Limit Theorems for Large Deviations

Mathematics and Its Applications (*Soviet Series*)

Volume 73

Limit Theorems for Large Deviations

by

L. Saulis

and

V. A. Statulevičius

Lithuanian Academy of Sciences,
Institute of Mathematics and Informatics,
Vilnius, Lithuania

KLUWER ACADEMIC PUBLISHERS
DORDRECHT / BOSTON / LONDON

Library of Congress Cataloging-in-Publication Data

"CIP-data available from the Publisher on request."

ISBN 0-7923-1475-1

Published by Kluwer Academic Publishers,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

Kluwer Academic Publishers incorporates
the publishing programmes of
D. Reidel, Martinus Nijhoff, Dr W. Junk and MTP Press.

Sold and distributed in the U.S.A. and Canada
by Kluwer Academic Publishers,
101 Philip Drive, Norwell, MA 02061, U.S.A.

In all other countries, sold and distributed
by Kluwer Academic Publishers Group,
P.O. Box 322, 3300 AH Dordrecht, The Netherlands.

Printed on acid-free paper

This book is a greatly revised and updated edition of
Limit Theorems for Large Deviations
Published by Mokslas Publishers,
Vilnius, Lithuania, 1989 (in Russian)

Printed in the Netherlands

SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and nonlinearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the non-linear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, p-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five subseries: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

The study of the probabilities of large deviations from the expected value comprises a most important part of probability for obvious reasons. There are several approaches to limit theorems for large deviations, and several books more or less completely devoted to this subject, testifying to the importance of the topic in general.

In this volume the authors present their new approach based on cumulants. This is a powerful new way that permits the systematic investigation of large deviations for sums of both independent and dependent random variables as well as for polynomial forms, multiple stochastic integrals, random fields, etc.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Amsterdam, August 1991

Michiel Hazewinkel

CONTENTS

PREFACE

Among the works on limit theorems of large deviations the investigation of sums of independent random variables comprises a major part. This is quite natural because the simplest case, to which powerful analytical methods are applicable, enables us to get a clear and complete picture of the behaviour of probabilities of large deviations.

Most often there have been considered these cases: when the Cramer condition is satisfied, i.e. when characteristic functions of summands are analytical in the zero neighbourhood; the Linnik case, when all the moments of summands are finite but their growth does not guarantee the analyticity of the corresponding characteristic functions in the zero neighbourhood; the case of so–called moderate deviations, when the summands have only the finite number of moments (the case studied first by H. Rubin and J. Sethuraman (1965)); the case when the Cramer and Linnik conditions are not fulfilled, but the behaviour of distribution tails of summands is regular enough (the case when summands belong to the domain of attraction of the stable law with the index $\alpha < 2$ is considered in the works by M.I. Fortus (1957), C.C. Heyde (1968), S.G. Tkachuk (1975), and the conditions for so–called superlarge deviations are investigated in the works of S.V. Nagaev (1963, 1965) and A.V. Nagaev (1969)).

Most of principal ideas and results in this field are rather exhaustively described in the monographs of I.A. Ibragimov and Yu.V. Linnik (1971), and V.V. Petrov (1975).

Another trend, drawing the attention of many investigators, is large deviations in functional limit theorems. Here one can hardly manage with analytical methods only and must combine them with direct methods. Most complete results are obtained for the so–called rough limit theorems of large deviations, i.e. for theorems, considering the behaviour of probabilities of large deviations with an accuracy of logarithmic equivalence. First of all we must refer to works of I.N. Sanov (1957), R.R. Bahadur (1960) for empirical distribution functions, of A.A. Borovkov and A.A. Mogulskii (1978, 1980) on large deviations in invariance principle for sums of independent random variables and for processes with independent increments, of M.D. Donsker and S.R.S. Varadhan (1975, 1976) on large deviations in ergodic theory of Markov processes, of A.D. Ventsel and M.I. Freidlin (1986) on large deviations for classes of families of Markov processes, and of R.Sh. Liptser and A.N. Shiryaev (1989) for martingales and semimartingales. Many results of this trend are presented in the books by A.D. Ventsel (1986), D.W. Stroock (1984), R.Sh. Liptser and A.N. Shiryaev (1989).

This book is devoted to the applications of the method of cumulants to limit theorems on large deviations. The reader can make sure that this method is good in the investigation of large deviations for sums of both independent and dependent random variables, polynomial forms, multiple stochastic integrals of random processes and fields, polynomial statistics. Only the case of normal approximation is considered here.

The authors deeply appreciate their colleagues A. Basalykas, A. Plikusas, D. Jakimavičius, who kindly presented their results for publication, J. Kazlauskaitė for thorough preparation of the manuscript, as well as J. Mačys and D. Surgailis who edited the book making many valuable remarks.

CHAPTER 1
THE MAIN NOTIONS

Let a *probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ be given, where Ω is a set of elements ω, $\mathcal{F}$ is a σ–algebra of subsets (events) of the set Ω, and $\mathbf{P}$ is a probability measure defined in $\mathcal{F}$. A measurable on $(\Omega, \mathcal{F}, \mathbf{P})$ real–valued function $\xi = \xi(\omega)$ is called a *random variable* (r.v.), and the measure

$$\mathbf{P}_\xi(B) := \mathbf{P}(\xi \in B) := \mathbf{P}\big(\{\omega : \xi(\omega) \in B\}\big) \tag{1.1}$$

defined on Borel sets of the real line R is called the *distribution of the r.v.* ξ.

The *distribution function* of the r.v. ξ

$$F_\xi(x) := \mathbf{P}(\xi < x) := \mathbf{P}_\xi\big((-\infty, x)\big) \tag{1.2}$$

uniquely defines the distribution $\mathbf{P}_\xi$ since

$$\mathbf{P}_\xi(B) = \int_B dF_\xi(x) \tag{1.3}$$

for any Borel set of the real line. In the sequel we shall not differentiate between the distribution and the distribution function corresponding to it.

R.v. $\xi_1, \xi_2, \ldots, \xi_k$, each of which is defined in $(\Omega, \mathcal{F}, \mathbf{P})$, form a random vector $\eta = (\xi_1, \ldots, \xi_k)$.

The measure

$$\mathbf{P}_\eta(B) := \mathbf{P}(\eta \in B) := \mathbf{P}\Big(\{\omega : ((\xi_1(\omega), \ldots, \xi_k(\omega)) \in B\}\Big) \tag{1.4}$$

defined on Borel sets of R^k is called the distribution of the random vector η (the joint distribution of r.v. $\xi_1, \ldots, \xi_k$). Sometimes we denote $\mathbf{P}_\eta(B)$ by $\mathbf{P}_{\xi_1,\ldots,\xi_k}(B)$. R.v. $\xi_1, \ldots, \xi_k$ are called independent if

$$\mathbf{P}(\xi_1 \in B_1, \ldots, \xi_k \in B_k) = \prod_{j=1}^{k} \mathbf{P}(\xi_j \in B_j) \tag{1.5}$$

for any Borel sets $B_1, \ldots, B_k$. In this case

$$F_{\xi_1,\ldots,\xi_k}(x_1, \ldots, x_k) = F_{\xi_1}(x_1)\cdot \ldots \cdot F_{\xi_k}(x_k), \tag{1.6}$$

where

$$F_{\xi_1,\ldots,\xi_k}(x_1, \ldots, x_k) := \mathbf{P}(\xi_1 < x_1, \ldots, \xi_k < x_k) \tag{1.7}$$

is the joint distribution function of the r.v. $\xi_1, \ldots, \xi_k$.

A family of r.v. $X(t, \omega)$, depending on a real parameter t taking values in some set T, is called a *random process* in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The set T is called the domain of definition of the process. The variables $X(t, \omega)$ can take real, complex or vector values. As before, ω is often omitted here and $X(t)$ or X_t is written instead of $X(t, \omega)$.The main characteristic of a random process is its finite–dimensional distributions, — the collection of functions

$$F_{t_1,\ldots,t_k}(B_1, \ldots, B_k) := \mathbf{P}\big(X(t_1) \in B_1, \ldots, X(t_k) \in B_k\big), \tag{1.8}$$

defined with any natural k, for $t_1, \ldots, t_k \in T$ and any Borel sets $B_1, \ldots, B_k$ from the state space of the process.

The answer to the question under what conditions a given set of functions $\{F_{t_1,\ldots,t_k}(B_1, \ldots, B_k)\}$ defines the random process with finite–dimensional distributions (1.8), is given by the well–known Kolmogorov theorem (Kolmogorov, 1933) about the extension of a family of measures on finite–dimensional spaces to a measure on an infinite–dimensional space.

We shall also consider random functions $X(t)$, as $T \subset R^m$, $m > 1$, i.e. random fields.

The distribution $\mathbf{P}_\xi$ on the real line is called *absolutely continuous* with respect to the Lebesgue measure, if

$$\mathbf{P}_\xi(B) = \int\limits_B p_\xi(x)\, dx \tag{1.9}$$

for any Borel set $B \subset R$. Then $p_\xi(x) = \frac{d}{dx}F_\xi(x)$ (with an accuracy of a set of the Lebesgue measure zero). The function $p_\xi(x)$ is called the *distribution density* of the r.v. ξ.

The distribution $\mathbf{P}_\xi$ is called *discrete* if it is concentrated on some denumerable set $\{x_k : k = 1, 2, \ldots\}$. Then

$$\mathbf{P}_\xi(B) = \sum_{x_k \in B} p_k, \quad F_\xi(x) = \sum_{x_k < x} p_k, \tag{1.10}$$

where

$$p_k = \mathbf{P}(\xi = x_k).$$

A discrete distribution $\mathbf{P}_\xi$ is called *lattice*, if the set $\{x_k : k = 1, 2, \dots\}$ is an arithmetic progression $\{a + kh : k = 0, \pm 1, \dots\}$. Random variable ξ with lattice distribution is a simple generalization of integer–valued random variable. The quantity h is called the *distribution step.* If for every pair of b and $h_1 > h$ all possible values of the r.v. ξ cannot be presented in the form $b + kh_1$, then the distribution step is called *maximal.*

A distribution, concentrated in one point a, is called *degenerate.* Its distribution function is $E(x - a)$, where

$$E(x) = \begin{cases} 0, & x \leqslant 0, \\ 1, & x > 0. \end{cases} \tag{1.11}$$

A distribution, concentrated on an uncountable set of the Lebesgue measure zero, is called *singular.*

Any distribution can be represented in the form

$$\mathbf{P} = a_1\mathbf{P}_1 + a_2\mathbf{P}_2 + a_3\mathbf{P}_3, \tag{1.12}$$

where $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ are continuous, discrete and singular distributions, respectively, $a_i \geqslant 0$, $i = 1, 2, 3$, $a_1 + a_2 + a_3 = 1$. The decomposition

$$F(x) = a_1F_1(x) + a_2F_2(x) + a_3F_3(x) \tag{1.13}$$

of an arbitrary distribution function into absolutely continuous, pure jump and singular components corresponds to decomposition (1.12).

A distribution function $F(x)$ is not decreasing, left continuous and has the limits $F(-\infty) = 0$, $F(\infty) = 1$. Conversely, any function, satisfying these requirements, is a distribution function.

Let ξ and η be independent r.v. with the distribution functions F_ξ and F_η. It is known that the distribution function of the sum $\xi + \eta$ is expressed by convolution of the distribution functions of the summands

$$F_{\xi+\eta}(x) = \int_{-\infty}^{\infty} F_\xi(x-y)\,dF_\eta(y) = \int_{-\infty}^{\infty} F_\eta(x-y)\,dF_\xi(y), \tag{1.14}$$

denoted in abbreviation

$$F_{\xi+\eta} = F_\xi * F_\eta. \tag{1.15}$$

We shall also write F^{*n} for the convolution of n identical distribution functions.

Note that if at least one of the components F_ξ or F_η is absolutely continuous, then the convolution $F_{\xi+\eta}$ is also absolutely continuous; moreover, if e.g. F_ξ is absolutely continuous, then

$$\frac{d}{dx}F_{\xi+\eta}(x) = \int_{-\infty}^{\infty} \frac{d}{dx}F_\xi(x-y)\,dF_\eta(y). \tag{1.16}$$

The integral

$$\mathbf{E}\xi := \int_{\Omega} \xi(\omega)\, \mathbf{P}(d\omega) \tag{1.17}$$

(if $\int_{\Omega} |\xi(\omega)|\, \mathbf{P}(d\omega)$ exists) is called the *mathematical expectation* or *mean* of r.v. For any measurable function $g(x)$, $x \in R$, $\mathbf{E}|g(\xi)| < \infty$, the relation

$$\mathbf{E}g(\xi) := \int_{\Omega} g(\xi(\omega))\, \mathbf{P}(d\omega) = \int_{R} g(x)\, \mathbf{P}_{\xi}(dx) = \int_{-\infty}^{\infty} g(x)\, dF_{\xi}(x) \tag{1.18}$$

is valid.

If for some $\nu > 0$ there exists the integral $\beta_\nu := \mathbf{E}|\xi^\nu|$, then β_ν, $\alpha_\nu := \mathbf{E}\xi^\nu$ and $\mu_\nu := \mathbf{E}(\xi - \mathbf{E}\xi)^\nu$ are called the *absolute moment*, the *moment* and the *central moment*, respectively, of order ν of the r.v. ξ.

The inequality for absolute moments*)

$$\beta_k \leqslant \beta_i^{(l-k)/(l-i)} \cdot \beta_l^{(k-i)/(l-i)}, \qquad 0 \leqslant i \leqslant k \leqslant l, \tag{1.19}$$

follows easily from Hölder's inequality. Hence, as $i = 0$, we have

$$\beta_k^{1/k} \leqslant \beta_l^{1/l}, \qquad 0 \leqslant k \leqslant l, \tag{1.20}$$

and when $i = k - 1$, $l = k + 1$

$$\beta_k^2 \leqslant \beta_{k-1} \cdot \beta_{k+1}, \qquad k \geqslant 1. \tag{1.21}$$

The *characteristic function* (c.f) of the r.v. ξ is defined by

$$f_\xi(t) := \mathbf{E}e^{it\xi} = \int_{-\infty}^{\infty} e^{itx}\, dF_\xi(x). \tag{1.22}$$

Its relation with the moments is determined by the following statement: if a r.v. ξ has the absolute moment of order k, then its c.f. has the k^{th} order derivatives and

$$\alpha_\nu = \frac{1}{i^\nu} \frac{d^\nu}{dt^\nu} f_\xi(t)\Big|_{t=0}, \qquad \nu = 1, \ldots, k, \tag{1.23}$$

*) See Appendix 1.

and consequently

$$f_\xi(t) = \sum_{\nu=0}^{k} \frac{i^\nu \alpha_\nu}{\nu!} t^\nu + o(|t|^k). \tag{1.24}$$

Characteristic function of the sum of independent r.v. equals the product of their c.f., i.e.

$$f_{S_n}(t) = \prod_{j=1}^{n} f_{\xi_j}(t), \tag{1.25}$$

where $S_n = \xi_1 + \cdots + \xi_n$, $\xi_1, \ldots, \xi_n$ are independent random variables.

A c.f. is defined uniquely by a distribution function. In its turn it completely determines the distribution function by the equality

$$F_\xi(x) = \frac{1}{2\pi} \lim_{y\to\infty} \lim_{c\to\infty} \int_{-c}^{c} \frac{e^{-ity} - e^{-itx}}{it} f_\xi(t)\, dt, \tag{1.26}$$

valid at all the points of continuity of $F_\xi(x)$. Thus, the correspondence (1.22) between distributions on the real line and characteristic functions is one–to–one.

Note that if $F_\xi(x)$ has density $p_\xi(x)$, then the c.f. $f_\xi(t)$ is the Fourier transform of the function $p_\xi(x)$:

$$f_\xi(t) = \int_{-\infty}^{\infty} e^{itx} p_\xi(x)\, dx. \tag{1.27}$$

In the latter case $\lim\limits_{t\to\infty} |f_\xi(t)| = 0$ according to the Riemann – Lebesgue theorem. Consequently, if the distribution function $F_\xi(x)$ has an absolutely continuous component, then

$$\overline{\lim_{t\to\infty}}\, |f_\xi(t)| < 1. \tag{1.28}$$

In case the distribution $F_\xi(x)$ is discrete, the function $f_\xi(t)$ is almost periodic and $\overline{\lim\limits_{t\to\infty}}\, |f_\xi(t)| = 1$.

Let ξ be a r.v. with the c.f. $f_\xi(t)$. Let us introduce the symmetrized r.v. $\bar{\xi} := \xi - \xi'$, where ξ' is a r.v. independent of ξ and having the same distribution as the r.v. ξ. The r.v. $\bar{\xi}$ has a real nonnegative c.f.

$$f_{\bar{\xi}}(t) = f_\xi(t) \cdot f_{\xi'}(t) = f_\xi(t) \cdot f_\xi(-t) = |f_\xi(t)|^2. \tag{1.29}$$

If an absolute moment $\beta_l = \mathbf{E}|\xi^l|$ exists, then for sufficiently small values of t (since $f_\xi(0) = 1$, $f_\xi(t)$ is continuous and $|f_\xi^{(k)}(t)| \leqslant \beta_k |t|^k$) the main part of $\log f_\xi(t)$ can be represented in the form

$$\log f_\xi(t) = \sum_{k=1}^{l} \frac{1}{k!} \gamma_k (it)^k + o(|t|^l), \tag{1.30}$$

where γ_k are the *cumulants*, defined by the formula

$$\gamma_k = \frac{1}{i^k} \frac{d^k}{dt^k} (\log f_\xi(t)) \Big|_{t=0}. \tag{1.31}$$

Existence of the moment β_k implies the existence of all cumulants up to the order k.

Making use of the expansion

$$\log(1+z) = \sum_{s=1}^{\infty} (-1)^{s+1} \frac{1}{s} z^s \qquad (|z| < 1), \tag{1.32}$$

we obtain the formal equality

$$\sum_{l=1}^{\infty} \frac{1}{l!} \gamma_l (it)^l = \sum_{s=1}^{\infty} (-1)^{s+1} \frac{1}{s} \left(\sum_{l=1}^{\infty} \frac{1}{l!} \alpha_l (it)^l \right)^s, \tag{1.33}$$

which allows to express the cumulant γ_k of arbitrary order k through the moments $\alpha_1, \alpha_2, \ldots, \alpha_k$ by equating the coefficients at t^k on both sides of equality (1.33):

$$\gamma_k = \sum_{\nu=1}^{k} \frac{(-1)^{\nu-1}}{\nu} \sum_{k_1+\ldots+k_\nu=k} \frac{k!}{k_1! \ldots k_\nu!} \alpha_{k_1} \ldots \alpha_{k_\nu}. \tag{1.34}$$

In particular,

$$\begin{aligned}
\gamma_1 &= \alpha_1, \qquad \gamma_2 = \alpha_2 - \alpha_1^2, \\
\gamma_3 &= \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3, \\
\gamma_4 &= \alpha_4 - 4\alpha_3\alpha_1 - 3\alpha_2^2 + 12\alpha_2\alpha_1^2 - 6\alpha_1^4, \\
\gamma_5 &= \alpha_5 - 5\alpha_4\alpha_1 - 10\alpha_3\alpha_2 + 20\alpha_3\alpha_1^2 + 30\alpha_2^2\alpha_1 - 60\alpha_2\alpha_1^3 + 24\alpha_1^5.
\end{aligned}$$

The cumulant γ_k of a r.v. ξ will be denoted by $\Gamma_k(\xi)$. Usually it is simpler to use cumulants than moments. For example, if $\xi_1, \ldots, \xi_n$ are independent r.v., then according to (1.25) and (1.31)

$$\Gamma_k(S_n) = \sum_{j=1}^{n} \Gamma_k(\xi_j). \tag{1.35}$$

This book is devoted to the development of the cumulant method for the investigation of asymptotical behaviour of distributions, mainly of probabilities of large deviations for different functionals of random processes and fields, and in the first of all for sums

$$S_n = \sum_{j=1}^{n} \xi_j \tag{1.36}$$

of independent r.v. and sums

$$S_n = \sum_{t=1}^{n} X_t \tag{1.37}$$

of dependent r.v., polynomial forms

$$\zeta_n^{(p)} = \sum_{1 \leqslant t_1 \leqslant \ldots \leqslant t_p \leqslant n} a_{t_1,\ldots,t_p} X_{t_1} \cdot \ldots \cdot X_{t_p}, \tag{1.38}$$

for multiple stochastic integrals

$$Y_n^{(p)} = \int \ldots \int a(t_1, \ldots, t_p)\, dX(t_1) \ldots dX(t_p) \tag{1.39}$$

with respect to the Wiener and Poisson processes, for the Pitman polynomial estimates, U–statistics, etc.

A normal distribution will be the principal approximating distribution in the sequel.

A r.v. η with the distribution density

$$p_\eta(x) := \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(x-a)^2}{2\sigma^2} \right\} \tag{1.40}$$

is called the *normal* random variable with the parameters a and $\sigma > 0$. The (0, 1)–normal distribution function (i.e. the distribution function of the r.v. η with the parameters 0 and 1) is usually denoted by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{ -\frac{1}{2} y^2 \right\} dy. \tag{1.41}$$

The characteristic function of the (a, σ)–normal r.v. η is

$$f_\eta(t) = \exp\left\{ -iat - \frac{1}{2}\sigma^2 t^2 \right\}, \tag{1.42}$$

and central moments and cumulants are equal

$$\begin{aligned}
&\mu_{2k+1} = 0, \qquad \forall k \geqslant 0, \\
&\mu_{2k} = 1 \cdot 3 \cdot \ldots \cdot (2k-1)\sigma^{2k} = (2k-1)!!\sigma^{2k}, \qquad \forall k \geqslant 1; \\
&\Gamma_1(\eta) = a, \qquad \Gamma_2(\eta) = \sigma^2, \\
&\Gamma_k(\eta) = 0, \qquad \forall k \geqslant 3.
\end{aligned} \tag{1.43}$$

In the investigation of random processes we shall make use of finite–dimensional distributions of a process, i.e. the distribution of a random vector $X = (X_{t_1}, \ldots, X_{t_k})$, $t_1, \ldots, t_k \in T$. If $\mathbf{E}|X_t^m| < \infty$, $t \in T$, then for all $k \leqslant m$ the functions

$$m_X(t_1, \ldots, t_k) := \mathbf{E}X_{t_1} \ldots X_{t_k} \tag{1.44}$$

are well–defined.

The function $m_X(t_1, \ldots, t_k)$ is called the k^{th} *moment function* or the *simple moment* of the k^{th} order of the random process X_t. Let

$$f_X(u_1, \ldots, u_k) := \mathbf{E}\exp\{i\,\langle u, X\rangle\} \tag{1.45}$$

be the *characteristic function of the random vector* X, where $\langle a, b\rangle = \sum_{j=1}^{k} a_j b_j$ is the scalar product of vectors $a = (a_1, \ldots, a_k) \in R^k$, $b = (b_1, \ldots, b_k) \in R^k$. Similarly to the one–dimensional case, if $\mathbf{E}|X_t^m| < \infty$, then for all k and $\nu = (\nu_1, \ldots, \nu_k)$, $\nu_i \geqslant 0$ and $k|\nu| \leqslant m$, where $|\nu| := |\nu_1| + \ldots + |\nu_k|$, there exist *mixed cumulants* of the random vector X

$$\Gamma_\nu(X) := \frac{1}{i^{|\nu|}} \frac{\partial^{\nu_1+\ldots+\nu_k}}{\partial u_1^{\nu_1} \ldots \partial u_k^{\nu_k}} (\ln f_X(u_1, \ldots, u_k))\bigg|_{u_1=0, \ldots, u_k=0} . \tag{1.46}$$

Sometimes we shall write $\Gamma_\nu(X)$ instead of $\Gamma_\nu(X_{t_1}, \ldots, X_{t_k})$.

If $\nu = (1, \ldots, 1) \in R^k$, then the corresponding cumulant $\Gamma_\nu(X_{t_1}, \ldots, X_{t_k})$ will be denoted by $\Gamma(X)$, $\Gamma(X_{t_1}, \ldots, X_{t_k})$ or $s_X(t_1, \ldots, t_k)$. The function $s_X(t_1, \ldots, t_k)$ is called the *correlation function* or *simple cumulant* of the k^{th} order of the random process X_t. If $S_n = \sum_{t=1}^{n} X_t$ or $S(T) = \int_0^T X_t\,dt$ (when the process $X(t)$ is measurable and for almost all ω there exists $\int_0^T X_t(\omega)\,dt$) then it follows from the definition that

$$\Gamma_k(S_n) = \sum_{1 \leqslant t_1, \ldots, t_k \leqslant n} \Gamma(X_{t_1}, \ldots, X_{t_k}) \tag{1.47}$$

and

$$\Gamma_k(S(T)) = \int_0^T \ldots \int_0^T \Gamma(X_{t_1}, \ldots, X_{t_k})\,dt_1 \ldots dt_k, \tag{1.48}$$

respectively.

We shall use the following notation: if $\nu = (\nu_1, \ldots, \nu_k)$ is an integer nonnegative vector and $a = (a_1, \ldots, a_k)$ is a real vector, then

$$a^\nu := a_1^{\nu_1} \ldots a_k^{\nu_k}, \qquad \nu! := \nu_1! \ldots \nu_k!, \qquad |\nu| := \nu_1 + \ldots + \nu_k. \tag{1.49}$$

Denote

$$\mathbf{E}_\nu(X) = \mathbf{E}X_{t_1}^{\nu_1}\dots X_{t_k}^{\nu_k}. \tag{1.50}$$

If $\mathbf{E}|X_t^m| < \infty$, $t = (t_1, \dots, t_k)$, for some integer $m \geqslant 1$, then the function $f_X(u)$ can be expanded by the Taylor formula

$$f_X(u) = \sum_{|\nu|\leqslant m} \frac{i^{|\nu|}}{\nu!}\mathbf{E}_\nu(X)u^\nu + o(|u|^m), \tag{1.51}$$

where $\sum\limits_{|\nu|\leqslant m}$ is taken over all nonnegative collections of $(\nu_1, \dots, \nu_k)$ such that $|\nu| \leqslant m$.

Analogously,

$$\log f_X(u) = \sum_{|\nu|\leqslant m} \frac{i^{|\nu|}}{\nu!}\Gamma_\nu(X)u^\nu + o(|u|^m) \tag{1.52}$$

in the neighbourhood $|u| < \delta$, $\delta > 0$. As in the one–dimensional case it is possible to derive formulas, connecting $\mathbf{E}_\nu(X)$ and $\Gamma_\nu(X)$:

$$\mathbf{E}_\nu(X) = \sum_{\lambda^{(1)}+\dots+\lambda^{(q)}=\nu} \frac{1}{q!}\frac{\nu!}{\lambda^{(1)}!\dots\lambda^{(q)}!}\prod_{p=1}^{q}\Gamma_{\lambda^{(p)}}(X), \tag{1.53}$$

$$\Gamma_\nu(X) = \sum_{\lambda^{(1)}+\dots+\lambda^{(q)}=\nu} \frac{(-1)^{q-1}}{q}\frac{\nu!}{\lambda^{(1)}!\dots\lambda^{(q)}!}\prod_{p=1}^{q}\mathbf{E}_{\lambda^{(p)}}(X), \tag{1.54}$$

where $\sum\limits_{\lambda^{(1)}+\dots+\lambda^{(q)}=\nu}$ stands for the summation over all ordered collections of integer nonnegative vectors $\lambda^{(p)}$, $|\lambda^{(p)}| > 0$, which equal ν in sum. These formulas can be written in the following form:

$$\mathbf{E}_\nu(X) = \sum_{\{r_1\lambda^{(1)}+\dots+r_\varkappa\lambda^{(\varkappa)}=\nu\}} \frac{1}{r_1!\dots r_\varkappa!}\frac{\nu!}{(\lambda^{(1)}!)^{r_1}\dots(\lambda^{(q)}!)^{r_\varkappa}}\times$$
$$\times\prod_{j=1}^{\varkappa}\left(\Gamma_{\lambda^{(j)}}(X)\right)^{r_j},$$

$$\begin{aligned}\Gamma_\nu(X) = &\sum_{\{r_1\lambda^{(1)}+\dots+r_\varkappa\lambda^{(\varkappa)}=\nu\}} \frac{(-1)^{q-1}(q-1)!}{r_1!\dots r_\varkappa!}\frac{\nu!}{(\lambda^{(1)}!)^{r_1}\dots(\lambda^{(q)}!)^{r_\varkappa}}\times\\ &\times\prod_{j=1}^{\varkappa}\left(\mathbf{E}_{\lambda^{(j)}}(X)\right)^{r_j},\end{aligned} \tag{1.55}$$

where $\sum\limits_{\{r_1\lambda^{(1)}+\ldots+r_\varkappa\lambda^{(\varkappa)}=\nu\}}$ stands for the summation over all unordered collections of different integer nonnegative vectors $\lambda^{(j)}$, $|\lambda^{(j)}| > 0$, and all ordered collections of integer positive numbers r_j such that $r_1\lambda^{(1)} + \ldots + r_\varkappa\lambda^{(\varkappa)} = \nu$.

Let $I = \{t_1, \ldots, t_k\}$ be a set of indices of vector X. An unordered collection of disjoint nonempty sets I_p, such that $\bigcup\limits_p I_p = I$, is called a partition of I. In this notation, the relations (1.53) and (1.54) between simple moments and cumulants can be rewritten as

$$\mathbf{E}X_{t_1}\ldots X_{t_k} = \sum_{\bigcup\limits_{p=1}^{q} I_p=I} (q-1)! \prod_{p=1}^{q} \Gamma(X_{I_p}), \tag{1.56}$$

$$\Gamma(X_{t_1}, \ldots, X_{t_k}) = \sum_{\bigcup\limits_{p=1}^{q} I_p=I} (-1)^{q-1}(q-1)! \prod_{p=1}^{q} \mathbf{E}(X_{I_p}). \tag{1.57}$$

Here and further we use the following notation: if $I' = \{t'_1, \ldots, t'_m\} \subset I$ then

$$X_{I'} := (X_{t'_1}, \ldots, X_{t'_m}),$$
$$\mathbf{E}(X_{I'}) := \mathbf{E}X_{t'_1}\ldots X_{t'_m},$$
$$\Gamma(X_{I'}) := \Gamma(X_{t'_1}, \ldots, X_{t'_m}).$$

The formulas (1.53) – (1.57) were proposed by A.N. Shiryaev and V.P. Leonov (Shiryaev, Leonov, 1959).

In the investigation and estimation of *cumulants* $\Gamma_k(S_n)$, $\Gamma_k\big(S(T)\big)$ and $\Gamma_k(\zeta_n^{(p)})$ it will be more convenient for us to express $\Gamma(X_{t_1}, \ldots, X_{t_k})$ through *centered moments*

$$\widehat{\mathbf{E}}\,(X_{I'}) := \mathbf{E}X_{t'_1}\widehat{X_{t'_2}\ldots \widehat{X_{t'_{m-1}}\widehat{X}_{t'_m}}}, \tag{1.58}$$

where the sign " $\frown$ " over the r.v. ξ means the centering:

$$\widehat{\xi} := \xi - \mathbf{E}\xi. \tag{1.59}$$

Sometimes $\widehat{\mathbf{E}}\,(X_{I'})$ will be used instead of $\widehat{\mathbf{E}}\,X_{t_1}\ldots X_{t_m}$. We have

$$\widehat{\mathbf{E}}\,X_t = \mathbf{E}X_t, \quad \widehat{\mathbf{E}}\,X_sX_t = \mathbf{E}X_sX_t - \mathbf{E}X_s\cdot \mathbf{E}X_t,$$
$$\widehat{\mathbf{E}}\,X_{t_1}X_{t_2}X_{t_3} = \mathbf{E}X_{t_1}X_{t_2}X_{t_3} - \mathbf{E}X_{t_1}\mathbf{E}X_{t_2}X_{t_3} - \tag{1.60}$$
$$- \mathbf{E}X_{t_1}X_{t_2}\cdot \mathbf{E}X_{t_3} + \mathbf{E}X_{t_1}\mathbf{E}X_{t_2}\mathbf{E}X_{t_3}.$$

Since $\Gamma(X_{t_1}, \dots, X_{t_k})$ does not change at any permutation of $t_1, \dots, t_k$, in $\widehat{\mathbf{E}}\, X_{t_1} \dots X_{t_k}$ we assume $t_1 \leqslant t_2 \leqslant \dots \leqslant t_k$. The formula

$$\widehat{\mathbf{E}}\, X_{t_1} \dots X_{t_k} = \sum_{\nu=1}^{k} (-1)^{\nu-1} \sum_{\bigcup_{p=1}^{\nu} I_p = I}^{*} \prod_{p=1}^{\nu} \mathbf{E}(X_{I_p}), \tag{1.61}$$

where $\mathbf{E}(X_{I_p}) = \mathbf{E} X_{t_1^{(p)}} \dots X_{t_{k_p}^{(p)}}$ and the summation $\sum\limits_{\bigcup_{p=1}^{\nu} I_p = I}^{*}$ is taken over partitionings $\{I_1, \dots, I_\nu\}$ of the set I such that $\max I_p \leqslant \min I_{p+1}$, $1 \leqslant p \leqslant \nu - 1$, gives an explicit expression of a centered moment through moments.

Using the centered moments L. Heinrich has obtained important results for sums of m-dependent random processes and fields (Heinrich, 1982, 1985).

Let X_t, $t = 0, 1, 2, \dots$, be r.v., related to a Markov chain ξ_t with initial probability distribution $\mathbf{P}_0(B)$ and transition probabilities $\mathbf{P}_{s,t}(x, B)$, i.e. $X_t = g_t\big((\xi(t)\big)$, where $g_t(x)$ are measurable functions. Then it is easy to see that

$$\begin{aligned} \widehat{\mathbf{E}}\, X_{t_1} \dots X_{t_k} &= \underbrace{\int \dots \int}_{k} g_{t_1}(x_1) \mathbf{P}_{t_1}(dx_1) \times \\ &\times \prod_{j=2}^{k} g_{t_j}(x_j) \big(\mathbf{P}_{t_{j-i}, t_j}(x_{j-1}, dx_j) - \mathbf{P}_{t_j}(dx_j) \big), \end{aligned} \tag{1.62}$$

where $\mathbf{P}_t(B) := \mathbf{P}(\xi \in B)$.

In the case of independent r.v. $X_{t_1}, \dots, X_{t_k}$, $\widehat{\mathbf{E}}\, X_{t_1} \dots X_{t_k}$ does not vanish only if $t_1 = t_2 = \dots = t_k$. The same is true for $\Gamma(X_{t_1}, \dots, X_{t_k})$.

Let us introduce the formula, expressing $\Gamma(X_{t_1}, \dots, X_{t_k})$ through centered moments (Statulevičius, 1969, 1970, 1979).

LEMMA 1.1. *The representation**)

$$\Gamma(X_{t_1}, \dots, X_{t_k}) = \sum_{\nu=1}^{k} (-1)^{\nu-1} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \dots, I_\nu) \prod_{p=1}^{\nu} \widehat{\mathbf{E}}\, X_{I_p} \tag{1.63}$$

---

*) The proof of Lemma 1.1 is presented in Appendix 2. In terms of graphs a geometric interpretation of numbers $N_\nu(I_1, \dots, I_\nu)$ is given in the proof of the Theorems 4.7 – 4.14. Other representations and estimates for $N_\nu(I_1, \dots, I_\nu)$ are also given there.

is valid, where $\sum\limits_{\bigcup\limits_{p=1}^{\nu} I_p = I}$ *denotes the summation over all* ν*–block partitionings* $\{I_1, \dots, I_\nu\}$ *of the set* I*. The integers* $N_\nu(I_1, \dots, I_\nu)$,

$$0 \leqslant N_\nu(I_1, \dots, I_\nu) \leqslant (\nu - 1)!, \tag{1.64}$$

depend on $\{I_1, \dots, I_\nu\}$ *only, and if* $N_\nu(I_1, \dots, I_\nu) > 0$*, then*

$$\sum_{p=1}^{\nu} \max_{t_j, t_j \in I_p} (t_j - t_i) \geqslant \max_{1 \leqslant i,j \leqslant k} (t_j - t_i). \tag{1.65}$$

Thus, for example,

$$\begin{aligned}
&\Gamma(X_t) = \widehat{\mathbf{E}}\, X_t = \mathbf{E} X_t, \quad \Gamma(X_s, X_t) = \widehat{\mathbf{E}}\, X_s X_t,\\
&\Gamma(X_{t_1}, X_{t_2}, X_{t_3}) = \widehat{\mathbf{E}}\, X_{t_1} X_{t_2} X_{t_3} - \widehat{\mathbf{E}}\, X_{t_2} \widehat{\mathbf{E}}\, X_{t_1} X_{t_3},\\
&\Gamma(X_{t_1}, X_{t_2}, X_{t_3}, X_{t_4}) = \widehat{\mathbf{E}}\, X_{t_1} X_{t_2} X_{t_3} X_{t_4} - \mathbf{E} X_{t_2} \widehat{\mathbf{E}}\, X_{t_1} X_{t_3} X_{t_4} -\\
&\qquad - \mathbf{E} X_{t_3} \widehat{\mathbf{E}}\, X_{t_1} X_{t_2} X_{t_4} - \widehat{\mathbf{E}}\, X_{t_1} X_{t_3} \widehat{\mathbf{E}}\, X_{t_2} X_{t_4} -\\
&\qquad - \widehat{\mathbf{E}}\, X_{t_1} X_{t_4} \widehat{\mathbf{E}}\, X_{t_2} X_{t_3} + \mathbf{E} X_{t_2} \mathbf{E} X_{t_3} \widehat{\mathbf{E}}\, X_{t_1} X_{t_4}, \dots .
\end{aligned}$$

Let a r.v. Z_T depends on a parameter T and there exist all moments $\mathbf{E}|Z_T^k| < \infty$, $k \geqslant 1$. If

$$\Gamma_k(Z_T) \to \Gamma_k(Z), \qquad T \to \infty, \tag{1.66}$$

for every k, where Z is a r.v. for which Carleman's test of the moment problem

$$\sum_{k=1}^{\infty} (\mathbf{E} Z^{2k})^{-\frac{1}{2k}} = \infty \tag{1.67}$$

is fulfilled, then, since moments can be expressed through cumulants, we obtain that

$$Z_T \xrightarrow{D} Z \qquad \text{as } T \to \infty, \tag{1.68}$$

i.e. the r.v. Z_T converges to the r.v. Z in distribution:

$$F_{Z_T}(x) \to F_Z(x) \qquad (T \to \infty)$$

at each point of continuity of $F_Z(x)$.

For convergence to the (0, 1)-normal distribution under conditions $\mathbf{E}Z_T = 0$, $\mathbf{E}Z_T^2 = 1$ it is sufficient that

$$\Gamma_k(Z_T) \to 0 \quad (T \to \infty) \tag{1.69}$$

for each $k \geqslant 3$ (Leonov, 1964).

If we are interested not only in the convergence to the normal distribution but also in a more accurate asymptotic analysis (rate of convergence, asymptotical expansions, probabilities of large deviations) of distribution $\mathbf{P}_{Z_T}$ (any normed r.v., defined by one of the relations (1.36) – (1.39), may be taken as Z_T), then we must find the accurate upper estimate for $\Gamma_k(Z_T)$, both with respect to T and to $k \geqslant 3$. After that we can use general lemmas on the behaviour of $\mathbf{P}_{Z_T}$.

The next chapter is dedicated to obtaining of general statements on the behaviour of $\mathbf{P}_{Z_T}$ if we have some information on cumulants of the r.v. Z_T.

Note that for abbreviation the sign ∎ is used at the end of the proof.

CHAPTER 2
THE MAIN LEMMAS

Let us consider a r.v. $\xi = \xi_\Delta$, depending on the parameter Δ, with the distribution function $F_\xi(x) = \mathbf{P}(\xi < x)$, the mean $\mathbf{E}\xi = 0$ and the variance $\mathbf{D}\xi = \mathbf{E}\xi^2 = 1$. Let

$$\varphi_\xi(z) = \mathbf{E}\exp\{z\xi\}$$

be a generating function of the moments of the r.v. ξ.

We say that the r.v. ξ satisfies condition (S_γ), if there exist $\gamma \geqslant 0$ and $\Delta > 0$ such that

$$|\Gamma_k(\xi)| \leqslant \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \qquad k = 3, 4, \ldots . \qquad (S_\gamma)$$

Condition (S_γ), as $\gamma = 0$, ensures analyticity of the generating function $\varphi_\xi(z)$ in the domain $|z| < \Delta$; to be more exact, if

$$|\Gamma_k(\xi)| \leqslant \frac{k!\,H}{\Delta^{k-2}}, \qquad k = 3, 4, \ldots ,$$

for some $H > 0$, then

$$|\ln \varphi_\xi(z)| \leqslant \frac{Hz^2}{1-\varrho}, \qquad |z| \leqslant \varrho\Delta, \quad 0 < \varrho < 1.$$

Conversely, if

$$|\ln \varphi_\xi(z)|_{|z|=\Delta^*} \leqslant H_1 \Delta^{*2},$$

i.e.

$$\left|\frac{\ln \varphi_\xi(z)}{\ln \varphi_\eta(z)}\right|_{|z|=\Delta^*} \leqslant \frac{1}{2} H_1,$$

where η is a $(0, 1)$–normal r.v., then from the Cauchy formula we obtain

$$|\Gamma_k(\xi)| \leqslant \frac{k!\,H_1}{\Delta^{*k-2}}, \qquad k = 3, 4, \ldots .$$

2.1. General lemmas on the approximation of distribution of an arbitrary random variable by the normal distribution

Denote

$$\Delta_\gamma = c_\gamma \Delta^{\frac{1}{1+2\gamma}}, \qquad c_\gamma = \frac{1}{6}\left(\frac{\sqrt{2}}{6}\right)^{\frac{1}{1+2\gamma}}. \tag{2.1}$$

Let θ (with or without an index) denote some variable, not always the same, not exceeding 1 in absolute value, $[m]$ is the integer part of m.

LEMMA 2.1. *If for a r.v. ξ with $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 = 1$ condition (S_γ) is satisfied, then $\forall T$, $T \geqslant \Delta_\gamma$, the inequality*

$$\begin{aligned} \sup_x \bigl|F_\xi(x) - \Phi(x)\bigr| \leqslant \frac{3}{\sqrt{2\pi}} \biggl\{ \frac{3\cdot 6^\gamma}{\Delta} + 100\Delta_\gamma e^{-\frac{3}{2}\sqrt{\Delta_\gamma}} + \frac{1.5}{T} + \\ + \frac{1}{\sqrt{2\pi}} \int\limits_{\Delta_\gamma}^{T} \Bigl| f_\xi(t) - \exp\Bigl\{-\frac{1}{2}t^2\Bigr\}\Bigr| \frac{dt}{t} \biggr\} \end{aligned} \tag{2.2}$$

holds.

COROLLARY 2.1. *If for an arbitrary r.v. ξ condition (S_γ) is fulfilled, then*

$$\sup_x \bigl|F_\xi(x) - \Phi(x)\bigr| \leqslant \frac{18}{\Delta_\gamma}. \tag{2.3}$$

LEMMA 2.2 (Rudzkis, Saulis, Statulevičius, 1978). *Let a r.v. ξ with $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 = 1$ satisfy the condition*

$$\bigl|\Gamma_k(\xi)\bigr| \leqslant \frac{(k-2)!}{\Delta^{k-2}}, \qquad k = 3, 4, \ldots, s+2, \tag{S^*}$$

where $s \leqslant 2\Delta^2$. Then in the interval

$$0 \leqslant x < \sqrt{s}/(3\sqrt{e})$$

the relations of large deviations

$$\frac{1 - F_\xi(x)}{1 - \Phi(x)} = \exp\bigl\{\tilde{L}(x)\bigr\}\Bigl(1 + \theta_1 \tilde{f}(x)\frac{x+1}{\sqrt{s}}\Bigr),$$

$$\frac{F_\xi(-x)}{\Phi(-x)} = \exp\{\tilde{L}(-x)\}\left(1 + \theta_2 \tilde{f}(x)\frac{x+1}{\sqrt{s}}\right) \tag{2.4}$$

hold.

Here

$$\tilde{f}(x) = \frac{117 + 96\, s\, \exp\left\{-\frac{1}{2}(1 - 3\sqrt{e}\, x/\sqrt{s})s^{1/4}\right\}}{(1 - 3\sqrt{e}\, x/\sqrt{s})}, \tag{2.5}$$

$\tilde{L}(x) = \sum\limits_{k=0}^{\infty} \tilde{l}_k x^{k+3}$ *is a power series, converging in* $|x| < \sqrt{2}\Delta/(3\sqrt{e})$. *In this circle* $|\tilde{L}(x)| \leqslant 5|x|^3/(4\Delta)$. *The coefficients* $\tilde{l}_k$, $k = 0, 1, 2, \ldots$, *are expressed through* $r_k = \min\{k+3, s\}$ *first cumulants of the r.v.* ξ, *and for* $k \leqslant s-3$ *coincide with the coefficients of the classical Cramer – Petrov series* (Petrov, 1972).

LEMMA 2.3 (Rudzkis, Saulis, Statulevičius, 1978). *If for an arbitrary r.v.* ξ *with* $\mathrm{E}\xi = 0$ *and* $\mathrm{E}\xi^2 = 1$ *condition* (S_γ) *is satisfied, then in the interval*

$$0 \leqslant x < \Delta_\gamma$$

the relations of large deviations

$$\frac{1 - F_\xi(x)}{1 - \Phi(x)} = \exp\{L_\gamma(x)\}\left(1 + \theta_1 f(x)\frac{x+1}{\Delta_\gamma}\right),$$

$$\frac{F_\xi(-x)}{\Phi(-x)} = \exp\{L_\gamma(-x)\}\left(1 + \theta_2 f(x)\frac{x+1}{\Delta_\gamma}\right) \tag{2.6}$$

are valid.

Here

$$f(x) = \frac{60\left(1 + 10\Delta_\gamma^2 \exp\{-(1 - x/\Delta_\gamma)\sqrt{\Delta_\gamma}\}\right)}{(1 - x/\Delta_\gamma)}, \tag{2.7}$$

$$L_\gamma(x) = \sum_{3 \leqslant k < p} \lambda_k x^k + \theta(x/\Delta_\gamma)^3, \qquad p = \begin{cases} (1/\gamma) - 1, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases} \tag{2.8}$$

The coefficients λ_k *(expressed by cumulants of the r.v.* ξ*) coincide with the coefficients of the Cramer – Petrov series* (Petrov, 1972) *given by the formula*

$$\lambda_k = -b_{k-1}/k, \tag{2.9}$$

where b_k *are determined successively from the equations*

$$\sum_{r=1}^{j} \frac{1}{r!}\Gamma_{r+1}(\xi) \sum_{\substack{j_1 + \cdots + j_r = j \\ j_i \geqslant 1}} \prod_{i=1}^{r} b_{j_i} = \begin{cases} 1, & j = 1, \\ 0, & j = 2, 3, \ldots. \end{cases} \tag{2.10}$$

In particular,

$$\begin{aligned}
b_1 &= 1,\\
b_2 &= -\frac{1}{2}\,\Gamma_3(\xi),\\
b_3 &= -\frac{1}{6}\big(\Gamma_4(\xi) - 3\,\Gamma_3^2(\xi)\big),\\
b_4 &= -\frac{1}{24}\,\big(\Gamma_5(\xi) - 10\,\Gamma_3(\xi)\Gamma_4(\xi) + 15\,\Gamma_3^3(\xi)\big), \ \dots .
\end{aligned}$$

For the coefficients λ_k the estimate

$$|\lambda_k| \leqslant \frac{2}{k}\left(\frac{16}{\Delta}\right)^{k-2}\big((k+1)!\big)^{\gamma} \tag{2.11}$$

holds, and therefore

$$L_\gamma(x) \leqslant \frac{x^2}{2}\cdot\frac{x}{x+8\,\Delta_\gamma}, \quad L_\gamma(-x) \geqslant -\frac{x^3}{3\Delta_\gamma}.$$

LEMMA 2.4 (Bentkus, Rudzkis, 1980). *Let for an arbitrary r.v. ξ with $\mathbf{E}\xi = 0$ there exist $\gamma \geqslant 0$, $H > 0$ and $\bar\Delta > 0$ such that*

$$|\Gamma_k(\xi)| \leqslant \left(\frac{k!}{2}\right)^{1+\gamma}\frac{H}{\bar\Delta^{k-2}}, \qquad k = 2, 3, \dots . \tag{2.12}$$

Then for all $x \geqslant 0$

$$\mathbf{P}(\pm\xi \geqslant x) \leqslant \exp\left\{-\frac{x^2}{2\big(H + (x/\bar\Delta^{1/(1+2\gamma)})\big)^{(1+2\gamma)/(1+\gamma)}}\right\}. \tag{2.13}$$

COROLLARY. *Under conditions of Lemma 2.4 we have*

$$\mathbf{P}(\pm\xi \geqslant x) \leqslant \begin{cases} \exp\left\{-\dfrac{x^2}{4H}\right\}, & 0 \leqslant x \leqslant (H^{1+\gamma}\bar\Delta)^{1/(1+2\gamma)},\\[2ex] \exp\left\{-\dfrac{1}{4}(x\bar\Delta)^{1/(1+\gamma)}\right\}, & x \geqslant (H^{1+\gamma}\bar\Delta)^{1/(1+2\gamma)}. \end{cases} \tag{2.14}$$

If $x \leqslant (H^{1+\gamma}\bar\Delta)^{1/(1+2\gamma)}$, then $H \geqslant (x/\bar\Delta^{1/(1+2\gamma)})^{(1+2\gamma)/(1+\gamma)}$, and, therefore, the right-hand side of (2.13) does not exceed $\exp\{-x^2/2(H+H)\}$. The second line of (2.14) is obtained analogously.

2.2. Proof of lemmas 2.1 – 2.4

Proof of Lemma 2.2. Denote

$$\tilde{\varphi}(z) = \exp\left\{\sum_{k=2}^{s} \frac{1}{k!}\Gamma_k(\xi)z^k\right\} = 1 + \sum_{k=2}^{\infty} \frac{1}{k!}\tilde{m}_k z^k, \tag{2.15}$$

where $\tilde{m}_k = m_k = \mathbf{E}\xi^k,\ 1 \leqslant k \leqslant s,\ s \leqslant 2\Delta^2$ by (S^*). We have

$$\tilde{m}_k = k! \sum_{j=1}^{[k/2]} \frac{1}{j!} \sum_{\substack{k_1+\ldots+k_j=k \\ k_i=2,3,\ldots,s}} \frac{\Gamma_{k_1}\{\xi\}\Gamma_{k_2}\{\xi\}\ldots\Gamma_{k_j}\{\xi\}}{k_1!\,k_2!\,\ldots\,k_j!}. \tag{2.16}$$

Using (S^*),

$$\begin{aligned} |\tilde{m}_k| &\leqslant k! \sum_{j=1}^{[k/2]} \frac{1}{j!}\left(\sum_{i=2}^{s} \frac{1}{i(i-1)}\right)^j \Delta^{2j-k} \leqslant \frac{k!}{a^k(k)}, \\ a(k) &= \min\{\sqrt{k/2e},\ \Delta/\sqrt{e}\}. \end{aligned} \tag{2.17}$$

Let $h = h(x)$ be defined by the equation

$$x = \tilde{m}(h) := \frac{d}{dh}\ln\tilde{\varphi}(h) = \sum_{k=2}^{s} \frac{1}{(k-1)!}\Gamma_k(\xi)h^{k-1}, \tag{2.18}$$

$$\tilde{\sigma}^2(h) := \frac{d^2}{dh^2}\ln\tilde{\varphi}(h) = \sum_{k=2}^{s} \frac{1}{(k-2)!}\Gamma_k(\xi)h^{k-2}. \tag{2.19}$$

Assume that $0 \leqslant h \leqslant \delta a$, where $0 < \delta < 1$ and $a = (s/4e)^{1/2}$. Since $s \leqslant 2\Delta^2$, so $a \leqslant \Delta/\sqrt{2e}$ and $h < \Delta/\sqrt{2e}$. Then, making use of condition (S^*), we obtain

$$x = h\Big(1 + \theta\big(\sqrt{2e}\,h/(3\Delta)\big)\Big) = h\big(1 + \theta(\delta/3)\big), \tag{2.20}$$

$$\tilde{\sigma}^2(h) = 1 + \theta\big(3\sqrt{2e}\,h/(4\Delta)\big) = 1 + \theta(3\delta/4). \tag{2.21}$$

Consequently, $\tilde{\sigma}^2(h) > 0$ and in the equation $x = \tilde{m}(h)$ a single value h corresponds to each value x. In order that $0 \leqslant h < a$, x must satisfy $0 \leqslant x < 2a/3$. Let

$$F_h(y) := \int_{-\infty}^{\tilde{\sigma}(h)y+\tilde{m}(h)} \tilde{\varphi}^{-1}(h)g_h(u)\,dF_\xi(u), \tag{2.22}$$

$$g_h(y) := \sum_{k=0}^{s} \frac{1}{k!}(hy)^k + y^2 \sum_{k=s+1}^{\infty} \frac{1}{k!}\,\tilde{m}_k h^k := \tilde{e}(hy) + y^2\tilde{r}(h). \tag{2.23}$$

Then

$$1 - F_\xi(x) = \tilde{\varphi}(h) \int_0^\infty g_h^{-1}\big(\tilde{\sigma}(h)y + x\big)\, dF_h(y). \tag{2.24}$$

Let us estimate $\sup_y \big|F_h(y) - \Phi(y)\big|$. According to the definition of the function $F_h(y)$, its Fourier transform has the form

$$\begin{gathered} f_h(t) = \int_{-\infty}^{\infty} e^{ity}\, dF_h(y) = \exp\Big\{-\frac{itx}{\tilde{\sigma}(h)}\Big\}\tilde{\varphi}^{-1}(h)\times \\ \times \int_{-\infty}^{\infty} \exp\Big\{\frac{itu}{\tilde{\sigma}(h)}\Big\} g_h(u)\, dF_\xi(u). \end{gathered} \tag{2.25}$$

Set

$$\tilde{f}_h(t) = \frac{\exp\big\{-\frac{itx}{\tilde{\sigma}(h)}\big\}}{\tilde{\varphi}(h)} \int_{-\infty}^{\infty} g_z(u)\, dF_\xi(u) = \frac{\exp\big\{-\frac{itx}{\tilde{\sigma}(h)}\big\}\tilde{\varphi}(z)}{\tilde{\varphi}(h)}, \tag{2.26}$$

where $z = h + it/\tilde{\sigma}(h)$. Then

$$\Big|f_h(t) - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big| \leqslant \big|f_h(t) - \tilde{f}_h(t)\big| + \Big|\tilde{f}_h(t) - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big|. \tag{2.27}$$

First we evaluate $\Big|\tilde{f}_h(t) - \exp\big\{-\frac{1}{2}t^2\big\}\Big|$. Expanding $\ln \tilde{f}_h(t)$ by the Taylor formula in the neighbourhood of the point h $|z-h| \leqslant \delta_2 a$, $0 < \delta < \delta_2 < 1$, i.e. for $|t| \leqslant (\delta_2 - \delta)a\tilde{\sigma}(h)$, we find

$$\begin{aligned} \ln \tilde{f}_h(t) &= -itx/\tilde{\sigma}(h) - \ln\tilde{\varphi}(h) + \ln\tilde{\varphi}(z) = \\ &= -\frac{1}{2}t^2 + \theta\Big(|t|^3/\big(4a(1-3\delta/4)\tilde{\sigma}(h)\big)\Big), \end{aligned}$$

since

$$\begin{aligned} \frac{d^3 \ln\tilde{\varphi}(z)}{dz^3}\bigg|_{z=h+\theta it/\tilde{\sigma}(h)} &= \sum_{k=3}^{s} \frac{1}{(k-3)!}\,\Gamma_k(\xi) z^{k-3} = \\ &= \theta\big((1-|z|/\Delta)^2\Delta\big)^{-1} = \theta\big(24/(17a)\big) \end{aligned}$$

and $\tilde{\sigma}^3(h) \geqslant (1-3\delta/4)\tilde{\sigma}(h)$. Put $T = (\delta_2 - \delta)a\tilde{\sigma}(h)$. Then, for all $|t| \leqslant T$

$$\left|\tilde{f}_h(t) - \exp\left\{-\frac{1}{2}t^2\right\}\right| \leqslant \frac{|t|}{T}\left(\exp\left\{-\frac{t^2}{4}\right\} - \exp\left\{-\frac{t^2}{2}\right\}\right). \tag{2.28}$$

It remains to evaluate $|f_h(t) - \tilde{f}_h(t)|$. In view of the definitions of $f_h(t)$ and $\tilde{f}_h(t)$ by (2.25) and (2.26) we have

$$|f_h(t) - \tilde{f}_h(t)| \leqslant I_1 + I_2, \tag{2.29}$$

where

$$I_1 = \tilde{\varphi}^{-1}(h)\left|\int_{-\infty}^{\infty} \left(e^{itu/\tilde{\sigma}(h)}\tilde{e}(hu) - \tilde{e}(zu)\right) dF_\xi(u)\right|,$$

$$I_2 = \tilde{\varphi}^{-1}(h)\left|\int_{-\infty}^{\infty} \left(e^{itu/\tilde{\sigma}(h)}r(h) - r(z)\right) dF_\xi(u)\right|.$$

Taking into account that $\tilde{\varphi}(h) \geqslant 1$, $r(z) = \sum_{k=s+1}^{\infty} \frac{1}{k!}\tilde{m}_k z^k$ and the fact that in (2.17) for $\tilde{m}_k$ the quantity $a(k) \geqslant \sqrt{2}\,a$ as $k > s$, we find

$$I_2 \leqslant 2\sum_{k=s+1}^{\infty} \frac{1}{k!}|\tilde{m}_k|\,|z|^k \leqslant 2\sum_{k=s+1}^{\infty} (\delta_2/\sqrt{2})^k. \tag{2.30}$$

Let us estimate the I_1. We have

$$I_1 \leqslant I_1^{(1)} + I_1^{(2)}, \tag{2.31}$$

where

$$I_1^{(1)} = \left|\int_{-b}^{b} \left(e^{itu/\tilde{\sigma}(h)}\tilde{e}(hu) - \tilde{e}(zu)\right) dF_\xi(u)\right|,$$

$$I_1^{(2)} = \left|\int_{|u|>b} \left(e^{itu/\tilde{\sigma}(h)}\tilde{e}(hu) - \tilde{e}(zu)\right) dF_\xi(u)\right|,$$

$$b := 4a = 4(s/4e)^{1/2}, \qquad z = h + it/\tilde{\sigma}(h).$$

Then, having in mind that $\tilde{e}(zu) = e^{zu} - \sum_{k=s+1}^{\infty} \frac{1}{k!}(zu)^k$, we have

$$I_1^{(1)} \leqslant 2\sum_{k=s+1}^{\infty} \frac{|z|^k}{k!}\int_{-b}^{b} |u|^k\, dF_\xi(u) \leqslant 2\sum_{k=s+1}^{\infty} \frac{(b|z|)^k}{k!} \leqslant 2\sum_{k=s+1}^{\infty} \delta_2^k, \tag{2.32}$$

since $b|z| \leqslant 4a^2\delta_2 = \delta_2 s/e$ and $k! > (s/e)^k$ for $k > s$. In order to evaluate the integral $I_1^{(2)}$, consider

$$m_k(b) := \int\limits_{|u|>b} |u|^k \, dF_\xi(u).$$

Making use of estimate (2.17),for $\tilde{m}_k$ we get

$$\frac{m_k(b)}{k!} = \begin{cases} a^{-k}(k) \leqslant a^{-k}, & s/2 < k \leqslant s, \\ \dfrac{(2k)!}{k! a^{2k}(2k)b^k} \leqslant \dfrac{\sqrt{2}}{a^k}, & b < k \leqslant s/2, \\ \dfrac{m_{[b]}(b)}{[b]!} \leqslant \dfrac{\sqrt{2}}{a^{[b]}}, & 0 \leqslant k \leqslant b. \end{cases} \tag{2.33}$$

Hence, under the assumption that $\delta_2 \geqslant 1/\sqrt{a}$, we obtain

$$\begin{aligned} I_1^{(2)} &\leqslant 2\Big(\sum_{k=0}^{[b]} \sqrt{2}\,(\delta_2 a)^k a^{-[b]} + \sqrt{2} \sum_{k=[b]+1}^{\infty} \delta_2^k\Big) \leqslant \\ &\leqslant 2\sqrt{2}\Big(\sum_{k=0}^{[b]} \delta_2^{2[b]-k} + \sum_{k=[b]+1}^{\infty} \delta_2^k\Big). \end{aligned} \tag{2.34}$$

The relations (2.29) to (2.32) and (2.34) enable us to conclude that for all $|t| \leqslant T$

$$|f_h(t) - \tilde{f}_h(t)| \leqslant l(\delta_2), \qquad l(\delta) = 4\sqrt{2}\,\delta^{[4a]}/(1-\delta). \tag{2.35}$$

Let us consider $f_h(t)$ in the neighbourhood of 0. We have

$$f_h(t) = 1 + it \int\limits_{-\infty}^{\infty} y \, dF_h(y) + \theta \frac{t^2}{2} \int\limits_{-\infty}^{\infty} y^2 \, |dF_h(y)|. \tag{2.36}$$

Further,

$$\int_{-\infty}^{\infty} y\,dF_h(y) = \big(\tilde{\varphi}(h)\tilde{\sigma}(h)\big)^{-1}\int_{-\infty}^{\infty}(t-x)g_h(t)\,dF_\xi(t) = -x/\tilde{\sigma}(h)+$$

$$+\big(\tilde{\varphi}(h)\tilde{\sigma}(h)\big)^{-1}\Big\{\sum_{k=1}^{s-1}\frac{1}{(k-1)!}m_k h^{k-1}+m_3 r(h)\Big\} = -x/\tilde{\sigma}(h)+$$

$$+\big(\tilde{\varphi}(h)\tilde{\sigma}(h)\big)^{-1}\Big\{\sum_{k=1}^{\infty}\frac{\tilde{m}_k h^{k-1}}{(k-1)!}-\sum_{k=s+1}^{\infty}\frac{\tilde{m}_k h^{k-1}}{(k-1)!}+m_{s+1}h^s/s!+m_3 r(h)\Big\} =$$

$$= -x/\tilde{\sigma}(h)+\big(\tilde{\varphi}(h)\tilde{\sigma}(h)\big)^{-1}\{-dr(h)/dh+d\tilde{\varphi}(h)/dh+m_{s+1}h^s/s!+m_3 r(h)\} =$$

$$= \theta\cdot\frac{2}{a}\left\{\frac{(s+1)\delta^s}{(\sqrt{2})^{s+1}(1-\delta/\sqrt{2})^2}+\frac{(s+1)\delta^s}{(\sqrt{2})^{s+1}}+\frac{\delta^{s+1}}{\sqrt{2e}\,(\sqrt{2})^{s+1}(1-\delta/\sqrt{2})}\right\} =$$

$$= \theta\cdot(0.02\,\delta^s/a)$$

for $s \geqslant 30$. Consequently,

$$\int_{-\infty}^{\infty} y\,dF_h(y) = \theta\cdot(0.02\,\delta^s/a), \qquad s\geqslant 30. \tag{2.37}$$

Now we have to evaluate $\int_{-\infty}^{\infty} y^2\,|dF_h(y)|$. It is easy to see that

$$|dF_h(y)| \leqslant \Big\{\tilde{e}\big(h\tilde{\sigma}(h)y+hx\big)+\big(\tilde{\sigma}(h)y+x\big)^2\,|r(h)|\Big\}\tilde{\varphi}^{-1}(h)\,dF_\xi\big(\tilde{\sigma}(h)y+x\big).$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} y^2\,|dF_h(y)| &\leqslant \big(\tilde{\varphi}(h)\tilde{\sigma}^2(h)\big)^{-1}\int_{-\infty}^{\infty}(t-x)^2\big(\tilde{e}(ht)+t^2|r(h)|\big)\,dF_\xi(t) = \\ &= J_1+J_2,\end{aligned} \tag{2.38}$$

where

$$J_1 = \big(\tilde{\varphi}(h)\tilde{\sigma}^2(h)\big)^{-1}\int_{-\infty}^{\infty}(t-x)^2\tilde{e}(ht)\,dF_\xi(t),$$

$$J_2 = \big(\tilde{\varphi}(h)\tilde{\sigma}^2(h)\big)^{-1}(m_4-2xm_3+x^2)|r(h)|.$$

We have

$$J_1 = \big(\tilde\varphi(h)\tilde\sigma^2(h)\big)^{-1}\Big\{\sum_{k=2}^{\infty}\frac{\tilde m_k h^{k-2}}{(k-2)!} - 2x\sum_{k=1}^{\infty}\frac{\tilde m_k h^{k-1}}{(k-1)!} +$$

$$+ x^2\sum_{k=0}^{\infty}\frac{\tilde m_k h^k}{k!} - \sum_{k=s+1}^{\infty}\frac{\tilde m_k h^{k-2}}{(k-2)!} - \sum_{k=s+1}^{s+2}\frac{m_k h^{k-2}}{(k-2)!} +$$

$$+ 2x\sum_{k=s+1}^{\infty}\frac{\tilde m_k h^{k-1}}{(k-1)!} - 2x\frac{m_{s+1}h^s}{s!} - x^2\sum_{k=s+1}^{\infty}\frac{\tilde m_k h^k}{k!}\Big\} =$$

$$= \frac{\tilde\varphi''(h) - 2x\tilde\varphi'(h) + x^2\tilde\varphi(h)}{\tilde\varphi(h)\tilde\sigma^2(h)} + \theta\big(\tilde\varphi(h)\tilde\sigma^2(h)\big)^{-1}\times$$

$$\times\Big\{\frac{(s+1)(s+2)h^s}{(\sqrt2\,a)^{s+2}} + \frac{s(s+1)h^{s-1}}{(\sqrt2\,a)^{s+1}} + \frac{2x(s+1)h^s}{(\sqrt2\,a)^{s+1}} - \frac{s(s+1)h^{s-1}}{(\sqrt2\,a)^{s+1}(1-h/\sqrt2\,a)^3} +$$

$$+ \frac{2x(s+1)h^s}{(\sqrt2\,a)^{s+1}(1-h/\sqrt2\,a)^2} + \frac{x^2h^{s+1}}{(\sqrt2\,a)^{s+1}(1-h/\sqrt2\,a)}\Big\} = 1 + \theta(2\delta^{s-1}/5) \qquad (2.39)$$

and

$$\begin{aligned}|J_2| &\leqslant 4|r(h)|(4 + 2x/\Delta + x^2) \leqslant\\ &\leqslant (\delta_2/\sqrt2)^{s+1}(1-\delta/\sqrt2)^{-1}(16 + 8x/\Delta + 4x^2) \leqslant \delta^{s+1}/32.\end{aligned} \qquad (2.40)$$

The relations (2.38) to (2.40) allow us to state that

$$\int_{-\infty}^{\infty} y^2\,|dF_h(y)| = 1 + \theta(\delta^{s-1}/2). \qquad (2.41)$$

Consequently, on the ground of the relations (2.36), (2.37) and (2.41) we obtain

$$f_h(t) = 1 + \theta\big\{0.02\,\delta^s|t| + (1 + \delta^{s-1}/2)(1/2)t^2\big\}.$$

Hence it follows that

$$\Big|f_h(t) - \exp\Big\{-\frac12 t^2\Big\}\Big| \leqslant 0.02\,\delta^s|t| + (1 + \delta^{s-1}/4)t^2. \qquad (2.42)$$

By employing the relations (2.27), (2.28), (2.35) and (2.42) we get

$$\begin{aligned}&\Big|f_h(t) - \exp\Big\{-\frac12 t^2\Big\}\Big| \leqslant \min\Big\{0.02\,\delta^s|t| + (1 + \delta^{s-1}/4)t^2,\\ &\frac{|t|}{T}\Big(\exp\Big\{-\frac14 t^2\Big\} - \exp\Big\{-\frac12 t^2\Big\}\Big) + l(\delta_2)\Big\}.\end{aligned} \qquad (2.43)$$

In order to estimate $\sup_y |F_h(y) - \Phi(y)|$ we shall make use of the following lemma.

LEMMA 2.5 (Zolotarev, 1965). *Let $L(x)$ be a distribution function, $H(x)$ be a function of bounded variation such that*

$$q = \sup_x |H'(x)| < \infty, \qquad H(-\infty) = 1 - H(\infty) = 0,$$

and $l(t)$, $h(t)$ be the Fourier – Stieltjes transforms corresponding to these functions. Further, let $p(x)$ be a density of a symmetric distribution and $\omega(t)$ its characteristic function. Put

$$\Delta = \sup_x |L(x) - H(x)|, \qquad \delta = l - h,$$

$$S(x) = x \int_0^x p(u)\, du, \qquad Q(T) = \frac{T}{2\pi q} \int_0^\infty |\omega(t)\delta(tT)| \frac{dt}{t},$$

and define A as a positive solution of the equation

$$4S(x) = x.$$

Then for all $T > 0$ and $x > A$ the inequality

$$\Delta \leqslant \frac{2qx\big(S(x) + Q(T)\big)}{T\big(4S(x) - x\big)} \tag{2.44}$$

holds.

Choose $p(x) = (1 - \cos x)/(\pi x^2)$. The characteristic function

$$\omega(t) = \begin{cases} 1 - |t|, & |t| \leqslant 1, \\ 0, & |t| > 1, \end{cases}$$

corresponds to this distribution. Making use of inequality (2.44), for any $\mu > A$ we have

$$\sup_y \big|F_h(y) - \Phi(y)\big| \leqslant \frac{2}{\sqrt{2\pi}} \frac{\mu\big(S(\mu) + Q(T)\big)}{T\big(4S(\mu) - \mu\big)}, \tag{2.45}$$

where $S(\mu) = \mu \int_0^\mu (1 - \cos u)/(\pi u^2)\, du$, A is a positive root of the equation $4S(\mu) = \mu$ and

$$Q(T) = \frac{T}{\sqrt{2\pi}} \int_0^T \Big(1 - \frac{|t|}{T}\Big) \Big| f_h(t) - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big| \frac{dt}{t}.$$

Set $t_0 = \min\{1, \sqrt{l(\delta_2)}\}$. Then

$$Q(T) \leqslant \frac{T}{\sqrt{2\pi}} \int_0^{t_0} \{0.02\,\delta^s t + (1+\delta^{s-1}/4)t^2\}\,\frac{dt}{t} + Tl(\delta_2)/\sqrt{2\pi} \int_{t_0}^{\max\{1,T\}} \frac{dt}{t} +$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{t_0}^{\max\{1,T\}} \Big(\exp\Big\{-\frac{1}{4}t^2\Big\} - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big)\,dt \leqslant$$

$$\leqslant \frac{T}{\sqrt{2\pi}}\Big\{l(\delta_2)\big(\ln(\max\{1,T\}) - \ln t_0\big) + 0.02\,\delta^s t_0 +$$

$$+ (1/2)(1+\delta^{s-1}/4)t_0^2 + (\sqrt{2}-1)/\sqrt{2}\Big\} \leqslant (T/\sqrt{2\pi})l(\delta_2)\Big\{0.02\,\delta^s t_0/l(\delta_2) +$$

$$+ (1+\delta^{s-1}/4)\big(t_0^2/l(\delta_2)\big) + \ln(3a/2) - 2a\ln\delta_2\Big\} + 0.23 \leqslant$$

$$\leqslant (T/\sqrt{2\pi})l(\delta_2)(\ln 2.1\sqrt{s} - 2a\ln\delta_2) + 0.23. \tag{2.46}$$

Let $\mu = 3.55$. Then it is easy to see that $S(\mu) \leqslant 1.28\,\mu/\pi$. Recalling the inequalities (2.45) and (2.46) we find

$$\sup_y |F_h(y) - \Phi(y)| \leqslant \frac{d}{\sqrt{2\pi}}, \tag{2.47}$$

where

$$d = 8.5/\big((\delta_2-\delta)a\big) + \big(9.1\,\delta_2^{[4a]}/(1-\delta_2)\big)(\ln 2.1\sqrt{s} - \sqrt{s/e}\,\ln\delta_2). \tag{2.48}$$

Considering (2.24) we obtain

$$1 - F_\xi(x) = \tilde{\varphi}(h)\int_0^\infty g_h^{-1}\big(\tilde{\sigma}(h)y + x\big)\,dF_h(y) = \tilde{\varphi}(h)\big(I_1(h) + I_2(h)\big). \tag{2.49}$$

Here

$$\begin{aligned} I_1(h) &= \int_0^\infty g_h^{-1}\big(\tilde{\sigma}(h)y + x\big)\,d\big(F_h(y) - \Phi(y)\big), \\ I_2(h) &= \int_0^\infty g_h^{-1}\big(\tilde{\sigma}(h)y + x\big)\,d\Phi(y). \end{aligned} \tag{2.50}$$

Now we evaluate the integral $I_1(h)$. The definition of the function $g_h(y)$ by (2.23) yields

$$g_h\big(\tilde\sigma(h)y+x\big)=\tilde e\Big(h\big(\tilde\sigma(h)y+x\big)\Big)+\big(\tilde\sigma(h)y+x\big)^2r(h)=$$
$$=\tilde e\Big(h\big(\tilde\sigma(h)y+x\big)\Big)\Big(1+\theta\big(2|r(h)|/h^2\big)\Big),$$

where

$$\frac{|r(h)|}{h^2}=\Big|\sum_{k=s+1}^{\infty}\frac{\tilde m_k h^{k-2}}{k!}\Big|=\theta\sum_{k=s+1}^{\infty}\frac{h^{k-2}}{(\sqrt2\,a)^k}=\theta 2(\delta/\sqrt2)^{s-1}.$$

Therefore the function $g_h^{-1}(y)\big(\tilde\sigma(h)y+x\big)$ of y in the interval $[0,\infty[$ is nonnegative and monotonically decreases. Consequently,

$$I_1(h)\leqslant\big(2\sup_y|F_h(y)-\Phi(y)|\big)\Big/\Big(\tilde e(hx)\big(1-(\delta/\sqrt2)^{s-5}\big)\Big)\leqslant$$
$$\leqslant 2\,d\exp\{-hx\}\Big/\Big(\sqrt{2\pi}\big(1-(\delta/\sqrt2)^{s-5}\big)\big(1-(\delta/6)^{s+1}\big)\Big),$$

since

$$\tilde e(hx)=e^{hx}\big(1+\theta(hx)^{s+1}/(s+1)!\big)=e^{hx}\big(1+(\delta/6)^{s+1}\big).$$

Hence, having in mind that for all $x\geqslant 0$

$$\big(1-\Phi(x)\big)\exp\left\{\frac12 x^2\right\}\geqslant 3.1/\big(4\sqrt{2\pi}\,(x+1)\big),$$

we obtain

$$I_1(h)\leqslant(8/3)d\exp\{-hx+x^2/2\}\big(1-\Phi(x)\big)(x+1). \tag{2.51}$$

It remains to evaluate the integral $I_2(h)$. According to (2.50),

$$I_2(h)=\int_0^{b_1}g_h^{-1}\big(\tilde\sigma(h)y+x\big)\,d\Phi(y)+\int_{b_1}^{\infty}g_h^{-1}\big(\tilde\sigma(h)y+x\big)\,d\Phi(y),$$

where $b_1=(b-x)/\tilde\sigma(h)\geqslant 5\,a/3$. Then for $0\leqslant y\leqslant b_1$

$$g_h\big(\tilde\sigma(h)y+x\big)=\exp\{-hx-h\tilde\sigma(h)y\}\Big(\big(1+\theta(\delta/\sqrt2)^{s-1}\big)\big(1+\theta\delta^{s+1}\big)\Big)^{-1}$$

and

$$
\begin{aligned}
I_2(h) = &\int_0^{b_1} \exp\big\{-h\big(x+\tilde\sigma(h)y\big)\big\}\, d\Phi(y)\times \\
&\times\Big(\big(1+\theta(\delta/\sqrt2)^{s-5}\big)\big(1+\theta\delta^{s+1}\big)\Big)^{-1}+ \\
&+\theta\int_{b_1}^{\infty} e^{-hx}\, d\Phi(y)\Big(\big(1-(\delta/6)^{s+1}\big)\big(1-(\delta/\sqrt2)^{s-5}\big)\Big)^{-1} = \\
&= \frac{\exp\big\{-hx+\frac12\big(h\tilde\sigma(h)\big)^2\big\}\Big(1-\Phi\big(h\tilde\sigma(h)\big)\Big)+\theta\exp\big\{-\frac12 b_1^2\big\}\big/\sqrt{2\pi}}{\big(1+\theta\delta^{s+1}\big)\big((1+\theta(\delta/\sqrt2)^{s-5}\big)}.
\end{aligned}
\tag{2.52}
$$

Further,

$$
\begin{aligned}
h\tilde\sigma^2(h)-x &= \sum_{k=3}^{s}\left(\frac{1}{(k-2)!}-\frac{1}{(k-1)!}\right)\Gamma_k\{\xi\}h^{k-1} = \\
&= \theta\sum_{k=3}^{s}\frac{k-2}{k-1}\,h(h/\Delta)^{k-2} = \theta\cdot(3/5)\,\delta h.
\end{aligned}
$$

Taking into consideration that $x = h(1+\theta\cdot\delta/3)$, we find

$$
h\tilde\sigma(h) = x\Big(1+\theta\cdot 3\delta/\big(5(1-\delta/3)\big)\Big) = x(1+\theta\cdot 0.09\,\delta). \tag{2.53}
$$

Set $\psi(y) = \exp\left\{\frac12 y^2\right\}\big(1-\Phi(y)\big)$, $\quad q(y) = y\psi(y)$. It is easy to see that the function $\psi(y)$ decreases while $q(y)$ increases in the interval $[0,\infty[$ and therefore for any $y>0$ and $z\in\,]-y,\,y[$ we have

$$
\psi(y+z) = \psi(y)y/(y+\theta z).
$$

Consequently, taking into account relations (2.52), (2.53) and the inequality $4b_1^2 \leqslant s$, we get

$$
I_2(h) = \exp\left\{-hx+\frac12 x^2\right\}\big(1-\Phi(x)\big)\left(1+\theta\frac{2\delta+(4/3)(x+1)e^{-s/8}}{1-\delta}\right). \tag{2.54}
$$

Let us estimate the quantity d. For this, take $\delta_2 = 1-(1-\delta)/(2\,s^{1/4})$. Then for $s\geqslant 30$, $\quad \delta_2\geqslant 1/\sqrt{a}$ we have $\delta_2-\delta = 1-1/(2s^{1/4})$ and $1-\delta_2 = (1-\delta)/(2s^{1/4})$. Consequently,

$$
\delta_2^{[4\,a]} \leqslant \big(1-(1-\delta)/(2s^{1/4})\big)^{\sqrt{s}}(1/\delta_2) \leqslant \exp\left\{-\frac12(1-\delta)s^{1/4}\right\}\delta_2^{-1}.
$$

Recalling the definition (2.48) of d, we have

$$d \leqslant \Big(8.5/a + 18.2\, s^{1/4} \exp\Big\{-\frac{1}{2}(1-\delta)s^{1/4}\Big\} \times$$
$$\times \big(\ln 2.1\sqrt{s} - \sqrt{s/e}\ln(1 - s^{1/4}/2)\big)\Big)\big((1-\delta)(1 - s^{1/4}/2)\big)^{-1}.$$

Since $\ln(1 - s^{1/4}/2) \geqslant -(2s^{1/4}-1)^{-1}$,

$$\begin{aligned} d &\leqslant 12/(a(1-\delta)) + \Big(35\sqrt{s}\exp\Big\{-\frac{1}{2}(1-\delta)s^{1/4}\Big\}\Big)\Big/(1-\delta) \leqslant \\ &\leqslant \Big(40 + 35\, s \exp\Big\{-\frac{1}{2}(1-\delta)s^{1/4}\Big\}\Big)\Big/\big(\sqrt{s}\,(1-\delta)\big). \end{aligned} \tag{2.55}$$

Having in mind that $\delta \leqslant 3x/(2a) < 5x/\sqrt{s}$, from (2.51), (2.55) and (2.54) we obtain

$$I_1(h) + I_2(h) = \exp\Big\{-hx + \frac{1}{2}x^2\Big\}(1 - \Phi(x)) \times$$
$$\times \Big\{1 + \theta \frac{117 + 96\, s\, \exp\big\{-\frac{1}{2}(1 - 3\sqrt{e}\,x/\sqrt{s})s^{1/4}\big\}}{(1 - 3\sqrt{e}\,x/\sqrt{s})} \frac{x+1}{\sqrt{s}}\Big\}.$$

Thus, in accordance with relation (2.49),

$$\frac{1 - F_\xi(x)}{1 - \Phi(x)} = \exp\{\tilde{L}(x)\}\Big(1 + \theta \tilde{f}(x)\frac{x+1}{\sqrt{s}}\Big) \tag{2.56}$$

for all $0 \leqslant x < (2/3)\sqrt{s/(4e)}$, where

$$\tilde{L}(x) = \frac{1}{2}x^2 + \ln\tilde{\varphi}(h) - hx, \tag{2.57}$$

$$\tilde{f}(x) = \frac{117 + 96\, s\, \exp\big\{-\frac{1}{2}(1 - 3\sqrt{e}\,x/\sqrt{s})s^{1/4}\big\}}{(1 - 3\sqrt{e}\,x/\sqrt{s})}. \tag{2.58}$$

The function $h = h(x)$ can be expanded in the series $h = x + \sum_{k=2}^{\infty} b_k x^k$, converging for $|x| < \sqrt{2}\Delta/(3\sqrt{e})$. Since $|h(z)|_{|z|=2\Delta/(3\sqrt{2e})} \leqslant \Delta/\sqrt{2e}$, by the Cauchy inequality one has

$$b_k = \theta\cdot(\Delta/\sqrt{2e})\big(3\sqrt{2e}/(2\Delta)\big)^k = \theta\cdot(3/2)^k(\sqrt{2e}/\Delta)^{k-1}, \qquad k \geqslant 2. \tag{2.59}$$

The coefficients b_k are expressed by $r_k = \min\{k+3,\, s\}$ first cumulants of the r.v. ξ. Further, it is easy to see that

$$\frac{d}{dx}\tilde{L}(x) = x - h.$$

Hence $\tilde{L}(x) = \sum\limits_{k=0}^{\infty} \tilde{l}_k x^{k+3}$, where

$$\tilde{l}_k = -b_{k+2}/(k+3) = \theta \frac{1.5^{k+2}(\sqrt{2e})^{k+1}}{(k+3)\Delta^{k+1}}, \qquad k = 0,\ 1,\ 2,\ \dots . \tag{2.60}$$

By employing (2.20) we have $h = h(z) = z/\big(1+\theta(\delta/3)\big)$, $\delta = \sqrt{2e}|h|/\Delta = 3\sqrt{e}/(\sqrt{2}\,\Delta) < 1$ and

$$\tilde{L}(z) = \frac{1}{2}(z-h)^2 + \sum_{k=3}^{s} \frac{1}{k!}\Gamma_k(\xi) z^k =$$

$$= \frac{1}{2}z^2\Big(1 - \big(1+\theta(\delta/3)\big)^{-1}\Big) + (9|z|^2/4)\sum_{k=1}^{\infty}(\delta/\sqrt{2e})^k/\big((k+1)(k+2)\big),$$

$$\tilde{L}(z) = \theta \cdot 5|z|^3/(4\Delta).$$

Further, for $0 \leqslant x < \sqrt{2}\,\Delta/(3\sqrt{e})$

$$\tilde{L}(x) = \inf_{0 \leqslant h < \Delta/\sqrt{2e}} \left\{ \frac{1}{2}(x-h)^2 + \sum_{k=3}^{s} \frac{1}{k!}\Gamma_k(\xi) h^k \right\}.$$

