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THE EXACT RATE OF APPROXIMATION IN ULAM'S METHOD

Christopher Bose[1]

Department of Mathematics and Statistics, University of Victoria
P.O. Box 3045, Victoria, BC, Canada V8W 3P4.

Rua Murray[2]

Department of Mathematics, University of Waikato
Private Bag 3105, Hamilton, New Zealand.

Abstract. This paper investigates the exact rate of convergence in Ulam's method: a well-known discretization scheme for approximating the invariant density of an absolutely continuous invariant probability measure for piecewise expanding interval maps. It is shown by example that the rate is no better than $O(\frac{\log n}{n})$, where n is the number of cells in the discretization. The result is in agreement with upper estimates previously established in a number of general settings, and shows that the conjectured rate of $O(\frac{1}{n})$ cannot be obtained, even for extremely regular maps.

SECTION 1

1.1. Introduction and Statement of Results. Throughout, we will be concerned with nonsingular transformations T from the unit interval $I = [0, 1]$ to itself that are piecewise monotone and continuous; specifically, those for which there is a finite partition of I given by $0 = a_0 < a_1 < a_2 < \cdots < a_N = 1$ such that for each (a_{i-1}, a_i), $T_i = T|_{(a_{i-1},a_i)}$ is continuous and (strictly) increasing. Nonsingular in this context means with respect to the ambient measure, Lebesgue measure λ on Borel subsets of I. In many instances, it is known that T admits an absolutely continuous invariant probability measure (a.c.i.p.m) $d\mu = \phi\, d\lambda$ which determines the statistical properties of T on I. It is therefore of great interest to know how to get ϕ.

Existence results for a.c.i.p.m require that additional conditions be imposed on T. Typical assumptions (but by no means necessary) are

(i) 'Expansiveness': there is a $\sigma > 1$ such that for every i and every pair x, y in (a_{i-1}, a_i) one has $|T_i(x) - T_i(y)| \geq \sigma|x - y|$, and

(ii) Additional 'regularity' of the branches, such as $T_i \in C^2$, or T_i' Hölder continuous or of bounded variation, and so on.

The role of the regularity condition (ii) is always the same, that being to control a 'distortion' parameter D for the map – a measure of the degree of deviation of T from local linearity. For example, in the C^2 case

$$D = \max \frac{T''(x)}{(T'(x))^2},$$

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after which a now classical result of Lasota and Yorke [LY73] states that if $\sigma > 2$ and $D < \infty$, then T admits an a.c.i.p.m.

Although there are now scores of results like this in the literature, we cite [LM94] and [BG97] and the references there as representative of both classical and modern investigations.

Suppose $d\nu = \phi\, d\lambda$ is an absolutely continuous probability measure on I (*ie.* $\phi \in L^1$, $\phi \geq 0$, and $\int \phi\, d\lambda = 1$). The fact that T is nonsingular implies that $\nu \circ T^{-1} << \lambda$ and we can define the Perron-Frobenius Operator P (variously known as the Transfer Operator or Ruelle Operator) associated to T by

$$P\phi = \frac{d\nu \circ T^{-1}}{d\lambda}.$$

The measure ν is invariant for T (i.e. $d\nu \circ T^{-1} = d\nu$) if and only if $P\phi = \phi$. In this way fixed points of P become the natural object of study in any search for a.c.i.p.m. for T.

The domain of P can be extended by linearity to all of L^1 yielding a Markov Operator with the following useful characterization

$$\int f\,(g \circ T) d\lambda = \int (Pf)\, g\, d\lambda,\ \forall f \in L^1,\ g \in L^\infty.$$

If the branches of T are differentiable, we have a well-known formula for P:

$$Pf(x) = \sum_{T(y)=x} \frac{f(y)}{|T'|(y)},\ f \in L^1,\ \text{a.e. } x \in I. \tag{1.1}$$

Unfortunately, in all but a few exceptional cases, it is impossible to determine the fixed points ϕ of P explicitly.

Motivated by this difficulty, S. Ulam in [Ul60] suggested that one could study the action of P on finite-dimensional subspaces of L^1 consisting of piecewise constant functions with respect to a finite partition η of I into intervals. Let π_η denote the natural projection of L^1 onto such a subspace (we will define this precisely in Section 1.2). Set

$$P_\eta = \pi_\eta \circ P.$$

P_η is a finite rank operator. Restricted to the finite-dimensional subspace $\pi_\eta(L^1)$ and with respect to the basis $v_h = (\lambda(I_h))^{-1}\chi_{I_h}$, for $I_h \in \eta$, P_η may be represented by a (row) stochastic matrix $\hat{P}_\eta$, so the classical Perron-Frobenius Theorem ensures that there is a probability vector $\phi_\eta > 0$ satisfying $\phi_\eta \hat{P}_\eta = \phi_\eta$. Taking a sequence of finite interval partitions η_n whose mesh size tends to zero, Ulam conjectured that the sequence of associated invariant probability vectors ϕ_{η_n} would converge to the invariant density ϕ.

