definition. $\square$

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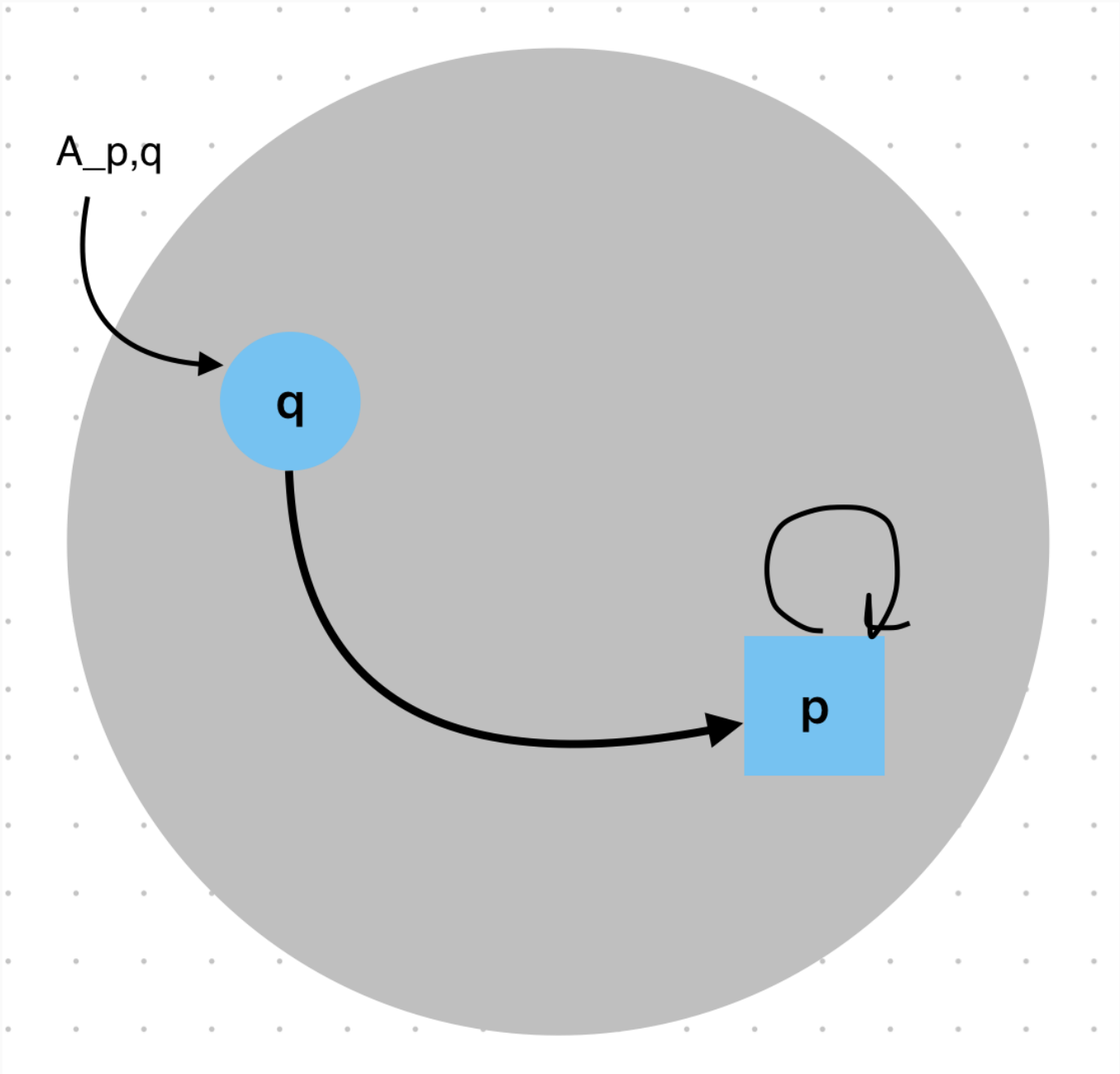

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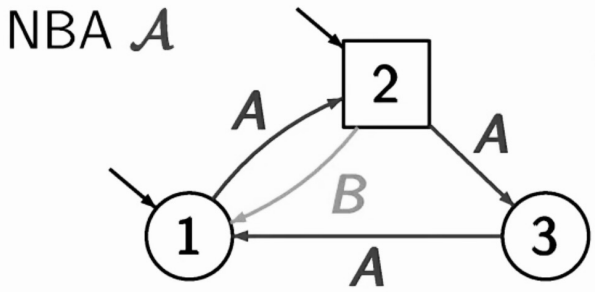

$$\mathcal{L}_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$$

$$L_{12} = \mathcal{L}(\mathcal{A}_{12})$$
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Example: Converting NBA to ω-regular expression