Put $h = \Delta x/(x+\Delta)$. Making use of condition (S^*) we obtain the estimate

$$|\tilde{L}(x)| \leqslant \frac{1}{2}x^2\big(x/(x+2\Delta)\big).$$

It completes the proof of Lemma 2.2, taking into account that for $s \leqslant 30$ the assertion of the lemma is trivial. ■

Proof of Lemma 2.3. First we show that condition (S_γ): $|\Gamma_k(\xi)| \leqslant (k!)^{1+\gamma}/\Delta^{k-2}$, $k = 2, 3, \dots$, yields the inequality

$$|\Gamma_k(\xi)| \leqslant (k-2)!/\Delta_s^{k-2}, \qquad k = 3,\ 4,\ \dots,\ s+2, \tag{S^*}$$

where s is even and does not exceed $2\Delta_s^2$. For $s \geqslant 4$ the estimate

$$(k!)^{1+\gamma}/\Delta^{k-2} \leqslant (k-2)!\big(6(s+2)^\gamma/\Delta\big)^{k-2}, \qquad k = 3,\ 4,\ \dots,\ s+2, \tag{2.61}$$

holds. Therefore, put $\Delta_s = \Delta/\big(6(s+2)^\gamma\big)$. Then for all even s, satisfying the inequalities

$$4 \leqslant s \leqslant \Delta^2/\big(18(s+2)^{2\gamma}\big), \tag{2.62}$$

(S_γ) implies (S^*). Consider the number

$$s = 2[(1/2)(\Delta^2/18)^{1/(1+2\gamma)}] - 2, \tag{2.63}$$

which is even and satisfies inequalities (2.62). We assume $\Delta > 10^{1+2\gamma}$, because in the opposite case the assertion of Lemma 2.2 is trivial. Directly from (2.62) we get

$$0.95(\sqrt{2}\Delta/6)^{1/(1+2\gamma)} < \sqrt{s} < (\sqrt{2}\Delta/6)^{1/(1+2\gamma)}. \tag{2.64}$$

Using Lemma 2.2 in the interval $0 \leqslant x < \Delta_\gamma$, where $\Delta_\gamma = \big((0.95/(3\sqrt{e})\big) \times (\sqrt{2}\Delta/6)^{1/(1+2\gamma)}$, we have

$$\frac{1-F_\xi(x)}{1-\Phi(x)} = \exp\{\tilde{L}(x)\}\left(1+\theta f(x)\frac{x+1}{\Delta_\gamma}\right), \tag{2.65}$$

where

$$f(x) = \big(24 + 500\,\Delta_\gamma^2 \cdot \exp\{-(1-x/\Delta_\gamma)\sqrt{\Delta_\gamma}\}\big)(1-x/\Delta_\gamma)^{-1}.$$

To complete the proof of Lemma 2.3 we have to truncate the series $\tilde{L}(x)$. Set

$$\tilde{L}(x) = L_1(x) + \tilde{L}_2(x),$$

$$L_1(x) = \sum_{k=0}^{s-3} l_k x^{k+3}, \qquad \tilde{L}_2(x) = \sum_{k=s-2}^{\infty} \tilde{l}_k x^{k+3},$$

where the coefficients l_k are the same as the corresponding coefficients of the Cramer - Petrov series. We shall use estimate (2.60) for $\tilde{l}_k$. Condition (S^*) under the assumptions of Lemma 2.3 is fulfilled as $\Delta = \Delta_s = \Delta/\big(6(s+2)^\gamma\big)$,where s is defined by (2.63). Then

$$\tilde{l}_k = \theta(1.5)^{k+2}(\sqrt{2e})^{k+1}/\big((k+3)\Delta_s^{k+1}\big)$$

and

$$\begin{aligned} \tilde{L}_2(x) &= \theta\big(3x^2/2(s+1)\big)\sum_{k=s-2}^{\infty}\big(3\sqrt{2e}\,x/(2\Delta_s)\big)^{k+1} = \\ &= \theta(3x^2/2s)\big(3\sqrt{2e}\,x/(2\Delta_s)\big)^{s-1}(1-3\sqrt{2e}\,x/\Delta_s)^{-1} = \\ &= \theta\cdot 0.05(x/\Delta_\gamma)^3. \end{aligned} \tag{2.66}$$

Next, let us evaluate the coefficients l_k. They are expressed through $(k+3)$ first cumulants of the r.v. ξ. It follows from condition (S_γ) that

$$|\Gamma_j(\xi)| \leqslant (j!)^{1+\gamma}/\Delta^{j-2}, \qquad j = 3,\ 4,\ \ldots,\ k+3.$$

Let us find Δ_k such that

$$(j!)^\gamma/\Delta^{j-2} \leqslant 1/\Delta_k^{j-2}, \qquad \text{i.e. } \Delta_k \leqslant \Delta/(j!)^{\gamma/(j-2)}.$$

Taking into consideration that $(j!)^{1/(j-2)} \leqslant \big((k+4)!\big)^{1/(k+1)}$, choose $\Delta_k = \Delta/\big((k+4)!\big)^{\gamma/(k+1)}$. Then

$$l_k = \theta\cdot 2(16/\Delta)^{k+1}\big((k+4)!\big)^\gamma/(k+3). \tag{2.67}$$

We have

$$\sum_{k=0}^{s-3} l_k x^{k+3} = \sum_{k=0}^{m-1} l_k x^{k+3} + \sum_{k=m}^{s-3} l_k x^{k+3}.$$

Making use of estimate (2.67) for $0 \leqslant m \leqslant s-3$, we get

$$\begin{aligned}
L_m(x) := \Big|\sum_{k=m}^{s-3} l_k x^{k+3}\Big| &\leqslant \frac{2x^2}{m+3} \sum_{k=m+3}^{s} \big((k+1)!\big)^\gamma (16\,x/\Delta)^{k-2} \leqslant \\
&\leqslant \frac{2x^2}{m+3}\big((m+4)!\big)^\gamma \sum_{k=m+3}^{s} \left(\frac{(s+1)^\gamma 16\,x}{\Delta}\right)^{k-m-3} \left(\frac{16\,x}{\Delta}\right)^{m+1} \leqslant \\
&\leqslant \frac{2\big((m+4)!\big)^\gamma x^2}{m+3}\left(\frac{16\,x}{\Delta}\right)^{m+1}\left(1-\frac{(s+1)^\gamma 16\,x}{\Delta}\right)^{-1} \leqslant \\
&\leqslant \frac{6\cdot 16^{m+1}\big((m+4)!\big)^\gamma}{(m+3)\Delta^{m+1}}\,x^{m+3} \leqslant \frac{16^{m+1}\big((m+4)!\big)^\gamma}{(m+3)6^{m+2}}\times \\
&\times(\sqrt{2}/6)^{(m+3)/(1+2\gamma)}(x/\Delta_\gamma)^{m+3}\cdot\Delta^{(m+3)/(1+2\gamma)}/\Delta^{m+1}.
\end{aligned} \tag{2.68}$$

Here we have employed of the fact that

$$\frac{16\,x(s+1)^\gamma}{\Delta} \leqslant \frac{8}{3}\cdot\frac{(\sqrt{2}/6)^{1/(1+2\gamma)}(\Delta/\sqrt{18})^{2\gamma/(1+2\gamma)}}{\Delta^{2\gamma/(1+2\gamma)}} < \frac{2}{3}.$$

Relation (2.68) shows that the integer m must be chosen so that $(m+3)/(1+2\gamma) \leqslant (m+1)$, i.e. $m \geqslant (1/\gamma) - 1$. Put $m = (\widehat{1/\gamma - 1})$, where $\hat{x} := \min\{n \geqslant$

$x|n$ is an integer$\} = -[-x]$. Let $\delta(\gamma) = (\widehat{1/\gamma - 1}) - (1/\gamma - 1)$. Then $0 \leqslant \delta(\gamma) < 1$. Obviously $m = 0$ for $\gamma \geqslant 1$. Therefore,

$$L_m(x) \leqslant \left(\frac{16\sqrt{2}}{6}\right)^{m+1} \frac{((m+4)!)^\gamma}{(m+3)6^{m+2}} \left(\frac{6}{\sqrt{2}\,\Delta}\right)^{\delta(\gamma)} \left(\frac{x}{\Delta_\gamma}\right)^{m+3}.$$

Using the estimate $(m+4)! \leqslant (m+3)^{m+3}$ and $(\Delta/\sqrt{18})^{2/(1+2\gamma)} \geqslant 30$ we obtain

$$L_m(x) = \begin{cases} (8/9)\cdot (x/\Delta_\gamma)^3, & \gamma \geqslant 1, \\ (5/9)\cdot (x/\Delta_\gamma)^{m+3}, & 0 \leqslant \gamma < 1. \end{cases} \tag{2.69}$$

The relations (2.66) and (2.69) permit to conclude that

$$L_\gamma(x) = \tilde{L}(x) = \sum_{0 \leqslant k < p} \lambda_k x^{k+3} + \theta\cdot 0.95\,(x/\Delta_\gamma)^3, \tag{2.70}$$

where $p = \min\{[1/\gamma] + 2,\ s - 3\}$. Making use of Lemma 2.2 and the fact that $\Delta_s \geqslant 4\Delta_\gamma$, we finally find that

$$\left(-\,x^3/(3\Delta_\gamma)\right) \leqslant L_\gamma(x) \leqslant (x^2/2)(x/(x + 8\Delta_\gamma)). \qquad \blacksquare$$

Proof of Lemma 2.1. If we put $h = 0$ in the relations (2.15), (2.18), (2.19), (2.22) and (2.23), then

$$\tilde{\varphi}(h)|_{h=0} = 1, \qquad x = \tilde{m}(h)|_{h=0} = 0,$$
$$F_h(y)|_{h=0} = F_\xi(y), \qquad f_h(t)|_{h=0} = f_\xi(t),$$

$$\tilde{f}_h(t)|_{h=0} = \tilde{\varphi}(it) = \exp\left\{\sum_{k=2}^{s} \frac{1}{k!}\,\Gamma_k(\xi)(it)^k\right\}.$$

As mentioned in the proof Lemma 2.3, for $s \geqslant 4$ we have

$$(k!)^{1+\gamma}/\Delta^{k-2} \leqslant (k-2)!/\Delta_s^{k-2}, \qquad k = 3,\ 4,\ \ldots,\ s+2,$$

where $\Delta_s = \Delta/\big(6(s+2)^\gamma\big)$, whereas $\Delta_s \geqslant \sqrt{s/2}$ and s is defined by equality (2.63). Expanding $\ln\tilde{\varphi}(it)$ by the Taylor formula in the interval $|t| \leqslant \delta_2\Delta_s/\sqrt{2e}$, $0 < \delta_2 < 1$, we have

$$\ln\tilde{\varphi}(it) = -\frac{1}{2}\,t^2 + \big((it)^3/6\big)\big(\ln\tilde{\varphi}(\tau)\big)'''_{\tau=\theta it} = -\frac{1}{2}\,t^2 + \theta\cdot 6^\gamma |t|^3/((1 - |t|/\Delta_s)\Delta),$$

since for $|\tau| \leqslant \Delta_s$

$$\left(\ln\tilde{\varphi}(\tau)\right)''' = \sum_{k=3}^{s}\frac{1}{(s-3)!}\,\Gamma_k(\xi)\tau^{k-3} = \theta\frac{1}{\Delta}\sum_{k=3}^{s}k(k-1)(k-2)(k!)^{\gamma}(|\tau|/\Delta)^{k-3} =$$
$$= \theta\,(6/\Delta)\sum_{k=3}^{s}(k!)^{\gamma}(6|\tau|/\Delta)^{k-3} = \theta\,(6^{1+\gamma}/\Delta)\sum_{k=3}^{s}\left(\frac{6\,s^{\gamma}|\tau|}{\Delta}\right)^{k-3} =$$
$$= \theta\left(6^{1+\gamma}/(\Delta(1-|\tau|/\Delta_s))\right).$$

Then

$$\begin{aligned}\left(\ln\tilde{\varphi}(\tau)\right)'''_{\tau=\theta it} &= \theta\sum_{k=3}^{s}\frac{1}{(k-3)!}\,\Gamma_k(\xi)|t|^{k-3} = \\ &= \theta\Delta_s^{-1}\sum_{k=3}^{s}(k-2)(|t|/\Delta_s)^{k-3} = \theta\Delta_s^{-1}(1-|t|/\Delta_s)^{-2},\end{aligned} \tag{2.71}$$

and therefore

$$\begin{aligned}&\left|\tilde{\varphi}(it) - \exp\left\{-\frac{1}{2}\,t^2\right\}\right| = e^{-\frac{1}{2}t^2}\left|\exp\left\{\ln\tilde{\varphi}(it) + \frac{1}{2}\,t^2\right\} - 1\right| = \\ &= \theta\frac{6^{\gamma}|t|^3}{\Delta(1-|t|/\Delta_s)}\exp\left\{-\frac{1}{2}\,t^2\left(1-|t|/(3(1-|t|/\Delta_s)^2\Delta_s)\right)\right\} = \\ &= \theta\frac{6^{\gamma}|t|^3}{\Delta(1-|t|/\Delta_s)}\exp\left\{-\frac{1}{4}\,t^2\right\}, \qquad |t| < \delta_2\Delta_s/\sqrt{2e}.\end{aligned} \tag{2.72}$$

Relation (2.53) yields

$$|f_\xi(t) - \tilde{\varphi}(it)| \leqslant l(\delta_2), \qquad |t| \leqslant \delta_2 a, \tag{2.73}$$

where $a = \sqrt{s/4e}$, s is defined by (2.63) and $l(\delta) = 4\sqrt{2}\,\delta^{[4a]}/(1-\delta)$. Further,

$$f_\xi(t) = \int_{-\infty}^{\infty} e^{\,ity}\,dF_\xi(y) = 1 + \theta\,\frac{t^2}{2},$$

$\exp\left\{-\frac{1}{2}\,t^2\right\} = 1+\theta\frac{t^2}{2}$ for all t. Consequently $\left|f_\xi(t) - \exp\left\{-\frac{1}{2}\,t^2\right\}\right| = \theta t^2$. Hence. in view of (2.72) and(2.73) for $|t| \leqslant T_2$, $\quad T_2 = \delta_2 a$, we get

$$\left|f_\xi(t) - \exp\left\{-\frac{1}{2}\,t^2\right\}\right| \leqslant \min\left\{t^2,\, 2\cdot 6^{\gamma}\frac{|t|^3}{\Delta}e^{-\frac{1}{4}t^2} + l(\delta_2)\right\}. \tag{2.74}$$

Further, set $\delta_2 = 1 - 1/(2s^{1/4}) \geqslant 1/\sqrt{a}$. Then $1 - \delta_2 = 1/(2s^{1/4})$ and

$$\begin{aligned} l(\delta_2) &= 4\sqrt{2}\,\delta_2^{[4a]}/(1-\delta_2) \leqslant 8\sqrt{2}\,s^{1/4}\delta_2^{[4a]}/\delta_2 \leqslant \\ &\leqslant (32\sqrt{2}\,/3)s^{1/4}\exp\{-(1/\sqrt{e})s^{1/4}\}. \end{aligned} \tag{2.75}$$

By (2.45), as $h = 0$, the quantity $Q(T)$ contained by the estimate of $\sup_x |F_\xi(x) - \Phi(x)|$ takes the form

$$\begin{aligned} Q(T) &= (T/\sqrt{2\pi})\int_0^T (1-t/T)\Big|f_\xi(t) - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big|\frac{dt}{t} \leqslant \\ &\leqslant (T/\sqrt{2\pi})(Q_1(T_2) + Q_2(T)), \end{aligned} \tag{2.76}$$

where

$$Q_1(T_2) = \int_0^{T_2} \Big|f_\xi(t) - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big|\frac{dt}{t},$$

$$Q_2(T) = \int_{T_2}^{T} \Big|f_\xi(t) - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big|\frac{dt}{t}.$$

Making use of estimate (2.73) and following the argument in the estimation of the quantity $Q(T)$ in Lemma 2.2, we obtain

$$\begin{aligned} Q_1(T_2) &= 4\sqrt{\pi}\,6^\gamma/\Delta + 42\sqrt{s}\,\exp\{-(1/\sqrt{e})s^{1/4}\} \leqslant \\ &\leqslant 4\sqrt{\pi}\,6^\gamma/\Delta + 294\,\Delta_\gamma \exp\Big\{-\frac{3}{2}\sqrt{\Delta_\gamma}\Big\}, \end{aligned} \tag{2.77}$$

where $\Delta_\gamma = c_\gamma \Delta^{1/(1+\gamma)}$, c_γ is defined by (2.1). Consequently, the estimate

$$\begin{aligned} Q(T) \leqslant (T/\sqrt{2\pi})\Big(&4\sqrt{\pi}\,6^\gamma/\Delta + 294\,\Delta_\gamma \exp\Big\{-\frac{3}{2}\sqrt{\Delta_\gamma}\Big\} + \\ &+ \int_{\Delta_\gamma}^{T} \Big|f_\xi(t) - \exp\Big\{-\frac{1}{2}t^2\Big\}\Big|\frac{dt}{t}\Big) \end{aligned} \tag{2.78}$$

holds. Substituting it into (2.45) and using the fact that $F_h(y)|_{h=0} = F_\xi(y)$, we obtain the assertion of Lemma 2.1. ■

Proof of Corollary 2.1. Using the estimate (2.71) we have

$$\left|\tilde{\varphi}(it) - \exp\left\{-\frac{1}{2}t^2\right\}\right| \leqslant |t|^3/(6(1-|t|/\Delta_s)^2\Delta_s)\times$$
$$\times\exp\left\{-\frac{1}{2}t^2\Big(1-|t|/(3(1-|t|/\Delta_s)^2\Delta_s)\Big)\right\} \leqslant 3|t|^3\exp\left\{-\frac{1}{4}t^2\right\}\Big/(4\Delta_s).$$

Recalling that $\Delta_s \geqslant \sqrt{s/2} \geqslant 4.4\,\Delta_\gamma$, we get

$$Q_1(T_2) \leqslant 7\sqrt{\pi}/\Delta_\gamma + 294\,\Delta_\gamma \exp\left\{-\frac{3}{2}\sqrt{\Delta_\gamma}\right\}.$$

Consequently, as $T = \Delta_\gamma$

$$Q(T) \leqslant (\Delta_\gamma/\sqrt{2\pi})\Big(7\sqrt{\pi}/\Delta_\gamma + 294\,\Delta_\gamma\exp\left\{-\frac{3}{2}\sqrt{\Delta_\gamma}\right\}\Big).$$

Using inequality (2.45) we have

$$\sup_x |F_\xi(x) - \Phi(x)| \leqslant \frac{1}{\sqrt{2\pi}}\left\{\frac{19.5}{\Delta_\gamma} + 300\,\Delta_\gamma\exp\left\{-\frac{3}{2}\sqrt{\Delta_\gamma}\right\}\right\} \leqslant \frac{18}{\Delta_\gamma}. \qquad \blacksquare$$

Proof of Lemma 2.4. Let m be a nonnegative integer, $g(x)$, $x \in R$, be an arbitrary real function, having m derivatives at $x = 0$. Denote

$$g_m(x) = \sum_{k=0}^{m} \frac{1}{k!}\, g^{(k)}(0)x^k, \qquad x \in R.$$

In particular,

$$\exp_m(x) = \sum_{k=0}^{m} \frac{1}{k!}x^k, \qquad x \in R, \quad m = 0,\ 1,\ \dots\ .$$

The function $\exp_{2n}(x)$ is positive for every $n = 0,\ 1,\ \dots$. For $x \geqslant 0$, this is obvious. Set

$$a_k = \frac{x^{2k-1}}{(2k-1)!} + \frac{x^{2k}}{(2k)!}, \qquad k = 1,\ 2,\ \dots\ .$$

If $x \leqslant -2n$, then $a_k \geqslant 0$, $\quad k = 1, 2, \dots, n$, and therefore

$$\exp_{2n}(x) = 1 + \sum_{k=0}^{n} a_k > 0.$$

If $-2n < x < 0$, then $a_k < 0$, $\quad k = n+1, n+2, \ldots$. Since

$$e^{x} = \exp_{2n}(x) + \sum_{k=n+1}^{\infty} a_k > 0,$$

this implies $\exp_{2n}(x) > 0$.

Furthermore, if $\mathbf{E}\xi^{2n} < \infty$, where $n \in \{1, 2, \ldots\}$, then for any $x \geqslant 0$

$$\mathbf{P}(\xi \geqslant x) \leqslant \inf_{h \geqslant 0} \exp_{2n}^{-1}(hx) \sum_{k=0}^{2n} \frac{1}{k!} m_k h^k, \tag{2.79}$$

where $m_k = \mathbf{E}\xi^k$. In fact, the function $\exp_{2n}(x)$ is nonnegative on R and increases monotonically in the interval $[0, \infty[$. By applying Chebyshev's inequality, for all $h \geqslant 0$ and $x \geqslant 0$ we have

$$\mathbf{P}(\xi \geqslant x) \leqslant \mathbf{P}\big(\exp_{2n}(hx) \geqslant \exp_{2n}(h\xi)\big) \leqslant \exp_{2n}^{-1}(hx) \sum_{k=0}^{2n} \frac{1}{k!} m_k h^k.$$

Denote

$$g(x) = \exp_n\left(\sum_{r=2}^{2n} \frac{1}{r!} \gamma_r x^r\right), \qquad x \in R. \tag{2.80}$$

For any $k \in \{1, 2, \ldots\}$

$$m_k = \sum_{r_1 + \ldots + r_q = k} \frac{1}{q!} \frac{k!}{r_1! \ldots r_q!} \gamma_{r_1} \ldots \gamma_{r_q},$$

where the summation is taken over all ordered partitions $r_1 + \ldots + r_q = k$, $r_j \geqslant 1$, $\quad q = 1, 2, \ldots k$. From (2.80), under the condition that $\mathbf{E}\xi = 0$ for any $k \in \{2, 3, \ldots 2n\}$ we obtain

$$\begin{aligned} m_k &= \sum_{q=1}^{k} \frac{1}{q!} \sum_{\substack{r_1 + \ldots + r_q = k \\ r_j \geqslant 2}} \frac{k!}{r_1! \ldots r_q!} \gamma_{r_1} \ldots \gamma_{r_q} = \\ &= \sum_{q=1}^{k} \frac{1}{q!} \frac{d^k}{dx^k} \left(\sum_{r=2}^{2n} \frac{1}{r!} \gamma_r x^r\right)^q \Bigg|_{x=0} = \frac{d^k}{dx^k} g(x)\Bigg|_{x=0}. \end{aligned}$$

Consequently,

$$\sum_{k=0}^{2n} \frac{1}{k!} m_k h^k = g_{2n}(h).$$

Evidently for any $h \geqslant 0$,

$$g_{2n}(h) \leqslant \exp_n \left(\sum_{r=2}^{2n} \frac{1}{r!} |\gamma_r| h^r \right). \tag{2.81}$$

Substituting (2.81) into (2.79) we obtain

$$\mathbf{P}(\xi \geqslant x) \leqslant \inf_{h \geqslant 0} \exp_{2n}^{-1}(hx) \exp_n \left(\sum_{k=2}^{2n} \frac{1}{k!} |\gamma_k| h^k \right) \tag{2.82}$$

for any $x \geqslant 0$.

We shall need the inequality

$$\exp_n(x) / \exp_{2n}(2x) \leqslant e^{-x}, \tag{2.83}$$

valid for $0 \leqslant x \leqslant 0.6$ if $n = 1$, for $0 \leqslant x \leqslant 1.4$ if $n = 2$, and for $0 \leqslant x \leqslant 0.8n$ if $n = 3, 4, \ldots$. Denote $\varepsilon = x/n$ and

$$\Delta_n(\varepsilon) = e^{-2\varepsilon n} \sum_{k=0}^{2n} \frac{1}{k!} (2\varepsilon n)^k - e^{-\varepsilon n} \sum_{k=0}^{n} \frac{1}{k!} (\varepsilon n)^k.$$

(2.83) is equivalent to the inequality $\Delta_n(\varepsilon) \geqslant 0$ for all $0 \leqslant \varepsilon \leqslant 0.6$ if $n = 1$, for all $0 \leqslant \varepsilon \leqslant 0.7$ if $n = 2$, and for all $0 \leqslant \varepsilon \leqslant 0.8$ if $n = 3, 4, \ldots$. It is easy to see

$$\frac{d}{d\varepsilon} \Delta_n(\varepsilon) = e^{-\varepsilon n} \frac{(\varepsilon n)^n}{(n-1)!} \left(1 - \frac{2^{2n-1} n!}{(2n)!} (\varepsilon n)^n e^{-\varepsilon n} \right).$$

By applying Stirling's formula

$$\sqrt{2\pi n}\, n^n e^{-n + \frac{1}{12n+1}} < n! < \sqrt{2\pi n}\, n^n e^{-n + \frac{1}{12n}}, \ .$$

we find

$$\frac{2^{2n+1} n!}{(2n)!} \leqslant n^{-n} e^n \sqrt{2} \exp \left\{ \frac{12n+1}{12n(24n+1)} \right\}.$$

Consequently, for all $\varepsilon \geqslant 0$ and $n = 1, 2, \ldots$

$$\frac{d\Delta_n(\varepsilon)}{d\varepsilon} \geqslant e^{-\varepsilon n} \left(1 - (\varepsilon e^{1-\varepsilon})^n \sqrt{2} \exp \left\{ \frac{12n+1}{12n(24n+1)} \right\} \right). \tag{2.84}$$

The function $f(\varepsilon) = \varepsilon e^{1-\varepsilon}$ is strictly increasing in the interval $[0, 1]$, $f(0) = 0$ and $f(1) = 1$. Therefore, for fixed n the right-hand side of (2.84) depend on $\varepsilon \in [0, 1]$ in such a way: it is positive in the interval $]\,0, \varepsilon_0\,[$, where $\varepsilon_0 \in]\,0, 1\,[$ is a unique point at which it equals to 0, and is negative in the interval $]\varepsilon_0, 1]$. Hence it follows that for every $n = 1, 2, \ldots$ the function Δ_n has the following property: if $0 < \varepsilon < 1$ and $\Delta_n(\varepsilon) \geqslant 0$ then $\Delta_n(\varepsilon') \geqslant 0$ for all $0 < \varepsilon' < \varepsilon$. Consequently, to prove (2.83) it suffices to show that $\Delta_1(0.6) \geqslant 0$, $\Delta_2(0.7) \geqslant 0$ and $\Delta_n(0.8) \geqslant 0$ if $n = 3, 4, \ldots$. If $n \geqslant 17$, then for all $x \in]0, 0.8]$ the right-hand side of (2.84) is positive, therefore $\Delta_n(0.8) \geqslant 0$. In the case $n < 17$, the necessary inequalities can be verified by direct calculations.

It suffices to consider the case $H = 1$, since if $H \neq 1$ we can introduce the random variable $\tilde{\xi} = \xi/H$ for which condition (2.12) is already fulfilled with $\tilde{H} = 1$ and $\tilde{\Delta} = \bar{\Delta}\sqrt{H}$. The estimate for $\mathbf{P}(\xi > x)$ can be obtained from the estimate for $\mathbf{P}(\tilde{\xi} > \tilde{x})$, by substituting $x/\sqrt{H}$ and $\bar{\Delta}\sqrt{H}$ for $\tilde{x}$ and $\tilde{\Delta}$, respectively.

Relation (2.82) and condition (2.12) imply that for all $x \geqslant 0$

$$\mathbf{P}(\xi \geqslant x) \leqslant \inf_{\substack{h \geqslant 0 \\ n=1,2,\ldots}} \frac{\exp_n\left(\frac{1}{2}h^2 \sum\limits_{k=2}^{2n} \left(\frac{h}{\bar{\Delta}}\right)^{k-2} \left(\frac{k!}{2}\right)^{\gamma}\right)}{\exp_{2n}(hx)}. \tag{2.85}$$

On the other hand, applying Chebyshev's inequality and considering that $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 \leqslant H = 1$, for all $x \geqslant 0$ we have

$$\mathbf{P}(\xi \geqslant x) \leqslant \mathbf{P}\big((1 + x\xi)^2 \geqslant (1 + x^2)^2\big) \leqslant \frac{1 + 2\mathbf{E}\xi x + \mathbf{E}(\xi x)^2}{(1 + x^2)^2} \leqslant \frac{1}{1 + x^2}. \tag{2.86}$$

Choose

$$h = \frac{x(x\bar{\Delta})^{1/(1+\gamma)}}{x^2 + (x\bar{\Delta})^{1/(1+\gamma)}}$$

and n so that the condition $0.8\,(n-1) < hx/2 \leqslant 0.8\,n$ is fulfilled. First consider the case $hx > 6.4$, i.e. $n \geqslant 5$. Then

$$\frac{k!}{2} \leqslant (hx)^{k-2}, \qquad k = 2, 3, \ldots, 2n. \tag{2.87}$$

In fact, since $hx > 1.6\,(n-1)$, (2.87) follows from the easily verifiable inequality $(2n)!/2 \leqslant \big(1.6\,(n-1)\big)^{2n-2}$, $n = 5, 6, \ldots$. By substituting the chosen values of h and n and taking (2.87) into account we have

$$\frac{1}{2}h^2 \sum_{k=2}^{2n} \left(\frac{h}{\bar{\Delta}}\right)^{k-2} \left(\frac{k!}{2}\right)^{\gamma} \leqslant \frac{1}{2}h^2 \sum_{k=2}^{2n} \left(\frac{h}{\bar{\Delta}}(hx)^{\gamma}\right)^{k-2} \leqslant \frac{h^2}{2}\frac{1}{1-q} = \frac{hx}{2}, \tag{2.88}$$

since

$$q = \left(\frac{h}{\bar{\Delta}}(hx)^{\gamma}\right)^{1/(1+\gamma)} = \frac{x^2}{x^2 + (x\bar{\Delta})^{1/(1+\gamma)}} < 1.$$

By substituting (2.88) into (2.85) and using (2.83) we obtain

$$\mathbf{P}(\xi \geqslant x) \leqslant \exp_{2n}^{-1}\{hx\}\exp_n\left\{\frac{hx}{2}\right\} \leqslant \exp\left\{-\frac{hx}{2}\right\},$$

or relation (2.13).

Now consider the case $0 < h \leqslant 6.4$. We split it into two subcases: 1) when the condition

$$\frac{1}{1+x^2} \leqslant \exp\left\{-\frac{hx}{2}\right\} \tag{2.89}$$

is fulfilled and 2) it is not fulfilled. In the first case (2.13) follows from (2.86). If condition (2.89) is not satisfied, then

$$\exp\left\{\frac{hx}{2}\right\} > 1 + x^2 > 1 + hx, \tag{2.90}$$

since $x > h$. It follows from (2.90) that $hx > 2.5$. Thus, it remains to consider the case $2.5 < hx \leqslant 6.4$ under the condition (2.90). In this case (2.13) is obtained from (2.85) by putting $n = 4$. By virtue of (2.90) we have $x^2 < \exp\{hx/2\} - 1$, therefore

$$(h/\bar{\Delta}) = q\big(1/(hx) - (1/x^2)\big)^\gamma < q\big(f(hx)\big)^\gamma,$$

where $f(t) = (1/t) - \left(e^{t/2} - 1\right)^{-1}$, $t > 0$. Since in the interval $2.5 < t \leqslant 6.4$ $e^{t/2} - 1 < t(t+1)/2$, for all $2.5 < t \leqslant 6.4$ the estimate

$$f(t) < \frac{t-1}{t(t+1)} < \left.\frac{t-1}{t(t+1)}\right|_{t=2.5} \leqslant 0.172$$

holds. Consequently, $h/\bar{\Delta} \leqslant q\,(0.172)^\gamma$. Therefore

$$\begin{aligned}\frac{1}{2}h^2\sum_{k=2}^{\infty}\left(\frac{h}{\bar{\Delta}}\right)^{k-2}\left(\frac{k!}{2}\right)^\gamma &\leqslant \frac{1}{2}h^2\sum_{k=2}^{\infty}q^{k-2}\left(0.172^{k-2}\frac{k!}{2}\right)^\gamma \leqslant \\ &\leqslant \frac{h^2}{2(1-q)} = \frac{hx}{2},\end{aligned} \tag{2.91}$$

since $0.172^{k-2}(k!/2) < 1$ for all $k = 2, 3, \ldots, 8$. Substituting (2.91) into (2.85) and using (2.83) we obtain (2.13). ■

CHAPTER 3
THEOREMS ON LARGE DEVIATIONS FOR THE DISTRIBUTIONS OF SUMS OF INDEPENDENT RANDOM VARIABLES

3.1. Theorems on large deviations under Bernstein's condition

We say that a r.v. ξ with $\mathbf{E}\xi = 0$ and $\sigma^2 = \mathbf{D}\xi > 0$ satisfies condition $(\overline{B}_\gamma)$, if there exist constants $\gamma \geqslant 0$ and $K > 0$ such that

$$|\mathbf{E}\xi^k| \leqslant (k!)^{1+\gamma} K^{k-2}\sigma^2, \qquad k = 3,\ 4,\ \dots\ . \tag{$\overline{B}_\gamma$}$$

Note that condition $(\overline{B}_\gamma)$ is a generalization of the well-known S.N. Bernstein's condition:

$$|\mathbf{E}\xi^k| \leqslant \frac{1}{2}\, k! K^{k-2}\sigma^2, \qquad k = 3,\ 4,\ \dots\ . \tag{B_0}$$

LEMMA 3.1. *Let a r.v. ξ satisfy condition $(\overline{B}_\gamma)$. Then for all $k = 2, 3, \dots$*

$$|\Gamma_k(\xi)| \leqslant (k!)^{1+\gamma}(2\max\{K,\ \sigma\})^{k-2}\sigma^2. \tag{3.1}$$

Proof. Let us prove first (3.1) for $\sigma^2 = 1$ (and $K \geqslant 1$). Let

$$\varphi_r(z) = \sum_{k=2}^{r} \frac{1}{k!}\, m_k z^k, \qquad m_k = \mathbf{E}\xi^k. \tag{3.2}$$

From the relation (1.33) between moments and cumulants we have

$$\Gamma_r(\xi) = \frac{d^r}{dz^r} \sum_{k=2}^{\infty} \frac{1}{k}\, (-1)^{k+1} \varphi_r^k(z)\Big|_{z=0}. \tag{3.3}$$

In the circle $|z| < \delta\big((r!)^{\gamma/r}K^{(r-2)/r}\big)^{-1}$, $0 < \delta < 1$, by virtue of condition $(\overline{B}_\gamma)$

$$|\varphi_r(z)| \leqslant \frac{1}{2}|z|^2 + \sum_{k=3}^{r}\frac{1}{k!}(k!)^{1+\gamma}K^{k-2}\delta^k\big((r!)^{k\gamma/r}K^{k(r-2)/r}\big)^{-1} \leqslant \frac{\delta^2}{2} + \frac{\delta^3}{1-\delta}$$

and

$$\Big|\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{k+1}\varphi_r^k(z)\Big| \leqslant \Big|\ln\Big(1 - \frac{\delta^2}{2} - \frac{\delta^3}{1-\delta}\Big)\Big|.$$

Further, let $\delta = 0.6$. Then for $r > 8$

$$\Big|\ln\Big(1 - \frac{\delta^2}{2} - \frac{\delta^3}{1-\delta}\Big)\Big| = |\ln 0.28| \leqslant 0.6^r \cdot 2^{r-2},$$

and with the help of Cauchy's formula we find that

$$|\Gamma_r(\xi)| \leqslant (r!)^{1+\gamma}(2K)^{r-2}, \qquad r > 8. \tag{3.4}$$

The latter estimate is also valid for $3 \leqslant r \leqslant 8$, what can be verified directly, expressing cumulants through moments. This proves (3.1) for $\sigma^2 = 1$.

In the general case $\sigma > 0$ put $\hat{\xi} = \xi/\sigma$, then $\mathbf{E}\hat{\xi} = 0$, $\mathbf{D}\hat{\xi} = 1$ and

$$|\mathbf{E}\hat{\xi}^k| \leqslant (k!)^{1+\gamma}(K/\sigma)^{k-2} \leqslant (k!)^{1+\gamma}(\max\{1, K/\sigma\})^{k-2}.$$

Since $\Gamma_k(\hat{\xi}) = \Gamma_k(\xi)/\sigma^k$, the assertion of Lemma 3.1 follows. ∎

a) *Sums of non-identically distributed random variables.* Let $\xi_1, \xi_2, \ldots, \xi_n$, $n \geqslant 1$, be independent r.v. with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{D}\xi_j > 0$, $j = 1, 2, \ldots, n$.

Set

$$S_n = \sum_{j=1}^{n}\xi_j, \qquad B_n^2 = \sum_{j=1}^{n}\sigma_j^2, \qquad Z_n = S_n/B_n, \tag{3.5}$$

$$F_{Z_n}(x) = \mathbf{P}(Z_n < x), \qquad p_{Z_n}(x) = \frac{d}{dx}F_{Z_n}(x),$$
$$L_{k,n} = \sum_{j=1}^{n}\mathbf{E}|\xi_j|^k/B_n^k. \tag{3.6}$$

The quantity $L_{k,n}$ is called the k^{th} Lyapunov fraction (Statulevičius, 1965).

We say that the r.v. ξ_j with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{D}\xi_j > 0$, $j = 1, 2, \ldots, n$, satisfy (B_γ), if there exist $\gamma \geqslant 0$ and $K > 0$ such that

$$|\mathbf{E}\xi_j^k| \leqslant (k!)^{1+\gamma}K^{k-2}\sigma_j^2, \qquad k = 3, 4, \ldots\,. \tag{B_γ}$$

THEOREM 3.1. *Let r.v. ξ_j, $j = 1, 2, \ldots, n$, satisfy condition (B_γ). Then*

$$|\Gamma_k(Z_n)| \leqslant \frac{(k!)^{1+\gamma}}{\Delta_n^{k-2}}, \qquad k = 3,\ 4,\ \ldots\ , \tag{3.7}$$

where

$$\Delta_n = \frac{B_n}{K_n}, \qquad K_n = 2\max\{K,\ \max_{1\leqslant j\leqslant n} \sigma_j\}. \tag{3.8}$$

Moreover, for r.v. $\xi = Z_n$ the relations (2.6), (2.13) *and estimates* (2.3), (2.2) *hold with the parameters*

$$\Delta_\gamma = c_\gamma \cdot \Delta_n^{1/(1+2\gamma)}, \qquad H = 2^{1+\gamma} \quad \text{and} \quad \bar{\Delta} = \Delta_n,$$

where c_γ and Δ_n are defined by (2.1) and (3.8), *respectively.*

COROLLARY 3.1. *Let r.v. ξ_j, $j = 1, 2, \ldots,$ satisfy condition (B_γ). Then the relations*

$$\lim_{n\to\infty} \frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} = 1, \quad \lim_{n\to\infty} \frac{F_{Z_n}(-x)}{\Phi(-x)} = 1 \tag{3.9}$$

hold for $x \geqslant 0$, $x = o(\Delta_n^\nu)$ as $\Delta_n \to \infty$, where $\nu = \nu(\gamma) = (1 + \max\{1, \gamma\})^{-1}$.

If, in addition, all moments of r.v. ξ_j of the order $r = 1, 2, \ldots, [1/\gamma] + 2$ coincide with the corresponding ones of a normal distribution, $j = 1, 2, \ldots,$ then for $x \geqslant 0$, $x = o(\Delta_n^{1/(1+2\gamma)})$, the relations (3.9) *hold.*

(It is clear that the last part of the assertion is meaningful only for $0 < \gamma < 1$.)

Proof of Theorem 3.1 *and of Corollary* 3.1. Using condition (B_γ) and Lemma 3.1 we get

$$|\Gamma_k(\xi_j)| \leqslant (k!)^{1+\gamma}(2\max\{K, \sigma_j\})^{k-2}\sigma_j^2, \qquad \forall k \geqslant 3. \tag{3.10}$$

It follows from the independence of the r.v. ξ_j, $j = 1, 2, \ldots,$ that

$$|\Gamma_k(S_n)| \leqslant (k!)^{1+\gamma} K_n^{k-2} \cdot B_n^2, \qquad \forall k \geqslant 3, \tag{3.11}$$

where K_n is defined in (3.8). Since $\Gamma_k(Z_n) = \Gamma_k(S_n)/B_n^k$, we get (3.7).

To complete the proof of Theorem 3.1 it suffices to make use of (3.7), Lemmas 2.3, 2.4 and Corollary 2.1.

The statement of the first part of Corollary 3.1 follows immediately if we use the definition of $L_\gamma(x)$ by relation (2.8). Taking into account estimate (2.11) and the fact that the moments of the r.v. ξ up to $(m+3)^{\text{th}}$ order, including $(m = [1/\gamma]-1)$, are the same as those of the normal law (whose cumulants, starting from the third, vanish) we find

$$L_\gamma(x) = \sum_{k=m+1}^{p} \lambda_k x^{k+3} + \theta\,(x/\Delta_\gamma)^3 =$$

$$= \theta_1 \frac{6\cdot 16^{m+2}\left((m+5)!\right)^\gamma}{m+4}\cdot \frac{x^{m+4}}{\Delta_n^{m+2}} + \theta_2 c_\gamma^3 (x/\Delta_n^{1/(1+2\gamma)})^3,$$

where Δ_n is defined in (3.8). Since $(m+4)/(1+2\gamma) \leqslant m+2$, $L_\gamma(x) \to 0$ for $x = o(\Delta_n^{1/(1+2\gamma)})$, $\Delta_n \to \infty$. ∎

We say that r.v. ξ_j, $j = 1, 2, \ldots$, satisfy condition (P), if there exist positive constants A, C, c_1, c_2, … , such that

$$\left|\frac{\ln \mathrm{E} \exp\{z\xi_j\}}{z^2}\right| \leqslant c_j^2, \qquad |z| < A \qquad (j = 1, 2, \ldots) \tag{P}$$

and

$$\varlimsup_{n\to\infty} \frac{1}{B_n^2} \sum_{j=1}^{n} c_j^2 \leqslant C. \tag{3.12}$$

THEOREM 3.2. *Let r.v. ξ_j with $\mathrm{E}\xi_j = 0$ and $\sigma_j^2 = \mathrm{D}\xi_j$ satisfy condition (P). Then*

$$|\Gamma_k(Z_n)| \leqslant \frac{k!\,C}{(A\,B_n)^{k-2}}, \qquad \forall k \geqslant 3, \tag{3.13}$$

and for r.v. $\xi = Z_n$ the relations of large deviations (2.6), (2.13) *and estimate* (2.3) *hold with*

$$\Delta_\gamma = \frac{1}{6}\cdot\frac{\sqrt{2}}{6}\,\frac{AB_n}{\max\{1, C\}}, \quad H = 2C \quad \text{and } \bar\Delta = AB_n.$$

Proof. By virtue of the equality

$$\Gamma_k(\xi_j) = \frac{d^k}{dz^k} \ln \mathrm{E} \exp\{z\xi_j\}\Big|_{z=0},$$

condition (P) and the Cauchy inequality for derivatives of analytical functions we find

$$|\Gamma_k(\xi_j)| \leqslant k!\, c_j^2/A^{k-2}, \qquad \forall k \geqslant 3.$$

Hence, making use of the independence of the r.v. ξ_j, $j = 1, 2, \ldots, n$, we obtain

$$|\Gamma_k(S_n)| \leqslant k! \sum_{j=1}^{n} c_j^2/A^{k-2}, \qquad \forall k \geqslant 3,$$

and

$$|\Gamma_k(Z_n)| \leqslant k!\, C/(A\cdot B_n)^{k-2}, \qquad \forall k \geqslant 3, \tag{3.14}$$

where C is defined by relation (3.12). By using (3.14), Lemmas 2.3, 2.4 and Corollary 2.1, we obtain the assertion of the theorem. ∎

b) *Sums of weighted random variables.* Let ξ_j, $j = 1, 2, \dots, n$, be independent r.v. with $\mathrm{E}\xi_j = 0$, $\sigma_j^2 = \mathrm{E}\xi_j^2$ and $\{a_{k,n}, 1 \leqslant k \leqslant n, 1 \leqslant n < \infty\}$ be an array of nonnegative numbers. Denote

$$\tilde{S}_n = \sum_{j=1}^{n} a_{j,n}\xi_j, \qquad \tilde{B}_n^2 = \sum_{j=1}^{n} a_{j,n}^2\sigma_j^2, \tag{3.15}$$

$$\tilde{Z}_n = \tilde{S}_n/\tilde{B}_n, \qquad \gamma_n = \max\{a_{j,n}, 1 \leqslant j \leqslant n\}. \tag{3.16}$$

THEOREM 3.3. *Let the r.v. ξ_j with $\mathrm{E}\xi_j = 0$ and $\sigma_j^2 = \mathrm{E}\xi_j^2$, $j = 1, 2, \dots, n$, satisfy condition (B_γ). Then*

$$|\Gamma_k(\tilde{Z}_n)| \leqslant \frac{(k!)^{1+\gamma}}{\tilde{\Delta}_n^{k-2}}, \qquad k = 3, 4, \dots, \tag{3.17}$$

where

$$\tilde{\Delta}_n = \frac{\tilde{B}_n}{K_n\gamma_n}, \qquad K_n = 2\max\{K, \max_{1\leqslant j\leqslant n} \sigma_j\}. \tag{3.18}$$

Moreover, for the r.v. $\xi = \tilde{Z}_n$ the relations of large deviations (2.6), (2.13) *and estimates* (2.3), (2.2) *are valid with*

$$\Delta_\gamma = c_\gamma \tilde{\Delta}_n^{1/(1+2\gamma)}, \qquad H = 2^{1+\gamma}, \quad \bar{\Delta} = \tilde{\Delta}_n,$$

where c_γ and $\tilde{\Delta}_n$ are determined by (2.1) *and* (3.18), *respectively.*

Proof. By independence of the r.v. ξ_j, $j = 1, 2, \dots, n$, we have

$$\Gamma_k(\tilde{S}_n) = \sum_{j=1}^{n} a_{j,n}^k \Gamma_k(\xi_j). \tag{3.19}$$

Using condition (B_γ) and (3.10) we obtain

$$|\Gamma_k(\tilde{S}_n)| \leqslant (k!)^{1+\gamma}(K_n\gamma_n)^{k-2}\cdot \tilde{B}_n^2, \tag{3.20}$$

where γ_n, K_n and $\tilde{B}_n^2$ are defined by relations (3.16), (3.18) and (3.15). As $\Gamma_k(\tilde{Z}_n) = \Gamma_k(\tilde{S}_n)/\tilde{B}_n^k$, we get estimate (3.17). With the help of Lemmas 2.3, 2.4 and estimates (2.3), (2.2), this proves the statement of Theorem 3.3. Note that in the equality (2.8) the coefficients λ_k are expressed in terms of cumulants of the r.v. $\tilde{Z}_n$, whereas

$$\Gamma_k(\tilde{Z}_n) = \sum_{j=1}^{n} a_{j,n}^k \Gamma_k(\xi_j)/\tilde{B}_n^k, \qquad \forall k \geqslant 2. \qquad \blacksquare$$

Let now r.v. ξ_j be identically distributed, $\mathbf{E}\xi_j = 0$ and $\sigma^2 = \mathbf{E}\xi_j^2 = 1$, $j = 1, 2, \ldots, n$. Let

$$b_n^2 = \sum_{j=1}^{n} a_{j,n}^2, \qquad \hat{Z}_n = \tilde{S}_n / b_n, \tag{3.21}$$

where $\tilde{S}_n$ is defined in (3.15).

We say that a r.v. ξ_1 satisfies *Linnik's condition* if there exists a constant C_γ such that

$$\mathbf{E}\exp\{|\xi_1|^{1/(1+2\gamma)}\} \leqslant C_\gamma < \infty, \qquad \gamma > 0. \tag{L}$$

It is easy to clarify the relation between conditions (L) and (S_γ). Put

$$C_\gamma^{(1)} = 4C_\gamma e^\beta \beta^{3\beta}, \qquad \beta = 1 + \gamma. \tag{3.22}$$

If condition (L) for the r.v. ξ_1 is fulfilled, then

$$|\Gamma_k(\xi_1)| \leqslant (C_\gamma^{(1)})^{k-2} (k!)^{1+\gamma}, \qquad k = 3, 4, \ldots . \tag{3.23}$$

Indeed, it follows from condition (L) that

$$\mathbf{E}(|\xi_1|^{m/\beta})/m! \leqslant C_\gamma, \qquad m = 1, 2, \ldots . \tag{3.24}$$

As $m = (\widehat{k\beta})$ $(\hat{x} := \min\{n \geqslant x | n \text{ is integer}\})$ we have

$$\mathbf{E}|\xi_1|^k \leqslant (\mathbf{E}|\xi_1|^{m/\beta})^{k\beta/m} \leqslant (m! C_\gamma)^{k\beta/m} \leqslant 2\, C_\gamma \beta^{k\beta} e^\beta (k!)^\beta.$$

Consequently, for $k \geqslant 3$ $\mathbf{E}|\xi_1|^k \leqslant (2\, C_\gamma e^\beta \beta^{3\beta})^{k-2} (k!)^\beta$. Hence, according to Lemma 3.1

$$|\Gamma_k(\xi_1)| \leqslant (k!)^\beta (C_\gamma^{(1)})^{k-2}, \qquad k = 3, 4, \ldots, \tag{3.25}$$

where $C_\gamma^{(1)}$ is defined by (3.22).

THEOREM 3.4. *Let identically distributed r.v. ξ_j with $\mathbf{E}\xi_j = 0$ and $\sigma^2 = \mathbf{E}\xi_j^2 = 1$, $j = 1, 2, \ldots, n$, satisfy condition (L). Then*

$$|\Gamma_k(\hat{Z}_n)| \leqslant \frac{(k!)^{1+\gamma}}{\hat{\Delta}_n^{k-2}}, \qquad k = 3, 4, \ldots, \tag{3.26}$$

where

$$\hat{\Delta}_n = \frac{b_n}{C_\gamma^{(1)} \gamma_n}, \tag{3.27}$$

and for r.v. $\xi = \tilde{Z}_n$ relations of large deviations (2.6), (2.13) *and estimates* (2.3), (2.2) *are valid with*

$$\Delta_\gamma = c_\gamma \hat{\Delta}_n^{1/(1+2\gamma)}, \qquad H = 2^{1+\gamma}, \quad \bar{\Delta} = \hat{\Delta}_n.$$

Proof. As the r.v. ξ_j have the same distribution, from (3.19) we have

$$\Gamma_k(\tilde{S}_n) = \Gamma_k(\xi_1) \sum_{j=1}^{n} a_{j,n}^k. \tag{3.28}$$

From inequality (3.23) we obtain

$$|\Gamma_k(\tilde{S}_n)| \leqslant (k!)^{1+\gamma} (C_\gamma^{(1)})^{k-2} \sum_{j=1}^{n} a_{j,n}^k.$$

Hence, (3.26) is valid for the r.v. $\hat{Z}_n = \tilde{S}_n / b_n$, where b_n is defined by (3.21). To complete the proof of the theorem it suffices to use Lemmas 2.3, 2.4 together with estimates (2.2) and (2.3). ■

REMARK. *In the summation theory of weighted r.v. the following condition was used* (Book, 1972)*: one can find two numbers $0 < \alpha \leqslant 1$ and $0 < q \leqslant 1$ such that among nonnegative numbers $a_{j,n}$, $j = 1, 2, \ldots, n$, there are αn numbers such that $a_{j,n} \geqslant q\gamma_n$, where $\gamma_n = \max\{a_{j,n}, 1 \leqslant j \leqslant n\}$.*

This condition means that a finite number of summands cannot determine the behavior of the sum $\tilde{S}_n$ as $n \to \infty$, where $\tilde{S}_n$ is defined by (3.15). In this case

$$b_n^2 = \sum_{j=1}^{n} a_{j,n}^2 \geqslant \alpha n (q\gamma_n)^2. \tag{3.29}$$

Then, according to (3.27) in Theorem 3.4 one should take $\hat{\Delta}_n = q(\alpha n)^{1/2} / C_\gamma^{(1)}$, where $C_\gamma^{(1)}$ is defined by (3.22).

3.2. A theorem of large deviations in terms of Lyapunov's fractions

As it is known, Lyapunov's fractions*)

$$L_{k,n} = \frac{1}{B_n^k} \sum_{j=1}^{n} \mathbf{E}|\xi_j|^k, \tag{3.30}$$

*) See Appendix 1.

$$\mathrm{E}\xi_j = 0, \qquad \sigma_j^2 = \mathrm{E}\xi_j^2, \qquad B_n^2 = \sum_{j=1}^{n} \sigma_j^2,$$

are convenient for constructing asymptotical expansions (Statulevičius, 1965, 1967) for distribution function $F_{Z_n}(x)$ of the r.v.

$$Z_n = \sum_{j=1}^{n} \xi_j / B_n.$$

It turns out that probabilities of large deviations in the Cramer – Petrov zone and Linnik power zones can also be investigated in terms of Lyapunov's fractions. Thus, probabilities of large deviations in such zones mainly depend not on individual but average properties of summands as emphasized in (Wolf, 1970, 1975).

THEOREM 3.5. *Assume, there exist $\gamma \geqslant 0$ and $\tau_n > 0$ such that*

$$L_{k,n} \leqslant \frac{(k!)^{1+\gamma}}{\tau_n^{k-2}}, \qquad k = 3,\ 4,\ \ldots\ . \tag{3.31}$$

Then, in the interval $0 \leqslant x < \tau_n^*$,

$$\tau_n^* = \begin{cases} \dfrac{c\tau_n}{|\ln \tau_n|}, & \gamma = 0, \\ c_\gamma^* \tau_n^{1/(1+2\gamma)}, & \gamma > 0, \end{cases} \tag{3.32}$$

the relations of large deviations

$$\begin{aligned} \frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} &= \exp\left\{L_{\gamma,n}(x)\right\}\left(1 + \theta_1 f(x)\frac{x+1}{\tau_n^*}\right), \\ \frac{F_{Z_n}(-x)}{\Phi(-x)} &= \exp\left\{L_{\gamma,n}(-x)\right\}\left(1 + \theta_2 f(x)\frac{x+1}{\tau_n^*}\right) \end{aligned} \tag{3.33}$$

hold and

$$\sup_x |F_{Z_n}(x) - \Phi(x)| \leqslant \frac{18}{\tau_n^*}, \tag{3.34}$$

where $L_{\gamma,n}(x)$, $f(x)$ are defined by relations (2.8) *and* (2.7) *with $\Delta_\gamma = \tau_n^*$ and*

$$c_\gamma^* \geqslant 96\sqrt{e}\, 3^{1/(1+2\gamma)} \big(e(1+\gamma)\big)^3 e^{\gamma/(1+2\gamma)} \Big(\big(3(1+2\gamma)^2/(e\gamma)\big)^{3(1+2\gamma)}\Big)^{-1}, \tag{3.35}$$
$$c \geqslant \sqrt{6}\,(36 \cdot 27)^{-1}.$$

In order to prove the theorem, first we shall verify the following statement.

PROPOSITION 3.1. *If the r.v. ξ_j with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{E}\xi_j^2$, $j = 1, 2, \dots, n$, satisfy condition (3.31) with the exponent $\gamma = 0$, then*

$$|\Gamma_k(Z_n)| \leqslant \frac{k!}{\big(\tau_n/27 \ln \tau_n)\big)^{k-2}}, \qquad k = 3, 4, \dots . \tag{3.36}$$

If condition (3.31) with the exponent $\gamma > 0$ is fulfilled, then

$$|\Gamma_k(Z_n)| \leqslant \max\left\{\frac{(k!)^{1+\gamma}}{\big(\tau_n/C_1^*(\gamma)\big)^{k-2}}, \frac{k!}{\tau_n^2\big(\tau_n^{1/(1+2\gamma)}/C_2^*(\gamma)\big)^{k-2}}\right\}, \qquad k = 3, 4, \dots , \tag{3.37}$$

where

$$\begin{aligned} C_1^*(\gamma) &= 48 \exp\{3(1+\gamma)\}, \\ C_2^*(\gamma) &= 16^{1/(1+2\gamma)}\big(e(1+\gamma)\big)^3 6^{\gamma/(1+2\gamma)}\left(\frac{3(1+2\gamma)^2}{e\gamma}\right)^{3(1+2\gamma)}. \end{aligned} \tag{3.38}$$

Proof. It follows from (3.31) that

$$\sum_{j=1}^n \mathbf{E}|\xi_j|^k/(k!)^{1+\gamma} \leqslant B_n^2(B_n/\tau_n)^{k-2}, \qquad \gamma \geqslant 0, \quad k \geqslant 3.$$

Let $\eta_j = \delta_n\xi_j$, where $\delta_n = (\tau_n/2B_n)$. Then

$$\sum_{j=1}^n \mathbf{E}|\eta_j|^k/(k!)^{1+\gamma} = \delta_n^k \sum_{j=1}^n \mathbf{E}|\xi_j|^k/(k!)^{1+\gamma} \leqslant$$
$$\leqslant (\delta_n B_n)^2(\delta_n B_n/\tau_n)^{k-2} = (\delta_n B_n)^2/2^{k-2}.$$

By summing over all $k \geqslant 3$, we get

$$\sum_{k=3}^\infty \sum_{j=1}^n \mathbf{E}|\eta_j|^k/(k!)^{1+\gamma} \leqslant \tilde{B}_n^2, \qquad \tilde{B}_n = \delta_n B_n.$$

Next,

$$\sum_{j=1}^n \mathbf{E}|\eta_j|^2/2^{1+\gamma} = B_n^2/2^{1+\gamma}.$$

Consequently

$$\sum_{k=2}^{\infty}\sum_{j=1}^{n} \mathbf{E}|\eta_j|^k/(k!)^{1+\gamma} \leqslant \tilde{B}_n^2(1+1/2^{1+\gamma}) \leqslant \frac{3}{2}\tilde{B}_n^2. \tag{3.39}$$

Put

$$C_j(\gamma) = \sum_{k=2}^{\infty} \mathbf{E}|\eta_j|^k/(k!)^{1+\gamma}. \tag{3.40}$$

It follows from (3.39) that

$$\sum_{j=1}^{n} C_j(\gamma) \leqslant (1+1/2^{1+\gamma})\tilde{B}_n^2. \tag{3.41}$$

Consider first the case $C_j(\gamma) \leqslant 1$. Let

$$\varphi_{r,j} = \sum_{k=2}^{r} \frac{1}{k!}\,\mathbf{E}\eta_j^k z^k.$$

By (3.3)

$$\Gamma_r(\eta_j) = \frac{d^r}{dz^r}\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{k+1}\varphi_{r,j}^k(z)\Big|_{z=0}.$$

In the circle $|z| < 1/\big(2(r!)^{\gamma/r}\big)$ one has

$$\begin{aligned}|\varphi_{r,j}(z)| &\leqslant \sum_{k=2}^{r}\frac{|\mathbf{E}\eta_j^k|}{k!\big(2(r!)^{\gamma/r}\big)^k} \leqslant \frac{1}{4}\sum_{k=2}^{r}\frac{|\mathbf{E}\eta_j^k|}{k!(r!)^{\gamma k/r}} \leqslant \\ &\leqslant \frac{1}{4}\sum_{k=2}^{r}\frac{\mathbf{E}|\eta_j|^k}{(k!)^{1+\gamma}} \leqslant \frac{1}{4}\,C_j(\gamma),\end{aligned}$$

since for $k < r$ the inequality $(k!)^r < (r!)^k$ holds. As $C_j(\gamma) \leqslant 1$,

$$\Big|\sum_{k=1}^{\infty}\frac{1}{k}(-1)^{k+1}\varphi_{r,j}(z)\Big| \leqslant |\ln\big(1-\varphi_{r,j}(z)\big)| \leqslant |\ln\big(1-C_j(\gamma)/4\big)|.$$

Using Cauchy's formula for derivatives of an analytic function, we find that

$$\begin{aligned}|\Gamma_r(\eta_j)| &\leqslant r!|\ln\big(1-C_j(\gamma)/4\big)|\,2^r(r!)^{\gamma} \leqslant (r!)^{1+\gamma}2^{r-1}C_j(\gamma) \leqslant \\ &\leqslant (r!)^{1+\gamma}3^{r-2}C_j(\gamma), \qquad r = 4,\ 5,\ \dots\ .\end{aligned} \tag{3.42}$$

Estimate (3.42) is true for $r = 2,\ 3$ as well, what can be verified easily by expressing the cumulants through moments and using inequality $\mathbf{E}|\eta_j|^k \leqslant (k!)^{1+\gamma}C_j(\gamma)$.

Consider now the case $C_j(\gamma) > 1,\ \gamma \geqslant 0$. There are two subcases: a) $\gamma = 0$; b) $\gamma > 0$.

Subcase a) $\gamma = 0$. Let $0 < \delta \leqslant 1$. Then

$$\mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\delta^k\Big) \leqslant \Big(\mathbf{E}\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big)^{\delta}. \tag{3.43}$$

We have

$$\mathbf{E}\,\exp\{\delta|\eta_j|\}/\mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\delta^k\Big) =$$

$$= 1+\mathbf{E}(1+|\eta_j|)/\mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big) \geqslant 1+\big(\mathbf{E}(1+\delta|\eta_j|)\big)\Big/\Big(\delta^2\mathbf{E}\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big) \geqslant$$

$$= 1+\mathbf{E}(1+|\eta_j|)\Big/\Big(\mathbf{E}\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big) = \mathbf{E}\,\exp\{|\eta_j|\}\Big/\Big(\mathbf{E}\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big).$$

Hence it follows that

$$\mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\delta^k\Big)\Big/\mathbf{E}\,\exp\{\delta|\eta_j|\} \leqslant \mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big)\Big/\mathbf{E}\,\exp\{|\eta_j|\}, \tag{3.44}$$

or

$$\mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\delta^k\Big) \leqslant \mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big)\mathbf{E}e^{\delta|\eta_j|}/\mathbf{E}e^{|\eta_j|} \leqslant$$

$$\leqslant \mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big)(\mathbf{E}e^{|\eta_j|})^{\delta}/\mathbf{E}e^{|\eta_j|} = \mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big)\Big/(\mathbf{E}e^{|\eta_j|})^{1-\delta} \leqslant$$

$$\leqslant \mathbf{E}\Big(\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big)\Big/\Big(\mathbf{E}\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|\Big)^{1-\delta} = \Big(\mathbf{E}\sum_{k=2}^{\infty}\frac{1}{k!}\,|\eta_j|^k\Big)^{\delta},$$

$0 < \delta \leqslant 1$, what proves (3.43). Using relations

$$\sum_{k=2}^{\infty}\frac{1}{k!}\,\mathbf{E}|\eta_j|^k = C_j(0), \qquad C_j(0) > 1$$

and having in mind inequality (3.43), we get

$$\sum_{k=2}^{\infty}\frac{1}{k!}\,\mathbf{E}|\eta_j|^k(\varepsilon\delta)^k \leqslant \varepsilon^2 C_j^{\delta}(0) \tag{3.45}$$

for $0 < \delta \leqslant 1$ and $0 < \varepsilon \leqslant 1$. Let $\varphi_j(z) = \sum\limits_{k=2}^{\infty} \frac{1}{k!} \mathbf{E}\eta_j^k z^k$. Then

$$\Gamma_k(\eta_j) = \frac{d^k}{dz^k} \ln\left(1 + \varphi_j(z)\right)\Big|_{z=0} = \frac{d^k}{dz^k} \sum_{l=1}^{\infty} \frac{1}{l} (-1)^{l+1} \varphi_j^l(z)\Big|_{z=0}.$$

Put $\delta = 2\Big(\ln\left(7.5\, C_j(0)\right)\Big)^{-1}$, $C_j(0) > 1$. Then it is easy to see that $C_j^\delta(0)\cdot \varepsilon^2 < e^2/9 < 1$. We have for $|\varphi_j(z)| < 1$

$$\Big|\sum_{l=2}^{\infty} \frac{1}{l} (-1)^{l+1} \varphi_j^l(z)\Big| \leqslant \big|\ln\left(1 - |\varphi_j(z)|\right)\big|.$$

By using the Cauchy formula as above together with (3.45), we find that

$$|\Gamma_k(\eta_j)| \leqslant \left(k! \ln(1 - e^2/9)/(\varepsilon\delta)^k \leqslant 1.7\, k!(3/2)^k \ln^k\left(7.5\, C_j(0)\right).\right.$$

Since $7.5\, C_j(0) \geqslant e^2$ and $(\ln^2 x)/x \leqslant 4/e^2$ for $x \geqslant 1$,

$$\begin{aligned} |\Gamma_k(\eta_j)| &\leqslant 7\, k!(3/2)^k C_j(0) \ln^{k-2}\left(7.5\, C_j(0)\right) \leqslant \\ &\leqslant k! 3^{k-2} C_j(0) \ln^{k-2}\left(7.5\, C_j(0)\right), \qquad k \geqslant 6, \end{aligned} \tag{3.46}$$

because $7(3/2)^k \leqslant 3^{k-2}$ for $k \geqslant 6$. Inequality (3.46) for $k = 3,\ 4,\ 5$ can be obtained directly from relations

$$\mathbf{E}|\eta_j|^k \leqslant k! C_j(0), \qquad k = 2,\ 3,\ \ldots\ , \qquad \mathbf{E}\eta_j^2 \leqslant (\mathbf{E}|\eta_j|^5)^{2/5},$$
$$\mathbf{E}|\eta_j|^3 \leqslant (\mathbf{E}|\eta_j|^5)^{3/5}, \qquad (\mathbf{E}\eta_j^2)^2 \leqslant \mathbf{E}\eta_j^4.$$

Consequently,

$$|\Gamma_k(\eta_j)| \leqslant k!\, 3^{k-2} C_j(0) \ln^{k-2}\left(7.5\, C_j(0)\right), \qquad k \geqslant 3. \tag{3.47}$$

Subcase b) $\gamma > 0$. We have

$$\exp\left\{|\eta_j(\omega)|^{1/(1+\gamma)}\right\} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} |\eta_j(\omega)|^{k/(1+\gamma)} = 1 + |\eta_j(\omega)|^{1/(1+\gamma)} +$$
$$+ \sum_{2 \leqslant k \leqslant k_0} \frac{1}{k!} |\eta_j(\omega)|^{k/(1+\gamma)} + \sum_{k > k_0} \frac{1}{k!} |\eta_j(\omega)|^{k/(1+\gamma)},$$

where $k_0 = k_0(\omega)$ for fixed $\omega \in \Omega$ is the least positive integer for which the inequality $|\eta_j(\omega)|^k/(k!)^{1+\gamma} \leqslant 1$ is satisfied. Consequently,

$$|\eta_j(\omega)|/(k!)^{1+\gamma} \begin{cases} \leqslant 1 & \text{for} \quad k > k_0(\omega), \\ > 1 & \text{for} \quad k \leqslant k_0(\omega). \end{cases} \tag{3.48}$$

Hence it follows that

$$|\eta_j(\omega)|^k/(k!)^{1+\gamma} > |\eta_j(\omega)|^{k/(1+\gamma)}/k!, \qquad k < k_0.$$

In view of relation (3.41), we find

$$\sum_{2 \leqslant k \leqslant k_0} \frac{1}{k!}\, |\eta_j|^{k/(1+\gamma)} \leqslant \sum_{2 \leqslant k \leqslant k_0} \frac{1}{(k!)^{1+\gamma}}\, |\eta_j|^k. \tag{3.49}$$

Next we'll estimate $\sum\limits_{k>k_0} \frac{1}{k!}|\eta_j|^{k/(1+\gamma)}$ bearing in mind that $|\eta_j(\omega)|^{k/(1+\gamma)}/k! \leqslant 1$ for $k > k_0(\omega)$. Since $k! < k^k/(1.65)^k$ for $k \geqslant 3$, it follows that $(1.65)^{1+\gamma}|\eta_j(\omega)| \leqslant 1$. Hence $(1.65)^{1+\gamma}|\eta_j|/k^{1+\gamma} \leqslant 1$ and $|\eta_j(\omega)|^{1+\gamma}/k < 5/8$ for $k > k_0$. Therefore,

$$\begin{aligned} \sum_{k>k_0} \frac{1}{k!}|\eta_j|^{k/(1+\gamma)} &\leqslant \frac{|\eta_j|^{k_0/(1+\gamma)}}{k_0!} \sum_{l=0}^{\infty} \left(\frac{|\eta_j|^{1/(1+\gamma)}}{k_0}\right)^l = \\ &= \frac{|\eta_j|^{k_0/(1+\gamma)}/k_0!}{1 - |\eta_j|^{1/(1+\gamma)}/k_0} \leqslant 2.55. \end{aligned} \tag{3.50}$$

It remains to evaluate $\mathbf{E}|\eta_j|^{1/(1+\gamma)}$. It follows from the inequalities (3.49) and (3.41) that $\mathbf{E}|\eta_j(\omega)|^{2/(1+\gamma)} \leqslant 2C_j(\gamma)$. Hence $\mathbf{E}|\eta_j(\omega)|^{1/(1+\gamma)} \leqslant \big(2C_j(\gamma)\big)^{1/2}$. Turning back to the relations (3.49) and (3.50) and recalling that $C_j(\gamma) \geqslant 1$ we find

$$\begin{aligned} \mathbf{E} \exp\{|\eta_j(\omega)|^{1/(1+\gamma)}\} &\leqslant C_j(\gamma) + \big(2C_j(\gamma)\big)^{1/2} + C_j(\gamma) + \\ &+ 2.55\, C_j(\gamma) \leqslant 6\, C_j(\gamma). \end{aligned} \tag{3.51}$$

From (3.41) we have $\mathbf{E}|\eta_j|^k \leqslant (k!)^{1+\gamma} C_j(\gamma), \quad k \geqslant 2$. Hence it follows that

$$\mathbf{E}|\eta_j|^k \leqslant (k!)^{1+\gamma} \exp\{k(1+\gamma)\} \qquad \text{for} \quad k \geqslant \big(\ln C_j(\gamma)\big)/(1+\gamma). \tag{3.52}$$

Further, let $k < \big(\ln C_j(\gamma)\big)/(1+\gamma)$. By using inequality (3.51), we get

$$\mathbf{E} \exp\{\delta |\eta_j|^{1/(1+\gamma)}\} \leqslant \big(6\, C_j(\gamma)\big)^{\delta}, \tag{3.53}$$

where $\delta > 0$ will be specified in the course of the proof. Inequality (3.51) yields that $\mathbf{E}|\eta_j|^{k/(1+\gamma)}\delta^k/k! \leqslant \bigl(6\,C_j(\gamma)\bigr)^\delta$. Setting $1+\gamma=\beta$ and $k/\beta = m$, we find

$$\mathbf{E}|\eta_j|^m \leqslant \bigl(6\,C_j(\gamma)\bigr)^\delta \cdot \frac{(m!)^\beta \beta^{m\beta}}{\delta^{m\beta}}.$$

Putting $\delta = \bigl(\ln 6\,C_j(\gamma)\bigr)^{-1}$, we get

$$\mathbf{E}|\eta_j|^m \leqslant (m!)^\beta \beta^{m\beta} e \ln^{m\beta} 6\,C_j(\gamma).$$

Taking into account that for $k \leqslant \bigl(\ln C_j(\gamma)\bigr)/\beta$, $\quad (k!)^\beta \leqslant (k!/\beta^{k\gamma}) \ln^{k\gamma} C_j(\gamma)$, we find

$$\mathbf{E}|\eta_j|^k \leqslant \begin{cases} (k!)^\beta C_1^k(\gamma), & k > \bigl(\ln C_j(\gamma)\bigr)/\beta, \\ k! C_2^k(\gamma) \ln^{k(1+2\gamma)} 6\,C_j(\gamma), & 3 \leqslant k \leqslant \ln C_j(\gamma)/\beta, \end{cases}$$

where $C_1(\gamma) = e^\beta$, $\quad C_2(\gamma) = e\beta$. Since

$$\ln^{k(1+2\gamma)} \bigl(6\,C_j(\gamma)\bigr) \leqslant C_3^{k-2}(\gamma)\bigl(C_j(\gamma)\bigr)^{\gamma(k-2)/(1+2\gamma)},$$

where $C_3(\gamma) = \bigl(3(1+2\gamma)^2/e\gamma\bigr)^{3(1+2\gamma)} \cdot 6^{\gamma/(1+2\gamma)}$, with the help of Lemma 3.1 we see that

$$|\Gamma_k(\eta_j)| \leqslant \max\Bigl\{(k!)^{1+\gamma} C_4^k(\gamma)\,C_j(\gamma),\ k!\,C_5^{k-2}(\gamma)\,\bigl(C_j(\gamma)\bigr)^{\gamma(k-2)/(1+2\gamma)}\Bigr\}, \qquad (3.54)$$

where

$$C_4(\gamma) = 2\,C_1(\gamma) = 2\,e^\beta,$$
$$C_5(\gamma) = 2\,C_2^3(\gamma)\,C_3(\gamma) = 2(e\beta)^3\bigl(3(1+2\gamma)^2/e\gamma\bigr)^{3/(1+2\gamma)} \cdot 6^{\gamma/(1+2\gamma)}.$$

Using the inequalities (3.41) and (3.42) for $\gamma = 0$ and relation (3.47) for the k^{th} order cumulant of the r.v. $\tilde S_n = \sum\limits_{j=1}^n \eta_j$, we obtain the following estimate:

$$\begin{aligned} |\Gamma_k(\tilde S_n)| &= \Bigl|\sum_{j=1}^n \Gamma_k(\eta_j)\Bigr| \leqslant \sum_{j:C_j(0)\leqslant 1} |\Gamma_k(\eta_j)| + \sum_{j:C_j(0)>1} |\Gamma_k(\eta_j)| \leqslant \\ &\leqslant k!\,3^{k-2} \sum_{j:C_j(0)\leqslant 1} C_j(0) + k! 3^{k-2} \sum_{j:C_j(0)>1} C_j(0) \ln^{k-2}\bigl(7.5\,C_j(0)\bigr) \leqslant \\ &\leqslant k!\,3^{k-2} \ln^{k-2}(11.25\,B_n^2) \sum_{j=1}^n C_j(0) \leqslant \\ &\leqslant k!\,3^{k-2}(3/2)\,\tilde B_n^2 \ln^{k-2}(11.25\,\tilde B_n^2), \qquad k = 2,\ 3,\ \ldots . \end{aligned}$$

Next,

$$\mathbf{D}\tilde{S}_n = \sum_{j=1}^{n} \mathbf{E}\eta_j^2 = \sum_{j=1}^{n} \mathbf{E}(\delta_n \xi_j)^2 = (\delta_n B_n)^2 = \tilde{B}_n^2.$$

Since $\tilde{S}_n = \delta_n S_n$, we have $\Gamma_k(\tilde{S}_n/\tilde{B}_n) = \Gamma_k(S_n/B_n) = \Gamma_k(Z_n)$. Consequently, if (3.31) with the exponent $\gamma = 0$ is fulfilled, then

$$\begin{aligned} |\Gamma_k(Z_n)| = |\Gamma_k(\tilde{S}_n/\tilde{B}_n)| &\leqslant \big(k!\, 3^{k-2} \tilde{B}_n^2 \ln^{k-2}(11.25\, \tilde{B}_n^2)\big)/\tilde{B}_n^k \leqslant \\ &\leqslant k!\, 3^{k-2}(3/2) \ln^{k-2}(3\,\tau_n^2)/(\tau_n/2)^{k-2} \leqslant k!/\tau_n^{*\,k-2} \end{aligned} \tag{3.55}$$

for all $k = 3, 4, \ldots$, where $\tau_n^* = \tau_n/(27 \ln \tau_n)$.

Now let us estimate $\Gamma_k(Z_n)$ in the case when condition (3.31) holds with $\gamma > 0$. Employing the relations (3.42), (3.54) and (3.41) we find

$$\begin{aligned} |\Gamma_k(\tilde{S}_n)| &\leqslant \sum_{j:C_j(\gamma)\leqslant 1} |\Gamma_k(\eta_j)| + \sum_{j:C_j(\gamma)>1} |\Gamma_k(\eta_j)| \leqslant (k!)^{\beta} 3^{k-2} \sum_{j:C_j(\gamma)\leqslant 1} C_j(\gamma) + \\ &+ \max\Big\{(k!)^{\beta} C_4^k(\gamma) \sum_{j:C_j(\gamma)>1} C_j(\gamma), \\ &k!\, C_5^{k-2}(\gamma) \sum_{j:C_j(\gamma)>1} \big(C_j(\gamma)\big)^{\gamma(k-2)/(1+2\gamma)}\Big\} \leqslant \\ &\leqslant (k!)^{\beta} C_4^k(\gamma) \sum_{j=1}^{n} C_j(\gamma) + k!\, C_5^{k-2}(\gamma) \Big(\sum_{j=1}^{n} C_j(\gamma)\Big)^{\gamma(k-2)/(1+2\gamma)} \leqslant \\ &\leqslant \max\{(k!)^{\beta} C_4^k(\gamma) \tilde{B}_n^2,\ k!\, C_5^{k-2}(\gamma) 2(3/2)^{\gamma(k-2)/(1+2\gamma)} \cdot \tilde{B}_n^{2\gamma(k-2)/(1+2\gamma)}\}. \end{aligned}$$

Consequently, if $\gamma > 0$, then

$$\begin{aligned} |\Gamma_k(Z_n)| &\leqslant \max\left\{ \frac{C_4^k(\gamma) 32^{k-2} (k!)^{\beta}}{\tau_n^{k-2}},\ \frac{C_5^{k-2}(\gamma) 3^{(k-2)/(1+2\gamma)} k!}{\tau_n^2 \tau_n^{(k-2)/(1+2\gamma)}} \right\} \leqslant \\ &\leqslant \max\left\{ \frac{(k!)^{1+\gamma}}{\big(\tau_n/C_1^*(\gamma)\big)^{k-2}},\ \frac{k!}{\tau_n^2 \big(\tau_n^{1/(1+2\gamma)}/C_2^*(\gamma)\big)^{k-2}} \right\}, \end{aligned} \tag{3.56}$$

where

$$C_1^*(\gamma) = 6\, C_4^3(\gamma), \qquad C_2^*(\gamma) = 8 \cdot 3^{1/(1+2\gamma)} \cdot C_5(\gamma). \qquad \blacksquare$$

Proof of Theorem 3.5. If $\gamma = 0$, then the assertion of the theorem follows from estimate (3.55) for the k^{th} order cumulant of the r.v. Z_n and Lemma 2.3.

In the case $\gamma > 0$ by applying estimate (3.56) we find

$$|\Gamma_k(Z_n)| \leqslant \frac{(k-2)!}{\left(\tau_n^{1/(1+2\gamma)}/C^*(\gamma)\right)^{k-2}}, \qquad k = 3,\, 4,\, \dots,\, s+2,$$

where

$$C^*(\gamma) = 6\ \max\left\{\left(C_1^*(\gamma)/18\right)^{1/(1+2\gamma)},\ C_2^*(\gamma)\right\} =$$
$$= 96\cdot 3^{1/(1+2\gamma)} 6^{\gamma/(1+2\gamma)} \left(e(1+\gamma)\right)^3 \cdot \left(3(1+2\gamma)^2/e\gamma\right)^{3(1+2\gamma)}$$

and

$$s = \min\{s_1,\, s_2\}.$$

Here

$$s_1 = 2\left[\left(\tau_n^*/(\sqrt{18}\,C_1^*(\gamma))\right)^{2/(1+2\gamma)}\right] - 2 \geqslant 0.9\left(\tau_n^*/(\sqrt{18}\,C_1^*(\gamma))\right)^{2/(1+2\gamma)},$$
$$s_2 \geqslant \tau_n^{2/(1+2\gamma)}/\left(2C_2^{*^2}(\gamma)\right).$$

Hence

$$\sqrt{s} = \min\{\sqrt{s_1},\, \sqrt{s_2}\} \geqslant \tau_n^{1/(1+2\gamma)} \times$$
$$\times \max\left\{\left(\sqrt{18}\,C_1^*(\gamma)/0.3\right)^{1/(1+2\gamma)},\ 2\,C_2^*(\gamma)\right\} = \tau_n^{1/(1+2\gamma)}/\left(2\,C_2^*(\gamma)\right).$$

Consequently, for the r.v. $\xi = Z_n$ the statement of Lemma 2.2 holds in the zone $0 \leqslant x < \sqrt{s}/(3\sqrt{e}) = \tau_n^{1/(1+2\gamma)}/\left(6\sqrt{e}\,C_2^*(\gamma)\right)$.

In order to complete the proof of the theorem, it is necessary to cut off the series $\tilde{L}(t)$, defined by relation (2.65). (It is realizable by using estimate (3.65) for the k^{th} order cumulant of the Z_n and literally following the proof of Lemma 2.3.) ∎

We shall mention the condition, which is equivalent to condition (3.31) for $\gamma = 0$, proposed by A.I. Sakhanenko (Sakhanenko, 1984): there exists $\lambda > 0$ such that

$$L_n(\lambda) := \lambda \sum_{j=1}^{n} \mathbf{E}|\xi_j|^3 e^{\lambda|\xi_j|} \leqslant \mathbf{D}S_n. \tag{B}$$

PROPOSITION 3.2. *Condition* (B) *implies* (3.31) *for* $\gamma = 0$ *and* $\tau_n = \lambda B_n$. *Conversely, condition* (3.31) *for* $\gamma = 0$ *implies* (B) *with* $\lambda = \tau_n/(12\,B_n)$.

Proof. To prove the first part of the proposition, write

$$\lambda^{k-2}\mathbf{E}|\xi_j|^k/(k-3)! = \lambda\mathbf{E}|\xi_j|^3(\lambda|\xi_j|)^{k-3}/(k-3)! < \lambda\mathbf{E}|\xi_j|^3 e^{\lambda|\xi_j|}, \qquad \forall k \geqslant 3.$$

Hence, taking into account condition (B),

$$\lambda^{k-2}\frac{L_{k,n}}{(k-3)!} \leqslant \frac{\lambda}{B_n^k}\sum_{j=1}^{n}\mathbf{E}|\xi_j|^3 e^{\lambda|\xi_j|} \leqslant B_n^{2-k}.$$

Therefore

$$L_{k,n} \leqslant \frac{(k-3)!}{(\lambda B_n)^{k-2}}, \qquad \forall k \geqslant 3.$$

Conversely, if condition (3.31) is fulfilled as $\gamma = 0$, then

$$L_n(\lambda) = \lambda\sum_{k=3}^{\infty}\frac{\lambda^{k-3}}{(k-3)!}L_{k,n}B_n^k \leqslant B_n^2\sum_{k=3}^{\infty}k(k-1)(k-2)(\lambda B_n/\tau_n)^{k-2} \leqslant$$
$$\leqslant B_n^2\sum_{k=3}^{\infty}(6\,\lambda B_n/\tau_n)^{k-2} \leqslant B_n^2$$

if $\lambda = \tau_n/(12 B_n)$. ■

Now let us consider an example (see (Sakhanenko, 1984)), illustrating that under condition (3.31) with $\gamma = 0$, estimate (3.36) for the k^{th} order cumulant of the r.v. Z_n is unimprovable with an accuracy to a constant. To be more precise, we shall give an example of the sequence satisfying the conditions (B) and (3.31) as $\gamma = 0$ with $\tau = \lambda$ and such that

$$|\Gamma_k(Z_n)| \leqslant \frac{k!}{R^{k-2}}, \qquad k = 3, 4, \ldots, \tag{3.57}$$

implies $R < \pi\tau/\ln\tau$.