Indeed, in [Li76] it was shown that under the hypothesis of T piecewise C^2 and $\sigma > 2$, $\phi_{\eta_n} \to \phi$ in the L^1 norm. It was quickly realized that similar techniques would give convergence of the scheme for maps where T' has bounded variation. All of this has been generalized considerably in recent years to transformations in multidimensional settings, and to different types of 'regularization' hypothesis on T. Since we are not really concerned with these more general results in this paper we direct the interested reader to [Mu98] where many relevant references may be found.

With asymptotic convergence established, many authors have considered numerical implementations of Ulam's scheme [Hu94, Li76, CDL92]. Thus, the question of error bounds becomes important. Since uniform partitions η_n into intervals of length $1/n$ are a natural choice for numerical work, we restrict our attention here to these. Given a uniform partition η_n we will simply write P_n and ϕ_n instead of P_{η_n} and ϕ_{η_n}. In this case we have a useful explicit formula for the matrix representation of P_n:

$$(\hat{P}_n)_{i,j} = \frac{\lambda(\alpha_i \cap T^{-1}\alpha_j)}{\lambda(\alpha_i)}$$

where we have chosen $\eta_n = \{\alpha_1, \dots, \alpha_n\}$.

The question of error bounds can be approached via estimates on the non-peripheral spectrum of P acting on an appropriate Banach space. In Section 1.5, we review such an argument, obtaining

$$||\phi_n - \phi||_1 \leq O\left(\frac{\log n}{n}\right).$$

The earliest result we know of confirming this upper estimate on the rate is [Ke82] where the spectral estimate comes from the Ionescu-Tulcea & Marinescu Theorem. Recent investigations yielding the same upper estimate in various contexts are [Hu96, Mu98, Mu00] – these are also notable for establishing constants in the rate.

In [CDL92], the rate $||\phi_n - \phi||_1 \leq O(1/n)$ was claimed for piecewise differentiable T with T' of bounded variation. Since, for bounded variation ϕ, $\|\pi_n\phi - \phi\|_1 = O\left(\frac{1}{n}\right)$ (see inequality (1.2) below) this rate is highly plausible. However, as noted in [DL98], the argument from [CDL92] cannot be applied to the piecewise constant approximation scheme, so up to this point the question of the optimal rate of convergence in Ulam's method has remained open.

In this article we show, by way of a specific example, that $O(\frac{\log n}{n})$ is indeed best possible (Theorem 1, Section 1.4), and that this rate cannot be improved even for extremely regular (in this case piecewise linear) and/or Markov maps.

The rest of Section 1 contains background material and a summary of the spectral argument mentioned above. In Section 2 we recall from [Mu98] a family of norms which allow us to establish strict contraction rates for integral zero functions; this analysis replaces the spectral decomposition and has the advantage of yielding computable constants for the rate. In Section 2.3 we work out an explicit upper bound on the rate of approximation for our map. The lower bound calculation appears in Section 3.

1.2. Basic Notation. Denote by L^1 the usual function space of integrable functions on $[0, 1]$. Since we will be working mostly with this space we will reserve $||\cdot||$ for its norm.

If f is measurable, $f^{\pm}$ will denote the positive and negative parts of f respectively.

Given an interval $J = [a, b]$ and a function f on J we will define the usual *variation* of f on J by

$$V_a^b f = \sup\left\{\sum |f(x_{i+1}) - f(x_i)| \,\middle|\, a = x_0 < x_1 < \cdots < x_k = b\right\}.$$

The space of functions on J of bounded variation will be denoted $\mathrm{BV}(J)$, or simply BV when $J = [0, 1]$. The subspace of functions in BV which integrate to zero will be denoted BV_0. Of course BV is a Banach space when equipped with norm $||f||_{\mathrm{BV}} \stackrel{\mathrm{def}}{=} V_0^1 f + ||f||$ and BV_0 is a closed subspace of BV.