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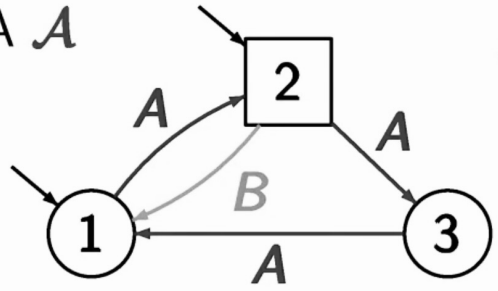

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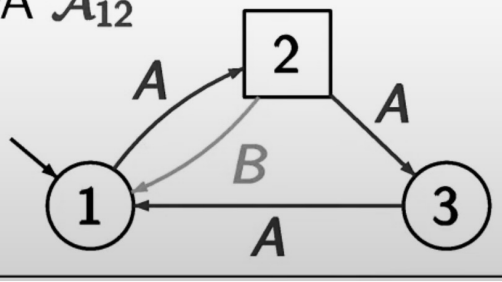
NFA $\mathcal{A}_{12}$



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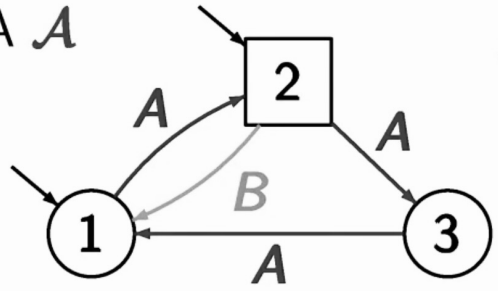

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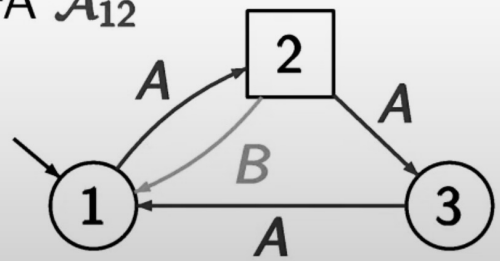
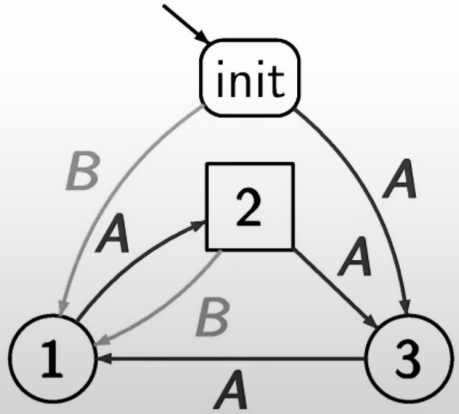
NFA $\mathcal{A}_{12}$





From ω-regular expression to NBA

Proof.

($\Longleftarrow$) Converting ω regular expression to NBA.

Let γ be an ω regular expression over Σ.

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \cdots + \alpha_n \cdot \beta_n^\omega$$

there exists an NBA A, with $\mathcal{L}\omega(A) = \mathcal{L}\omega(\gamma)$

Proof: consider NFA $\mathcal{A}_i$ for α_i and $\mathcal{B}_i$ for β_i

- construct NBA $\mathcal{B}_i^\omega$ for β_i^ω
- construct NBA $\mathcal{C}_i = \mathcal{A}_i \cdot \mathcal{B}_i^\omega$ for $\alpha_i \cdot \beta_i^\omega$
- construct NBA for $\bigcup_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$

□

We proceed the proof by showing that the set of Buchi recognisable languages satisy the following closure properties:

Closure under Finite Union

Finite Union

Let $A_1 = (Q_1, \Sigma, \delta_1, Q_{01}, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, Q_{02}, F_2)$ be two NBA's. Then the language $\mathcal{L}(A_1) \cup \mathcal{L}(A_2)$ is Buchi recognisable.

Proof.

Just take the union of the two automata !