Let the r.v. ξ_0 take two values $\pm\sigma$ with equal probabilities, whereas $\sigma \leqslant 1/e$. Then $\mathbf{E}\xi_0 = 0$ and $\mathbf{D}\xi_0 = \sigma^2$. Further, let $\xi_1, \xi_2, \ldots, \xi_n$ and ξ_0 be independent r.v. with a common normal distribution function. Let

$$S = \xi_0 + S', \quad S' = \xi_1 + \ldots + \xi_n.$$

Suppose $\mathbf{D}S = 1$, then $\mathbf{D}S' = 1 - \sigma^2$ and $\mathbf{D}\xi_1 = (1-\sigma^2)/n$. In this case

$$L_{k,n} = \mathbf{E}|\xi_0|^k + \sum_{j=1}^{n}\mathbf{E}|\xi_j|^k \to \sigma^k \qquad (n\to\infty)$$

and

$$L_n(\lambda) = \lambda\mathbf{E}|\xi_0|^3 e^{\lambda|\xi_0|} + \lambda\sum_{j=1}^{n}\mathbf{E}|\xi_j|^3 e^{\lambda|\xi_j|} \to \lambda\mathbf{E}|\xi_0|^3 e^{\lambda|\xi_0|} \qquad (n\to\infty).$$

In turn, for $\lambda = (1/\sigma)\ln(1/\sigma)$

$$\lambda \mathbf{E}|\xi_0|^3 e^{\lambda|\xi_0|} = \lambda\sigma^3 e^{\lambda\sigma} = (1/\sigma)\ln(1/\sigma)\sigma^3 \exp\{\sigma(1/\sigma)\ln(1/\sigma)\} = \\ = \sigma\ln(1/\sigma) < \sigma(1/\sigma) = 1,$$

since $\ln(1/\sigma) \leqslant (1/\sigma) - 1 < 1/\sigma$, $\quad \sigma < 1/e$. Consequently, condition (B) is fulfilled. Thus, in accordance with Proposition 3.2, condition (3.31) holds with $\gamma = 0$ and $\tau = \lambda = (1/\sigma)\ln(1/\sigma)$.

Next, for any complex number z,

$$\varphi(z) := \ln \mathbf{E}e^{zS} = \ln \mathbf{E}e^{z\xi_0} + \ln \mathbf{E}e^{zS'} = \ln \mathbf{E}e^{z\xi_0} + \sum_{k=2}^{\infty} \frac{1}{k!}\Gamma_k(S')z^k = \\ = \ln \mathbf{E}e^{z\xi_0} + \Gamma_2(S')z^2/2 = \ln \mathbf{E}e^{z\xi_0} + (1-\sigma^2)z^2/2.$$

Moreover,

$$\mathbf{E}e^{z\xi_0} = (e^{\sigma z} + e^{-\sigma z})/2 = \cos(i\sigma z) = 0$$

as $z = i(\pi/2\sigma)$, since $\cos(i\sigma z) = \cos(i^2\pi\sigma/2\sigma) = \cos(-\pi/2) = 0$. Put $R_0 = \pi/2\sigma$, then for $z = iR_0$ we have $\mathbf{E}e^{z\xi_0} = 0$, or $\ln \mathbf{E}e^{z\xi_0} = -\infty$.

Since condition (3.57) implies analyticity of the function $\varphi(z)$ in the circle $|z| < R$, we have proved the following fact: if in the case considered condition (3.57) is true, then $R < R_0$.

Since $\tau = \lambda = (1/\sigma)\ln(1/\sigma)$,

$$R_0 = \pi/(2\sigma) = \pi(1/\sigma)\ln(1/\sigma)/\big(2\ln(1/\sigma)\big) = \pi\tau/\big(2\ln(1/\sigma)\big).$$

As $\ln(1/\sigma) \leqslant (1/\sigma) - 1 < 1/\sigma$, it follows that $\tau = (1/\sigma)\ln(1/\sigma) < (1/\sigma)(1/\sigma) < \sigma^{-2}$. Hence $\ln\tau \leqslant \ln(1/\sigma^2)$, i.e. $\ln\tau < 2\ln(1/\sigma)$. Therefore $R_0 < \pi\tau/\ln\tau$. ∎

CHAPTER 4
THEOREMS OF LARGE DEVIATIONS FOR SUMS OF DEPENDENT RANDOM VARIABLES

Let X_t, $t = 1, 2, \ldots$, be a random process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\{\mathcal{F}_s^t, \ 1 \leqslant s \leqslant t < \infty\}$ be a family of σ–algebras such that

$$
\begin{aligned}
&1)\ \mathcal{F}_s^t \subset \mathcal{F}, \qquad \forall s \leqslant t;\\
&2)\ \mathcal{F}_{s_1}^{t_1} \subset \mathcal{F}_{s_2}^{t_2}, \qquad \forall [s_1, t_1] \subset [s_2, t_2];\\
&3)\ \mathcal{F}_s^t \supset \sigma\{X_u, \ s \leqslant u \leqslant t\}.
\end{aligned}
$$

As usual, the functions of α–, φ– and ψ–mixing are defined by the following relations:

$$\alpha(s, t) = \sup_{A \in \mathcal{F}_1^s, \ B \in \mathcal{F}_t^\infty} |\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)|$$

(Rosenblatt, 1956),

$$\varphi(s, t) = \sup_{A \in \mathcal{F}_1^s, \ B \in \mathcal{F}_t^\infty} \left| \frac{\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)}{\mathbf{P}(A)} \right|$$

(Ibragimov, 1959),

$$\psi(s, t) = \sup_{A \in \mathcal{F}_1^s, \ B \in \mathcal{F}_t^\infty} \left| \frac{\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)}{\mathbf{P}(A)\mathbf{P}(B)} \right|$$

(Blum, Hanson, Koopmans, 1963).

Combinatorial analysis is often used in this chapter. Let us introduce standard notation.

1. *Constants.* Positive constants will be denoted by letters C, K, a, b with or without indices.

2. *Sets* (definitions are taken from (Sachkov, 1982)). Finite sets will be denoted by letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{I}$, $\mathcal{J}$ eventually in the square brackets [ ], with or without

indices, taking into account that the sets below have a fixed structure throughout this chapter:

$$\mathcal{A} = \{a_1, \ldots, a_s\}, \qquad a_1 \leqslant \ldots \leqslant a_s,$$
$$\mathcal{N} = \{1, 2, \ldots, n\},$$
$$I = \{t_1, \ldots, t_k | t_j \in \mathcal{N}\}, \qquad t_1 \leqslant \ldots \leqslant t_k,$$

$\mathcal{J} = \{l_1,, \ldots, l_r\}$ is the support of the k–set I, i.e. $l_1 < \ldots < l_r$ are the points of growth of the sequence $t_1, \ldots, t_k$,

$\{I_1, \ldots, I_\nu\}$ is a partition of the set I,

$$I_p = \{t_1^{(p)}, \ldots, t_{k_p}^{(p)}\}, \qquad t_1^{(p)} \leqslant \ldots \leqslant t_{k_p}^{(p)}, \quad 1 \leqslant p \leqslant \nu, \quad k_1 + \ldots + k_\nu = k,$$

$\{\mathcal{J}_1, \ldots, \mathcal{J}_\nu\}$ is a partition of the set $\mathcal{J}$, corresponding to the partition $\{I_1, \ldots, I_\nu\}$ of the set I, so that $\mathcal{J}_p$ is the support of the k_p–set I_p,

$$\mathcal{J}_p = \{l_1^{(p)}, \ldots, l_{r_p}^{(p)}\}, \qquad l_1^{(p)} < \ldots < l_{r_p}^{(p)}, \quad 1 \leqslant p \leqslant \nu, \quad r_1 + \ldots + r_\nu \geqslant r,$$

$(m_1, \ldots, m_r)$ is the vector of primary specification indices of the set I, generated by the set $\mathcal{J}$, $m_1 + \ldots + m_r = k$, i.e. $m_j = \big|\{i | t_i = l_j\}\big|$, $1 \leqslant j \leqslant r$,

$(m_1^{(p)}, \ldots, m_{r_p}^{(p)})$ is the vector of primary specification indices of the set I_p, generated by the set $\mathcal{J}_p$, $m_1^{(p)} + \ldots + m_{r_p}^{(p)} = k_p$, $1 \leqslant p \leqslant \nu$. Let us define the operation $[\mathcal{A}]_I$ for an arbitrary A and the set I:

$$[\mathcal{A}] := [\mathcal{A}]_I = \{t \in I | a_1 \leqslant t \leqslant a_s\} = [a_1, a_s] \cap I.$$

It is evident that

$$[I] = [\mathcal{J}] = I,$$
$$[I_p] = [\mathcal{J}_p].$$

Let us introduce the following notation:

$$\mathcal{N}_u^v = \{u, u+1, \ldots, v\}, \qquad u, v \in \mathcal{N}, \quad u \leqslant v \leqslant n,$$
$$\mathcal{N}_u = \{u, u+1, \ldots, n\}, \qquad u \in \mathcal{N}, \quad u \leqslant n,$$
$$\mathcal{A}^{(i)} = \mathcal{A} \backslash \{a_1, \ldots, a_i\}, \qquad i \leqslant s.$$

$|\mathcal{A}|$ will stand for the number of elements of the set $\mathcal{A}$.

3. *Sums.* The s–th Cartesian power of the set $\mathcal{A}$ is

$$_{(s)}\mathfrak{A} = \underbrace{\mathcal{A} \times \ldots \times \mathcal{A}}_{s \text{ times}}.$$

In a Cartesian power the indices are retained. For example,

$$_{(s)}[\mathfrak{A}]^{(i)} = \underbrace{[\mathcal{A}]^{(i)} \times \ldots \times [\mathcal{A}]^{(i)}}_{s \text{ times}},$$

$$_{(s)}\mathfrak{N}_u^v = \underbrace{\mathcal{N}_u^v \times \ldots \times \mathcal{N}_u^v}_{s \text{ times}}.$$

Taking into account the introduced notation we shall write

$\sum\limits_{\mathfrak{a} \in {}_{(s)}\mathfrak{A}}$ for the sum over all $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_s) \in {}_{(s)}\mathfrak{A}$,

$\sum\limits_{\mathfrak{a} \in {}_{(s)}\mathfrak{A}}^{(\leqslant)}$ for the sum over all $\mathfrak{a} \in {}_{(s)}\mathfrak{A}$ such that $\mathfrak{a}_1 \leqslant \ldots \leqslant \mathfrak{a}_s$,

$\sum\limits_{\mathfrak{a} \in {}_{(s)}\mathfrak{A}}^{(<)}$ for the sum over all $\mathfrak{a} \in {}_{(s)}\mathfrak{A}$ such that $\mathfrak{a}_1 < \ldots < \mathfrak{a}_s$.

Sometimes in $_{(s)}\mathfrak{A}$ we shall omit s and write simply $\mathfrak{A}$, if the number of coordinates of the vector $\mathfrak{a}$ which defines the value of index (s) is known.

Wherever convenient, the vector $\mathfrak{a}$ will be replaced by a set $\mathcal{B}$ in the notation of domains of summation; in this case the summation will be taken over all elements of the set $\mathcal{B}$ with values from $\mathcal{A}$ for each of them, the value of index (s) being equal to the cardinality of the set $\mathcal{B}$;

$\sum\limits_{\bigcup_{p=1}^{\nu} I_p = I}$ will denote the sum over all ν–block partitions $\{I_1, \ldots, I_\nu\}$ of the set I.

Following (Andrews, 1976), any finite sequence of positive integers $k_1, \ldots, k_\nu$ will be called a decomposition of a positive integer k, if $\sum\limits_{p=1}^{\nu} k_p = k$.

By $c(k, \nu)$ let us denote the number of all such decompositions of the number k in ν components and by $\sum\limits_{\lambda_1+\ldots+\lambda_\nu=k}$ the sum over all such decompositions. Then

$$c(k, \nu) = \binom{k-1}{\nu-1} = \frac{(k-1)!}{(\nu-1)!(k-\nu)!}, \tag{4.1}$$

$$c(k, \nu) = \sum_{k_1+\ldots+k_\nu=k} 1. \tag{4.2}$$

Some facts of the theory of partitions ((Sachkov, 1982), (Andrews, 1976)) will be needed.

The number $s(k, \nu)$ of ways of partitioning a k–element set into ν nonempty subsets is called the Stirling number of the second kind. In our notations they can be expressed as either

$$s(k, \nu) = |\{I_1, \ldots, I_\nu\}|, \tag{4.3}$$

either

$$s(k, \nu) = \sum_{\bigcup\limits_{p=1}^{\nu} I_p = I} 1, \tag{4.4}$$

or

$$s(k, \nu) = \sum_{k_1+\ldots+k_\nu=k} \frac{k!}{k_1! \ldots k_\nu! \nu!}. \tag{4.5}$$

One must note that the Stirling numbers of the second kind $s(k, \nu)$ serve as coefficients in the expansion of x^k in the basis $1, (x)_1, (x)_2, \ldots, (x)_k$ $\big((x)_k = x(x-1) \ldots (x-k+1)\big)$, namely

$$x^k = \sum_{\nu=0}^{k} s(k, \nu)(x)_\nu. \tag{4.6}$$

4. *Classes of generalized mixing functions.* Let

$$\begin{aligned} \mathcal{K} := \{f \in L_\infty(R^2) | 0 \leqslant f(s_1, t_1) \leqslant f(s, t) \leqslant f(s_2, t_2),& \\ [s_2, t_2] \subset [s, t] \subset [s_1, t_1]\}.& \end{aligned} \tag{4.7}$$

Every function $f \in \mathcal{K}$ will be called a generalized mixing function. Let

$$d(f, g) := \sup_{(s,t)\in R^2} |f(s, t) - g(s, t)|$$

be the distance in $\mathcal{K}$. Denote

$$\mathcal{K}^{(\leqslant 1)} = \{f \in \mathcal{K} | d(0, f) \leqslant 1\}, \tag{4.8}$$

$$\mathcal{K}^{(\geqslant 1)} = \{f \in \mathcal{K} | d(0, f) \geqslant 1\}. \tag{4.9}$$

Then

$$\alpha,\ \varphi,\ \psi,\ \bar{m} \in \mathcal{K},$$
$$\alpha,\ \varphi,\ \bar{m} \in \mathcal{K}^{(\leqslant 1)},$$
$$4\alpha,\ \varphi,\ \psi,\ \bar{m} \in \mathcal{K}^{(\geqslant 1)},$$

where $\bar{m}(s, t) = \mathbf{1}_{\{t-s\leqslant m\}}(s, t)$ is the function of m–dependence.

Theorems 4.1 – 4.31 of this chapter were obtained by D. Jakimavičius and V. Statulevičius (Jakimavičius, Statulevičius, 1987, 1988)*) .

*) All the assertions are also valid for an arbitrary $t \in T \subset R$.

4.1. Estimates of the k^{th} order centered moments of random processes with mixing

The upper estimates of $\widehat{\mathbf{E}}X_sX_t = \mathbf{E}X_sX_t - \mathbf{E}X_s\mathbf{E}X_t$, expressed through the functions α, φ or ψ, are very important in limit theorems for sums

$$S_n = \sum_{t=1}^{n} X_t$$

of dependent random variables under different mixing conditions. Remind the basic ones:

$$|\widehat{\mathbf{E}}X_sX_t| \leqslant 4C^2\alpha(s,\, t), \tag{A}$$

if $|X_s| \leqslant C$ and $|X_t| \leqslant C$ with probability 1 (Volkonskii, Rozanov, 1959);

$$|\widehat{\mathbf{E}}X_sX_t| \leqslant 6\alpha^{1-\frac{1}{u}-\frac{1}{v}}(s,\, t)\mathbf{E}^{\frac{1}{u}}|X_s|^u\mathbf{E}^{\frac{1}{v}}|X_t|^v \tag{B}$$

for any $u \geqslant 1,\ v \geqslant 1,\ (1/u) + (1/v) \leqslant 1$, if $\mathbf{E}|X_s|^u$ and $\mathbf{E}|X_t|^v$ are finite (Davydov, 1968);

$$|\widehat{\mathbf{E}}X_sX_t| \leqslant 2\varphi^{\frac{1}{p}}(s,\, t)\mathbf{E}^{\frac{1}{p}}|X_s|^p\mathbf{E}^{\frac{1}{q}}|X_t|^q \tag{C}$$

for any $p \geqslant 1,\ q \geqslant 1,\ (1/p) + (1/q) = 1$, if $\mathbf{E}|X_s|^p$ and $\mathbf{E}|X_t|^q$ are finite (Ibragimov, 1959);

$$|\widehat{\mathbf{E}}X_sX_t| \leqslant \psi(s,\, t)\mathbf{E}|X_s|\mathbf{E}|X_t|, \tag{D}$$

if there exist $\mathbf{E}|X_s|$ and $\mathbf{E}|X_t|$ (Philipp, 1969).

Let us generalize these estimates for $\widehat{\mathbf{E}}X_{t_1}\,\ldots\,X_{t_k}$.

THEOREM 4.1. *If* $|X_{t_j}| \leqslant C$ *with probability* 1, $\quad j = 1, \ldots, k, \quad k = 2, 3, \ldots,$ *then for all* $i = 1, \ldots, k-1$

$$1)\ |\widehat{\mathbf{E}}X_{t_1}\,\ldots\,X_{t_k}| \leqslant 2^kC^k\alpha(t_i,\, t_{i+1}),$$

$$2)\ |\widehat{\mathbf{E}}X_{t_1}\,\ldots\,X_{t_k}| \leqslant 2^{k-1}C^k\varphi(t_i,\, t_{i+1}),$$

$$3)\ |\widehat{\mathbf{E}}X_{t_1}\,\ldots\,X_{t_k}| \leqslant 2^{k-2}C^k\psi(t_i,\, t_{i+1}).$$

THEOREM 4.2. *If for some collection* $p_j \geqslant 1,\ j = 1, \ldots, k$, *such that*

$$\sum_{j=1}^{k} \frac{1}{p_j} \leqslant 1, \qquad k = 2, 3, \ldots,$$

there exist $\mathbf{E}|X_{t_j}|^{p_j}$, $j = 1, \ldots, k$, *then for all* $i = 1, \ldots, k-1$

$$1)\ |\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 3\cdot 2^{k-1} \alpha^{1-\sum_{j=1}^{k} \frac{1}{p_j}} (t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{p_j}} |X_{t_j}|^{p_j},$$

$$2)\ |\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-1} \varphi^{\sum_{j=1}^{i} \frac{1}{p_j}} (t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{p_j}} |X_{t_j}|^{p_j},$$

$$3)\ |\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-2} \psi(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{p_j}} |X_{t_j}|^{p_j}.$$

COROLLARIES of Theorem 4.2. 1) a) *If for some* $k \in \{2, 3, \ldots\}$ *and* $\delta \geqslant 0$ *there exist* $\mathbf{E}|X_{t_j}|^{(1+\delta)k}$, $j = 1, \ldots, k$, *then for all* $i = 1, \ldots, k-1$

$$|\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 3\cdot 2^{k-1} \alpha^{\frac{\delta}{1+\delta}} (t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{(1+\delta)k}} |X_{t_j}|^{(1+\delta)k},$$

b) *if for some* $\gamma_1 \geqslant 0$, $H_1 > 0$

$$\mathbf{E}|X_{t_j}|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad j = 1, \ldots, k,\ k = 2, 3, \ldots,^{*)}$$

then for all $i = 1, \ldots, k-1$, $\delta \geqslant 0$

$$|\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 3(k!)^{1+\gamma_1} 2^{k-1} H_1^k \widehat{(1+\delta)}^{(1+\gamma_1)k} \alpha^{\frac{\delta}{1+\delta}} (t_i, t_{i+1}),$$

where $\hat{u} = \min\{v \geqslant u | v$ *is integer*$\}$*;*

2) a) *if for some* $k \in \{2, 3, \ldots\}$ *and* $\delta \geqslant 0$ *there exist* $\mathbf{E}|X_{t_j}|^{(1+\delta)k}$, $j = 1, \ldots, k$, *then for all* $i = 1, \ldots, k-1$

$$|\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-1} \varphi^{\frac{\delta}{1+\delta}} (t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{(1+\delta)k}} |X_{t_j}|^{(1+\delta)k},$$

---

*) The estimates of $\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}$, $\Gamma(X_{t_1}, \ldots, X_{t_k})$ and $\Gamma_k(S_n)$ under the condition of the type $\mathbf{E}|X_t^k| \leqslant (k!)^{1+\gamma} H^k$ for the case $\gamma > 0$ were generalized by D. Jakimavičius (Jakimavičius, 1988).

b) *if for some* $\gamma_1 \geqslant 0$, $H_1 > 0$ $\mathbf{E}|X_{t_j}|^k \leqslant (k!)^{1+\gamma_1} H_1^k$, $j = 1, \ldots, k$, $k = 2, 3, \ldots$, *then for all* $i = 1, \ldots, k-1$, $\delta \geqslant 0$

$$|\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant (k!)^{1+\gamma_1} 2^{k-1} H_1^k \widehat{(1+\delta)}^{(1+\gamma_1)k} \varphi^{\frac{\delta}{1+\delta}}(t_i, t_{i+1}).$$

Let $l_1 < \ldots < l_r$ be the points of growth of the sequence $t_1 \leqslant \ldots \leqslant t_k$, m_j be the number of elements in $\{t_1, \ldots, t_k\}$, equal to l_j. In the sequel while formulating Theorems 4.3, 4.5, 4.6, 4.10, 4.12, 4.14 and their corollaries the numbers m_j will be considered determined and the explanation of their structure will not be given.

THEOREM 4.3. *If for some* $k \in \{2, 3, \ldots\}$

$$|\mathbf{E}(X_{l_j}^{m_j} | \mathcal{F}_1^{l_j - 1})| < \infty \qquad \text{with probability } 1, \quad j = 1, 2, \ldots, r,$$

then for all $i = 1, \ldots, k-1$

$$|\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 2^k \alpha(t_i, t_{i+1}) \prod_{j=1}^{r} \operatorname{ess\,sup} |\mathbf{E}(X_{l_j}^{m_j} | \mathcal{F}_1^{l_j - 1})|.$$

COROLLARY of Theorem 4.3. *If for some* $\gamma_2 \geqslant 0$, $H_2 > 0$ $|\mathbf{E}(X_{l_j}^k | \mathcal{F}_1^{l_j - 1})| \leqslant (k!)^{1+\gamma_2} H_2^k$ *with probability* 1, $j = 1, \ldots, r$, $k = 2, 3, \ldots$, *then for all* $i = 1, \ldots, k-1$

$$|\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant \prod_{j=1}^{r} (m_j!)^{1+\gamma_2} 2^k H_2^k \alpha(t_i, t_{i+1}).$$

Let us consider a case when the variables X_t are related to a Markov chain ξ_t (i.e. $X_t = g_t(\xi_t)$, where $g_t(x)$ is a measurable function for each t) with transition probabilities

$$\mathbf{P}_{s,t}(x, A) = \mathbf{P}(\xi_t \in A | \xi_s = x), \qquad \mathbf{P}_t(A) = \mathbf{P}(\xi_t \in A).$$

Let $\mathcal{F}_s^t = \sigma\{\xi_u,\ s \leqslant u \leqslant t\}$. Then

$$\varphi(s, t) = \sup_{x, A \in \mathcal{F}_s^t} |\mathbf{P}_{s,t}(x, A) - \mathbf{P}_t(A)| \leqslant 1 - \alpha_{s,t},$$

where $\alpha_{s,t}$ is the ergodicity coefficient

$$\alpha_{s,t} = 1 - \sup_{x, y, A \in \mathcal{F}_s^t} |\mathbf{P}_{s,t}(x, A) - \mathbf{P}_{s,t}(y, A)|$$

(see (Dobrushin, 1953, 1956)).

Let $\alpha^{(n)} = \min_{1\leqslant s<n} \alpha_{s,s+1}$ be the ergodicity coefficient of the chain. It is known that $1-\alpha_{s,t} \leqslant (1-\alpha^{(n)})^{t-s} \leqslant \exp\{-\alpha^{(n)}(t-s)\}$ for all $1 \leqslant s \leqslant t \leqslant n$.

THEOREM 4.4. *Let X_t be related to a Markov chain ξ_t. If $|X_{l_j}| \leqslant C$ with probability* 1, $j=1,\dots,r$, $r=2,3,\dots$, *then*

$$1)\ |\widehat{\mathbf{E}}\, X_{t_1}\dots X_{t_k}| \leqslant 2^{k-1} C^k \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}),$$

$$2)\ |\widehat{\mathbf{E}}\, X_{t_1}\dots X_{t_k}| \leqslant 2^{k-r} C^k \prod_{j=1}^{r-1} \psi(l_j, l_{j+1}).$$

THEOREM 4.5. *Let X_t be related to a Markov chain ξ_t.*

1. *If for some collection $q_j \geqslant 1$, $j=1,\dots,r$, $r=2,3,\dots$, such that $\sum_{j=1}^{r} \frac{1}{q_j} =$ 1, there exist $\mathbf{E}|X_{l_j}|^{m_j q_j}$, $j=1,\dots,r$, then*

$$|\widehat{\mathbf{E}}\, X_{t_1}\dots X_{t_k}| \leqslant 2^{k-1} \prod_{j=1}^{r-1} \varphi^{\sum_{i=1}^{j}\frac{1}{q_i}}(l_j, l_{j+1}) \prod_{j=1}^{r} \mathbf{E}^{\frac{1}{q_j}} |X_{l_j}|^{m_j q_j}.$$

2. *If for some $r \in \{2,3,\dots\}$ there exist $\mathbf{E}|X_{l_j}|^{m_j}$, $j=1,\dots,r$, then*

$$|\widehat{\mathbf{E}}\, X_{t_1}\dots X_{t_k}| \leqslant 2^{k-r} \prod_{j=1}^{r-1} \psi(l_j, l_{j+1}) \prod_{j=1}^{r} \mathbf{E}|X_{l_j}|^{m_j}.$$

COROLLARY of Theorem 4.5. *Let X_t be related to a Markov chain ξ_t.*

1) a) *If for some $k \in \{2,3,\dots\}$ and $\delta \geqslant 0$ there exist $\mathbf{E}|X_{t_j}|^{(1+\delta)k}$, $j=1,\dots,r$, then*

$$|\widehat{\mathbf{E}}\, X_{t_1}\dots X_{t_k}| \leqslant 2^{k-1} \prod_{j=1}^{r-1} \varphi^{\frac{\delta}{1+\delta}}(l_j, l_{j+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{(1+\delta)k}} |X_{t_j}|^{(1+\delta)k}.$$

b) *If for some $\gamma_1 \geqslant 0$, $H_1 > 0$*

$$\mathbf{E}|X_{l_j}|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad j=1,\dots,r, \quad r=2,3,\dots, \quad k=2,3,\dots,$$

then for any $\delta \geqslant 0$

$$|\widehat{\mathbf{E}}\, X_{t_1} \, \ldots \, X_{t_k}| \leqslant 2^{k-1}(k!)^{1+\gamma_1}(\widehat{1+\delta})^{(1+\gamma_1)k} H_1^k \prod_{j=1}^{r-1} \varphi^{\frac{\delta}{1+\delta}}(l_j, \, l_{j+1}).$$

2) *If for some* $\gamma_1 \geqslant 0, \quad H_1 > 0$

$$\mathbf{E}|X_{l_j}|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad j = 1, \ldots, r, \quad r = 2, 3, \ldots, \quad k = 2, 3, \ldots,$$

then

$$|\widehat{\mathbf{E}}\, X_{t_1} \, \ldots \, X_{t_k}| \leqslant 2^{k-r} H_1^k \prod_{j=1}^{r}(m_j!)^{1+\gamma_1} \prod_{j=1}^{r-1} \psi(l_j, \, l_{j+1}).$$

THEOREM 4.6. *Let* X_t *be related to a Markov chain* ξ_t. *If for some* $r \in \{2, 3, \ldots\}$

$$|\mathbf{E}(X_{l_j}^{m_j} | \mathcal{F}_1^{l_{j-1}})| < \infty \qquad \text{with probability } 1, \quad j = 1, \, 2, \, \ldots, \, r,$$

then

$$|\widehat{\mathbf{E}}\, X_{t_1} \, \ldots \, X_{t_k}| \leqslant 2^{k-1} \prod_{j=1}^{r-1} \varphi(l_j, \, l_{j+1}) \prod_{j=1}^{r} \operatorname{ess\,sup} |\mathbf{E}(X_{l_j}^{m_j} | \mathcal{F}_1^{l_{j-1}})|.$$

COROLLARY of Theorem 4.6. *Let* X_t *be related to a Markov chain* ξ_t. *If for some* $\gamma_2 \geqslant 0, \quad H_2 > 0$

$$|\mathbf{E}(X_{l_j}^k | \mathcal{F}_1^{l_{j-1}})| \leqslant (k!)^{1+\gamma_2} H_2^k \qquad \text{with probability } 1,$$

$j = 1, \ldots, r, \quad r = 2, 3, \ldots, \quad k = 2, 3, \ldots,$ *then*

$$|\widehat{\mathbf{E}}\, X_{t_1} \, \ldots \, X_{t_k}| \leqslant 2^{k-1} H_2^k \prod_{j=1}^{r}(m_j!)^{1+\gamma_2} \prod_{j=1}^{r-1} \varphi(l_j, \, l_{j+1}).$$

Proof of Theorems 4.1 – 4.6. Remind that

$$I_p \in \{I_1, \ldots, I_\nu\}, \quad I_p = \{t_1^{(p)}, \ldots, t_{k_p}^{(p)}\}, \quad \mathcal{J}_p = \{l_1^{(p)}, \ldots, l_{r_p}^{(p)}\}$$

is the support of the set I_p, $\quad (m_1^{(p)}, \ldots, m_{r_p}^{(p)})$ is a vector of primary specification indices of the set I_p with respect to $\mathcal{J}_p$, $\quad 1 \leqslant p \leqslant \nu, \quad 1 \leqslant \nu \leqslant k$.

Within the limits of the present section we fix the index p and omit it, hoping that this will not lead to ambiguity.

Let f be a some mixing function corresponding to the random process (i.e. its arguments are the arguments of the process (points of time)).

In the stationary case f depends only on $t-s$, therefore the relations

$$t_{i+1}-t_i=\max_{1\leqslant j<k}(t_{j+1}-t_j),$$

$$f(t_i,\,t_{i+1})=\min_{1\leqslant j<k}f(t_j,\,t_{j+1})$$

are equivalent. The interval $[t_i,\,t_{i+1}]$ is called the maximal interval.

In the nonstationary case these relations are not equivalent, however, following a tradition, we shall adhere to the term "maximal interval" (Zhurbenko, 1972), (Zuev, 1973), assuming that it depends on the mixing function and is defined by the second relation.

As it is known, estimates of the cumulants of random variables related to a Markov chain (Statulevičius, 1969, 1970) (or satisfying the more general condition RMT (Zhurbenko, 1972, 1982), (Zuev, 1981), (Statulevičius, 1970, 1979, 1983), (Rosenblatt, 1979)) are expressed by the product of mixing functions. In the case of arbitrary variables it is impossible to achieve this. In this case one has to restrict oneself to estimates of the "maximal interval".

One should note that the inequalities

$$\alpha(s,\,t)\leqslant\varphi(s,\,t)\leqslant\psi(s,\,t)$$

make possible passage from estimates in terms of $\alpha(s,\,t)$ to $\varphi(s,\,t)$ and from $\varphi(s,\,t)$ to $\psi(s,\,t)$ (see (Iosifescu, 1980)) by means of direct change of mixing functions.

The estimates for the centered moments will be proved in the following manner.

Associate random variables $Y_{t_1},\ldots,Y_{t_k}$ with the random variables $X_{t_1},\ldots,X_{t_k}$ by the relations

$$\begin{aligned}Y_{t_j}&=X_{t_j}\widehat{Y_{t_{j+1}}},\qquad 1\leqslant j<k,\\ Y_{t_k}&=X_{t_k},\end{aligned}\tag{4.10}$$

where the symbol " $\frown$ " denotes the operation of centering of a random variable by its mean: $\widehat{\xi}=\xi-\mathbf{E}\xi$.

Obviously, for all $i,\ j,\ 1\leqslant j\leqslant i<k$,

$$Y_{t_j}=X_{t_j}\widehat{X_{t_{j+1}}\ldots\widehat{X_{t_i}\widehat{Y}_{t_{i+1}}}},\tag{4.11}$$

in particular,

$$Y_{t_j} = X_{t_j}\widehat{X_{t_{j+1}} \dots \widehat{X_{t_{k-1}} X_{t_k}}}, \tag{4.12}$$

$$\mathbf{E}Y_{t_1} = \widehat{\mathbf{E}}\, X_{t_1} \dots X_{t_k}. \tag{4.13}$$

Due to measurability of $X_{t_j} \dots X_{t_i}$ with respect to $\mathcal{F}_1^{t_i}$ we obtain from (4.11)

$$\mathbf{E}(Y_{t_j} | \mathcal{F}_1^{t_i}) = X_{t_j}\widehat{X_{t_{j+1}} \dots \widehat{X_{t_i} \mathbf{E}(Y_{t_{i+1}} | \mathcal{F}_1^{t_i}}}), \tag{4.14}$$

$$\mathbf{E}Y_{t_1} = \mathbf{E}X_{t_1}\widehat{X_{t_2} \dots \widehat{X_{t_i} \mathbf{E}(Y_{t_{i+1}} | \mathcal{F}_1^{t_i}}}). \tag{4.15}$$

Now, in order to estimate $\mathbf{E}^{\frac{1}{u}} |\mathbf{E}(Y_{t_{i+1}} | \mathcal{F}_1^{t_i})|^u$, $\quad u \geqslant 1$, in terms of mixing functions we need following inequalities.

LEMMA 4.1. *If a r.v. Y is $\mathcal{F}_t^\infty$-measurable, then for any u and v, $1 \leqslant u \leqslant v$,*

$$1)\ \mathbf{E}^{1/u} |\mathbf{E}(Y|\mathcal{F}_1^s) - \mathbf{E}Y|^u \leqslant 2(1 + 2^{1/u})\big(\alpha(s,\, t)\big)^{1/u - 1/v} \mathbf{E}^{1/v} |Y|^v \tag{4.16}$$

(McLeish, 1975)*;*

$$2)\ \mathbf{E}^{1/u} |\mathbf{E}(Y|\mathcal{F}_1^s) - \mathbf{E}Y|^u \leqslant 2\big(\varphi(s,\, t)\big)^{1 - 1/v} \mathbf{E}^{1/v} |Y|^v \tag{4.17}$$

(Serfling, 1968)*;*

$$3)\ \mathbf{E}^{1/u} |\mathbf{E}(Y|\mathcal{F}_1^s) - \mathbf{E}Y|^u \leqslant \psi(s,\, t)\mathbf{E}|Y|. \tag{4.18}$$

Note that the inequality (4.18) also holds as $u \to \infty$, i.e.

$$\operatorname{ess\,sup} |\mathbf{E}(Y|\mathcal{F}_1^s) - \mathbf{E}Y| \leqslant \psi(s,\, t)\mathbf{E}|Y|,$$

and it is a slight modification of assertion (D), where $Y_t = Y$.

A method for estimating of centered moments is based on successive application of the Hölder and Minkowski inequalities to equality (4.15) as well on relation (4.13). Namely,

$$\begin{aligned} |\widehat{\mathbf{E}}\, X_{t_1} \dots X_{t_k}| &\leqslant 2^{i-1} \mathbf{E}^{1/u_1} |X_{t_1}|^{u_1} \mathbf{E}^{1/(v_1 u_2)} |X_{t_2}|^{v_1 u_2} \times \dots \times \\ &\times \mathbf{E}^{1/(v_1 \dots v_{i-2} u_{i-1})} |X_{t_{i-1}}|^{v_1 \dots v_{i-2} u_{i-1}} \mathbf{E}^{1/(v_1 \dots v_{i-1})} |X_{t_i} \mathbf{E}\widehat{(Y_{t_{i+1}} | \mathcal{F}_1^{t_i}})|^{v_1 \dots v_{i-1}}, \end{aligned} \tag{4.19}$$

where $1/u_j + 1/v_j = 1$, $\quad u_j,\, v_j \geqslant 1$, $\quad j = 1, \dots, i$, $\quad i = 1, \dots, k-1$.

Proof of Theorem 4.1. By virtue of boundedness of variables X_t, $t = t_1, \dots, t_k$, for any $i = 1, \dots, k-1$ as $u_1, \dots, u_{i-1} \to 0$ we obtain

$$|\widehat{\mathbf{E}}\, X_{t_1} \dots X_{t_k}| \leqslant 2^{i-1} C^{i-1} \mathbf{E}|X_{t_i}(\mathbf{E}(Y_{t_{i+1}} | \mathcal{F}_1^{t_i}) - \mathbf{E}Y_{t_{i+1}})|.$$

Under the conditions of the theorem it follows from (4.12) that $|Y_{t_{i+1}}| \leqslant 2^{k-i-1}C^{k-i}$ with probability 1. The proof of theorem 4.1 is completed by applying the estimates (A), (C), and (D). ■

REMARK to Theorem 4.1. *Under conditions of theorem 4.1*

$$|\widehat{\mathrm{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 2\, C^k \varphi^{1/2}(t_i,\, t_{i+1}). \tag{4.20}$$

Proof. Since

$$\begin{gathered}\mathrm{E}^{1/2}|Y_{t_j}|^2 \leqslant C\,\mathrm{E}^{1/2}|Y_{t_{j+1}} - \mathrm{E}Y_{t_{j+1}}|^2 \leqslant C\,\mathrm{E}^{1/2}|Y_{t_{j+1}}|^2, \\ j = 1,\, \ldots,\, i-1, \quad i+1,\, \ldots,\, k-1,\end{gathered} \tag{4.21}$$

we obtain

$$|\widehat{\mathrm{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant C^{i-1}\mathrm{E}^{1/2}|X_{t_i}(\mathrm{E}(Y_{t_{i+1}}|\mathcal{F}_1^{t_i}) - \mathrm{E}Y_{t_{i+1}})|^2,$$

and taking into account (4.17) and (4.21)

$$|\widehat{\mathrm{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 2\, C^{i-1}\varphi^{1/2}(t_i,\, t_{i+1}) C\,\mathrm{E}^{1/2}|Y_{t_{i+1}}|^2 \leqslant 2\, C^k\varphi^{1/2}(t_i,\, t_{i+1}). \quad ■$$

*Proof of Theorem 4.2.1.**) By applying (4.16) to estimate (4.19) we obtain

$$\begin{aligned}|\widehat{\mathrm{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant{}& 2^{k-2}\mathrm{E}^{1/u_1}|X_{t_1}|^{u_1}\mathrm{E}^{1/(v_1u_2)}|X_{t_2}|^{v_1u_2} \times \ldots \times \\ &\times \mathrm{E}^{1/(v_1\ldots v_{i-1}u_i)}|X_{t_i}|^{v_1\ldots v_{i-1}u_i} 2(1 + 2^{1/(v_1\ldots v_i)})\times \\ &\times \alpha^{1/(v_1\ldots v_i) - 1/(v_1\ldots v_i(1+\varepsilon))}(t_i,\, t_{i+1})\times \\ &\times \mathrm{E}^{1/((1+\varepsilon)v_1\ldots v_iu_{i+1})}|X_{t_{i+1}}|^{(1+\varepsilon)v_1\ldots v_iu_{i+1}} \times \ldots \times \\ &\times \mathrm{E}^{1/((1+\varepsilon)v_1\ldots v_{k-2}u_{k-1})}|X_{t_{k-1}}|^{(1+\varepsilon)v_1\ldots v_{k-2}u_{k-1}}\times \\ &\times \mathrm{E}^{1/((1+\varepsilon)v_1\ldots v_{k-1})}|X_{t_k}|^{(1+\varepsilon)v_1\ldots v_{k-1}},\end{aligned} \tag{4.22}$$

where $u_j,\ v_j \geqslant 1, \quad (1/u_j) + (1/v_j) = 1, \quad j = 1, \ldots, k-1, \quad \varepsilon \geqslant 0.$

---

*) We write "Theorem 4.3.1.b)" instead of "section b) of assertion 1 of theorem 4.3".

Put

$$\begin{aligned} p_1 &= u_1, \\ p_2 &= v_1 u_2, \\ &\cdots\cdots\cdots\cdots \\ p_i &= v_1 \ldots v_{i-1} u_i, \\ p_{i+1} &= (1+\varepsilon) v_1 \ldots v_i u_{i+1}, \\ &\cdots\cdots\cdots\cdots\cdots \\ p_{k-1} &= (1+\varepsilon) v_1 \ldots v_{k-2} u_{k-1}, \\ p_k &= (1+\varepsilon) v_1 \ldots v_{k-1}. \end{aligned}$$

Since

$$\begin{aligned} 1/(v_1 \ldots v_i) &= 1 - \sum_{j=1}^{i} \frac{1}{p_j}, \\ 1/\big((1+\varepsilon)v_1 \ldots v_i\big) &= \sum_{j=i+1}^{k} \frac{1}{p_j}, \end{aligned} \tag{4.23}$$

then

$$1/(v_1 \ldots v_i) - 1/\big((1+\varepsilon)v_1 \ldots v_i\big) = 1 - \sum_{j=1}^{k} \frac{1}{p_j}.$$

Theorem 4.2.1 is proved. ■

Proof of Theorem 4.2.2. Analogously to the proof of Theorem 4.2.1, applying (4.17) to estimate (4.19) we obtain

$$\begin{aligned} |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant{} & 2^{k-2} \mathbf{E}^{1/u_1} |X_{t_1}|^{u_1} \mathbf{E}^{1/(v_1 u_2)} |X_{t_2}|^{v_1 u_2} \times \ldots \times \\ & \times \mathbf{E}^{1/(v_1 \ldots v_{i-1} u_i)} |X_{t_i}|^{v_1 \ldots v_{i-1} u_i} 2\varphi^{1-1/(v_1 \ldots v_i)}(t_i,\, t_{i+1}) \times \\ & \times \mathbf{E}^{1/(v_1 \ldots v_i u_{i+1})} |X_{t_{i+1}}|^{v_1 \ldots v_i u_{i+1}} \times \ldots \times \\ & \times \mathbf{E}^{1/(v_1 \ldots v_{k-2} u_{k-1})} |X_{t_{k-1}}|^{v_1 \ldots v_{k-2} u_{k-1}} \times \\ & \times \mathbf{E}^{1/(v_1 \ldots v_{k-1})} |X_{t_k}|^{v_1 \ldots v_{k-1}}, \end{aligned} \tag{4.24}$$

where $u_j,\ v_j \geqslant 1$, $\quad (1/u_j) + (1/v_j) = 1, \quad j = 1, \ldots, k-1.$

Putting

$$
\begin{aligned}
&p_1 = u_1, \\
&p_2 = v_1 u_2, \\
&\dots\dots\dots\dots\dots \\
&p_{k-1} = v_1 \dots v_{k-2} u_{k-1}, \\
&p_k = v_1 \dots v_{k-1}
\end{aligned}
\tag{4.25}
$$

and making use of (4.23) we obtain the desired result.Theorem 4.2.2 is proved. ■

Proof of Theorem 4.2.3. Similarly to the proof of the above theorems by applying (4.18) to estimate (4.19) we obtain

$$
\begin{aligned}
|\widehat{\mathrm{E}}\, X_{t_1} \dots X_{t_k}| &\leqslant 2^{k-2} \mathrm{E}^{1/u_1} |X_{t_1}|^{u_1} \mathrm{E}^{1/(v_1 u_2)} |X_{t_2}|^{v_1 u_2} \times \dots \times \\
&\times \mathrm{E}^{1/(v_1 \dots v_{i-1} u_i)} |X_{t_i}|^{v_1 \dots v_{i-1} u_i} \psi(t_i,\, t_{i+1}) \mathrm{E}^{1/u_{i+1}} |X_{t_{i+1}}|^{u_{i+1}} \times \dots \times \\
&\times \mathrm{E}^{1/(v_{i+1} \dots v_{k-2} u_{k-1})} |X_{t_{k-1}}|^{v_{i+1} \dots v_{k-2} u_{k-1}} \times \\
&\times \mathrm{E}^{1/(v_{i+1} \dots v_{k-1})} |X_{t_k}|^{v_{i+1} \dots v_{k-1}}.
\end{aligned}
\tag{4.26}
$$

Increasing $v_1 \dots v_i$ times the order of moments of random variables $X_{t_{i+1}}, \dots, X_{t_k}$ and using relations (4.25) we complete the proof of theorem 4.2.3. ■

Proof of the Corollaries of Theorem 4.2.1. In case a) it suffices to put $p_j = (1+\delta)k, \quad j = 1, \dots, k$, in the inequality of Theorem 4.2.1; in case b) restriction on the growth of moments enables us to estimate the centered moment in the following manner:

$$
|\widehat{\mathrm{E}}\, X_{t_1} \dots X_{t_k}| \leqslant 3 \cdot 2^{k-1} \alpha^{\delta/(1+\delta)}(t_i,\, t_{i+1}) \Big(\big(k(\widehat{1+\delta})\big)! \Big)^{(1+\gamma_1)/(\widehat{1+\delta})} H_1^k.
$$

Because of the estimate

$$
(n_1 n_2)! \leqslant (n_1!)^{n_2} \cdot n_2^{n_1 n_2},
\tag{4.27}
$$

holding for any positive integers n_1 and n_2, we get the stated result. ■

Proof of the Corollaries of Theorem 4.2.2. In case a) we put $p_1 = (1+\delta)k/(1+k\delta), \quad p_j = (1+\delta)k, \quad j = 2, \dots, k$, in the assertion of Theorem 4.2.2, then we use the inequalities

$$
\mathrm{E}^{(1+k\delta)/(1+\delta)k} |X_{t_1}|^{(1+\delta)k/(1+k\delta)} \leqslant \mathrm{E}^{1/(1+\delta)k} |X_{t_1}|^{(1+\delta)k},
$$

$$
\varphi^{\sum_{j=1}^{i} \frac{1}{p_j}}(t_i,\, t_{i+1}) \leqslant \varphi^{(1+k\delta)/(1+\delta)k}(t_i,\, t_{i+1}) \leqslant \varphi^{\delta/(1+\delta)}(t_i,\, t_{i+1}).
$$

The proof of case b) is analogous to that of Corollary 1 b) of Theorem 4.2, therefore, it is omitted. ■

Proof of Theorem 4.3. Analogously to formula (4.10) let us define the random variables

$$\begin{aligned} Y_{t_j} &= \mathbf{E}(X_{t_j}|\mathcal{F}_1^{t_j-1})\widehat{Y}_{t_{j+1}}, \qquad j=1,\,\ldots,\,k-1, \\ Y_{t_k} &= \mathbf{E}(X_{t_k}|\mathcal{F}_1^{t_k-1}), \end{aligned} \tag{4.28}$$

where $\mathcal{F}_1^{t_0} \subset \mathcal{F}_1^{t_1}$. Then in the formula (4.15)

$$\mathbf{E}Y_{t_1} = \mathbf{E}\mathbf{E}(X_{t_1}|\mathcal{F}_1^{t_0})\widehat{\mathbf{E}(X_{t_2}|\mathcal{F}_1^{t_1})\ldots\widehat{\mathbf{E}(X_{t_i}|\mathcal{F}_1^{t_i-1})\widehat{\mathbf{E}(Y_{t_{i+i}}|\mathcal{F}_1^{t_i})}}}. \tag{4.29}$$

Obviously $\mathbf{E}Y_{t_1} = \widehat{\mathbf{E}}\,X_{t_1}\,\ldots\,X_{t_k}$ and in special case $\mathbf{E}Y_{t_1}=0$ if $\mathcal{F}_1^{t_0}=\{\varnothing,\,\Omega\}$.

Let $t_i = l_{i'}$, $t_{i+1} = l_{i'+1}$, i.e. the selected "maximal interval" does not degenerate into a point. Then

$$\begin{aligned} |\mathbf{E}Y_{t_1}| \leqslant 2^{m_1+\ldots+m_{i'-1}-1} \prod_{j=1}^{i'-1} \operatorname{ess\,sup} |\mathbf{E}(X_{l_j}^{m_j}|\mathcal{F}_1^{l_j-1})| \times \\ \times\,\mathbf{E}\big(|\mathbf{E}(X_{l_{i'}}^{m_{i'}}|\mathcal{F}_1^{l_{i'}-1})|\,|\mathbf{E}(Y_{l_{i'+1}}|\mathcal{F}_1^{l_{i'}}) - \mathbf{E}Y_{l_{i'+1}}|\big). \end{aligned} \tag{4.30}$$

Since

$$\operatorname{ess\,sup} |\mathbf{E}(Y_{l_{i'+1}}|\mathcal{F}_1^{l_{i'}})| \leqslant 2^{m_{i'+1}+\ldots+m_r-1} \prod_{j=i'+1}^{r} \operatorname{ess\,sup} |\mathbf{E}(X_{l_j}^{m_j}|\mathcal{F}_1^{l_j-1})|, \tag{4.31}$$

substituting (A) and (4.31) into inequality (4.30) and denoting i' by i we complete the proof of Theorem 4.3. ■

The proof of the corollary of Theorem 4.3 is obtained directly from Theorem 4.3.

Now we shall estimate centered moments of the random variables X_t, related to a Markov chain ξ_t. In this case recurrent equations of type (4.10) take the form

$$\begin{aligned} Y_{t_j} &= X_{t_j}\widehat{\mathbf{E}(Y_{t_{j+1}}|\mathcal{F}_1^{t_j})}, \qquad j=1,\,\ldots,\,k-1, \\ Y_{t_k} &= X_{t_k}. \end{aligned} \tag{4.32}$$

By virtue of Markov condition Y_{t_j} is $\mathcal{F}_{t_j}^{t_j}$–measurable for any $j=1,\,\ldots,\,k$,and the relation

$$\begin{aligned} \mathbf{E}^{1/u}|Y_{t_j}|^u &\leqslant \mathbf{E}^{1/up}|X_{t_j}|^{up}\mathbf{E}^{1/uq}|\mathbf{E}(Y_{t_{j+1}}|\mathcal{F}^{t_j}) - \mathbf{E}Y_{t_{j+1}}|^{uq} \leqslant \\ &\leqslant 2\varphi^{1-1/uq}(t_j,\,t_{j+1})\mathbf{E}^{1/up}|X_{t_j}|^{up}\mathbf{E}^{1/uq}|Y_{t_{j+1}}|^{uq} \end{aligned} \tag{4.33}$$

holds for any $u \geqslant 1$ and any $t_1, \ldots, t_k, \quad 1/p + 1/q = 1, \quad p,\ q \geqslant 1$.

If we apply repeatedly inequalities (4.33) we must exclude the coincidence of σ-algebras for different i and take into account the mixing only between essentially different points $l_1, \ldots, l_r$.

Proof of Theorem 4.4. Assertion 1 follows from the assertion of Theorem 4.5.1, proved below, as $q_2, \ldots, q_r \to \infty$, assertion 2 follows directly from Theorem 4.5.2.■

Proof of Theorem 4.5.1. The proof is analogous to the proof of Theorem 4.2.2. Relation (4.33) is applied to those i, for which $i = m_1 + \ldots + m_j, \quad j = 1, \ldots, r-1$, while for remaining i the estimates, obtained with the help of Hölder's and Minkovski's inequalities, are used. ■

The proof of Theorem 4.5.2 is analogous to that of Theorem 4.5.1, only estimates (4.18) are used instead of (4.17).

Proof of the Corollaries of Theorem 4.5.1. In case a) we introduce $u_1, \ldots, u_k$ by the relations

$$\begin{aligned}
&u_1 = u_2 = \ldots = u_{m_1} = m_1 q_1, \\
&u_{m_1+1} = u_{m_1+2} = \ldots = u_{m_1+m_2} = m_2 q_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&u_{m_1+\ldots+m_{r-1}+1} = u_{m_1+\ldots+m_{r-1}+2} = \ldots = u_k = m_r q_r,
\end{aligned}$$

so that

$$\sum_{j=1}^{k} \frac{1}{u_j} = \sum_{j=1}^{r} \sum_{i=1}^{m_j} \frac{1}{u_i} = \sum_{j=1}^{r} \frac{1}{q_j} = 1,$$

and afterwards we put

$$u_1 = (1+\delta)k/(1+k\delta), \qquad u_j = (1+\delta)k, \quad j = 2, \ldots, k,$$

meanwhile we increase the order of the moment of the random variable X_{t_1} to $(1+\delta)k$ and decrease the exponent of the mixing function to $\delta/(1+\delta)$.

Case b) is a direct consequence of case a) if we make use of the restrictions on the growth of moments and formula (4.27). ■

The proof of the Corollaries of Theorem 4.5.2 is reduced to the application of condition on the growth of moments to the result of Theorem 4.5.2.

Proof of Theorem 4.6. Now let $Y_{t_1}, \ldots, Y_{t_k}$ be defined as follows:

$$\begin{aligned}
Y_{t_j} &= \mathbf{E}(X_{t_j} | \mathcal{F}_1^{t_{j-1}}) \big(\mathbf{E}(Y_{t_{j+1}} | \mathcal{F}_1^{t_j}) - \mathbf{E} Y_{t_{j+1}} \big), \\
Y_{t_k} &= \mathbf{E}(X_{t_k} | \mathcal{F}_1^{t_{k-1}}).
\end{aligned} \tag{4.34}$$

Here as in (4.32) due to the Markovian property the behaviour of the process at the moment l_{i+1} depends on the information only from $\mathcal{F}_{l_i}^{l_i}$. Hence the estimates follow:

$$\begin{aligned}
&\mathbf{E}^{1/u}|\mathbf{E}(Y_{l_j}|\mathcal{F}_1^{l_{j-1}})|^u \leqslant \\
&\leqslant 2^{m_j-1}\mathbf{E}^{1/u}\big(|\mathbf{E}(X_{l_j}^{m_j}|\mathcal{F}_1^{l_{j-1}})|\,|\mathbf{E}(Y_{l_{j+1}}|\mathcal{F}_1^{l_j}) - \mathbf{E}Y_{l_{j+1}}|\big)^u, \\
&j = 1, \dots, r-1, \\
&\mathbf{E}^{1/u}|\mathbf{E}(Y_{l_r}|\mathcal{F}_1^{l_{r-1}})|^u \leqslant 2^{m_r-1}\mathbf{E}^{1/u}|\mathbf{E}(X_{l_r}^{m_r}|\mathcal{F}_1^{l_{r-1}})|^u,
\end{aligned} \tag{4.35}$$

where $u \geqslant 1$ is arbitrary, $\mathcal{F}_1^{l_0} = \mathcal{F}_1^{t_0}$. Now we take ess sup $|\mathbf{E}(X_{l_j}^{m_j}|\mathcal{F}_1^{l_{j-1}})|$ out of the right–hand side of inequality and afterwards we apply (4.17) as $v = u$. Due to arbitrariness of u we get the result as $u \to \infty$. ■

The corollary of Theorem 4.6 follows from the result of Theorem 4.6, taking into account the condition on the growth of moments.

4.2. Estimates of mixed cumulants of random processes with mixing

Having estimates of theorems 4.1 – 4.6 and their corollaries, from relation (1.63) and taking into account the behaviour of $N_\nu(I_1, \dots, I_\nu)$ we obtain the estimates for the cumulants $\Gamma(X_{t_1}, \dots, X_{t_k})$.

THEOREM 4.7. *If* $|X_{t_j}| \leqslant C$ a.s., $j = 1, \dots, k$, $k = 2, 3, \dots$, *then for all* $i = 1, 2, \dots, k-1$

$$\begin{aligned}
&1)\ |\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)!2^k C^k \alpha(t_i, t_{i+1}), \\
&2)\ |\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)!2^{k-1} C^k \varphi(t_i, t_{i+1}), \\
&3)\ |\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)!2^{k-2} C^k \psi(t_i, t_{i+1}).
\end{aligned}$$

THEOREM 4.8. *If for a collection* $p_j \geqslant 1$, $j = 1, \dots, k$, *such that*

$$\sum_{j=1}^k \frac{1}{p_j} \leqslant 1, \qquad k = 2, 3, \dots,$$

there exist $\mathbf{E}|X_{t_j}|^{p_j}$, $j = 1, \ldots, k$, *then for all* $i = 1, \ldots, k-1$

$$1)\ |\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant 3\,(k-1)!2^{k-1}\alpha^{1-\sum\limits_{j=1}^{k}\frac{1}{p_j}}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{p_j}}|X_{t_j}|^{p_j},$$

$$2)\ |\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!2^{k-1}\varphi^{\sum\limits_{j=1}^{i}\frac{1}{p_j}}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{p_j}}|X_{t_j}|^{p_j},$$

$$3)\ |\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!2^{k-2}\psi(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{p_j}}|X_{t_j}|^{p_j}.$$

COROLLARIES of Theorem 4.8. *If for some* $k \in \{2, 3, \ldots\}$ *and* $\delta \geqslant 0$ *there exist* $\mathbf{E}|X_{t_j}|^{(1+\delta)k}$, $j = 1, \ldots, k$, *then for all* $i = 1, \ldots, k-1$

$$1)\ |\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant 3\,(k-1)!2^{k-1}\alpha^{\frac{\delta}{1+\delta}}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{(1+\delta)k}}|X_{t_j}|^{(1+\delta)k},$$

$$2)\ |\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!2^{k-1}\varphi^{\frac{\delta}{1+\delta}}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{(1+\delta)k}}|X_{t_j}|^{(1+\delta)k}.$$

THEOREM 4.9. *If for some* $\gamma_1 \geqslant 0$, $H_1 > 0$ $\mathbf{E}|X_{t_j}|^k \leqslant (k!)^{1+\gamma_1}H_1^k$, $j = 1, \ldots, k$, $k \in \{2, 3, \ldots\}$, *then for all* $i = 1, \ldots, k-1$, $\delta \geqslant 0$

$$1)\ |\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant 3\,(k!)^{1+\gamma_1}4^{k-1}H_1^k(\widehat{1+\delta})^{(1+\gamma_1)k}\alpha^{\frac{\delta}{1+\delta}}(t_i, t_{i+1}),$$

$$2)\ |\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k!)^{1+\gamma_1}4^{k-1}H_1^k(\widehat{1+\delta})^{(1+\gamma_1)k}\varphi^{\frac{\delta}{1+\delta}}(t_i, t_{i+1}).$$

THEOREM 4.10. *If for some* $k \in \{2, 3, \ldots\}$ $\operatorname{ess\,sup} |\mathbf{E}(X_{l_j}^{m_j}|\mathcal{F}_1^{l_{j-1}})| < \infty$ a.s., $j = 1, \ldots, r$, *then for all* $i = 1, \ldots, k-1$

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!2^k\alpha(t_i, t_{i+1}) \prod_{j=1}^{r} \operatorname{ess\,sup} |\mathbf{E}(X_{l_j}^{m_j}|\mathcal{F}_1^{l_{j-1}})|.$$

COROLLARY of Theorem 4.10. *If for some* $\gamma_2 \geqslant 0$, $H_2 > 0$ $|\mathbf{E}(X_{l_j}^k|\mathcal{F}_1^{l_{j-1}})| \leqslant (k!)^{1+\gamma_2}H_2^k$ a.s., $j = 1, \ldots, r$, $k = 2, 3, \ldots$, *then for all* $i = 1, \ldots, k-1$

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k!)^{1+\gamma_2}2^{2k-1}H_2^k\alpha(t_i, t_{i+1}).$$

THEOREM 4.11. *Let X_t be related to a Markov chain ξ_t. If $|X_{t_j}| \leqslant C$ with probability 1, $j = 1, \dots, k$, $k \in \{2, 3, \dots\}$, then*

$$1)\ |\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)! 2^{k-1} C^k \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}),$$

$$2)\ |\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)! 2^{k-r} C^k \prod_{j=1}^{r-1} \psi(l_j, l_{j+1}).$$

THEOREM 4.12. *Let X_t be related to a Markov chain ξ_t.*

1) *If for a collection $q_j \geqslant 1$, $j = 1, \dots, r$, $r \in \{2, 3, \dots\}$, such that $\sum\limits_{j=1}^{r} \frac{1}{q_j} = 1$, there exist $\mathbf{E}|X_{l_j}|^{m_j q_j}$, then*

$$|\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)! 2^{k-1} \prod_{j=1}^{r-1} \varphi^{\sum_{i=1}^{j} \frac{1}{q_i}}(l_j, l_{j+1}) \prod_{j=1}^{r} \mathbf{E}^{\frac{1}{q_j}} |X_{l_j}|^{m_j q_j}.$$

2) *If for some $r \in \{2, 3, \dots\}$ there exist $\mathbf{E}|X_{l_j}|^{m_j}$, $j = 1, \dots, r$, then*

$$|\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)! 2^{k-r} \prod_{j=1}^{r-1} \psi(l_j, l_{j+1}) \prod_{j=1}^{r} \mathbf{E}|X_{l_j}|^{m_j}.$$

COROLLARY of Theorem 4.12. *Let X_t be related to a Markov chain ξ_t. If for some $k \in \{2, 3, \dots\}$ and $\delta \geqslant 0$ there exist $\mathbf{E}|X_{t_j}|^{(1+\delta)k}$, $j = 1, \dots, k$, then*

$$|\Gamma(X_{t_1}, \dots, X_{t_k})| \leqslant (k-1)! 2^{k-1} \prod_{j=1}^{r-1} \varphi^{\frac{\delta}{1+\delta}}(l_j, l_{j+1}) \prod_{j=1}^{k} \mathbf{E}^{\frac{1}{(1+\delta)k}} |X_{t_j}|^{(1+\delta)k}.$$

THEOREM 4.13. *Let X_t be related to a Markov chain ξ_t. If for some $\gamma_1 \geqslant 0$, $H_1 > 0$*

$$\mathbf{E}|X_{l_j}|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad j = 1, \dots, r, \quad r = 2, 3, \dots, \quad k = 2, 3, \dots,$$

then for any $\delta \geqslant 0$

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k!)^{1+\gamma_1} 4^{k-1} H_1^k (\widehat{1+\delta})^{(1+\gamma_1)k} \prod_{j=1}^{r-1} \varphi^{\frac{\delta}{1+\delta}}(l_j, l_{j+1}).$$

THEOREM 4.14. *Let X_t be related to a Markov chain ξ_t. If for some $r \in \{2, 3, \ldots\}$*

$$|\mathbf{E}(X_{l_j}^{m_j} | \mathcal{F}_1^{l_j-1})| \leqslant \infty \qquad \text{with probability } 1, \quad j = 1, \ldots, r,$$

then

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)! 2^{k-1} \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}) \prod_{j=1}^{r} \operatorname{ess\,sup} |\mathbf{E}(X_{l_j}^{m_j} | \mathcal{F}_1^{l_j-1})|.$$

COROLLARY of Theorem 4.14. *Let X_t be related to a Markov chain ξ_t. If for some $\gamma_2 \geqslant 0$, $H_2 > 0$*

$$|\mathbf{E}(X_{l_j}^{k} | \mathcal{F}_1^{l_j-1})| \leqslant (k!)^{1+\gamma_2} H_2^k \text{ a.s.}, \ j = 1, \ldots, r, \ r = 2, 3, \ldots, \ k = 2, 3, \ldots,$$

then

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k!)^{1+\gamma_2} 4^{k-1} H_2^k \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}).$$

Proof of Theorems 4.7 – 4.14. It has been already mentioned that in the proof of these theorems relations (1.63), (1.65) and the estimate

$$0 \leqslant N_\nu(I_1, \ldots, I_\nu) \leqslant (\nu - 1)!$$

will be basic. We shall give some additional information about $N_\nu(I_1, \ldots, I_\nu)$.

Its explicit form is defined by formula (A15) in Appendix. We present its equivalent but a more obvious representation.

Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_\mu\}$ be a system of subsets of the set I. We say that the system $\{\mathcal{A}_1, \ldots, \mathcal{A}_\mu\}$ essentially covers the point $t \in I$, if

$$\{q | t \in [\mathcal{A}_q \backslash \{t\}], \quad 1 \leqslant q \leqslant \mu\} \neq \varnothing.$$

In other words, there must exist $\mathcal{A}_p \in \{\mathcal{A}_1, \ldots, \mathcal{A}_\mu\}$ such that $t \in [\mathcal{A}_p \backslash \{t\}]$.

We shall explain this by means of the following example. Let $\mu = 8$ and $\{\mathcal{A}_1, \ldots, \mathcal{A}_8\}$ be the partition $\{I_1, \ldots, I_8\}$ of the set $\{t_1, \ldots, t_{14}\}$ of the following form: $I_1 = \{t_1, t_5\}$, $I_2 = \{t_2, t_9\}$, $I_3 = \{t_3, t_6\}$, $I_4 = \{t_4\}$, $I_5 = \{t_7\}$, $I_6 = \{t_8, t_{13}\}$, $I_7 = \{t_{10}, t_{14}\}$, $I_8 = \{t_{11}, t_{12}\}$. If we choose $t = t_{11}$, then

$$\{q | t \in [I_q \backslash \{t\}], \quad 1 \leqslant q \leqslant 8\} = \{6, 7\}$$

since

$$t_{11} \in [I_6 \backslash \{t_{11}\}] = \{t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}\},$$

$$t_{11} \in [I_7 \backslash \{t_{11}\}] = \{t_{10}, t_{11}, t_{12}, t_{13}, t_{14}\}.$$

The number

$$n_t(\mathcal{A}_1, \ldots, \mathcal{A}_\mu) = |\{q | t \in [\mathcal{A}_q \backslash \{t\}], \quad 1 \leqslant q \leqslant \mu\}| \tag{4.36}$$

will be called the number of maximal covering of a point $t \in I$ by the system $\{\mathcal{A}_1, \ldots, \mathcal{A}_\mu\}$. In our example

$$n_{t_{11}}(I_1, \ldots, I_8) = |\{6, 7\}| = 2.$$

It turns out that the numbers $N_\nu(I_1, \ldots, I_\nu)$ occurring in formula (1.63) can be expressed by the following relations:

$$\begin{aligned} &N_1(I) = 1, \\ &N_\nu(I_1, \ldots, I_\nu) = \prod_{j=2}^{\nu} n_{t_1^{(j)}}(I_1, \ldots, I_\nu). \end{aligned} \tag{4.37}$$

Formula (4.37) is not legible enough, and we shall illustrate by an example a number of its graphic representations, explaining the structure of the numbers $N_\nu(I_1, \ldots, I_\nu)$. The terms of graph theory are taken from (Zykov, 1987).

Let us return to our example. Due to unordering of the partition blocks we have deliberately arranged them according to the increase of the first (leftmost) elements.

Two graphs $G^{(1)}_{\{I_1, \ldots, I_\nu\}}$ (fig. 1) and $G^{(2)}_{\{I_1, \ldots, I_\nu\}}$ (fig. 2) correspond to each partition $\{I_1, \ldots, I_\nu\}$.

Graph $G^{(1)}_{\{I_1, \ldots, I_\nu\}}$ is constructed in following manner. Its vertices are the points of the set I, arranged according to increase, i.e. vertex 1 of the graph corresponds to the point t_1, vertex 2 to the point t_2, etc. The vertices in some block of partition are connected in pairs by arcs (oriented edges) according to increase of their numbers. If in a certain block there is only one point, then the vertex, corresponding to it, is connected by a loop.

Graph $G^{(2)}_{\{I_1, \ldots, I_\nu\}}$ consists of the vertices, corresponding to the blocks of partition and numbered in order of increase of the leftmost points. The vertices of the graph i and j, for which $[I_i] \cap [I_j] \neq \varnothing$, are connected by links (nonoriented edges).

In our example graph $G^{(1)}_{\{I_1,\ldots,I_8\}}$ will be as follows:

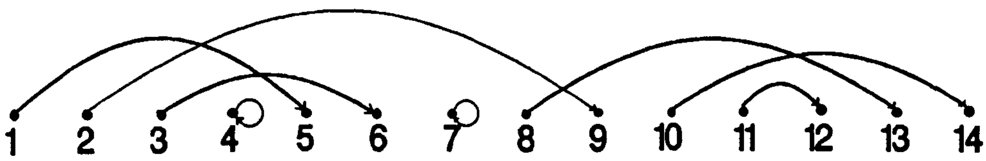

Fig. 1

The graph $G^{(2)}_{\{I_1,\ldots,I_8\}}$ will be as follows:

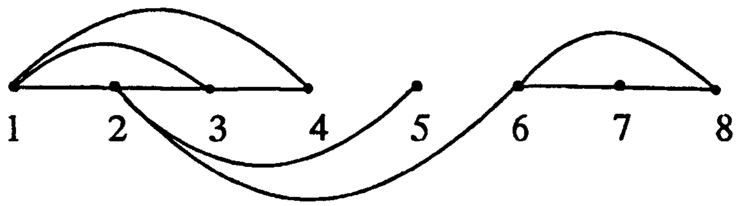

Fig. 2

Graphically the numbers $N_\nu(I_1, \ldots, I_\nu)$ can be determined in the following manner. We take the vertices of graph $G^{(1)}_{\{I_1,\ldots,I_\nu\}}$, incidental only to the arcs and loops, starting from them, and sort them out, beginning with second one. Each vertex is marked with the corresponding number, equal to the number of arcs,overpassing it. It is easy to see that this number has already been determined by formula (4.36), if we replace $\{\mathcal{A}_1, \ldots, \mathcal{A}_\mu\}$ by $\{I_1, \ldots, I_\nu\}$. Thus, the number $N_\nu(I_1, \ldots, I_\nu)$ is obtained by multiplying out all these numbers found.

Here the estimate

$$N_\nu(I_1, \ldots, I_\nu) \leqslant (\nu-1)!,$$

i.e. estimate (1.64) is a simple consequence of the inequalities

$$n_{t_1^{(p)}}(I_1, \ldots, I_\nu) \leqslant p-1, \qquad 2 \leqslant p \leqslant \nu.$$

Moreover, a stronger inequality

$$N_\nu(I_1, \ldots, I_\nu) \leqslant \min\{(\nu-1)!, [k/2]!\}$$

also holds what can be seen from the structure of graph $G^{(1)}_{\{I_1,\ldots,I_\nu\}}$.

The structure of the graph $G^{(2)}_{\{I_1,\ldots,I_\nu\}}$ is discussed in (Malyshev, Minlos, 1985). It turns out that the numbers $N_\nu(I_1, \ldots, I_\nu)$ are zeroes on those partitions $(I_1, \ldots, I_\nu)$ of the set I for which the graph $G^{(2)}_{\{I_1,\ldots,I_\nu\}}$ is disconnected. Conversely, for the partitions corresponding to the connected graph $G^{(2)}_{\{I_1,\ldots,I_\nu\}}$, the numbers N_ν are strictly positive.

Let us call the partition $\{I_1, \ldots, I_\nu\}$ *connected*, if the corresponding graph $G^{(2)}_{\{I_1,\ldots,I_\nu\}}$ is connected.

It is evident that for any connected partition $\{I_1, \ldots, I_\nu\}$ the condition

$$\sum_{p=1}^{\nu} \max_{i,j} \{t_i^{(p)} - t_j^{(p)}\} \geqslant \max_{i,j} \{t_i - t_j\}$$

is fulfilled, i.e. relation (1.65) holds.

An inverse assertion is not valid in a general case.

Recalling formula (1.63) it is easy to see that in fact summation is taken over the set of all connected ν–block partitions $\{I_1, \ldots, I_\nu\}$ of the set I.

A vector with nonnegative integer components whose dimension is equal to cardinality of the set of vertices, can be put in correspondence to graph $G^{(1)}_{\{I_1,\ldots,I_\nu\}}$. The vertices, incidental only to arcs and loops starting from them, are denoted by zeroes and the others by the number of the vertex originating the arc, which enters the considered one. In our example the vector is (0, 0, 0, 0, 1, 3, 0, 0, 2, 0, 0, 11, 8, 10).

LEMMA 4.2.

$$\sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) = \sum_{j=0}^{\nu-1} (-1)^j \binom{k-\nu+j}{k-\nu} (\nu - j - 1)!\, s(k, \nu - j). \tag{4.38}$$

Proof. Let us consider a representation of the mixed cumulant in the form (1.57). Direct comparison of the right–hand sides of the formulas (1.63) and (1.57) leads us to the relations

$$s(k, \nu) = \frac{1}{(\nu-1)!} \sum_{j=0}^{\nu-1} \binom{k-\nu+j}{k-\nu} N(k, \nu - j), \qquad 1 \leqslant \nu \leqslant k, \tag{4.39}$$

where

$$N(k, \nu) = \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu).$$

Note, that $N(k,\ \nu)$ is the number of members of the type $\prod_{p=1}^{\nu} \widehat{\mathrm{E}}(X_{I_p})$ present in the expansion (1.63) of the mixed cumulant. For example, if $k = 6$, then

$$0!\, s(6,\ 1) = \binom{5}{5} N(6,\ 1),$$

$$1!\, s(6,\ 2) = \binom{5}{4} N(6,\ 1) + \binom{4}{4} N(6,\ 2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$5!\, s(6,\ 6) = \binom{5}{0} N(6,\ 1) + \binom{4}{0} N(6,\ 2) + \ldots + \binom{0}{0} N(6,\ 6).$$

Hence

$$N(6,\ 1) = 1, \qquad N(6,\ 2) = 26, \qquad N(6,\ 3) = 66,$$
$$N(6,\ 4) = 26, \qquad N(6,\ 5) = 1, \qquad N(6,\ 6) = 0.$$

Putting (4.39) into the identity considered we obtain

$$N(k,\ \nu) = \sum_{j=0}^{\nu-1} (-1)^j \binom{k-\nu+j}{k-\nu} (\nu - j - 1)!\, s(k,\ \nu - j) =$$

$$= \sum_{j=0}^{\nu-1} (-1)^j \sum_{i=0}^{\nu-j-1} \frac{(k-\nu+i+j)!}{(k-\nu)!i!j!} N(k,\ \nu - i - j) =$$

$$= \sum_{\alpha=0}^{\nu-1} \sum_{\substack{i+j=\alpha \\ i,\, j\geqslant 0}} \frac{(k-\nu+\alpha)!}{(k-\nu)!i!j!} (-1)^j N(k,\ \nu - \alpha) =$$

$$= N(k,\ \nu) + \sum_{\alpha=1}^{\nu-1} \frac{(k-\nu+\alpha)!}{(k-\nu)!\alpha!} N(k,\ \nu-\alpha) \sum_{\substack{i+j=\alpha \\ i,\, j\geqslant 0}} (-1)^j \frac{\alpha!}{i!j!}.$$

Since

$$\sum_{\substack{i+j=\alpha \\ i,\, j\geqslant 0}} (-1)^j \frac{\alpha!}{i!j!} = 0, \qquad \forall \alpha \geqslant 1,$$

the proof of the lemma is completed. ■

Let us clarify relation of $N(k,\ \nu)$ and Euler's number $A(k,\ \nu)$ (see (Aigner, 1979), (Sachkov, 1982)). We remind that the Euler number $A(k,\ \nu)$ denotes the cardinality of the set of substitutions $s \in S_k$, having exactly $\nu - 1$ descents.

LEMMA 4.3.

$$N(k,\ \nu) = A(k-1,\ \nu). \tag{4.40}$$

Proof. By virtue of the property of Stirling's numbers of the second kind

$$s(k,\,\nu)=s(k-1,\,\nu-1)+\nu s(k-1,\,\nu),$$

$$N(k,\,\nu)=\sum_{j=0}^{\nu-2}(-1)^j\binom{k-\nu-j}{j}(\nu-j-1)!\,s(k-1,\,\nu-j-1)+$$

$$+\nu\sum_{j=0}^{\nu-1}(-1)^j\binom{k-1-\nu+j}{j}(\nu-j-1)!\,s(k-1,\,\nu-j)+$$

$$+\sum_{j=0}^{\nu-2}(-1)^j\binom{k-\nu+j}{j+1}(j+1)(\nu-j-2)!\,s(k-1,\,\nu-j-1)+$$

$$+\sum_{j=0}^{\nu-2}(-1)^{j+1}\binom{k-\nu+j}{j}(\nu-j-1)!\,s(k-1,\,\nu-j-1).$$

Now it ought to be noted that

$$\sum_{j=0}^{\nu-2}(-1)^j\binom{k-\nu+j}{j+1}(j+1)(\nu-j-2)!\,s(k-1,\,\nu-j-1)=$$

$$=(k-\nu)\sum_{j=0}^{\nu-2}(-1)^j\binom{k-\nu+j}{j}(\nu-j-2)!\,s(k-1,\,\nu-j-1).$$

Thus, we have proved the equality

$$N(k,\,\nu)=\nu N(k-1,\,\nu)+(k-\nu)N(k-1,\,\nu-1).$$

It is known that for Euler's numbers the relation (Aigner, 1979)

$$A(k,\,\nu)=\nu A(k-1,\,\nu)+(k-\nu+1)A(k-1,\,\nu-1).$$

holds. So, $N(k,\,\nu)$ and $A(k-1,\,\nu)$ are subject to one and the same recurrent equation with the same initial conditions what completes the proof of Lemma 4.3.■

COROLLARIES of Lemma 4.3.

$$1)\ N(k,\,\nu)=N(k,\,k-\nu), \tag{4.41}$$

$$2)\ N(k,\,\nu)=\sum_{j=0}^{\nu-1}(-1)^j\binom{k}{j}(\nu-j)^{k-1}, \tag{4.42}$$

$$3)\ \sum_{\nu=1}^{k}N(k,\,\nu)=(k-1)!. \tag{4.43}$$

Proof of the corollaries to Lemma 4.3. Equality (4.40) and relation $A(k,\nu) = A(k, k-\nu+1)$ implies directly (4.41); formula (4.42) is obtained after the corresponding transformation (see (3.24) (Sachkov, 1982)); relation (4.43) expresses the well-known fact that the cardinality of the set of permutations of the set $\{1, \ldots, k-1\}$ is $(k-1)!$. Besides, (4.43) follows directly from (4.38), what has been proved in the corollaries to Lemma 3.1 (Statulevičius, Jakimavičius, 1988).

For proving Theorems 4.7 – 4.14 we need the following lemmas.

LEMMA 4.4.

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)! \max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}\, X_{I_p}|. \tag{4.44}$$

Proof. Formula (4.43) implies

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant \sum_{\nu=1}^{k} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}\, X_{I_p}| \leqslant$$

$$\leqslant \max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}\, X_{I_p}| \sum_{\nu=1}^{k} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \leqslant (k-1)! \max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}\, X_{I_p}|.$$

LEMMA 4.5.

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant k! 2^{k-1} \max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} \frac{|\widehat{\mathrm{E}}\, X_{I_p}|}{k_p!}. \tag{4.45}$$

Proof.

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant \sum_{\nu=1}^{k} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}\, X_{I_p}| \leqslant$$

$$\leqslant \max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} \frac{|\widehat{\mathrm{E}}\, X_{I_p}|}{k_p!} \sum_{\nu=1}^{k} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \prod_{p=1}^{\nu} k_p!.$$

Since $\prod_{p=1}^{\nu} k_p!$ does not depend on the partition $\{I_1, \ldots, I_\nu\}$ but only on the cardinality of its blocks, we have

$$\sum_{\bigcup_{p=1}^{\nu} I_p = I} \prod_{p=1}^{\nu} k_p! = \sum_{k_1+\ldots+k_\nu=k} \prod_{p=1}^{\nu} k_p! T_k(k_1, \ldots, k_\nu),$$

where $T_k(k_1, \ldots, k_\nu)$ is the cardinality of the set $\{\{I_1, \ldots, I_\nu\} \big| |I_j| = k_j\}$ and (see formula (4.33) in (Sachkov, 1982))

$$T_k(k_1, \ldots, k_\nu) = \frac{1}{\nu!} \cdot \frac{k!}{k_1! \ldots k_\nu!}.$$

Consequently, having in the mind the condition $N_\nu(I_1, \ldots, I_\nu) \leqslant (\nu-1)!$ as well as (4.1) and (4.2) we get

$$\sum_{\nu=1}^{k} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \prod_{p=1}^{\nu} k_p! \leqslant k! 2^{k-1}. \tag{4.46}$$

Lemma 4.5 is proved. ■

The method of proof of the estimates for mixed cumulants rests on Lemmas 4.4 and 4.5. Its essence is as follows.

For any connected partition $\{I_1, \ldots, I_\nu\}$, $1 \leqslant \nu \leqslant k$, and for all t_i, t_{i+1}, $1 \leqslant i < k$, because of (4.36) and (4.37) there must exist blocks $I_{j_1}, \ldots, I_{j_{k-1}}$ such that $t_i, t_{i+1} \in [I_{j_i}]$, and number of different indices from $j_1, \ldots, j_{k-1}$ is not more than ν.

In the general case the estimates of centered moments are applied only to the obtained block I_{j_i}, corresponding to the selected i while for the remaining blocks Hölder's and Minkowski's inequalities are used.

In a case of variables related to a Markov chain the estimates of centered moments are applied to all blocks of the partition $\{I_1, I_2, \ldots, I_\nu\}$.

Note that if $t_i, t_{i+1} \in [I_{j_i}]$, but $t_i, t_{i+1} \notin I_{j_i}$, then the monotonicity of the mixing function makes possible the direct passage from estimation by the block I_{j_i} to the estimation by the block $[I_{j_i}]$, i.e., to the estimation by the "shortened" interval. Note that the worst estimate for $\prod_{p=1}^{\nu} \widehat{\mathrm{E}}(X_{I_p})$ is obtained when $\nu = 1$. With the help of Lemma 4.4 the estimates of the rest $(k-1)! - 1$ members of expansion (1.63) are equated to it. A more subtle Lemma 4.5 takes advantage of the fact that multiplicity of occurrence of the product $\prod_{p=1}^{\nu} \widehat{\mathrm{E}}(X_{I_p})$ in expansion (1.63) (number N_ν) does not exceed $(\nu-1)!$.

The proof of Theorem 4.7 is obtained directly from Theorem 4.1 and Lemma 4.4 since

$$\max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^k C^k \alpha(t_i, t_{i+1}),$$

$$\max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1} C^k \varphi(t_i, t_{i+1}),$$

$$\max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-2} C^k \psi(t_i, t_{i+1}).$$

REMARK to Theorem 4.7. *Under conditions of Theorem 4.7*

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant 2(k-1)! C^k \varphi^{1/2}(t_i, t_{i+1}).$$

In fact, this follows from the estimate (4.20) since

$$\max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2\, C^k \varphi^{1/2}(t_i, t_{i+1}).$$

Proof of the Theorem 4.8.1. The estimates of Theorem 4.2.1 imply

$$\max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 3\cdot 2^{k-1} \alpha^{1-\sum_{j=1}^{k} \frac{1}{p_j}}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathrm{E}^{\frac{1}{p_j}} |X_{t_j}|^{p_j},$$

as $\alpha \leqslant 1$. Since the conditions of Lemma 4.4 are fulfilled, the Theorem is proved.■

The proof of Theorem 4.8.2 is analogous to that of Theorem 4.8.1. Only note that in this case due to the condition $\varphi \leqslant 1$

$$\max_{1\leqslant\nu\leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1} \varphi^{\sum_{j=1}^{i} \frac{1}{p_j}}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathrm{E}^{\frac{1}{p_j}} |X_{t_j}|^{p_j}.$$

Proof of the corollaries to Theorem 4.8. Using the estimates of corollaries 1 a) and 2 a) of Theorem 4.2 we obtain

$$\max_{1\leqslant\nu\leqslant k}\prod_{p=1}^{\nu}|\widehat{\mathrm{E}}(X_{I_p})| \leqslant 3\cdot 2^{k-1}\alpha^{\frac{\delta}{1+\delta}}(t_i,\, t_{i+1})\prod_{j=1}^{k}\mathrm{E}^{\frac{1}{(1+\delta)k}}|X_{t_j}|^{(1+\delta)k},$$

$$\max_{1\leqslant\nu\leqslant k}\prod_{p=1}^{\nu}|\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1}\varphi^{\frac{\delta}{1+\delta}}(t_i,\, t_{i+1})\prod_{j=1}^{k}\mathrm{E}^{\frac{1}{(1+\delta)k}}|X_{t_j}|^{(1+\delta)k}.$$

The corollaries of Theorem 4.8 are proved. ■

Proof of Theorem 4.9.1. Let $t_i,\, t_{i+1}\in[I_p]$ for some p, $1\leqslant p\leqslant\nu$. The centered moment is evaluated by the obtained block I_p in accordance with corollary 1 b) of Theorem 4.2:

$$|\widehat{\mathrm{E}}(X_{I_p})| \leqslant 3(k_p!)^{1+\gamma_1}2^{k_p-1}H_1^{k_p}(\widehat{1+\delta})^{(1+\gamma_1)k_p}\alpha^{\frac{\delta}{1+\delta}}(t_i,\, t_{i+1}).$$

The estimate

$$|\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k_q-1}H_1^{k_q}(k_q!)^{1+\gamma_1}, \qquad q=1,\ldots,\nu, \quad q\neq p,$$

is applied to the remaining $q\neq p$. Using the inequality

$$(k_1!\ldots k_\nu!)^\beta\leqslant(k!)^\beta, \tag{4.47}$$

valid for any ν, $1\leqslant\nu\leqslant k$, $\beta\geqslant 0$, the estimate

$$\max_{1\leqslant\nu\leqslant k}\prod_{p=1}^{\nu}\frac{|\widehat{\mathrm{E}}(X_{I_p})|}{k_p!}\leqslant(k!)^{\gamma_1}3\cdot 2^{k-1}\cdot H_1^k(\widehat{1+\delta})^{(1+\gamma_1)k}\alpha^{\frac{\delta}{1+\delta}}(t_i,\, t_{i+1})$$

and the result of Lemma 4.5, we complete the proof of Theorem 4.9.1. ■

The proof of Theorem 4.9.2 is similar to the previous one if one uses corollary 2 b) of Theorem 4.2 and Lemma 4.5.