Given and integer n, the uniform interval partition of order n will be denoted $\eta_n = \{\alpha_j\}_{j=1}^n$ where $\alpha_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$. Besides the Perron-Frobenius operator defined above, we need the n-th order projection π_n associated to η_n and defined on L^1 by

$$\pi_n f = \sum_{j=1}^{n} f_j \chi_{\alpha_j},$$

where $f_j = n \int_{\alpha_j} f d\lambda$. π_n is positive, preserves integrals (hence is a contraction on L^1), and satisfies the following condition on BV functions

$$||\pi_n f - f|| \leq \frac{V_0^1 f}{n}. \tag{1.2}$$

We also note that π_n cannot increase variation

$$V_0^1 \pi_n f \leq V_0^1 f, \tag{1.3}$$

an elementary estimate which we shall use a number of times.

1.3. Lasota Yorke Inequalities and the Ionescu-Tulcea & Marinescu Theorem. The most important analytical tool linking the map T with its associated Perron-Frobenius operator P is the following inequality.

LASOTA YORKE INEQUALITY (L-Y) *A map T is said to admit a Lasota-Yorke inequality for P if there are constants $\alpha < 1$ and $K < \infty$ such that for all $f \in BV$*

$$V_0^1 Pf \leq \alpha V_0^1 f + K||f||.$$

This estimate was first derived in [LY73] for piecewise C^2 maps where $\alpha = 2/\sigma$ ($\sigma > 2$ being the coefficient of expansion for T) and K depends on the distortion coefficient D. Since then many generalizations have appeared, for maps with different regularity assumptions, and for multidimensional transformations [BG89]. We will derive a L-Y inequality tailored to our purpose below in Section 2.2.

As a prototype application, observe that if

$$\mathcal{C}_L = \left\{ f \in \mathrm{BV} \,\middle|\, f \geq 0, \ \int f \, d\lambda = 1, \ V_0^1 f \leq L \right\},$$

then the L-Y inequality implies $P : \mathcal{C}_L \to \mathcal{C}_L$ provided $L \geq \frac{K}{1-\alpha}$. Since $\mathcal{C}_L$ is convex and compact in L^1 it contains a fixed point for P, thus establishing the existence of an invariant density for T.

Observe that if T admits a L-Y inequality for P, then it also does for P_n for every n by virtue of (1.3):

$$V_0^1 P_n f = V_0^1 (\pi_n P f) \leq V_0^1 Pf \leq \alpha V_0^1 f + K||f|| \tag{1.4}$$

A much deeper consequence of the L-Y inequality is the celebrated

IONESCU-TULCEA & MARINESCU THEOREM (I-T&M) *Let T be weak-mixing and admit a Lasota-Yorke inequality for P. Then there are constants $C < \infty$ and $0 < d < 1$ so that for all k and $f \in BV_0$*

$$||P^k f||_{\mathrm{BV}} \leq C d^k ||f||_{\mathrm{BV}}.$$

More precisely, the I-T &M Theorem [ITM50] describes a spectral decomposition for P. The norm estimate on the integral zero subspace is a Corollary, but it is the part of the Theorem we need for approximation.

1.4. Example and The Main Result. We construct a map for which the Ulam approximation proceeds at rate $O\left(\frac{\log n}{n}\right)$.

EXAMPLE Define T as follows (see Figure 1.1):

$$T(x) = \begin{cases} 4x & \text{if } 0 \leq x < 1/6; \\ 3(x-1/6)+2/3 & \text{if } 1/6 \leq x < 5/18; \\ 3(x-5/18) & \text{if } 5/18 \leq x < 11/18; \\ 3(x-11/18) & \text{if } 11/18 \leq x < 2/3; \\ 4(x-2/3)+1/6 & \text{if } 2/3 \leq x < 21/24; \\ 8(x-21/24) & \text{if } 21/24 \leq x \leq 1. \end{cases}$$

REMARKS *We collect below a few simple properties of this example which we will use; a graph of T is also displayed below.*

a) The map is piecewise affine on 6 intervals which we will denote I_1 through I_6 counting left to right, and where the slopes of T (counting left to right) are 4, 3, 3, 3, 4 and 8 respectively. In particular the map is expanding with $\sigma = 3$. Each branch is orientation preserving.

b) The points $\{\frac{1}{6}, \frac{2}{3}\}$ form a periodic orbit of length 2.

c) The unique invariant probability density for T is

$$\phi = \frac{32}{31}\chi_{[0,1/6]} + \frac{30}{31}\chi_{[1/6,2/3]} + \frac{32}{31}\chi_{[2/3,1]}$$

and $P^k f \xrightarrow{\text{BV}} \phi$ as $k \to \infty$ for any $f \in BV$ with $\int f\, d\lambda = 1$.

d) For all sufficiently large n the nonnegative, stochastic $n \times n$ matrix $\hat{P}_n$ representing $\pi_n P$ is irreducible and aperiodic having a unique invariant probability vector ϕ_n.