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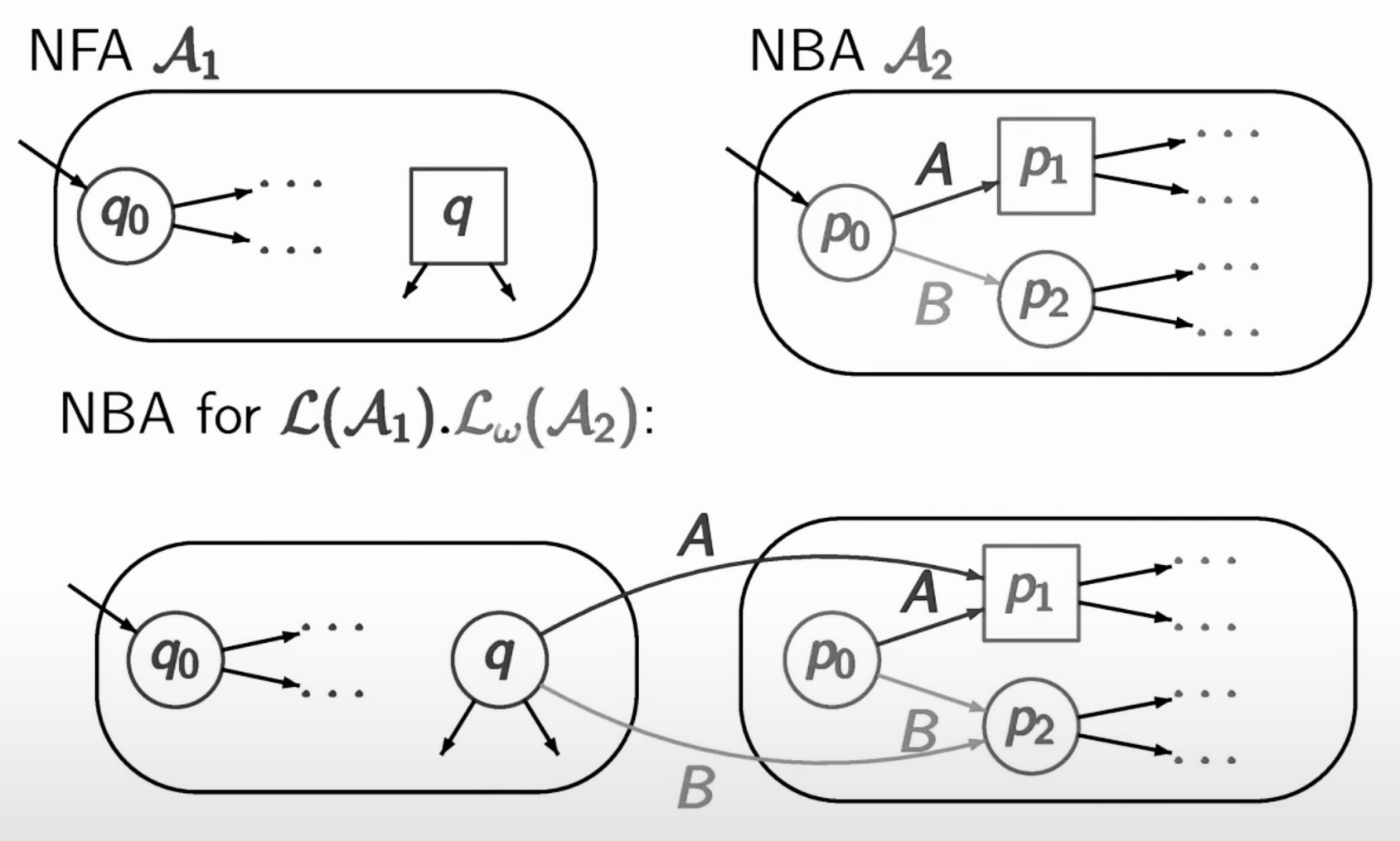
Concatenation of NFA and NBA

Concatenation

If $L(A_1)$ is regular and $L_\omega(A_2)$ is Buchi recognisable then $L(A_1)L_\omega(A_2)$ is Buchi recognisable.