The proof of Theorem 4.10 follows from Theorem 4.3 and Lemma 4.4 employing the estimate

$$\max_{1\leqslant\nu\leqslant k}\prod_{p=1}^{\nu}|\widehat{\mathrm{E}}(X_{I_p})|\leqslant 2^k\alpha(t_i,\, t_{i+1})\prod_{j=1}^{r}\operatorname{ess\,sup}|\mathrm{E}(X_{l_j}^{m_j}|\mathcal{F}_1^{l_j-1})|.$$

Proof of the corollary to Theorem 4.10. It is proved in a similar way as Theorem 4.9.1 by using the inequality

$$(m_1^{(p)}!\ldots m_{r_p}^{(p)}!)^\beta\leqslant(k_p!)^\beta, \tag{4.48}$$

which is valid for any p, $1 \leqslant p \leqslant \nu$, $\beta \geqslant 0$.

Theorems 4.11 – 4.14, their corollaries and remarks on them are proved in the same way as the previous ones, therefore, only the estimates for

$$\max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})|, \qquad \max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} \frac{|\widehat{\mathrm{E}}(X_{I_p})|}{k_p!}$$

are presented here.

The proof of Theorem 4.11 follows from the estimates

$$1) \max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1} C^k \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}),$$

$$2) \max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-r} C^k \prod_{j=1}^{r-1} \psi(l_j, l_{j+1}).$$

The proof of Theorem 4.12.1 follows from the estimate

$$\max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1} \prod_{j=1}^{r-1} \varphi^{\sum_{i=1}^{j} \frac{1}{q_i}}(l_j, l_{j+1}) \prod_{j=1}^{r} \mathrm{E}^{\frac{1}{q_j}} |X_{l_j}|^{m_j q_j}.$$

The proof of Theorem 4.12.2 follows from the estimate

$$\max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1} \prod_{j=1}^{r-1} \psi(l_j, l_{j+1}) \prod_{j=1}^{r} \mathrm{E}|X_{l_j}|^{m_j}.$$

The proof of the corollary of Theorem 4.12 follows from the estimate

$$\max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1} \prod_{j=1}^{r-1} \varphi^{\frac{\delta}{1+\delta}}(l_j, l_{j+1}) \prod_{j=1}^{k} \mathrm{E}^{\frac{1}{(1+\delta)k}} |X_{t_j}|^{(1+\delta)k}.$$

The proof of Theorem 4.13 follows from the estimate

$$\max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} \frac{|\widehat{\mathrm{E}}(X_{I_p})|}{k_p!} \leqslant (k!)^{\gamma_1} 2^{k-1} H_1^k (\widehat{1+\delta})^{(1+\gamma_1)k} \prod_{j=1}^{r-1} \varphi^{\frac{\delta}{1+\delta}}(l_j, l_{j+1}).$$

The proof of Theorem 4.14 follows from the estimate

$$\max_{1 \leqslant \nu \leqslant k} \prod_{p=1}^{\nu} |\widehat{\mathrm{E}}(X_{I_p})| \leqslant 2^{k-1} \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}) \prod_{j=1}^{r} \operatorname{ess\,sup} |\mathrm{E}(X_{l_j}^{m_j} | \mathcal{F}_1^{l_j-1})|.$$

The proof of the corollary of Theorem 4.14 follows from the estimate

$$\max_{1\leqslant \nu\leqslant k} \prod_{p=1}^{\nu} \frac{|\widehat{\mathbf{E}}(X_{I_p})|}{k_p!} \leqslant (k!)^{\gamma_2} 2^{k-1} H_2^k \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}).$$

4.3. Estimates of cumulants of sums of dependent random variables

As in Chapter 1, let $S_n = \sum_{t=1}^{n} X_t$ and $\Gamma_k(S_n)$ be the k^{th} order cumulant of the sum S_n and

$$\Lambda_n(f, u) := \max\Big\{1, \max_{1\leqslant s\leqslant n} \sum_{t=s}^{n} f^{1/u}(s, t)\Big\},$$

where $f(s, t)$ is one of the mixing functions α, φ or ψ, and $u > 0$.

THEOREM 4.15. *If $|X_t| \leqslant C$ with probability 1, $t = 1, 2, \ldots, n$, then for all $k = 2, 3 \ldots$, $\beta > 0$, $\delta > 0$*

$$1)\ |\Gamma_k(S_n)| \leqslant 2\,k!8^{k-1} C^k \Lambda_n^{k-1}(\alpha, k-1)n,$$

$$2)\ |\Gamma_k(S_n)| \leqslant k!8^{k-1} C^{k-2} \Lambda_n^{k-2}\big(\varphi, (1+\beta)(1+1/\delta)(k-2)\big)\times$$
$$\times \sum_{1\leqslant s\leqslant t\leqslant n} \varphi^{\frac{\beta\delta}{(1+\beta)(1+\delta)}}(s, t)\mathbf{E}^{\frac{\delta}{1+\delta}}|X_s|^{1+\frac{1}{\delta}}\mathbf{E}^{\frac{1}{1+\delta}}|X_t|^{1+\delta}.$$

THEOREM 4.16. *If for some $k \in \{2, 3, \ldots\}$ and $\delta > 0$ there exist $\mathbf{E}|X_t|^{(1+\delta)k}$, $t = 1, \ldots, n$, then for all $\beta > 0$*

$$1)\ |\Gamma_k(S_n)| \leqslant 2\,k!12^{k-1} \Lambda_n^{k-1}\big(\alpha, (1+1/\delta)(k-1)\big) \max_{1\leqslant t\leqslant n} \mathbf{E}^{\frac{1}{1+\delta}}|X_t|^{(1+\delta)k}\cdot n,$$

$$2)\ |\Gamma_k(S_n)| \leqslant k!8^{k-1} \Lambda_n^{k-2}\big(\varphi, (1+\beta)(1+1/\delta)(k-2)\big) \max_{1\leqslant t\leqslant n} \mathbf{E}^{\frac{k-2}{(1+\delta)k}}|X_t|^{(1+\delta)k}\times$$
$$\times \sum_{1\leqslant s\leqslant t\leqslant n} \varphi^{\frac{\beta\delta}{(1+\beta)(1+\delta)}}(s, t)\mathbf{E}^{\frac{1}{(1+\delta)k}}|X_s|^{(1+\delta)k}\mathbf{E}^{\frac{1}{(1+\delta)k}}|X_t|^{(1+\delta)k}.$$

THEOREM 4.17. *If for some $\gamma_1 \geqslant 0$, $H_1 > 0$*

$$\mathbf{E}|X_t|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad t = 1, \ldots, n, \quad k = 2, 3, \ldots,$$

then for all $\delta > 0$

$$|\Gamma_k(S_n)| \leqslant 2\,(k!)^{2+\gamma_1}12^{k-1}H_1^k\widehat{(1+\delta)}^{(1+\gamma_1)k}\Lambda_n^{k-1}\big(\alpha,\,(1+1/\delta)(k-1)\big)\cdot n.$$

THEOREM 4.18. *If for some* $\gamma_2 \geqslant 0,\ H_2 > 0$

$$|\mathbf{E}(X_t^k|\mathcal{F}_1^{t-1})| \leqslant (k!)^{1+\gamma_2}H_2^k \quad \textit{with probability } 1, \quad t = 1, \dots, n,$$

$k = 2, 3, \dots,$ *then*

$$|\Gamma_k(S_n)| \leqslant 2\,(k!)^{1+\gamma_2}16^{k-1}H_2^k\Lambda_n^{k-1}(\alpha,\,k-1)\cdot n.$$

THEOREM 4.19. *Let* X_t *be related to a Markov chain* ξ_t. *If* $|X_t| \leqslant C$ *with probability* 1, $t = 1, \dots, n$, *then for all* $k = 2, 3, \dots,\ \delta > 0$

$$1)\ |\Gamma_k(S_n)| \leqslant k!8^{k-1}C^k\Lambda_n^{k-1}(\varphi,\,1)n,$$

$$2)\ |\Gamma_k(S_n)| \leqslant k!8^{k-1}C^{k-2}\Lambda_n^{k-2}(\varphi,\,1+1/\delta)\times$$
$$\times \sum_{1\leqslant s\leqslant t\leqslant n}\varphi^{\frac{\delta}{(1+\delta)}}(s,\,t)\mathbf{E}^{\frac{\delta}{1+\delta}}|X_s|^{1+\frac{1}{\delta}}\mathbf{E}^{\frac{1}{1+\delta}}|X_t|^{1+\delta}.$$

THEOREM 4.20. *Let* X_t *be related to a Markov chain* ξ_t. *If for some* $k \in \{2, 3, \dots\}$ *and* $\delta > 0$ *there exist* $\mathbf{E}|X_t|^{(1+\delta)k}$, $t = 1, \dots, n$, *then*

$$1)\ |\Gamma_k(S_n)| \leqslant k!8^{k-1}\Lambda_n^{k-1}(\varphi,\,1+1/\delta)\max_{1\leqslant t\leqslant n}\ \mathbf{E}^{\frac{1}{1+\delta}}|X_t|^{(1+\delta)k}\cdot n,$$

$$2)\ |\Gamma_k(S_n)| \leqslant k!8^{k-1}\Lambda_n^{k-2}(\varphi,\,1+1/\delta)\max_{1\leqslant t\leqslant n}\ \mathbf{E}^{\frac{k-2}{(1+\delta)k}}|X_t|^{(1+\delta)k}\times$$
$$\times \sum_{1\leqslant s\leqslant t\leqslant n}\varphi^{\frac{\delta}{(1+\delta)}}(s,\,t)\mathbf{E}^{\frac{1}{(1+\delta)k}}|X_s|^{(1+\delta)k}\mathbf{E}^{\frac{1}{(1+\delta)k}}|X_t|^{(1+\delta)k}.$$

THEOREM 4.21. *Let* X_t *be related to a Markov chain* ξ_t. *If for some* $\gamma_1 \geqslant 0$, $H_1 > 0$

$$\mathbf{E}|X_t|^k \leqslant (k!)^{1+\gamma_1}H_1^k, \qquad t = 1, \dots, n, \quad k = 2, 3, \dots,$$

then for all $\delta > 0$

$$1)\ |\Gamma_k(S_n)| \leqslant (k!)^{2+\gamma_1} 8^{k-1} H_1^k (\widehat{1+\delta})^{(1+\gamma_1)k} \Lambda_n^{k-1}(\varphi,\, 1+1/\delta) n,$$

$$2)\ |\Gamma_k(S_n)| \leqslant (k!)^{1+\gamma_1} 16^{k-1} H_1^k \Lambda_n^{k-1}(\psi,\, 1) n.$$

THEOREM 4.22. *Let* X_t *be related to a Markov chain* ξ_t*. If for some* $\gamma_2 \geqslant 0$, $H_2 > 0$

$$|\mathbf{E}(X_t^k | \mathcal{F}_1^{t-1})| \leqslant (k!)^{1+\gamma_2} H_2^k \qquad \textit{with probability } 1,$$

$t = 1, \ldots, n,\ k = 2, 3, \ldots,$ *then*

$$|\Gamma_k(S_n)| \leqslant (k!)^{1+\gamma_2} 16^{k-1} H_2^k \Lambda_n^{k-1}(\varphi,\, 1) n.$$

Proof of Theorems 4.15 – 4.22. Applying the results of theorems 4.7 – 4.14 and relation (1.47)

$$\Gamma_k(S_n) = \sum_{1 \leqslant t_1, \ldots, t_k \leqslant n} \Gamma(X_{t_1}, \ldots, X_{t_k}),$$

the estimates for $\Gamma_k(S_n)$ can immediately be obtained under different mixing conditions and restrictions on behaviour of the moments $\mathbf{E}|X_t|^k$. However, they would not be optimal with respect to order of the cumulant k. For this necessary to investigate and to estimate more exactly dependence $\Gamma(X_{t_1}, \ldots, X_{t_k})$ on $t_1, \ldots, t_k$. This is achieved with the help of the relations (1.47) and (1.63) and the estimates for $\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}$ given in Theorems 4.1 – 4.6: we shall show how the estimates for the sums $\sum_{1 \leqslant t_1, \ldots, \leqslant t_k \leqslant n} \Gamma(X_{t_1}, \ldots, X_{t_k})$ can be made to be exact in order if preliminary estimates for $\widehat{\mathbf{E}}\,(X_{I_p})$, obtained in § 4.1, are known.

It is easy to assure ourselves that the estimates of § 4.1 let us obtain the product of ν mixing functions from $\prod_{p=1}^{\nu} \widehat{\mathbf{E}}\, X_{I_p}$; each of them is evaluated in terms of arbitrary neighbouring points of the corresponding block, i.e. the points can be chosen so that the estimate over these points will be minimal for the block. If the process is related to a Markov chain, the mixing function is estimated by the complete lengths of blocks, taking advantage of the fact that the sum of lengths of blocks of the partition $\{I_1, \ldots, I_\nu\}$ is not less than the length of the set I itself.

It is clear, that upon refinement of the partition $\{I_1, \ldots, I_\nu\}$ the cardinality of the set $\big\{\{I_1, \ldots, I_\nu\} | N_\nu(I_1, \ldots, I_\nu) > 0\big\}$ increases together with ν, however, enlarging the number of blocks stops this growth due the sufficiently high rate of decrease of the mixing function.

We need the following formulas, valid for any nonnegative symmetric function $g(a_1, \ldots, a_s)$, $\{a_1, \ldots, a_s\} = \mathcal{A}_s$:

$$\sum_{\mathcal{A}_s \in \mathfrak{N}} g(a_1, \ldots, a_s) = \sum_{r=1}^{s} \sum_{m_1+\ldots+m_r=s} \frac{s!}{m_1! \ldots m_r!} \times \sum_{\mathcal{A}_r \in \mathfrak{N}}^{(<)} g(\underbrace{a_1, \ldots, a_1}_{m_1}, \ldots, \underbrace{a_r, \ldots, a_r}_{m_r}), \tag{4.49}$$

and its inference

$$\sum_{\mathcal{A}_s \in \mathfrak{N}} g(a_1, \ldots, a_s) \leqslant s! \sum_{\mathcal{A}_r \in \mathfrak{N}}^{(\leqslant)} g(a_1, \ldots, a_s), \tag{4.50}$$

$$\sum_{\mathcal{A}_s \in \mathfrak{N}} g(a_1, \ldots, a_s) \geqslant s! \sum_{\mathcal{A}_r \in \mathfrak{N}}^{(<)} g(a_1, \ldots, a_s). \tag{4.51}$$

One can note that for $g \equiv 1$ formula (4.49) takes the form

$$n^s = \sum_{r=1}^{s} \sum_{m_1+\ldots+m_r=s} \frac{s!}{m_1! \ldots m_r!} \binom{n}{r}.$$

It can be easily obtained from formula (4.6), if in the latter one replaces x by n, k by s and takes into account (4.5).

We fix a partition $\{I_1, \ldots, I_\nu\}$. Without loss of generality we regard the blocks $I_1, \ldots, I_\nu$ of the partition to be arranged with respect to their first element, i.e. $t_1^{(p)} \leqslant t_1^{(p+1)}$, $1 \leqslant p \leqslant \nu - 1$. It is easy to see that the connectedness condition of partitions $N_\nu(I_1, \ldots, I_\nu) > 0$ (see formulas (4.36) and (4.37)) will be equivalent to

$$\forall p:\ 2 \leqslant p \leqslant \nu \quad \exists q:\ 1 \leqslant q \leqslant \nu - 1\,,\ t_1^{(p)} \in [I_q \backslash \{t_1^{(p)}\}].$$

Let

$$q_j = \max\left\{q | t_1^{(j+1)} \in [I_q \backslash \{t_1^{(j+1)}\}],\ 1 \leqslant q \leqslant \nu\right\}, \qquad 1 \leqslant j \leqslant \nu - 1. \tag{4.52}$$

Obviously,

$$q_1 = 1, \qquad q_j \leqslant q_{j+1}, \quad 1 \leqslant j \leqslant \nu - 2,$$
$$q_j \leqslant j, \qquad 1 \leqslant j \leqslant \nu - 1.$$

For the given partition $\{I_1, \ldots, I_\nu\}$ let q_j takes exactly $\mu - 1$ different values at the points $n_1, \ldots, n_{\mu-1}$, i.e.

$$\begin{aligned} &n_1 = 1, \\ &n_{i+1} = \min\{q_j | q_j > q_{n_i}\}, \qquad 1 \leqslant i \leqslant \mu - 2, \\ &n_\mu - 1 = \nu. \end{aligned} \tag{4.53}$$

Hence

$$\bigcup_{p=n_i+1}^{n_{i+1}-1} I_p \subset [I_{n_i}], \qquad 1 \leqslant i \leqslant \mu - 1,$$
$$t_1^{(n_{i+1})} \in [I_{n_i}], \qquad 1 \leqslant i \leqslant \mu - 2. \tag{4.54}$$

Geometric interpretation of the numbers q_j is apparent: q_j is the largest cardinal number of the blocks covering $t_1^{(j+1)}$.

Let us explain the meaning of μ. It is evident that all the first elements of the blocks $I_2, \ldots, I_\nu$ of the connected partition belong to the set $\bigcup\limits_{p=1}^{\nu-1} [I_p \backslash \{t_1^{(p)}\}]$. Then the number $\mu - 1$ denotes the least cardinality of the system $\{I_{n_1}, \ldots, I_{n_{\mu-1}}\} \subset \{I_1, \ldots, I_\nu\}$ such that all the leftmost elements of the blocks belong to the set $\bigcup\limits_{p=1}^{\mu-1} [I_{n_p} \backslash \{t_1^{(n_p)}\}]$. It follows from (4.53) that $I_{n_1} = I_1$, therefore, the condition $|I_1| \geqslant 2$ is natural.

Geometrical meaning of the number μ is closely related with the graph $G^{(2)}_{\{I_1, \ldots, I_\nu\}}$, considered in § 4.2 (see p. 81, fig.2).

Let us construct the minimal, in a sense of a number of vertices, connected subgraph, where the first and the last vertices are not subject to removal. The minimal connected subgraph of the original graph will have exactly μ vertices. In our example it is obtained from the original graph $G^{(2)}_{\{I_1, \ldots, I_\nu\}}$ by removing the 3rd, 4th 5th and 7th vertices and incidental to them parts.

Fig. 3

Thus, keeping in mind linearity and symmetry of a centered moment and previous remarks we have

$$\sum_{I \in \mathfrak{N}} \Gamma(X_I) = \sum_{\nu=1}^{k} (-1)^{\nu-1} \sum_{\bigcup\limits_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \times$$
$$\times \prod_{i=1}^{\mu-1} \Bigg\{ \sum_{\substack{I_{n_i} \in \mathfrak{N} \\ t_1^{(n_i)} \in [I_{n_{i-1}}]}} \widehat{\mathrm{E}}(X_{I_{n_i}}) \prod_{p=n_i+1}^{n_{i+1}-1} \sum_{I_p \in [\mathfrak{I}_{n_i}]} \widehat{\mathrm{E}}(X_{I_p}) \Bigg\}, \tag{4.55}$$
$$[I_{n_0}] = \mathcal{N}, \qquad \Gamma(X_I) = \Gamma(X_{t_1}, \ldots, X_{t_k}).$$

Formula (4.55) will be basic for our further investigations. Applying the formulas (4.49) – (4.51) to it one can get sufficiently exact estimates for the k^{th} order cumulant of the sum of random variables.

Let $T_i = \{t_1^{(n_i+1)}, \ldots, t_1^{(n_{i+1})}\}$. The estimate

$$\Big|\sum_{I\in\mathfrak{N}} \Gamma(X_I)\Big| \leqslant \sum_{\nu=1}^{k} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \times$$

$$\times \sum_{t_1=1}^{n} \prod_{i=1}^{\mu-1} \Big\{ k_{n_i} \sum_{I_{n_i}^{(1)} \in \mathfrak{N}_{t_1^{(n_i)}}} \sum_{T_i \in [\mathfrak{J}_{n_i}]}^{(<)} |\widehat{\mathbf{E}}(X_{I_{n_i}})| \prod_{p=n_i+1}^{n_{i+1}-1} k_p \sum_{I_p^{(1)} \in \mathfrak{N}_{t_1^{(p)}}} |\widehat{\mathbf{E}}(X_{I_p})| \Big\}$$

follows from (4.55).

We remark that if for some partition $\{I_1, \ldots, I_\nu\}$ there exists $i \in \{1, \ldots, \mu-1\}$ such that $T_i = \varnothing$, then this partition is not connected. Applying (4.51) and (4.50) in turn to a part of the expression in braces we obtain

$$\Big|\sum_{I\in\mathfrak{N}} \Gamma(X_I)\Big| \leqslant \sum_{\nu=1}^{k} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \times$$

$$\times \sum_{t_1=1}^{n} \prod_{i=1}^{\mu-1} \Big\{ \frac{(k_{n_i} - 1 + n_{i+1} - n_i)! k_{n_i}}{(n_{i+1} - n_i)!} \sum_{I_{n_i}^{(1)} \cup T_i \in \mathfrak{N}_{t_1^{(n_i)}}}^{(\leqslant)} |\widehat{\mathbf{E}}(X_{I_{n_i}})| \times$$

$$\times \prod_{p=n_i+1}^{n_{i+1}-1} k_p! \sum_{I_p^{(1)} \in \mathfrak{N}_{t_1^{(p)}}} |\widehat{\mathbf{E}}(X_{I_p})| \Big\}.$$

If we notice that

$$\sum_{i=1}^{\mu-2} (k_{n_i} - 1 + n_{i+1} - n_i) + k_{n_{\mu-1}} - 1 + n_\mu - 1 - n_{\mu-1} \leqslant k - 1,$$

$$\frac{(k_{n_i} - 1 + n_{i+1} - n_i)! k_{n_i}!}{(k_{n_i} - 1)!(n_{i+1} - n_i)!} \leqslant k_{n_i}! 2^{k_{n_i} - 1 + n_{i+1} - n_i},$$

we arrive at the estimate

$$\Big|\sum_{I\in\mathfrak{N}}\Gamma(X_I)\Big| \leqslant 2^{k-1}\sum_{\nu=1}^{k}\sum_{\bigcup_{p=1}^{\nu} I_p=I} N_\nu(I_1,\ldots,I_\nu)\times$$

$$\times\sum_{t_1=1}^{n}\prod_{i=1}^{\mu-1}\Big\{k_{n_i}!\sum^{(\leqslant)}_{I^{(1)}_{n_i}\cup T_i\in\mathfrak{N}_{t_1^{(n_i)}}}|\widehat{\mathbf{E}}(X_{I_{n_i}})|\prod_{p=n_i+1}^{n_{i+1}-1}k_p!\sum^{(\leqslant)}_{I^{(1)}_p\in\mathfrak{N}_{t_1^{(p)}}}|\widehat{\mathbf{E}}(X_{I_p})|\Big\}. \tag{4.56}$$

LEMMA 4.6. *If*

$$|\widehat{\mathbf{E}}(X_{I_p})| \leqslant C_0^{k_p-\varepsilon}C_2^{k_p}\min_{1\leqslant i<k_p} f^{1/u}(t_i^{(p)},t_{i+1}^{(p)}), \tag{4.57}$$

where $0\leqslant\varepsilon\leqslant k_p$, $u\geqslant 1$, $C_0\geqslant 1$, $C_2>0$, $f\in\mathcal{K}^{*)}$, $1\leqslant p\leqslant\nu$, $1\leqslant\nu\leqslant k$, *then*

$$\Big|\sum_{I\in\mathfrak{N}}\Gamma(X_I)\Big| \leqslant nk!4^{k-1}C_0^{k-\varepsilon}C_2^{k}\max_{1\leqslant s<k}\Lambda_n^{k-1}(f,su); \tag{4.58}$$

if

$$|\widehat{\mathbf{E}}(X_{I_p})| \leqslant C_0^{k_p-\varepsilon}C_2^{k_p}\prod_{j=1}^{r_p-1} f^{1/u}(l_j^{(p)},l_{j+1}^{(p)}), \tag{4.59}$$

where $0\leqslant\varepsilon\leqslant k_p$, $u\geqslant 1$, $C_0\geqslant 1$, $C_2>0$, $f\in\mathcal{K}^{(\geqslant 1)}$, $1\leqslant p\leqslant\nu$, $1\leqslant\nu\leqslant k$, *then*

$$\Big|\sum_{I\in\mathfrak{N}}\Gamma(X_I)\Big| \leqslant nk!4^{k-1}C_0^{k-\varepsilon}C_2^{k}\Lambda_n^{k-1}(f,u). \tag{4.60}$$

Proof. Since for any monotone sequence $s_1,\ldots,s_v$ such that $I_p\subset\{s_1,\ldots,s_v\}$ and $s_1=t_1^{(p)}$, $s_v=t_{k_p}^{(p)}$

$$\min_{1\leqslant i<k_p} f(t_i^{(p)},t_{i+1}^{(p)}) \leqslant \prod_{j=1}^{v-1} f^{\frac{1}{v-1}}(s_j,s_{j+1}),\qquad f\in\mathcal{K},$$

then

$$\sum^{(\leqslant)}_{I_p^{(1)}\in\mathfrak{N}_{t_1^{(p)}}}|\widehat{\mathbf{E}}(X_{I_p})| \leqslant C_0^{k_p-\varepsilon}C_2^{k_p}\Lambda_n^{k_p-1}(f,u(k_p-1)),$$

---

*) We remind that the classes of generalized mixing functions $\mathcal{K}$, $\mathcal{K}^{(\leqslant 1)}$, $\mathcal{K}^{(\geqslant 1)}$ are determined by the relations (4.7), (4.8) and (4.9), respectively.

$$\sum_{I_{n_i}^{(1)}\cup T_i\in \mathfrak{N}_{t_1^{(n_i)}}}^{(\leqslant)} |\widehat{\mathbf{E}}(X_{I_{n_i}})| \leqslant C_0^{k_{n_i}-\varepsilon} C_2^{k_{n_i}} \Lambda_n^{k_{n_i}-1+n_{i+1}-n_i}(f,\, u(k_{n_i}-1+n_{i+1}-n_i)).$$

Applying the inequality (4.46) we complete the proof of relation (4.58).

Relation (4.60) is proved analogously, if we note that for any monotone sequence $s_1, \ldots, s_v$ such that

$$\mathcal{J}_p \subset \{s_1, \ldots, s_v\}, \qquad s_1 = l_1^{(p)}, \quad s_v = l_{r_p}^{(p)},$$
$$\prod_{j=1}^{r_p-1} f^{1/u}(l_j^{(p)}, l_{j+1}^{(p)}) \leqslant \prod_{j=1}^{v-1} f^{1/u}(s_j, s_{j+1})$$

for any $f \in \mathcal{K}^{(\leqslant 1)}$. Therefore

$$\sum_{I_p^{(1)}\in \mathfrak{N}_{t_1^{(p)}}}^{(\leqslant)} |\widehat{\mathbf{E}}(X_{I_p})| \leqslant C_0^{k_p-\varepsilon} C_2^{k_p} \Lambda_n^{k_p-1}(f,\, u),$$

$$\sum_{I_{n_i}^{(1)}\cup T_i\in \mathfrak{N}_{t_1^{(n_i)}}}^{(\leqslant)} |\widehat{\mathbf{E}}(X_{n_i})| \leqslant C_0^{k_{n_i}-\varepsilon} C_2^{k_{n_i}} \Lambda_n^{k_{n_i}-1+n_{i+1}-n_i}(f,\, u).$$

Lemma 4.6 is proved. ■

COROLLARY of Lemma 4.6. *If*

$$|\widehat{\mathbf{E}}(X_{I_p})| \leqslant C_0^{k_p-\varepsilon} C_2^{k_p} \min_{1\leqslant i<k_p} f^{1/u}(t_i^{(p)}, t_{i+1}^{(p)}), \tag{4.61}$$

where $0 \leqslant \varepsilon \leqslant k_p$, $u \geqslant 1$, $C_0 \geqslant 1$, $C_2 > 0$, $f \in \mathcal{K}^{(\leqslant 1)}$, $1 \leqslant p \leqslant \nu$, $1 \leqslant \nu \leqslant k$, *then*

$$\Big|\sum_{I\in\mathfrak{N}} \Gamma(X_I)\Big| \leqslant nk!4^{k-1} C_0^{k-\varepsilon} C_2^k \Lambda_n^{k-1}(f,\, (k-1)u). \tag{4.62}$$

The proof follows directly from (4.58), since the condition $f \in \mathcal{K}^{(\leqslant 1)}$ implies the inequality

$$\Lambda_n(f,\, v_1) \leqslant \Lambda_n(f,\, v_2), \qquad 1 \leqslant v_1 \leqslant v_2.$$

Proof of Theorem 4.15. In assertion 1 it suffices in (4.61) to put $f = \alpha$, $C_0 = 2$, $C_2 = C$, $\varepsilon = 0$, $u = 1$. Asserttion 2 is proved analogously to the Corollary of Lemma 4.6, however, instead of Theorem 4.1.2 a more subtle inequality

$$|\widehat{\mathbf{E}}\, X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-1} C^{k-2} \varphi^{\frac{\delta}{1+\delta}}(t_i,\, t_{i+1}) \mathbf{E}^{\frac{\delta}{1+\delta}} |X_{t_i}|^{1+\frac{1}{\delta}} \mathbf{E}^{\frac{1}{1+\delta}} |X_{t_{i+1}}|^{1+\delta},$$

valid for $|X_t| \leqslant C$ with probability 1, $\delta > 0$, is used. The boundedness of the random variables $X_1, \ldots, X_n$ is used in the estimate

$$\sum_{t=s}^{n} f^{1/u}(s, t)\mathbf{E}^{1/v}|X_t|^v \leqslant C \sum_{t=s}^{n} f^{1/u}(s, t) \leqslant C\,\Lambda_n(f, u).$$

The stronger inequality

$$\sum_{t=s}^{n} f^{1/u}(s, t)\mathbf{E}^{1/v}|X_t|^v \leqslant \max_{s \leqslant t \leqslant n} \mathbf{E}^{1/v}|X_t|^v \Lambda_n(f, u),$$

on which the proof of Theorem 4.16 rests, holds too. ■

Proof of Theorem 4.16. In case 1) we use the result of corollary 1 a) to Theorem 4.2 to get the estimate of the type of (4.61), where $f = \alpha$,

$$C_0 = \frac{3}{2}, \quad C_2 = 2 \max_{1 \leqslant t \leqslant n} \mathbf{E}^{\frac{1}{(1+\delta)k}} |X_t|^{(1+\delta)k}, \quad \varepsilon = 1, \quad u = 1 + \frac{1}{\delta}.$$

The proof of 4.16.2 is analogous. ■

REMARK to Theorem 4.16. *It is easy to see that the result of Theorem 4.15.1 follows from 4.16.1 letting $\delta \to \infty$.*

Proof of Theorem 4.17. Since under conditions of the theorem

$$\mathbf{E}|X_t|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad t = 1, \ldots, n, \quad k = 2, 3, \ldots,$$

for some $\gamma_1 \geqslant 0$, $H_1 > 0$, then

$$\mathbf{E}^{\frac{1}{1+\delta}} |X_t|^{(1+\delta)k} \leqslant (k!)^{1+\gamma_1} H_1^k (\widehat{1+\delta})^{(1+\gamma_1)k}, \qquad \delta \geqslant 0,$$

and due to Theorem 4.16.1

$$|\Gamma_k(S_n)| \leqslant 2\,(k!)^{2+\gamma_1} 12^{k-1} H_1^k (\widehat{1+\delta})^{(1+\gamma_1)k} \Lambda_n^{k-1}\big(\alpha, (1 + 1/\delta)(k-1)\big)\, n. \quad ■$$

Proof of Theorems 4.19 – 4.21.1. The scheme of the proof is the same as in the theorems 4.15 – 4.17 with the only difference that (4.59) is used instead of (4.61) and (4.60) is obtained instead of (4.62).

Now only Theorems 4.18, 4.21.2 and 4.22 are to be proved, as far as on the one hand, the estimates of corollaries to Theorems 4.3, 4.5.2 and 4.6 are not contained by the scheme of Lemma 4.6, and on the other hand, the estimate

$$\prod_{p=1}^{\nu} \prod_{j=1}^{r_p} m_j^{(p)}! \leqslant k!,$$

true for all ν, $1 \leqslant \nu \leqslant k$, is too rough. In order to avoid this difficulty we return to formula (4.55). After applying (4.49) to it, as it was done above, we arrive at the estimate

$$\Big|\sum_{I\in\mathfrak{N}}\Gamma(X_I)\Big| \leqslant \sum_{\nu=1}^{k}\sum_{\bigcup\limits_{p=1}^{\nu} I_p=I} N_\nu(I_1,\ldots,I_\nu)\sum_{t_1=1}^{n}\prod_{i=1}^{\mu-1}\times$$

$$\times\Bigg\{\sum_{r_{n_i}=1}^{k_{n_i}}\sum_{m_1^{(n_i)}+\ldots+m_{r_{n_i}}^{(n_i)}=k_{n_i}}\frac{k_{n_i}!}{m_1^{(n_i)}!\ldots m_{r_{n_i}}^{(n_i)}!}\sum^{(<)}_{\mathcal{J}_{n_i}^{(1)}\in\mathfrak{N}_{l_1^{(n_i)}}}\sum^{(<)}_{T_i\in[\mathfrak{J}_{n_i}]}|\widehat{\mathbf{E}}(X_{I_{n_i}})|\times$$

$$\times\prod_{p=n_i+1}^{n_{i+1}-1}\sum_{r_p=1}^{k_p}\sum_{m_1^{(p)}+\ldots+m_{r_p}^{(p)}=k_p}\frac{k_p!}{m_1^{(p)}!\ldots m_{r_p}^{(p)}!}\sum^{(<)}_{\mathcal{J}_p^{(1)}\in\mathfrak{N}_{l_1^{(p)}}}|\widehat{\mathbf{E}}(X_{I_p})|\Bigg\}.$$

Next, applying (4.51) and (4.50) to the sum

$$\sum^{(<)}_{\mathcal{J}_{n_i}^{(1)}\in\mathfrak{N}_{l_1^{(n_i)}}}\sum^{(<)}_{T_i\in[\mathfrak{J}_{n_i}]}|\widehat{\mathbf{E}}(X_{I_{n_i}})|$$

we obtain the estimate

$$\Big|\sum_{I\in\mathfrak{N}}\Gamma(X_I)\Big| \leqslant \sum_{\nu=1}^{k}\sum_{\bigcup\limits_{p=1}^{\nu} I_p=I} N_\nu(I_1,\ldots,I_\nu)\sum_{t_1=1}^{n}\prod_{i=1}^{\mu-1}\times$$

$$\times\Bigg\{\sum_{r_{n_i}=1}^{k_{n_i}}\sum_{m_1^{(n_i)}+\ldots+m_{r_{n_i}}^{(n_i)}=k_{n_i}}\frac{k_{n_i}!}{m_1^{(n_i)}!\ldots m_{r_{n_i}}^{(n_i)}!}\,\frac{(k_{n_i}-1+n_{i+1}-n_i)!}{(n_{i+1}-n_i)!(k_{n_i}-1)!}\times$$

$$\times\sum^{(\leqslant)}_{\mathcal{J}_{n_i}^{(1)}\cup T_i\in\mathfrak{N}_{l_1^{(n_i)}}}|\widehat{\mathbf{E}}(X_{I_p})|\prod_{p=n_i+1}^{n_{i+1}-1}\sum_{r_p=1}^{k_p}\times \tag{4.63}$$

$$\times\sum_{m_1^{(p)}+\ldots+m_{r_p}^{(p)}=k_p}\frac{(k_p)!}{m_1^{(p)}!\ldots m_{r_p}^{(p)}!}\sum_{\mathcal{J}_p^{(1)}\in\mathfrak{N}_{l_1^{(p)}}}|\widehat{\mathbf{E}}(X_{I_p})|\Bigg\}.$$

Estimate (4.63) allows to weaken the restriction (4.57). The following assertion is true.

LEMMA 4.7. *If*

$$|\widehat{\mathbf{E}}(X_{I_p})| \leqslant m_1^{(p)}! \ldots m_{r_p}^{(p)}! C_0^{k_p-\varepsilon} C_2^{k_p} \min_{1 \leqslant i < k_p} f^{1/u}(t_i^{(p)}, t_{i+1}^{(p)}), \tag{4.64}$$

where $0 \leqslant \varepsilon \leqslant k_p,\ u \geqslant 1,\ C_0 \geqslant 1,\ C_2 > 0,\ f \in \mathcal{K},\ 1 \leqslant p \leqslant \nu,\ 1 \leqslant \nu \leqslant k$, *then*

$$\Big|\sum_{I \in \mathfrak{N}} \Gamma(X_I)\Big| \leqslant nk! 8^{k-1} C_0^{k-\varepsilon} C_2^k \max_{1 \leqslant s < k} \Lambda_n^{k-1}(f, su); \tag{4.65}$$

if

$$|\widehat{\mathbf{E}}(X_{I_p})| \leqslant m_1^{(p)}! \ldots m_{r_p}^{(p)}! C_0^{k_p-\varepsilon} C_2^{k_p} \prod_{j=1}^{r_p-1} f^{1/u}(l_j^{(p)}, l_{j+1}^{(p)}), \tag{4.66}$$

where $0 \leqslant \varepsilon \leqslant k_p,\ u \geqslant 1,\ C_0 \geqslant 1,\ C_2 > 0,\ f \in \mathcal{K}^{(\geqslant 1)},\ 1 \leqslant p \leqslant \nu,\ 1 \leqslant \nu \leqslant k$, *then*

$$\Big|\sum_{I \in \mathfrak{N}} \Gamma(X_I)\Big| \leqslant nk! 8^{k-1} C_0^{k-\varepsilon} C_2^k \Lambda_n^{k-1}(f, u). \tag{4.67}$$

The proof is analogous to that of Lemma 4.7, only the relations (4.1), (4.2) and

$$\sum_{r_p=1}^{k_p} \sum_{m_1^{(p)}+\ldots+m_{r_p}^{(p)}=k_p} 1 = \sum_{r_p=1}^{k_p} \binom{k_p-1}{r_p-1} = 2^{k_p-1}$$

are used.

COROLLARY of Lemma 4.7. *If*

$$|\widehat{\mathbf{E}}(X_{I_p})| \leqslant m_1^{(p)}! \ldots m_{r_p}^{(p)}! C_0^{k_p-\varepsilon} C_2^{k_p} \min_{1 \leqslant i \leqslant k_p} f^{1/u}(t_i^{(p)}, t_{i+1}^{(p)}), \tag{4.68}$$

where $0 \leqslant \varepsilon \leqslant k_p,\ u \geqslant 1,\ C_0 \geqslant 1,\ C_2 > 0,\ f \in \mathcal{K}^{(\leqslant 1)},\ 1 \leqslant p \leqslant \nu,\ 1 \leqslant \nu \leqslant k$, *then*

$$\Big|\sum_{I \in \mathfrak{N}} \Gamma(X_I)\Big| \leqslant nk! 8^{k-1} C_0^{k-\varepsilon} C_2^k \Lambda_n^{k-1}(f, (k-1)u). \tag{4.69}$$

The proof is analogous to that of the corollary to Lemma 4.7, therefore it is omitted.

Now we are able to give exact (in a sense of order) estimates of Theorems 4.18, 4.21.2 and 4.22.

Proof of Theorem 4.18. In (4.68) we assume $f = \alpha,\ C_0 = 2,\ C_2 = H_2,\ \varepsilon = 0,\ u = 1$.

Proof of Theorem 4.21.2. In this case it suffices to put $f = \psi,\ C_0 = 2,\ C_2 = H_1,\ \varepsilon = 1,\ u = 1$.

Proof of Theorem 4.22. In (4.66) we set $f = \varphi,\ C_0 = 2,\ C_2 = H_2,\ \varepsilon = 1,\ u = 1$.

4.4. Theorems and inequalities of large deviations for sums of dependent random variables

The estimates of $\Gamma_k(S_n)$, obtained in Theorems 4.15 – 4.22, and basic lemmas from Chapter 2 enable us to get theorems and inequalities of large deviations for the distribution $\mathbf{P}(Z_n < x)$ of the normed sum $Z_n = S_n/B_n$, $B_n^2 = \mathbf{E}S_n^2$ (everywhere $\mathbf{E}X_t = 0, \quad t = 1, \dots, n$).

In order not to complicate the proof too much, theorems of large deviations for $\mathbf{P}(Z_n < x)$ will be presented only for the case of a stationary sequence X_t, $t = 1, 2, \dots$. In the case of a general nonstationary sequence in Theorems 4.15 – 4.22 it is better to estimate $\Gamma_k(S_n)$ with the help of $\Lambda_n^{k-2} L_{k,\,n}$ instead of

$$\Lambda_n^{k-2} n \max_{1 \leqslant t \leqslant n} \mathbf{E}|X_t|^k / B_n^k,$$

where

$$L_{k,\,n} = \sum_{t=1}^{n} \mathbf{E}|X_t|^k / B_n^k.$$

To this end one needs to express the sum S_n in terms of enlarged summands and to study the behavior of $\Gamma_k(S_n)$ with respect to B_n.

Thus, in the following theorems we consider a stationary sequence X_t with $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = 1$, $B_n^2 = \mathbf{E}S_n^2 \geqslant \sigma_0^2 n$, $\sigma_0 > 0$.

THEOREM 4.23. *If* $|X_1| \leqslant C$ *with probability* 1,

$$\alpha(s, t) \leqslant K_1 \, exp\{-b_1(t - s)\}, \qquad K_1 > 0, \quad b_1 > 0,$$

then

$$|\Gamma_k(Z_n)| \leqslant (k!)^2 B_1 \left(\frac{8\,Ce}{b_1 B_n}\right)^{k-2},$$

$k = 2, 3, \dots$, $B_1 = 8\,C^2 K \exp\{1 + b_1\}/(b_1 \sigma_0^2)$, $K = \max\{1, K_1\}$, *and for* $\xi = Z_n$ *the relations of large deviations* (2.6), (2.13) *and estimates* (2.2), (2.3) *with*

$$\gamma = 1, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0}\right)^{1/3}, \quad \bar{\Delta} = \frac{b_1 B_n}{8\,eC}$$

are valid, where

$$H_0 = (8\,eC/b_1) \max\{1, B_1\}, \quad H = 4\,B_1$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0}\right)^{1/3} (\sqrt{n})^{1/3}.$$

THEOREM 4.24. *If for some* $\gamma_1 \geqslant 0$, $H_1 > 0$

$$\mathbf{E}|X_1|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad k = 2, 3, \dots, \quad \alpha(s, t) \leqslant K_1 \exp\{-b_1(t-s)\},$$

$K_1 > 0$, $b_1 > 0$, *then*

$$|\Gamma_k(Z_n)| \leqslant (k!)^{3+\gamma_1} B_2 \left(\frac{48\, eH_1 2^{\gamma_1}}{b_1 B_n}\right)^{k-2},$$

$k = 2, 3, \dots$, $B_2 = 96\, H_1^2 4^{\gamma_1} \sqrt{K} \exp\{1 + b_1/2\}/(b_1 \sigma_0^2)$, $K = \max\{1, K_1\}$, *and for* $\xi = Z_n$ *the relations of large deviations* (2.6), (2.13) *and estimates* (2.2), (2.3) *with*

$$\gamma = 2 + \gamma_1, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0}\right)^{1/(5+2\gamma_1)}, \quad \bar{\Delta} = \frac{b_1 B_n}{48\, e \cdot 2^{\gamma_1} H_1}$$

hold, where

$$H_0 = (48\, e 2^{\gamma_1} H_1/b_1) \max\{1, B_2\}, \quad H = 2^{3+\gamma_1} B_2$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0}\right)^{1/(5+2\gamma_1)} \cdot (\sqrt{n})^{1/(5+2\gamma_1)}.$$

THEOREM 4.25. *If for some* $\gamma_2 \geqslant 0$, $H_2 > 0$ $|\mathbf{E}(X_t^k | \mathcal{F}_1^{t-1})| \leqslant (k!)^{1+\gamma_2} H_2^k$ *with probability* 1, $k = 2, 3, \dots$, $t = 1, \dots, n$, $\alpha(s, t) \leqslant K_1 \exp\{-b_1(t-s)\}$, $K_1 > 0$, $b_1 > 0$, *then*

$$|\Gamma_k(Z_n)| \leqslant (k!)^{2+\gamma_2} B_3 \left(\frac{16\, eH_2}{b_1 B_n}\right)^{k-2},$$

$k = 2, 3, \dots$, $B_3 = 16\, H_2^2 K \exp\{1 + b_1\}/(b_1 \sigma_0^2)$, *and for* $\xi = Z_n$ *the relations of large deviations* (2.6), (2.13) *and estimates* (2.2), (2.3) *with*

$$\gamma = 1 + \gamma_2, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0}\right)^{1/(3+2\gamma_2)}, \quad \bar{\Delta} = \frac{b_1 B_n}{16\, eH_2}$$

hold, where

$$H_0 = (16\, eH_2/b_1) \max\{1, B_3\}, \quad H = 2^{2+\gamma_2} B_3$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0}\right)^{1/(3+2\gamma_2)} (\sqrt{n})^{1/(3+2\gamma_2)}.$$

THEOREM 4.26. *Let random variables X_t be related to a Markov chain ξ_t. If $|X_1| \leqslant C$ with probability 1, $\varphi(s, t) \leqslant \exp\{-b_2(t-s)\}$, $b_2 > 0$, then*

$$|\Gamma_k(Z_n)| \leqslant k!\, B_4 \left(\frac{8(1+b_2)C}{b_2 B_n}\right)^{k-2},$$

$k = 2, 3, \ldots$, $B_4 = 8C^2(1+b_2)/(b_2\sigma_0^2)$, and for $\xi = Z_n$ the relations of large deviations (2.6), (2.13) and estimates (2.2), (2.3) with

$$\gamma = 0, \quad \Delta_\gamma = c_\gamma \frac{B_n}{H_0}, \quad \bar{\Delta} = \frac{b_2 B_n}{8(1+b_2)C}$$

hold, where

$$H_0 = \big(8C(1+b_2)/b_2\big) \max\{1, B_4\}, \quad H = 2B_4$$

and

$$\Delta_\gamma \geqslant c_\gamma \frac{\sigma_0}{H_0}\sqrt{n}.$$

THEOREM 4.27. *Let random variables X_t be related to a Markov chain ξ_t. If for some $\gamma_1 \geqslant 0$, $H_1 > 0$*

$$\mathbf{E}|X_1|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad k = 2, 3, \ldots, \qquad \varphi(s, t) \leqslant \exp\{-b_2(t-s)\},$$
$$\psi(s, t) \leqslant K_3 \exp\{-b_3(t-s)\}, \qquad K_3 > 0, \quad b_2 > 0, \quad b_3 > 0,$$

then

$$1)\ |\Gamma_k(Z_n)| \leqslant (k!)^{2+\gamma_1} B_5 \left(\frac{16(2+b_2)2^{\gamma_1} H_1}{b_2 B_n}\right)^{k-2},$$

$k = 2, 3, \ldots$, $B_5 = 32 \cdot 4^{\gamma_1} H_1^2 (2+b_2)/(b_2\sigma_0^2)$, and for $\xi = Z_n$ the relations of large deviations (2.6), (2.13) and estimates (2.2), (2.3) with

$$\gamma = 1 + \gamma_1, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0}\right)^{1/(3+2\gamma_1)}, \quad \bar{\Delta} = \frac{b_2 B_n}{16(2+b_2)2^{\gamma_1} H_1}$$

hold, where

$$H_0 = \big(16(2+b_2)2^{\gamma_1} H_1/b_2\big) \max\{1, B_5\}, \quad H = 2^{2+\gamma_1} B_5$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0}\right)^{1/(3+2\gamma_1)} (\sqrt{n})^{1/(3+2\gamma_1)};$$

$$2)\ |\Gamma_k(Z_n)| \leqslant (k!)^{1+\gamma_1} B_6 \left(\frac{16(1+b_3)KH_1}{b_3 B_n}\right)^{k-2}, \quad k = 2, 3, \dots,$$

$K = \max\{1, K_3\}$, $B_6 = 16H_1^2 K(1+b_3)/(b_3\sigma_0^2)$, *and for* $\xi = Z_n$ *the relations of large deviations* (2.6), (2.13) *and estimates* (2.2), (2.3) *with*

$$\gamma = \gamma_1, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0}\right)^{1/(1+2\gamma_1)}, \quad \bar{\Delta} = \frac{b_3 B_n}{16(1+b_3)KH_1}$$

are valid, where

$$H_0 = (H_1 B_6/\sigma_0^2) \max\{1, B_6\}, \quad H = 2^{1+\gamma_1} B_6$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0}\right)^{1/(1+2\gamma_1)} (\sqrt{n})^{1/(1+2\gamma_1)}.$$

THEOREM 4.28. *Let random variables* X_t *be connected to a Markov chain* ξ_t. *If for some* $\gamma_2 \geqslant 0$, $H_2 > 0$ $|\mathrm{E}(X_t^k|\mathcal{F}_1^{t-1})| \leqslant (k!)^{1+\gamma_2} H_2^k$ *with probability* 1, $k = 2, 3, \dots,$ $t = 1, \dots, n$, $\varphi(s,t) \leqslant \exp\{-b_2(t-s)\}$, $b_2 > 0$, *then*

$$|\Gamma_k(Z_n)| \leqslant (k!)^{1+\gamma_2} B_7 \left(\frac{16(1+b_2)H_2}{b_2 B_n}\right)^{k-2}, \quad k = 2, 3, \dots,$$

$B_7 = 16\, H_2^2(1+b_2)/(b_2\sigma_0^2)$, *and for* $\xi = Z_n$ *the relations of large deviations* (2.6), (2.13) *and estimates* (2.2), (2.3) *with*

$$\gamma = \gamma_2, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0}\right)^{1/(1+2\gamma_2)}, \quad \bar{\Delta} = \frac{b_2 B_n}{16(1+b_2)H_2}$$

hold, where

$$H_0 = (B_7\sigma_0^2/H_2) \max\{1, B_7\}, \quad H = 2^{1+\gamma_2} B_7$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0}\right)^{1/(1+2\gamma_2)} (\sqrt{n})^{1/(1+2\gamma_2)}.$$

THEOREM 4.29. *Let random variables X_t be m-dependent. If $|X_1| \leqslant C$ with probability 1, then*

$$|\Gamma_k(Z_n)| \leqslant k! B_8 \left(\frac{8(m+1)C}{B_n} \right)^{k-2}, \quad k = 2, 3, \dots ,$$

$B_8 = 16(m+1)C^2/\sigma_0^2$, and for $\xi = Z_n$ the relations of large deviations (2.6), (2.13) and estimates (2.2), (2.3) with

$$\gamma = 0, \quad \Delta_\gamma = c_\gamma \frac{B_n}{H_0}, \quad \bar{\Delta} = \frac{B_n}{8(m+1)C}$$

are valid, where

$$H_0 = 8(m+1)C \max\{1, B_8\}, \quad H = 2 B_8$$

and

$$\Delta_\gamma \geqslant c_\gamma \frac{\sigma_0}{H_0} \sqrt{n}.$$

THEOREM 4.30. *Let random variables X_t be m-dependent. If for some $\gamma_1 \geqslant 0$, $H_1 > 0$*

$$\mathbf{E}|X_1|^k \leqslant (k!)^{1+\gamma_1} H_1^k, \qquad k = 2, 3, \dots ,$$

then

$$|\Gamma_k(Z_n)| \leqslant (k!)^{2+\gamma_1} B_9 \left(\frac{16(m+1)2^{\gamma_1} H_1}{B_n} \right)^{k-2}, \quad k = 2, 3, \dots ,$$

$B_9 = 96(m+1)4^{\gamma_1} H_1^2/\sigma_0^2$, and for $\xi = Z_n$ the relations of large deviations (2.6), (2.13) and estimates (2.2), (2.3) with

$$\gamma = 1 + \gamma_1, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0} \right)^{1/(3+2\gamma_1)}, \quad \bar{\Delta} = \frac{B_n}{16(m+1)2^{\gamma_1} H_1}$$

hold, where

$$H_0 = 16(m+1)2^{\gamma_1} H_1 \max\{1, B_9\}, \quad H = 2^{2+\gamma_1} B_9$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0} \right)^{1/(3+2\gamma_1)} (\sqrt{n})^{1/(3+2\gamma_1)}.$$

THEOREM 4.31. *Let random variables X_t be m–dependent. If for some $\gamma_2 \geqslant 0$, $H_2 > 0$*

$$|\mathbf{E}(X_t^k|\mathcal{F}_1^{t-1})| \leqslant (k!)^{1+\gamma_2} H_2^k \qquad \text{with probability } 1,$$

$k = 2, 3, \ldots,\quad t = 1, \ldots, n$, *then*

$$|\Gamma_k(Z_n)| \leqslant (k!)^{1+\gamma_2} B_{10} \left(\frac{16(m+1)H_2}{B_n}\right)^{k-2}, \quad k = 2, 3, \ldots,$$

$B_{10} = 32(m+1)H_2^2/\sigma_0^2$, *and for $\xi = Z_n$ the relations of large deviations (2.6), (2.13) and estimates (2.2), (2.3) with*

$$\gamma = \gamma_2, \quad \Delta_\gamma = c_\gamma \left(\frac{B_n}{H_0}\right)^{1/(1+2\gamma_2)}, \quad \bar{\Delta} = \frac{B_n}{16(m+1)2^{\gamma_1} H_2}$$

are valid, where

$$H_0 = 16(m+1)H_2 \max\{1, B_{10}\}, \quad H = 2^{1+\gamma_2} B_{10}$$

and

$$\Delta_\gamma \geqslant c_\gamma \left(\frac{\sigma_0}{H_0}\right)^{1/(1+2\gamma_2)} (\sqrt{n})^{1/(1+2\gamma_2)}.$$

Proof of Theorems 4.23 – 4.31. Theorems 4.23 – 4.31 are proved by direct calculating γ, Δ_γ, $\bar{\Delta}$, H and applying the results of Theorems 4.15 – 4.22 in basic lemmas of Chapter 2. It is important to notice that in the case $f(s, t) \leqslant K \exp\{-b(t-s)\}$, $K \geqslant 1$,

$$\begin{aligned}
&\Lambda_n(f, 1) \leqslant K\big(1 + \exp\{-b\} + \ldots + \exp\{-b(n-s)\}\big) \leqslant \\
&\qquad \leqslant K/(1 - \exp\{-b\}) = K\big(1 + 1/(\exp\{b\} - 1)\big) \leqslant K(1 + 1/b), \\
&\Lambda_n(f, k-1) \leqslant K^{\frac{1}{k-1}}\big(1 + (k-1)/b\big), \\
&\Lambda_n(f, 1 + 1/\delta) \leqslant K^{\frac{\delta}{1+\delta}}\big(1 + (1+\delta)/b\delta\big), \qquad \delta > 0, \\
&\Lambda_n(f, (1+1/\delta)(k-1)) \leqslant K^{\frac{\delta}{(1+\delta)(k-1)}}\big(1 + (1+\delta)(k-1)/b\delta\big), \qquad \delta > 0.
\end{aligned}$$

Due to the inequality $k^k \leqslant k! \exp\{k\}$ and to condition $k \geqslant 2$,

$$\begin{aligned}
\Lambda_n^{k-1}(f,\, k-1) &\leqslant K\big(1+(k-1)/b\big)^{k-1} = \\
&= K\big((k-1)/b\big)^{k-1}\big(1+b/(k-1)\big)^{k-1} \leqslant K(e/b)^{k-1}(k-1)!e^b \leqslant \\
&\leqslant k!(K/2b)\exp\{1+b\}(e/b)^{k-2},
\end{aligned}$$

$$\begin{aligned}
&\Lambda_n^{k-1}\big(f,\,(1+1/\delta)(k-1)\big) \leqslant K^{\frac{\delta}{1+\delta}}\big(1+(1+\delta)(k-1)/b\delta\big)^{k-1} = \\
&= K^{\frac{\delta}{1+\delta}}\big((1+\delta)(k-1)/b\delta\big)^{k-1}\Big(1+b\delta/\big((1+\delta)(k-1)\big)\Big)^{k-1} \leqslant \\
&\leqslant K^{\frac{\delta}{1+\delta}}\big((1+\delta)(k-1)/b\delta\big)^{k-1}\exp\{b\delta/(1+\delta)\} \leqslant \\
&\leqslant k!K^{\frac{\delta}{1+\delta}}\big((1+\delta)/2b\delta\big)\exp\{1+b\delta/(1+\delta)\}\big((1+\delta)e/b\delta\big)^{k-2}.
\end{aligned}$$

Theorems 4.29 – 4.31 are proved analogously to Theorems 4.23 – 4.25 if we note that in the case of m–dependent random variables the estimate

$$\Lambda_n\big(\bar{m},\,(1+1/\delta)(k-1)\big) \leqslant m+1, \qquad \delta > 0,$$

is valid, where $\bar{m}(s,\,t)$ is the function of m–dependence.

CHAPTER 5
THEOREMS OF LARGE DEVIATIONS FOR POLYNOMIAL FORMS, MULTIPLE STOCHASTIC INTEGRALS AND STATISTICAL ESTIMATES

5.1. Estimates of cumulants and theorems of large deviations for polynomial forms, polynomial Pitman estimates and U–statistics

Consider a polynomial form

$$\zeta_n^{(p)} = \sum_{1 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_p \leqslant n} a_{\alpha_1, \ldots, \alpha_p} X_{\alpha_1} \cdot \ldots \cdot X_{\alpha_p} \tag{5.1}$$

of order $p \geqslant 1$, where $X_1, \ldots, X_n$ are values of the random process X_t at the points $t = 1, 2, \ldots, n$, and the coefficients $a_{\alpha_1, \ldots, \alpha_p}$ do not change under any permutation of indices $\alpha_1, \ldots, \alpha_p$. Sum (5.1) can be regarded as the sum of dependent random variables, and for the study of the k^{th} order cumulant $\Gamma_k(\zeta_n^{(p)})$ one can apply the technique, presented in Chapter 4.

Throughout this chapter we assume that $X_1, X_2, \ldots$ are independent identically distributed r.v. with $\mathrm{E}X_1 = 0$ and $\mathrm{E}X_1^2 = \sigma^2 > 0$. For brevity by $\max\limits_{\alpha}$, $\sum\limits_{\alpha}$ denote the corresponding operation over all collections $\alpha = \{\alpha_1, \ldots, \alpha_p\}$, $1 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_p \leqslant n$, and $\max\limits_{\alpha(1,s)} \sum\limits_{\alpha(s+1,p)} a_{\alpha_1, \ldots, \alpha_p}$ will denote summation over any indices $1 \leqslant \alpha_{s+1} \leqslant \ldots \leqslant \alpha_p \leqslant n$ of the collection $\{\alpha_1, \ldots, \alpha_p\}$ afterwards taking maximum over the remaining indices $1 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_s \leqslant n$. Put

$$X_\alpha = X_{\alpha_1} \cdot \ldots \cdot X_{\alpha_p}, \qquad a_\alpha = a_{\alpha_1, \ldots, \alpha_p}.$$

Then

$$\begin{aligned} \zeta_n^{(p)} &= \sum_\alpha a_\alpha X_\alpha, \\ B_n^2 = \mathrm{D}\zeta_n^{(p)} &= \sum_{\substack{\alpha, \alpha' \\ \alpha \cap \alpha' \neq \varnothing}} a_\alpha a_{\alpha'} \mathrm{E}(X_\alpha - \mathrm{E}X_\alpha)(X_{\alpha'} - \mathrm{E}X_{\alpha'}). \end{aligned} \tag{5.2}$$

The results of this section for polynomial forms were obtained by A. Basalykas and V. Statulevičius (Basalykas, Plikusas, Statulevičius, 1987), for polynomial Pitman estimates by A. Basalykas (Basalykas, 1985) and for U-statistics by A. Aleškevičienė (Aleškevičienė, 1990).

Denote

$$A_n^2 = \max_{\substack{1 \leqslant s_1, s_2 \leqslant p \\ s_1+s_2=p}} \Big(\max_{\alpha(1,s_1)} \sum_{\alpha(s_1+1,p)} |a_\alpha| \Big) \Big(\max_{\alpha(1,s_2)} \sum_{\alpha(s_2+1,p)} |a_\alpha| \Big).$$

We prove the asymptotic normality of the r.v.

$$Z_n^{(p)} := \frac{\zeta_n^{(p)} - \mathrm{E}\zeta_n^{(p)}}{B_n}, \tag{5.3}$$

as $A_n/B_n \to 0$ $(n \to \infty)$; in the case of bounded r.v. $X_1, X_2, \ldots$ the theorem of probabilities of large deviations for $Z_n^{(p)}$ can be obtained in the zone B_n/A_n, while under Bernstein's condition on the moments of the r.v. X_j it holds in the zone $(B_n/A_n)^{1/(2p-1)}$.

As above, $\beta_k = \mathrm{E}|X_1|^k$, and $\Gamma_k(Y)$ will denote the k^{th} cumulant of the r.v. Y with $\mathrm{E}|Y|^k < \infty$.

LEMMA 5.1. *Let $\beta_k < \infty$ for all $k \geqslant 1$. Then for $k \geqslant 2$*

$$|\Gamma_k(Z_n^{(p)}| \leqslant k!\, \beta_{kp} 4^k (A_n/B_n)^{k-2}. \tag{5.4}$$

If there exist constants $H_0 > 0$ and $\sigma > 0$ such that

$$\mathrm{E}|X_1^k| \leqslant (k!/2) H_0^{k-2} \sigma^2, \qquad k = 2, 3, \ldots,$$

then

$$|\Gamma_k(Z_n^{(p)}| \leqslant (k!)^p 2^k p^{2p} H_0^{2p-2} \sigma^2 H_1^{4k-2} (p^p H_0^p A_n/B_n)^{k-2}, \tag{5.5}$$

where $H_1 = \max\{1, \sigma/H_0\}$.

Proof. Consider r.v. $Z_N = a_1Y_1 + \ldots + a_NY_N$, where $a_i \in R$ are nonrandom coefficients, and $Y_1, \ldots, Y_N$ are some r.v. with $\mathrm{E}|Y_j|^k < \infty$ for $j = 1, 2, \ldots, N$ and $k = 1, 2, \ldots$. If $\Gamma(Y_{\alpha_1}, \ldots, Y_{\alpha_k})$ is a simple cumulant of the vector $(Y_{\alpha_1}, \ldots, Y_{\alpha_k})$, then it is easy to see that

$$\begin{aligned} \Gamma_k(Z_N) &= \sum_{1 \leqslant \alpha_1, \ldots, \alpha_k \leqslant N} a_{\alpha_1} \ldots a_{\alpha_k} \Gamma(Y_{\alpha_1}, \ldots, Y_{\alpha_k}) = \\ &= \sum_{1 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_k \leqslant N} \sum_{i_1, \ldots, i_k}^{\nabla} a_{i_1} \ldots a_{i_k} \Gamma(Y_{i_1}, \ldots, Y_{i_k}), \end{aligned} \tag{5.6}$$

where $\sum^{\nabla}_{i_1,\ldots,i_k}$ stands for summation over all the permutations of $i_1, \ldots, i_k$ in the collection $\alpha_1, \ldots, \alpha_k$. Equating members with $a_{\alpha_1} \ldots a_{\alpha_k}$, from the relation

$$\Gamma_k(Z_N) = k! \sum_{\nu=1}^{k} \frac{(-1)^{\nu-1}}{\nu} \sum_{\substack{1 \leqslant k_1, \ldots, k_\nu \leqslant k \\ k_1+\ldots+k_\nu=k}} \frac{\mathrm{E}Z_N^{k_1} \ldots \mathrm{E}Z_N^{k_\nu}}{k_1! \ldots k_\nu!} \tag{5.7}$$

we obtain

$$\begin{aligned} &\sum^{\nabla}_{i_1,\ldots,i_k} a_{i_1} \ldots a_{i_k} \Gamma(Y_{i_1}, \ldots, Y_{i_k}) = \\ &= k! \sum_{\nu=1}^{k} \frac{(-1)^{\nu-1}}{\nu} \sum_{\substack{1 \leqslant k_1, \ldots, k_\nu \leqslant k \\ k_1+\ldots+k_\nu=k}} \prod_{j=1}^{\nu} \frac{\mathrm{E}(a_{\alpha_1} Y_{\alpha_1} + \ldots a_{\alpha_k} Y_{\alpha_k})^{k_j}}{k_j!}. \end{aligned} \tag{5.8}$$

Note that if the multiplicity among $\alpha_1, \ldots, \alpha_k$ is equal to r, $r \geqslant 2$, for some $\alpha_s \in \{\alpha_1, \ldots, \alpha_k\}$, then in the right-hand side of relation (5.8) instead of

$$\mathrm{E}(a_{\alpha_1} Y_{\alpha_1} + \ldots + a_{\alpha_k} Y_{\alpha_k})^{k_j}$$

the member

$$\mathrm{E}(a_{\alpha_1} Y_{\alpha_1} + \ldots + a_{\alpha_s} Y_{\alpha_s} + a_{\alpha_{s+r}} Y_{\alpha_{s+r}} + \ldots + a_{\alpha_k} Y_{\alpha_k})^{k_j}$$

will occur. It is clear that

$$\begin{aligned} \mathrm{E}(a_{\alpha_1} Y_{\alpha_1} + \ldots + a_{\alpha_k} Y_{\alpha_k})^{k_j} \leqslant k_j! \sum_{\nu=1}^{k_j} \sum_{\substack{1 \leqslant r_1, \ldots, r_\nu \leqslant k_j \\ r_1+\ldots+r_\nu=k_j}} (r_1! \ldots r_\nu!)^{-1} \times \\ \times \sum^{\nabla}_{i_1,\ldots,i_\nu} |a_{i_1}|^{r_1} \ldots |a_{i_\nu}|^{r_\nu} |\mathrm{E}(Y_{i_1}^{r_1} \ldots Y_{i_\nu}^{r_\nu})|. \end{aligned} \tag{5.9}$$

Now, let $\sum\limits_{j=1}^{N} a_j Y_j$ represent $\zeta_n^{(p)}/B_n$. In this case

$$|\mathrm{E}Y_{i_1}^{r_1} \ldots Y_{i_\nu}^{r_\nu}| \leqslant \beta_{pk_j}. \tag{5.10}$$

Since $\mathrm{E}X_1 = 0$, from relations (5.6) and (5.8) – (5.10) we obtain

$$|\Gamma_k(Z_n^{(p)}| \leqslant k! \beta_{kp} 4^k B_n^{-k} \sum^{*}_{\alpha^{(1)}, \ldots, \alpha^{(k)}} |a_{\alpha^{(1)}}| \ldots |a_{\alpha^{(k)}}|, \tag{5.11}$$

where $\sum\limits^{\ *}_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ means summation over all connected with each other sets $\alpha^{(1)}, \ldots, \alpha^{(k)}$, $\alpha^{(s)} = \{\alpha_1^{(s)}, \ldots, \alpha_p^{(s)}\}$, $1 \leqslant \alpha_1^{(s)} \leqslant \ldots \alpha_p^{(s)} \leqslant n$, where each $\alpha_j^{(s)}$, $1 \leqslant j \leqslant p$, $1 \leqslant s \leqslant k$, must recur at least twice in the collections $\alpha^{(1)}, \ldots, \alpha^{(k)}$ (for unconnected sets $\alpha^{(1)}, \ldots, \alpha^{(k)}$ $\Gamma(Y_{\alpha^{(1)}}, \ldots, Y_{\alpha^{(k)}}) = 0$).

By $(y^{(j)}_{r_{j-1}}, y^{(j+1)}_{r_j})$ denote a partition of the vector $\{\alpha_1^{(j)}, \ldots, \alpha_p^{(j)}\}$ such that $y^{(j)}_{r_{j-1}} \cup y^{(j+1)}_{r_j} = \{\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_p^{(j)}\}$ and $y^{(j)}_{r_{j-1}} \cap y^{(j+1)}_{r_j} = \varnothing$. Here r_{j-1} and r_j stands for the number of elements in the sets $y^{(j)}_{r_{j-1}}$ and $y^{(j+1)}_{r_j}$, respectively. It is clear that $r_{j-1} + r_j = p$. As the sets $\alpha^{(1)}, \ldots, \alpha^{(k)}$ are connected, it follows

$$\sum_{\alpha^{(1)},\ldots,\alpha^{(k)}}^{\ *} \prod_{s=1}^{k} |a_{\alpha^{(s)}}| \leqslant \max_{1 \leqslant r_0, \ldots, r_{k-1} \leqslant p} \Bigg\{ \max_{y^{(2)}_{r_1}} \sum_{y^{(1)}_{r_0}} \sqrt{|a_{(y^{(1)}_{r_0}, y^{(2)}_{r_1})}|} \times$$
$$\times \Bigg(\sum_{y^{(k)}_{r_{k-1}}, y^{(k-1)}_{r_{k-2}}}^{\ *} |a_{(y^{(1)}_{r_0}, y^{(k)}_{r_{k-1}})}|\, |a_{(y^{(k-1)}_{r_{k-2}}, y^{(k)}_{r_{k-1}})}| \Bigg)^{1/2} \Bigg\} \times$$
$$\times \Bigg\{ \max_{y^{(k-2)}_{r_{k-3}}} \sum_{y^{(k-1)}_{r_{k-2}}} \sqrt{|a_{(y^{(k-2)}_{r_{k-3}}, y^{(k-1)}_{r_{k-2}})}|} \times \tag{5.12}$$
$$\times \Bigg(\sum_{y^{(1)}_{r_0}, y^{(k)}_{r_{k-1}}}^{*} |a_{(y^{(1)}_{r_0}, y^{(k)}_{r_{k-1}})}|\, |a_{(y^{(k-1)}_{r_{k-2}}, y^{(k)}_{r_{k-1}})}| \Bigg)^{1/2} \Bigg\} \times$$
$$\times \prod_{j=2}^{k-2} \Bigg\{ \max_{y^{(j-1)}_{r_{j-2}}, y^{(j+1)}_{r_j}} \sum_{y^{(j)}_{r_{j-1}}} \sqrt{|a_{(y^{(j-1)}_{r_{j-2}}, y^{(j)}_{r_{j-1}})}|} \cdot \sqrt{|a_{(y^{(j)}_{r_{j-1}}, y^{(j+1)}_{r_j})}|} \Bigg\}.$$

Since

$$\sum_{\alpha} |a_\alpha|\, |b_\alpha| \leqslant \sqrt{\sum_{\alpha} a_\alpha^2 \sum_{\alpha} b_\alpha^2} \tag{5.13}$$

and

$$B^2 = \sum_{\alpha^{(1)}, \alpha^{(2)}}^{*} a_{\alpha^{(1)}} a_{\alpha^{(2)}} \mathrm{E}(X_{\alpha^{(1)}} - \mathrm{E}X_{\alpha^{(1)}})(X_{\alpha^{(2)}} - \mathrm{E}X_{\alpha^{(2)}}), \tag{5.14}$$

applying inequality (5.13) to the sums over all braces of relation (5.12) and considering (5.14) we get

$$|\Gamma_k(Z_n^{(p)})| \leqslant k! 4^k \beta_{kp} (A_n/B_n)^{k-2}. \tag{5.15}$$

If $|\mathrm{E}X_1^k| \leqslant (k!/2) H_0^{k-2} \sigma^2$, then the estimate

$$|\mathrm{E}Y_{i_1}^{r_1} \ldots Y_{i_\nu}^{r_\nu}| \leqslant ((p-1)k_j)! r_1! \ldots r_\nu! (\sigma^2/2\, H_0^2)^{\nu+1} H_0^{k_j p}$$

implies

$$|\Gamma_k(Z_n^{(p)})| \leqslant (k!)^p (pH_0)^{kp} \cdot 2^k (\sigma/H_0)^2 H_1^{4k-2} (A_n/B_n)^{k-2}. \tag{5.16}$$

Lemma 5.1 is proved.

Estimate (5.16) also yields asymptotic normality $Z_n^{(p)}$, if $A_n/B_n \to 0$ $(n \to \infty)$.

THEOREM 5.1. *Let for a sequence of identically distributed* r.v. $X_1, X_2, \ldots$ *there exist constants* $H_0 > 0$ *and* $\sigma > 0$ *such that Bernstein's condition*

$$|\mathrm{E}X_1^k| \leqslant \frac{1}{2}\, k! H_0^{k-2} \sigma^2, \qquad k = 2, 3, \ldots,$$

is satisfied. Then for a r.v. $\xi = Z_n^{(p)}$, *determined by equality* (5.3) *in the interval*

$$0 \leqslant x < \Delta_\gamma, \qquad \gamma = p - 1,$$

the relations of large deviations (2.6) *with*

$$\Delta_\gamma = \frac{1}{6} \left(\frac{3 B_n}{4\sqrt{2}\, (pH_0)^{3p} H_1^8 A_n} \right)^{\frac{1}{2p-1}}$$

and estimates (2.3), (2.14) *with* $\gamma = p - 1$, $H = (H_0 p)^{2p} 2^p 4 \sigma^2 H_1^6 / H_0^2$,

$$\bar{\Delta} = \frac{B_n}{2(pH_0)^{2p} H_1^4 A_n}, \qquad H_1 = \max\left\{1, \frac{\sigma}{H_0}\right\}$$

are valid.

If $|X_1| \leqslant C$ *a.s., then for the* r.v. $\xi = Z_n^{(p)}$ *in the interval*

$$0 \leqslant x < \Delta_\gamma, \qquad \gamma = 0,$$

the relations (2.6) *with*

$$\Delta_\gamma = \frac{\sqrt{2}\, B_n}{144\, A_n C^p \max\{1,\ 16\, C^{2p-2}\}}, \tag{5.17}$$

and estimates (2.3), (2.14) *with* $\gamma = 0$, $\bar{\Delta} = B_n/(4\, C^p A_n)$, $H = 32\sigma^2 C^{2p-2}$, *are valid.*

The proof of the theorem follows immediately from estimates (5.16) and Lemmas 2.3, 2.1, 2.4 (or their corollaries). ■

Further, let there exist observations $X_1, \ldots, X_n$ of the type $X_i = \theta + \xi_i$, where $\theta \in R$ is a (shift) parameter to be estimated and $\xi_1, \ldots, \xi_n$ are independent identically distributed r.v. with the distribution function $F(x)$ such that

$$\mu_s = \int x^s \, dF(x) < \infty, \qquad s = 1, 2, \ldots, 2p.$$

Let

$$\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j, \qquad m_j = \frac{1}{n}\sum_{j=1}^{n}(X_i - \bar{X})^j, \tag{5.18}$$

and by V_p denote a space of all polynomials in $X_1 - \bar{X}, \ldots, X_n - \bar{X}$ of the degree, not exceeding p. In order to estimate the parameter θ one can employ the so-called polynomial Pitman estimate $t_{n,p}^{(1)}$ or modified one $t_{n,p}^{(2)}$ having a more simple form than $t_{n,p}^{(1)}$ ((Kagan, 1966), (Kagan, Klebanov, Fintushal, 1974)), possessing a number of good properties. These estimates are determined there as follows:

$$t_{n,p}^{(1)} = \bar{X} - \hat{\mathbf{E}}(\bar{X}|V_p), \tag{5.19}$$

$$t_{n,p}^{(2)} = \bar{X} - A_1 + \sum_{j=2}^{p} A_j m_j, \tag{5.20}$$

where $\hat{\mathbf{E}}(\,\cdot\,|V_p)$ is an operator projecting on the space V_p, and $A_1, \ldots, A_p$ are expressed through $\mu_1, \mu_2, \ldots, \mu_{2p}$. Note that $\mathbf{E}_\theta t_{n,p}^{(1)} = \theta$, and $\mathbf{E}_\theta t_{n,p}^{(2)} = \theta + O(1/n)$.

Since the estimates $t_{n,p}^{(j)}$ can be expressed in the form (5.1) with coefficients

$$|a^{(j)}_{\alpha^{(1)},\ldots,\alpha^{(s)}}| \leqslant C_j(s, \mu_1, \ldots, \mu_{2p})/n^s, \qquad s = 1, \ldots, p,$$

it is possible for them to get the analog of Lemma 5.1.

LEMMA 5.2 (Basalykas, 1985). *Let* $\mathbf{E}\xi_1 = 0$, $\sigma^2 = \mathbf{E}\xi_1^2$, $|\xi_i| \leqslant L$ a.s., $i = 1, \ldots, n$, $L > 0$. *Then for all* $k = 3, 4, \ldots$

$$|\Gamma_k((t_{n,p}^{(j)} - \theta)/\sqrt{\mathbf{D}t_{n,p}^{(j)}})| \leqslant k!(H_{1,j}/\sqrt{n})^{k-2}, \qquad j = 1, 2, \tag{5.21}$$

where $H_{1,j} = H_{1,j}(\sigma, L, p)$ *are constants, depending on the quantities in parentheses, whose explicit form is given in the paper* (Basalykas, 1984).

If $\mathbf{E}\xi_1 = 0$, $\sigma^2 = \mathbf{E}\xi_1^2 > 0$ *and there exists a constant* $H_0 > 0$ *such that* $|\mathbf{E}\xi_1^k| \leqslant \frac{1}{2}\, k! H_0^{k-2}\sigma^2$, $k = 2, 3, \ldots$, *then for every* $k = 3, 4, \ldots$

$$|\Gamma_k((t_{n,p}^{(j)} - \theta)/\sqrt{\mathbf{D}t_{n,p}^{(j)}})| \leqslant (k!)^p(H_{2,j}/\sqrt{n})^{k-2}, \qquad j = 1, 2, \tag{5.22}$$

where $H_{2,j} = H_{2,j}(H_0, \mu_2, \mu_3, \ldots, \mu_{2p}, p)$.

The proof of the lemma will not be given in detail here since it follows from the proof of Lemma 5.1. It ought to be noted only that the quantity

$$J = \max_{\substack{0 \leqslant k_1 \leqslant k \\ 2 \leqslant v_1, \ldots, v_{k_1} \leqslant p}} \sum^{*}_{\substack{\alpha^{(1)}(1, v_1) \\ \cdots\cdots \\ \alpha^{(k_1)}(1, v_{k_1})}} \prod_{s=1}^{k_1} |a^{(j)}_{\alpha^{(s)}(1, v_s)}| \prod_{s=k_1+1}^{k} |a^{(j)}_{\alpha_s}|$$

appears here instead of $\sum\limits^{*}_{\alpha^{(1)},\ldots,\alpha^{(k)}} \prod\limits_{s=1}^{k} |a_{\alpha^{(s)}}|$. The connectedness of the collections of indices and the estimates of coefficients enables to obtain

$$J \leqslant \max_{\substack{0\leqslant k_1\leqslant k\\ 2\leqslant v_1,\ldots,v_{k_1}\leqslant p}} \frac{C_{1,j}^{k} n^{\frac{v_1+\ldots+v_{k_1}}{2}+1}}{n^{v_1+\ldots+v_{k_1}-k_1/2}(\sqrt{n})^{k-k_1}} \leqslant \frac{C_{1,j}^{k}}{(\sqrt{n})^{k-2}},$$

where $C_{1,j} = \max\limits_{1\leqslant s\leqslant p} C_j(s, \mu_1, \ldots, \mu_{2p})$.

THEOREM 5.2 (Basalykas, 1985). *Let observations $X_1, \ldots, X_n$ have the form $X_j = \theta + \xi_j$, where $\xi_1, \ldots, \xi_n$ are independent identically distributed r.v. with $\mathbf{E}\xi_1 = 0$, $\mathbf{E}\xi_1^2 = \sigma^2 > 0$, and let there exist a constant $H_0 > 0$ such that*

$$|\mathbf{E}\xi_1^k| \leqslant \frac{1}{2}\, k! H_0^{k-2}\sigma^2, \qquad k = 2, 3, \ldots .$$

Then for the estimates $t_{n,p}^{(j)}$, $j = 1, 2$, defined by equalities (5.19), (5.20), in the interval

$$0 \leqslant x < \Delta_\gamma^{(j)}, \qquad \gamma = p - 1,$$

relations of large deviations (2.6) with

$$\Delta_\gamma = \Delta_\gamma^{(j)} = \frac{1}{6}\left(\frac{\sqrt{2n}}{6H_{2,j}}\right)^{\frac{1}{2p-1}}, \qquad j = 1, 2,$$

and estimates (2.3), (2.14) with $\gamma = p-1$, $H = 2^p$, $\bar\Delta = \Delta^{(j)} = \sqrt{n}/H_{2,j}$, $j = 1, 2$, are valid.

If $\mathbf{E}\xi_1 = 0$, $\mathbf{E}\xi_1^2 = \sigma^2 > 0$, $|\xi_i| \leqslant L$ $(L > 0)$ a.s., $i = 1, 2, \ldots, n$, then for the estimates $t_{n,p}^{(j)}$ in the interval

$$0 \leqslant x < \Delta_\gamma^{(j)}, \qquad \gamma = 0,$$

relations of large deviations (2.6) with

$$\Delta_\gamma = \Delta_\gamma^{(j)} = \frac{\sqrt{2n}}{36H_{1,j}}, \qquad j = 1, 2,$$

and estimates (2.3), (2.14) with $\gamma = 0$, $H = 2$, $\bar\Delta = \sqrt{n}/H_{1,j}$, $j = 1, 2$, are valid.