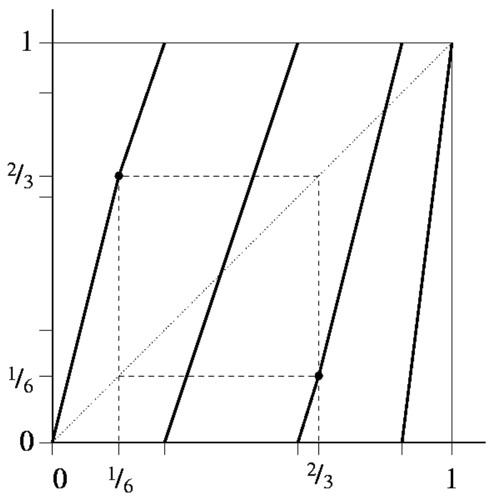

Figure 1.1: Graph of T, illustrating period–2 orbit.

The reader may wish to note that Remark c) above follows easily from contractive estimates on P similar to those obtained for P_n in Lemma 2.4 and the calculation in Corollary 2.5 (the argument for P is also given explicitly in [Mu98].) Similar considerations lead to d), but we provide the following self-contained argument, since it is highly illustrative of the relationship between P and P_n. First, notice that for any $f \geq 0$, $\text{supp}(\pi_n f) \supseteq \text{supp}(f)$. By induction, $\text{supp}(P_n^k f) \supseteq \text{supp}(P^k f)$ (where both P, P_n are regarded as operators in L^1). Given a discretization interval α_i, let $f = n\chi_{\alpha_i}$. Provided k is such that $\sigma^k \lambda(\alpha_i) > 1$, $T^k(\alpha_i) \supseteq [0,1]$ (by expansivity). Thus, $\text{supp}(P^k f) = [0,1]$, so that also $\text{supp}(P_n^k f) = [0,1]$. On a matrix level, f corresponds to the vector with a 1 in the ith entry, and 0 everywhere else, so after k right multiplications by the matrix $\hat{P}_n$, we obtain the ith row of the matrix $\hat{P}_n^k$. Since, this vector is the vector representation of the function $P_n^k f$ (which is supported on all of L^1), all elements of this vector must be positive. Thus, whenever $\sigma^k/n > 1$, $\hat{P}_n^k$ is a positive matrix. The uniqueness of the invariant probability vector ϕ_n now follows from the classical Perron-Frobenius theorem.

Our main result is the following.

THEOREM 1 *For T as defined above, for all $n \geq 162$*

$$||\phi_n - \phi|| \leq \frac{25}{100}\left(\frac{\log n}{n}\right)$$

while for infinitely many n

$$||\phi_n - \phi|| \geq \frac{1}{100}\left(\frac{\log n}{n}\right).$$

1.5. Obtaining Approximation Rates: Why $O(\frac{\log n}{n})$? In general, one seeks estimates on

$$||\phi_n - \phi||$$

where neither ϕ nor the ϕ_n are known explicitly.[3] The following argument is reasonably standard:

Let us assume that the Ionescu–Tulcea and Marinescu ergodic theorem holds uniformly for the operators P_n (in the sense that the constants C, d are independent of n.) By Remark (d), $\phi_n = \lim_{k\to\infty} P_n^k \pi_n \phi$ and since $\pi_n\phi = P_n\phi$ one obtains

$$\begin{aligned}\phi_n - \phi &= \lim_{k\to\infty} P_n^k \pi_n\phi - \phi \\ &= \sum_{l=0}^{\infty} P_n^l(\pi_n\phi - \phi).\end{aligned} \tag{1.5}$$

In the second line of (1.5) we reiterate that we are considering $P_n = \pi_n P$ as a finite-rank operator on all of L^1 rather than a finite-dimensional operator (*ie.* a matrix) as would be obtained by restricting its domain to η_n-step functions as discussed in Section 1.1.

Set $\Psi_n = \pi_n\phi - \phi$. Then $||\Psi_n|| = ||\pi_n\phi - \phi|| = O(\frac{1}{n})$ by equation (1.2). Since $\Psi_n \in \text{BV}_0$, The I-T & M Theorem applied to the infinite sum in (1.5) gives a simple estimate on its BV-norm (and hence an upper bound on its L^1 norm through

[3]For the map T which is the subject of this investigation, we are slightly ahead of the game, knowing ϕ in advance, but having no closed-form expression for the ϕ_n.