Concatenation of NFA and NBA



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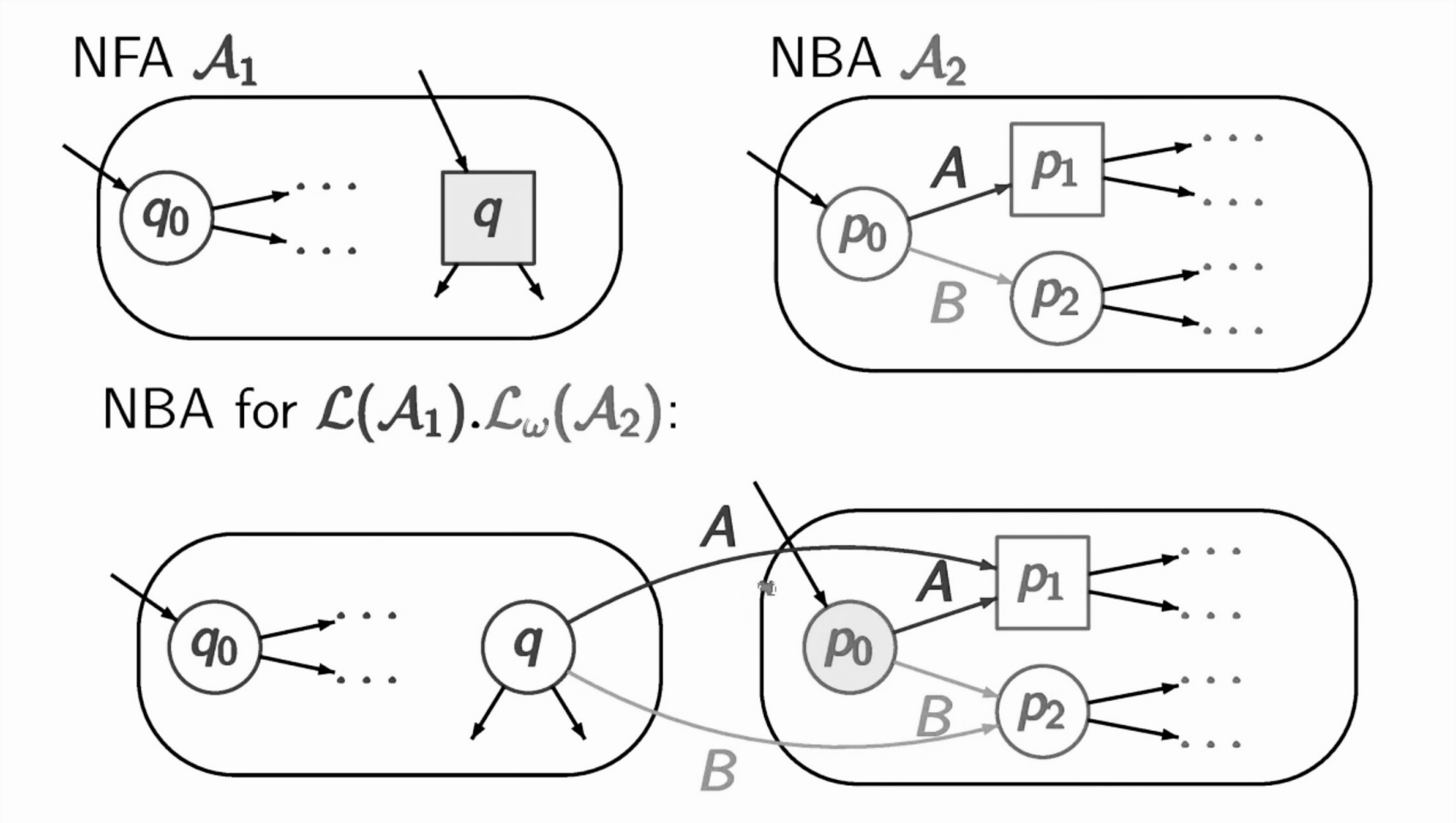
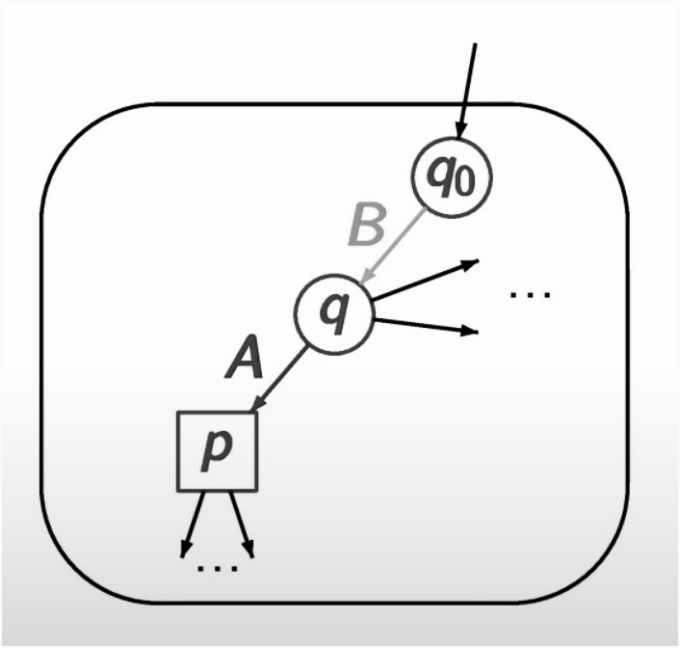
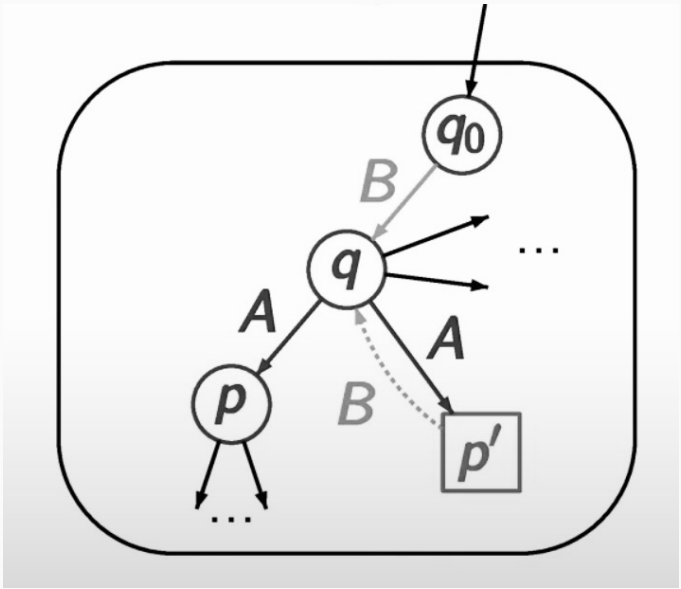