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed r.v. with the common distribution function F, and $\varphi(x_1, x_2)$ be any symmetric relative to its arguments function. Consider U–statistic of the second order with the kernel $\varphi(x_1, x_2)$:

$$U_n := \binom{n}{2}^{-1} \sum_{1\leqslant i<j\leqslant n} \varphi(X_i, X_j).$$

Assume that

$$\mathbf{E}\varphi(X_1, X_2) = 0 \tag{5.23}$$

and

$$\sigma_g^2 = \mathbf{D}g(X_1) > 0, \tag{5.24}$$

where

$$g(x) = \mathbf{E}\big(\varphi(X_1, X_2)|X_1 = x\big).$$

One can easily verify that

$$\begin{aligned} \sigma_U^2 := \mathbf{D}U_n = \mathbf{E}U_n^2 = \frac{4\,\sigma_g^2}{n} + \\ + \frac{2}{n(n+1)}\,\mathbf{E}\psi^2(X_1, X_2) = \frac{4\,\sigma_g^2}{n} + O\Big(\frac{1}{n^2}\Big), \end{aligned} \tag{5.25}$$

where

$$\psi(X_1, X_2) := \varphi(X_1, X_2) - g(X_1) - g(X_2).$$

The study of probabilities of large deviations for U–statistics was begun with the work (Hoeffding, 1963), in which the Bernstein type inequality for U–statistics with a bounded kernel was obtained. The paper (Malevich, Abdalimov, 1979) is devoted to the study of Cramer and Linnik type large deviations. It is shown here that if conditions (5.23), (5.24) are satisfied and if there exist constants $K > 0$ and $\gamma \geqslant 0$ such that

$$\mathbf{E}|\varphi(X_1, X_2)|^2 \leqslant K^k k^{\gamma' k} \tag{5.26}$$

for all $k = 1, 2, \ldots,$ then

$$\mathbf{P}(\sqrt{n}\,U_n/2\,\sigma_g \geqslant x) \sim 1 - \Phi(x), \qquad n \to \infty, \tag{5.27}$$

uniformly in x in the domain

$$0 < x \leqslant \varrho(n)n^{1/\left(2(5+2\,\gamma')\right)}, \qquad \varrho(n) \to 0 \quad (n \to \infty).$$

M.Vandemaele (Vandemaele, 1982) has shown that relation (5.27) also holds under condition (5.26) in a wider interval

$$0 < x \leqslant o\big(n^{1/\left(2(3+2\,\gamma')\right)}\big).$$

R. Dasgupta (Dasgupta, 1984) has dealt with large deviations of Chernov type for U–statistics, formed by non–identically distributed r.v. In these papers U–statistics of the m^{th} order were considered, when the kernel $\varphi(x_1, \ldots, x_m)$ is a symmetric function of m arguments, $m \geqslant 2$.

We present here the estimates for cumulants and a theorem of large deviations for the second order U–statistics, which imply that under condition (5.26) relations of large deviations (2.6) for $\sigma_U^{-1} U_n$ hold in the interval $0 < x \leqslant cn^{1/2(1+2\gamma')}$.

LEMMA 5.3 (Aleškevičienė, 1990). *If conditions* (5.23) *and* (5.24) *are fulfilled and if for all* $k = 3, 4, \ldots$

$$\mathbf{E}|\varphi(X_1, X_2)|^k \leqslant (k!)^{1+\gamma} C^k, \qquad \gamma \geqslant 0,$$

then for $k = 3, 4, \ldots, n-1$

$$|\Gamma_k(U_n)| < 2\, e^{2(k-2)} \frac{2^k - 1}{k} C^k (k!)^{2+\gamma} \frac{1}{n^{k-1}}, \qquad n \geqslant 7,$$

and, consequently, for $n \geqslant n_0 \geqslant 7$

$$|\Gamma_k(U_n/\sqrt{\mathbf{D}U_n})| \leqslant (k!)^{2+\gamma} \left(\frac{2\sqrt{2}\, eC(\sigma)}{\sqrt{n}} \right)^{k-2}, \qquad k = 3, \ldots, n-1, \qquad (5.28)$$

where $C(\sigma) = C/\sigma$ *as* $C \leqslant \sigma$ *and* $C(\sigma) = C^3/\sigma^3$ *as* $C > \sigma$ *and* $n_0 \geqslant 7$ *is defined so that for all* $n \geqslant n_0$ *the inequality* $\mathbf{D}U_n \geqslant e^2\sigma^2/(2n)$ *is satisfied.*

For the proof of Lemma 5.3 we need one more lemma.

Let $X = (X_1, X_2, \ldots, X_k)$ be a r.vector, $f_X(t) = \mathbf{E} e^{i(t, X)}$, $t = (t_1, \ldots, t_k)$, be its characteristic function and

$$\Gamma(X_1^{\nu_1}, \ldots, X_k^{\nu_k}) := \frac{1}{i^\nu} \frac{\partial^\nu}{\partial t_1^{\nu_1} \ldots \partial t_k^{\nu_k}} \ln f_X(t) \Big|_{t_1=0, \ldots, t_k=0} \qquad (5.29)$$

be a mixed cumulant of order $\nu = \nu_1 + \ldots + \nu_k$ of $X = (X_1, \ldots, X_k)$. If all coordinates of $X = (X_1, \ldots, X_k)$ are the same, then

$$\Gamma(X_1^{\nu_1}, \ldots, X_k^{\nu_k}) = \Gamma_\nu(X_1) = \frac{1}{i^\nu} \frac{\partial^\nu}{\partial t_1^\nu} \ln f_{X_1}(t_1) \Big|_{t_1=0}, \qquad (5.30)$$

where $\Gamma_\nu(X_1)$ is the ν^{th} order cumulant of the r.v. X_1.

LEMMA 5.3a (Aleškevičienė, 1990). *If all the coordinates* X_l, $l = 1, \ldots, k$, *of the random vector* $X = (X_1, \ldots, X_k)$ *are identically distributed and for all* $\nu = 3, 4, \ldots, s$

$$\mathbf{E}|X_1|^\nu \leqslant (\nu!)^{1+\gamma} K^\nu, \qquad K > 0, \quad \gamma \geqslant 0, \qquad (5.31)$$

then for $3 \leqslant \nu = \nu_1 + \ldots + \nu_k \leqslant s,\ 0 \leqslant \nu_k \leqslant s,$

$$|\Gamma(X_1^{\nu_1}, \ldots, X_k^{\nu_k})| \leqslant (\nu!)^{1+\gamma} \frac{2^\nu - 1}{\nu} K^\nu. \tag{5.32}$$

Proof of Lemma 5.3a. First consider the case when all coordinates of $X = (X_1, \ldots, X_k)$ are the same, i.e. first estimate the cumulant $\Gamma_\nu(X_1)$.

According to (1.54)

$$\Gamma_\nu(X_1) = \sum{}' \frac{(-1)^{q-1}}{q} \frac{\nu!}{\lambda_1! \ldots \lambda_q!} \prod_{p=1}^{q} m_{\lambda_p} =$$
$$= \sum_{q=1}^{\nu} \sum{}'' \frac{(-1)^{q-1}}{q} \frac{\nu!}{\lambda_1! \ldots \lambda_q!} m_{\lambda_1} \ldots m_{\lambda_q}, \qquad m_k = \mathbf{E}X_1^k,$$

where $\sum'$ denotes summation over all ordered integer nonnegative solutions of the equation $\lambda_1 + \ldots + \lambda_q = \nu,\ 0 \leqslant \lambda_i \leqslant \nu,\ 1 \leqslant q \leqslant \nu$, and $\sum''$ denotes summation over all ordered integer positive solutions of the same equation.

From (5.33) taking into account (5.31) and the obvious inequality

$$\lambda_1! \lambda_2! \ldots \lambda_q! \leqslant \nu!$$

(here $1 \leqslant \lambda_i \leqslant \nu,\ i = 1, \ldots, q,\ 1 \leqslant q \leqslant \nu$, such that $\lambda_1 + \lambda_2 + \ldots + \lambda_q = \nu$), we obtain

$$\begin{aligned} |\Gamma_\nu(X_1)| &< \sum_{q=1}^{\nu} \sum_{\lambda_1+\ldots+\lambda_q=\nu} \frac{\nu!}{q} (\lambda_1!)^\gamma \ldots (\lambda_q!)^\gamma K^\nu = \\ &= (\nu!)^{1+\gamma} K^\nu \sum_{q=1}^{\nu} \frac{1}{q} \sum_{\lambda_1+\ldots+\lambda_q=\nu} 1. \end{aligned} \tag{5.33}$$

The exact number of ordered solutions of the equation

$$\lambda_1 + \ldots + \lambda_q = \nu, \qquad 1 \leqslant \lambda_i \leqslant \nu, \quad i = 1, \ldots, q,$$

as known, is equal to

$$C_{\nu-1}^{q-1} = \binom{\nu-1}{q-1}.$$

Consequently,

$$|\Gamma_\nu(X_1)| < (\nu!)^{1+\gamma} K^\nu \sum_{q=1}^{\nu} \frac{1}{q} \binom{\nu-1}{q-1} = (\nu!)^{1+\gamma} K^\nu \sum_{q=0}^{\nu-1} \frac{1}{q+1} \binom{\nu-1}{q}.$$

Hence, using the relation

$$\sum_{q=0}^{\nu-1} \frac{1}{q+1}\binom{\nu-1}{q} = \frac{2^\nu - 1}{\nu},$$

we finally get

$$|\Gamma_\nu(X_1)| < (\nu!)^{1+\gamma}\frac{2^\nu - 1}{\nu} K^\nu. \tag{5.34}$$

Let us proceed with the estimation of mixed cumulants. It follows from definition of a mixed cumulant and a cumulant of one random variable (cf. relations (5.29) and (5.30)) that a general number of summands in the representation of the ν^{th} order mixed cumulant in terms of moments is equal to the general number of summands in representation (5.33) of the ν^{th} order cumulant of one random variable X_1. In addition, the corresponding summands of mixed and ordinary (i.e. of one random variable X_1) cumulants of the same order will differ only by the fact that in the representation of ordinary cumulants (cf. formula (5.33)) there will be moments $\mathbf{E}X_{i_1}^{l_1}X_{i_2}^{l_2} \ldots X_{i_r}^{l_r}$, $l_1 + l_2 + \ldots + l_r = l$, $1 \leqslant r \leqslant l$, instead of the moments m_l, $2 \leqslant l \leqslant \nu$, in the representation of mixed cumulants. For illustration of this statement compare, for example, the representations of the cumulants $\Gamma_4(X_1)$ and $\Gamma(X_1^2, X_2, X_3)$:

$$\begin{aligned}
\Gamma_4(X_1) = \Gamma(X_1^4) &= m_4 - 4\,m_3m_1 - 3\,m_2^2 + 12\,m_2m_1^2 - 6\,m_1^4, \quad m_i = \mathbf{E}X_1^i,\\
\Gamma(X_1^2, X_2, X_3) &= \mathbf{E}X_1^2X_2X_3 - (2\,\mathbf{E}X_1X_2X_3 \cdot \mathbf{E}X_1 + \mathbf{E}X_1^2X_2 \cdot \mathbf{E}X_3 +\\
&+ \mathbf{E}X_1^2X_3 \cdot \mathbf{E}X_2) - (2\,\mathbf{E}X_1X_2 \cdot \mathbf{E}X_1X_3 + \mathbf{E}X_1^2 \cdot \mathbf{E}X_1X_3) + \cdot\\
&+ 2(2\,\mathbf{E}X_1X_2 \cdot \mathbf{E}X_1 \cdot \mathbf{E}X_3 + 2\,\mathbf{E}X_1X_3 \cdot \mathbf{E}X_1 \cdot \mathbf{E}X_2 +\\
&+ (\mathbf{E}X_1)^2\mathbf{E}X_2X_3 + \mathbf{E}X_1^2\mathbf{E}X_2\mathbf{E}X_3) - 6(\mathbf{E}X_1)^2\mathbf{E}X_2 \cdot \mathbf{E}X_3
\end{aligned}$$

Next, recall that in order to obtain estimates for ordinary cumulants all the members of representation (5.33) were estimated by absolute value. Then, by virtue of the inequality

$$\begin{aligned}
|\mathbf{E}X_{i_1}^{l_1}X_{i_2}^{l_2} \ldots X_{i_r}^{l_r}| &\leqslant (\mathbf{E}|X_{i_1}|^l)^{l_1/l}(\mathbf{E}|X_{i_2}|^l)^{l_2/l} \ldots (\mathbf{E}|X_{i_r}|^l)^{l_r/l} =\\
&= \mathbf{E}|X_1|^l, \qquad l_1 + \ldots + l_r = l, \quad 1 \leqslant r \leqslant l,
\end{aligned}$$

all summands of the representation of mixed cumulants can be estimated in the same way as the corresponding summands in representation (5.33) of the cumulants of one random variable. Consequently, mixed cumulants

$$|\Gamma_\nu(X_{i_1}^{l_1}, X_{i_2}^{l_2}, \ldots, X_{i_r}^{l_r})|, \qquad l_1 + \ldots + l_r = l, \quad 1 \leqslant l_i \leqslant \nu, \quad r < k,$$

will not exceed the obtained estimate (5.34) for the ν^{th} order cumulant of one random variable X_1. Thus, if condition (5.31) is fulfilled and $l_1 + l_2 + \ldots + l_r = \nu$, then for $\nu \geqslant 3$

$$|\Gamma_\nu(X_{i_1}^{l_1}, X_{i_2}^{l_2}, \ldots, X_{i_r}^{l_r})| \leqslant \frac{2^\nu - 1}{\nu} K^\nu (\nu!)^{1+\gamma}.$$

Lemma 5.3a is proved.

Proof of Lemma 5.3. In accordance with (1.47)

$$\Gamma_k(C_n^2 U_n) = \sum_{1 \leqslant i_1 < j_1 \leqslant n} \cdots \sum_{1 \leqslant i_k < j_k \leqslant n} \Gamma_k\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big), \tag{5.35}$$

where $\Gamma\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big)$ is the k^{th} order mixed cumulant of the vector $\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big)$.

Recall a property of mixed cumulants, which will be used in the estimation of $\Gamma_k(C_n^2 U_n)$. Let $\Gamma_k(X_1, \ldots, X_k)$ be a mixed k^{th} order cumulant of the random vector $(X_1, \ldots, X_k)$. If some group of coordinates of $(X_1, \ldots, X_k)$ does not depend on the remaining group of coordinates, then $\Gamma_k(X_1, \ldots, X_k) = 0$.

In order to estimate the quantity $\Gamma_k(C_n^2 U_n)$ first of all it is necessary to determine how many and which summands on the right-hand side of relation (21) are not equal trivially to zero. To this end we note that if the set of all coordinates of the k-dimensional vector

$$\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big) \tag{5.36}$$

can be divided into separate independent groups, then the mixed cumulant

$$\Gamma_k\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big) \tag{5.37}$$

is zero. Note also that if among random variables $X_{i_1}, X_{j_1}, \ldots, X_{i_k}, X_{j_k}$ there are more than $k+1$ different then the coordinates of the vector (5.36) can be divided into two or more groups and, consequently, the corresponding mixed cumulant (5.37) will be equal to zero.

Decompose the sum on the right-hand side of relation (5.35) into k separate sums as follows:

$$\Gamma_k(C_n^2 U_n) = S^{(k)} + S^{(k-1)} + \ldots + S^{(2)} + S^{(1)}. \tag{5.38}$$

Those summands $\Gamma\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big)$ on the right-hand side of relation (5.35), whose all the coordinates $\varphi(X_{i_1}, X_{j_1})$ are different and indecomposable into independent groups, are included in the sum $S^{(k)}$. All the summands (5.37) of the sum (5.35), the corresponding vector (5.36) of which consists of $k-1$ different coordinates, one of which has multiplicity 2, are included in the sum $S^{(k-1)}$. Further, all the summands (5.37) of the sum (5.35) the correspondig vector (5.36) of which consists of $k-2$ different coordinates, indecomposable into independent groups, and either one of these coordinates has multiplicity 3 or two of them has multiplicity 2, are included in the sum $S^{(k-2)}$. Analogously, all the summands of the sum (5.35), the corresponding vector of which (5.36) consists of $k-r$ different coordinates, indecomposable into independent groups, and sum of whose multiplicities equals k, are included in the sum $S^{(k-r)}$, $r = 3, \dots, k-1$.

Let us return to the sum $S^{(k)}$ and estimate its number of summands (5.37). Remind that all the coordinates $\varphi(X_{i_l}, X_{j_l})$, $l = 1, \dots, k$, of the vector (5.36), corresponding to the summand (5.37) $S^{(k)}$, are different and indecomposable into independent groups. Then, according to the graph theory, each such vector (5.36) can be associated with the connected graph having k edges and not more than $k+1$ vertex. First consider the case when $\big(\varphi(X_{i_1}, X_{j_1}), \dots, \varphi(X_{i_k}, X_{j_k})\big)$ can be associated with the connected graph having exactly $k+1$ vertex and k edges, i.e. when among the r.v. $X_{i_1}, X_{j_1}, \dots, X_{i_k}, X_{j_k}$ there will be exactly $k+1$ different real variable–vertex and exactly k different pairs–edges (X_i, X_j), $i < j$. For example, the graph with $k+1$ vertices, corresponding to the r.v. $X_1, X_2, \dots, X_k, X_{k+1}$ and with k different edges, corresponding to k different pairs

$$(X_1, X_2), (X_2, X_3), \dots, (X_{k-1}, X_k), (X_k, X_{k+1}),$$

will be associated with vector

$$\big(\varphi(X_1, X_2), \dots, \varphi(X_{k-1}, X_k), \varphi(X_k, X_{k+1})\big).$$

A connected graph having $k+1$ vertex and k edges is called tree, or a $k+1$–tree. The number of different trees which can be constructed from $k+1$ vertex, is equal to $(k+1)^{k-1}$ (Sachkov, 1977). One can choose $k+1$ different vertex from n possible vertices in total by C_n^{k+1} ways.

According to the polynomial formula

$$(a_1 + a_2 + \dots + a_r)^k = \sum c_k(k_1, k_2, \dots, k_n) a_1^{k_1} \dots a_n^{k_n},$$

$$c_k(k_1, k_2, \dots, k_n) = \frac{k!}{k_1!\, k_2! \dots k_n!}$$

(summation is taken here over all collections of nonnegative integers $k_1, k_2, \dots, k_n$, for which $k_1 + k_2 + \dots + k_n = k$) there will be $k!$ identical members

$$\Gamma\big(\varphi(X_{i_1}, X_{j_1}), \dots, \varphi(X_{i_k}, X_{j_k})\big)$$

in sum $S^{(k)}$.

Consequently, in sum $S^{(k)}$ there will be not more than $C_n^{k+1}(k+1)^{k-1}k!$ not equal trivially to zero summands $\Gamma\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big)$ corresponding to vectors (5.36) with k different coordinates $\varphi(X_{i_l}, X_{j_l})$, $l = 1, \ldots, k$, constructed by $k+1$ different random variable–vertice.

Since from k different random variables–vertices it is possible to construct C_k^2 different pairs–edges, all in all there will be $C^k_{C_k^2}$ different k–dimensional vectors (5.36), constructed in total by k different random variables–vertices. One can choose k vertices by C_n^k ways. There will be $k!$ identical members (5.37). Thus, the sum $S^{(k)}$ will contain not more than $C_n^k C^k_{C_k^2} k!$ summands (5.37), constructed by k different random variables–vertices. The cases, when vectors (5.36) are constructed by $k-1$, $k-2, \ldots, [\sqrt{2}\,k]$ different random variables–vertices are treated analogously. Consequently, the sum $S^{(k)}$ will consist of not more than

$$\Big(C_n^{k+1}(k+1)^{k-1} + C_n^k C^k_{C_k^2} + C_n^{k-1} C^k_{C^2_{k-1}} + \ldots + C_n^{[\sqrt{2k}]} C^k_{C^2_{[\sqrt{2k}]}}\Big) k!$$

summands $\Gamma\big(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k})\big)$ unequal trivially to zero; here $C_l^m = 0$ as $l < m$.

From the above–mentioned, taking into account the Stirling's formula

$$n! = \sqrt{2\pi} n^{n+1/2} \exp\Big\{-n + \frac{\theta}{12}\Big\}, \qquad 0 < \theta < 1,$$

and the inequalities

$$x^\alpha e^{-x} \leqslant \alpha^\alpha, \qquad x > 0, \quad \alpha > 0,$$
$$(x-l)^x < x^x e^{-l}, \qquad x > l > 0,$$

we obtain the estimate

$$\begin{aligned} N(S^{(k)}) \leqslant &\Big(C_n^{k+1}(k+1)^{k-1} + \sum_{j=0}^{q_{k,0}} C_n^{k-j} C^k_{C^2_{k-j}}\Big) k! < \\ <&0.032\, e^{2(k-2)} 2^{-(k-2)} n(n-1)^k, \\ &7 \leqslant k \leqslant \min\{s, n\}, \quad q_{k,0} = k - [\sqrt{2}\,k], \end{aligned} \tag{5.39}$$

for the number $N(S^{(k)})$ of not equal trivially to zero summands (5.37) in the sum $S^{(k)}$.

Let us go over to the sum $S^{(k-1)}$. Its summands (5.37) can be associated with $(k-1)$–dimensional vectors with in total not independent coordinates, one of which has multiplicity 2, and which are constructed by k random variables–vertices. The k–tree can be associated with each such vector, constructed by exactly k real variables–vertices. The number of different trees, which can be constructed from the k vertices given, is equal to k^{k-2}. One can choose k different vertices from n possible by C_n^k ways, and a multiple one by $(k-1)$ way. However, such $(k-1)$–dimensional vector can also be constructed by $k-1$, $k-2$, ..., $[\sqrt{2(k-1)}\,]$ random variables–vertices. In addition, in each of all these cases there will be $k!/2!$ identical members (5.37). Consequently,

$$\begin{aligned} N(S^{(k-1)}) &\leqslant \left\{ C_n^k k^{k-2} + \sum_{j=1}^{q_{k,1}} C_n^{k-j} C_{C_{k-j}^2}^{k-1} \right\} \frac{(k-1)k!}{2} < \\ &< 0.037\, e^{2(k-2)} n(n-1)^k 2^{-(k-2)}, \qquad 7 \leqslant k \leqslant \min\{s, n\}, \\ &q_{k,1} = k - [\sqrt{2(k-1)}]. \end{aligned} \tag{5.40}$$

Quite analogously

$$\begin{aligned} N(S^{(k-r)}) &\leqslant \left\{ C_n^{k-r+1}(k-r+1)^{k-r-1} + \sum_{j=r}^{q_{k,r}} C_n^{k-j} C_{C_{k-j}^2}^{k-r} \right\} \frac{(k-r)^r k!}{2^r} < \\ &< e^{2(k-2)} 2^{-(k-2)} n(n-1)^k k! \left(\frac{e^{1/12}}{k^2\sqrt{2\pi}}\, e^{-2(r-2)} + \right. \\ &\left. + \frac{1}{4\pi k^2}\, e^{-2(r-2)} \left(1 + \frac{1}{e^2 - 1} \right) \right), \end{aligned} \tag{5.41}$$

where $q_{k,r} = k - [\sqrt{2(k-r)}]$, $2 \leqslant r \leqslant k-1$, $7 \leqslant k \leqslant \min\{s, n\}$.

Taking into account relation (5.38) and estimates (5.39) – (5.41) we obtain the estimate for a number of not equal trivially to zero summands (5.37) of the sum (5.35):

$$N(\Gamma_k(C_n^2 U_n)) < 0.089\, e^{2(k-2)} 2^{-(k-2)} n(n-1)^k\, k!, \qquad 7 \leqslant k \leqslant n-1. \tag{5.42}$$

The estimates of $N(\Gamma_k(C_n^2 U_n))$ for $k = 3, 4, 5, 6$ are calculated directly, namely, for $n \geqslant 7$

$$\begin{aligned} N(\Gamma_3(C_n^2 U_n)) &< 0.468\, e^2 2^{-1} n(n-1)^3 3!, \\ N(\Gamma_4(C_n^2 U_n)) &< 0.169\, e^4 2^{-2} n(n-1)^4 4!, \\ N(\Gamma_5(C_n^2 U_n)) &< 0.071\, e^6 2^{-3} n(n-1)^5 5!, \\ N(\Gamma_6(C_n^2 U_n)) &< 0.072\, e^8 2^{-4} n(n-1)^6 6!. \end{aligned}$$

Hence and from (5.42) we find

$$N\big(\Gamma_k(C_n^2 U_n)\big) < 0.468\, e^{2(k-2)} 2^{-(k-2)} n(n-1)^k\, k!, \qquad 3 \leqslant k \leqslant n-1. \tag{5.43}$$

On the other hand, under condition (5.28) and Lemma 5.3a

$$\big|\Gamma(\varphi(X_{i_1}, X_{j_1}), \ldots, \varphi(X_{i_k}, X_{j_k}))\big| \leqslant \frac{2^k-1}{k} C^k (k!)^{1+\gamma}, \qquad \gamma \geqslant 0. \tag{5.44}$$

Next, it follows from (5.35), (5.43) and (5.44) that

$$|\Gamma_k(C_n^2 U_n)| < 0.468\, e^{2(k-1)} \frac{2^k-1}{k} 2^{-(k-2)} C^k (k!)^{2+\gamma} n(n-1)^k,$$

or

$$|\Gamma_k(U_n)| < 2\, e^{2(k-2)} \frac{2^k-1}{k} C^k (k!)^{2+\gamma} n^{-(k-1)}. \tag{5.45}$$

It follows from relations (5.24) and (5.25) that there exists an integer $n_0 \geqslant 7$ such that for all $n \geqslant n_0$

$$\mathbf{D}U_n = \frac{4\sigma^2}{n} + O\left(\frac{1}{n^2}\right) \geqslant \frac{e^2\sigma^2}{2n}.$$

Hence and from (5.45) we find that for $n \geqslant n_0$

$$\begin{aligned}|\Gamma_k(U_n/\sqrt{\mathbf{D}U_n})| &< \frac{C^2}{\sigma^2}\left(\frac{2\sqrt{2}\,eC}{\sigma\sqrt{n}}\right)^{k-2}(k!)^{2+\gamma} \leqslant \\ &\leqslant \left(\frac{2\sqrt{2}\,eC(\sigma)}{\sqrt{n}}\right)^{k-2}(k!)^{2+\gamma},\end{aligned}$$

where $C(\sigma) = C/\sigma$ for $C \leqslant \sigma$ and $C(\sigma) = C^3/\sigma^3$ for $C > \sigma$. ∎

Using Lemmas 5.3 and 2.3, we get the following result.

THEOREM 5.3 (Aleškevičienė, 1990). *If conditions* (5.23) *and* (5.24) *are fulfilled and there exist constants* $C > 0$ *and* $\gamma > 0$ *such that*

$$\mathbf{E}|\varphi(X_1, X_2)|^k \leqslant C^k (k!)^{1+\gamma},$$

then in the interval $0 < x < \Delta_\gamma$,

$$\Delta_\gamma = \frac{1}{6}\left(\frac{\sqrt{n}}{3\sqrt{2}\max\{2,\ 2\sqrt{2}\,eC(\sigma)\}}\right)^{1/(3+2\gamma)},$$

for r.v. $\xi = U_n/\sqrt{\mathbf{D}U_n}$ *relations of large deviations* (2.6) *hold.*

5.2. Cumulants of multiple stochastic integrals and theorems of large deviations *)

Consider a complex Gaussian random measure $\beta(\Lambda)$, $\Lambda \subset R^1$, with the standard properties:

$$\text{a) } \mathbf{E}\beta(\Lambda) = 0,$$

$$\text{b) } \beta(\Lambda) = \beta(-\Lambda),$$

$$\text{c) } \mathbf{E}\beta(\Lambda_1)\overline{\beta(\Lambda_2)} = F(\Lambda_1 \cap \Lambda_2).$$

Here Λ, Λ_1, Λ_2 are measurable sets in R^1, F is the spectral measure of β. It will be assumed that the measure F is continuous (has no atoms). We introduce a space $L_2(F)$ of even complex-valued functions defined in R^m:

$$L_2^{(m)}(F) := \Big\{ \varphi : ||\varphi||^2 = \int\limits_{R^m} |\varphi(\lambda_1, \ldots, \lambda_m)|^2 \prod_{j=1}^{m} F(d\lambda_j) < \infty,$$

$$\varphi(-\lambda_1, \ldots, -\lambda_m) = \overline{\varphi(\lambda_1, \ldots, \lambda_m)}, \qquad m = 1, 2, \ldots \Big\}.$$

The case of the real measure β and the function φ is considered analogously with natural simplifications.

For functions $\varphi \in L_2^{(m)}(F)$ one can determine a multiple stochastic integral with respect to measure β (see, for example, (Ito, 1951))

$$I^{(m)}(\varphi) = \int \ldots \int \varphi(\lambda_1, \ldots, \lambda_m)\, \beta(d\lambda_1) \ldots \beta(d\lambda_m).$$

A definition can be formulated in a usual way, starting from step-functions, equal to zero on diagonals of the space R^m. Let

$$\tilde{\varphi}(\lambda_1, \ldots, \lambda_m) = \frac{1}{m!} \sum{}' \varphi(\lambda_{i_1}, \ldots, \lambda_{i_m}),$$

$\sum'$ denote summation over all permutations of the set $\{1, \ldots, m\}$. Then $I^{(m)}(\varphi) = I^{(m)}(\tilde{\varphi})$.

Let us pay attention to some other properties of the multiple stochastic integrals.

1. $I^{(m)}(\varphi)$ is a real random variable.

---

*) The results of this paragraph were obtained by A. Plikusas (Plikusas, 1980, 1981), (Basalykas, Plikusas, Statulevičius, 1987).

2. $\mathbf{E}I^{(m)}(\varphi) = 0$.

3. $\mathbf{E}I^{(m)}(\varphi)I^{(n)}(\psi) = 0$, if $m \neq n$, and

$$\mathbf{E}I^{(m)}(\varphi)I^{(m)}(\psi) = m! \int \overline{\tilde{\varphi}(\lambda_1, \ldots, \lambda_m)\tilde{\psi}(\lambda_1, \ldots, \lambda_m)} \prod_{j=1}^{m} F(d\lambda_j).$$

4. Let $\varphi_1, \ldots, \varphi_n$ be an orthonormal system of functions in $L_2^{(m)}(F)$, and $H_k(x) = (-1)^k e^{x^2/2} \dfrac{d^k}{dx^k} e^{-x^2/2}$ be the Hermite polynomial of the degree k. Then

$$\underbrace{\int \cdots \int}_{n} \varphi_1(\lambda_1) \cdot \ldots \cdot \varphi_1(\lambda_{p_1})\varphi_2(\lambda_{p_1+1}) \cdot \ldots \cdot \varphi_2(\lambda_{p_1+p_2}) \cdot \ldots \cdot$$

$$\cdot \varphi(\lambda_{p_1+\ldots+p_{n-1}+1}) \cdot \ldots \cdot \varphi(\lambda_{p_1+\ldots+p_n}) \prod_{i=1}^{r} \beta(d\,\lambda_i) =$$

$$= \prod_{i=1}^{n} H_{p_j}\Big(\int \varphi_j(\lambda)\,\beta(d\lambda)\Big), \qquad r = p_1 + \ldots + p_n.$$

5. Let $X_t = \int e^{it\lambda}\beta(d\lambda)$ and $\mathcal{F} = \sigma\{X_t,\, t \in R^1\}$. Then any $\mathcal{F}$-measurable r.v. ξ with $\mathbf{E}\xi^2 < \infty$ can be expanded in a series

$$\xi = \mathbf{E}\xi + \sum_{m=1}^{\infty} I^{(m)}(\varphi_m),$$

convergent in square mean. A collection of functions $\varphi_m \in L_2^{(m)}(F)$ is unique, if one restricts oneself by symmetric functions.

Let us derive a formula to calculate cumulants of the r.v. $I^{(m)}(\varphi)$. By D or $D_{k,m}$ denote a table of pairs of the indices $D_{k,m} = \{(i, j),\ i = \overline{1, k},\ j = \overline{1, m}\}$, with k rows and m columns. A partition of the set $D = D' \cup D''$ will be called a row partition if any row from D belongs either to D' or D''.

Definition 5.1 (Leonov, Shiryaev, 1959). *A partition* $D_{k,m} = \bigcup_{j=1}^{r} D_j$ *will be called indecomposable, if there exists no row partition* $D = D' \cup D''$ *for which any* D_j *belongs either to* D' *or* D''.

Lemma 5.4. *Let* $\varphi \in L_2^{(m)}(F)$. *Then* $\Gamma_k(I^{(m)}(\varphi)) = 0$, *if* km *is odd, and*

$$\Gamma_k(I^{(m)}(\varphi)) = \sum\nolimits^{*} \int_{R^r} \prod_{j=1}^{k} \varphi(\lambda_{j,1}, \ldots, \lambda_{j,m}) \prod_{s=1}^{r} F(d\lambda_s), \tag{5.46}$$

if km, $km = 2r$, is even. $\sum^*$ *denotes summation over all indecomposable partitions of the set* $D_{k,m}$ *into subsets of two elements. The product* $\prod_{j=1}^{k} \varphi(\lambda_{j,1}, \ldots, \lambda_{j,m})$ *is defined by an indecomposable partition* $D_{k,m} = \bigcup_{j=1}^{r} D_j$ *as follows: if* $D_j = \{(p, q), (s, t)\}$, *then one should set* $\lambda_{p,q} = \lambda_j$, $\lambda_{s,t} = -\lambda_j$.

Proof. We shall prove the lemma under the assumption that φ is symmetric, and the measure F is finite. Let us introduce a space of step–functions $S^{(m)}$ in the following manner. We take a partition of the set $(0, \infty) = \bigcup_{j=1}^{N} \Delta_j$ into nonintersecting measurable subsets. Define $\Delta_{-j} = -\Delta_j$. Thus we get a partition of the straight line $R^1 = \bigcup_{|j|=1}^{N} \Delta_j$. Step–functions of the type

$$\varphi(\lambda_1, \ldots, \lambda_m) = \begin{cases} a_{j_1, \ldots, j_m}, & \text{if } \forall s, t, \ |j_s| \neq |j_t|, \\ 0 & \text{in the opposite case,} \end{cases}$$

belong to the space $S^{(m)}$.

Set $(j) = (j_1, \ldots, j_m)$, $a_{(j)} = a_{j_1, \ldots, j_m}$, $(j)_s$ are distinct collections of indices of (j) type. The property of linearity of a cumulant for $\varphi \in S^{(m)}$ implies

$$\begin{aligned} \Gamma_k\big(I^{(m)}(\varphi)\big) &= \Gamma_k\Bigg(\sum_{|j_1|, \ldots, |j_m|=1}^{N} a_{j_1, \ldots, j_m} \beta(\Delta_{j_1}) \ldots \beta(\Delta_{j_m}) \Bigg) = \\ &= \sum_{(j)_1, \ldots, (j)_k} a_{(j)_1}, \ldots, a_{(j)_k} \Gamma\big(\beta(\Delta_{(j)_1}), \ldots, \beta(\Delta_{(j)_k})\big). \end{aligned} \tag{5.47}$$

Here $\beta(\Delta_{(j)_s}) = \beta(\Delta_{j_1}) \cdot \ldots \cdot \beta(\Delta_{j_m})$, $(j_s) = (j_1, \ldots, j_m)$. By applying Leonov – Shiryaev's formula (cf. formula (A20) of Appendix 3), we obtain

$$\Gamma\big(\beta(\Delta_{(j)_1}), \ldots, \beta(\Delta_{(j)_k})\big) = \sum{}^* \prod_{j=1}^{r} \Gamma(\beta(D_j)). \tag{5.48}$$

Summation in $\sum^*$ is taken over all indecomposable partitions of the set $D_{k,m}$ into two–element subsets, since only the second order cumulants of the variables $\beta(\Delta_i)$ are not equal to zero. Put $(j)_s = (j_1^s, \ldots, j_m^s)$, $s = 1, \ldots, k$, and

$$\Gamma(\beta(D_j)) = \Gamma\big(\beta(\Delta_{j_q^p}), \beta(\Delta_{j_t^s})\big) \qquad \text{if} \quad D_j = \{(p, q), (s, t)\}.$$

Since

$$F(D_j) = \Gamma\big(\beta(\Delta_{j_q^p}), \beta(\Delta_{j_t^s})\big) = \begin{cases} F(\Delta_{j_q^p}), & \text{if } j_q^p = -j_t^s, \\ 0 & \text{in the opposite case,} \end{cases}$$

substituting (5.48) into (5.47) we obtain

$$\Gamma_k(I^{(m)}(\varphi)) = \sum_{(j_1),\dots,(j_k)} \prod_{i=1}^{k} a_{(j)_i} \sum\nolimits^* \prod_{j=1}^{r} F(D_j). \tag{5.49}$$

The equality in proof for step–functions is obtained by equating the right–hand side of (5.46) to that of (5.49).

To complete the proof of Lemma 5.4 we need the following assertion.

PROPOSITION 5.1. *Let $\varphi_j \in L_2^{(m)}(F)$, $j = 1, 2, \dots, k$, and the number $km = 2r$ be even. Then the inequality*

$$\left| \int_{R^r} \prod_{j=1}^{k}{}^* \varphi_j(\lambda_{j_1}, \dots, \lambda_{j_m}) \prod_{s=1}^{r} F(d\lambda_s) \right| \leqslant \prod_{j=1}^{k} \|\varphi_j\| \tag{5.50}$$

holds. The product $\prod^*$ is taken over collections of different indices $(j_1, \dots, j_m)$, so that each index j_i occurs exactly in two collections.

Proof. Take from the product $\prod^*$ any two functions $\varphi_i(\lambda_{i_1}, \dots, \lambda_{i_m})$ and $\varphi_j(\lambda_{j_1}, \dots, \lambda_{j_m})$. Two cases are possible:

a) there are no equal indices in the collections $(i_1, \dots, i_m)$ and $(j_1, \dots, j_m)$;

b) some of indices, indicated in the collections, are the same. Without loss of generality we shall assume in this case that the first s indices coincide.

Define the function

$$\psi_1(\lambda_{i_{s+1}}, \dots, \lambda_{i_m}, \lambda_{j_{s+1}}, \dots, \lambda_{j_m}) = \begin{cases} \varphi_i\varphi_j & \text{in case a)}, \\ \displaystyle\int_{R^s} \varphi_i\varphi_j \prod_{\nu=1}^{s} F(d\lambda_{i_\nu}) & \text{in case b)}. \end{cases}$$

Note that under condition of the lemma the product $\prod\limits_{\nu\neq i,j}^{*} \varphi_\nu$ does not depend on of the variables $\lambda_{i_1}, \dots, \lambda_{i_s}$. Evaluate

$$\|\psi_1\|^2 = \int_{R^{2(m-s)}} |\psi_1|^2 \prod_{\nu=s+1}^{m} F(d\lambda_{i_\nu}) \prod_{q=s+1}^{m} F(d\lambda_{j_q}) \leqslant$$

$$\leqslant \int_{R^{2(m-s)}} \left[\left(\int_{R^s} |\varphi_i|^2 \prod_{\nu=1}^{s} F(d\lambda_{i_\nu}) \right)^{1/2} \times \right.$$

$$\left. \times \left(\int_{R^s} |\varphi_i|^2 \prod_{\nu=1}^{s} F(d\lambda_{j_\nu}) \right)^{1/2} \right]^2 \prod_{\nu=s+1}^{m} F(d\lambda_{i_\nu}) F(d\lambda_{j_\nu}) = \|\varphi_i\|^2 \|\varphi_j\|^2 < \infty.$$

Now choose another factor $\varphi_l(\lambda_{l_1}, \dots, \lambda_{l_m})$, $l \neq i, j$, and define a function ψ_2 by integrating the product $\varphi_l \psi_1$ with respect to the variables whose indices are contained in both collections $(l_1, \dots, l_m)$ and $(i_{s+1}, \dots, i_m, j_{s+1}, \dots, j_m)$. If in these collections there are no equal indices, then we set $\psi_2 = \varphi_l \psi_1$. Analogously, we see that

$$||\psi_2|| \leqslant ||\varphi_l||\ ||\psi_1|| \leqslant ||\varphi_l||\ ||\varphi_i||\ ||\varphi_j||.$$

Continuing the procedure, we prove the validity of the proposition.

Let us take a sequence of step-functions $\varphi_n \in S^{(m)}$, converging to the function $\varphi \in L_2^{(m)}(F)$ in $L_2^{(m)}(F)$. To complete the proof it remains to see that each of the integrals on the right-hand side of equality (5.46) with φ replaced by φ_n converges to the corresponding integral from (5.46) as $n \to \infty$:

$$\left|\int_{R^r} \underbrace{\varphi \dots \varphi}_{k} - \int_{R^r} \varphi_n \dots \varphi_n\right| = \left|\int \underbrace{\varphi \dots \varphi}_{k} - \int \underbrace{\varphi \dots \varphi}_{k-1} \varphi_n + \right.$$

$$\left. + \int \underbrace{\varphi \dots \varphi}_{k-1} \varphi_n - \int \underbrace{\varphi \dots \varphi}_{k-2} \varphi_n \cdot \varphi_n + \dots + \int \varphi \underbrace{\varphi_n \dots \varphi_n}_{k-1} - \int \varphi_n \dots \varphi_n \right| \leqslant$$

$$\leqslant \left|\int \underbrace{\varphi \dots \varphi}_{k-1} (\varphi - \varphi_n)\right| + \left|\int \underbrace{\varphi \dots \varphi}_{k-2} (\varphi - \varphi_n)\varphi_n\right| + \dots +$$

$$\left.\int (\varphi - \varphi_n) \underbrace{\varphi_n \dots \varphi_n}_{k-1}\right| \leqslant ||\varphi||^{k-1} ||\varphi_n - \varphi|| +$$

$$||\varphi||^{k-2} ||\varphi - \varphi_n||\ ||\varphi_n|| + \dots + ||\varphi_n - \varphi||\ ||\varphi_n||^{k-1} \to 0, \qquad n \to \infty.$$

In the latter inequality proposition 5.1 was used. Note that Lemma 5.4 holds also in the case of the σ-finite measure F.

COROLLARY 5.1. *Let $\varphi \in L_2^{(2)}(F)$ be symmetric. Then*

$$\Gamma_k(I^{(2)}(\varphi)) =$$

$$= 2^{k-1}(k-1)! \int_{R^k} \varphi(\lambda_1, -\lambda_2)\varphi(\lambda_2, -\lambda_3) \dots \varphi(\lambda_k, -\lambda_1) \prod_{j=1}^{k} F(d\lambda_j). \tag{5.51}$$

Proof. It is obvious that all summands on the right-hand side of (5.46) are equal. It remains to determine the number of indecomposable partitions of the set $D_{k,2}$, what is elementary.

COROLLARY 5.2. *Let $\varphi \in L_2^{(2)}(F)$ be symmetric. Then*

$$|\Gamma_k(I^{(2)}(\varphi))| \leqslant 2^{k-1}(k-1)!||\varphi||^k.$$

The proof follows from inequality (5.50).

The estimate obtained allows one to assert that the distribution of r.v. $I^{(2)}(\varphi)$ is determined by its moments or cumulants.

It is known that a symmetric and almost everywhere (with respect to the measure F) nonzero function φ from $L_2^{(2)}(F)$ defines a self-adjoint operator from $L_2^{(1)}(F)$ to $L_2^{(1)}(F)$ with nonzero eigenvalues $\{\mu_j\}$ by the equality

$$\varphi_1(\lambda_1) = \int \varphi_2(\lambda_2)\varphi(\lambda_1, \lambda_2)\, F(d\lambda_2), \qquad \varphi_2 \in L_2^{(1)}(F).$$

If $\{\psi_j\} \subset L_2^{(1)}(F)$ is an orthonormal sequence of eigenfunctions, then one has the following decomposition in the sense of convergence in $L_2^{(2)}(F)$:

$$\varphi(\lambda_1, \lambda_2) = \sum_j \mu_j \psi_j(\lambda_1)\psi_j(\lambda_2). \tag{5.52}$$

PROPOSITION 5.2. *Let $\varphi \in L_2^{(2)}(F)$ be symmetric. Then*

$$\Gamma_k(I^{(2)}(\varphi)) = 2^{k-1}(k-1)! \sum_j \mu_j^k, \qquad k = 2, 3, \dots . \tag{5.53}$$

The proof is obtained by substituting (5.52) into (5.51).

It follows from (5.53) that the r.v. $I^{(2)}(\varphi)$ is distributed just like $\sum_j \mu_j(X_j^2 - 1)$, where X_j are independent standard Gaussian variables.

LEMMA 5.5. *The estimate*

$$|\Gamma_k(I^{(m)}(\varphi))| \leqslant M(k, m)||\varphi||^k, \qquad k = 1, 2, \dots , \tag{5.54}$$

is valid, where $M(k, m) = \Gamma_k(H_m(X))$, and X is a standard Gaussian r.v. The equality in (5.54) *is achieved as $\varphi(\,\cdot\,) = \psi(\lambda_1) \cdot \ldots \cdot \psi(\lambda_m)$.*

Proof. Using (5.46) and (5.50) we get (5.54), where $M(k, m)$ is the number of indecomposable partitions of the table $D_{k,m}$. As $\varphi = 1$ and $F(R^1) = 1$ we have $I^{(m)}(\varphi) = H_m(X)$. By substituting into (5.46) we obtain $M(k, m) = \Gamma_k(H_m(X))$.

PROPOSITION 5.3. *There exist positive constants H_i and C_i, $i = 1, 2$, depending only on m such that*

$$H_1 C_1^k (k!)^{m/2} \leqslant M(k, m) \leqslant H_2 C_2^k (k!)^{m/2}, \qquad k = 2, 3, \dots .$$

In particular, one can put $C_1 = \sqrt{2}$ as $H_1 = 1/8$ and $C_2 = m^{m/2}$ as $H_2 = 1$.

Proof. Obviously $M(k, m)$ does not exceed the number of all possible partitions of the set $D_{k,m}$ into subsets of two elements. By applying Stirling's formula we obtain

$$M(k, m) < (mk-1)!! < (m^{m/2})^k (k!)^{m/2}.$$

Estimate the number $M(k, m)$ from below. Let the number of columns m be even. Group the columns in pairs, what can be done in $(m-1)!!$ ways. Each pair of the columns is indecomposably partitioned in $2^{k-1}(k-1)!$ ways. The partitions of $D_{k,m}$ obtained thereby are also indecomposable. Consequently,

$$M(k, 2) = 2^{k-1}(k-1)! \geqslant (\sqrt{2})^k k!/8, \qquad k \geqslant 2,$$
$$M(k, m) > (\sqrt{2})^k (k!)^{m/2}, \qquad k \geqslant 2, \quad m = 4, 6, 8, \dots .$$

Now let m be odd. We arrange $m-1$ column in pairs (by $(m-2)!!$ ways) and divide each of the pairs indecomposably into two–element subsets. Here the m^{th} column can be divided into two–element subsets arbitrarily. Consequently, as $m = 5, 7, 9, \dots$

$$M(k, m) > (m-2)!![2^{k-1}(k-1)!]^{(m-1)/2}(k-1)!! \geqslant$$
$$\geqslant (\sqrt{2})^k \sqrt{k}\,(k!)^{(m-1)/2}[(k-1)^2(k-3)^2 \cdot \dots \cdot 1]^{1/2} > (\sqrt{2})^k (k!)^{m/2},$$
$$M(k, 3) \geqslant (\sqrt{2})^k (k!)^{3/2}/8.$$

The proposition is proved.

Thus, we have the estimate

$$|\Gamma_k(I^{(m)}(\varphi))| \leqslant (m^{m/2}\|\varphi\|)^k (k!)^{m/2}. \tag{5.55}$$

Using (5.55) and exponential estimates (2.14) we get the following result.

THEOREM 5.4. *The inequality*

$$\mathbf{P}(I^{(m)}(\varphi) \geqslant \sigma_m x) \leqslant \begin{cases} \exp\{-c_1 x^2\}, & x \leqslant A_m, \\ \exp\{-c_2 x^{2/m}\}, & x > A_m, \end{cases}$$

is valid, where

$$\sigma_m^2 = \Gamma_2\big(I^{(m)}(\varphi)\big),$$

$$c_1 = \frac{1}{4}\left(\frac{2}{3}e^2\right)^{-m/2}\sqrt{\pi m}, \qquad c_2 = \frac{1}{4e}(2\pi m)^{1/(2m)},$$

$$A_m = \frac{1}{(\pi m)^{1/4}} 2^{(m^2+1)/(4(m-1)} 3^{m^2/4(m-1)} e^{m/2}.$$

Let us consider a stationary Gaussian process

$$X_t = \int_{R^1} e^{it\lambda}\,\beta(d\lambda), \qquad t \in R^1,$$

defined on the probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$. We assume that the measure F is absolutely continuous with respect to Lebesgue measure on the real line, i.e. $F(d\lambda) = f(\lambda)d\lambda$ and $F(R^1) = 1$. The process X_t defines the shift operator $S_t : \Omega \to \Omega$ by the equality $X_0\big(S_t(\omega)\big) = X_t(\omega)$. It is known that then

$$I^{(m)}\big(S_t(\omega)\big) = \int \ldots \int e^{it(\lambda_1 + \ldots + \lambda_m)} \varphi(\lambda_1, \ldots, \lambda_m) \times$$
$$\times\, \beta(d\lambda_1) \ldots \beta(d\lambda_m) := I_t^{(m)}(\varphi).$$

Let

$$Y_T^{(m)} = Y_T^{(m)}(\varphi) = \int_0^T I_t^{(m)}(\varphi)\,dt, \qquad T > 0.$$

Our goal is to obtain limit theorems for $Y_T^{(m)}$ as $T \to \infty$, in which the rate of convergence to normal distribution is established as well as probabilities of large deviations are studied.

Let us define a polynomial form $\Phi_{k,m}(x_1, \ldots, x_{k-1})$ of the function $\varphi \in L_2^{(2)}(F)$ through which we shall express the k^{th} cumulant of the r.v. $Y_T^{(m)}(\varphi)$. If km is odd then we assume $\Phi_{k,m}(x) = 0$. If km is even, then

$$\varphi^*(\lambda_1, \ldots, \lambda_r) = \prod_{s=1}^{k} \varphi(\lambda_{s,1}, \ldots, \lambda_{s,m}), \qquad r = km/2.$$

Here the product on the right-hand side of the equality is determined by the indecomposable partition $D_{k,m} = \bigcup_{j=1}^{r} D_j$ into subsets of two elements in the same way as in the proof of Lemma 5.4. (The variables $\lambda_1, \ldots, \lambda_r$ occur in the product twice: the first time with the sign plus, and the second with the sign minus.) Let us make the linear transformation

$$\sum_{j=1}^{m} \lambda_{s,j} = x_s, \qquad s = 1, \ldots, k-1,$$

$$\lambda_{j_k} = x_k,$$

$$\cdots\cdots\cdot$$

$$\lambda_{j_r} = x_r.$$

The first $k-1$ equation are linearly independent by virtue of indecomposability of the partition of the set $D_{k,m}$. The remaining equations are chosen in the simplest way so that the transformation be nondegenerate. It is easy to see that the modulus of Jacobian of this transformation is equal to 1. Now we take the function

$$\varphi^*(\lambda_1, \ldots, \lambda_r) = \prod_{j=1}^{r} f(\lambda_j),$$

and by $\varphi_1^*(x_1, \ldots, x_r)$ denote the expression, obtained after the change of variables. Now we put

$$\Phi_{k,m}(x) = \Phi_{k,m}(x_1, \ldots, x_{k-1}) = \sum\nolimits^* \int\limits_{R^{r-k+1}} \varphi_1^*(x_1, \ldots, x_r)\, dx_k \ldots dx_r.$$

Here, as before, $\sum^*$ denotes summation over all indecomposable partitions of $D_{k,m}$ into subsets of two elements.

Introduce one more function

$$\Psi_T^{(n)}(x) = \frac{2}{\pi^n T} \cdot \frac{\sin \frac{Tx_1}{2}}{x_1} \cdot \ldots \cdot \frac{\sin \frac{Tx_n}{2}}{x_n} \cdot \frac{\sin \frac{T(x_1 + \ldots + x_n)}{2}}{x_1 + \ldots + x_n},$$

$$n = 1, 2, \ldots, \quad T > 0, \quad x = (x_1, \ldots, x_n).$$

The function $\Psi_T^{(n)}(x)$ as $n = 1$ is usually called the Fejer kernel. Note, that the functions $\Psi_T^{(n)}(x)$, $n \geqslant 2$, introduced in the paper (Bentkus, 1972), are one of possible generalizations of the Fejer kernel for a multidimensional case. The most important properties of the functions $\Psi_T^{(n)}(x)$ were investigated in the paper mentioned. Now we can formulate another result.

LEMMA 5.6. *The equality*

$$\Gamma_k\Bigl(\int\limits_0^T I_t^{(m)}(\varphi)\Bigr) = (2\pi)^{k-1} T \int\limits_{R^{k-1}} \Psi_T^{(k-1)}(x) \Phi_{k,m}(x)\, dx, \qquad \varphi \in L_2^{(m)}(F),$$

holds.

The proof mainly resembles the proof of Lemma 5.4. First the lemma is proved for step–functions and afterwards the inequality $|\Psi_T^{(k-1)}(x)| \leqslant T^{k-1}$ is applied.

We shall give some corollaries to the lemma.

PROPOSITION 5.4. *Let $\varphi \in L_2^{(2)}(F)$ be symmetric. Then*

$$\Gamma_k\Big(\int_0^T I_t^{(2)}(\varphi)\Big) = (4\pi)^{k-1}(k-1)!T \int_{R^{k-1}} \Psi_T^{(k-1)}(x)\varphi^*(x)\,dx,$$

where

$$\varphi^*(x) = \int_{R^1} \prod_{j=1}^{k} \varphi(y_j, -y_{j+1}) f(y_j)\,dx_k,$$

$$x = (x_1, \dots, x_{k-1}), \quad y_j = \sum_{s=j}^{k} x_s, \quad y_{k+1} = -y_1, \quad k = 2, 3, \dots .$$

PROPOSITION 5.5. *Let*

$$v_T^2 := \Gamma_2\Big(\int_0^T I_t^{(m)}(\varphi)\Big) = 4\,m! \int_{R^1} \frac{1}{\lambda} \sin^2 \frac{T\lambda}{2} \times$$

$$\times \int_{R^{m-1}} \Big|\varphi\Big(\lambda - \sum_{j=1}^{m-1} \mu_j, \mu_1, \dots, \mu_{m-1}\Big)\Big|^2 f\Big(\lambda - \sum_{j=1}^{m-1} \mu_j\Big) \prod_{j=1}^{m-1} f(\mu_j)\,d\mu_j\,d\lambda.$$

In particular, if $\varphi = C$, then

$$v_T^2 = 4\,m!\,C^2 \int_{R^1} \frac{1}{\lambda} \sin^2 \frac{T\lambda}{2} f^{*m}(\lambda)\,d\lambda,$$

where f^{*m} is the m^{th} order convolution of the spectral density $f(\lambda)$.

PROPOSITION 5.6. *Let $\Phi_{2,m}(\lambda)$ be continuous at the point $\lambda = 0$. Then*

$$v_T^2 = 2\pi\Phi_{2,m}(0)T + o(T), \qquad T \to \infty.$$

This assertion follows from Proposition 5.5 and theorem 18.3.1 from the book (Ibragimov, Linnik, 1971).

PROPOSITION 5.7. *Let $\Phi_{k,m}(x)$, $k, m = 2, 3, \dots,$ be bounded, continuous at the point $x = (x_1, \dots, x_{k-1}) = (0, \dots, 0)$. Then*

$$\Gamma_k\left(\int_0^T I_t^{(m)}(\varphi)\,dt\right) = (2\pi)^{k-1}\Phi_{k,m}(0)T + o(T), \qquad T \to \infty.$$

Proof. Note three useful properties of the kernels $\Psi_T^{(k)}$ (Bentkus, 1972):

$$1.\ \int_{R^k} \Psi_T^{(k)}(x)\,dx = 1, \qquad k = 1, 2, \dots .$$

$$2.\ \sup_{T>0} \int_{R^k} |\Psi_T^{(k)}(x)|\,dx < \infty, \qquad k = 1, 2, \dots .$$

$$3.\ \forall\delta > 0, \int_{R^k\setminus\{|x|<\delta\}} |\Psi_T^{(k)}(x)|\,dx \to 0, \qquad T \to \infty,$$

where $\{|x| < \delta\} = \{|x_1| < \delta, \dots, |x_k| < \delta\}$.

The continuity of $\Phi_{k,m}$ at the point 0 means that for any $\varepsilon > 0$ one can find $\delta > 0$ such that

$$|\Phi_{k,m}(x) - \Phi_{k,m}(0)| < \varepsilon \qquad \text{as} \quad |x| < \delta.$$

Let $\sup_x |\Phi_{k,m}(x)| \leqslant C_\Phi$. Then, in view of properties 1 – 3 of the kernel $\Psi_T^{(k-1)}(x)$

$$\left|\Gamma_k\left(\int_0^T I_t^{(m)}(\varphi)\,dt\right) - (2\pi)^{k-1}T\Phi_{k,m}(0)\right| =$$

$$= (2\pi)^{k-1}T\left|\int_{R^{k-1}} \Psi_T^{(k-1)}(x)\Phi_{k,m}(x)\,dx - \int_{R^{k-1}} \Psi_T^{(k-1)}(x)\Phi_{k,m}(0)\,dx\right| \leqslant$$

$$\leqslant (2\pi)^{k-1} T \Bigg(\int\limits_{|x|<\delta} |\Psi_T^{(k-1)}(x)| \, |\Phi_{k,m}(x) - \Phi_{k,m}(0)| \, dx +$$

$$+ \int\limits_{R^{k-1} \setminus \{|x|<\delta\}} |\Psi_T^{(k-1)}(x)| \, |\Phi_{k,m}(x) - \Phi_{k,m}(0)| \, dx \Bigg) \leqslant$$

$$\leqslant (2\pi)^{k-1} T \Bigg(\varepsilon \int\limits_{R^{k-1}} |\Psi_T^{(k-1)}(x)| \, dx + 2\, C_\Phi \int\limits_{R^{k-1} \setminus \{|x|<\delta\}} |\Psi_T^{(k-1)}(x)| \, dx \Bigg) =$$

$$= o(T), \qquad T \to \infty.$$

The proposition is proved. ■

Let us proceed to the estimation of cumulants from above. By $R(t) = \mathbf{E} X_0 X_t$ we denote the correlation function of the original Gaussian process.

LEMMA 5.7. *Let*

$$\operatorname{ess\,sup} |\varphi(\lambda_1, \ldots, \lambda_m)| = A_\varphi < \infty.$$

Then

$$\big|\Gamma_k\big(Y_T^{(m)}(\varphi)\big)\big| \leqslant A_\varphi^k M(k, m) T \Bigg(\int\limits_{-T}^{T} |R(t)| \, dt \Bigg)^{k-2} \int\limits_{-T}^{T} |R(t)|^2 \, dt,$$

$$k = 2, 3, \ldots .$$

Proof. For a step-function φ_s it is easy to calculate that

$$\big|\Gamma_k\big(Y_T^{(m)}(\varphi_s)\big)\big| \leqslant \sum\nolimits^* \int\limits_{[0,T]^k} \int\limits_{R^r} \prod_{s=1}^{k} \varphi_s(\lambda_{s,1}, \ldots, \lambda_{s,m}) \times$$

$$\times \prod_{j=1}^{r} \exp\{it(D_j)\lambda_j\} f(\lambda_j) \, d\lambda_j \, dt.$$

Here

$$t(D_j) = \begin{cases} t_p - t_u, & \text{if } \lambda_{p,q} = \lambda_j, \\ t_u - t_p, & \text{if } \lambda_{p,q} = -\lambda_j, \end{cases}$$

as $D_j = \{(p, q), (u, v)\}$. Hence

$$\big|\Gamma_k\big(Y_T^{(m)}(\varphi)\big)\big| \leqslant$$

$$\leqslant A_\varphi^k \sum\nolimits^* \int\limits_{[0,T]^k} \Bigg| \int\limits_{R^r} \prod_{j=1}^{k} \exp\{it(D_j)\lambda_j\} f(\lambda_j) \, d\lambda_j \Bigg| \, dt. \tag{5.56}$$

Set $t(D_j) = u_j$, $j = 1, \ldots, r$. It is easy to see that for any indecomposable $D_{k,m} = \bigcup_{j=1}^{r} D_j$ in the system of linear equations $t(D_j) = u_j$ one can find $k-1$ linearly independent equation. After change of variables

$$t(D_{j_\nu}) = z_\nu, \qquad \nu = 1, \ldots, k-1,$$
$$t_{j_k} = z_k,$$

(5.56) can be evaluated by the quantity

$$A_\varphi^k T M(k, m) \prod_{\nu=1}^{k-2} \int_{[-T,T]} |R(z_\nu)|\, dz_\nu \int_{[-T,T]} |R(z)|^2\, dz.$$

For $\varphi \in L_2^{(m)}(F)$ the proof is concluded as in Lemma 5.4. ■

Now it is possible to obtain the estimate of cumulants of the required type.

LEMMA 5.8. *Let the conditions*

1) $\operatorname{ess\,sup} |\varphi(\lambda_1, \ldots, \lambda_m)| \leqslant A_\varphi$,

2) $\int_{R^1} |R(t)|\, dt = c_R < \infty$,

3) *there exists a constant* $c_1 > 0$ *such that*

$$v_T \geqslant c_1\sqrt{T}$$

be fulfilled. Then

$$\left|\Gamma_k\big(Y_T^{(m)}(\varphi)/v_T\big)\right| \leqslant (k!)^{m/2}/(A_1\sqrt{T})^{k-2}.$$

One can choose $A_1 = c_1(A_\varphi c_R m^{m/2})^{-1}$, *if* $A_\varphi^2 m^m c_R \leqslant c_1^2$, *and in the opposite case*

$$A_1 = \begin{cases} \left(c_1(A_\varphi m^{m/2} c_R)^{-1}\right)^3 & \text{as } c_R > 1, \\ \left(c_1(A_\varphi m^{m/2})^{-1}\right)^3 & \text{as } c_R \leqslant 1. \end{cases}$$

Denote the distribution function of the r.v. $Y_T^{(m)}(\varphi)/v_T$ by $F_T(x)$. The following assertion is true.

THEOREM 5.5. *Under the conditions of Lemma 5.8 the estimate*

$$\sup_x |F_T(x) - \Phi(x)| \leqslant cT^{-1/(2(m-1))}$$

holds, and relations (2.6) of the probabilities of large deviations are valid in the zone $(1/6)(\sqrt{2}\,A_1\sqrt{T}\,/6)^{1/(m-1)}$.

Let us consider a Poisson process $X(t)$, $t \in [0, \infty)$ with mean $\mathbf{E}X(t) = m(t)$, $X(0) = 0$, i.e. a nondecreasing process with nonnegative integer values and independent increments. The distribution of increments is defined by probabilities

$$\mathbf{P}\big(X(t) - X(s) = k\big) = \frac{\big(m(t) - m(s)\big)^k}{k!} e^{-\left(m(t)-m(s)\right)}, \qquad t > s,$$
$$k = 0, 1, 2, \dots ,$$

where $m(t)$ is assumed continuous. Introduce a space of real functions in q variables

$$L_T^p := \left\{ a(t_1, \dots, t_q) : \int \dots \int_{0 \leqslant t_1 < \dots < t_q \leqslant T} |a(t_1, \dots, t_q)|^p \, dm(t_1) \dots dm(t_q) < \infty \right\}.$$

It is known (Engel, 1982) that for $a \in L_T^2$ it is possible to determine a multiple stochastic integral with respect to the process $X(t)$

$$Y_T^{(q)} = \int \dots \int_{0 \leqslant t_1 < \dots < t_q \leqslant T} a(t_1, \dots, t_q) \, dX(t_1) \dots dX(t_q).$$

Note that for $a \equiv 1$

$$Y_T^{(q)} = (q!)^{-1} X(T)\big(X(T) - 1\big) \dots \big(X(T) - q + 1\big).$$

Somewhat other approach to the definition of the multiple stochastic integral by the Poisson process is considered in (Surgailis, 1981).

By τ_k denote the moment of the k^{th} jump of the process $X(T)$. So $Y_T^{(q)}$ can be written in the form

$$Y_T^{(q)} = \sum_{1 \leqslant i_1 < \dots < i_q \leqslant \tilde{T}} a(\tau_{i_1}, \dots, \tau_{i_q}).$$

Here $\tilde{T}$ stands for the number of steps of the process $X(t)$ up to the moment T. Thus, $Y_T^{(q)}$ represents U–statistics of special type related to the queueing theory. Multiple stochastic integrals of distinct types are also applied in the construction of new classes of self–similar fields (Surgailis, 1981).

By $D_{k,\,q}$ denote a table of pairs of the indices:

$$D_{k,\,q} = \big\{(m, i) \,:\, m \in \{1, 2, \dots, k\},\; i \in \{1, 2, \dots, q\}\big\}.$$

We say that the element $(m, i) \in D_{k,q}$ belongs to the m^{th} row and to the i^{th} column.

DEFINITION 5.2. *A partition of the set $D_{k,q} = \bigcup_{r=1}^{j} D_r$ is called P–indecomposable if the next two conditions are satisfied:*

1) *each subset D_r, $r = 1, \ldots, j$, contains not more than one element of one row;*

2) *there exists no group from r, $r = 1, \ldots, k-1$, rows, having its own partition, generated by the partition $D_{k,q} = \bigcup_{r=1}^{j} D_r$.*

LEMMA 5.9. *For $a \in L_T^k$ the equality*

$$\Gamma_k(Y_T^{(q)}) = \sum_{j=q}^{k(q-1)+1} \sum\nolimits^{(j)} \int\limits_{0 \leqslant t_1 < \ldots < t_q \leqslant T} \!\!\cdots\! \int \prod_{m=1}^{k} a(t_{m1}, \ldots, t_{mq}) \times \tag{5.57}$$
$$\times \prod_{i=1}^{j} dm(t_i), \qquad T > 0, \quad q \geqslant 2, \quad k \geqslant 1,$$

holds, where the summation in $\sum^{(j)}$ is taken over all P–indecomposable partitions of $D_{k,q} = \bigcup_{r=1}^{j} D_r$ into j subsets. In the product under the sign of integral we suppose $t_{mi} = t_r$ as $(m, i) \in D_r$, $r = 1, 2, \ldots, j$.

Let us explain formula (5.57) by examples. Find the variance $Y_T^{(2)}$. Denote $A_{ij} = \{(t_i, t_j) \in R^2 : 0 \leqslant t_i < t_j \leqslant T\}$. Then

$$\Gamma_2(Y_T^{(2)}) = \int\limits_{A_{12}} a^2(t_1, t_2)\, dm(t) + \int\limits_{A_{12} \cap A_{13}} a(t_1, t_2) a(t_1, t_3)\, dm(t) +$$
$$+ \int\limits_{A_{12} \cap A_{31}} a(t_1, t_2) a(t_3, t_1)\, dm(t) + \int\limits_{A_{13} \cap A_{23}} a(t_1, t_3) a(t_2, t_3)\, dm(t) +$$
$$+ \int\limits_{A_{12} \cap A_{32}} a(t_1, t_2) a(t_3, t_2)\, dm(t)$$

Here $dm(t) = \prod_i dm(t_i)$. Define a function $\tilde{a}(t_1, \ldots, t_q) = a(t_{i_1}, \ldots, t_{i_q})$, where $(i_1, \ldots, i_q)$ is such a permutation of indices $(1, \ldots, q)$, that $t_{i_1} < \ldots < t_{i_q}$. If $t_i = t_j$ for some i, j, then we put $\tilde{a} = 0$. Thus

$$\Gamma_2(Y_T^{(2)}) = \int\limits_{A_{12}} a^2(t_1, t_2)\, dm(t) + \int\limits_{[0, T]^3} \tilde{a}(t_1, t_2) \tilde{a}(t_2, t_3)\, dm(t).$$

By applying Hölder's inequality to the second integral we obtain

$$\Gamma_2(Y_T^{(2)}) \leqslant ||\tilde{a}_T||^2\Big(\frac{1}{2}+m(T)\Big).$$

Here $||\tilde{a}_T||^2 = \int\limits_{[0,T]^2} \tilde{a}^2(t_1, t_2)\, dm(t)$.

So it is easy to see that

$$\Gamma_3(Y_T^{(2)}) = \int\limits_{A_{12}} a^3(t_1, t_2)\, dm(t) + 3\int\limits_{[0,T]^3} \tilde{a}^2(t_1, t_2)\tilde{a}(t_1, t_3)\, dm(t) +$$
$$+ \int\limits_{[0,T]^3} \tilde{a}(t_1, t_2)\tilde{a}(t_2, t_3)\tilde{a}(t_3, t_1)\, dm(t) + \int\limits_{[0,T]^4} \tilde{a}(t_1, t_2)\tilde{a}(t_1, t_3)\times$$
$$\times\, \tilde{a}(t_1, t_4)\, dm(t) + 3\int\limits_{[0,T]^4} \tilde{a}(t_1, t_2)\tilde{a}(t_2, t_3)\tilde{a}(t_3, t_4)\, dm(t).$$

Repeatedly applying Hölder's inequality we establish that the fourth integral does not exceed

$$\int\limits_{[0,T]}\Big(\int\limits_{[0,T]} |\tilde{a}(t_1, t_2)|\, dm(t_2)\Big)^3 dm(t_1) \leqslant m^3(T)\int\limits_{[0,T]^2} |\tilde{a}(t_1, t_2)|^2\, dm(t).$$

Doing the same with the other integrals we obtain the estimate

$$|\Gamma_3(Y_T^{(2)})| \leqslant \frac{1}{2}\int\limits_{[0,T]^2} |\tilde{a}(t_1, t_2)|^3\, dm(t) + 3\,m(t)\int\limits_{[0,T]^2} |\tilde{a}(t_1, t_2)|^3\, dm(t) +$$
$$+\Big(\int\limits_{[0,T]^2} \tilde{a}^2(t_1, t_2)\, dm(t)\Big)^{3/2} + m^2(T)\int\limits_{[0,T]^2} |\tilde{a}(t_1, t_2)|^3\, dm(t) +$$
$$+ 3\,m(T)\Big(\int\limits_{[0,T]^2} \tilde{a}^2(t_1, t_2)\, dm(t)\Big)^{3/2}.$$

It is obvious that due to (5.57) the cumulant $\Gamma_k(Y_T^{(q)})$ for any k can be estimated by linear combination of

$$\int\limits_{[0,T]^q} |\tilde{a}(t_1, \ldots, t_q)|^j\, dm(t), \quad j = 1, 2, \ldots, k.$$

Proof of Lemma 5.9. It is sufficient to prove the lemma for step–functions. Divide a segment $[0, T]$ into subsets Δ_i, $i = 1, \ldots, N$ ($\Delta_i < \Delta_j$ for $i < j$) and define the step–function

$$a_N(t_1, \ldots, t_q) = \sum_{1 \leqslant i_1 < \ldots < i_q \leqslant N} a_{i_1 \ldots i_q} \mathbf{1}_{\Delta_{i_1}}(t_1) \ldots \mathbf{1}_{\Delta_{i_q}}(t_q).$$

By $X(\Delta) = X(t) - X(s)$ as $\Delta = [s, t]$ denote a random measure, generated by the process $X(t)$, and correspondingly $m(\Delta) = m(t) - m(s)$. Then, for $a = a_N$ the multiple stochastic integral is equal to

$$Y_T^{(q)} = \sum_{1 \leqslant i_1 < \ldots < i_q \leqslant N} a_{i_1 \ldots i_q} X(\Delta_{i_1}) \ldots X(\Delta_{i_q}).$$

Note that each summand is a product of independent random variables. Introduce the notation $i(\cdot) = i_1, \ldots, i_q$ for different collections of indices, satisfying the condition $1 \leqslant i_1 < \ldots < i_q \leqslant N$,

$$\Gamma_k(Y_T^{(q)}) = \Gamma(Y_T^{(q)}, \ldots, Y_T^{(q)}) = \\ = \sum_{i(1), \ldots, i(k)} \Gamma\big(X(\Delta_{i(1)}), \ldots, X(\Delta_{i(k)})\big).$$

Here $a_{i(j)} = a_{i_1 \ldots i_q}$, as $i(j) = i_1, \ldots, i_q$, and

$$X(\Delta_{i(j)}) = \prod_{r=1}^{q} X(\Delta_{i_r}).$$

We shall use formula ($A20$) (see Appendix 3). Put the set of indices $i(1), \ldots, i(k)$ to correspondence with the table $D_{k,q}$:

$$i(1) = i_1, \ldots, i_q \to (1, 1), \ldots, (1, q),$$

$$-----------------$$

$$i(k) = j_1, \ldots, j_q \to (k, 1), \ldots, (k, q).$$

Correspondingly, $X(\Delta_{(i,j)}) = X(\Delta_{i_j})$, if $i_j \to (i, j)$. Therefore,

$$\Gamma\big(X(\Delta_{i(1)}), \ldots, X(\Delta_{i(k)})\big) = \Gamma\Big(\prod_{j=1}^{q} X(\Delta_{(1,j)}), \ldots, \prod_{j=1}^{q} X(\Delta_{(k,j)})\Big) = \\ = \sum_{r=q}^{k(q-1)+1} \sum_{D=\bigcup_{j=1}^{r} D_j}^{(r)} \Gamma(D_1) \cdot \ldots \cdot \Gamma(D_r). \tag{5.58}$$

Here $\sum^{(r)}$ denotes the summation over all partitions of the table $D_{k,q}$ into r subsets satisfying item 2 of our definition of the P–indecomposable partition, and

$$\Gamma(D_j) = \Gamma\big(X(\Delta_{(s,t)}), \ldots, X(\Delta_{(u,v)})\big),$$

if $D_j = \{(s, t), \ldots, (u, v)\}$. If two elements of one row occur in D_j, then $\Gamma(D_j) = 0$ by virtue of independence of corresponding values of the measure $X(\Delta)$. Thus, it is sufficient to sum over the partitions, satisfying both conditions in the definition of the P–indecomposable partition. The lower bound of the summation in (5.58) is equal to q, since under $r < q$ in any partition of $D_{k,q} = \bigcup_{j=1}^{r} D_j$ one can find a subset D_j, containing at least two elements of one row, whence $\Gamma(D_j) = 0$. The upper bound of summation is equal to $r = k(q-1)+1$, because for $r > k(q-1)+1$ it is impossible to form a P–indecomposable partition, satisfying condition 2. Now it remains to recall that for the Poisson process $X(t)$ $\Gamma_k\big(X(\Delta)\big) = m(\Delta)$, $\Delta \subset [0, T]$, $k = 1, 2, \ldots$. Therefore, $\Gamma(D_j) = m(\Delta)$ or $\Gamma(D_j) = 0$. The first case occurs if one and the same quantity $X(\Delta)$ corresponds to all the elements of D_j. Thus, we get integral sums, occurring in the formulation of the lemma. ∎

By $B_q(k, r)$ denote the number of P–indecomposable partitions of the table $D = D_{k,q}$ into r subsets. Such an assertion follows directly from Lemma 5.9.

LEMMA 5.10. *If $|a| \leqslant c_a$, then*

$$|\Gamma_k(Y_T^{(q)})| \leqslant c_a^k \max_{2 \leqslant r \leqslant k(q-1)+1} \{B_q(k, r)\, m^r(T)\}, \qquad k = 2, 3, \ldots.$$

THEOREM 5.6. *Let $|a| \leqslant c_a$ and there exist a constant $c_0 > 0$ such that*

$$\mathbf{D}Y_T^{(q)} \geqslant c_0\big(m(T)\big)^{2q-1}, \qquad T > 0.$$

Then the relations of large deviations (2.6) for the r.v.

$$Z_T^{(q)} := \frac{Y_T^{(q)} - \mathbf{E}Y_T^{(q)}}{\sqrt{\mathbf{D}Y_Y^{(q)}}}$$

are valid in the zone

$$0 \leqslant x < C_q\big(m(T)\big)^{1/2},$$

$$C_q = c_a^{-2} c_0^2\, 2\, e^{-q} \inf_{0 \leqslant \alpha \leqslant q-1} \left\{ \frac{1}{6}\left(\frac{\sqrt{2}}{6}\right)^{1/(2q-2\alpha-1)} \cdot \alpha^{q-\alpha} \right\}.$$

Proof. Consider the extreme case $r = k(q-1)+1$. Obviously $B_q(k,\, k(q-1)+1)$ does not exceed the number of all possible partitions of the set $D_{k,\,q}$ into $k(q-1)+1$ subsets, what, in its turn, is Stirling's number of the second type $S\big(qk,\, k(q-1)+1\big)$. Using the well-known from combinatorial analysis representation of numbers $S(\,\cdot\,,\,\cdot\,)$, we can write

$$S\big(qk,\, k(q-1)+1\big) =$$
$$= \frac{1}{\big(k(q-1)+1\big)!} \sum_{j=0}^{k(q-1)+1} (-1)^j \binom{k(q-1)+1}{j} \big(k(q-1)+1-j\big)^{kq}. \tag{5.59}$$

Making use of Stirling's formula,

$$S\big(qk,\, k(q-1)+1\big) \leqslant \big(2\,e^q(q-1)\big)^k k!, \qquad k = 2, 3, \dots .$$

Estimate $B_q(k,\, \alpha k)$ from above as $\alpha \in [0,\, q-1]$. Suppose that $k\alpha$ is integer. Then (5.59) and Stirling's formula yield

$$B_q(k,\, \alpha k) < S(kq,\, \alpha k) \leqslant (2\,e^q \alpha^{q-\alpha})^k (k!)^{q-\alpha}.$$

From Lemma 5.10 and the condition of the theorem on the growth of variance we deduce

$$|\Gamma_k(Z_T^{(q)})| \leqslant (c_a c_0^{-1})^k \max_{2 \leqslant r \leqslant k(q-1)+1} \{B_q(k,\, r) m(T)^{r-\left(q-\frac{1}{2}\right)k}\}. \tag{5.60}$$

Assume that the maximum in (5.60) is achieved as $r = \bar{\alpha}(k-2) + 2\,q - 1$. Then

$$|\Gamma_k(Z_T^{(q)})| \leqslant \bar{c}_\alpha^k (k!)^{q-\bar{\alpha}} \big[\big(m(T)\big)^{q-\bar{\alpha}-1/2}\big]^{2-k}, \qquad k = 3, 4, \dots .$$

Here $\bar{c}_a = (2\,e)^q \bar{\alpha}^{q-\alpha} c_\alpha c_0^{-1}$. Put

$$c_\alpha = \begin{cases} \bar{c}_\alpha, & \text{if } \bar{c}_\alpha \leqslant 1, \\ \bar{c}_\alpha^2, & \text{if } \bar{c}_\alpha > 1. \end{cases}$$

Then

$$|\Gamma_k(Z_T^{(q)})| \leqslant c_\alpha^{k-2} \Big(\big(m(T)\big)^{q-\bar{\alpha}-1/2}\Big)^{2-k} (k!)^{q-\bar{\alpha}}.$$

As it follows from general Lemma 2.3, the interval of values of x, in which the relations of probabilities of large deviations are hold, is equal to

$$\Delta_T = \frac{1}{6}\left(\frac{\sqrt{2}}{6}\right)^{1/(2\,q-2\,\bar{\alpha}-1)} c_\alpha^{-1/(2\,q-2\,\bar{\alpha}-1)} \big(m(T)\big)^{\left(q-\bar{\alpha}-\frac{1}{2}\right)\left(1+2(q-1-\bar{\alpha})^{-1}\right)} \geqslant$$

$$\geqslant c_q \big(m(T)\big)^{1/2}, \qquad c_q = \inf_{0 \leqslant \alpha \leqslant q-1} \left\{ \frac{1}{6}\left(\frac{\sqrt{2}}{6}\right)^{1/(2\,q-2\,\alpha-1)} c_\alpha^{-1/(2\,q-2\,\alpha-1)} \right\}.$$

Note that in the case of a standard Poisson process $m(T) = aT$, and we have a theorem of large deviations in the zone $0 \leqslant x < c\sqrt{T}$. The theorem proved improves the result (Basalykas, Plikusas, Statulevičius, 1987), where the order of increase of the zone depends on the multiplicity of the integral q. ∎

Let us consider a multiple stochastic integral with respect to the centered Poisson process $\tilde{X}(t) = X(t) - m(t)$. The multiple stochastic integral

$$\tilde{Y}_T^{(q)} = \int\limits_{0\leqslant t_1<\ldots<t_q\leqslant T}\!\!\!\ldots\int a(t_1, \ldots, t_q)\, d\tilde{X}(t_1)\ldots d\tilde{X}(t_q),$$

$$T > 0, \quad q \geqslant 2, \quad a \in L_T^2.$$

is defined analogously to the integral $Y_T^{(q)}$. We have $\tilde{Y}_T^{(q)} = K_q\big(X(T), m(T)\big)$ for $a \equiv 0$, where

$$K_q(u, t) = \frac{1}{q!}\sum_{r=0}^{q}\binom{q}{r}(-t)^r u_{(q-r)},$$

$$u_{(n)} = u(u-1)\ldots(u-n+1).$$

As known, the Poisson – Charlier polynomials $K_q(u, t)$, $q = 1, 2, \ldots$, are orthogonal with respect to the Poisson measure:

$$\sum_{r=0}^{\infty} K_n(r, t)K_m(r, t)\,\frac{t^r e^{-r}}{r!} = \frac{t^n}{n!}\,\delta_{nm}, \qquad \delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

In contrast to $Z_T^{(q)}$, the normed r.v. $\tilde{Z}_T^{(q)} := \tilde{Y}_T^{(q)}(\mathbf{D}Y_T^{(q)})^{-1/2}$ does not converge to the Gaussian r.v. as $m(T) \to \infty$.

LEMMA 5.11. *Let $a \in L_T^k$. Then*

$$\Gamma_k(\tilde{Y}_T^{(q)}) = \sum_{j=q}^{[kq/2]} \sum\nolimits^{(j)} \int\limits_{0\leqslant t_{m1}<\ldots<t_{mq}\leqslant T}\!\!\!\ldots\int\ \prod_{m=1}^{k} a(t_{m1}, \ldots, t_{mq}) \prod_{i=1}^{j} dm(t_i),$$

$k \geqslant 2$, $T > 0$, $q \geqslant 2$, $[kq/2]$ is an integer part of the number $kq/2$. The summation in $\sum^{(j)}$ is taken over all P–indecomposable partitions of $D_{k,q} = \bigcup_{r=1}^{j} D_r$ into j parts. Obviously it suffices to sum over P–indecomposable partitions into subsets containing at least two elements of $D_{k,q}$.

The proof of Lemma 5.11 is just like the proof of Lemma 5.9 with only difference, that $\Gamma_1\big(\tilde{X}(\Delta)\big) = 0$, $\Delta \subset R^1$, and, consequently, the summands, corresponding to the partitions containing one–element parts, are excluded. Obviously, $\Gamma_k\big(\tilde{X}(\Delta)\big) = m(\Delta)$, $k = 2, 3, \ldots$, and $\mathbf{E}\tilde{Y}_T^{(q)} = 0$. The variance

$$\mathbf{E}(\tilde{Y}_T^{(q)})^2 = \Gamma_2(\tilde{Y}_T^{(q)}) = \int\limits_{0\leqslant t_1<\ldots<t_q\leqslant T}\!\!\!\!\ldots\int a^2(t_1, \ldots, t_q)\, dm(t).$$

Hence it follows that the order of growth of the variance is $\big(m(T)\big)^q$ as $a = \mathrm{const}$. In this case Lemma 5.11 shows that the cumulants of even order of the r.v. $\tilde{Z}_T^{(q)}$ do not converge to zero as $m(T) \to \infty$.

When estimating cumulants in the same way as in Theorem 5.6 we establish that the basic contribution is made by the summands, corresponding to partition of the table into $kq/2$ parts. Cumulants of the r.v. $\tilde{Y}_T^{(q)}$ will be estimated through $C^k(k!)^{q/2}$. Using exponential inequality (2.13) for the distribution function we get the following assertion.