$||\cdot|| \leq ||\cdot||_{\mathrm{BV}}$) in terms of $||\Psi_n||_{\mathrm{BV}}$. However, the latter need not converge to zero with $n \to \infty$! See [DL98].

To get around this we split the sum at some N:

$$\phi_n - \phi = \sum_{l=0}^{N} P_n^l \, \Psi_n + \sum_{l=1}^{\infty} P_n^l (P_n^N \, \Psi_n). \tag{1.6}$$

Iterative application of the L-Y inequality (1.4) implies that

$$V_0^1 P_n^N \, \Psi_n \leq \alpha^N V_0^1 \, \Psi_n + \frac{1-\alpha^N}{1-\alpha} K ||\Psi_n||.$$

Choosing N in (1.6) to satisfy

$$N \geq -\frac{\log n}{\log \alpha} = O(\log n)$$

ensures that

$$\alpha^N \leq \frac{1}{n}.$$

Noting that

$$||P_n^N \, \Psi_n|| \leq ||\Psi_n|| = O\left(\frac{1}{n}\right),$$

we have

$$||P_n^N \Psi_n||_{\mathrm{BV}} \leq \frac{1}{n} V_0^1 \, \Psi_n + \frac{K}{1-\alpha} ||\Psi_n|| + ||P_n^N \, \Psi_n|| \leq O\left(\frac{1}{n}\right).$$

This, combined with the contraction from I-T&M gives the desired estimate on the second sum in (1.6):

$$\left\| \sum_{l=1}^{\infty} P_n^l (P_n^N \, \Psi_n) \right\|_{\mathrm{BV}} \leq O\left(\frac{1}{n}\right).$$

The remainder of the computation has now been transferred entirely to the first sum

$$\begin{aligned} ||\phi_n - \phi|| &\leq \sum_{l=0}^{N} ||P_n^l \, \Psi_n|| + O\left(\frac{1}{n}\right) \\ &\leq O\left(\frac{N}{n}\right) + O\left(\frac{1}{n}\right) \sim O\left(\frac{\log n}{n}\right). \end{aligned}$$

The above argument provides a clue as to when $O(\frac{\log n}{n})$ is going to give the optimal rate.

Writing

$$||\phi_n - \phi|| \geq \left\| \sum_{l=0}^{N} P_n^l \, \Psi_n \right\| - O\left(\frac{1}{n}\right)$$

we wish to show that for T as defined above,

$$|| \sum_{l=0}^{N} P_n^l \, \Psi_n || \geq O\left(\frac{\log n}{n}\right).$$

We will do this in Section 3 by showing that for some n, the first $N = O(\log n)$ iterates of Ψ_n^+ and Ψ_n^- move on disjoint orbits, yielding a sum satisfying

$$\left\| \sum_{l=0}^{N} P_n^l \, \Psi_n \right\| = \sum_{l=0}^{N} \left\| P_n^l \, \Psi_n \right\| = (N+1)||\Psi_n||.$$

1.6. When is the Ulam Approximation as Bad as Possible: Definition of $\mathcal{N}$. In order to have the norm of Ψ_n relatively large, we need to make sure that η_n is badly behaved with respect to the jumps in ϕ. This cannot happen uniformly in n; for example, when $n = 6k$, $\Psi_n \equiv 0$.

It will be convenient to examine the following sequence of integers:

$$\mathcal{N} = \{n = 3(2k-1) \mid k = 1, 2, \dots\}.$$

Then for every $n \in \mathcal{N}$, $2/3$ lies on the boundary of an n-th order interval, while $1/6$ lies in the centre. The upper estimate in Theorem 1 will be established for all n while the lower estimate will be computed only for $n \in \mathcal{N}$.

1.7. Some Practical Considerations. The argument sketched in Section 1.5 will not identify the constants in the rate, since these depend on the constants in the I-T & M Theorem (non-constructive) and in the L-Y inequality (seldom optimal in general form). Since we are interested in these constants, and since it requires no more effort, we will derive an optimal L-Y inequality for our T and replace the use of I-T & M with a direct estimate of the contractive strength (*ie.* the Spectral Gap) of P using an idea reminiscent of Doeblin's Condition in the theory of Markov chains. This is discussed in Sections 2.1 and 2.2. Details of the upper estimate appear in 2.3. More delicate computations verifying the lower bound are delayed until Section 3.