Construct NBA for NFA

NFA A for a language $\Longrightarrow$ $L \subseteq \Sigma^+$



NFA $\mathcal{B}$ for L such that all final states are terminal.



$$\mathcal{L}(A)^\omega = \mathcal{L}_w(\mathcal{B}^\omega)$$

Construct NBA for NFA

The final states (e.g. p) must be terminal states. If not make them.

WHY ? Because, if not, then there might be other possible paths introduced when we make the final state an "initial state" in the NBA.

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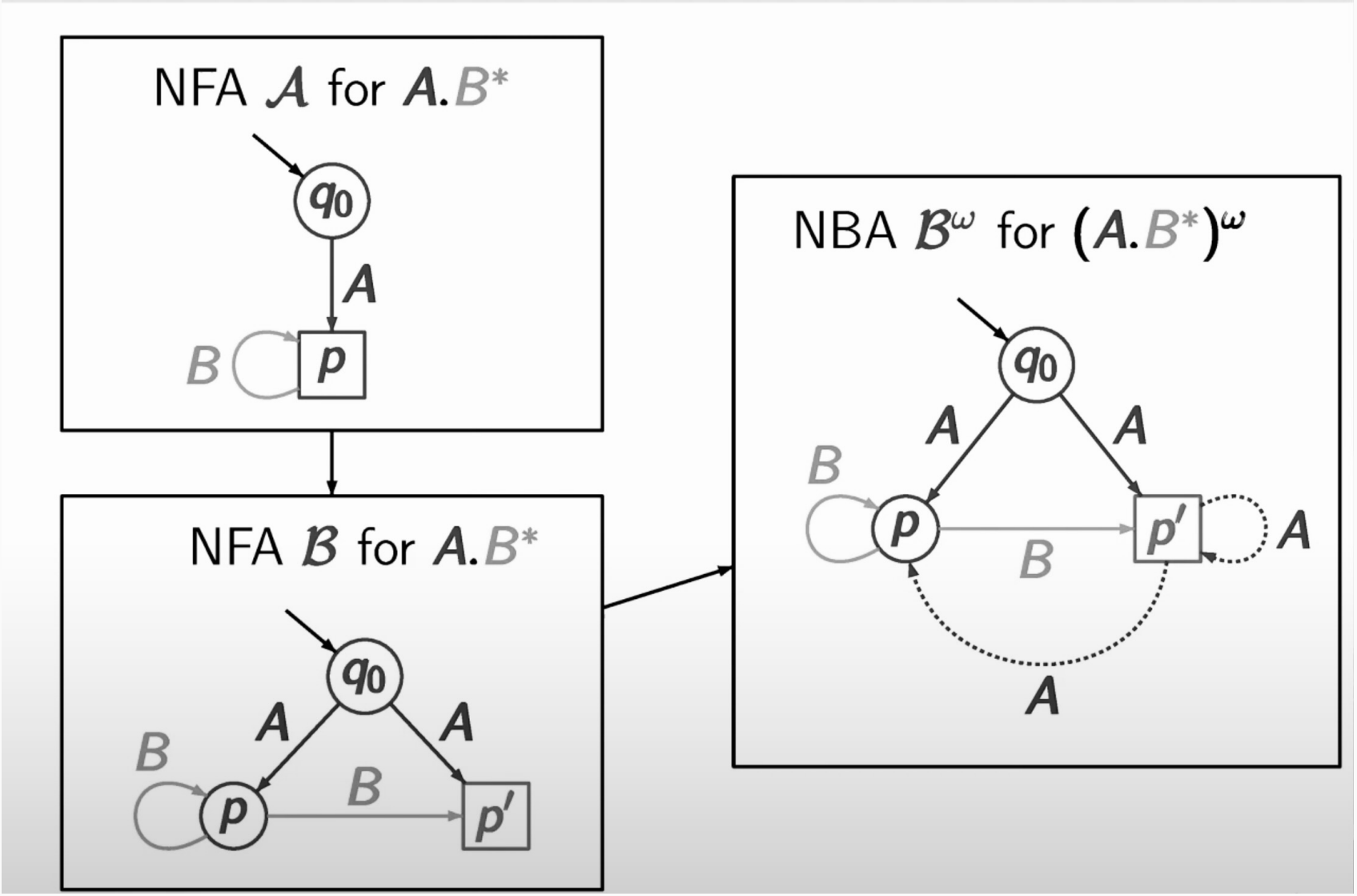
Example: Construct NBA for NFA