THEOREM 5.7. *Let there exists a constant* $\tilde{c}_0$ *such that*

$$\mathbf{E}(\tilde{Y}_T^{(q)})^2 \geqslant \tilde{c}_0\big(m(T)\big)^q,$$

$|a| \leqslant c_a$, $m(T) > 1$. *Then*

$$\mathbf{P}(\tilde{Z}_T^{(q)} \geqslant x) \leqslant \exp\{-hx^{2/q}\}$$

for $x > 2\,e^{-q}\tilde{c}_0/c_a$ *and* $h \leqslant \tilde{c}_0/(2\,e^q c_a)$.

Note that analogous results have been obtained for multiple stochastic integrals with respect to the Gaussian process in (Plikusas, 1981).

5.3. Large deviations for estimates of the spectrum of a stationary sequence

Consider the estimate

$$\hat{A}(\varphi) = \int_{-\pi}^{\pi} \varphi(x) I_N(x)\, dx,$$

where

$$I_N(x) := \frac{1}{2\pi N}\sum_{s,\,t=1}^{N} X_s X_t \exp\{-i(s-t)x\}$$

is a periodogram of the second order, constructed by the sample of size N from a stationary in a wide sense sequence $\{X(t),\ t = \dots, -1, 0, 1, \dots\}$ with the spectral density $f(\lambda)$, and φ is any function. If $\varphi = W_N(\,\cdot - \lambda)$, where W_N is some kernel (asymptotically delta–shaped function), then $\hat{A}(\varphi)$ is the estimate (of second order) of spectral density f at the point λ.

It is known that

$$\mathbf{E}\hat{A}(\varphi) = \int_{-\pi}^{\pi} \Phi_N(u)\,du \int_{-\pi}^{\pi} \varphi(x) f(x+u)\,dx,$$

where

$$\Phi_N(x) := \frac{1}{2\pi N} \cdot \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

is the Fejer kernel. Using Leonov–Shiryaev's formula we get the following representation (Bentkus, 1976) for the mixed cumulant:

$$\Gamma(\hat{A}(\varphi_1), \dots, \hat{A}(\varphi_k)) = \frac{1}{N^{k-1}} \int_{\Pi^{2k-1}} G(u) \Phi_N^{(2k)}(u)\,du$$

(the proof is analogous to that of Lemma 5.4 and Lemma 5.6), where

$$\Phi_N^{(n)}(u) := \frac{2}{(2\pi)^{n-1}N} \frac{\sin(Nu_1/2)}{\sin(u_1/2)} \cdots \frac{\sin(Nu_{n-1}/2)}{\sin(u_{n-1}/2)} \times$$
$$\times \frac{\sin \frac{N}{2}(u_1 + \dots + u_{n-1})}{\sin \frac{1}{2}(u_1 + \dots + u_{n-1})}$$

is a generalized Fejer kernel (Bentkus, 1972), $\Pi := [-\pi, \pi]$, and the function G is defined by the functions $\varphi_1, \dots, \varphi_k$ and spectral densities up to order $2\,k$ inclusive (naturally, if they exist).

For simplicity, let X_t be a stationary Gaussian sequence with mean 0 and spectral density $f(\lambda)$, $-\pi \leqslant \lambda \leqslant \pi$. Investigate the probabilities of large deviations for statistical estimates of the type

$$\hat{f}(\lambda) = \int_{-\pi}^{\pi} W(x-\lambda) I_N(x)\,dx,$$

where $W \in L_1(-\pi, \pi)$. The function $W(x) = W_N(x)$ is usually called a spectral window.

LEMMA 5.12 (Bentkus, Rudzkis, 1980). *Let* $\varphi \in L_{p_1}$ *and* $f \in L_{p_2}$, *where* $p_1,\ p_2 \in [1,\ \infty]$. *Then for all* $k = 3,\ 4,\ \dots$ *and* $N = 1,\ 2,\ \dots$

$$\left|\Gamma_k(\hat{A}(\varphi))\right| \leqslant (k-1)!\,\mathbf{D}\hat{A}(\varphi)\left(4\pi\|\varphi\|_{p_1}\cdot\|f\|_{p_2}N^{-1+\frac{1}{p_1}+\frac{1}{p_2}}\right)^{k-2}. \tag{5.61}$$

Proof. It is easy to see that

$$\hat{A}(\varphi) = \frac{1}{2\pi N}\sum_{s,\,t=1}^{N} a(s-t)X_sX_t = \frac{1}{2\pi N}(T_\varphi X,\ X),$$

where

$$a(u) = \int_{-\pi}^{\pi}\varphi(x)\cos ux\,dx,$$

$$T_\varphi = \|a(s-t)\|_{s,\,t=\overline{1,\,N}},\qquad X = (X_1,\ \dots,\ X_N).$$

Thus, by simultaneously reducing the matrix T_φ and the covariance matrix T_f of the vector X to a diagonal form we obtain that the r.v. $\hat{A}(\varphi)$ is distributed just as the sum $\frac{1}{2\pi N}\sum\limits_{j=1}^{N}\mu_j\eta_j^2$, where η_j are independent Gaussian r.v. with mean 0 and variance 1 and μ_j are eigenvalues of the matrix $T_\varphi T_f$. Consequently,

$$\Gamma_k(\hat{A}(\varphi)) = \frac{(k-1)!}{2(\pi N)^k}\sum_{j=1}^{N}\mu_j^k \tag{5.62}$$

and

$$\mathbf{D}\hat{A}(\varphi) = \Gamma_2(\hat{A}(\varphi)) = \frac{1}{2(\pi N)^2}\sum_{j=1}^{N}\mu_j^2. \tag{5.63}$$

It is easy to see that μ_j are real. Actually, since for any real function φ the matrix T_φ is Hermitian, for $x \in C^N$ always $(T_\varphi x,\ x) \in R^1$. Besides, for $z \in C^N$

$$\begin{aligned}(T_\varphi z,\ z) &= \int\varphi(x)\sum_{s,\,t=1}^{N} z_t\bar{z}_s\cos(s-t)x\,dx = \\ &= \frac{1}{2}\int\varphi(x)\left\{\left|\sum_{t=1}^{N} z_t e^{-itx}\right|^2 + \left|\sum_{t=1}^{N} z_t e^{itx}\right|^2\right\}dx.\end{aligned} \tag{5.64}$$

Using representation (5.64) for $(T_f z,\ z)$ we find that T_f is positively defined, since the spectral density f is nonnegative and it is not equal to 0 a.e. Hence it follows that there exists a positively defined inverse matrix T_f^{-1}. If λ is the eigenvalue of the matrix $T = T_f T_\varphi$ and x is a characteristic vector ($x \neq 0$), corresponding to the number λ, then $Tx = \lambda x = \lambda T_f T_f^{-1} x = T_f \lambda T_f^{-1} x$ and $T_\varphi x = \lambda T_f^{-1} x$. Consequently,

$$\lambda = \frac{(T_\varphi x,\, x)}{(T_f^{-1} x,\, x)}, \tag{5.65}$$

i.e., λ is real.

From (5.62) it follows

$$\left|\Gamma_k(\hat{A}(\varphi))\right| \leqslant (k-1)!\,\mathbf{D}\hat{A}(\varphi)\big(\max_j |\mu_j|/(\pi N)\big)^{k-2}. \tag{5.66}$$

Let us estimate $|\mu_j|$. We have

$$\max |\mu_j| \leqslant \sup_{||z||=1} |(Tz,\, z)| \leqslant ||T|| \leqslant ||T_\varphi|| \cdot ||T_f||.$$

Since T_φ is a Hermite matrix, in view of (5.64) for $p_1 > 1$,

$$||T_\varphi|| = \sup_{||z||=1} (Tz,\, z) \leqslant ||\varphi||_{p_1} \sup_{||z||=1} \left(\int \left| \sum_{t=1}^N z_t e^{itx} \right|^{2q_1} dx \right)^{1/q_1} \leqslant$$

$$\leqslant ||\varphi||_{p_1} \sup_{||z||=1} N \left(\int \frac{1}{N} \left| \sum_{t=1}^N z_t e^{itx} \right|^2 dx \right)^{1/q_1} \leqslant (2\pi)^{1-\frac{1}{p_1}} ||\varphi||_{p_1} N^{1/p_1},$$

as far as

$$\frac{1}{N} \left| \sum_{t=1}^N z_t e^{itx} \right|^2 \leqslant ||z|| = 1$$

and

$$\int \left| \sum_{t=1}^N z_t e^{itx} \right|^2 dx = 2\pi ||z||^2 = 2\pi.$$

Here $\dfrac{1}{p_1} + \dfrac{1}{q_1} = 1$. It is easy to see that for $p_1 = 1$ $||T_\varphi|| \leqslant ||\varphi||_1 N$. Analogously,

$$||T_f|| \leqslant (2\pi)^{1-\frac{1}{p_2}} ||f||_{p_2} N^{1/p_2}.$$

Consequently,

$$\max_j |\mu_j| \leqslant (2\pi)^{2-\frac{1}{p_1}-\frac{1}{p_2}} ||\varphi||_{p_1} \, ||f||_{p_2} N^{\frac{1}{p_1}+\frac{1}{p_2}}. \tag{5.67}$$

The lemma follows from (5.63), (5.66) and (5.67). ∎

Further we shall assume that the functions φ and f are bounded.

LEMMA 5.13 (Bentkus, Rudzkis, 1976). *If the functions φ and f are bounded, then for all $N = 1, 2, \ldots$*

$$\mathbf{D}\hat{A}(\varphi) \leqslant 4\pi \,||\varphi||_1\, ||\varphi||_\infty\, ||f||_\infty^2\, N^{-1}. \tag{5.68}$$

Proof. First show that

$$\mathbf{D}\hat{A}(\varphi) \leqslant \mathbf{D}\hat{A}(|\varphi|). \tag{5.69}$$

Since $\sum\limits_{j=1}^{N} \mu_j^k = \mathrm{Sp}\,[(T_f T_\varphi)^k]$, in accordance with (5.63) it suffices to prove that

$$\mathrm{Sp}\,[(T_f T_\varphi)^2] \leqslant \mathrm{Sp}\,[(T_f T_{|\varphi|})^2]. \tag{5.70}$$

Let $\varphi = \varphi^+ - \varphi^-$ be a representation of φ in the form of the difference of two nonnegative functions. Then

$$T_f T_\varphi = T_f T_{\varphi^+} - T_f T_{\varphi^-}, \quad T_f T_{|\varphi|} = T_f T_{\varphi^+} + T_f T_{\varphi^-},$$

$$\mathrm{Sp}\,[(T_f T_\varphi)^2] = \mathrm{Sp}\,[(T_f T_{\varphi^+})^2] - 2\,\mathrm{Sp}\,[(T_f T_{\varphi^+} T_f T_{\varphi^-})] + \mathrm{Sp}\,[(T_f T_{\varphi^-})^2]$$

and

$$\mathrm{Sp}\,[(T_f T_{|\varphi|})^2] = \mathrm{Sp}\,[(T_f T_{\varphi^+})^2] + 2\,\mathrm{Sp}\,[(T_f T_{\varphi^+} T_f T_{\varphi^-})] + \mathrm{Sp}\,[(T_f T_{\varphi^-})^2].$$

Further, let λ and x be the eigenvalue and corresponding to it characteristic vector $(x \neq 0)$ of the matrix $T_f T_{\varphi^+} T_f T_{\varphi^-}$. As far as $\mathbf{D}\hat{A}(-\varphi) = \mathbf{D}\hat{A}(\varphi)$, one can assume that φ^+ is not equal to zero a.e. Then the matrix $T_f T_{\varphi^+} T_f$ is positively defined. As earlier,

$$\lambda = \frac{(T_{\varphi^-} x,\, x)}{((T_f T_{\varphi^+} T_f)^{-1} x,\, x)} \geqslant 0,$$

whence it follows that $\mathrm{Sp}\,[(T_f T_{\varphi^+} T_f T_{\varphi^-})] \geqslant 0$. The last inequality yields (5.70), and, consequently, (5.69) as well.

Similarly to (5.63)

$$\mathbf{D}\hat{A}(|\varphi|) = \frac{1}{2(\pi n)^2} \sum_{j=1}^{N} \lambda_j^2,$$

where λ_j are eigenvalues of the matrix $T_f T_{|\varphi|}$. Equality (5.65) yields that all $\lambda_j > 0$. As $k = 1$ formula (5.62) gives

$$\mathbf{E}\hat{A}(|\varphi|) = \frac{1}{2\pi N} \sum_{j=1}^{N} \lambda_j.$$

Formula (5.67) implies

$$\mathbf{D}\hat{A}(|\varphi|) \leqslant \frac{1}{\pi N} \max_j \lambda_j \mathbf{E}\hat{A}(|\varphi|) \leqslant \frac{4\pi}{N} \,||\varphi||_\infty \,||f||_\infty \,\mathbf{E}\hat{A}(|\varphi|). \tag{5.71}$$

It is not difficult to verify that

$$\mathbf{E}\hat{A}(|\varphi|) = \int \Phi_N(u)\, du \int |\varphi(x)| f(x+u)\, dx.$$

It means that $\mathbf{E}\hat{A}(|\varphi|) \leqslant ||\varphi||_1 \,||f||_\infty$. Taking that into consideration we get from (5.71)

$$\mathbf{D}\hat{A}(|\varphi|) \leqslant 4\pi \,||\varphi||_1 \,||\varphi||_\infty \,||f||_\infty^2 \,N^{-1},$$

what together with (5.69) yields (5.68). The lemma is proved. ■

Let the functions φ and f be bounded. Set

$$\sigma_N^2 = N\,\mathbf{D}\hat{A}(\varphi),$$
$$Z_N = \sqrt{N}\,\big(\hat{A}(\varphi) - \mathbf{E}\hat{A}(\varphi)\big)/\sigma_N.$$

Then, $\mathbf{E}Z_N = 0$, $\mathbf{E}Z_N^2 = 1$, and according to Lemma 5.13

$$|\Gamma_k(Z_N)| \leqslant (k-1)!/\Delta_N^{k-2}, \quad \forall k \geqslant 1,$$

where

$$\Delta_N = \sqrt{N}\,\sigma_N/(4\pi ||\varphi||_\infty \,||f||_\infty).$$

Applying Lemmas 2.3 and 2.4 we can prove the following assertion.

THEOREM 5.8 (Bentkus, Rudzkis, 1980). *If $\sigma_N > 0$ and the functions φ and f are bounded, then in the interval*

$$0 \leqslant x \leqslant \frac{\sqrt{2}}{36}\,\Delta_N$$

the relations of large deviations

$$\frac{\mathbf{P}\Big(\sqrt{N}(\hat{A}(\varphi) - \mathbf{E}\hat{A}(\varphi)) \geqslant \sigma_N x\Big)}{1 - \Phi(x)} = \exp\{L_0(x)\}\Big(1 + \tilde{\theta}_1 f(x)\frac{x+1}{\Delta_N}\Big),$$

$$\frac{\mathbf{P}\Big(\sqrt{N}(\hat{A}(\varphi) - \mathbf{E}\hat{A}(\varphi)) \leqslant -\sigma_N x\Big)}{\Phi(-x)} = \exp\{L_0(-x)\}\Big(1 + \tilde{\theta}_2 f(x)\frac{x+1}{\Delta_N}\Big)$$

and the inequality

$$\mathbf{P}\Big(\pm\sqrt{N}\,(\hat{A}(\varphi)-\mathbf{E}\hat{A}(\varphi))\geqslant\sigma_N x\Big)\leqslant\exp\left\{-\frac{x^2}{2(1+x/\Delta_n)}\right\}$$

are valid for all $x\geqslant 0$. Here $|\theta_i|\leqslant 36/\sqrt{2}$, and the functions L_0, f are defined in Lemma 2.3.

As mentioned $\hat{f}(\lambda)=\hat{A}(\varphi_N(\cdot\,-\lambda))$ is usually taken as the estimate for the spectral density $f(\lambda)$, where φ_N is some kernel. As earlier, we assume that φ_N and f are bounded and not equal to zero a.e. When investigating large deviations for the distribution of r.v.

$$Z_N=\frac{\hat{f}(\lambda)-\mathbf{E}\hat{f}(\lambda)}{\sqrt{\mathbf{D}\hat{f}(\lambda)}}$$

one can apply theorem 5.8 with $\hat{A}(\varphi):=\hat{A}(\varphi_N(\cdot\,-\lambda)):=\hat{f}(\lambda)$. The behaviour of

$$\Delta_N=\frac{N\sqrt{\mathbf{D}\hat{f}(\lambda)}}{4\pi||\varphi_N||_\infty\,||f||_\infty}$$

is essential here. Note that for the majority of actually applicable estimates of spectral density $||\varphi_N||_\infty\to\infty$, however $||\varphi_N||_\infty N^{-1}\to 0$ as $N\to\infty$ and $\sup\limits_N||\varphi_N||_1<\infty$.

If the spectral density f is continuous at the point λ and if the kernel φ_N satisfies the conditions:

1) $\displaystyle\int\varphi_N(x)\,dx=1,\quad\forall N,$

2) $\sup\limits_N||\varphi_N||_1<\infty,$

3) $\forall\delta>0\quad\displaystyle\int\limits_{[-\pi,\,\pi]\setminus[-\delta,\,\delta]}|\varphi_N(x)|\,dx\to 0\ (N\to\infty),$

4) $||\varphi_N||_\infty\to\infty,\quad||\varphi_N||_\infty N^{-1}\to 0\ (N\to\infty),$

5) there exist limits

$$\Lambda_1=\lim_{N\to\infty}\frac{2\pi}{||\varphi_N||_\infty}\int\varphi_N^2(x)\,dx,$$

$$\Lambda_2=\lim_{N\to\infty}\frac{2\pi}{||\varphi_N||_\infty}\int\varphi_N(x)\varphi_N(-x)\,dx,$$

then (Bentkus, 1976)

$$\varrho^2(\lambda) = \lim_{N\to\infty} N\,\mathbf{D}\hat{f}(\lambda)/||\varphi_N||_\infty = \begin{cases} \Lambda_1 f^2(\lambda), & \lambda \notin \{-\pi,\, 0,\, \pi\}, \\ (\Lambda_1 + \Lambda_2) f^2(\lambda), & \lambda \in \{-\pi,\, 0,\, \pi\}. \end{cases}$$

Consequently, Δ_N behaves as $\sqrt{N}\,\varrho(\lambda)/(4\pi\,||f||_\infty)$.

For more details on asymptotic behaviour of spectral estimates see (Bentkus, 1976) and (Bentkus, Rudzkis, 1983).

CHAPTER 6
ASYMPTOTIC EXPANSIONS IN THE ZONES OF LARGE DEVIATIONS

Chapters 6 and 7 of the monograph by V.V. Petrov "Sums of independent random variables" (1975) are devoted to asymptotic expansions in integral and local limit theorems with uniform and nonuniform estimates of the remainder terms. Extensive bibliography on these subjects is also presented in the book.

The works (Cramer, 1938), (Linnik, 1960, 1961), (Petrov, 1954, 1965, 1975), (Ibragimov, 1967), (Borovkov, 1962, 1964), (Zolotarev, 1962), (Prokhorov, 1962), (S.V. Nagaev, 1963), (A.V. Nagaev, 1967, 1969), (Osipov, 1978) and others paved the way for asymptotic expansions in the theorems of large deviations. The works (Saulis, 1969), (Bikelis, Žemaitis, 1974), (Wolf, 1970, 1975), (Jakševičius, 1983) and other are devoted namely to this problem. Note that asymptotic expansions for large deviations were first obtained in the probability number theory by J. Kubilius (Kubilius, 1964).

6.1. Asymptotic expansion for distribution density of an arbitrary random variable

Let $p_\xi(x)$ be the distribution density of a r.v. ξ with $\mathbf{E}\xi = 0$ and $\mathbf{E}\xi^2 = 1$ such that

$$\sup_x p_\xi(x) < \infty. \tag{A}$$

By $\mathcal{E}$ denote the set of all points on the line, at which $p_\xi(x)$ either is continuous or has a discontinuity of the first kind, and in the latter case assume

$$p(x) = \frac{p(x-0) + p(x+0)}{2}.$$

Suppose that there exist $\gamma \geqslant 0$ and $\Delta > 0$ such that the k^{th} order cumulants of r.v. ξ satisfy the condition:

$$|\Gamma_k(\xi)| \leqslant \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \qquad k = 3, 4, \dots . \tag{S_γ}$$

Denote

$$f_\xi(t) = \mathbf{E}\exp\{it\xi\}, \quad g(x) = \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}x^2\Big\},$$

$$\Delta_\gamma = c_\gamma \Delta^{1/(1+2\gamma)}, \quad c_\gamma = \frac{1}{6}\left(\frac{\sqrt{2}}{6}\right)^{1/(1+2\gamma)}, \tag{6.1}$$

$$\varepsilon(\gamma, \Delta) = \frac{1}{12}\left(1 - \frac{x}{\Delta_\gamma}\right)\Delta_\gamma, \qquad 0 \leqslant x < \Delta_\gamma; \tag{6.2}$$

$$f^*(t) = \begin{cases} \displaystyle\sum_{k=0}^{s}\left(\frac{3}{2}\right)^k \frac{x^k}{k!}\,|f_\xi^{(k)}(t)| & \text{for } \gamma > 0, \\ \displaystyle\int_{-\infty}^{\infty} \exp\{itu\}p_h(u)\,du & \text{for } \gamma = 0, \end{cases} \tag{6.3}$$

s is defined by equality (2.63),

$$p_h(u) = \frac{\exp\{hu\}p_\xi(u)}{\int_{-\infty}^{\infty}\exp\{hu\}p_\xi(u)\,du}, \tag{6.4}$$

h is the root of the equation

$$x = \sum_{k=2}^{\infty}\frac{1}{(k-1)!}\,\Gamma_k(\xi)h^{k-1}; \tag{6.5}$$

$$c_1 = 2\pi + 6^\gamma \cdot 25^3\sqrt{2\pi}/\Delta, \tag{6.6}$$

$$q(l, \gamma) = \left(\frac{3\sqrt{2e}}{2}\right)^l 8(l+2)^2 6^{\gamma(l-1)}4^{3(l+1)}\big((l+1)!\big)^{\gamma(l-1)}\Gamma\left(\frac{3l+1}{2}\right), \tag{6.7}$$

and

$$\begin{aligned} r^*(x, \Delta) = \left(1 + 9\big((m+2)!\big)^\gamma 16^{m-1}c_\gamma^{m+1-l}\frac{1}{m+1}\left(\frac{x}{\Delta}\right)^l\right)\times \\ \times\left(1 + 46\,\Delta_\gamma\exp\left\{-\frac{1}{2}\left(1-\frac{x}{\Delta_\gamma}\right)\sqrt{\Delta_\gamma}\right\}\Big/\left(1-\frac{x}{\Delta_\gamma}\right)\right), \end{aligned} \tag{6.8}$$

where $m = [1/\gamma] + l + 1, \ \gamma > 0, \ l \geqslant 1$, and

$$r^*(x, \Delta) \equiv 0 \quad \text{as } \gamma = 0.$$

Let θ (with or without an index) denote some quantity, not always one and the same, not exceeding 1 in absolute value.

LEMMA 6.1. *If for a r.v. ξ conditions (S_γ) and (A) are fulfilled, then $\forall l,\ l \geqslant 1$, in the interval*

$$0 \leqslant x < \Delta_\gamma$$

the relation

$$\begin{aligned}\frac{p_\xi(x)}{g(x)} = \exp\{L_m(x)\}\Big(1 + \sum_{\nu=0}^{l-1} M_\nu(x) + \theta_1 q(l,\,\gamma)\Big(\frac{x+1}{\Delta}\Big)^l + \\ + \theta_2 c_1(\gamma)\Delta_\gamma^{3/2}\exp\Big\{-\frac{1}{72}\Big(1-\frac{x}{\Delta_\gamma}\Big)\sqrt{\Delta_\gamma}\Big\} + \\ + \theta_3 \int\limits_{|t|\geqslant\varepsilon(\gamma,\,\Delta)} |f^*(t)|\,dt\Big)\left(1+\theta_4 r^*(x,\,\Delta)\right)\end{aligned} \tag{6.9}$$

holds. Here the quantities $q(l,\,\gamma)$, $c_1(\gamma)$, $r^*(x,\,\Delta)$, $f^*(t)$ and $\varepsilon(\gamma,\,\Delta)$ are defined by relations (6.7), (6.6), (6.8), (6.3) and (6.2), respectively;

$$L_m(x) = \sum_{3\leqslant k<m} \lambda_k x^k, \qquad m = \begin{cases}(1/\gamma)+l+1, & \gamma>0,\\ \infty, & \gamma=0,\end{cases} \tag{6.10}$$

where $\lambda_k = -b_{k-1}/k$ and b_k are determined from equations (2.10). Moreover,

$$|\lambda_k| \leqslant (2/k)(16/\Delta)^{k-2}\left((k+1)!\right)^\gamma, \qquad k=3,\,4,\,\dots\,. \tag{6.11}$$

For polynomials $M_\nu(x)$ the following formula holds:

$$M_\nu(x) = \sum_{k=0}^{\nu} K_k(x) q_{\nu-k}(x), \tag{6.12}$$

where

$$K_\nu(x) = \sum \prod_{m=1}^{\nu} \frac{1}{k_m!}(-\lambda_{m+2}x^{m+2})^{k_m}, \qquad K_0(x)\equiv 1,$$

$$q_\nu(x) = \sum H_{\nu+2l}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!}\left(\frac{\Gamma_{m+2}(\xi)}{(m+2)!}\right)^{k_m},$$

$q_0(x) \equiv 1$, $H_l(x)$ are Chebyshev – Hermite polynomials, and the summation is taken over all integer solutions of the equation $k_1 + 2k_2 + \dots + \nu k_\nu = \nu$.

In particular,

$$M_0(x) \equiv 0, \qquad M_1(x) = -\frac{1}{2}\Gamma_3(\xi)x,$$
$$M_2(x) = (1/8)\big(5\Gamma_3^2(\xi) - 2\Gamma_4(\xi)\big)x^2 + (1/24)\big(3\Gamma_4(\xi) - 5\Gamma_3^2(\xi)\big),$$
$$M_3(x) = (1/48)\big(34\Gamma_3(\xi)\Gamma_4(\xi) - 4\Gamma_5(\xi) - 45\Gamma_3^3(\xi)\big)x^3 +$$
$$+ (1/48)\big(6\Gamma_5(\xi) - 35\Gamma_3(\xi)\Gamma_4(\xi) + 35\Gamma_3^2(\xi)\big)x, \ldots;$$

LEMMA 6.2. *If for a r.v. ξ conditions (S_γ) and (A) are fulfilled, then*

$$\sup_{x\in\mathcal{E}} |p_\xi(x) - g(x)| \leqslant \frac{1}{\sqrt{2\pi}}\left\{\frac{16\cdot 6^\gamma}{\Delta} + 364\,\Delta_\gamma^{3/2}e^{-\frac{3}{2}\sqrt{\Delta_\gamma}} + \int\limits_{|t|\geqslant\Delta_\gamma}\left|f_\xi(t) - \exp\left\{-\frac{1}{2}t^2\right\}\right|dt\right\}. \tag{6.13}$$

If $p_\xi''(x)$ $\forall x \in R$ exists and belongs to $L_1(-\infty, \infty)$, then $|f_\xi(t)| \sim t^{-2}$ as $t \to \infty$. Consequently,

$$\sup_{x\in\mathcal{E}} |p_\xi(x) - g(x)| \leqslant \frac{\text{const}}{\Delta_\gamma}.$$

Proof of Lemma 6.1. Since distribution density of the r.v. ξ exists, according to the definition of the function $F_h(y)$ by equality (2.22) we have

$$F_h(y) = \tilde{\varphi}^{-1}(h)\int\limits_{-\infty}^{\tilde{\sigma}(h)y+\tilde{m}(h)} g_h(u)p_\xi(u)\,du, \tag{6.14}$$

where $\tilde{m}(h)$, $\tilde{\sigma}^2(h)$, $\tilde{\varphi}(h)$ and $g_h(y)$ are determined by (2.18), (2.19), (2.15) and (2.23), respectively. Then the Fourier transform of the function $F_h(y)$ has the form

$$f_h(t) := \int\limits_{-\infty}^{\infty} e^{ity}\,dF_h(y) = \tilde{\varphi}^{-1}(h)\exp\{-it\tilde{m}(h)/\tilde{\sigma}(h)\}\times \int\limits_{-\infty}^{\infty}\exp\{-itu/\tilde{\sigma}(h)\}g_h(u)p_\xi(u)\,du. \tag{6.15}$$

From (6.14) we obtain

$$p_h(y) = \frac{dF_h(y)}{dy} = \tilde{\varphi}^{-1}(h) g_h\big(\tilde{\sigma}(h)y + \tilde{m}(h)\big)\tilde{\sigma}(h) p_\xi\big(\tilde{\sigma}(h)y + \tilde{m}(h)\big),$$

$$\tilde{\sigma}(h)y + \tilde{m}(h) \in \mathcal{E} \quad \text{or} \quad p_h\Big(\big(u - \tilde{m}(h)\big)/\tilde{\sigma}(h)\Big) = \tilde{\varphi}^{-1}(h) g_h(u)\tilde{\sigma}(h) p_\xi(u).$$

The quantity h is defined as the solution of the equation $x = \tilde{m}(h)$, where $\tilde{m}(h)$ is defined by relation (2.18). Consequently, we have

$$p_\xi(x) = \tilde{\varphi}(h)\big(\tilde{\sigma}(h) g_h(x)\big)^{-1} \cdot p_h(0), \qquad x \in \mathcal{E}. \tag{6.16}$$

It is easy to see that $p_h(u) \in L_1$. Actually,

$$\int_{-\infty}^{\infty} p_h(y)\, dy = \tilde{\varphi}^{-1}(h) \int_{-\infty}^{\infty} g_h(y) p_\xi(y)\, dy = \tilde{\varphi}^{-1}(h) \times$$

$$\times \left(\sum_{k=0}^{s} \frac{1}{k!} h^k \int_{-\infty}^{\infty} y^k p_\xi(y)\, dy + \sum_{k=s+1}^{\infty} \frac{1}{k!} \tilde{m}_k h^k \int_{-\infty}^{\infty} y^2 p_\xi(y)\, dy \right) = 1.$$

Since $\tilde{\sigma} u + \tilde{m}(h) \in \mathcal{E}$, by (Titchmarsh, 1948, Theorem 27)

$$p_h(u) = \frac{1}{2\pi} \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} \exp\{itu\}(1 - |t|/\lambda) f_h(t)\, dt. \tag{6.17}$$

Then according to (6.16) and (6.17) we have

$$p_\xi(x) = \big(2\pi \tilde{\sigma}(h) g_h(x)\big)^{-1} \tilde{\varphi}(h)(I_1 + I_2), \tag{6.18}$$

where

$$I_1 = \lim_{\lambda \to \infty} \int_{-\varepsilon}^{\varepsilon} (1 - |t|/\lambda) f_h(t)\, dt, \tag{6.19}$$

$$I_2 = \lim_{\lambda \to \infty} \left(\int_{-\lambda}^{-\varepsilon} + \int_{\varepsilon}^{\lambda} \right) (1 - |t|/\lambda) f_h(t)\, dt,$$

and $\varepsilon > 0$ will be specified later. By passing to the limit under the sign of integral in (6.19) (conditions for that evidently satisfied)

$$I_1 = \int_{|t| \leqslant \varepsilon} f_h(t)\, dt = I_1^{(1)} + I_1^{(2)}, \tag{6.20}$$

where

$$I_1^{(1)} = \int\limits_{|t|\leqslant\varepsilon} \tilde{f}_h(t)\,dt, \qquad I_1^{(2)} = \int\limits_{|t|\leqslant\varepsilon} \big(f_h(t) - \tilde{f}_h(t)\big)\,dt,$$

$$\begin{aligned} \tilde{f}_h(t) :&= \tilde{\varphi}^{-1}(h)\exp\{-itu/\tilde{\sigma}(h)\}\int\limits_{-\infty}^{\infty} g_z(u)p_\xi(u)\,du = \\ &= \exp\{-itu/\tilde{\sigma}(h)\}\tilde{\varphi}\big(h + it/\tilde{\sigma}(h)\big)/\tilde{\varphi}(h). \end{aligned} \tag{6.21}$$

It is evident that

$$\big(2\pi\tilde{\sigma}(h)g_h(x)\big)^{-1}\tilde{\varphi}(h)\int\limits_{|t|\leqslant\varepsilon}\tilde{f}_h(t)\,dt = \frac{\exp\{hx\}}{2\pi i g_h(x)}\int\limits_{h-i\varepsilon/\tilde{\sigma}(h)}^{h+i\varepsilon/\tilde{\sigma}(h)}\exp\{\tilde{K}(z) - zx\}\,dz,$$

where $\tilde{K}(z) = \ln\tilde{\varphi}(z) = \sum\limits_{k=2}^{s}\frac{1}{k!}\Gamma_k(\xi)z^k$. Denote

$$K_l(h) = \sum_{k=3}^{l-1}\frac{1}{k!}\tilde{K}^{(k)}(h)\big(it/\tilde{\sigma}(h)\big)^k. \tag{6.22}$$

By expanding $\tilde{K}(z)$ in the neighbourhood of the point h we get

$$\begin{aligned} &\tilde{K}(z) - zx - \tilde{K}(h) + hx + \frac{1}{2}t^2 - K_l(h) - \frac{1}{l!}\tilde{K}^{(l)}\Big(h + \theta\frac{it}{\tilde{\sigma}(h)}\Big)\big(it/\tilde{\sigma}(h)\big)^l = \\ &\quad = \theta(l!)^\gamma\big((1 - |\tau|/\Delta_s)^{l-1}\Delta^{l-2}\big)^{-1}\big(it/\tilde{\sigma}(h)\big)^l, \end{aligned} \tag{6.23}$$

because of condition (S_γ)

$$\begin{aligned} \tilde{K}^{(m)}(z) &= \sum_{k=m}^{s}\frac{1}{(k-m)!}\Gamma_k(\xi)z^{k-m} = \\ &= \theta\Delta^{-(m-1)}\sum_{k=m}^{s}k(k-1)\dots\big(k-(m-1)\big)\,(k!)^\gamma(|z|/\Delta)^{k-m} = \\ &= \theta(m!)^\gamma\Delta^{2-m}\cdot\sum_{k=m}^{s}k(k-1)\dots\big(k-(m-1)\big)\Big(\frac{s^\gamma|z|}{\Delta}\Big)^{k-m} = \\ &= \theta\,\frac{m(m-1)(m!)^\gamma}{\Delta^{m-2}}\sum_{k=m}^{s}k(k-2)\dots\big(k-(m-1)\big)(|z|/\Delta_s)^{k-m} = \\ &= \theta(m!)^{1+\gamma}/\big((1-|z|/\Delta_s)^{m-1}\Delta^{m-2}\big), \quad 3\leqslant m\leqslant s,\ \Delta_s = \Delta/6\,s^\gamma, \end{aligned} \tag{6.24}$$

$K^{(m)}(z) \equiv 0$, $m \geqslant s+1$. As it has already been noted in the proof of Lemma 2.3, $(k!)^{1+\gamma}/\Delta^{k-2} \leqslant (k-2)!/\Delta_s^{k-2}$, $k = 3, 4, \ldots, s+2$, where $\Delta_s \geqslant \Delta/6(s+2)^\gamma \geqslant (s/2)^{1/2}$ and s is defined by equality (2.63). Then for $|t| \leqslant (\delta_2 - \delta)\tilde{\sigma}(h)\Delta_s/\sqrt{2e}$, $0 < \delta < \delta_2 < 1$, according to (6.22) we have

$$\begin{aligned} K_l(h) &= \theta t^2\big(6(1-h/\Delta_s)\tilde{\sigma}^2(h)\big)^{-1}\times \\ &\times \sum_{k=3}^{l-1}\Big(|t|/\big((1-h/\Delta_s)\tilde{\sigma}(h)\Delta_s\big)\Big)^{k-2} = \theta c t^2 \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} &\Big((m!)^\gamma/\big((1-|\tau|/\Delta_s)^{m-1}\Delta^{m-2}\big)\Big)\big(|t|/\tilde{\sigma}(h)\big)^m = \theta|t|^3\times \\ &\times\big((1-|\tau|/\Delta_s)\tilde{\sigma}^3(h)\Delta_s\big)^{-1}\cdot\Big(|t|/\big((1-|\tau|/\Delta_s)\tilde{\sigma}(h)\Delta_s\big)\Big)^{m-3} = \theta c t^2, \end{aligned} \tag{6.26}$$

where $c = 3/13$, because $(\delta_2 - \delta)(\sqrt{2e} - \delta_2)^{-1} \leqslant 4/5$ and

$$\tilde{\sigma}^2(h) = 1 + \theta(3\delta/4), \qquad 0 \leqslant h \leqslant \delta\Delta_s/\sqrt{2e}. \tag{6.27}$$

Let

$$\tilde{K}_l^{(m)}(z) := \sum_{k=m}^{l-1}\frac{1}{(k-m)!}\Gamma_k(\xi)z^{k-m} \tag{6.28}$$

and

$$\tilde{K}_l(h) := \sum_{m=3}^{l-1}\frac{1}{m!}\tilde{K}_l^{(m)}(h)\big(it/\tilde{\sigma}(h)\big)^m. \tag{6.29}$$

Then, on the basis of (6.24) we have

$$\begin{aligned} K_l(h) &= \tilde{K}_l(h) + \sum_{m=3}^{l-1}\frac{1}{m!}\left(\sum_{k=l}^{s}\frac{1}{(k-m)!}\Gamma_k(\xi)h^{k-m}\right)\left(\frac{it}{\tilde{\sigma}(h)}\right)^m = \\ &= \tilde{K}_l(h) + \theta r_1(t, h, l, \gamma)/\Delta^{l-2}, \end{aligned} \tag{6.30}$$

as far as

$$\begin{aligned} &\sum_{m=3}^{l-1}\frac{1}{m!}\left(\sum_{k=l}^{s}\frac{1}{(k-m)!}\Gamma_k(\xi)h^{k-m}\right)\left(\frac{it}{\tilde{\sigma}(h)}\right)^m = \\ &= \theta\sum_{k=l}^{s}\sum_{m=3}^{l-1}\frac{(k!)^{1+\gamma}h^{k-m}}{m!(k-m)!\Delta^{k-2}}\left(\frac{it}{\tilde{\sigma}(h)}\right)^m = \\ &= \theta(3/2)r_1(t, h, l, \gamma)/\Delta^{l-2}\sum_{k=l}^{s}\left(\frac{2s^\gamma h}{\Delta}\right)^{k-l} = \theta r_1(t, h, l, \gamma)/\Delta^{l-2}, \end{aligned}$$

where

$$r_1(t,\, h,\, l,\, \gamma) = (2/3)(l-3)2^l(l!)^\gamma\Big(\big(|t|/\tilde{\sigma}(h)\big)^3 h^{l-3} + \big(|t|/\tilde{\sigma}(h)\big)^{l-1}\cdot h\Big). \tag{6.31}$$

Recalling (6.23), (6.27) and using the inequality $|e^x - 1| \leqslant |x|e^{|x|}$, we find

$$\begin{aligned}
&\exp\Big\{-\frac{1}{2}t^2 + K_l(h) + \theta(l!)^\gamma\big((1-|\tau|/\Delta_s)^{l-1}\Delta^{l-2}\big)^{-1}\big(it/\tilde{\sigma}(h)\big)^l\Big\} = \\
&= \exp\Big\{-\frac{1}{2}t^2 + \tilde{K}_l(h) + \theta r_2(t,\, h,\, l,\, \gamma)/\Delta^{l-2}\Big\} = \\
&= \exp\Big\{-\frac{1}{2}t^2 + \tilde{K}_l(h)\Big\} + \exp\Big\{-\frac{1}{2}t^2 + \tilde{K}_l(h)\Big\}\times \\
&\times \big(\exp\{\theta r_2(t,\, h,\, l,\, \gamma)/\Delta^{l-2}\} - 1\big) = \exp\Big\{-\frac{1}{2}t \,+ \tilde{K}_l(h)\Big\} + \\
&+ \theta\big(r_2(t,\, h,\, l,\, \gamma)/\Delta^{l-2}\big)\exp\Big\{-\Big(\frac{1}{2} - 2c\Big)t^2\Big\},
\end{aligned} \tag{6.32}$$

where

$$r_2(t,\, h,\, l,\, \gamma) = r_1(t,\, h,\, l,\, \gamma) + \big((l!)^\gamma/(1-|\tau|/\Delta_s)^{l-1}\big)\big(|t|/\tilde{\sigma}(h)\big)^l. \tag{6.33}$$

It remains to investigate the first summand on the right-hand side of (6.32). Let y be a real parameter, satisfying the condition $|y| \leqslant 1$. Put

$$\begin{aligned}
L(y) &= \frac{1}{y^2}\sum_{k=3}^{l-1}\frac{1}{k!}\tilde{K}_l^{(k)}(h)\Big(\frac{ity}{\tilde{\sigma}(h)}\Big)^k = \\
&= \sum_{k=1}^{l-3}\frac{1}{(k+2)!}\tilde{K}_l^{(k+2)}(h)\Big(\frac{it}{\tilde{\sigma}(h)}\Big)^{k+2}y^k.
\end{aligned} \tag{6.34}$$

Then

$$\exp\{L(y)\} = 1 + \sum_{\nu=1}^{l-3}P_\nu(h,\, it)y^\nu + R(y), \tag{6.35}$$

where $R(y)$ is power series in y, beginning with the $(l-2)^{\text{nd}}$ power. Estimate $R(y)$, using the relation

$$e^\omega = \sum_{\nu=0}^{l-3}\frac{1}{\nu!}\omega^\nu + \theta|\omega|^{l-2}e^{|\omega|}.$$

According to the definition $\tilde{K}_l^{(m)}(z)$ by (6.28) we have

$$\tilde{K}_l^{(m)}(z) = \theta\big((l-1)!\big)^{\gamma}/\Delta^{m-2}\cdot \sum_{k=m}^{l-1} k(k-1)\dots\big(k-(m-1)\big)\times \\ \times (|z|/\Delta)^{k-m} = \theta\big((l-1)!\big)^{\gamma} m!/\big((1-|z|/\Delta)^{m+1}\cdot\Delta^{m-2}\big) \tag{6.36}$$

for $|z| < \Delta$, or

$$\tilde{K}_l^{(m)}(z) = \theta \sum_{k=m}^{l-1} \frac{(k!)^{1+\gamma}|z|^{k-m}}{(k-m)!\,\Delta^{k-2}} = \theta m(m-1)m!/\Delta^{m-2}\times \\ \times \sum_{k=m}^{l-1}(k-2)\dots\big(k-(m-1)\big)\left(\frac{(l-1)|z|}{\Delta}\right)^{k-m} = \\ = \theta\frac{(k!)^{1+\gamma}}{(1-|z|/\Delta_s)^{m-1}\Delta^{m-2}} \tag{6.37}$$

for $|z| < \Delta_s$. Consequently, similarly to relation (6.25) we have

$$L(y) = \theta c|y|t^2, \tag{6.38}$$

$$L^{l-2}(y) = \theta|y|^{l-2}\big(|t|/\tilde{\sigma}(h)\big)^{3(l-2)}\cdot\left(\sum_{k=1}^{l-3}\frac{((k+2)!)^{\gamma}}{\Delta^{*k}}\left(\frac{|t|}{\tilde{\sigma}(h)}\right)^{k-1}\right)^{l-2} = \\ = \theta\left(\frac{6^{\gamma}|y|}{\Delta}\left(\frac{|t|}{\tilde{\sigma}(h)}\right)^{3}\right)^{l-2}\cdot\left(\sum_{k=1}^{l-3}\frac{(l-1)^{\gamma}|t|^{k-1}}{\tilde{\sigma}(h)\Delta^{*}}\right)^{l-2} = \\ = \theta\left(\frac{6^{1+\gamma}|y|}{\Delta}\left(\frac{|t|}{\tilde{\sigma}(h)}\right)^{3}\right)^{l-2}, \tag{6.39}$$

$\Delta^* = \Delta/4$, because $\big((l-1)^{\gamma}|t|/(\tilde{\sigma}(h)\Delta^*)\big) \leqslant 4(l-1)^{\gamma}(\delta_2-\delta)\Delta_s/\sqrt{2e}\,\Delta \leqslant 1/3$, $l \leqslant s$. Using (6.34) for $y = 1$ as well as (6.38) and (6.39) we obtain

$$\exp\left\{-\frac{1}{2}t^2 + \tilde{K}_l(h)\right\} = \exp\left\{-\frac{1}{2}t^2\right\}\Big(1 + \sum_{\nu=1}^{l-3}P_{\nu}(h, it) + \\ + \theta_1\big(c(l,\gamma)/\Delta^{l-2}\big)\Big(\big(|t|/\tilde{\sigma}(h)\big)^{l+2} + \big(|t|/\tilde{\sigma}(h)\big)^{3(l-3)+1}\Big) + \\ + \theta_2(6^{(1+\gamma)(l-2)}/\Delta^{l-2}\big(|t|/\tilde{\sigma}(h)\big)^{3(l-2)}\exp\{ct^2\}\Big), \tag{6.40}$$

where $c(l,\gamma) \leqslant 2^{l-3}6^{\gamma(l-3)}\big((l-2)!\big)^{\gamma}$. Hence and from (6.32) we conclude that

$$\begin{aligned}
&\exp\Big\{-\frac{1}{2}t^2 + K_l(h) + \theta(l!)^{\gamma}/\big((1-|\tau|/\Delta_s)^{l-1}\Delta^{l-2}\big)\big(|t|/\tilde{\sigma}(h)\big)^{l}\Big\} = \\
&= \exp\Big\{-\frac{1}{2}t^2\Big\}\Big(1+\sum_{\nu=1}^{l-3}P_\nu(h,\,it) + \theta_1\big(c(l,\,\gamma)/\Delta^{l-2}\big)\times \\
&\times\Big(\big(|t|/\tilde{\sigma}(h)\big)^{l+2} + \big(|t|/\tilde{\sigma}(h)\big)^{3(l-3)+1}\Big) + \\
&+\theta_2(6^{(1+\gamma)(l-2)}/\Delta^{l-2})\big(|t|/\tilde{\sigma}(h)\big)^{3(l-2)}\exp\{ct^2\} + \\
&+\theta_3 r_2(t,\,h,\,l,\,\gamma)\exp\{2\,ct^2\}/\Delta^{l-2}\Big),
\end{aligned} \tag{6.41}$$

where

$$\begin{aligned}
P_r(h,\,it) &= \frac{\tilde{K}_l^{(r+2)}(h)}{(r+2)!}\left(\frac{it}{\tilde{\sigma}(h)}\right)^{r+2} + \\
&+\sum_{k=1}^{r-1}\frac{(r-k)\tilde{K}_l^{(r-k+2)}(h)}{r(r-k+2)!}\left(\frac{it}{\tilde{\sigma}(h)}\right)^{r-k+2}P_k(h,\,it).
\end{aligned}$$

Making use of estimate (6.36) for $\tilde{K}_l^{(m)}(z)$ by induction we obtain the estimate

$$\begin{aligned}
|P_r(h,\,it)| &\leqslant \frac{\big((r+2)!\big)^{\gamma}+4^r6^{\gamma r}}{\Delta^r}\left(\frac{|t|}{\tilde{\sigma}(h)}\right)^{r+2}\sum_{k=0}^{r-1}\frac{1}{(k+1)!}C_{r-1}^{k}\big(t^2/\tilde{\sigma}(h)\big)^{k} \leqslant \\
&\leqslant \Big(c_1(\gamma,\,r)/\big((1-3\,\delta/4)^{r-1}\Delta^r\big)\Big)\cdot\big(|t|/\tilde{\sigma}(h)\big)^{r+2}\exp\{2\,ct^2\},
\end{aligned} \tag{6.42}$$

where $c_1(\gamma,\,r) \leqslant 3\cdot 18^{r-1}\Big(6^{\gamma r}+\big((r+2)!\big)^{\gamma}\Big)$, $1 \leqslant r \leqslant l-1$. Hence it follows that

$$\begin{aligned}
&\Big|\sum_{r=1}^{l-3}P_r(h,\,it)\Big| \leqslant \\
&\leqslant \exp\{ct^2\}\sum_{r=1}^{l-3}\Big(c_1(\gamma,\,r)/\big((1-3\,\delta/4)^{r-1}\Delta^r\big)\Big)\big(|t|/\tilde{\sigma}(h)\big)^{r+2} \leqslant \\
&\leqslant (3\cdot 6^{\gamma}/\Delta)\big(|t|/\tilde{\sigma}(h)\big)^{3}\exp\{2\,ct^2\}\times \\
&\times\sum_{r=1}^{l-3}\Big(18(l-1)^{\gamma}|t|/\big((1-3\,\delta/4)\tilde{\sigma}(h)\Delta\big)\Big)^{r-1} \leqslant \\
&\leqslant (6^{1+\gamma}/\Delta)\big(|t|/\tilde{\sigma}(h)\big)^{3}\exp\{2\,ct^2\}.
\end{aligned} \tag{6.43}$$

Employing (6.23), (6.41) and (6.42), we obtain

$$\frac{\tilde{\varphi}(h)}{2\pi\tilde{\sigma}(h)g_h(x)}\int\limits_{|t|\leqslant\varepsilon}\tilde{f}_h(t)\,dt=$$

$$\frac{\exp\{-hx+\tilde{K}(h)\}}{2\pi\tilde{\sigma}(h)g_h(x)}\int\limits_{|t|\leqslant\varepsilon}e^{-\frac{1}{2}t^2}\Big\{1+\sum_{\nu=1}^{l-3}P_\nu(h,\,it)+$$
$$+\theta_1\big(c(l,\,\gamma)/\Delta^{l-2}\big)\Big(\big(|t|/\tilde{\sigma}(h)\big)^{l-2}+\big(|t|/\tilde{\sigma}(h)\big)^{3l-8}\Big)+$$
$$+\theta_2(6^{(1+\gamma)(l-2)}/\Delta^{l-2}\big(|t|/\tilde{\sigma}(h)\big)^{3(l-2)}\exp\{ct^2\}+ \tag{6.44}$$
$$+\theta_3\big(r_2(t,\,h,\,l,\,\gamma)/\Delta^{l-2}\big)\exp\{2\,ct^2\}\Big\}\,dt=$$
$$=\exp\{\tilde{K}(h)\}\cdot J(h,\,x)+\theta_1\big(c_2(l,\,\gamma)/\Delta^{l-2}\big)\,(1+h+h^{l-3})+$$
$$+\theta_2c_1(\gamma)\exp\,\big\{-\big((\delta_2-\delta)\tilde{\sigma}(h)\Delta_s\big)^2/60\,e\big\}\times$$
$$\times\,\big(2\pi\tilde{\sigma}(h)g_h(x)\big)^{-1}\exp\{-hx+\tilde{K}(h)\},$$

where

$$J(h,\,x)=\frac{\exp\{hx\}}{2\pi\tilde{\sigma}(h)g_h(x)}\int\limits_{-\infty}^{\infty}e^{-\frac{1}{2}t^2}\Big(1+\sum_{\nu=1}^{l-3}P_\nu(h,\,it)\Big)\,dt, \tag{6.45}$$

$$c_2(l,\,\gamma)\leqslant 4^{3(l-1)}6^{\gamma(l-3)}(l!)^\gamma\Gamma\big((3\,l-5)/2\big),$$
$$c_1(\gamma)\leqslant 2\sqrt{\pi}+6^\gamma 25^3\,\sqrt{2\pi}/\Delta.$$

We used here the estimates

$$\int\limits_{|t|\geqslant\varepsilon}e^{-\frac{1}{2}t^2}\Big(1+\sum_{\nu=1}^{l-3}P_\nu(h,\,it)\Big)\,dt\leqslant c_1(\gamma)\exp\,\big\{-\big((\delta_2-\delta)\tilde{\sigma}(h)\Delta_s\big)^2/60\,e\big\}$$

and

$$\big(c(l,\,\gamma)/\Delta^{l-2}\big)\int\limits_{|t|\geqslant\varepsilon}e^{-\frac{1}{2}t^2}\Big(\big(|t|/\tilde{\sigma}(h)\big)^{l-2}+\big(|t|/\tilde{\sigma}(h)\big)^{3l-8}\Big)\,dt+$$
$$+\,(6^{(1+\gamma)(l-2)}/\Delta^{l-2}\int\limits_{|t|\leqslant\varepsilon}\exp\{-t^2/4\}\big(|t|/\tilde{\sigma}(h)\big)^{3(l-2)}\,dt$$
$$+\,(1/\Delta^{l-2})\int\limits_{|t|\leqslant\varepsilon}r_2(t,\,h,\,l,\,\gamma)\exp\{-t^2/26\}\,dt\leqslant\big(c_2(l,\,\gamma)/\Delta^{l-2}\big)\,(1+h+h^{l-3}).$$

It remains to investigate $J(h, x)$ defined by equality (6.45). Recalling (2.20) and (2.59), we have

$$x = \tilde{m}(h) = h\bigl(1 + \theta(\delta/3)\bigr), \qquad 0 \leqslant h \leqslant \delta\Delta_s/\sqrt{2e}, \tag{6.46}$$

and

$$h = x + \sum_{k=2}^{\infty} b_k x^k, \quad \text{where } b_k = \theta(3/2)^k(\sqrt{2e}/\Delta_s)^{k-1},$$

$k = 1, 2, \ldots$. The coefficients b_k are expressed through the $r_k = \min\{k+3, s\}$ first cumulants of the r.v. ξ and are calculated by formula (2.10). Applying estimate (6.31) to $\tilde{K}_l^{(m)}(z)$, analogously to estimate (6.42) for $P_r(h, it)$ we find

$$\begin{aligned} |P_r(h, it)| \leqslant \Bigl(\bigl((l-1)!\bigr)^{\gamma}/\bigl((1-h/\Delta)^{r+3}\Delta^r\bigr)\Bigr)\bigl(|t|/\tilde{\sigma}(h)\bigr)^{r+2} \times \\ \times \sum_{k=0}^{r-1} \frac{1}{(k+1)!} C_{r-1}^k \Bigl(\bigl((l-1)!\bigr)^{\gamma} t^2/\bigl((1-h/\Delta)^3\tilde{\sigma}^2(h)\bigr)\Bigr)^k. \end{aligned} \tag{6.47}$$

Recalling that $x = h\bigl(1 + \theta(\delta/3)\bigr)$ we obtain

$$P_\nu(h, it) = \sum_{k=0}^{\infty} P_{\nu,k}^*(it)h^k = \sum_{k=0}^{\infty} \overline{P}_{\nu,k}(it)x^k,$$

where

$$P_{\nu,k}^*(it) = \frac{1}{k!}\bigl(P_\nu(h, it)\bigr)^{(k)}\Big|_{h=0}.$$

Making use of estimate (6.47) for $P_\nu(h, it)$ with the help of Chauchy's formula we find

$$\begin{aligned} P_{\nu,k}^*(it) = \Bigl(\bigl((l-1)!\bigr)^{\gamma}/\bigl((1-h/\Delta)^{\nu+3}\Delta^\nu\bigr)\Bigr)\bigl(|t|/\tilde{\sigma}(h)\bigr)^{\nu+2}(\sqrt{2e}/\Delta)^k \times \\ \times \sum_{m=0}^{\nu-1} \frac{1}{(m+1)!} C_{\nu-1}^m \Bigl(\bigl((l-1)!\bigr)^{\gamma} t^2/\bigl((1-h/\Delta)^3\tilde{\sigma}^2(h)\bigr)\Bigr)^m. \end{aligned}$$

Hence it follows that $P_{\nu,k}^*(it)h^k = P_{\nu,k}^*(it)x^k \Big/ \Bigl(\bigl(1+\theta(\delta/3)\bigr)\Bigr)$. Consequently,

$$\begin{aligned} &|\overline{P}_{\nu,k}(it)| \leqslant \\ &\leqslant \Bigl(\bigl((l-1)!\bigr)^{\gamma}/\bigl((1-h/\Delta)^{\nu+3}\Delta^\nu\bigr)\Bigr)\bigl(|t|/\tilde{\sigma}(h)\bigr)^{\nu+2}\Bigl(\sqrt{2e}/\bigl(1+\theta(\delta/3)\bigr)\Delta\Bigr)^k \times \\ &\times \sum_{m=0}^{\nu-1} \frac{1}{(m+1)!} C_{\nu-1}^m \Bigl(\bigl((l-1)!\bigr)^{\gamma} t^2/\bigl((1-h/\Delta)^3\tilde{\sigma}^2(h)\bigr)\Bigr)^m. \end{aligned}$$

Then

$$\sum_{\nu=1}^{l-3} P_\nu(h, it) = \sum_{\nu=1}^{l-3} \sum_{k=0}^{l-3-\nu} \overline{P}_{\nu,k}(it)x^k + \sum_{\nu=1}^{l-3} \sum_{k=l-2-\nu}^{\infty} \overline{P}_{\nu,k}(it)x^k =$$

$$= \sum_{\nu=1}^{l-3} \sum_{k=0}^{l-3-\nu} \overline{P}_{\nu,k}(it)x^k + \theta\left(((l-1)!)^{\gamma(l-3)}(l-3)^2 2^{5(l-3)}/\Delta^{l-2}\right)\times \tag{6.48}$$

$$\times (3\sqrt{2e}/\Delta)^{l-3}\left(\left(|t|/\tilde\sigma(h)\right)^3 + \left(|t|/\tilde\sigma(h)\right)^{3(l-3)}\right),$$

because

$$\sum_{\nu=1}^{l-3} \sum_{k=l-2-\nu}^{\infty} \overline{P}_{\nu,k}(it)x^k = \theta \sum_{\nu=1}^{l-3} ((l-1)!)^\gamma \left((1-h/\Delta)^{\nu+3}\Delta^\nu\right)^{-1}\times$$

$$\times \left(\frac{|t|}{\tilde\sigma(h)}\right)^{\nu+2} \sum_{m=0}^{\nu-1} \frac{1}{(m+1)!} C_{\nu-1}^m \left(((l-1)!)^\gamma t^2/((1-h/\Delta)^3\tilde\sigma^2(h))\right)^m \times$$

$$\times \sum_{k=l-2-\nu}^{\infty} \left(\sqrt{2e}\,x/(1+\theta\delta/3)\Delta\right)^k =$$

$$= \theta((l-1)!)^\gamma \sum_{\nu=1}^{l-3}\Big\{ \left(\sqrt{2e}\,x\left((1+\theta\delta/3)\Delta\right)^{-1}\right)^{l-2-\nu}\times$$

$$\times \left(1/((1-h/\Delta)^{\nu+3}\Delta^\nu)\right)\cdot\left(1-\sqrt{2e}\,x/(1+\theta\delta/3)\Delta\right)^{-1}\left(|t|/\tilde\sigma(h)\right)^{\nu+2}\times$$

$$\times \sum_{m=0}^{\nu-1} \frac{1}{(m+1)!} C_{\nu-1}^m \left(((l-1)!)^\gamma t^2/((1-h/\Delta)^3\tilde\sigma^2(h))\right)^m \Big\} =$$

$$= \theta\left(((l-1)!)^{\gamma(l-3)}(l-3)^2 2^{5(l-3)}/\Delta^{l-2}\right)\cdot(3\sqrt{2e}\,x/2)^{l-3}\times$$

$$\times\left(\left(|t|/\tilde\sigma(h)\right)^3 + \left(|t|/\tilde\sigma(h)\right)^{3(l-3)}\right).$$

Further, according to (6.28)

$$\tilde\sigma^2(h) = \tilde K_l^{(2)}(h) + \sum_{k=l}^{s} \frac{1}{(k-2)!}\Gamma_k(\xi)h^{k-2} =$$

$$= \tilde K_l^{(2)}(h) + \theta\left((l!)^\gamma l(l-1)h^{l-2}/\Delta^{l-2}\right)\sum_{k=l}^{s}(6\,s^\gamma h/\Delta)^{k-l} = \tag{6.49}$$

$$= \overline{\sigma}^2(h)\left(1+\theta(l!)^\gamma 8\,l(l-1)h^{l-2}/\Delta^{l-2}\right),$$

where

$$\overline{\sigma}^2(h) = \tilde{K}_l^{(2)}(h) = 1 + \theta((l-1)!)^\gamma \sum_{k=3}^{l-1} k(k-1)(h/\Delta)^{k-2} = \\ = 1 + \theta\Big(48\,((l-1)!)^\gamma (h/\Delta)\Big), \qquad 0 \leqslant h \leqslant \Delta/\sqrt{2\,e}. \tag{6.50}$$

Then

$$(1/\overline{\sigma}(h)) = \sum_{k=0}^{\infty} a_k h^k = \sum_{k=0}^{\infty} d_k x^k, \tag{6.51}$$

where $a_k = (1/k!)(1/\overline{\sigma}(h))^{(k)}\big|_{h=0}$. Using Cauchy's formula we have

$$|a_k| \leqslant \sup_{0\leqslant h\leqslant \Delta/\sqrt{2\,e}} (1/\overline{\sigma}(h))\,(\sqrt{2\,e}/\Delta)^k \leqslant 48\,((l-1)!)^\gamma (\sqrt{2\,e}/\Delta)^k.$$

Hence

$$|d_k| \leqslant 48\,((l-1)!)^\gamma (3\sqrt{2\,e}/\Delta)^k,$$

$$(1/\overline{\sigma}(h)) = \sum_{k=0}^{l-3} d_k x^k + \theta\, 96\,((l-1)!)^\gamma (3\sqrt{2\,e}\,x/2\,\Delta)^{l-2}, \qquad x \leqslant 2\,\Delta_s/(3\sqrt{2\,e}).$$

Now it follows from (6.49) that

$$(1/\overline{\sigma}(h)) = \sum_{k=0}^{l-3} d_k x^k + \theta\, 92\,(l!)^\gamma 8\, l(l-1)(3\sqrt{2\,e}\,x/\Delta)^{l-2} \tag{6.52}$$

for $0 \leqslant x \leqslant 2\,\Delta_s/\sqrt{2\,e}$.

Next, in accordance with (2.23) and (2.50) we have

$$g_h(x) = \exp\{hx\}\,\big(1 + \theta\,(\delta/6)^{s+1}\big)\,\big(1 + \theta 2\,(\delta/\sqrt{2})^{s-1}\big)$$

and

$$g_h^{-1}(x)\exp\{hx\} = 1 + \theta\, 3\,(\delta/\sqrt{2})^{s-1} = 1 + \theta\, 5\,\exp\Big\{-\frac{1}{3}\,s\Big\}, \tag{6.53}$$

where s is defined by equality (2.63).

Relations (6.44), (6.48), and (6.53) enable us to conclude that

$$J(h,\,x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{l-3} d_k x^k + \sum_{\nu=1}^{l-3} \sum_{k=0}^{l-3-\nu} \frac{x^k}{2\pi} \int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2}\,t^2\Big\} \times \\ \times \overline{P}_{\nu,\,k}(it)\,dt + \theta_1 u_1(x,\,\Delta)\left(1 + \theta_2 5\,\exp\Big\{-\frac{1}{3}\,s\Big\}\right) \tag{6.54}$$

for $1 \leqslant x \leqslant \min\{\delta a,\ 2\Delta_s/2\sqrt{2e}\}, a = (s/4e)^{1/2},\ \Delta_s = \Delta/\big(6(s+2)^\gamma\big)$. Here

$$u_1(x,\Delta) = \Big(\big((l-1)!\big)^{\gamma(l-3)}(l-3)^2 2^{\gamma(l-3)}\Gamma\big((3l-8)/2\big)/\Delta^{l-2}\Big)\times \\ \times (3\sqrt{2e}\,x/2)^{l-3} + 92\,(l!)^\gamma 8\,l(l-1)(3\sqrt{2e}\,x/2\,\Delta)^{l-2}. \tag{6.55}$$

Hence it follows that

$$\frac{\tilde\varphi(h)}{2\pi\tilde\sigma(h)g_h(x)}\int\limits_{|t|\leqslant\varepsilon}\tilde f_h(t)\,dt = \exp\{-hx+\tilde K(h)\}\times \\ \times\Big\{\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{l-3}d_k x^k + \sum_{\nu=1}^{l-3}\sum_{k=0}^{l-3-\nu}\frac{x^k}{2\pi}\int\limits_{-\infty}^{\infty}e^{-\frac12 t^2}\overline{P}_{\nu,k}(it)\,dt + \theta_1 u_2(x,\Delta) + \\ + \theta_2 c_1(\gamma)\exp\Big\{-\big((\delta_2-\delta)\tilde\sigma(h)\Delta_s\big)^2/(60\,e)\Big\}\Big\}\Big(1+\theta\,5\,e^{-\frac13 s}\Big), \tag{6.56}$$

where

$$u_2(x,\Delta) = u_1(x,\Delta) + (3/2)^{l-3}c_2(l,\gamma)x^{l-3}/\Delta^{l-2}. \tag{6.57}$$

Now estimate the integral $I_1^{(2)}$ from relation (6.20). Relation (2.35) yields

$$|f_h(t)-\tilde f_h(t)| \leqslant l(\delta_2), \qquad l(\delta) := 4\sqrt2\,\delta^{[4a]}/(1-\delta)$$

for $|t|\leqslant\varepsilon,\ \varepsilon = (\delta_2-\delta)\tilde\sigma(h)a$. Choose $\delta_2 = 1 - 1/2\,s^{1/4} + \delta/2\,s^{1/4}$ and note that $\delta < 3\,x/a = 6\sqrt e\,x/\sqrt s$. Then

$$l(\delta_2) \leqslant 16\,s^{1/4}\exp\Big\{-\frac12(1-6\sqrt e\,x/\sqrt s)s^{1/4}\Big\}\Big/(1-6\sqrt e\,x/\sqrt s),$$

where s is defined by equality (2.63). Relation (6.53) implies

$$\frac{\tilde\varphi(h)}{2\pi\tilde\sigma(h)g_h(x)}\int\limits_{|t|\leqslant\varepsilon}|f_h(t)-\tilde f_h(t)|\,dt \leqslant \frac{1}{\sqrt{2\pi}}\exp\{-hx+\ln\tilde\varphi(h)\}\,7\,s^{3/4}\times \\ \times\exp\Big\{-\frac12(1-6\sqrt e\,x/\sqrt s)s^{1/4}\Big\}\Big(1+\theta\,5\,e^{-\frac13 s}\Big). \tag{6.58}$$

From (6.20), (6.54) and (6.53) we get

$$\big(2\pi\tilde\sigma(h)g_h(x)\big)^{-1}\tilde\varphi(h)I_1 = \exp\{-hx+\ln\tilde\varphi(h)\}\,(1+P_{l-3}(x)+ \\ + \theta_1 u_3(x,\Delta) + \theta_2 7\,s^{3/4}\exp\Big\{-\frac12(1-6\sqrt e\,x/\sqrt s)s^{1/4}\Big\}\Big(1+\theta\,5\,e^{-\frac13 s}\Big). \tag{6.59}$$

Here

$$u_3(x,\,\Delta) = u_2(x,\,\Delta) + c_1(\gamma)\exp\left\{-\left((\delta_2-\delta)\tilde{\sigma}(h)\Delta_s\right)^2/(60\,e)\right\}, \tag{6.60}$$

where $u_2(x,\,\Delta)$, $c_1(\gamma)$ are defined by (6.57), (6.60) and

$$1 + P_{l-3}(x) = \sum_{k=0}^{l-3} d_k x^k + \sum_{\nu=1}^{l-3}\ \sum_{k=0}^{l-3-\nu} \frac{x^k}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\overline{P}_{\nu,\,k}(it)\,dt.$$

Hence

$$\begin{aligned} P_{l-3}(x) = \sum_{j=1}^{l-3} A_j x^j + \sum_{r=1}^{l-3} \frac{1}{r!} \sum_{\{q:3\,r\leqslant 2\,q\leqslant l-3+2\,r\}} (-1)^q(2\,q-1)!!\times \\ \times \sum_{j=0}^{s-3+2(r-q)} A_{rj}^{(2\,q)} x^j, \end{aligned} \tag{6.61}$$

where

$$A_j = \sum_{r=1}^{j} (-1)^r \frac{1}{r!}(2\,r-1)!! \sum_{\substack{j_1+\dots+j_r=j\\ j_i\geqslant 1}} \prod_{i=1}^{r} b_{j_i}^{(r)},$$

$$b_j^{(l)} = \frac{1}{l!}\sum_{r=1}^{j}\frac{1}{r!}\Gamma_{l+r}(\xi) \sum_{\substack{j_1+\dots+j_r=j\\ j_i\geqslant 1}} \prod_{i=1}^{r} b_{j_i}, \quad b_0^{(l)} = \frac{1}{l!}\Gamma_l(\xi),$$

$$A_{rk}^{(q)} = \sum_{j=0}^{k} A_{q+1,\,k-j} \sum_{\substack{q_1+\dots+q_r=q\\ q_i\geqslant 3}}\ \sum_{\substack{j_1+\dots+j_r=j\\ j_i\geqslant 0}} \prod_{i=1}^{r} b_{j_i}^{(q_i)},$$

$$A_{rj} = \sum_{\substack{j_1+\dots+j_r=j\\ j_i\geqslant 0}} \prod_{i=1}^{r} A_{j_i}, \quad A_0 = 1.$$

Recall that the coefficients b_k of the expansions $h = \sum_{k=1}^{\infty} b_k x^k$ are found from equations (2.10).

Finally, investigate the integral

$$I_2 = \lim_{\lambda\to\infty}\left(\int_{-\lambda}^{-\varepsilon} + \int_{\varepsilon}^{\lambda}\right)(1-|t|/\lambda)f_h(t)\,dt.$$

In view of (6.15) and (6.53)

$$\begin{aligned}
&\big(2\pi\tilde\sigma(h)g_h(x)\big)^{-1}\tilde\varphi(h)|I_2| \leqslant \big(2\pi\tilde\sigma(h)\big)^{-1}\exp\{-hx+\ln\tilde\varphi(h)\}\times\\
&\times\int\limits_{|t|\geqslant\varepsilon}\left|\tilde\varphi^{-1}(h)\exp\left\{-\frac{itx}{\tilde\sigma(h)}\right\}\int\limits_{-\infty}^{\infty}\exp\left\{\frac{itu}{\tilde\sigma(h)}\right\}g_h(u)p_\xi(u)\,du\right|dt\times\\
&\times\big(1+\theta 3(\delta/\sqrt2)^{s-1}\big) = (1/2\pi)\exp\{-hx+\ln\tilde\varphi(h)\}\times\\
&\times\int\limits_{|t|\geqslant\varepsilon}|f_1^*(t)|\,dt\big(1+\theta 3(\delta/\sqrt2)^{s-1}\big)\big(1+\theta 4(\delta/\sqrt2)\big),
\end{aligned}\tag{6.62}$$

where

$$|f_1^*(t)| = \left|\sum_{k=0}^{s}\frac{1}{k!}h^k\int\limits_{-\infty}^{\infty}u^k\exp\{itu\}p_\xi(u)\,du\right| \leqslant \sum_{k=0}^{s}(3/2)^k\frac{1}{k!}|x|^k|f_\xi^{(k)}(t)|,\tag{6.63}$$

$$f_\xi^{(0)}(t) = f_\xi(t) = \mathbf{E}\exp\{it\xi\}$$

as $\gamma > 0$.

Let the condition (S_γ) with $\gamma = 0$ be satisfied. Then, putting $g_h(u) = \exp\{hu\}$, we have $\Delta_s = \Delta/6$ and $f_h(t) = \tilde f_h(t)$. In this case $p_h(u)$ is defined by (6.4) and $f^*(t)$ by (6.3).

Consider $\tilde L(x) := -\frac12 x^2 - hx + \ln\tilde\varphi(h)$. According to (2.60) we have

$$\tilde L(x) = \sum_{k=3}^{\infty}\lambda_k x^k = \sum_{k=3}^{r(l,\gamma)}\lambda_k x^k + \sum_{k>r}\lambda_k x^k,\tag{6.64}$$

where $\lambda_k = -b_{k-1}/k$, and b_k are the coefficients of the expansions $h = \sum\limits_{k=1}^{\infty} b_k x^k$. Making use of estimate (2.59) we find

$$\begin{aligned}
&\sum_{r<k\leqslant p}\lambda_k x^k = \theta\sum_{r<k\leqslant p}(2/k)(16/\Delta)^{k-2}\big((k+1)!\big)^\gamma x^k = \theta(2/r+1)\times\\
&\times(16/\Delta)^{r-1}\big((r+2)!\big)^\gamma x^{r+1}\sum_{r<k\leqslant p}\Big(16\big((p+1)!\big)^\gamma x/\Delta\Big)^{k-r-1} =\\
&= \theta\,\frac{2\big((r+2)!\big)^\gamma 16^{r-1}}{(r+1)\Big(1-16\big((p+1)!\big)^\gamma x/\Delta\Big)}\,\frac{x^{r+1}}{\Delta^{r-1}} = \theta\, 2\cdot 16^{r-1}\times\\
&\times\big((r+2)!\big)^\gamma c_\gamma^{r+1-l}/(r+1)\Big(1-16\big((p+1)!\big)^\gamma x/\Delta\Big)\cdot(x/\Delta)^l,\\
&l\geqslant 1,\quad r\geqslant(1/\gamma)+l+1,
\end{aligned}\tag{6.65}$$

where c_γ is defined by (2.1). Taking into consideration estimate (2.60) for $\tilde{l}_k$ and observing that $\tilde{l}_{k-3} = \lambda_k$, we obtain

$$\sum_{k>p} \lambda_k x^k = \theta(3/2)\big(x^2/(p+1)\big) \sum_{k=p+1}^{\infty} \big(3\sqrt{2e}\,x/(2\,\Delta_s)\big)^{k-2} =$$

$$= \theta(3/2)\big(x^2/(p+1)\big)\big(3\sqrt{2e}\,x/(2\,\Delta_s)\big)^{p-1}\cdot\big(1-3\sqrt{2e}\,x/(2\,\Delta_s)\big)^{-1} =$$

$$= \theta\big(3\,x^2/(p+1)\big)\big(3\sqrt{2e}\,x/(2\,\Delta_s)\big)^{p-1} =$$

$$= \theta\Big(48\,x^2/\big(e(p+1)\big)\Big)\exp\Big\{-\frac{1}{2}(p+1)\Big\}.$$

Let $p+1 = v(l,\gamma)\ln\Delta$. Then for $v \geqslant 2(l+2)/(1+2\gamma)$

$$\sum_{k>p} \lambda_k x^k = \theta\Big(48\,c_\gamma^2/\big(e(p+1)\big)\Big)\Delta^{-l} = \theta(12\,e\Delta^l)^{-1},$$

and

$$\sum_{r<k\leqslant p} \lambda_k x^k = \theta\, 2\big((r+2)!\big)^\gamma 16^{r-1} c_\gamma^{r+1-l} \times$$

$$\times\Big((r+1)\Big(1-16\big((p+1)!\big)^\gamma x/\Delta\Big)\Big)^{-1}(x/\Delta)^l = \tag{6.66}$$

$$= \theta\, 9\big((r+2)!\big)^\gamma 16^{r-1} c_\gamma^{r+1-l}(r+1)^{-1}(x/\Delta)^l,$$

as

$$16\big((p+1)!\big)^\gamma x/\Delta \leqslant (4/3)(\sqrt{2}/6)^{1/(1+2\gamma)}\Big(2\big(l+2/(1+2\gamma)\big)\ln\Delta/\Delta^{2/(1+2\gamma)}\Big)^\gamma \leqslant$$

$$\leqslant (4/3)(\sqrt{2}/6)^{1/(1+2\gamma)}\Big(\Big(2\big(l(1+2\gamma)\big)^2 + 4\,l(1+2\gamma)\Big)/e^{2/(1+2\gamma)}\Big)^\gamma \leqslant$$

$$\leqslant (4/3)(\sqrt{2}/6)^{1/(1+2\gamma)}(6/e^2)^\gamma \leqslant 10/13$$

for $l \geqslant 1$, $\gamma \geqslant 0$ and $\Delta \geqslant \exp\{l(1+2\gamma)^2\}$.

Further,

$$16^{r-1}9\big((r+2)!\big)^\gamma c_\gamma^{r+1-l}(r+1)^{-1}(x/\Delta)^l \leqslant$$

$$\leqslant 9\cdot 16^{l+1/\gamma} c_\gamma^{l+2+1/\gamma}\cdot\big((l+2+1/\gamma)!\big)^\gamma\big((l+2+1/\gamma)\Delta^{2\,l\gamma/(1+2\gamma)}\big)^{-1} \leqslant$$

$$\leqslant 9\big(16(c_1+1/\gamma)\big)^{-1}(\sqrt{2}/6)^{(c_1+1/\gamma)/(1+2\gamma)}\times$$

$$\times(c_1+1/\gamma)^{1+c_1\gamma}\exp\{-2\,l^2\gamma(1+2\gamma)\} \leqslant$$

$$\leqslant (3/16)(\sqrt{2}/6)^{(c_1+1/\gamma)/(1+2\gamma)}(c_1+1/\gamma)^{c_1\gamma}\exp\{-c_1^2\gamma(1+2\gamma)/2\} \leqslant 1/2,$$

where $c_1 = l+2$.

To obtain the last estimate one has to consider the cases: 1) $0 \leqslant \gamma < 1$ and 2) $\gamma \geqslant 1$. In the first case the possibilities $c_1\gamma \geqslant 1$ and $0 \leqslant c_1\gamma < 1$ should be considered separately. On the basis of (6.64) – (6.66) we get

$$\exp\{\tilde{L}(x)\} = \exp\{L_m(x)\Big(1+\theta\, 9((r+2)!)^{\gamma}16^{m-1}c_{\gamma}^{r+1-m}(m+1)^{-1}(x/\Delta)\Big), \quad (6.67)$$

where $L_m(x)$ is defined by equality (6.2).

It ought to be noted that the polynomial $P_{l-3}(x)$ can be represented in the form

$$P_{l-3}(x) = \sum_{\nu=1}^{l-3} M_\nu(x),$$

where explicit expressions for the polynomials $M_\nu(x)$ are found from the following formal equalities:

$$p_\xi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2 + \frac{x^3}{\Delta_s}\lambda\left(\frac{x}{\Delta_s}\right)\right\} \sum_{\nu=0}^{\infty} M_\nu(x), \qquad M_0(x) \equiv 0,$$

and

$$p_\xi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} \sum_{\nu=0}^{\infty} q_\nu(x), \qquad q_0(x) \equiv 0,$$

where the polynomials $q_\nu(x)$ are defined by (6.12). Using the expansions

$$\exp\left\{-\frac{x^3}{\Delta_s}\lambda\left(\frac{x}{\Delta_s}\right)\right\} = \exp\left\{-\frac{x^3}{\Delta_s}\sum_{k=0}^{\infty}\lambda_k\left(\frac{x}{\Delta_s}\right)^k\right\}$$

$$= \exp\left\{\sum_{k=1}^{\infty} -\lambda_{k-1}x^{k+2}/\Delta_s^k\right\} = \sum_{\nu=0}^{\infty} K_\nu(x)/\Delta_s^\nu,$$

where

$$K_\nu(x) = \sum \prod_{m=1}^{\nu} \frac{1}{k_m!}(-\lambda_{m-1}x^{m+2})^{k_m},$$

$\nu = 1, 2, \ldots$, and the summation is taken over all integer nonnegative solutions $k_1, k_1, \ldots, k_\nu$ of the equation $k_1 + 2k_2 + \ldots + \nu k_\nu = \nu$, we have

$$M_\nu(x) = \sum_{k=0}^{\nu} K_k(x) q_{\nu-k}(x). \qquad (6.68)$$

From (6.18), (6.59) and (6.62) we obtain the assertion of the lemma. ■

Proof of Lemma 6.2. Since $p_\xi(x) \in L_1$, from (6.16) we have $\forall x \in \mathcal{E}$

$$p_\xi(x) = \frac{1}{2\pi} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} e^{-itx}(1 - |t|/\lambda) f_\xi(t)\, dt.$$

Then $\forall x \in \mathcal{E}$

$$\begin{aligned} p_\xi(x) - g(x) &= \frac{1}{2\pi} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} e^{-itx}(1 - |t|/\lambda)\Big(f_\xi(t) - \exp\Big\{ -\frac{1}{2} t^2 \Big\}\Big)\, dt = \\ &= \frac{1}{2\pi}(I_1 + I_2), \end{aligned} \tag{6.69}$$

where

$$I_1 = \lim_{\lambda\to\infty} \int_{-\varepsilon}^{\varepsilon} e^{-itx}(1 - |t|/\lambda)\Big(f_\xi(t) - \exp\Big\{ -\frac{1}{2} t^2 \Big\}\Big)\, dt,$$

$$I_2 = \lim_{\lambda\to\infty} \Big(\int_{-\lambda}^{-\varepsilon} + \int_{\varepsilon}^{\lambda} \Big)(1 - |t|/\lambda)\Big(f_\xi(t) - \exp\Big\{ -\frac{1}{2} t^2 \Big\}\Big)\, dt,$$

$\varepsilon = T_2 := (1 - 1/2\, s^{1/4})(s/4\, e)^{1/2}$, s is defined by equality (2.68). Making use of estimate (2.74) we find

$$\begin{aligned} I_1 &\leqslant 2 \int_0^{T_2} \Big| f_\xi(t) - \exp\Big\{ -\frac{1}{2} t^2 \Big\}\Big|\, dt \leqslant \\ &\leqslant 2 \int_0^{t_0} \Big| f_\xi(t) - \exp\Big\{ -\frac{1}{2} t^2 \Big\}\Big|\, dt + 2 \int_{t_0}^{T_2} \Big| f_\xi(t) - \exp\Big\{ -\frac{1}{2} t^2 \Big\}\Big|\, dt \leqslant \\ &\leqslant 2 \int_0^{t_0} t^2\, dt + \frac{2 \cdot 6^\gamma}{\Delta} \int_{t_0}^{T_2} |t| e^{-\frac{1}{4} t^2}\, dt + 2\, l(\delta_2) \int_{t_0}^{T_2} dt \leqslant \\ &\leqslant 2\, l(\delta_2)(1 + T_2 - t_0) + 16 \cdot 6^\gamma/\Delta \leqslant \\ &\leqslant (64\sqrt{2}\, s^{1/4}/3) \exp\{-(1/\sqrt{e}) s^{1/4}\}\, (1 + \sqrt{s/4\, e} - (64\sqrt{2}/3)^{1/2} \cdot s^{1/8} \times \\ &\times \exp\{-(1/2\sqrt{e}) s^{1/4}\}) + 16 \cdot 6^\gamma/\Delta \leqslant 364\, \Delta_\gamma^{3/2} e^{-\frac{3}{2}\sqrt{\Delta_\gamma}} + 16 \cdot 6^\gamma/\Delta, \end{aligned} \tag{6.70}$$

where $t_0 = \min\big\{ \big(l(\delta_2)\big)^{1/2}, 1 \big\}$, $l(\delta)$ and Δ_γ are defined by equalities (2.75) and (2.1). Hence and from (6.69) follows the assertion of Lemma 6.2. ■

6.2. Estimates for characteristic functions

Let for a r.v. ξ with $\mathbf{E}\xi = 0$, $\sigma^2 = \mathbf{E}\xi^2 < \infty$ and the distribution function $F_\xi(x)$ there exist density $p_\xi(x)$, and $p_{\tilde\xi}(x)$ is the density of a symmetrized r.v. $\tilde\xi = \xi - \xi'$, where ξ' is independent of ξ and has the same distribution as ξ.