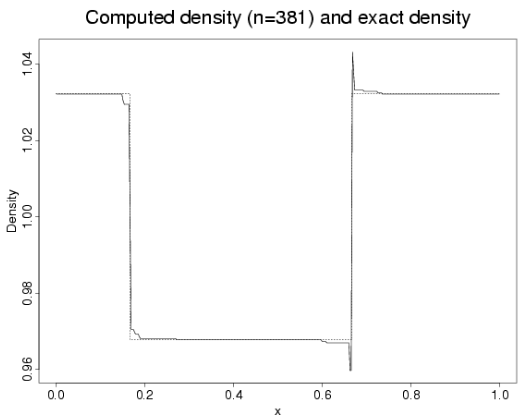

Figure 1.2: ϕ *vs.* ϕ_{381}

We have implemented Ulam's method to illustrate the results of Theorem 1. In Figure 1.2 above we display the results of a sample computation with $n = 381$. The broken line represents the exact density ϕ, while the solid line is ϕ_{381}. This computation yields the expected difficulties at the two jump points and the error integrates numerically to

$$||\phi - \phi_{381}|| \approx 4.2 \times 10^{-4}.$$

Theorem 1 estimates

$$||\phi - \phi_{381}|| \leq 3.89 \times 10^{-3}.$$

While the lower bound computed in Theorem 1 is approximately 1.5×10^{-4} the details following will show that 'sufficiently large' is a great deal larger than $n = 381$, so it would be inappropriate to make a direct comparison between our lower estimate and the numerical example above.

SECTION 2

2.1. Preliminaries About Cones and Norms. The set $\mathcal{C}$ of non-negative BV functions equipped with the partial order $\leq$ is a *cone*. That is,

$$f \in \mathcal{C} \text{ implies } \mu f \in \mathcal{C}\ \forall \mu \geq 0 \text{ and } f, g \in \mathcal{C} \text{ implies } f + g \in \mathcal{C}.$$

Of particular interest for us will be the cones

$$\mathcal{C}_a = \{f \in L^1 \mid f \geq 0,\ V_0^1 f \leq a||f||\}$$

where $a > 0$ is fixed. Following [Mu98], we can use the cones $\mathcal{C}_a$ to define a family of norms on BV_0:

$$\|f\|_a = \inf\left\{\|f^{(1)}\| = \|f^{(2)}\| \;\middle|\; f = f^{(1)} - f^{(2)},\ f^{(1)}, f^{(2)} \in \mathcal{C}_a\right\}.$$

The functions $f^{(i)}$ $(i = 1, 2)$ are somewhat analogous to the positive and negative parts f^+ and f^- of f. Indeed, if both $f^\pm \in \mathcal{C}_a$, then the infimum in the definition will be attained for $f^{(1)} = f^+$ and $f^{(2)} = f^-$. Otherwise, for any $a > 0$ it is easy to see that if $f \in \mathrm{BV}_0$ then $f^{(i)} \in \mathcal{C}_a$ can be constructed by adding a sufficiently large constant to each of the BV functions f^+ and f^-. Thus, for any $a > 0$, $\|\cdot\|_a$ defines a norm on BV_0, and from [Mu98] we have:

LEMMA 2.1 *For any $a > 0$ and $g \in BV_0$, we have:*

(i) $\|g\| \leq 2\|g\|_a$;

(ii) $V_0^1 g \leq 2a\|g\|_a$;

(iii) $\|g\|_a \leq \max\left\{\frac{\|g\|}{2}, \frac{V_0^1 g}{a}\right\}$;

(iv) If P is a Markov operator and there exist constants $0 < k < 1$ and $a > 0$ such that for any $f \in \mathcal{C}_a$ both
1/ $Pf \geq k\|f\|$ *and*
2/ $V_0^1(Pf) \leq (1-k)a\|f\|$,
then

$$\|Pg\|_a \leq (1-k)\|g\|_a$$

for any $g \in BV_0$.

REMARK. *Parts (i)–(iii) of this Lemma are extracted from Lemma 2.3 of [Mu98], and part (iv) essentially reports the content of Theorem 1 of the same paper. To see why it is true, let $g^{(1)}$ and $g^{(2)}$ be as in the definition of $\|\cdot\|_a$ and such that*

$\|g\|_a = \|g^{(i)}\|$. *Then conditions 1/ and 2/ of the Lemma 2.1 (iv) imply that both the functions* $Pg^{(i)} - k\|g\|_a$ *are in the cone* $\mathcal{C}_a$. *Thus*

$$\|Pg\|_a \le \left\|Pg^{(i)} - k\|g\|_a\right\| = (1-k)\|g\|_a.$$

2.2. Lasota Yorke Inequality and the Spectral Gap For P_n. From now on P will be the Perron–Frobenius operator for the map T defined in Section 1.4, and P_n the corresponding Ulam approximation. We will now apply Lemma 2.1 to get a spectral gap for P_n. We need another Lemma, whose proof is deferred to the end of this section.