Logic of Sequences and Büchi Automata

Logic of Sequences

Buchi's original motivation for studying automata on infinite inputs was to solve a decision problem from logic. He discovered a deep and beautiful connection between ω-regular languages and sets of models of formulas in certain logics.

S1S

The logic that Buchi considered was the monadic second-order theory of one successor, abbreviated as S1S. This logic is interpreted over the set N0 of natural numbers. In general, second-order logic permits quantification over relations and functions, unlike first-order logic, which permits quantification over just individual elements.

- Second-order means that we allow quantifications over relations.
- Monadic means that quantifications is restricted to monadic relations, namely sets.

S1S (contd.)

Fix a logical structure $(\omega, s, \in)$

- s is the successor function $x \to x + 1$
- $\in$ is the standard memebrship relation between elements and sets.

Variables

- First-order variables (x, y, z, etc.) range over natural numbers.
- Second-order variables (X, Y, Z, etc.) range over sets of natural numbers.

Terms

- First-order variables are terms.
- If t is a term, then so is s(t)

S1S (contd.)

Formulas

- Atomic formulas are of the form $t \in X$ where t is a term and X is a second-order variable.
- S1S formulas are built up from atomic formulas using standard boolean connectives, with $\forall$ and $\exists$ quantifications over first-order and second-order variables.

Satisfiability and Buchi's Theorem

An S1S formula is said to be satisfiable if we can choose M such that $M \vDash \phi$

Buchi's Theorem

Buchi showed how to associate an ω-regular language $L_{(\phi)}$ with each S1S formula ϕ, such that every word in L_ϕ represents an interpretation for the free variables in ϕ under which the formula ϕ evaluates to true. Moreover, every interpretation that makes ϕ true is represented by some word in L_ϕ. Thus, ϕ is satisfiable iff there is some interpretation that makes it true iff L_ϕ is non-empty.

Example

Let ϕ be a invariant with invariant condition, $a \vee \neg b$. Then the ω- regular language associated with the invariant condition is $(\phi + \{a\} + \{a, b\})^\omega$ where ϕ is the empty set.

Motivation

- Büchi automata's primary application is in formal verification of hardware and software systems.
- They are used to verify the correctness of systems that are infinite-state, such as communication protocols, distributed systems, and embedded systems.
- By infinite state, we mean the system runs indefinitely.
- Büchi automata are used to verify the correctness of such systems by checking if the system satisfies a given property.
- The property is usually specified as a formula in a temporal logic, such as LTL or CTL.

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The end

References:

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The end

Thank You!