LEMMA 6.3. *Let ξ be an arbitrary r.v. with density $p_\xi(x)$. Then for any collection $\mathfrak{M} = \{\Delta_i,\ C_i,\ i = 1, 2, \ldots\}$ of nonoverlapping intervals Δ_i and positive constants $C_i \leqslant \infty$ for any $-\infty < t < \infty$ the estimate*

$$|f_\xi(t)| \leqslant \exp\left\{ -\frac{t^2}{3} \sum_{i=1}^{\infty} \frac{Q_i^3}{(|\Delta_i|\,|t| + 2\pi)^2 C_i^2} \right\} \tag{6.71}$$

holds, where

$$Q_i = \int_{\Delta_i} \min\{C_i,\ p_{\tilde\xi}(x)\}\, dx. \tag{6.72}$$

COROLLARY. *If $p_\xi(x) \leqslant C < \infty$ and $\sigma^2 = \mathbf{E}\xi^2 < \infty$, then*

$$|f_\xi(t)| \leqslant \exp\left\{ -\frac{t^2}{96} \cdot \frac{1}{(2\sigma|t| + \pi)^2 C^2} \right\} \tag{6.73}$$

for all $-\infty < t < \infty$.

Proof. We have

$$|f_\xi(2\pi t)| \leqslant \exp\{-I(t)\}, \tag{6.74}$$

where

$$I(t) = \frac{1}{2}\left(1 - |f_\xi(2\pi t)|^2\right) = \int_{-\infty}^{\infty} \sin^2(\pi t x) p_{\tilde\xi}(x)\, dx.$$

As $|\sin \pi\alpha| \geqslant 2(\alpha)$, where (α) denotes the distance of number α to the nearest integer, we write

$$I(t) \geqslant 4 \sum_{i=1}^{\infty} I_{\Delta_i}(t), \tag{6.75}$$

where

$$I_{\Delta_i}(t) = \int_{\Delta_i} (xt)^2 q_i(x)\, dx = t^2 \sum_{r=-\infty}^{\infty} \int_{|x| \leqslant 1/2|t|} x^2 q_i(x + r/|t|)\, dx. \tag{6.76}$$

Here

$$q_i(x) = \begin{cases} \min\{C_i,\, p_\xi(x)\} & \text{as } x \in \Delta_i, \\ 0 & \text{as } x \notin \Delta_i. \end{cases}$$

For fixed t the value of the sum (6.76) depends on the function $q_i(x)$. It is minimal if $q_i(x)$ is the indicator function of the interval $[(r/|t|) - d,\, (r/|t|) + d]$ multiplied by C_i. Thus

$$I_{\Delta_i}(t) \geqslant t^2(R+1)\int_{-d}^{d} x^2 C_i\,dx = \frac{2}{3}\,t^2 d^3 C_i(R+1) = \frac{1}{3}\,t^2 d^2 Q_i,$$

because $2\,d(R+1)C_i = Q_i$. Here R is the number of integer points r which satisfy the condition $r/|t| \in \Delta_i$. In view of the fact that $R \leqslant |t|\,|\Delta_i|$ we have

$$d \geqslant Q_i\big(2(|t|\,|\Delta_i| + 1)C_i\big)^{-1}$$

and

$$I_{\Delta_i}(t) \geqslant (1/12)t^2 Q_i^3\big((|t|\,|\Delta_i| + 1)^2 C_i^2\big)^{-1},$$

or

$$I(t) \geqslant \frac{1}{3}\,t^2 \sum_{i=1}^{\infty} Q_i^3\big((|\Delta_i|\,|t| + 1)^2 C_i^2\big)^{-1}. \tag{6.77}$$

Substituting t for $2\pi t$, from (6.74) and (6.77) we obtain the assertion of the lemma.

Relation (6.73) follows easily from (6.71), if we put $\mathfrak{M} = \{\Delta,\, C\}$, $\Delta =]-2\sigma,\, 2\sigma\,[$. Then the corresponding $Q \geqslant 1/2$. ■

LEMMA 6.4. *Let a nonnegative function $g(t)$, defined on the interval $[a, \infty\,[$, satisfy the Lipschitz condition*

$$|g(t+s) - g(t)| \leqslant K|s|. \tag{6.78}$$

Moreover, let

$$V := \int_a^{\infty} g(t)\,dt < \infty.$$

Then for any $\varepsilon > 0$ and any partition

$$a = t_0 < t_1 < t_2 < \ldots$$

of the interval $[a, \infty[$ *with* $\max\limits_{0 \leqslant k < \infty} (t_{k+1} - t_k) \leqslant \varepsilon$ *we have the inequality*

$$\sum_{k=0}^{\infty} \big(\max_{t_k \leqslant t \leqslant t_{k+1}} g^2(t) \big) \Delta t_k \leqslant V\big(2K\varepsilon + 4 \sup_{a \leqslant t < \infty} g(t)\big), \tag{6.79}$$

where $\Delta t_k = t_{k+1} - t_k$.

Proof. Set $Q_k = \int\limits_{t_k}^{t_{k+1}} g(t)\, dt$ and define the function

$$\psi(t) = \max\{0,\ h_k - K(t - t_k)\} \tag{6.80}$$

for $t_k \leqslant t \leqslant t_{k+1}$ in the interval $[a, \infty[$, where

$$h_k = \begin{cases} \sqrt{2KQ_k} & \text{if } k \in \mathcal{R}_1, \\ (Q_k + K(\Delta t_k)^2/2)/\Delta t_k & \text{if } k \in \mathcal{R}_2, \end{cases}$$

and

$$\mathcal{R}_1 = \{k = 0, 1, 2, \dots : Q_k \leqslant (K/2)(\Delta t_k)^2\},$$
$$\mathcal{R}_2 = \{k = 0, 1, 2, \dots : Q_k > (K/2)(\Delta t_k)^2\}.$$

By definition of the function $\psi(t)$,

$$\int\limits_{t_k}^{t_{k+1}} \psi(t)\, dt = \int\limits_{t_k}^{t_{k+1}} g(t)\, dt = Q_k \tag{6.81}$$

and

$$|\psi(t+s) - \psi(t)| = K|s|, \tag{6.82}$$

whenever t and $t+s$ belong to that part of the interval $[t_k, t_{k+1}]$ where $\psi(t) > 0$. Consequently, it follows from (6.78) – (6.82) that

$$\max_{t_k \leqslant t \leqslant t_{k+1}} g(t) \leqslant \max_{t_k \leqslant t \leqslant t_{k+1}} \psi(t) = \psi(t_k) = h_k.$$

Therefore,

$$\sum_{k=0}^{\infty} \big(\max_{t_k \leqslant t \leqslant t_{k+1}} g^2(t) \big) \Delta t_k \leqslant \sum_{k=0}^{\infty} h_k^2 \Delta t_k \leqslant 2K \sum_{k \in \mathcal{R}_1} Q_k \Delta t_k +$$

$$+ 4 \sum_{k \in \mathcal{R}_2} Q_k^2 / \Delta t_k \leqslant 2K\varepsilon V + 4V \sup_{a \leqslant t < \infty} g(t) = V\big(2K\varepsilon + 4 \sup_{a \leqslant t < \infty} g(t)\big)$$

as $Q_k \leqslant \Delta t_k \sup\limits_{a\leqslant t<\infty} g(t)$ and $h_k \leqslant 2\,Q_k/\Delta t_k$ for $k \in \mathcal{R}_2$. ∎

Now let

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots \tag{6.83}$$

be a sequence of independent r.v. with $\mathbf{E}\xi_j = 0$ and finite variances $\sigma_j^2 = \mathbf{E}\xi_j^2$, $j = 1, 2, \ldots$. Denote

$$S_n = \sum_{j=1}^{n} \xi_j, \quad B_n^2 = \sum_{j=1}^{n} \sigma_j^2, \quad Z_n = S_n/B_n,$$

$$l_n(N_n) = \frac{1}{B_n^2} \sum_{j=1}^{n} \int\limits_{|x|\leqslant N_n} x^2\, dF_{\tilde{\xi}_j}(x), \quad N_n > 0. \tag{6.84}$$

LEMMA 6.5. *If* $\mathbf{E}|\xi_j|^3 < \infty$, $j = 1, 2, \ldots$, *and* $l = \varliminf\limits_{n\to\infty} l_n(N_n) > 0$, *then*

$$|f_{Z_n}(t)| \leqslant \exp\left\{-\frac{l}{\pi^2}t^2\right\} \quad \text{for } |t| \leqslant \pi B_n/N_n. \tag{6.85}$$

COROLLARY. *If* $\mathbf{E}|\xi_j|^3 < \infty$, $j = 1, 2, \ldots, n$, *then*

$$|f_{Z_n}(t)| \leqslant \exp\left\{-\frac{1}{\pi^2}t^2\right\} \quad \text{for } |t| \leqslant (\pi/4)L_{3,n}^{-1}.$$

Proof. According to (6.74) we have

$$|f_{\xi_j}(t)| \leqslant \exp\{-I_j(t/2\pi)\}, \tag{6.86}$$

where

$$I_j(t) = \frac{1}{2}\left(1 - |f_{\xi_j}(2\pi t)|^2\right) = \int\limits_{-\infty}^{\infty} \sin^2(\pi t x)\, dF_{\tilde{\xi}_j}(x).$$

Hence

$$|f_{S_n}(t)| \leqslant \exp\{-I_n(t/2\pi)\}, \tag{6.87}$$

where

$$\begin{aligned} I_n(t) &= \sum_{j=1}^{n} \int\limits_{-\infty}^{\infty} \sin^2(\pi t x)\, dF_{\tilde{\xi}_j}(x) \geqslant \\ &\geqslant 4t^2 \sum_{j=1}^{n} \int\limits_{|x|\leqslant 1/2|t|} x^2\, dF_{\tilde{\xi}_j}(x) = 4t^2 B_n^2 l_n(1/2|t|). \end{aligned} \tag{6.88}$$

Recalling that $l = \varliminf_{n\to\infty} l_n(N_n) > 0$, we get

$$|f_{S_n}(t)| \leqslant \exp\{-(l/\pi^2)t^2 B_n^2\} \quad \text{for } |t| \leqslant \pi/N_n,$$

or

$$|f_{Z_n}(t)| \leqslant \exp\{-(l/\pi^2)t^2\} \quad \text{for } |t| \leqslant \pi B_n/N_n.$$

According to the definition of the function $l_n(N_n)$ by (6.84) we have

$$l_n(N_n) \geqslant 2(1 - 2\,B_n L_{3,\,n}/N_n). \tag{6.89}$$

Put $N_n = 4\,B_n L_{3,\,n}$, then $l_n(N_n) \geqslant 1$. Consequently,

$$|f_{Z_n}(t)| \leqslant \exp\{-t^2/\pi^2\} \quad \text{for } |t| \leqslant (\pi/4)L_{3,\,n}^{-1}. \qquad \blacksquare$$

Further, assume that the distribution of the r.v. ξ_j has density $p_{\xi_j}(x)$. By $\mathfrak{M}_k = \{\Delta_{ki},\ C_{ki},\ i = 1,\ 2,\ \ldots\}$ denote a collection of nonoverlapping intervals Δ_{ki} of the length $|\Delta_{ki}|$ and positive constants $C_{ki} \leqslant \infty$, $k = 1,\ 2,\ \ldots$. Let

$$\alpha(\mathfrak{M}_k,\ N) = \sum_{i=1}^{\infty} \mathcal{Q}_{k1}^3\left((|\Delta_{ki}| + 2\,N)^2 C_{ki}^2\right)^{-1}, \tag{6.90}$$

where $N > 0$ and

$$\mathcal{Q}_{ki} = \int_{\Delta_{ki}} \min\{C_{ki},\ p_{\tilde{\xi}_k}(x)\}\,dx.$$

Lemma 6.6. *If for a r.v. ξ_j with $\mathrm{E}\xi_j = 0$ and $\sigma_j^2 = \mathrm{E}\xi_j^2 < \infty$, $j = 1,\ 2,\ \ldots,\ n$, there exists density $p_{\xi_j}(x)$, then*

$$|f_{Z_n}(t)| \leqslant \exp\{-M_n\} \quad \text{for } |t| \geqslant \pi/N_n,$$

where

$$M_n = \frac{1}{3}\sum_{k=1}^{n} \alpha(\mathfrak{M}_k,\ N_n). \tag{6.91}$$

Moreover, if $p_{\xi_j}(x) \leqslant C_j \leqslant \infty$, $j = 1,\ 2,\ \ldots,\ n$, then

$$\alpha(\mathfrak{M}_k,\ N_n) \geqslant (1/256)\left((\sigma_k^2 + N_n^2)C_k^2\right)^{-1}. \tag{6.92}$$

Proof. Lemma 6.3 implies

$$I_n(t) \geqslant \frac{1}{3}t^2 \sum_{k=1}^{n} \alpha\big(\mathfrak{M}_k,\ 1/(2|t|)\big) \tag{6.93}$$

for all $-\infty < t < \infty$. The minimal value of the right-hand side of (6.93) in the interval $|t| \geqslant 1/(2N_n)$ equals M_n, where M_n is defined by (6.91). Taking into consideration (6.87) we find

$$|f_{S_n}(t)| \leqslant \exp\{-M_n\} \quad \text{for } |t| \geqslant \pi/N_n,$$

or

$$|f_{Z_n}(t)| \leqslant \exp\{-M_n\} \quad \text{for } |t| \geqslant \pi B_n/N_n.$$

In addition, if $p_{\xi_j}(x) \leqslant C_j,\ j = 1, 2, \ldots$, then with

$$\mathfrak{M}_k = \{\Delta_{k1}, C_k\}, \quad \Delta_{k1} =]-2\sigma_k, 2\sigma_k[,$$

we obtain that

$$\alpha(\mathfrak{M}_k,\ N_n) \geqslant \mathcal{Q}_{k1}^3\left((4\sigma_k + N_n)^2 C_k^2\right)^{-1} \geqslant (1/256)\left((\sigma_k^2 + N_n^2)C_k^2\right)^{-1}, \tag{6.94}$$

because

$$\mathcal{Q}_{k1} = \int_{-2\sigma_k}^{2\sigma_k} p_{\tilde{\xi}_1}(x)\,dx \geqslant \frac{1}{2}. \qquad \blacksquare$$

Next we find for $f_{Z_n}(t)$ an estimate depending on t, which enables us to evaluate $f_{Z_n}(t)$ for $|t| \geqslant \pi B_n/N_n$.

Since $|\sin \pi\alpha| \geqslant 2(\alpha)$, according to (6.88)

$$I_n(t) \geqslant 4\,J_n(t), \tag{6.95}$$

where

$$J_n(t) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} (xy)^2 p_{\tilde{\xi}_k}(x)\,dx. \tag{6.96}$$

LEMMA 6.7. *For any $n \geqslant 1$ and $N_n > 0$ there exists a partition*

$$\ldots < t_{-1}^{(n)} < t_0^{(n)} = 0 < t_1^{(n)} < t_2^{(n)} < \ldots \tag{6.97}$$

of the interval $]-\infty, \infty[$ satisfying the condition

$$\frac{1}{6\,N_n} \leqslant t_{k+1}^{(n)} - t_k^{(n)} \leqslant \frac{1}{4\,N_n}, \tag{6.98}$$

such that

$$J_n(t) \geqslant \frac{1}{2}\, l_n(N_n)(t - t_{k0}^{(n)})^2 B_n^2 \tag{6.99}$$

if $t \in [t_k^{(n)}, t_{k+1}^{(n)}]$, *where for a given* n, $t_{k0}^{(n)}$ *equals either* $t_k^{(n)}$ *or* $t_{k+1}^{(n)}$ *depending on* t .

Proof. Let t_0 be an arbitrary value of t. We set

$$A^+(t_0) = \bigcup_{r=-\infty}^{\infty} \{x : r \leqslant xt_0 < r + 1/2,\ 0 \leqslant x \leqslant N_n\},$$

$$A^-(t_0) = \bigcup_{r=-\infty}^{\infty} \{x : r - 1/2 \leqslant xt_0 < r,\ 0 \leqslant x \leqslant N_n\},$$

$$D_n^+(t_0) = \sum_{k=1}^{n} \int_{A^+(t_0)} x^2\, dF_{\tilde{\xi}_k}(x),$$

$$D_n^-(t_0) = \sum_{k=1}^{n} \int_{A^-(t_0)} x^2\, dF_{\tilde{\xi}_k}(x).$$

Since $A^+(t_0) + A^-(t_0) = [0, N_n]$, due to the symmetry of $p_{\tilde{\xi}_k}(x)$

$$D_n^+(t_0) + D_n^-(t_0) = \frac{1}{2}\, l_n(N_n) B_n^2. \tag{6.100}$$

If $x \in A^+(t_0)$, for example, $r \leqslant xt_0 < r + 1/2$, $t \in [t_0, t_0 + \delta_n]$, $\delta_n = 1/(4N_n)$, then

$$(xt) \begin{cases} = (xt_0) + (t - t_0)x & \text{as } r \leqslant xt_0 \leqslant r + \dfrac{1}{2} - (t - t_0)x, \\ \geqslant \dfrac{1}{2} - (t - t_0)x & \text{as } r + \dfrac{1}{2} - (t - t_0)x \leqslant xt_0 \leqslant r + \dfrac{1}{2}. \end{cases}$$

In this case

$$(xt) \geqslant (t - t_0)x. \tag{6.101}$$

For $x \in A^-(t_0)$, $t \in [t_0 - \delta_n, t_0]$ analogously

$$(xt) \geqslant (t_0 - t)x. \tag{6.102}$$

From (6.101), (6.102) together with (6.96) and (6.100) it follows

$$J_n(t) \geqslant 2(t - t_0)^2 D_n^+(t_0) \quad \text{as } t \in [t_0, t_0 + \delta_n] \tag{6.103}$$

and

$$J_n(t) \geqslant 2(t_0 - t)^2 D_n^-(t_0) \quad \text{as } t \in [t_0 - \delta_n, t_0]. \tag{6.104}$$

Now construct the required partition (6.97). Let $\tau_k^{(n)} = 2\,k\delta_n/3,\ k = 0, \pm 1, \pm 2, \dots$. If for each pair $(k,\ k+1),\ 0 \leqslant k \leqslant m$, at least one of the relations

$$D_n^+(\tau_k^{(n)}) \geqslant D_n^-(\tau_k^{(n)}) \quad \text{or } D_n^+(\tau_k^{(n)}) \leqslant D_n^-(\tau_k^{(n)}), \tag{6.105}$$

is fulfilled, then we put $t_k^{(n)} = \tau_k^{(n)},\ k = 1, 2, \dots, m$. Moreover, if for the pair $k,\ k+1$ the first inequality is fulfilled, we assume $t_{k0}^{(n)} = t_k^{(n)}$, and if the second, then $t_{k0}^{(n)} = t_{k+1}^{(n)}$. If for $k = m+1$ and $k = m+2$ both inequalities (6.105) are violated, then by virtue of continuity of $D_n^+(t)$ and $D_n^-(t)$ there exists a point $\tau_{m+1,0}^{(n)} \in]\,\tau_{m+1}^{(n)},\ \tau_{m+2}^{(n)}\,[$ such that

$$D_n^+(\tau_{m+1,\,0}^{(n)}) = D_n^-(\tau_{m+1,0}^{(n)}). \tag{6.106}$$

In the case $\tau_{m+1,0}^{(n)} < \tau_{m+1}^{(n)} + \delta_n/3$ we put $t_{m+1}^{(n)} = \tau_{m+1,0}^{(n)},\ t_{m+2}^{(n)} = \tau_{m+2}^{(n)}$ and in the case $\tau_{m+1,0}^{(n)} \geqslant \tau_{m+1}^{(n)} + \delta_n/3$ we put $t_{m+1}^{(n)} = \tau_{m+1}^{(n)},\ t_{m+2}^{(n)} = \tau_{m+1,0}^{(n)},\ t_{m+3}^{(n)} = \tau_{m+3}^{(n)}$. Due to (6.106) at least one inequality (6.105) is fulfilled for the formed pairs, therefore, we determine the points $t_{k0}^{(n)}$ just like for the previous pairs.

For $k = -1, -2, \dots$, the partition is defined trivially due to the symmetry of $J_n(t)$.

Obviously $2\delta_n/3 \leqslant t_{k+1}^{(n)} - t_k^{(n)} < \delta_n$, so that (6.98) is satisfied. According to the choice of $t_{k0}^{(n)}$, from (6.103) and (6.104) for $t \in [t_k^{(n)},\ t_{k+1}^{(n)}]$ we have

$$J_n(t) \geqslant 2(t - t_{k0}^{(n)})^2 \max\{D_n^+(t_{k0}^{(n)}),\ D_n^-(t_{k0}^{(n)})\},$$

whence, taking into consideration (6.100), we obtain (6.99). ∎

6.3. Asymptotic expansion in the Cramer zone for distribution density of sums of independent random variables

Suppose that for a r.v. ξ_j with $\mathbf{E}\xi_j = 0$ and $\sigma_j^2 = \mathbf{E}\xi_j^2 < \infty,\ j = \overline{1,n}$, there exist densities $p_{\xi_j}(x)$ such that

$$\sup_x p_{\xi_j}(x) \leqslant C_j \leqslant \infty. \tag{D}$$

In the case ξ_j has no density, then $C_j = \infty$ by definition.

Moreover, let there exist a quantity $K > 0$ such that

$$|\mathbf{E}\xi_j^k| \leqslant k!\,K^{k-2}\sigma_j^2, \qquad k = 3, 4, \dots . \tag{B_0}$$

Put

$$K_n = 2\max\{K, \max_{1\leqslant j\leqslant n} \sigma_j\}, \quad \Delta_{0,n} = \frac{B_n}{24 K_n}.$$

THEOREM 6.1. *If for a r.v. ξ_j conditions (B_0) and (D) are satisfied, then $\forall l \geqslant 1$ in the interval*

$$0 \leqslant x < \Delta_{0,n}$$

the relation of large deviations

$$\begin{aligned}\frac{p_{Z_n}(x)}{g(x)} = \exp\{L_n(x)\}\Bigg(1 + \sum_{\nu=1}^{l-1} M_{\nu,n}(x) + \theta_1 q(l)\left(\frac{x+1}{\Delta_{0,n}}\right)^l + \\ + \theta_2\, 825\pi^2 \exp\left\{-\frac{1}{72}\left(1 - \frac{x}{\Delta_{0,n}}\right)^2 \Delta_{0,n}^2\right\} + \\ + \theta_3\, 342\, e^4 \sqrt{2\pi}\, K_n \prod_{i=1}^{4} C_i^{1/4} \exp\left\{-\frac{c}{K_n^2}\sum_{j=1}^{n}\frac{1}{C_j^2}\right\}\Bigg)\end{aligned}$$

holds. Here $0 < c < (1/729)\cdot 10^{-3}$,

$$L_n(x) = \sum_{k=3}^{\infty} \lambda_{k,n} x^k$$

is the Cramer – Petrov series, where the coefficients $\lambda_{k,n}$ are expressed in terms of cumulants of the r.v. Z_n and are found from the recurrent equations (2.10), being

$$|\lambda_{k,n}| \leqslant \frac{2}{k}\left(\frac{16}{\Delta_{0,n}}\right)^{k-2}, \qquad k = 3, 4, \dots;$$

$q(l) = q(l, 0)$, and $q(l, \gamma)$ is defined by (6.9). For the polynomials $M_{\nu,n}(x)$ (6.8) is valid, with cumulants of Z_n instead of the cumulants of the r.v. ξ.

In particular,

$$\begin{aligned}M_{1,n}(x) &= -\frac{1}{2}\Gamma_3(Z_n)x,\\ M_{2,n}(x) &= \frac{1}{8}\left(5\Gamma_3^2(Z_n) - 2\Gamma_4(Z_n)\right)x^2 + \frac{1}{24}\left(3\Gamma_4(Z_n) - 5\Gamma_3^2(Z_n)\right).\end{aligned}$$

Proof. Let condition (B) be satisfied. Lemma 2.1 implies

$$|\Gamma_k(\xi_j)| \leqslant k!\, K^{k-2}\sigma_j^2, \qquad k = 3, 4, \dots, \tag{6.107}$$

where $K_n = 2\max\{K, \max_{1\leqslant j\leqslant n} \sigma_j\}$. Hence

$$|\Gamma_k(\xi_j)| \leqslant k!\,/\Delta_n^{k-2}, \qquad \Delta_n = B_n/K_n. \tag{6.108}$$

Let $\eta_j = \xi_j/B_n$, then $\Gamma_k(\eta_j) = \Gamma_k(\xi_j)/B_n^k$. Consequently,

$$\begin{aligned}\tilde\varphi_{\eta_j}(z) = \tilde\varphi_{\xi_j}(z/B_n) &:= \exp\left\{\sum_{k=2}^{\infty}\frac{1}{k!}\Gamma_k(\xi_j)(z/B_n)^k\right\} = \\ &= \exp\left\{\frac{1}{2}\,\sigma_j^2(z/B_n)^2\left(1+\theta\,2\sum_{k=3}^{\infty}(|z|K_n/B_n)^{k-2}\right)\right\} = \\ &= \exp\left\{\frac{1}{2}\,\sigma_j^2(|z|/B_n)^2\left(1+\theta(1/4)\right)\right\}, \qquad |z|\leqslant \Delta_n/9.\end{aligned} \tag{6.109}$$

Hence,

$$\exp\left\{\frac{3}{8}\,\sigma_j^2|z|^2\right\} \leqslant |\tilde\varphi_{\xi_j}(z)| \leqslant \exp\left\{\frac{5}{8}\,\sigma_j^2|z|^2\right\} \tag{6.110}$$

for $|z|\leqslant A_n$, $A_n = \Delta_n/9$.

Let $\xi_j(h)$, $j = 1, 2, \ldots, n$, be r.v. with the distribution density

$$p_{\xi_j(h)}(y) := \frac{e^{\,hy}p_\xi(y)}{\int\limits_{-\infty}^{\infty} e^{\,hy}p_{\xi_j(y)}\,dy} \tag{6.111}$$

and the characteristic function $f_{\xi_j(h)}(t) = \mathbf{E}\exp\{it\xi_j(h)\}$. Then

$$\begin{aligned} m_j(h) :&= \mathbf{E}\xi_j(h) = \frac{d}{dh}\ln\tilde\varphi_{\xi_j}(h) = \sum_{k=2}^{\infty}\frac{1}{(k-1)!}\Gamma_k(\xi_j)h^{k-1}, \\ \sigma_j^2(h) :&= \mathbf{D}\xi_j(h) = \frac{d^2}{dh^2}\ln\tilde\varphi_{\xi_j}(h) = \sum_{k=2}^{\infty}\frac{1}{(k-2)!}\Gamma_k(\xi_j)h^{k-2},\end{aligned}$$

and the k^{th} order cumulant of the r.v. $\xi_j(h)$ is

$$\Gamma_k\big(\xi_j(h)\big) = \frac{d^k}{dh^k}\ln\tilde\varphi_{\xi_j}(h) = \sum_{l=k}^{\infty}\frac{1}{(l-k)!}\Gamma_l(\xi_j)h^{l-k}. \tag{6.112}$$

Put

$$S_n(h) = \sum_{j=1}^{n} \xi_j(h), \qquad B_n^2(h) = \sum_{j=1}^{n} \sigma_j^2(h),$$

$$M_n(h) = \mathbf{E} S_n(h) = \sum_{j=1}^{n} m_j(h), \qquad Z_n(h) = \big(S_n(h) - M_n(h)\big)/B_n(h),$$

$$L_{k,n}(h) = \sum_{j=1}^{n} \mathbf{E}|\xi_j(h) - m_j(h)|^k / B_n^k(h).$$

The quantity h is determined from the equation

$$x = \frac{M_n(h)}{B_n} = \frac{1}{B_n} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \Gamma_k(S_n) h^{k-1}. \tag{6.113}$$

If we put the conjugate r.v. $\xi_j(h)$ instead of ξ_j in relation (6.84), then

$$\begin{aligned} l_n\big(N_n(h)\big) &= \frac{1}{B_n^2(h)} \sum_{j=1}^{n} \Bigg(\int_{-\infty}^{\infty} x^2\, dF_{\bar{\xi}_j(h)}(x) - 2 \int_{N_n(h)}^{\infty} x^2\, dF_{\bar{\xi}_j(h)}(x) \Bigg) = \\ &= 2 \Bigg(1 - \frac{1}{B_n^2(h)} \sum_{j=1}^{n} \int_{N_n(h)}^{\infty} x^2\, dF_{\xi_j(h)}(x) \Bigg) \geqslant \\ &\geqslant 2\big(1 - L_{4,n}(h) B_n^2(h)/N_n^2(h)\big). \end{aligned} \tag{6.114}$$

Using the relation $\mathbf{E}\big(\xi_j(h) - m_j(h)\big)^4 = \Gamma_4\big(\xi_j(h)\big) + 3\sigma_j^4(h)$, we find

$$L_{4,n}(h) \leqslant \Gamma_4\big(S_n(h)\big)/B_n^4(h) + 3 \max_{1 \leqslant j \leqslant n} \sigma_j^2(h)/B_n^2(h). \tag{6.115}$$

Next, (6.111) and (6.107) yield

$$\sigma_j^2(h) = \sigma_j^2 \left(1 + \theta \sum_{k=3}^{\infty} k(k-1)(hB_n/\Delta_n)^{k-2} \right) = \sigma_j^2 \left(1 + \theta(5/8)\right)$$

for $0 \leqslant h \leqslant \Delta_n/(12\, B_n) = 1/(12\, K_n)$. By (6.112)

$$\begin{aligned} \big|\Gamma_4\big(S_n(h)\big)\big| &= \Bigg| \sum_{k=4}^{\infty} \frac{1}{(k-4)!} \Gamma_k(S_n) h^{k-4} \Bigg| \leqslant \\ &\leqslant (3.5\, K_n B_n)^2 \sum_{k=4}^{\infty} (k-2)(k-3)(3.5\, K_n h)^{k-4} \leqslant 81\, (3.5\, K_n B_n)^2. \end{aligned}$$

Hence

$$\left|\Gamma_4\big(S_n(h)\big)/B_n^4(h)\right| \leqslant 18.2\,(B_n/\Delta_n)^2/B_n^2(h).$$

Employing (6.115) we obtain

$$L_{4,\,n}(h)B_n^2(h) \leqslant 18.2\,(B_n/\Delta_n)^2.$$

Put $N_n(h) = 6.2\,(B_n/\Delta_n)$, then (6.114) implies $l_n\big(N_n(h)\big) \geqslant 1$. Taking into account (6.87) we have

$$|f_{S_n(h)}(t)| \leqslant \exp\{-I_{h,\,n}(t/2\pi)\}, \tag{6.116}$$

where

$$I_{h,\,n}(t) = \sum_{j=1}^{n}\int_{-\infty}^{\infty} \sin^2(\pi t x) p_{\bar{\xi}_j(h)}(x)\,dx \geqslant$$

$$\geqslant 4\,t^2 \sum_{j=1}^{n} \int_{|x|\leqslant 1/2\,|t|} x^2 p_{\bar{\xi}_j(h)}(x)\,dx = 4\,t^2 B_n^2 l_n(1/2\,|t|).$$

Hence

$$I_{h,\,n}(t/2\pi) \geqslant t^2 B_n^2(h) l_n(\pi/|t|)/\pi^2,$$

where, according to (6.114), $l_n(\pi/|t|) \geqslant 1$ as $|t| \leqslant N_n(h)$. Consequently,

$$|f_{Z_n(h)}(t)| \leqslant \exp\{-t^2/\pi^2\} \tag{6.117}$$

for $|t| \leqslant T_n,\ T_n = \pi B_n(h)\Delta_n/(6.2\,B_n)$.

Let $R_n = (1/12)(1 - x/\Delta_n)\Delta_n$. Then

$$\begin{aligned} \int_{R_n\leqslant|t|\leqslant T_n} |f_{Z_n(h)}(t)|\,dt &\leqslant \int_{R_n\leqslant|t|\leqslant T_n} \exp\{-t^2/\pi^2\}\,dt \leqslant \\ &\leqslant \big(\pi^2/(2\,R_n)\big)\exp\{-R_n^2/\pi^2\}. \end{aligned} \tag{6.118}$$

Further, by Lemma 6.7 for $t \in [t_k^{(n)},\, t_{k+1}^{(n)}]$

$$I_{h,\,n}(t) \geqslant 2\,l_n\big(N_n(h)\big)(t - t_{k0}^{(n)})^2 \cdot B_n^2(h), \tag{6.119}$$

where

$$\big(6\,N_n(h)\big)^{-1} \leqslant t_{k+1}^{(n)} - t_k^{(n)} \leqslant \big(4\,N_n(h)\big)^{-1},$$

and $t_{k0}^{(n)}$ for fixed n equals $t_k^{(n)}$ or $t_{k+1}^{(n)}$ depending on t . On the other hand, employing (6.93) we have

$$I_{h,n}(t) \geqslant \frac{1}{3}\sum_{k=1}^{n} \alpha(\mathfrak{M}_k,\, 1/2\,|t|) \tag{6.120}$$

for all $-\infty < t < \infty$, where $\alpha(\mathfrak{M}_k,\, 1/2\,|t|)$ is defined by (6.90). The least value of the right–hand side of (6.120) in the interval $|t| \geqslant 1/(2N_n(h))$ is

$$M_n(h) = \frac{1}{3}\sum_{k=1}^{n} \alpha\big(\mathfrak{M}_k,\, N_n(h)\big). \tag{6.121}$$

Estimating the first summand on the right–hand side of the equality $I_{h,n}(t) = \frac{1}{4}I_{h,n}(t) + \frac{3}{4}I_{h,n}(t)$ with the help of (6.119) and the second one with the help of (6.21), we obtain

$$\begin{aligned}
&\int\limits_{T_n\leqslant|t|<\infty} |f_{Z_n(h)}(t)|\,dt = 2\pi B_n(h) \int\limits_{(2N_n(h))^{-1}\leqslant|t|<\infty} |f_{S_n(h)}(2\pi t)|\,dt \leqslant \\
&\leqslant 2\pi B_n(h) \int\limits_{(2N_n(h))^{-1}\leqslant|t|<\infty} \exp\{-I_{h,n}(t) - I_{h,4}(t)\}|f_{S_4(h)}(2\pi t)|\,dt \leqslant \\
&\leqslant 2\pi e^4 B_n(h) \exp\Big\{-\frac{3}{4}M_n(h)\Big\}\times \qquad\qquad (6.122)\\
&\times \sum_k \int\limits_{t_k^{(n)}}^{t_{k+1}^{(n)}} \exp\Big\{-\frac{1}{2}(t-t_{k0}^{(n)})^2 B_n^2(h)\Big\}|f_{S_4(h)}(2\pi t)|\,dt \leqslant \\
&\leqslant 2\pi e^4 \sqrt{2\pi}\, \exp\Big\{-\frac{3}{4}M_n(h)\Big\}\cdot U_n(h),
\end{aligned}$$

where, by Cauchy's inequality,

$$U_n(h) = \sum_k \sup_{t_k^{(n)}<t<t_{k+1}^{(n)}} |f_{S_4(h)}(2\pi t)| \leqslant \prod_{i=1}^{4}\Big(\sum_k \sup_{t_k^{(n)}<t<t_{k+1}^{(n)}} |f_{\xi_i(h)}(2\pi t)|^4\Big)^{1/4}$$

For estimation of $U_n(h)$ and $M_n(h)$, find τ_j such that

$$Q_j(h) = \int\limits_{|y|\leqslant\tau_j} p_{\bar\xi_j(h)}(y)\,dy \geqslant 1 - e^{-c}, \qquad c > 0.$$

Using (6.110), we get

$$\int\limits_{|y|\geqslant \tau_j} p_{\tilde{\xi}_j(h)}(y)\,dy = \tilde{\varphi}_{\xi_j}(h) \int\limits_{|y|\geqslant \tau_j} \exp\{hy\} p_{\tilde{\xi}_j}(y)\,dy \leqslant$$
$$\leqslant \exp\{-(A_n - h)\tau_j\}\tilde{\varphi}_{\xi_j}(A_n)\tilde{\varphi}_{\tilde{\xi}_j}^{-1}(h) \leqslant$$
$$\leqslant \exp\{-(1/36)A_n\tau_j + (A_n\sigma_j)^2\}, \qquad A_n = \Delta_n/(9\,B_n).$$

As

$$\tau_j \geqslant 36\,c/A_n + 36\,A_n \cdot \sigma_j^2, \tag{6.123}$$

it is easy to see that the inequality

$$\exp\{-A_n\tau_j/36 + (A_n\sigma_j)^2\} \leqslant \exp\{-c\}, \qquad c > 0$$

is fulfilled. Consequently,

$$\mathcal{Q}_j(h) \geqslant 1 - \exp\{-c\}. \tag{6.124}$$

Now estimate the density $p_{\tilde{\xi}_j(h)}(y)$ in the interval $]-\tau_j,\,\tau_j[$. As $p_{\xi_j}(x) \leqslant C_j$, $j = 1, 2, \ldots, n$, by condition (D), then $p_{\tilde{\xi}_j(h)}(y) \leqslant C_j$, $j = 1, 2, \ldots, n$, too. Hence

$$\begin{aligned} p_{\tilde{\xi}_j(h)}(y) &= \tilde{\varphi}_{\xi_j}^{-1}(h)\exp\{hy\}p_{\tilde{\xi}_j}(y) \leqslant \\ &\leqslant \exp\left\{\frac{3}{4}\,A_n\tau_j + \frac{1}{4}\,(A_n\sigma_j)^2\right\} C_j \leqslant c_1 C_j \end{aligned} \tag{6.125}$$

for $|h| \leqslant A_n$, where $c_1 = \exp\{27\,c + (1/4)(1/18)^2\}$.

Let $\mathfrak{M}_j = \{\Delta_{j1},\, C_{j1}\}$, $\Delta_{j1} =]-\tau_j,\, \tau_j[$, $C_{j1} = c_1 C_j$. Then (6.90) and (6.121) imply

$$\begin{aligned} \alpha\big(\mathfrak{M}_k,\, N_n(h)\big) &= \mathcal{Q}_j^3(h)\left(\big(2\tau_j + 2\,N_n(h)\big)^2 C_{j1}^2\right)^{-1} \geqslant \\ &\geqslant \mathcal{Q}_j^3(h)(c_2 C_j^2 K_n^2)^{-1} \end{aligned} \tag{6.126}$$

and

$$M_n(h) \geqslant \frac{c_3}{K_n^2} \sum_{j=1}^{n} \frac{1}{C_j^2}, \tag{6.127}$$

where

$$c_2 = (36 \cdot 18\,c + 27)^2 \exp\{27\,c\},$$
$$c_3 = (1/3\,c_0), \qquad c_0 = 729/c^3, \qquad c < 10^{-3}.$$

To conclude the proof of the theorem, it remains to evaluate the quantity $U_n(h)$. Let $g_j(t) := |f_{\xi_j(h)}(2\pi t)|^2$. Then

$$\begin{aligned} |g_j(t+s) - g_j(t)| &= \Big| \int_{-\infty}^{\infty} \exp\{2\pi i(t+s)y\} p_{\tilde\xi_j(h)}(y)\,dy - \\ &\quad - \int_{-\infty}^{\infty} \exp\{2\pi i t y\} p_{\tilde\xi_j(h)}(y)\,dy \Big| = \\ &= \Big| \int_{-\infty}^{\infty} \exp\{2\pi i t y\} (\exp\{2\pi i s y\} - 1) p_{\tilde\xi_j(h)}(y)\,dy \Big| \leqslant \\ &\leqslant 2\pi s \Big(\int_{-\infty}^{\infty} y^2 p_{\tilde\xi_j(h)}(y)\,dy \Big)^{1/2} = 2\sqrt{2}\,\pi\sigma_j(h) s. \end{aligned} \tag{6.128}$$

Consequently, Lemma 6.4 holds with $K_i(h) = 2\sqrt{2}\,\pi\sigma_i(h)$ and

$$V_i(h) = \int_{-\infty}^{\infty} g_i(t)\,dt = p_{\tilde\xi_i(h)}(0) \leqslant C_i, \qquad i = 1, 2, 3, 4.$$

Recalling that $N_n(h) = 6.2(B_n/\Delta_n)$, $\sigma_j(h) \leqslant 1.75\sigma_j$ for $|h| \leqslant \Delta_n/(12B_n)$, in view of Lemma 2.4 we obtain

$$\begin{aligned} U_n(h) &\leqslant \prod_{i=1}^{4} \Big(\sum_{i=1}^{4} \sup_{t_k^{(n)} < t < t_{k+1}^{(n)}} |f_{\xi_i(h)}(2\pi t)|^4 \Big)^{1/4} \leqslant \\ &\leqslant 37.2\,K_n \prod_{i=1}^{4} \Big((4 + 1.75\sqrt{2}\,\pi\sigma_i(6.7\,K_n)^{-1}) C_i \Big)^{1/4} \leqslant \\ &\leqslant 172\,K_n \prod_{i=1}^{4} C_i^{1/4}. \end{aligned} \tag{6.129}$$

Returning to (6.127) and (6.129) we find

$$\begin{aligned} \int_{T_n \leqslant |t| < \infty} |f_{Z_n(h)}(t)|\,dt &= 2\pi e^4 \sqrt{2\pi} \exp\Big\{ -\frac{3}{4} M_n(h) \Big\} U_n(h) \leqslant \\ &\leqslant 684\,e^4 \pi \sqrt{2\pi}\,K_n \prod_{i=1}^{4} C_i^{1/4} \exp\Big\{ -\frac{c_3}{K_n^2} \sum_{j=1}^{n} \frac{1}{C_j^2} \Big\}. \end{aligned} \tag{6.130}$$

Employing (6.113) and (6.108) we get

$$x = B_n h \left(1 + \theta \sum_{k=3}^{\infty} k(B_n|h|/\Delta_n)^{k-2}\right) =$$
$$= B_n h \left(1 + \theta(3\,B_n(h)/\Delta_n)(1 - 3\,B_n|h|/\Delta_n)^{-1}\right)$$

for $|h| < \Delta_n/(3\,B_n)$. Consequently, for $|h| \leqslant \Delta_n/(12\,B_n)$ we have $|x| \leqslant \Delta_n/18$.

Theorem 6.1. follows now from (6.1), (6.2), (6.18) and (6.130). ■

6.4. Asymptotic expansions in integral theorems with large deviations

Let

$$\xi_1,\ \xi_2,,\ \ldots,\ \xi_n,\ \ldots \tag{6.131}$$

be a sequence of independent identically distributed r.v. with $\mathbf{E}\xi_j = 0$ and $\sigma^2 = \mathbf{E}\xi_j^2$, $j = 1, 2, \ldots$. Denote

$$S_n = \sum_{j=1}^{n} \xi_j, \quad F_n(x) = \mathbf{P}(S_n < x), \quad f_{\xi_1}(t) = \mathbf{E}\exp\{it\xi_1\},$$

$$p_z(y) = \begin{cases} \exp\left\{-\dfrac{1}{2}y^2\right\} \Big/ \displaystyle\int_z^{\infty} \exp\left\{-\dfrac{1}{2}y^2\right\} dy, & y \geqslant z, \\ 0, \quad y < z; \end{cases} \tag{6.132}$$

$$\mu_k(z) = \int_{-\infty}^{\infty} y^k p_z(y)\,dy = \int_z^{\infty} y^k e^{-\frac{1}{2}y^2}\,dy \Big/ \int_z^{\infty} e^{-\frac{1}{2}y^2}\,dy, \tag{6.133}$$

$$\begin{aligned} \omega_k(z) &= \int_z^{\infty} (y-z)^k \exp\left\{-\frac{1}{2}y^2\right\} dy \Big/ \int_z^{\infty} \exp\left\{-\frac{1}{2}y^2\right\} dy = \\ &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \mu_{k-l}(z). \end{aligned} \tag{6.134}$$

Suppose that the r.v. ξ_1 satisfies the conditions

$$\mathbf{E}\exp\{a|\xi_1|\} \leqslant B < \infty, \qquad 0 < a < A, \tag{K}$$

and

$$\overline{\lim_{|t|\to\infty}} |f_{\xi_1}(t)| < 1 \tag{C}$$

(Cramer's conditions).

THEOREM 6.2. *Let a r.v. ξ_1 satisfy conditions (K) and (C). Then there exists a positive constant ε such that for any integer $q \geqslant 1$ in the interval*

$$1 \leqslant x \leqslant \varepsilon\sqrt{n}$$

the relations

$$\begin{gathered}
\frac{\mathbf{P}(S_n > x\sqrt{n})}{1-\Phi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}}\lambda\left(\frac{x}{\sqrt{n}}\right)\right\}\times \\
\times\left(1 + L(x;\, q) + O\Big(\Big(\frac{x}{\sqrt{n}}\Big)^q\Big)\right), \\
\frac{\mathbf{P}(S_n < -x\sqrt{n})}{\Phi(-x)} = \exp\left\{-\frac{x^3}{\sqrt{n}}\lambda\left(-\frac{x}{\sqrt{n}}\right)\right\}\times \\
\times\left(1 + L(-x;\, q) + O\Big(\Big(\frac{x}{\sqrt{n}}\Big)^q\Big)\right)
\end{gathered} \tag{6.135}$$

are valid. Here

$$\begin{aligned}
L(z;\, q) = &\sum_{\nu=1}^{q-1} N_\nu(z)\Big(\frac{z}{\sqrt{n}}\Big)^\nu + \\
&+\sum_{\nu=1}^{q-1}\sum_{l=1}^{\nu}\sum_{i=0}^{[3\,l/2]} e_{i,\,l,\,\nu-l} n^{-\nu/2} z^{\nu-l}\omega_{3\,l-2\,i}(z) + \\
&+\sum_{\nu=1}^{q-3}\sum_{l=1}^{\nu}\sum_{i=0}^{[3\,l/2]} e_{i,\,l,\,\nu-l} n^{-\nu/2} z^{\nu-l}\sum_{\tilde{l}=1}^{q-\nu-1} M_{\tilde{l}}(z)\left(\frac{z}{\sqrt{n}}\right)^{\tilde{l}},
\end{aligned} \tag{6.136}$$

where

$$N_\nu(z) = \sum_{l=1}^{\nu}(-1)^l\frac{1}{l!}\,\omega_l(z) z^l b_{l,\,\nu},$$

$$M_{\tilde{l}} = \sum_{r=1}^{\tilde{l}}(-1)^r\frac{1}{r!}\,\omega_{r+3\,l-2\,i}(z)\cdot z^r b_{r,\,\tilde{l}}, \quad b_{l,\,k} = \sum_{\substack{k_j\geqslant 1\\ k_1+\dots+k_l=k}}\ \prod_{j=1}^{l} b_{k_j},$$

the coefficients being defined by recurrent equations (2.10); $\omega_k(z)$ *are the functions, defined by* (6.134), *and satisfy the recurrent relations*

$$\begin{aligned} &\omega_k(x) = -x\omega_{k-1}(x) + (k-1)\omega_{k-2}(x), \qquad k = 2, 3, \ldots, \\ &\omega_0(x) \equiv 1; \end{aligned} \tag{6.137}$$

moreover, $\omega_k(x) = O(x^{-k})$.

Further,

$$\lambda(t) = \sum_{k=0}^{\infty} \lambda_k t^k \tag{6.138}$$

is the Cramer power series convergent for $|t| < \varepsilon$, $\varepsilon > 0$. *For the coefficients* λ_k *the formula*

$$\lambda_k = -b_{k+2}/(k+3)$$

holds.

In particular,

$$\begin{aligned} &b_1 = 1, \quad b_2 = -(1/2)\Gamma_3(\xi_1), \\ &b_3 = -(1/6)\left(\Gamma_4(\xi_1) - 3\,\Gamma_3^2(\xi_1)\right), \\ &b_4 = -(1/24)\left(\Gamma_5(\xi_1) - 10\,\Gamma_3(\xi_1)\Gamma_4(\xi_1) + 15\,\Gamma_3^3(\xi_1)\right), \end{aligned}$$

and

$$\begin{aligned} &L(z;1) \equiv 0, \\ &L(z;2) = -\frac{\Gamma_3(\xi_1)}{6\sqrt{n}}\left(\omega_3(z) - 3\,\omega_1(z)\right) = -\frac{\Gamma_3(\xi_1)}{6\sqrt{n}}\left(\frac{1}{z} - \frac{4}{z^3} + O\Big(\frac{1}{z^5}\Big)\right) \end{aligned}$$

as $z \to \infty$.

Let us consider large deviations in the Linnik power zones.

Let $\{g(\cdot)\}$ denote the class of continuous functions $g(x)$, $x \in R$, such that $g(x)$ increases, $g(x)x^{-1}$ strictly decreases and

$$\varrho(x)\ln x \leqslant g(x) \leqslant C(g)x^{\alpha}, \qquad 0 < \alpha < 1, \tag{6.139}$$

Here $\varrho(x)$ is an arbitrary increasing function, satisfying the condition

$$\lim_{n\to\infty} \varrho(n) = +\infty, \tag{6.140}$$

and $C(g)$ is a positive constant depending on g.

By $\Lambda(n)$ denote the solution of the equation

$$kx^2 = ng(x), \tag{6.141}$$

where the constant k is larger than 1. It follows from (6.139) and (6.141) that

$$\Lambda(n) \leqslant [C^*(g)n]^{1/(2-\alpha)}. \tag{6.142}$$

Let m be a nonnegative integer. By $\lambda^{[m]}(t)$ denote the segment of the Cramer series, consisting of its first m terms:

$$\lambda^{[m]}(t) = \sum_{k=0}^{m-1} \lambda_k t^k. \tag{6.143}$$

For the r.v. ξ_1 introduce S.V. Nagaev's condition

$$\mathbf{E}\exp\{g(|\xi_1|)\} < \infty. \tag{A}$$

(Nagaev, 1965).

THEOREM 6.3 (Wolf, 1977). *Let a r.v. ξ_1 satisfy conditions (A) and (C). Then for an arbitrary integer $q \geqslant 1$ in the interval*

$$\sqrt{n} < x \leqslant \Lambda(n)$$

as $n \to \infty$ the relations

$$\begin{aligned} \frac{1-F_n(x)}{1-\Phi(x/\sqrt{n})} &= \exp\left\{\frac{x^3}{n^2}\lambda^{[s+q]}\left(\frac{x}{n}\right)\right\}\times \\ &\times\left(1+L\left(\frac{x}{\sqrt{n}};\, q\right)+O\left(\left(\frac{x}{n}\right)^q\right)\right), \end{aligned} \tag{6.144}$$

$$\begin{aligned} \frac{F_n(-x)}{\Phi(-x/\sqrt{n})} &= \exp\left\{-\frac{x^3}{n^2}\lambda^{[s+q]}\left(-\frac{x}{n}\right)\right\}\times \\ &\times\left(1+L\left(-\frac{x}{\sqrt{n}};\, q\right)+O\left(\left(\frac{x}{n}\right)^q\right)\right) \end{aligned}$$

hold. Here $\Lambda(n)$ is the solution of equation (6.141), $s = [\alpha/(1-\alpha)-0]$, *and $L(z;\, q)$ is defined by* (6.136).

Recall Linnik's condition (see p. 47)

$$\mathbf{E}\exp\{|\xi_1|^{1/(1+2\gamma)}\} < \infty, \qquad \gamma > 0. \tag{L}$$

THEOREM 6.4 (Wolf, 1977). *Let a r.v. ξ_1 satisfy conditions (L) and (C). Then for an arbitrary integer $q \geqslant 1$ in the interval*

$$1 \leqslant x \leqslant (\sqrt{n})^{1/(1+2\gamma)}$$

as $n \to \infty$ the relations

$$\begin{aligned}\frac{\mathbf{P}(S_n > x\sqrt{n})}{1-\Phi(x)} &= \exp\left\{\frac{x^3}{\sqrt{n}}\,\lambda^{[s+q]}\left(\frac{x}{\sqrt{n}}\right)\right\}\times \\ &\times\left(1 + L(x;\, q) + O\Big(\Big(\frac{x}{\sqrt{n}}\Big)^q\Big)\right),\end{aligned} \tag{6.145}$$

$$\begin{aligned}\frac{\mathbf{P}(S_n < -x\sqrt{n})}{\Phi(-x)} &= \exp\left\{-\frac{x^3}{\sqrt{n}}\,\lambda^{[s+q]}\left(-\frac{x}{\sqrt{n}}\right)\right\}\times \\ &\times\left(1 + L(-x;\, q) + O\Big(\Big(\frac{x}{\sqrt{n}}\Big)^q\Big)\right),\end{aligned} \tag{6.146}$$

hold, where $s = [1/\gamma]$.

For the interval

$$1 \leqslant x \leqslant (\sqrt{n})^{1/(1+2\gamma)}/\varrho(n), \tag{6.147}$$

where $\varrho(n)$ is any function satisfying the condition

$$\lim_{n\to\infty} \varrho(n) = +\infty,$$

relations (6.145) and (6.146) have been obtained under the conditions of Theorem 6.4 in (Saulis, 1973).

COROLLARY 6.1. *Let the condition of Theorem 6.3 be fulfilled and $\gamma_3 = \alpha_3 = 0$. Then in the interval*

$$\sqrt{n} < x \leqslant \Lambda(n)$$

as $n \to \infty$

$$\frac{1-F_n(x)}{1-\Phi(x/\sqrt{n})} = \exp\left\{\frac{x^3}{n^2}\,\lambda^{[s+2]}\left(\frac{x}{n}\right)\right\}\left(1 + O\Big(\Big(\frac{x}{n}\Big)^2\Big)\right), \tag{6.148}$$

and for $\alpha \leqslant 1/2$

$$1 - F_n(x) = \left(1 - \Phi\Big(\frac{x}{\sqrt{n}}\Big)\right)\exp\left\{\frac{\gamma_4}{24}\frac{x^4}{n^3}\right\}\left(1 + O\Big(\Big(\frac{x}{n}\Big)^2\Big)\right). \tag{6.149}$$

Analogous relations hold also for negative values of x.

COROLLARY 6.2. *Let the conditions of Theorem 6.3 be fulfilled and*

$$\gamma_k = \Gamma_k(\xi_1) = 0 \quad \text{for } k = 3, 4, \ldots, s+q+2.$$

Then in the interval

$$\sqrt{n} < x \leqslant \Lambda(n)$$

as $n \to \infty$

$$1 - F_n(x) = \left(1 - \Phi\left(\frac{x}{n}\right)\right)\left(1 + O\left(\left(\frac{x}{n}\right)^q\right)\right),$$

$$F_n(-x) = \Phi\left(-\frac{x}{n}\right)\left(1 + O\left(\left(\frac{x}{n}\right)^q\right)\right). \tag{6.150}$$

For $\alpha = r/(1+r)$, where r is a positive integer, (6.146) *hold in the interval* $\sqrt{n} < x \leqslant \left(C^*(g)n\right)^{(1+r)/(2+r)}$.

It follows from (6.150) that in the interval

$$1 < |x| \leqslant \left(C^*(g)n\right)^{(1+r)/(2+r)} n^{r/2(r+2)}$$

as $n \to \infty$

$$|F_n(x/\sqrt{n}) - \Phi(x)| = O\left(\frac{|x|^{q-1}}{(\sqrt{n})^q} \exp\left\{-\frac{1}{2}x^2\right\}\right). \tag{6.151}$$

If we abandon condition (C), then we obtain only the first term of asymptotic expansion.

THEOREM 6.5 (Wolf, 1977). *Let a nonnegative r.v. ξ_1 satisfy condition (A). Then, in the interval*

$$\sqrt{n} < x \leqslant \Lambda(n)$$

as $n \to \infty$

$$\frac{1 - F_n(x)}{1 - \Phi(x/\sqrt{n})} = \exp\left\{\frac{x^3}{n^2}\lambda^{[s+2]}\left(\frac{x}{n}\right)\right\}\left(1 + L\left(\frac{x}{\sqrt{n}}; 2\right) + o\left(\frac{x}{n}\right)\right). \tag{6.152}$$

An analogous relation also holds for negative values of x.

THEOREM 6.6 (Wolf, 1977). *Let a nonlattice r.v. ξ_1 satisfy condition (L). Then in the interval*

$$1 \leqslant x \leqslant (\sqrt{n})^{1/(1+2\gamma)}$$

as $n \to \infty$

$$\frac{\mathbf{P}(S_n > x\sqrt{n})}{1-\Phi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}}\lambda^{[s+1]}\left(\frac{x}{\sqrt{n}}\right)\right\}\left(1 + L\left(\frac{x}{\sqrt{n}}; 2\right) + o\left(\frac{x}{\sqrt{n}}\right)\right), \quad (6.153)$$

where $s = [1/\gamma]$.

In (Saulis, 1973), the relation (6.153) was obtained for the interval $1 \leqslant x \leqslant (\sqrt{n})^{1/(1+2\gamma)}/\varrho(n)$, where $\varrho(n)$ is an arbitrary function, satisfying the condition $\lim\limits_{n\to\infty} \varrho(n) = +\infty$.

The complete proofs of Theorems 6.2 – 6.6 can be found in (Saulis, 1969, 1973), (Wolf, 1977). Since the proofs do not use the method of cumulants, they are not included in the present book.

CHAPTER 7 PROBABILITIES OF LARGE DEVIATIONS FOR RANDOM VECTORS

Integral large deviation limit theorems for random vectors taking values in a k-dimensional Euclidean space were investigated in (Borovkov, Rogozin, 1965), (Vilkauskas, 1965), (W. Richter, 1957), (Bahr, 1967), (W.-D. Richter, 1978), (Rozovskii, 1982) and other papers. Asymptotic formulas for special classes of sets, as shown in (Osipov, 1982), (W.-D. Richter, 1982), (Saulis, 1983), can be essentially simplified. It was established in (Aleškevičienė, 1983), (Svetulevičienė, 1981), (Saulis, 1984, 1987) that in theorems of large deviations for convex Borel sets, it suffices to study a multidimensional analog of the Cramer – Petrov series at the closest to the origin point of the set.

As a rule, asymptotic behaviour of the probability $\mathbf{P}(S_n/\sqrt{n} \in D_n)$, where $S_n = X^{(1)} + \ldots + X^{(n)}$ is the sum of independent identically distributed random vectors, has been considered in all these works. The paper (Borovkov, Mogulskii, 1985) also is devoted to the investigation of this probability using the deviation function. Large deviations for dependent random vectors were discussed by Sh.K. Formanov (Formanov, 1973), and for Banach space valued independent random vectors by V. Bentkus (Bentkus, 1986).

It is shown in this chapter that the method of cumulants, proposed in the paper (Statulevičius, 1966) and developed in (Rudzkis, Saulis, Statulevičius, 1978), applies to the multidimensional case, too.

7.1. General lemmas on large deviations for a random vector with regular behaviour of cumulants

Consider a random vector X in the k-dimensional Euclidean space R^k with mean $\mathbf{E}X = 0$ and the unit covariance matrix (c.m.) E. Let $(\cdot,\ \cdot)$ denote a scalar product in R^k, $||x|| = (x,\ x)^{1/2}$, $|x| = |x_1| + \ldots + |x_k|$, $\Phi(x)$ is the standard k-dimensional Gaussian distribution function. Everywhere below θ (with or without an index) denotes a quantity not exceeding 1 in absolute value.

Denote $f_X(t) = \mathbf{E}\exp\{i(t, X)\}$. Suppose that $\mathbf{E}|X|^l < \infty$. Let

$$\gamma_{j_1,\dots,j_l}(0) := \frac{1}{i^l} \frac{\partial^l \ln f_X(t)}{\partial t_{j_1} \dots \partial t_{j_l}}\bigg|_{t=0}, \tag{7.1}$$

where $j_1, \dots, j_l$ take values from 1 to k. Then the l^{th} order cumulant $\Gamma_l\big((X, z)\big)$ of the r.v. (X, z) equals:

$$\begin{aligned}\Gamma_l\big((X, z)\big) &= \frac{1}{i^l}\left(\frac{\partial}{\partial t}, z\right)^l \ln f_X(t)\Big|_{t=0} = \\ &= \sum_{j_1,\dots,j_l=1}^{k} \gamma_{j_1,\dots,j_l}(0) z_{j_1} \dots z_{j_l}.\end{aligned} \tag{7.2}$$

We say that a random vector X satisfies condition (S_γ^*) if there exist $\gamma \geqslant 0$, $H > 0$, and $\Delta > 0$ such that $\forall z \in R^k$ with $||z|| = 1$

$$\big|\Gamma_l\big((X, z)\big)\big| \leqslant H(l!)^{1+\gamma}/\Delta^{l-2}, \qquad l = 3, 4, \dots . \tag{S_γ^*}$$

Let

$$\Delta_\gamma = c_\gamma \Delta^{1/(1+2\gamma)}, \quad c_\gamma = \frac{1}{3\sqrt{e}}\left(\frac{\sqrt{2}}{6}\right)^{1/(1+2\gamma)}, \tag{7.3}$$

$$\Delta_\gamma^* = \frac{\Delta_\gamma}{2\,k\,\max\{1, H\}(1 + kH)}, \quad s = 2\left[\frac{1}{2}\left(\frac{\Delta^2}{18}\right)^{1/(1+2\gamma)}\right] - 2, \tag{7.4}$$

$[m]$ is the integer part of m. Let

$$\tilde{\varphi}_X(z) := \exp\left\{\sum_{l=2}^{s} \frac{1}{l!}\Gamma_l\big((X, z)\big)\right\}, \tag{7.5}$$

and $h = h(x)$ is the solution of the equation

$$\tilde{\varphi}_X^{-1}(h) \operatorname{grad} \tilde{\varphi}_X(h) = x. \tag{7.6}$$

By $\tilde{E}(h) = \big(e_{i,j}(h)\big)$ denote the matrix with the entries

$$e_{i,j}(h) = \partial^2 \ln \tilde{\varphi}_X(h)/(\partial h_i \partial h_j), \qquad i, j = \overline{1, k}.$$

Note that $\det \tilde{E}(h)$ is the Jacobian of equation (7.6), which is continuous in h and equals 1 at $h = 0$. As $h = 0$ is the inner point of the set $\tilde{O} = \{h \in R^k : \tilde{\varphi}_X(h) < \infty\}$, there is a neighbourhood of $h = 0$ in which $\det \tilde{E}(h) \neq 0$, the mapping $h \to x$ is 1–1 and mutually continuous and the image of which under this mapping is an open set in the x–plane containing $x = 0$ (which corresponds to $h = 0$).

Let $A \subset R^k$ be a closed convex Borel set, whose point a is the closest to the origin point. We say that the set A belongs to the class $\mathfrak{U}_1$ if:

1) the point $h(a)$ is the closest to the origin point of the set $A - a + h(a)$;

2) $\Phi(A) \geqslant \exp\left\{-\frac{1}{4}\Delta_\gamma^{*2}\right\}$.

Let us introduce a class of convex Borel sets, contained in some nondegenerate positive convex cone Ω in R^k with vertex at the origin. We say that a closed convex Borel set A, whose point a is the closest to the origin point, belongs to the class $\mathfrak{U}_2 = \mathfrak{U}_2(\Omega, \eta)$, $\eta > 0$, if condition 2) in the definition of $\mathfrak{U}_1$ is satisfied and

1′) $A \subset a + \Omega$;

3) $\max\{(\widehat{x, a}),\ x - a \in \Omega\} \leqslant (\pi/2) - \eta$,

where $\eta \geqslant \alpha_\Delta$, $\quad \alpha_\Delta = \big(\widehat{a, h(a)}\big)$ and $(\widehat{x, y})$ is the angle between vectors x and y.

Denote

$$\mathfrak{U}_i(c) := \{A \in \mathfrak{U}_i : 1 \leqslant ||a|| < c\}, \quad i = 1, 2. \tag{7.7}$$

LEMMA 7.1. *Let condition (S_γ^*) be fulfilled for the random vector X with* $\mathbf{E}X = 0$ *and the unit c.m. E. If $A \in \mathfrak{U}_1(\Delta_\gamma^*)$, the relation*

$$\mathbf{P}(X \in A) = \Phi\big(A + b(a)\big)\exp\{\lambda(a)\}\big(1 + \theta_1 q_1(k, H, r)r/\Delta_\gamma^*\big) \tag{7.8}$$

holds, where

$$b(a) = \big(||a||/||h(a)||\big)h(a) - a, \tag{7.9}$$

$$r = \max\{||a||, (\ln \Delta_\gamma^*)^{1/2}\}, \tag{7.10}$$

$$\begin{aligned}\lambda(a) &= \frac{1}{2}||a||^2 + \ln\tilde{\varphi}_X\big(h(a)\big) - \big(h(a), a\big) = \\ &= \sum_{3\leqslant\nu<p} Q_\nu(a) + \theta_3(||a||/\Delta_\gamma^*)^3, \quad p = (1/\gamma) + 2,\end{aligned} \tag{7.11}$$

and $Q_\nu(x)$ is a ν–linear form in the variables x_l with coefficients depending on cumulants of order ν of the random vector X. In particular,

$$\begin{aligned}Q_3(x) &= \frac{1}{6}\Gamma_3\big((X, x)\big) = \frac{1}{6}\Big(\frac{\partial}{\partial t}, x\Big)^3 \ln\tilde{\varphi}_X(t)\Big|_{t=0} = \\ &= \sum_{j_1, j_2, j_3 = 1}^{k} \gamma_{j_1, j_2, j_3}(0) x_{j_1} x_{j_2} x_{j_3},\end{aligned}$$

$$Q_4(x) = -\frac{1}{8} \sum_{l,m,p,r=1}^{k} \Big(\gamma_{l,m,p,r}(0) - 3 \sum_{k_1=1}^{k} \gamma_{l,m,k_1}(0)\gamma_{p,r,k_1}(0)\Big) \times$$
$$\times\, x_l x_m x_p x_r,\ \ldots .$$

The explicit form of $q_1(k, H, r)$ *is*

$$q_1(k, H, r) = c_0\big\{c_1(k, H) + \big(c_2(k) + c_3(k, H)\Delta_\gamma^{*\,k+2} \times \\ \times \exp\{-(1 - r/\Delta_\gamma^*)\sqrt{\Delta_\gamma}\}\big)/\big(r(1 - r/\Delta_\gamma^*)\big)\big\}, \tag{7.12}$$

where c_0, $c_1(k, H)$, $c_2(k)$ *and* $c_3(k, H)$ *depend on the quantities according to* (7.15) - (7.18).

LEMMA 7.2. *Let condition* (S_γ^*) *be fulfilled for the random vector* X *with* $\mathbf{E}X = 0$ *and the unit c.m.* E. *If* $A \in \mathfrak{U}_2(\Delta_\gamma^*)$, *then*

$$\mathbf{P}(X \in A) = \Phi(A) \exp\{\lambda(a)\} \big(1 + \theta_2 q_2(k, H, \eta, r) r/\Delta_\gamma^*\big). \tag{7.13}$$

Here

$$q_2(k, H, \eta, r) = c_0^*\big\{c_1(k, H) + \big(c_2(k) + c_3(k, H)\Delta_\gamma^{*\,k+2} \times \\ \times \exp\{-(1 - r/\Delta_\gamma^*)\sqrt{\Delta_\gamma}\}\big)/\big(r(1 - r/\Delta_\gamma^*)\big)\big\}, \tag{7.14}$$

where c_0^*, $c_1(k, H)$, $c_2(k)$ *and* $c_3(k, H)$ *are defined by* (7.19), (7.16) - (7.18), *respectively.*

The quantities c_0, c_0^*, $c_1(k, H)$, $c_2(k)$, and $c_3(k, H)$ occurring in the definition of $q_1(k, H, r)$, $q_2(k, H, \eta, r)$ by (7.12) and (7.14) have the form

$$c_0 = \Big(1 + \big((\sqrt{2})^{k-5}/c(H)\big)\big(r/\Delta_\gamma^*\big)\Big)\big(1 + 5\exp\{-6\Delta_\gamma^*\}\big), \tag{7.15}$$

$$c_1(k, H) = 2.3\sqrt{e}\, k(3/2)^{k/2-1} + 2.3\, e^4 2^{k+5} c(k) c(H)(1 + kH) \times \\ \times \Big(1 + \big(\sqrt{k}\, c(k)/40\big) r/\Delta_\gamma^*\Big)\Big(1 + \big(c(k)/72\big) r/\Delta_\gamma^*\Big), \tag{7.16}$$

$$c_2(k) = 21\, e^4 4^{k+1} \pi^{k/2} \Gamma\big((k+1)/2\big) \times \\ \times \exp\big\{\big(\sqrt{k}\, c(k)/30\big) r/\Delta_\gamma^*\big\}/\big(\sqrt{k}\,\Gamma(k/2)\big), \tag{7.17}$$

$$c_3(k, H) = 90\, e^4 4^{k+4} \pi^{k/2} (1 + kH) k^{k+1} \times \\ \times \exp\big\{\big(\sqrt{k}\, c(k)/30\big) r/\Delta_\gamma^*\big\}/\Gamma\big((k/2) + 1\big), \tag{7.18}$$

$$c_0^* = \Big(1 + \big((k+1)!/(8\,c(H)\eta^{k+1})\big)r/\Delta_\gamma^*\Big)\,(1 + 5\exp\{-6\Delta_\gamma^*\}), \tag{7.19}$$

$c(H) = \max\{1,\ H\}$, and $c(k)$ is defined by the inequality

$$K\big(\{x : |x| < c(k)\}\big) \geqslant 3/4, \tag{7.20}$$

where the distribution K is the six-tuple convolution of the uniform distribution in the k-dimensional sphere with radius $1/6$ and has the density

$$q(x) = \alpha_k\big(||x||^{-k/6} I_{k/2}(||x||/6)\big)^6,$$

$I_t(x)$ is the Bessel function, α_k is the normalizing constant.

In what follows let $\Delta \to \infty$. Put $\tilde{\Delta}_\gamma = \Delta^{1/(1+2\gamma)}$, $h(a) = \nu^0 e_a$, $\nu^0 \in R^1$, $e_a = a/||a||$, ν^0 is the solution of the equation

$$\big(a,\ \tilde{m}(\nu e_a)\big) = (a,\ a), \tag{7.21}$$

where $\tilde{m}(h) := \tilde{\varphi}_X^{-1}(h)\mathrm{grad}\,\tilde{\varphi}_X(h)$. Next, denote $b^*(a) = a - \tilde{m}(\nu^0 e_a)$, $\tilde{r} = \max\{||a||,\ (\ln\tilde{\Delta}_\gamma)^{1/2}\}$, $b^0_{\tilde{m}}$ is the closest to the origin point of the set $A + b^*(a)$.

Suppose that the closed convex Borel set A, whose point a is the closest point to the origin, is subject to the condition:

$$\Phi(A) \geqslant \exp\Big\{-\frac{1}{4}\tilde{\Delta}_\gamma^2\Big\}. \tag{G}$$

LEMMA 7.3. *Let condition (S_γ^*) be fulfilled for the random vector X with $\mathbf{E}X = 0$ and the unit c.m. E. Then:*

1) *if A satisfies condition (G) and $1 \leqslant ||a|| = o(\tilde{\Delta}_\gamma^{2/3})$, as $\Delta \to \infty$,*

$$\mathbf{P}(X \in A) = \Phi\big(A + b^*(a)\big)\exp\{L(a)\}\,\big(1 + O(\tilde{r}/\tilde{\Delta}_\gamma^{2/3}\big); \tag{7.22}$$

2) *if A satisfies condition (G) and $||b^0_{\tilde{m}} - a|| \leqslant \tilde{r}^{1/3}$, $1 \leqslant ||a|| = o(\tilde{\Delta}_\gamma)$, as $\Delta \to \infty$,*

$$\mathbf{P}(X \in A) = \Phi(A)\exp\{L(a)\}\,\big(1 + O(\tilde{r}/\tilde{\Delta}_\gamma)\big). \tag{7.23}$$

Here $L(a) = \frac{1}{2}||a||^2 + \ln\tilde{\varphi}_X(\nu^0 e_a) - (\nu^0 e_a,\ a)$, and ν^0 is the solution of equation (7.21).

COROLLARY 7.1. *Let for a random vector X with $\mathbf{E}X = 0$ and the c.m. E condition (S_γ^*) be fulfilled and $A = \{y : (a,\ y - a) \geqslant 0\}$ be the halfspace at the* *distance* $||a||$ *from the origin. If, moreover, condition* (G) *is satisfied and* $1 \leqslant ||a|| = o(\tilde{\Delta}_\gamma)$, *then*

$$\mathbf{P}(X \in A) = \Phi(A) \exp\{L(a)\} \big(1 + O(\tilde{r}/\tilde{\Delta}_\gamma)\big). \tag{7.24}$$

Note that the point $\tilde{m}(\nu^0 e_a)$ belongs to the hyperplane $\{y : (a,\, y - a) = 0\}$. It is easy to see that $||b^0_{\tilde{m}} - a|| \uparrow$, if curvature of the set A increases at the point a, whereas

$$||b^0_{\tilde{m}} - a|| \leqslant ||\tilde{m}(\nu^0 e_a) - a|| = ||b^*(a)||.$$

Consequently, the condition $||b^0_{\tilde{m}} - a|| \leqslant \tilde{r}^{1/3}$ is fulfilled for convex Borel sets with sufficiently small curvature at the point a.

A complete proof of Lemmas 7.1 – 7.3 is given in (Saulis, 1987).

7.2. Theorems on large deviations for sums of random vectors and quadratic forms

a) *Sums of non-identically distributed random vectors.* Let $X^{(1)}, \ldots, X^{(n)}$, $n \geqslant 1$, be independent random vectors with $\mathbf{E}X^{(j)} = 0$ and the c.m. $D^{(j)}$. The largest eigenvalue (e.v.) of the operator $D^{(j)}$ is denoted by $\lambda_1^{(j)}$, $S_n = X^{(1)} + \ldots + X^{(n)}$, $\lambda_n = \lambda_n(R_n)$ is the smallest e.v. of the covariance matrix R_n of the random vector S_n, $Z_n = K_n S_n$, where the matrix K_n is such that $K_n' R_n K_n = E$ (K_n' is the transposed matrix).

We say that the random vector $X^{(j)}$ with $\mathbf{E}X^{(j)} = 0$ and the c.m. $D^{(j)}$ satisfies condition (B_γ^*) if there exist $\gamma \geqslant 0$ and $K > 0$ such that

$$|\mathbf{E}(X^{(j)},\, z)^l| \leqslant (l!)^{1+\gamma} (K||z||)^{l-2} \mathbf{D}(X^{(j)},\, z), \qquad \forall l \geqslant 3. \tag{B_γ^*}$$

PROPOSITION 7.1. *Let condition* (B_γ^*) *be fulfilled for a random vector* $X^{(j)}$ *with* $\mathbf{E}X^{(j)} = 0$ *and c.m.* $D^{(j)}$. *Then* $\forall z \in R^k$ *with* $||z|| = 1$ *the estimate*

$$\big|\Gamma_l\big((Z_n,\, z)\big)\big| \leqslant (l!)^{1+\gamma}/\Delta_n^{l-2}, \qquad l = 3, 4, \ldots, \tag{7.25}$$

holds, where

$$\Delta_n = \lambda_n^{1/2}(R_n)/M_n, \qquad M_n = 2\max\{K,\, \max_{1 \leqslant j \leqslant n} {\lambda_1^{(j)}}^{1/2}\}. \tag{7.26}$$

If $D^{(j)}$ is a diagonal matrix, then

$$\lambda_1^{(j)} = \max_{1 \leqslant i \leqslant k} \mathbf{E} X_i^{(j)^2}, \quad \lambda_n^{1/2}(R_n) = \min_{1 \leqslant i \leqslant k} B_{n,i},$$
$$B_{n,i}^2 = \mathbf{E} X_i^{(1)^2} + \ldots + \mathbf{E} X_i^{(n)^2}.$$

Proof. As $D^{(j)}$ is the c.m. of the random vector $X^{(j)}$, so

$$\begin{aligned}\mathbf{D}(X^{(j)}, z) = (D^{(j)}z, z) = z' D^{(j)} z = ||z||^2 e_z' D^{(j)} e_z \leqslant \\ \leqslant ||z||^2 \sup_{||z||=1} z' D^{(j)} z = \lambda_1^{(j)} ||z||^2,\end{aligned} \tag{7.27}$$

where $e_z = z/||z||$, z is a column vector, z' is a row vector. Making use of the independence of random vectors $X^{(j)}$ we get

$$\mathbf{D}(S_n, z) = \sum_{j=1}^{n} \mathbf{D}(X^{(j)}, z) = (R_n z, z) = z' R_n z, \tag{7.28}$$

where R_n is the c.m. of the random vector S_n. Let H^* be the dual of an operator H. As R_n is a real matrix and $R_n^* = R_n' = R_n$, the matrix R_n is Hermitian. Then

$$\begin{aligned}(K_n z)' R_n (K_n z) = (R_n K_n z, K_n z) = (K_n z, R_n^* K_n z) = (K_n z, R_n K_n z) = \\ = (z, K_n^* R_n K_n z) = (z, K_n' R_n K_n z) = (z, Ez) = ||z||^2\end{aligned} \tag{7.29}$$

and

$$\begin{aligned}||K_n z||^2 = (K_n z, K_n z) = (z, K_n^* K_n z) = \\ = (z, K_n' K_n z) = (z, R_n^{-1} z) = z' R_n^{-1} z,\end{aligned} \tag{7.30}$$

because $K_n' K_n = R_n^{-1}$, where R_n^{-1} is the converse matrix. Using (B_γ^*), Lemma 3.1 and (7.29) we obtain

$$\begin{aligned}\left|\Gamma_l\big((X^{(j)}, z)\big)\right| \leqslant (l!)^{1+\gamma} \Big(2 \max\{K||z||, \sqrt{\mathbf{D}(X^{(j)}, z)}\}\Big)^{l-2} \mathbf{D}(X^{(j)}, z) \leqslant \\ \leqslant (l!)^{1+\gamma} ||z||^{l-2} (2 \max\{K, \lambda_1^{(j)^{1/2}}\})^{l-2} \mathbf{D}(X^{(j)}, z).\end{aligned}$$

Hence it follows

$$\left|\Gamma_l\big((S_n, z)\big)\right| \leqslant (l!)^{1+\gamma} ||z||^{l-2} M_n^{l-2} \mathbf{D}(S_n, z), \tag{7.31}$$

where M_n is defined by (7.26).

Formulas (7.29) – (7.31) imply

$$\left|\Gamma_l\big((Z_n,\, z)\big)\right| = |\Gamma_l(S_n,\, K_n z)| \leqslant (l!)^{1+\gamma}\|K_n z\|^{l-2}(K_n z)' R_n (K_n z) \leqslant \\ \leqslant (l!)^{1+\gamma}\|z\|^2 (z' R_n z)^{(l-2)/2} M_n^{l-2}.$$

Hence $\forall z \in R^k$ with $\|z\| = 1$

$$\left|\Gamma_l\big((Z_n,\, z)\big)\right| \leqslant (l!)^{1+\gamma}(z' R_n^{-1} z)^{(l+2)/2} M_n^{l-2} \leqslant \\ \leqslant (l!)^{1+\gamma} / \big((\operatorname{Sp} R_n^{-1})^{-1/2}/M_n\big)^{l-2} \tag{7.32}$$

As $z' R_n^{-1} z \leqslant \lambda_{\max}(R_n^{-1}) = \lambda_{\min}^{-1}(R_n)$, we obtain Proposition 7.1.

THEOREM 7.1. *Let condition* (B_γ^*) *be fulfilled for a random vector* $X^{(j)}$ *with* $\mathbf{E}X^{(j)} = 0$ *and c.m.* $D^{(j)}$. *Then for the random vector* Z_n

1) *if* $A \in \mathfrak{U}_1(\Delta_{n,\gamma}^*)$, *relation* (7.8) *is valid with* $H = 1$ *and* $\Delta = \Delta_n$, *where* Δ_n *is defined by* (2.2);

2) *if* $A \in \mathfrak{U}_2(\Delta_{n,\gamma}^*)$, *relation* (7.13) *is valid with* $H = 1$ *and* $\Delta = \Delta_n$;

3) *if a convex Borel set* A *satisfies a corresponding condition of Lemma 7.3, relations* (7.22) *and* (7.23) *are valid with* $H = 1$ *and* $\Delta = \Delta_n$.

Here $\Delta_{n,\gamma}^* = c_\gamma \Delta_n^{1/(1+2\gamma)}/\big((2\,k(k+1)\big)$, *and* c_γ *is defined by* (7.3).

The proof of the theorem follows from Proposition 7.1 and Lemmas 7.1 – 7.3.

b) *Sums of weighted random vectors.* Let $a_n^{(j)} = (a_{n,1}^{(j)},\, \ldots,\, a_{n,k}^{(j)})$ be a vector with nonnegative components. Denote

$$\hat{S}_n = \sum_{j=1}^n a_n^{(j)} X^{(j)}, \quad a_n^{(j)} X^{(j)} = (a_{n,1}^{(j)} X_1^{(j)},\, \ldots,\, a_{n,k}^{(j)} X_k^{(j)}),$$

$$b_n^2 = \sum_{j=1}^n a_n^{(j)2}, \quad \gamma_n = \max_{1\leqslant j\leqslant n}\ \max_{1\leqslant i\leqslant k} a_{n,i}^{(j)}, \tag{7.33}$$

$$\hat{Z}_n = \hat{S}_n/b_n = (\hat{S}_{n,1}/b_{n,1},\, \ldots,\, \hat{S}_{n,k}/b_{n,k}).$$

PROPOSITION 7.2. *Let condition* (B_γ^*) *be fulfilled for a random vector* $X^{(j)}$, $j = 1, \ldots, n$, *with* $\mathbf{E}X^{(j)} = 0$ *and the unit c.m.* E. *Then* $\forall z \in R^k$ *with* $\|z\| = 1$

$$\left|\Gamma_l\big((\hat{Z}_n,\, z)\big)\right| \leqslant \frac{(l!)^{1+\gamma}}{\hat{\Delta}_n^{l-2}}, \qquad \forall l \geqslant 3, \tag{7.34}$$

where

$$\hat{\Delta}_n = \frac{\min\limits_{1\leqslant i\leqslant k} b_{n,\,i}}{2\max\{1,\,K\}\gamma_n}. \tag{7.35}$$

Proof. As $\mathbf{E}X^{(j)} = 0$ and the c.m. of $X^{(j)}$ is unit, so $\mathbf{E}\hat{Z}_n = 0$, $\mathbf{E}(\hat{S}_{n,\,l}/b_{n,\,l})^2 = 1$, $l = \overline{1,\,k}$, and $\mathbf{E}\hat{S}_{n,\,i}\hat{S}_{n,\,j}/(b_{n,\,i}b_{n,\,j}) = 0$, $i \neq j$. Therefore, the c.m. of the random vector $\hat{Z}_n$ is unit. Next,

$$(\hat{S}_n,\,z) = \Big(\sum_{j=1}^{n} a_n^{(j)}X^{(j)},\,z\Big) = \sum_{j=1}^{n}(a_n^{(j)}X^{(j)},\,z) = \sum_{j=1}^{n}(X^{(j)},\,a_n^{(j)}z). \tag{7.36}$$

Condition (B_γ^*) in this case is written as

$$|\mathbf{E}(X^{(j)},\,z)^l| \leqslant (l!)^{1+\gamma}(K||z||)^{l-2}||z||^2, \qquad \forall l \geqslant 3.$$

Hence, in view of Lemma 3.1

$$\big|\Gamma_l\big((X^{(j)},\,z)\big)\big| \leqslant (l!)^{1+\gamma}(K^*||z||)^{l-2}||z||^2$$

and

$$\begin{aligned}\big|\Gamma_l\big((X^{(j)},\,a_n^{(j)}z)\big)\big| &\leqslant (l!)^{1+\gamma}K^{*\,l-2}||a_n^{(j)}z||^{l-2}||a_n^{(j)}z||^2 \leqslant \\ &\leqslant (l!)^{1+\gamma}K^{*\,l-2}(\gamma_n||z||)^{l-2}||a_n^{(j)}z||^2,\end{aligned} \tag{7.37}$$

where $K^* = 2\max\{1,\,K\}$ and γ_n is defined by (7.33). Consequently,

$$\begin{aligned}\big|\Gamma_l\big((\hat{S}_n,\,z)\big)\big| &\leqslant (l!)^{1+\gamma}(K^*\gamma_n)^{l-2}||z||^{l-2}\sum_{j=1}^{n}||a_n^{(j)}z||^2 = \\ &= (l!)^{1+\gamma}(K^*\gamma_n)^{l-2}||z||^{l-2}||b_nz||^2,\end{aligned} \tag{7.38}$$

b_n being defined by (7.33). Hence

$$\begin{aligned}\big|\Gamma_l\big((\hat{Z}_n,\,z)\big)\big| = \big|\Gamma_l\big((\hat{S}_n,\,z/b_n)\big)\big| &\leqslant (l!)^{1+\gamma}(K^*\gamma_n)^{l-2}||z||^{l-2}||z/b_n||^{l-2}||z||^2 \leqslant \\ &\leqslant (l!)^{1+\gamma}(K^*\gamma_n/\min_{1\leqslant i\leqslant k} b_{n,\,i})^{l-2}||z||^l.\end{aligned}$$

Consequently, $\forall z \in R^k$ with $||z|| = 1$ we obtain

$$\big|\Gamma_l\big((\hat{Z}_n,\,z)\big)\big| \leqslant (l!)^{1+\gamma}/\hat{\Delta}_n, \qquad l = 3,\,4,\,\ldots\,,$$

where $\hat{\Delta}_n$ is defined by (7.35). ∎

THEOREM 7.2. *Let condition* (B_γ^*) *be fulfilled for a random vector* $X^{(j)}$, $j = 1, \ldots, n$, *with* $\mathbf{E}X^{(j)} = 0$ *and the unit c.m.* E. *Then for the random vector* $X = \hat{Z}_n$

1) *if* $A \in \mathfrak{U}_1(\hat{\Delta}_{n,\gamma}^*)$, *relation* (7.8) *is valid with* $H = 1$ *and* $\Delta = \hat{\Delta}_n$;

2) *if* $A \in \mathfrak{U}_2(\hat{\Delta}_{n,\gamma}^*)$, *relation* (7.13) *is valid with* $H = 1$ *and* $\Delta = \hat{\Delta}_n$;

3) *if a convex Borel set* A *satisfies a corresponding condition of Lemma 7.3, relations* (7.22) *and* (7.23) *are valid with* $H = 1$ *and* $\Delta = \hat{\Delta}_n$.