LEMMA 2.2. *Let* $f \ge 0$ *be a BV function. Then*
(i) (L-Y Inequality) $V_0^1 Pf \le \frac{1}{3} V_0^1 f + \frac{1}{2}||f||$; *and*
(ii) (Lower Estimate) *If* $f \in \mathcal{C}_{a=3/2}$ *then* $Pf \ge 3/8||f||$.

COROLLARY 2.3 *Lemma 2.2 holds with P replaced by* P_n.

Proof. Part (i) holds because $P_n = \pi_n \circ P$ and π_n is non–increasing of variation (*cf.* equation (1.3)). For part (ii), notice that

$$\pi_n Pf \ge \pi_n(3/8||f||) = 3/8||f||$$

by Lemma 2.2 (ii). □

We can now give a Lemma about the spectral gap for P_n. This, combined with the norm comparison in Lemma 2.1, will let us control the second sum in (1.6).

LEMMA 2.4. (Spectral Gap) *If* $g \in BV_0$ *then* $||P_n\, g||_{a=3/2} \le 2/3||g||_{a=3/2}$.

Proof. Let $f \ge 0$ be a BV function such that $f \in \mathcal{C}_{a=3/2}$. Then, by Corollary 2.3 (i) and the definition of $\mathcal{C}_{a=3/2}$,

$$V_0^1(P_n f) \le \frac{1}{3} V_0^1 f + \frac{1}{2}\|f\| \le \frac{1}{3}\frac{3}{2}\|f\| + \frac{1}{2}\|f\| = \|f\|,$$

and by part (ii) of Corollary 2.3,

$$P_n f \ge \frac{3}{8}\|f\| \ge \frac{1}{3}\|f\|.$$

The lemma now follows from Lemma 2.1 (iv) by taking $k = 1/3$. □

COROLLARY 2.5 (Convergence to equilibrium) *For any* $n > 0$ *and* $f \in BV$,

$$P_n^k f \xrightarrow{\mathrm{BV}} \left(\int f\, d\lambda\right)\phi_n \qquad \text{as } k \to \infty.$$

Proof of Corollary 2.5. Since $f, \phi_n \in$ BV, the function $g \stackrel{\text{def}}{=} f - (\int f\, d\lambda)\phi_n \in \mathrm{BV}_0$. Thus, by Lemma 2.1 (i) and (ii),

$$\left\|P_n^k g\right\|_{\mathrm{BV}} = V_0^1(P_n^k g) + \|P_n^k g\| \le 2\frac{3}{2}\|P_n^k g\|_{a=3/2} + 2\|P_n^k\|_{a=3/2} = 5\|P_n^k g\|_{a=3/2}.$$

Then, since $P_n^k \phi_n = \phi_n$, we can apply Lemma 2.4 (k times) to obtain

$$\left\|P_n^k f - \left(\int f\, d\lambda\right)\phi_n\right\|_{\mathrm{BV}} = \left\|P_n^k g\right\|_{\mathrm{BV}} \le 5\left(\frac{2}{3}\right)^k \|g\|_{a=3/2} \to 0$$

as $k \to \infty$. □

Proof of Lemma 2.2 (i). We proceed directly:

$$
\begin{aligned}
Pf = &\{1/4 f(T_1^{-1}x)\chi_{[0,2/3]}(x) + 1/3 f(T_2^{-1}x)\chi_{[2/3,1]}(x)\}\ (=\mathrm{I}))\\
&+ 1/3 f(T_3^{-1}x)\\
&+ \{1/3 f(T_4^{-1}x)\chi_{[0,1/6]}(x) + 1/4 f(T_5^{-1}x)\chi_{[1/2,1]}(x)\}\ (=\mathrm{II}))\\
&+ 1/8 f(T_6^{-1}x).
\end{aligned}
$$

A standard argument upper bounds the variation of the two terms in braces as follows:

$$\mathrm{Var}\,\{\mathrm{I})\} \leq 1/4 V_{I_1} f + 1/3 V_{I_2} f + (1/3 - 1/4) f(1/6)$$

and

$$\mathrm{Var}\,\{\mathrm{II})\} \leq 1/3 V_{I_4} f + 1/4 V_{I_5} f + (1/3 - 1/4) f(2/3)$$

respectively. Jump terms are then estimated

$$f(1/6) \leq 6 \int_{I_1} f + V_{I_1} f,\ \ f(2/3) \leq 24/5 \int_{I_5} f + V_{I_5} f.$$