Here $\hat{\Delta}_{n,\gamma}^* = c_\gamma \hat{\Delta}_n^{1/(1+2\gamma)} / \big((2\,k(k+1)\big)$, *and* c_γ *is defined by* (7.3).

The proof of the theorem follows from Proposition 7.2 and Lemmas 7.1 – 7.3.

c) *Sums of random number of random vectors.* Let $X^{(j)}$, $j = 1, 2, \ldots$, be independent random vectors with $\mathbf{E}X^{(j)} = 0$ and c.m. V. Denote $S_\omega = \sum_{j=1}^{\omega} X^{(j)}$, where the integer random variable ω does not depend on $X^{(j)}$, $j = 1, 2, \ldots$, and $\alpha = \alpha(t) = \mathbf{E}\omega$.

Introduce the following condition for the random variable ω: there exist $K_2 > 0$ and $q \geqslant 0$ such that

$$|\Gamma_l(\omega)| \leqslant l!\, K_2^{l-1} \alpha^{1+(l-1)q}, \qquad \forall l \geqslant 1. \tag{7.39}$$

Denote the covariance matrix of the random vector S_ω by R_t. It is easy to see that $R_t = \alpha V$, if $\mathbf{E}X^{(j)} = 0$.

We say that the random vector $X^{(j)}$ satisfies condition (B_0^*), if there exists $K > 0$ such that $\forall z \in R^k$

$$|\mathbf{E}(X^{(j)}, z)^l| \leqslant (l!/2)(K||z||)^{l-2} z'Vz, \qquad \forall l \geqslant 2. \tag{B_0^*}$$

Denote the normed sum by $Z_\omega = K_t S_\omega$, where K_t is such that $K_t' R_t K_t = E$.

PROPOSITION 7.3. *Let condition* (B_0^*) *be fulfilled for random vectors* $X^{(j)}$, $j = 1, \ldots, n$, *with* $\mathbf{E}X^{(j)} = 0$ *and the c.m.* V, *and condition* (7.39) *be satisfied for the r.v.* ω. *Then* $\forall z \in R^k$ *with* $||z|| = 1$

$$\big|\Gamma_l\big((Z_\omega, z)\big)\big| \leqslant \frac{3}{2}\, \frac{l!}{\Delta_t^{l-2}}, \qquad \forall l \geqslant 3, \tag{7.40}$$

where

$$\Delta_t = \frac{\sqrt{\alpha}\, \lambda_{\min}^{1/2}(V)}{\max\big\{4 \max\big(K,\ \lambda_{\max}^{1/2}(V)\big),\ \big(6\, K_2 \lambda_{\max}(V) \alpha^q\big)^{1/2}\big\}}. \tag{7.41}$$

Here $\lambda_{\min}(V)$ and $\lambda_{\max}(V)$ are the minimal and maximal eigenvalues of the c.m. V.

For example, if $\mathbf{P}(\omega = l) = t^l e^{-t}/l!$, then $\Gamma_l(\omega) = t$, $l = 1, 2, \ldots$, and in formula (2.17) one should put $K_2 = 1$ and $q = 0$.

If $\mathbf{P}(\omega = l) = \binom{l}{n} p^l (1-p)^{n-l}$, then $\mathbf{E}\omega = np$ and $|\Gamma_l(\omega)| \leqslant (l!)\, 6^{l-1} np$. Consequently, in formula (7.4) one should take $K_2 = 6$ and $q = 0$.

Proof. Let $\mathbf{E}X^{(j)} = a$. Then

$$\mathbf{E}S_\omega = \sum_l \mathbf{E}S_l \mathbf{P}(\omega = l) = \sum_l (\mathbf{E}X^{(1)}) l \mathbf{P}(\omega = l) = a\mathbf{E}\omega = a\alpha$$

and

$$\begin{aligned}
\mathbf{E}(S_{\omega,i} - \alpha a_i)^2 &= \sum_l \mathbf{E}S_{l,i}^2 \mathbf{P}(\omega = l) - (\alpha a_i)^2 = \\
&= \sum_l \left(l\mathbf{E}X_i^{(1)^2} + l(l-1)\right)\mathbf{E}X_i^{(1)}\mathbf{E}X_i^{(2)}\mathbf{P}(\omega = l) - (\alpha a_i)^2 = \\
&= \alpha\sigma_i^2 - \alpha a_i^2 + a_i^2(\beta^2 + \alpha^2) - (\alpha a_i)^2 = \alpha\sigma_i^2 + a_i^2\beta^2,
\end{aligned}$$

where $\beta^2 = \mathbf{D}\omega = \mathbf{E}\omega^2 - \alpha^2$. Further,

$$\begin{aligned}
\mathbf{E}S_{\omega,i}S_{\omega,j} &= \sum_l \mathbf{E}S_{l,i}S_{l,j}\mathbf{P}(\omega = l) = \\
&= \sum_l \left(l\mathbf{E}X_i^{(1)}X_j^{(1)} + l(l-1)a_i a_j\right)\mathbf{P}(\omega = l) = \\
&= \alpha(\mathbf{E}X_i^{(1)}X_j^{(1)} - \mathbf{E}X_i^{(1)}\mathbf{E}X_j^{(1)}) + a_i a_j(\beta^2 + \alpha^2).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{E}(S_{\omega,i} - \alpha a_i)(S_{\omega,j} - \alpha a_j) &= \mathbf{E}S_{\omega,i}S_{\omega,j} - \alpha a_i\mathbf{E}S_{\omega,j} - \alpha a_j\mathbf{E}S_{\omega,i} + \alpha^2 a_i a_j = \\
&= \alpha(\mathbf{E}X_i^{(1)}X_j^{(1)} - \mathbf{E}X_i^{(1)}\mathbf{E}X_j^{(1)}) + a_i a_j\beta^2.
\end{aligned}$$

Denote

$$A = \begin{pmatrix} a_1^2 & a_1a_2 & \ldots & a_1a_k \\ a_2a_1 & a_2^2 & \ldots & a_2a_k \\ \vdots & \vdots & \ddots & \vdots \\ a_ka_1 & a_ka_2 & \ldots & a_k^2 \end{pmatrix}.$$

It is easy to see that the c.m. of S_ω is equal to $R_t = \alpha(t)V + A\beta^2$. Hence it follows that $R_t = \alpha V$, if $a = 0$. Next, let

$$\varphi_{S_\omega}(z) := \mathbf{E}\exp\{(S_\omega, z)\}. \tag{7.42}$$

We have

$$\begin{aligned}\varphi_{S_\omega}(z) &= \sum_l \mathbf{E}\exp\{(S_l,\, z)\}\mathbf{P}(\omega = l) = \\ &= \sum_{l=1}^{\infty} \bigl(\mathbf{E}\exp\{(X^{(1)},\, z)\}\bigr)^l \mathbf{P}(\omega = l) = \\ &= \sum_{l=1}^{\infty} \exp\{l \ln \varphi_{X^{(1)}}(z)\}\mathbf{P}(\omega = l) = \mathbf{E}\exp\{\omega \ln \varphi_{X^{(1)}}(z)\}.\end{aligned} \tag{7.43}$$

Hence, using condition (7.39), we find

$$\begin{aligned}\ln \varphi_{S_\omega}(z) &= \sum_{l=1}^{\infty} \frac{1}{l!}\Gamma_l\bigl((S_\omega,\, z)\bigr) = \sum_{l=1}^{\infty}\Gamma_l(\omega)\bigl(\ln \varphi_{X^{(1)}}(z)\bigr)^l = \\ &= \alpha \ln \varphi_{X^{(1)}}(z)\left(1 + \theta\sum_{l=2}^{\infty}\bigl(K_2\alpha^q|\ln \varphi_{X^{(1)}}(z)|\bigr)^{l-1}\right) = \\ &= \theta(3/2)\alpha|\ln \varphi_{X^{(1)}}(z)|\end{aligned}$$

for $|\ln \varphi_{X^{(1)}}(z)| \leqslant (3K_2\alpha^q)^{-1}$. The condition (B_0^*) and Lemma 3.1 imply

$$\begin{aligned}\bigl|\Gamma_l\bigl((X^{(j)},\, z)\bigr)\bigr| &\leqslant l!\,\Bigl(2\max\bigl\{K||z||,\, \sqrt{\mathbf{D}(X^{(j)},\, z)}\,\bigr\}\Bigr)^{l-2}\mathbf{D}(X^{(j)},\, z) \leqslant \\ &\leqslant l!\, M^{l-2}||z||^{l-2}\mathbf{D}(X^{(j)},\, z),\end{aligned}$$

where $M = 2\max\{K,\, \lambda_{\max}^{1/2}(V)\}$. Hence

$$\begin{aligned}\ln \varphi_{X^{(1)}}(z) &= \sum_{l=2}^{\infty}\frac{1}{l!}\Gamma_l\bigl((X^{(1)},\, z)\bigr) = \theta\,\mathbf{D}(X^{(1)},\, z)\sum_{l=2}^{\infty}(M||z||)^{l-2} = \\ &= \theta\,\mathbf{D}(X^{(1)},\, z)(1 - M||z||)^{-1}\end{aligned}$$

for $M||z|| < 1$. Consequently, either the inequality $\lambda_{\max}(V)||z||^2(1 - M||z||)^{-1} \leqslant (3K_2\alpha^q)^{-1}$ holds, or

$$||z|| \leqslant \Bigl(\max\bigl\{2M,\ \bigl(6K_2\lambda_{\max}(V)\alpha^q\bigr)^{1/2}\bigr\}\Bigr)^{-1}.$$

In the last case

$$\Gamma_l\bigl((S_\omega,\, z)\bigr) = \theta(3/2)\alpha\Gamma_l\bigl((X^{(1)},\, z)\bigr). \tag{7.44}$$

As $\mathbf{D}(X^{(1)}, z) = z'Vz$ and $R_t = \alpha V$, so $\alpha\mathbf{D}(X^{(1)}, z) = \alpha z'Vz = z'R_t z$. Therefore,

$$\left|\Gamma_l\big((S_\omega, z)\big)\right| \leqslant (3/2)l!\, M^{l-2}||z||^{l-2} z' R_t z.$$

Considering that $Z_\omega = K_t S_\omega$, where $K_t' R_t K_t = E$, we find

$$\begin{aligned}\left|\Gamma_l\big((Z_\omega, z)\big)\right| &= \left|\Gamma_l\big((S_\omega, K_t z)\big)\right| \leqslant \\ &\leqslant (3/2)l!\, ||K_t z||^{l-2} M^{l-2} (K_t z)' R_t (K_t z) \leqslant \\ &\leqslant (3/2)l!\, ||z||^2 (z' R_t^{-1} z)^{(l-2)/2} M^{l-2}.\end{aligned}$$

Hence $\forall z \in R^k$ with $||z|| = 1$

$$\left|\Gamma_l\big((Z_\omega, z)\big)\right| \leqslant (3/2)l!\, /\Delta_t^{l-2}, \qquad l = 3, 4, \ldots,$$

where Δ_t is defined by (7.41). ■

THEOREM 7.3. *Let condition* (B_0^*) *be fulfilled for independent random vectors* $X^{(j)}$ *with* $\mathbf{E}X^{(j)} = 0$ *and the c.m.* V *and condition* (7.39) *be fulfilled for the r.v.* ω. *Then for the random vector* $X = Z_\omega$

1) *if* $A \in \mathfrak{U}_1(\Delta_{t,0}^*)$, *relation* (7.8) *is valid with* $H = 3/2$, $\gamma = 0$ *and* $\Delta = \Delta_t$, *where* Δ_t *is defined by* (2.17);

2) *if* $A \in \mathfrak{U}_2(\Delta_{t,0}^*)$, *relation* (7.13) *is valid with* $H = 3/2$, $\gamma = 0$ *and* $\Delta = \Delta_t$;

3) *if a convex Borel set* A *satisfies a corresponding condition of Lemma 7.3, relations* (7.22) *and* (7.23) *are valid with* $H = 3/2, \gamma = 0$ *and* $\Delta = \Delta_t$.

Here $\Delta_{t,0}^* = \big(1/(3\sqrt{e})\big)(\sqrt{2}/6)\Delta_t/\big((3\,k(1 + 3\,k/2)\big)$.

The proof of the theorem follows from Proposition 7.3 and Lemmas 7.1 – 7.3.■

d) *Quadratic forms.* Let $X(t) = \big(X_1(t), \ldots, X_k(t)\big)$ be a k–dimensional Gaussian random process with $\mathbf{E}X(t) = 0$ and for $t = 1, 2, \ldots, n$, $n \geqslant 1$, $X(t)$ has the c.m. $R_t = E$.

Denote

$$\begin{aligned}\zeta(n) &= \big(\zeta_1(n), \ldots, \zeta_k(n)\big) = \\ &= \sum_{l,m=1}^{n} \big(a_1(l, m)X_1(l)X_1(m), \ldots, a_k(l, m)X_k(l)X_k(m)\big),\end{aligned}$$

where, without losing generality, the matrices $A_i = [a_i(l, m)]_{l=\overline{1,n}}^{m=\overline{1,n}}$, $i = 1, \ldots, k$, are symmetric, because the quadratic form with a nonsymmetric matrix $\tilde{A}_i =$

$[\tilde{a}_i(l,\, m)]_{l=\overline{1,\, n}}^{m=\overline{1,\, n}}$ can be reduced to the quadratic form with a symmetric matrix A_i, i.e.

$$\sum_{l,\, m=1}^{n} \tilde{a}_i(l,\, m)X_i(l)X_i(m) = \sum_{l,\, m=1}^{n} a_i(l,\, m)X_i(l)X_i(m),$$

where $a_i(l,\, m) = (1/2)\big(\tilde{a}_i(l,\, m) + \tilde{a}_i(m,\, l)\big)$.

In the sequel let $R_i^*(n \times n)$ be the c.m. of the random vector

$$X_i' = \big(X_i(1),\, X_i(2),\, \ldots,\, X_i(n)\big), \quad i = 1,\, 2,\, \ldots,\, k,$$

where $\det R_i^* \neq 0$. Due to the fact that the quadratic forms $X_i'A_iX_i$ and $X_i'R_i^*X_i$ are symmetric, $X_i'R_i^*X_i$ being a positive definite form, one can find a real transformation, which simultaneously diagonalizes the forms $X_i'R_i^*X_i$ and $X_i'A_iX_i$, where X_i is the column vector. Consequently, the random variable

$$\zeta_i(n) = \sum_{l,\, m=1}^{n} a_i(l,\, m)X_i(l)X_i(m)$$

is distributed identically with the r.v. $\zeta_i^*(n) = \sum_{l=1}^{n} \mu_i(l)Y_i^2(l)$, where $Y_i(l)$, $l = 1,\, 2,\, \ldots,\, n$ are independent r.v. with mean 0 and variance 1, $\mu_i(l)$ are the eigenvalues of the matrix $R_i^*A_i$, satisfying the n^{th} degree algebraic equation $\det(A_i - \mu_i R_i^{*\,-1}) = 0$. Then

$$\mathbf{E}\zeta_i(n) = \sum_{l=1}^{n} \mu_i(l), \quad \nu_i^2(n) = \mathbf{D}\zeta_i(n) = 2\sum_{l=1}^{n} \mu_i^2(l). \tag{7.45}$$

Let $K_n(k \times k)$ be the diagonal matrix with elements $\nu_i^2(n)$, $i = 1,\, \ldots,\, k$. Denote

$$\tilde{\zeta}(n) = K_n^{-1/2}\big(\zeta(n) - \mathbf{E}\zeta(n)\big), \tag{7.46}$$

$$\varkappa_i(n) = \Big(\sum_{l=1}^{n} \mu_i^2(l)\Big)^{1/2} \Big/ \big(\sqrt{2} \max_{1 \leqslant l \leqslant n} |\mu_i(l)|\big),$$

$$\varkappa(n) = \min_{1 \leqslant i \leqslant k} \varkappa_i(n), \tag{7.47}$$

$$B_i = R_i^*A_i = [b_i(l,\, m)]_{l=\overline{1,\, n}}^{m=\overline{1,\, n}},$$

$$Q_i = \max_{1 \leqslant l \leqslant n} \sum_{m=1}^{n} |b_i(l,\, m)|,$$

$$P_i = \max_{1 \leqslant m \leqslant n} \sum_{l=1}^{n} |b_i(l,\, m)|.$$

It is known that

$$\max_{1\leqslant l\leqslant n} |\mu_i(l)| \leqslant \min\{P_i,\ Q_i\}.$$

PROPOSITION 7.4. *Let $X(t) = \big(X_1(t), \ldots, X_k(t)\big)$ be a k-dimensional Gaussian random process with $\mathbf{E}X(t) = 0$ and for $t = 1, 2, \ldots, n$, $n \geqslant 1$, $X(t)$ has the the c.m. $R_t \equiv E$. Moreover, let $\det R_i^* \neq 0$, where R_i^* is the c.m. of the random vector $X_i' = \big(X_i(1), \ldots, X_i(n)\big)$. Then $\forall z \in R^k$ with $\|z\| = 1$*

$$\Big|\Gamma_p\Big(\big(\tilde\zeta(n),\ z\big)\Big)\Big| \leqslant (p-1)!\,/\varkappa^{p-2}(n), \qquad p \geqslant 3, \tag{7.48}$$

where $\tilde\zeta(n)$ and $\varkappa(n)$ are defined by (7.46) and (7.47).

Proof. Since the c.m. of $X(t)$ is equal to $R(t) = E$, $\mathbf{E}Y_i(l)Y_j(m) = 0$ as $(i,\ l) \neq (j,\ m)$, $i,\ j = \overline{1,\ k}$; $l,\ m = \overline{1,\ n}$. Considering that r.v. $Y_i(l)$ are Gaussian, we have $0 = \mathbf{E}Y_i(l)Y_j(m) = \mathbf{E}Y_i(l)\mathbf{E}Y_j(m)$, i.e. the r.v. $Y_i(l)$ and $Y_j(m)$ are independent. Consequently, components of the vector are independent.

Further, let $\overline\zeta(n) = \zeta(n) - \mathbf{E}\zeta(n)$. Then the c.m. $K_n(k\times k)$ of the random vector $\overline\zeta(n)$ is a diagonal one with elements $\nu_i^2(n)$, where $\nu_i(n)$ are defined by (7.45). The characteristic function of $\overline\zeta(n)$

$$f_{\overline\zeta(n)}(t) = \mathbf{E}e^{\,i(t,\ \overline\zeta(n))} = e^{-i(t,\ \mathbf{E}\zeta(n))}\prod_{i=1}^{k} f_{\zeta_i(n)}(t_i), \tag{7.49}$$

whereas

$$f_{\zeta_i(n)}(t) = \prod_{j=1}^{n}\big(1 - 2\,it\mu_i(j)\big)^{-1/2}. \tag{7.50}$$

Let

$$\tilde\zeta(n) := K_n^{-1/2}\overline\zeta(n) = \big(\overline\zeta_1(n)/\nu_1(n),\ \ldots,\ \overline\zeta_k(n)/\nu_k(n)\big). \tag{7.51}$$

Employing relation (7.49) we have

$$\ln f_{\overline\zeta(n)}(t) = -i\big(t/\nu(n),\ \mathbf{E}\zeta(n)\big) + \sum_{i=1}^{k}\ln f_{\zeta_i(n)}\big(t_i/\nu_i(n)\big).$$

Since

$$\ln f_{\zeta_i(n)}\big(t_i/\nu_i(n)\big) = -\frac{1}{2}\sum_{j=1}^{n}\ln\big(1 - 2\,it_i\mu_i(j)/\nu_i(n)\big),$$

it follows

$$\frac{d^l}{dt_i^l}\ln f_{\zeta_i(n)}\big(t_i/\nu_i(n)\big)\Big|_{t_i=0} = (l-1)!\,2^{l-1}\nu_i^{-l}(n)i^l\sum_{j=1}^{n}\mu_i^l(j).$$

Next

$$\Gamma_l((\tilde{\zeta}(n), z)) = \frac{1}{i^l}\Big(\frac{\partial}{\partial t}, z\Big)^l \ln f_{\tilde{\zeta}(n)}(t)\Big|_{t=0} =$$

$$= \Big(\frac{\partial}{\partial t}, z\Big)^l\Big(\sum_{i=1}^k \ln f_{\zeta_i(n)}\big(t_i/\nu_i(n)\big)\Big)\Big|_{t_1=0,\dots,t_k=0} =$$

$$= (l-1)!\,2^{l-1}\sum_{i=1}^k\Big(\big(z_i/\nu_i(n)\big)^l\sum_{j=1}^n \mu_i^l(j)\Big).$$

Hence

$$\Big|\Gamma_l\Big((\tilde{\zeta}(n), z)\Big)\Big| \leqslant (l-1)!\,2^{l-1}\sum_{i=1}^k\big(\max_{1\leqslant j\leqslant n}|\mu_i(j)|\big)^{l-2}(1/2)\big(z_i/\nu_i(n)\big)^l\nu_i^2(n) =$$

$$= (l-1)!\,2^{l-2}\sum_{i=1}^k\big(\max_{1\leqslant j\leqslant n}|\mu_i(j)|/\nu_i(n)\big)^{l-2}z_i^l \leqslant$$

$$\leqslant (l-1)!\,||z||^l\max_{1\leqslant i\leqslant k}\big(2\max_{1\leqslant j\leqslant n}|\mu_i(j)|/\nu_i(n)\big)^{l-2} \leqslant (l-1)!\,||z||^l/\varkappa^{l-2}(n),\quad l\geqslant 3,$$

where $\varkappa(n)$ is defined by (7.47). ■

THEOREM 7.4. *Let $X(t) = \big(X_1(t), \dots, X_k(t)\big)$ be a k-dimensional Gaussian random process with $\mathbf{E}X(t) = 0$ and for $t = 1, 2, \dots, n$, $n \geqslant 1$, $X(t)$ have the c.m. $R_t = E$. Let $\det R_i^* \neq 0$, where R_i^* is the c.m. of the random vector $X_i' = \big(X_i(1), \dots, X_i(n)\big)$. Then for $X = \tilde{\zeta}(n)$, defined by (7.46),*

1) *if $A \in \mathfrak{U}_1\big(\varkappa_0^*(n)\big)$, relation (7.8) is valid with $H = 1$, $\gamma = 0$ and $\Delta = \varkappa(n)$;*

2) *if $A \in \mathfrak{U}_2\big(\varkappa_0^*(n)\big)$, relation (7.13) is valid with $H = 1$, $\gamma = 0$ and $\Delta = \varkappa(n)$;*

3) *if a convex Borel set A satisfies a corresponding condition of Lemma 7.3, relations (7.22) and (7.23) are valid with $H = 1, \gamma = 0$ and $\Delta = \varkappa(n)$, where $\varkappa(n)$ is defined by (7.47).*

Here $\varkappa_0^*(n) = (1/3\sqrt{e})(\sqrt{2}/6)\varkappa(n)/\big(2k(1+k)\big)$.

The proof of the theorem follows from Proposition 7.4 and Lemmas 7.1 – 7.3.

APPENDICES

Appendix 1
The proof of inequalities for moments and Lyapunov fractions

The inequality for absolute moments of the r.v. ξ

$$\beta_k \leqslant \beta_i^{(l-k)/(l-i)} \cdot \beta_l^{(k-i)/(l-i)}, \qquad 0 \leqslant i \leqslant k \leqslant l, \tag{1.19}$$

can be proved employing convexity of the function $g(t) = |\ln \beta_t|$ in the domain $0 \leqslant t \leqslant l$, i.e. $g(\lambda_1 t_1 + \lambda_2 t_2) \leqslant \lambda_1 g(t_1) + \lambda_2 g(t_2)$ for any nonnegative λ_1 and λ_2 under the condition $\lambda_1 + \lambda_2 = 1$ and for any t_1 and t_2 in $[0, l]$. Putting $t_1 = i$, $t_2 = l$, $\lambda_1 = (l-k)/(l-i)$, $\lambda_2 = (k-i)/(l-i)$, we obtain the inequality (1.19).

Now, let $\xi_1, \xi_2, \ldots, \xi_n$ be independent r.v. with $\mathbf{E}|\xi_j|^l < \infty$ for some $l > 2$ and $\mathbf{E}\xi_j = 0$, $j = 1, \ldots, n$. Consider the Lyapunov fraction of order k, $2 \leqslant k \leqslant l$,

$$L_{k,n} = \sum_{j=1}^{n} \mathbf{E}|\xi_j|^k / B_n^k,$$

where

$$B_n^2 = \mathbf{E}S_n^2, \quad S_n = \sum_{j=1}^{n} \xi_j.$$

Since

$$\overline{F}(x) = \frac{1}{n} \sum_{j=1}^{n} F_{\xi_j}(x)$$

is the distribution function, then

$$B_{k,n} = \int_{-\infty}^{\infty} |x|^k \, d\overline{F}(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{E}|\xi_j|^k = \frac{B_n^k}{n} L_{k,n}.$$

Applying (1.19) to the moments $B_{k,n}$ as $i = 2$, $2 \leqslant k \leqslant l$ we obtain

$$L_{k,n}^{1/(k-2)} \leqslant L_{l,n}^{1/(l-2)}.$$

Appendix 2
Proof of the lemma on the representation of cumulants

Formula (1.63) from Lemma 1.1 can be rewritten in the following manner:

$$\Gamma(X_{t_1}, \ldots, X_{t_k}) = \sum_{\nu=1}^{k} (-1)^{\nu-1}(\nu-1)! \sum_{\bigcup_{p=1}^{\nu} I_p = I} \prod_{p=1}^{\nu} \mathbf{E}(X_{I_p}), \tag{A1}$$

where $\sum_{\bigcup_{p=1}^{\nu} I_p=I}$ stands for summation over all ν-block partitions $\{I_1, \ldots, I_\nu\}$ of the set $I = \{t_1, \ldots, t_k\}$ into subsets I_p, $p = 1, \ldots, \nu$. Since $\Gamma(X_{t_1}, \ldots, X_{t_k})$ is a symmetric function, without loss of generality one may assume $t_1 \leqslant \ldots \leqslant t_k$. Let the multipliers $X_{t_1^{(p)}}, \ldots, X_{t_{k_p}^{(p)}}$ in

$$\mathbf{E}(X_{I_p}) = \mathbf{E}X_{t_1^{(p)}} \cdot \ldots \cdot X_{t_{k_p}^{(p)}}$$

be arranged so that $t_1^{(p)} \leqslant \ldots \leqslant t_{k_p}^{(p)}$, $p = 1, \ldots, \nu$. The expression $\prod_{p=1}^{\nu} \mathbf{E}(X_{I_p})$ represents the product of mathematical expectations. Partition of the set I into subsets I_p shows how the random variables $X_{t_1}, \ldots, X_{t_k}$, are distributed into multipliers of the product $\prod_{p=1}^{\nu} \mathbf{E}(X_{I_p})$. Introduce the numbers m_r, $r = 0, 1, \ldots, k-1$, as follows. The r.v. $X_{t_{r+1}}$ belongs to one of the multipliers $\mathbf{E}(X_{I_p})$, $p = 1, \ldots, \nu$. If the r.v. X_{t_i} precedes the r.v. $X_{t_{r+1}}$ in this multiplier, then we assume $m_r = i$. However, if $X_{t_{r+1}}$ is leftmost in this multiplier, then we assume $m_r = 0$. The numbers m_r, $r = 0, 1, \ldots, k-1$, are defined uniquely. Obviously $m_0 = 0$. Put

$$\prod_{p=1}^{\nu} \mathbf{E}(X_{I_p}) = W_{\nu-1}(\mathfrak{m}). \tag{A2}$$

The set $\mathfrak{m} = \{m_1, \ldots, m_{k-1}\}$ in $W_{\nu-1}(\mathfrak{m})$ contains the $(k-\nu)^{\text{th}}$ order permutation of the numbers $1, \ldots, k-1$, while the remaining $\nu - 1$ numbers are equal to zero. Besides, $m_r \leqslant r$, $r = 1, \ldots, k-1$. A set of such sets denote by $\mathfrak{M}_{\nu-1}$. Then, in accord with $(A1)$ and $(A2)$

$$\Gamma(X_{t_1}, \ldots, X_{t_k}) = \sum_{\nu=0}^{k-1} (-1)^{\nu}\nu! \sum_{\mathfrak{m}\in\mathfrak{M}_\nu} W_\nu(\mathfrak{m}). \tag{A3}$$

Thus, for example, in the new notation

$$\Gamma(X_{t_1}, X_{t_2}, X_{t_3}) = W_0(1, 2) - W_1(0, 2) - W_1(1, 0) - \\ - W_1(0, 1) + 2W_2(0, 0),$$

$$\widehat{\mathbf{E}}\, X_{t_1} X_{t_2} X_{t_3} = W_0(1, 2) - W_1(0, 2) - W_1(1, 0) + W_2(0, 0),$$

$$\mathbf{E} X_{t_2} \widehat{\mathbf{E}}\, X_{t_1} X_{t_3} = W_1(0, 1) - W_2(0, 0),$$

and, consequently,

$$\Gamma(X_{t_1}, X_{t_2}, X_{t_3}) = \widehat{\mathbf{E}}\, X_{t_1} X_{t_2} X_{t_3} - \mathbf{E} X_{t_2} \widehat{\mathbf{E}}\, X_{t_1} X_{t_3}.$$

Let us take some member of the sum $(A3)$

$$W_\nu(\overline{\mathfrak{m}}), \quad \overline{\mathfrak{m}} = \{\overline{m}_1, \ldots, \overline{m}_{k-1}\} \in \mathfrak{M}_\nu. \tag{A4}$$

There are $\nu!$ such members, and they are included in the sum $(A3)$ with the sign $(-1)^\nu$. Among $\overline{m}_r$, $r = 1, \ldots, k-1$, there are ν zeroes. Let

$$\overline{m}_{r_1} = \overline{m}_{r_2} = \ldots = \overline{m}_{r_\nu} = 0, \qquad r_1 < r_2 < \ldots < r_\nu.$$

For brevity set $m_i^j = \{m_i, \ldots, m_j\}$, $1 \leqslant i \leqslant j \leqslant k-1$. By $\mathfrak{m}_r(s)$ denote a vector, whose coordinate m_r equals s, and let $q_r(\mathfrak{m})$ be the number of coordinates of the vector $\mathfrak{m}_{r+1}^{k-1}$, equal to one of the numbers $1, \ldots, r$.

Among summands of the sum

$$(-1)^{\nu-1}(\nu-1)! \sum_{\mathfrak{m} \in \mathfrak{M}_{\nu-1}} W_{\nu-1}(\mathfrak{m}) \tag{A5}$$

there are summands, corresponding to the values $\mathfrak{m} \in \mathfrak{M}_{\nu-1}$ of type $\mathfrak{m}_{r_i}(s)$, $s > 0$. Denote a set of such vectors by $\mathfrak{M}_{\nu-1}(\mathfrak{m}_{r_i}(\cdot))$. Let $N(A)$ be the number of elements of a finite set A. Then

$$N\Big(\mathfrak{M}_{\nu-1}(\overline{\mathfrak{m}}_{r_i}(\cdot))\Big) = i - q_{r_i}(\overline{\mathfrak{m}}), \tag{A6}$$

as far as in $\overline{\mathfrak{m}}_{r_i}(s)$ may assume all values except for coinciding with $\overline{m}_{r_i-1}, \ldots, \overline{m}_{k-1}$ (there will be $q_{k_i}(\mathfrak{m})$ of them), and those assumed by the coordinates $\overline{m}_1, \ldots, \overline{m}_{r_i-1}$ (there will be $k_i - 1 - (i-1) = k_i - 1$ of them, because among $\overline{m}_1, \ldots, \overline{m}_{r_i-1}$ there are $i-1$ zeroes).

Similarly among the summands of

$$(-1)^{\nu+1}(\nu+1)! \sum_{\mathfrak{m} \in \mathfrak{M}_{\nu+1}} W_{\nu+1}(\mathfrak{m}) \tag{A7}$$

there are summands, corresponding to the values $\mathfrak{m} \in \mathfrak{M}_{\nu+1}$ of type $\mathfrak{m} = \overline{\mathfrak{m}}_l(0)$, $l = 1, \ldots, k-1$, $l \neq r_1, \ldots, r_\nu$. The number of such summands for the given l is equal to $(\nu+1)!$, and they are included with the sign $(-1)^{\nu-1}$ opposite to the sign of the summand $(A4)$. Denote

$$W_\nu\big(\mathfrak{m}_{k-1}(\widehat{m}_{k-1})\big) := W_\nu(\mathfrak{m}_1^{k-2}, \widehat{m}_{k-1}) = W_\nu(\mathfrak{m}) - W_{\nu+1}\big(\mathfrak{m}_{k-1}(0)\big),$$

$$W_\nu(\mathfrak{m}_1^{k-3}, \widehat{m}_{k-2}, \widehat{m}_{k-1}) = W_\nu\big(\mathfrak{m}_{k-2,\,k-1}(\widehat{m}_{k-2}, \widehat{m}_{k-1})\big) =$$
$$= W_\nu\big(\mathfrak{m}_{k-1}(\widehat{m}_{k-1})\big) - W_{\nu+1}\big(\mathfrak{m}_{k-2,\,k-1}(0, \widehat{m}_{k-1})\big),$$

and in general

$$W_\nu(\mathfrak{m}_1^{r-1}, \widehat{m}_r, \widehat{\mathfrak{m}}_{r+1}^{k-1}) = W_\nu(\mathfrak{m}_1^r, \widehat{\mathfrak{m}}_{r+1}^{k-1}) - W_{\nu+1}(\mathfrak{m}_1^{r-1}, 0, \widehat{\mathfrak{m}}_r^{k-1}), \qquad (A8)$$

only the numbers m_r unequal to zero being marked by the sign " $\frown$". The sign " $\frown$" over the vector $\mathfrak{m}$ (i.e. $\widehat{\mathfrak{m}}$) means that all the coordinates unequal to zero are already marked.

We shall show that either all nonzero coordinates of the vector $\overline{\mathfrak{m}}$ in the member $(A4)$ will be marked by the sign " $\frown$" after the calculations indicated, or this member $(A4)$ will vanish before that.

Note, that in order to form

$$W_\nu(\overline{\mathfrak{m}}_1^{r_\nu-1}, 0, \widehat{\overline{\mathfrak{m}}}_{r_\nu+1}^{k-1}) \qquad (A9)$$

it is quite enough of the summands

$$W_{\nu+1}\big(\overline{\mathfrak{m}}_1^{r_\nu}, \widehat{\overline{\mathfrak{m}}}_{r_\nu+1}^{k-1}(m_l)\big), \qquad (A10)$$

$l = r_\nu + 1, \ldots, k-1$, as far as there are $(\nu+1)!$ summands $(A8)$ all in all in the sum $(A7)$ for the given l, while in $W_\nu(\overline{\mathfrak{m}})$ the coordinate m_l can assume not more than $(\nu+1)$ positive value and there are $\nu!$ summands of $W_\nu(\overline{\mathfrak{m}})$. Some summands of type $(A9)$ will be necessary for us to form the members

$$W_{\nu-1}(\overline{\mathfrak{m}}_1^{r_\nu-1}, \widehat{m}_{r_\nu}, \widehat{\overline{\mathfrak{m}}}_{r_\nu+1}^{k-1}), \qquad m_{r_\nu} > 0. \qquad (A11)$$

In accord with $(A6)$ $N\Big(\mathfrak{M}_{\nu-1}\big(\overline{\mathfrak{m}}_r(\,\cdot\,)\big)\Big) = \nu - q_{r_\nu}(\overline{\mathfrak{m}})$ here. Consequently, after forming $\widehat{m}_{r_\nu}$ in $(A11)$ the number of the rest members $(A9)$ will be equal to

$$N_{\nu,\,r_\nu}(\overline{\mathfrak{m}}) = \nu! - (\nu-1)!\big(\nu - q_{r_\nu}(\overline{\mathfrak{m}})\big) = (\nu-1)!q_{r_\nu}(\overline{\mathfrak{m}}). \qquad (A12)$$

If we denote by $N_{\nu,\,r_i}(\overline{\mathfrak{m}})$ the number of members of $(A9)$, remaining after forming

$$W_{\nu-1}(\overline{\mathfrak{m}}_1^{r_i-1}, \widehat{\overline{\mathfrak{m}}}_{r_i}, \widehat{\overline{\mathfrak{m}}}_{r_i+1}^{k-1}),$$

then for $N_{\nu,\,r_i}(\overline{\mathfrak{m}})$ we obtain a recurrent relation

$$N_{\nu,\,r_i}(\overline{\mathfrak{m}}) = N_{\nu,\,r_{i+1}}(\overline{\mathfrak{m}}) - N_{\nu-1,\,r_{i+1}}(\overline{\mathfrak{m}}_{r_i}(m>0))N\Big(\mathfrak{M}_{\nu-1}(\mathfrak{m}_{r_i}(\,\cdot\,))\Big), \qquad (A13)$$
$$i = 1,\,\dots,\,\nu-1.$$

Here m is some value $0 < m \leqslant r_i$ with the only condition $\overline{\mathfrak{m}}_{r_i}(m) \in \mathfrak{M}_{\nu-1}$. It is obvious that if m and m' are two such values, then

$$N_{\nu-1,\,r_{i+1}}(\overline{\mathfrak{m}}_{r_i}(m')) = N_{\nu-1,\,r_{i+1}}(\overline{\mathfrak{m}}_{r_i}(m)),$$

therefore, from $(A6)$, $(A12)$ and $A(13)$ we derive

$$N_{\nu,\,r_i}(\overline{\mathfrak{m}}) = (i-1)!\,\prod_{j=1}^{\nu} q_{r_j}(\overline{\mathfrak{m}}). \qquad (A14)$$

If there exists a number i_0 such that $q_{r_{i_0}}(\overline{\mathfrak{m}}) = 0$ and $q_{r_i}(\mathfrak{m}) > 0$ as $i > i_0$, then, as it follows from relation $(A14)$, $N_{\nu,\,r_{i_0}}(\overline{\mathfrak{m}}) = 0$. But summands of the type

$$W_{\nu-1}\big(\overline{\mathfrak{m}}_{1,\;r_1,\dots,\,r_{i_0-1}}^{r_{i_0}-1}(m_{r_1},\,\dots,\,m_{r_{i_0-1}}),\,\widehat{\overline{\mathfrak{m}}}_{r_{i_0}}^{k-1}\big)$$

are already missing, because

$$q_{r_{i_0}}\big(\overline{\mathfrak{m}}_{r_1,\dots,\,r_{i_0-1}}(m_{r_1},\,\dots,\,m_{r_{i_0-1}})\big) = q_{r_{i_0}}(\overline{\mathfrak{m}}) = 0$$

will enter formula $(A14)$ for

$$N_{\nu-1,\,r_{i_0}}\big(\overline{\mathfrak{m}}_{r_1,\dots,\,r_{i_0-1}}(m_{r_1},\,\dots,\,m_{r_{i_0-1}})\big)$$

as a multiplier.

Thus, the summands $W_{\nu+1}(\mathfrak{m})$ necessary to form $W_\nu(\widehat{\overline{\mathfrak{m}}})$ will not vanish until $W_\nu(\overline{\mathfrak{m}})$ vanishes or until all the coordinates of the vector $\overline{\mathfrak{m}}$ are marked. So it is proved that all the nonzero coordinates of the vector $\overline{\mathfrak{m}}$ in the member $(A4)$ will be marked by the sign " $\frown$" or the member $(A4)$ itself will vanish before that.

Let $N_\nu(\mathfrak{m})$ be the number of summands $W_\nu(\mathfrak{m})$ remained after marking by the sign " $\frown$". Let $r_{i_0} = r_{i_0}(\mathfrak{m})$ be the leftmost of zeroes of the vector $\mathfrak{m}$, used to form $W_{\nu-1}(\widehat{\mathfrak{m}})$. Then, according to $(A14)$ we have

$$N_\nu(\mathfrak{m}) = (i_0-1)\prod_{j=i_0}^{\nu} q_{r_j}(\mathfrak{m}), \qquad (A15)$$

where $r_i = r_i(\mathfrak{m})$, $i = 1, \ldots, \nu$, are the indices of zero coordinates of the vector $\mathfrak{m} \in \mathfrak{M}_\nu$.

If $N_\nu(\mathfrak{m}) > 0$, then it follows from $(A13)$ that $N\Big(\mathfrak{M}_{\nu-1}\big(\mathfrak{m}_{r_{i_0-1}}(\cdot)\big)\Big) = 0$ or, according to $(A6)$,

$$q_{r_{i_0-1}}(\mathfrak{m}) = i_0 - 1. \qquad (A16)$$

Denote

$$d(\mathfrak{m}) = \sum_{r=1}^{k-1} d(m_r),$$

where

$$d(m_r) = \begin{cases} t_{r+1} - t_{m_r}, & \text{if } m_r > 0, \\ 0, & \text{if } m_r = 0. \end{cases} \qquad (A17)$$

It is easy to show that $d(\mathfrak{m}) = \max\limits_{1 \leqslant i,\ j \leqslant k} (t_i - t_j)$ as $N_\nu(\mathfrak{m}) > 0$.

In fact, let

$$U_r = \begin{cases} r + 1 - m_r & \text{for } m_r > 0, \\ 0 & \text{for } m_r = 0. \end{cases} \qquad (A18)$$

Then, it follows from $(A16)$ that among $m_{r_{i_0-1}+1}, \ldots, m_{k-1}$ there is $i_0 - 1$ number from the sequence $1, 2, \ldots, r_{i_0-1}$. By virtue of

$$N_\nu(\mathfrak{m}) > 0$$

it follows from $(A15)$ that $q_{r_j}(\mathfrak{m}) \geqslant 1$, $j = j_0, \ldots, \nu$.

There are ν zeroes among U_r, $r = 1, \ldots, k-1$. Consequently

$$\sum_{r=1}^{k-1} U_r \geqslant \sum_{r=2}^{i_0} r + 2(\nu + 1 - i_0) + (k - 1 - 2\nu) =$$
$$= k - 1 + (i_0 - 1)\big((i_0 + 2)/2 - 2\big) \geqslant k - 1.$$

Hence it follows the validity of relation $(A18)$.

Thus

$$\Gamma(X_{t_1}, \ldots, X_{t_k}) = \sum_{\nu=0}^{k-1} (-1)^\nu \sum_{m \in \mathfrak{M}_\nu} N_\nu(\mathfrak{m}) W_\nu(\widehat{\mathfrak{m}}),$$

where $\mathfrak{M}_\nu$ is the set of all vectors $\mathfrak{m} = (m_1, \ldots, m_{k-1})$ such that $(m_1, \ldots, m_{k-1})$ is permutation of order $k - 1 - \nu$ of numbers $1, 2, \ldots, k$, and the remaining ν numbers m_i are equal to zero, besides $m_r \leqslant r$, $r = 1, \ldots, k-1$. The number $N_\nu(\mathfrak{m})$ is defined by relation $(A15)$, while $W_\nu(\mathfrak{m})$, $W_\nu(\widehat{\mathfrak{m}})$ by relations $(A2)$ and

$(A8)$. If $N_\nu(\mathfrak{m}) > 0$, then $d(\mathfrak{m})$ in $(A17)$ is equal to $\max\limits_{1\leqslant i,\, j\leqslant k}(t_i - t_j)$. It follows from $(A15)$ that $N_\nu(\mathfrak{m}) \leqslant \nu!$. Taking into account $(A1)$ and the sense of operation " $\frown$", it is easy to get convinced that

$$W_{\nu-1}(\widehat{\mathfrak{m}}) = \prod_{p=1}^{\nu} \widehat{\mathbf{E}}(X_{I_p}).$$

Denoting $N_\nu(I_1, \ldots, I_p) = N_{\nu-1}(\mathfrak{m})$, we obtain

$$\sum_{\mathfrak{m}\in\mathfrak{M}_{\nu-1}} N_{\nu-1}(\mathfrak{m}) W_{\nu-1}(\widehat{\mathfrak{m}}) = \sum_{\bigcup\limits_{p=1}^{\nu} I_p = I} N_\nu(I_1, \ldots, I_\nu) \prod_{p=1}^{\nu} \widehat{\mathbf{E}}(X_{I_p}),$$

$$N_\nu(I_1, \ldots, I_\nu) \leqslant (\nu-1)!, \qquad 1 \leqslant \nu \leqslant k,$$

and (1.63) follows from $(A3)$. The lemma is proved.

Appendix 3
Leonov – Shiryaev's formula

Let

$$\eta_j = \sum_{\nu_j} a_j(\nu_j)\xi_j^{\nu_j}, \qquad j = 1, 2, \ldots, n,$$

where $\nu_j = (\nu_{j1}, \ldots, \nu_{jm_j})$, $\nu_{jk} \in \{0, 1\}$, $m_j \in \{1, 2, \ldots\}$, $a_j(\nu_j)$ are real coefficients, $\xi_j^{\nu_j} = \xi_{j1}^{\nu_{j1}} \cdot \ldots \cdot \xi_{jm_j}^{\nu_{jm_j}}$, ξ_{jk} are real random variables, $\sum\limits_{\nu_j}$ is the sum over all possible ν_j.

Denote

$$D = \{(1, 1), \ldots, (1, m_1)(2, 1), \ldots, (2, m_2), \ldots, (n, 1), \ldots, (n, m_n)\}.$$

Unordered collection $\{D_1, \ldots, D_q\}$, $1 \leqslant q \leqslant m_1 + \ldots + m_n$, of unordered sets $D_p \subset D$ is called a *partition* of the set D, if $\bigcup\limits_{p=1}^{q} D_p = D$, $D_p \neq \varnothing$ and $D_p \cap D_r = \varnothing$ as $p \neq r$. The partition of the set D will be written as follows:

$$D = D_1 + \ldots + D_q.$$

Partition $D = D_1 + D_2$ is called a *row partition* in case for every row of the table

$$\begin{array}{c} (1, 1), \ldots, (1, m_1) \\ \ldots\ldots\ldots\ldots\ldots \\ (n, 1), \ldots, (n, m_n) \end{array}$$

we have: either all its elements belong to D_1 or all of its elements belong to D_2. Partition $D = D_1 + \ldots + D_q$ is called *indecomposable* if there does not exist a row partition $D = D' + D''$ such that for any p either $D_p \subset D'$, or $D_p \subset D''$.

The sets D_p and D_r *engage* if there exist (j, k) and (j, l) such that (j, k) belongs to one of them and (j, l) to the other. The sets $D_{p'}$ and $D_{p''}$ in partition $D = D_1 + \ldots + D_q$ *communicate* if in this partition there exist sets $D_{p_1} = D_{p'}, D_{p_2}, \ldots, D_{p_r} = D_{p''}$ such that D_{p_j} and $D_{p_{j+1}}$ engage.

Obviously partition $D = D_1 + \ldots + D_q$ is indecomposable if and only if every two sets in it communicate.

Next, let $\nu = (\nu_1, \ldots, \nu_n)$, $a(\nu) = a_1(\nu_1)\ldots a_n(\nu_n)$ and $\sum\limits_{\nu} = \sum\limits_{\nu_1, \ldots, \nu_n}$. The following two formulas (Leonov, Shiryaev, 1959):

$$\mathrm{E}\eta_1 \ldots \eta_n = \sum_{\nu} a(\nu) \sum_{D(\nu)=D_1+\ldots+D_q} \Gamma(\xi_{D_1}) \ldots \Gamma(\xi_{D_q}) \tag{A19}$$

and

$$\Gamma(\eta_1, \ldots, \eta_n) = \sum_{\nu} a(\nu) \sum_{D(\nu)=D_1+\ldots+D_q}^{*} \Gamma(\xi_{D_1}) \ldots \Gamma(\xi_{D_q}) \tag{A20}$$

are valid, where

$D(\nu) = \{(i, j) \in D : \nu_{ij} = 1\}$,

$D_p = (D_{p1}, \ldots, D_{p\mu_p})$, $\quad D_{pr} \in D$,

$\xi_{D_p} = (\xi_{D_{p1}}, \ldots, \xi_{D_{p\mu_p}})$, $\quad \xi_{D_{pr}} = \xi_{jk}$, if $D_{pr} = (j, k)$,

$\sum\limits_{D(\nu)=D_1+\ldots+D_q}$ is the sum over all possible partitions of the set $D(\nu)$,

$\sum\limits_{D(\nu)=D_1+\ldots+D_q}^{*}$ is the sum over all possible indecomposable partitions of the set $D(\nu)$. In both sums summation is taken over $q = 1, 2, \ldots$.

REFERENCES

Achmedov S. A. (1990). On nonuniform estimations in the central limit theorem for dependent random variables. *Lithuanian Math. J.*, **30**, 623–629.

Aigner M. (1979). *Combinatorial Theory*, Springer–Verlag, Heidelberg, New York.

Aleškevičienė A. (1983). Multidimensional integral limit theorems for large deviations. *Theory Probab. Appl.*, **28**, 62–82.

Aleškevičienė A. (1990). Probabilities of large deviations for U–statistics and von Mises functionals. *Theory Probab. Appl.*, **35**, 3–14.

Amosova N. N. (1990). On the necessity of Cramer's condition in local limit theorems. *Probab. Theory and Math. Statist. Proceedings of the Fifth Vilnius Conference*, **1**, VSP, Utrecht, The Netherlands, Mokslas, Vilnius, Lithuania, 52–56.

Andrews G. E. (1976). *The Theory of Partitions*, Addison–Wesley, Reading, Mass.

Arak T. V. and Zaitsev A. Yu. (1986). Uniform limit theorems for sums of independent random variables. *Proc. Steklov Inst. Math.*, **174**, Nauka, Leningrad.

Bahadur R. R. (1960). On the asymptotic efficiency of tests and estimates. *Sankhya*, **22**, 229–252.

Bahr B. von (1967). Multidimensional integral limit theorems for large deviations. *Arkiv Math.*, **7**, 89–99.

Basalykas A., Plikusas A. and Statulevičius V. (1987). Theorems of large deviations for multinomial forms and multiple stochastic integrals. *Proceedings of the First World Congress of the Bernoulli Soc.*, VNU Science Press, Utrecht, The Netherlands, 629–639.

Basalykas A. (1984). Some asymptotical properties of polynomial Pitman–Linnik estimators. *Lithuanian Math. J.*, **24**, 16–29.

Basalykas A. (1985). Some asymptotical properties of the polynomial and modified polynomial Pitman–Linnik estimators. *Soviet Math. Dokl.*, **280**, 1037–1039.

Basalykas A. (1988). Some asymptotical properties for distributions of the polynomial forms. *Lithuanian Math. J.*, **28**, 644–654.

Bentkus R. (1972). On the error of the estimate of the spectral function of a stationary process. *Lithuanian Math. J.*, **12**, 55–71.

Bentkus R. (1976). On cumulants of the spectrum estimation of a stationary time series. *Lithuanian Math. J.*, **16**, 37–61.

Bentkus R. and Rudzkis R. (1976). The large deviations for estimates of spectrum of the Gaussian stationary time series. *Lithuanian Math. J.*, **16**, 63–77.

Bentkus R. and Rudzkis R. (1980). On exponential estimates of the distribution of random variables. *Lithuanian Math. J.*, **20**, 15–30.

Bentkus R. and Rudzkis R. (1983). On the distribution of some statistical estimates of spectral density. *Theory Probab. Appl.*, **27**, 795–814.

Bentkus V. (1986). On large deviations in Banach spaces. *Theory Probab. Appl.*, **31**, 710–716.

Bhattacharya R. N. and Ranga Rao R. (1976). *Normal Approximation and Asymptotic Expansions*, Wiley, New York, London, Sydney, Toronto.

Bhattacharya R. N. and Puri M. L. (1983). On the order of magnitude of cumulants of von Mises functionals and related statistics. *Ann. Probab.*, **11**, 346–359.

Bikelis A. (1967). On the remainder terms of the asymptotic expansions of the characteristic functions and their derivatives. *Lithuanian Math. J.*, **7**, 571–582.

Bikelis A. (1968). The asymptotic expansions of the distribution function and of density function for the sums of independent identically distributed random vectors. *Lithuanian Math. J.*, **8**, 405–422.

Bikelis A. (1971, 1972). On the central limit theorem in R^k I, II. *Lithuanian Math. J.*, **11**, 17–58, **12**, 53–84.

Bikelis A. and Žemaitis A. (1974). Asymptotische Entwicklung in Grenzwertsätze für grosse Abweichungen II. *Lithuanian Math. J.*, **14**, 45–52.

Bikelis A. and Žemaitis A. (1976). Asymptotische Entwicklung in Grenzwertsätze für grosse Abweichungen. Normalische Approximation III. *Lithuanian Math. J.*, **16**, 31–50.

Blum J. R., Hanson D. L. and Koopmans L. H. (1963). On the strong law of large numbers for a class of stochastic processes. *Z. Wahrsch. verw. Geb.*, **2**, 1–11.

Book S. A. (1972). Large deviation probabilities for weighted sums. *Ann. Math. Statist.*, **43**, 1221–1234.

Book S. A. (1973). A large deviation theorem for weighted sums. *Z. Wahrsch. verw. Geb.*, **26**, 43–49.

Borovkov A. A. (1962). New limit theorems in boundary problems for sums of independent random variables. *Sib. Math. J.*, **3**, 645–694.

Borovkov A. A. (1964). Investigation on large deviations in boundary problems with arbitrary boundaries I, II. *Sib. Math. J.*, **2**, 253–289, **4**, 750–767.

Borovkov A. A. and Rogozin B. A. (1965). On the multi–dimensional central limit theorem. *Theory Probab. Appl.*, **10**, 55–62.

Borovkov A. A. (1967). Boundary–value problems for random walks and large deviations in function spaces. *Theory Probab. Appl.*, **12**, 575–595.

Borovkov A. A. (1976). *Probability Theory*, Nauka, Moscow.

Borovkov A. A. and Mogulskii A. A. (1978, 1980). On probabilities of large deviations in topologic spaces I, II. *Sib. Math. J.*, **19**, 988–1004, **21**, 12–26.

Borovkov A. A. (1983). Boundary problems, invariance principle and large deviations. *Usp. Math. Nauk,* **38**, 227–254.

Borovkov A. A. (1984). *Mathematical Statistics,* Nauka, Moscow.

Borovkov A. A. and Mogulskii A. A. (1985). Uniform theorems on large deviations for sums of random vectors. *Preprint,* **21**, Inst. Math. Sib. Branch Acad. Sci. USSR, Novosibirsk, 3–48.

Borovskikh Yu. V. (1980). Approximation problem for distributions of U–statistics and von Mises functionals. *Preprint,* **7**, Inst. Math. Acad. Sci. Ukrainian SSR, Kiev, 31–36.

Brown B. M. (1970). Characteristic functions, moments and the central limit theorem. *Ann. Math. Statist.,* **41**, 658–664.

Cramér H. (1938). Sur un nouveau théorème–limite de la théorie de probabilités. Act. Sci. Ind., **736**.

Dasgupta R. (1984). On large deviation probabilities of U–statistics in non i.i.d. case. *Sankhya,* **A46**, 110–116.

Davydov Yu. A. (1968). On convergence of distributions generated by stationary stochastic processes. *Theory Probab. Appl.,* **13**, 730–737.

Dobrushin R. L. (1953). Limit laws for Markov chains. *Izvestija AN SSSR. Ser. Math.,* **17**, 291–330.

Dobrushin R. L. (1956). Central limit theorem for non–stationary Markov chains I, II. *Theory Probab. Appl.,* **1**, 72–89, **1**, 365–425.

Donsker M. D. and Varadhan S. R. S. (1975, 1976). Asymptotic evaluation of certain Markov process expectations for large time. *Comm. Pure Appl. Math.,* **28**, 1–47, **28**, 279–301, **29**, 389–461.

Engel D. (1982). The multiple stochastic integral. *Memoirs of the American Math. Soc.,* **38**, 1–82.

Esseen C.–G. (1944). Fourier analysis of distribution functions. A mathematical study of the Laplace–Gaussian law. *Acta Math.,* **77**, 1–125.

Feller W. (1969). Limit theorems for probabilities of large deviations. *Z. Wahrsch. verw. Geb.,* **14**, 1–20.

Formanov Sh. K. (1973). Some limit theorems on large deviations for homogeneous Markov chains. *Random processes and statistical decisions,* **3**, Tashkent, 173–185.

Fuk D. Kh. and Nagajev S. V. (1971). Probability inequalities for the sums of independent random variables. *Theory Probab. Appl.,* **26**, 660–675.

Gečiauskas E. (1974). Large deviations in the case of domain. *Lithuanian Math. J.,* **14**, 71–77.

Gnedenko B. V. and Kolmogorov A. N. (1968). *Limiting Distributions for Sums of Independent Random Variables,* Addison–Wesley, Reading, Mass.

Groeneboom P. and Shorack G. R. (1981). Large deviations of fit statistics and linear combinations of order statistics. *Ann. Probab.,* **9**, 971–987.

Heinrich L. (1980). Integrale Grenzwertsätze für Wahrscheinlichkeiten grosser Abweichungen im Falle homogen Markovsch verbundener Zufallsgrössen. *Lithuanian Math. J.*, **20**, 193–214.

Heinrich L. (1982). Über eine Faktorisierung der charakteristischen Funktion einer Summe abhängiger Zufallsgrössen. *Lithuanian Math. J.*, **22**, 190–202.

Heinrich L. (1983). Über Wahrscheinlichkeiten grosser Abweichungen von Summen Markovsch verbundener Zufallsgrössen bei Nichterfüllung der Cramerschen Bedingung. *Lithuanian Math. J.*, **23**, 211–223.

Heinrich L. (1985). Non-uniform estimates, moderate and large deviations in the central limit theorem for m-dependent random variables. *Math. Nachr.*, **121**, 107–121.

Heinrich L. (1986). Beitrage zur Summationstheorie von Folgen und Feldern m-abhängiger Zufallsgrössen. *Dissertation*, Fakultät für Math., Bergakademie, Freiberg.

Heyde C. C. (1968). On large deviation probabilities in the case of attraction to a nonnormal stable law. *Sankhya*, **A30**, 253–258.

Hoeffding W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.*, **19**, 293–325.

Hoeffding W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Ass.*, **58**, 13–32.

Hoeffding W. (1967). On probabilities of large deviations. *Proceedings of the Fifth Berkeley Symp. on Math. Statist. and Probab.*, **1**, Univ. Calif. Press, Berkeley, Los Angeles, 203–219.

Ibragimov I. A. (1959). Some limit theorems for strongly stationary random process. *Sov. Math. Dokl.*, **125**, 711–714.

Ibragimov I. A. (1966). On the accuracy of Gaussian approximation to the distribution functions of sums of independent random variables. *Theory Probab. Appl.*, **11**, 632–655.

Ibragimov I. A. (1967). On the Chebyshev–Cramer asymptotic expansions. *Theory Probab. Appl.*, **12**, 455–469.

Ibragimov I. A. and Linnik Yu. V. (1971). *Independent and Stationary Sequences of Random Variables*, Wolters–Noordhoff, Groningen.

Iosifescu M. (1980). Recent advances in mixing sequences of random variables. *Proceedings of the Third International Summer School on Probab. Theory and Math. Statist.*, Sofia, Varna, 111–138.

Ito K. (1951). Multiple Wiener integral. *J. Math. Soc. Japan*, **3**, 157–169.

Jakimavičius D. and Statulevičius V. (1987). Estimates of cumulants and centered moments of mixing random process. *Preprint*, **3**, Inst. Math. Cybern. Lithuanian Acad. Sci., Vilnius.

Jakimavičius D. (1988). On the estimates of cumulants and centered moments of mixing random process. *Lithuanian Math. J.*, **28**, 614–626.

Jakševičius Š. (1983 – 1985). Asymptotic expansions for probability distributions I – IV. *Lithuanian Math. J.*, **23**, 73–83, **23**, 196–213, **24**, 216–223, **25**, 194–208.

Kagan A. M. (1966). On the estimation theory of location parameters. *Sankhya*, **28**, 335–352.

Kagan A. M., Klebanov L. B. and Fintushal S. M. (1974). Asymptotic behaviour of Pitman polynomial estimates. *Zapiski Nauch. Semin. Leningrad Math. Institute,* **43**, 24–39.

Kolmogorov A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung,* Springer, Berlin.

Kolmogorov A. N. and Fomin S. V. (1968). *Elements of the Theory of Functions and Functional Analysis,* Nauka, Moscow.

Kubilius J. (1964). *Probabilistic Methods in the Theory of Numbers,* American Mathematical Society, Providence.

Leonov V. P. and Shiryaev A. N. (1959). On a method of calculating semi–invariants. *Theory Probab. Appl.,* **4**, 319–329.

Leonov V. P. (1964). *Some Applications of Higher Cumulants in Random Process Theory,* Nauka, Moscow.

Linnik Yu. V. (1960). The new limit theorems for sums of independent random variables. *Sov. Math. Dokl.,* **133**, 1291–1293.

Linnik Yu.V. (1961). On the probability of large deviations for the sums of independent variables. *Proceedings of the Fourth Berkeley Symp. on Math. Statist. and Probab.,* **1**, Univ. California Press, Berkeley, Los Angeles, 289–306.

Linnik Yu. V. (1961, 1962). Limit theorems for sums of independent variables taking into account large deviations I – III. *Theory Probab. Appl.,* **6**, 131–148, **6**, 345–360, **7**, 115–120.

Liptser R. Sh. and Shiryaev A. N. (1989). *Theory of Martingales,* Reidel, Dordrecht, Holland.

Malevich T. L. and Abdalimov B. (1979). Large deviation probabilities for U–statistics. *Theory Probab. Appl.,* **24**, 215–219.

Malyshev V. A. and Minlos R. A. (1985). *Gibbs Random Fields,* Nauka, Moscow.

McDonald D. (1979). A local limit theorem for large deviations of sums of independent nonidentically distributed random variables. *Ann. Probab.,* **7**, 526–531.

McLeish D. L. (1975). Invariance principles for dependent variables. *Z. Wahrsch. verw. Geb.,* **32**, 165–178.

Misevičius E. and Saulis L. (1973). On asymptotic expansions for large deviation. *Lithuanian Math. J.,* **13**, 129–136.

Mogulskii A. A. (1975). Large deviations in trajectory space for sequences and processes with stationary increments. *Sib. Math. J.,* **16**, 314–327.

Nagaev A. V. (1967). Local limit theorems with respect of large deviations. *Limit theorems and random processes,* Tashkent, 71–88.

Nagaev A. V. (1969). Integral limit theorems taking into account large deviations when Cramer's condition does not hold I, II. *Theory Probab. Appl.,* **14**, 51–63, **14**, 203–216.

Nagaev S. V. (1961). More exact statements of limit theorems for homogeneous Markov chains. *Theory Probab. Appl.,* **6**, 67–86.

Nagaev S. V. (1962). Local limit theorems for large deviations. *Vestnik Leningrad Univ.*, **1**, 80–88.

Nagaev S. V. (1963). Large deviations for one class of distributions. *Probability theory limit theorems*, Tashkent, 56–68.

Nagaev S. V. (1965). Some limit theorems for large deviations. *Theory Probab. Appl.*, **10**, 214–235.

Nagaev S. V. and Fuk D. Kh. (1971). Probability inequalities for the sums of independent random variables. *Theory Probab. Appl.*, **16**, 643–650.

Nagaev S. V. and Pinelis I. F. (1974). Some estimates for large deviations and their applications to the strong law of large numbers. *Sib. Math. J.*, **15**, 212–218.

Nagaev S. V. and Sakoyan S. D. (1976). On an estimate for the probability of large deviations. *Limit Theorems and Mathematical Statistics*, Fan, Tashkent, 132–140.

Nagaev S. V. and Pinelis I. F. (1977). Some inequalities for the distributions of the sums of independent random variables. *Theory Probab. Appl.*, **22**, 254–263.

Nagaev S. V. (1979). Large deviations of independent random variables. *Ann. Probab.*, **7**, 745–789.

Osipov L. V. (1975). Multidimensional limit theorem for large deviations. *Theory Probab. Appl.*, **20**, 38–56.

Osipov L. V. (1977). On large deviations for sums of independent random vectors. *Abstracts of Reports of the Second Vilnius Conference on Probab. Theory and Math. Statist.*, 2, IMC, Vilnius, 95–96.

Osipov L. V. (1978). On large deviation probabilities for sums of independent random vectors. *Theory Probab. Appl.*, **23**, 490–505.

Osipov L. V. (1982). Probabilities of large deviations in certain classes of sets for sums of independent random vectors. *Math. zametki*, **32**, 147–153.

Petrov V. V. (1953). An extension of Cramer's limit theorem for independent nonidentically distributed random variables. *Vestnik Leningrad Univ.*, **8**, 13–25.

Petrov V. V. (1954). A generalization of Cramer's limit theorem. *Uspekhi Math. Nauk*, **9**, 195–202.

Petrov V. V. (1961). On large deviations for sums of random variables. *Vestnik Leningrad Univ.*, **1**, 25–37.

Petrov V. V. (1963, 1964). Limit theorem for large deviations violating Cramer's condition I, II. *Vestnik Leningrad Univ.*, **19**, 49–68, **1**, 58–75.

Petrov V. V. (1965). On the probabilities of large deviations for sums of independent random variables. *Theory Probab. Appl.*, **10**, 287–298.

Petrov V. V. (1968). The asymptotic behaviour of large deviation probabilities. *Theory Probab. Appl.*, **13**, 432–444.

Petrov V. V. (1975). *Sums of Independent Random Variables*, Springer–Verlag, Berlin, New York.

Petrov V. V. (1987). *Limit Theorems for Sums of Independent Random Variables,* Nauka, Moscow.

Philipp W. (1969). The central limit problem for mixing sequences of random variables. *Z. Wahrsch. verw. Geb.,* **12**, 155–171.

Pinelis I. F. and Utev S. A. (1984). Estimates of the moments of sums of independent random variables. *Theory Probab. Appl.,* **29**, 574–577.

Pinelis I. F. and Sakhanenko A. I. (1985). Remarks on inequalities for large deviation probabilities. *Theory Probab. Appl.,* **30**, 143–148.

Plikusas A. (1980). The estimations of the cumulants of some nonlinear transformations of stationary Gaussian process. *Lithuanian Math. J.,* **20**, 119–128.

Plikusas A. (1981). Some properties of the multiple Ito integral. *Lithuanian Math. J.,* **21**, 163–173.

Prokhorov Yu. V. (1956). Convergence of random processes and limit theorems in probability theory. *Theory Probab. Appl.,* **1**, 157–214.

Prokhorov Yu. V. (1960). The method of characteristic functionals. *Proceedings of the Fourth Berkeley Symp. on Math. Statist. and Probab.,* **2**, Univ. Calif. Press, Berkeley, Los Angeles, 403–410.

Prokhorov Yu. V. (1962). Extremal problems in limit theorems. *Proceedings of the Sixth All-Union Conference on Probab. Theory and Math. Statist.,* Vilnius, 77–94.

Prokhorov Yu. V. (1968). An extension of S. N. Bernstein inequalities to multidimensional distributions. *Theory Probab. Appl.,* **13**, 260–267.

Prokhorov Yu. V. (1972). Multidimensional distribution inequalities and limit theorems. *Probab. Theory and Math. Statist.,* **10**, 5–24.

Prokhorov Yu. V. and Rozanov Yu. A. (1973). *Probability Theory. Limit Theorems. Random Processes,* Nauka, Moscow.

Puri M. L. and Seon M. (1988). On the rate convergence in normal approximation and large deviations probabilities for a class of statistics. *Theory Probab. Appl.,* **33**, 735–750.

Ramachandran B. (1967). *Advanced Theory of Characteristic Functions,* Statistical Publishing Soc., Calcutta.

Richter W. (1957). Local limit theorem for large deviations. *Sov. Math. Dokl.,* **115**, 53–56.

Richter W. (1957). Lokale Grenzwertsätze für grosse Abweichungen. *Theory Probab. Appl.,* **11**, 214–229.

Richter W. -D. (1978). Über Wahrscheinlichkeiten grosser Abweichungen standartisierter Summen unabhängiger Zufallsvektoren. *Math. Nachr.,* **84**, 345–358.

Richter W. -D. (1979). Wahrscheinlichkeiten grosser Abweichungen von Summen unabhängiger, identisch verteilter Zufallsvektoren. *Lithuanian Math. J.,* **19**, 55–68.

Richter W. -D. (1982). Grosse Abweichungen in endlichdimensionalen Parallelepipeden. *Lithuanian Math. J.,* **22**, 162–169.

Rosenblatt M. (1956). A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U.S.A.*, **42**, 43–47.

Rosenblatt M. (1979). Some remarks on a mixing condition. *Ann. Probab.*, **7**, 170–172.

Rozovskii L. V. (1982). On probabilities of large deviations in convex Borel sets in R^k. *Fourth USSR–Japan Symposium on Probab. Theory and Math. Statist. Abstracts of Communications*, **2**, Tbilisi, 191.

Rubin H. and Sethuraman J. (1965). Probabilities of moderate deviations. *Sankhya*, **A27**, 325–346.

Rudzkis R. (1977). On the lemma of V. Statulevičius. *Lithuanian Math. J.*, **17**, 179–185.

Rudzkis R., Saulis L. and Statulevičius V. (1978). A general lemma of large deviations. *Lithuanian Math. J.*, **18**, 99–116.

Rudzkis R., Saulis L. and Statulevičius V. (1979). On large deviations for sums of independent random variables. *Lithuanian Math. J.*, **19**, 169–179.

Rudzkis R. (1983). On probability of large deviations of random vectors. *Lithuanian Math. J.*, **23**, 195–204.

Sachkov V. N. (1977). *Combinatorial Methods in Discrete Mathematics*, Nauka, Moscow.

Sachkov V. N. (1982). *Introduction to Combinatorial Methods of Discrete Mathematics*, Nauka, Moscow.

Sakhanenko A. I. (1984). Convergence rate in invariance principle for nonidentically distributed variables with exponential moments. *Tr. Inst. Math. Sib. Branch Acad. Sci. USSR*, **3**, 3–49.

Sanov I. N. (1957). On the probability of large deviations of random variables. *Math. Sb.*, **42**, 11–44.

Saulis L. (1969). On the asymptotic expansions for the probabilities of large deviations. *Lithuanian Math. J.*, **9**, 605–625.

Saulis L. and Statulevičius V. (1970). Asymptotic expansions for the probabilities of large deviations for sums of random variables related to a Markov chain. *Lithuanian Math. J.*, **10**, 359–366.

Saulis L. (1973). Limit theorems on the large deviations under Yu. V. Linnik's condition. *Lithuanian Math. J.*, **13**, 173–196.

Saulis L. and Nakas A. (1973). Asymptotic expansions for large deviations when Cramer's condition fails. *Lithuanian Math. J.*, **13**, 199–219.

Saulis L. and Statulevičius V. (1976). Local large deviation theorem for the sums of random variables related to the Markov chain. *Lithuanian Math. J.*, **16**, 151–159.

Saulis L. and Statulevičius V. (1976). On large deviations for a distribution density function. *Math. Nachr.*, **70**, 111–132.

Saulis L. (1978). On the large deviations for the maximum of random–sums of random variables. *Lithuanian Math. J.*, **18**, 147–164.

Saulis L. (1979). On large deviations for sums of independent weighted random variables. *Lithuanian Math. J.*, **19**, 179–187.

Saulis L. (1980). A general lemma for the distribution density function. *Lithuanian Math. J.*, **20**, 165–185.

Saulis L. (1981). On large deviations for a sum of random number of random vectors. *Stochastic Processes and their Applications. Resumés Tenth Conference,* Montreal.

Saulis L. (1981). On large deviations for random vectors. *Third International Vilnius Conference on Probab. Theory and Math. Statist. Abstracts of Communications,* **3**, IMC, Vilnius, 304-307.

Saulis L. (1981). General lemmas in approximation by a normal law. *Lithuanian Math. J.*, **21**. 175–189.

Saulis L. (1983). On random vector large deviation in certain classes of sets I, II. *Lithuanian Math. J.*, **23**, 142–154, **23**, 50-57.

Saulis L. (1984). On large deviations in Euclidean space R^k. *Sov. Math. Dokl.*, **29**, 443–446.

Saulis L. (1985). An asymptotic expansions and large deviations of the density of distribution of sums of independent random variables. *Fourth International Vilnius Conference on Probab. Theory and Math. Statist. Abstracts of Communications,* **4**, IMC, Vilnius, 267–269.

Saulis L. (1985). On approximation by the normal distribution in R^k. *Fourth USSR–Japan Symposium on Probab. Theory and Math. Statist. Abstracts of Communications,* **2**, Tbilisi 192–193.

Saulis L. (1986). On large deviations for the probability density of sums of independent random variables. *Probab. Theory and Math. Statist. Proccedings of the Fourth Vilnius Conference,* **2**, VNU Science Press, Utrecht, The Netherlands, 541–559.

Saulis L. (1987, 1988). General large deviation lemmas for random vector with semiinvariants of regular growth I – III. *Lithuanian Math. J.*, **27**, 535–549, **27**, 747–758, **28**, 99–111.

Saulis L. and Statulevičius V. (1989). *Limit Theorems for Large Deviations,* Mokslas, Vilnius.

Saulis L. (1990). General lemmas on the distribution density for large deviations of a random vector. *Probab. Theory and Math. Statist. Proceedings of the Fifth Vilnius Conference,* **2**. VSP, Utrecht, The Netherlands, Mokslas, Vilnius, Lithuania, 383–393.

Sazonov V. V. (1974). On the estimation of moments of sums of independent random variables *Theory Probab. Appl.*, **19**, 383–386.

Serfling R. J. (1968). Contributions to central limit theorem for dependent variables. *Ann Math. Statist.*, **39**, 1158–1175.

Serfling R. J. (1980). Approximation Theorems for Linear Combinations of Order Statistics Wiley, New York.

Sethuraman J. (1964). On the probability of large deviations of families of sample means. *Ann Math. Statist.*, **35**, 1304–1316.

Sethuraman J. (1970). Probabilities of deviations. *S. N. Roy Memorial Volume,* Univ. North Carolina Press, 655–672.

Shiryaev A. N. (1960). Some problems in the spectral theory of higher order moments I. *Theory Probab. Appl.,* **5**, 265–284.

Shorgin S. Ya. (1978). A nonclassical estimate of the rate of convergence in the multidimensional central limit theorem taking into account large deviations. *Theory Probab. Appl.,* **23**, 667–671.

Shorgin S. Ya. (1983). Nonclassical estimates of the rate of convergence in the central limit theorem taking into account large deviations. *Theory Probab. Appl.,* **27**, 324–337.

Sievers G. L. (1969). On the probabilities of large deviations and exact slopes. *Ann. Math. Statist.,* **40**, 1908–1921.

Sievers G. L. (1975). Multivariate probabilities of large deviations. *Ann. Math. Statist.,* **3**, 897–905.

Statulevičius V. (1965). Limit theorems for the density functions and asymptotic expansions for the distributions of sums of independent random variables. *Theory Probab. Appl.,* **10**, 645–659.

Statulevičius V. (1966). On large deviations. *Z. Wahrsch. verw. Geb.,* **6**, 133–144.

Statulevičius V. (1967). On the probabilities of large deviations for sums of random number of independent random variables. *Lithuanian Math. J.,* **7**, 513–516.

Statulevičius V. (1969, 1970). Limit theorems for the sums of random variables related to a Markov chain I, II. *Lithuanian Math. J.,* **9**, 345–362, **10**, 161–169.

Statulevičius V. (1970). On the limit theorems for the random functions. *Lithuanian Math. J.,* **10**, 583–592.

Statulevičius V. (1979). Limit theorems of large deviations for sums of dependent random variables I. *Lithuanian Math. J.,* **19**, 199–208.

Statulevičius V. (1983). On Markov type regularity conditions. *Theory Probab. Appl.,* **28**, 358–362.

Statulevičius V. and Jakimavičius D. (1988). Estimates of cumulants and centered moments of mixing random processes I, II. *Lithuanian Math. J.,* **28**, 112–129, **28**, 360–375.

Stroock D. W. (1984). *An Introduction to the Theory of Large Deviations,* Springer–Verlag, New York.

Surgailis D. (1981). On infinitely divisible self-similar random fields. *Z. Wahrsch. verw. Geb.,* **58**, 453–477.

Survila P. (1966). On large deviations for a distribution density function. *Lithuanian Math. J.,* **6**, 591–600.

Svetulevičienė V. (1981). On probabilities of large deviations for sums of random vectors. *Lithuanian Math. J.,* **21**, 191–199.

Titchmarsh E. C. (1937). *Introduction to the Theory of Fourier Integrals,* Oxford Univ. Press.

Tkachuk S. G. (1975). Theorem on large deviations in case of distributions with tails of regular growth. *Random processes and statistical decisions,* **5**, Tashkent, 164–174.

Utev S. A. (1989). Cumulants and moment inequalities. *Theory Probab. Appl.*, **34**, 810–815

Utev S. V. (1990). Central limit theorem for dependent random variables. *Probab. Theory an Math. Statist. Proceedings of the Fifth Vilnius Conference*, 2, VSP, Utrecht, The Netherlands Mokslas, Vilnius, Lithuania, 519–528.

Vandemaele M. (1982). On probabilities of large deviations for U–statistics. *Theory Probab Appl.*, **27**, 573–574.

Vandemaele M. and Veraverbeke N. (1982). Cramer type large deviations for linear combination of order statistics. *Ann. Probab.*, **10**, 423–434.

Varadhan S. R. S. (1984). *Large Deviations and Applications*, SIAM, Philadelphia.

Ventsel A. D. (1976). Rough limit theorems on large deviations for Markov stochastic processe I, II. *Theory Probab. Appl.*, **21**, 227–242, **21**, 499–512.

Ventsel A. D. (1986). *Limit Theorems on Large Deviations for Markov Stochastic Processes* Nauka, Moscow.

Ventsel A. D. (1990). *Limit Theorems on Large Deviations for Markov Stochastic Processes* Kluwer, Dordrecht.

Vilkauskas L. (1963). Two integral theorems on large deviations in the multidimensional case *Lithuanian Math. J.*, **3**, 53–67.

Vilkauskas L. (1965). The multi–dimensional large deviations of Linnik type on some regions *Lithuanian Math. J.*, **5**, 25–43.

Volkonskii V. A. and Rozanov Yu. A. (1959). Some limit theorems for random functions I *Theory Probab. Appl.*, **4**, 178–197.

Wolf W. (1970). Some large deviation limit theorems for sums of independent random variables *Sov. Math. Dokl.*, **191**, 1209–1211.

Wolf W. (1974). Über Wahrscheinlichkeiten grosser Abweichungen. *Math. Nachr.*, **62**, 261–288

Wolf W. (1975). Über Wahrscheinlichkeiten grosser Abweichungen bei Nichterfüllung der Cramer schen Bedingung. *Math. Nachr.*, **70**, 197–215.

Wolf W. (1977). Asymptotische Entwicklungen für Wahrscheinlichkeiten grosser Abweichungen *Z. Wahrsch. verw. Geb.*, **40**, 239–256.

Wolf W. and Mikosch T. (1983). On probabilities of large deviations in triangular array. *Lithuanian Math. J.*, **23**, 43–48.

Wolf W. (1990). Some large deviation results for U–statistics. *Probab. Theory and Math. Statist. Proceedings of the Fifth Vilnius Conference*, 2, VSP, Utrecht, The Netherlands, Mokslas Vilnius, Lithuania, 571–584.

Yurinskii V. V. (1974). Exponential bounds for large deviations. *Theory Probab. Appl.*, **19** 154–155.

Yurinskii V. V. (1976). Exponential inequalities for sums of random vectors. *J. Multivariat Anal.*, **6**, 473–499.

Yurinskii V. V. (1988). On the asymptotics of large deviations in a Hilbert space . *Preprint,* **23**, Inst. Math. Sib. Branch Acad. Sci. USSR, Novosibirsk.

Zaitsev A. Yu. (1984). On the Gaussian approximation of convolutions when satisfying the multidimensional analogs of the Bernstein inequality conditions. *Preprint,* **9**, Leningrad.

Zaitsev A. Yu. (1989). On approximation by Gaussian distributions under multidimensional analogues of the Bernstein conditions. *Sov. Math. Dokl.,* **276**, 1046–1048.

Zhurbenko I. G. (1972). On the strong estimates of mixed semiinvariants. *Sib. Math. J.,* **13**, 293–308.

Zhurbenko I. G. (1982). *Spectral Analysis of Time Series,* MGU, Moscow.

Zolotorev V. M. (1961). On a certain probability problem. *Theory Probab. Appl.,* **6**, 201–204.

Zolotarev V. M. (1962). On a new view point to limit theorems admitting large deviations. *Proceedings of the Sixth All-Union Conference on Probab. Theory and Math. Statist.,* Vilnius, 43–47.

Zolotarev V. M. (1965). On the nearness of distributions of two independent random variables. *Theory Probab. Appl.,* **10**, 519–526.

Zuev N. M. (1973). On estimates of mixed cumulants of random processes. *Math. Zametki,* **13**, 581–586.

Zuev N. M. (1981). The estimate of mixed semiinvariants of random processes satisfying the almost Markovian type mixing condition. *Lithuanian Math. J.,* **21**, 81–85.

Zykov A. A. (1987). *Foundations of Graph Theory,* Nauka, Moscow.

Žemaitis A. (1973). Die asymptotischen Entwicklungen von Tchebyschew für die Verteilungsfunktionen. *Lithuanian Math. J.,* **13**, 91–96.

Žemaitis A. (1974). Asymptotische Entwicklungen in Grenzwertsätze für grossen Abweichungen. *Lithuanian Math. J.,* **14**, 13–25.

Index