Putting these together

$$
\begin{aligned}
V_0^1 Pf &\leq 1/3 V_0^1 f + 1/2 \int_{I_1} f + 2/5 \int_{I_5} f\\
&\leq 1/3 V_0^1 f + 1/2 ||f||.
\end{aligned}
$$

Proof of Lemma 2.2 (ii). Apply Proposition 1 of [Mu98] with $\mathcal{L} = P$, $n = 1$, $a = 3/2$, $\lambda = \sigma = 3$ and $D = 4/3$ ($D = \sup |T'(x)|/|T'(y)|$ where x and y are under the same branch of T; thus $D = 4/3$.) One then has the formula:

$$Pf \geq \|f\| \left(1 - \tfrac{a}{\sigma}\right)/D = \tfrac{3}{8}\|f\|. \qquad \square$$

2.3. An Application – Establishing the Upper Estimate. Following the argument from Section 1.5 we estimate the error in approximation as in (1.6):

$$\|\phi_n - \phi\| \leq \sum_{l=0}^{N} \left\|P_n^l\, \Psi_n\right\| + \left\| \sum_{l=N+1}^{\infty} P_n^l\, \Psi_n \right\|. \tag{2.1}$$

The first sum is majorized by $(N+1)||\Psi_n||$ while the second sum is estimated using Lemma 2.1 (i), Lemma 2.4 (spectral gap) and Lemma 2.1 (iii):

$$
\begin{aligned}
\left\| \sum_{l=N+1}^{\infty} P_n^l\, \Psi_n \right\| &\leq 2 \left\| \sum_{l=N+1}^{\infty} P_n^l\, \Psi_n \right\|_{a=3/2}\\
&\leq 2 \sum_{l=N+1}^{\infty} \left(\frac{2}{3}\right)^l \left\|P_n^l\, \Psi_n\right\|_{a=3/2}\\
&\leq 6 \left(\frac{2}{3}\right)^{N+1} ||\Psi_n||_{a=3/2}\\
&\leq 6 \left(\frac{2}{3}\right)^{N+1} \max\left\{ \frac{||\Psi_n||}{2}, \frac{V_0^1\, \Psi_n}{3/2} \right\}.
\end{aligned}
$$

Since $\Psi_n = \pi_n\phi - \phi$ we have

$$V_0^1\,\Psi_n \le 4/31 \text{ and } ||\Psi_n|| \le \frac{2}{31n}$$

(the latter inequality being a simple exercise).

Choose $N = \lfloor \frac{\log 8n}{\log(3/2)} \rfloor$ to obtain the bound

$$(N+1)||\Psi_n|| \le \frac{2\left\{\left[\frac{\log 8n}{\log(3/2)}\right]+1\right\}}{31n}$$

while

$$||\sum_{l=N+1}^{\infty} P_n^l\,\Psi_n|| \le \frac{16}{31}\left(\frac{2}{3}\right)^{N+1} \le \frac{2}{31n}.$$

Combining the two estimates yields our upper estimate for $n \ge 162$ (a restriction which will be in place in the next section anyway)

$$||\phi_n - \phi|| \le \left\{\frac{\frac{2}{\log(3/2)} + \frac{2(\log 8)(\log 3/2)^{-1}+2+2}{\log 162}}{31}\right\}\frac{\log n}{n} \le 0.25\frac{\log n}{n}.$$

SECTION 3

3.1. The Lower Estimate – Preliminaries. Recall from the discussion of Section 1.6 that we will look for nontrivial lower estimates only for values of $n \in \mathcal{N}$, so this hypothesis is assumed throughout this section. We will also assume that n is chosen sufficiently large such that $1/n << 1/18$, the length of the smallest interval over which T is linear. The exact restriction on n will be made clear as we proceed with the details in Section 3.2.

As in the argument for the upper estimate, we begin with the expansion (1.6), followed by the triangle inequality

$$||\phi_n - \phi|| \ge \left\|\sum_{l=0}^{N} P_n^l\Psi_n\right\| - \left\|\sum_{l=N+1}^{\infty} P_n^l\Psi_n\right\|$$

where $\Psi_n = \pi_n\phi - \phi$. We reiterate that the goal is to choose $n = N(n) = O(\log n)$ such that the second sum is $O(\frac{1}{n})$ while the first sum is $O\left(\frac{\log n}{n}\right)$.

3.2. The Functions Ψ_n, $P_n\Psi_n$ and $P_n^2\Psi_n$. It is necessary to carefully follow the first two iterates of Ψ_n by P_n. After that we can proceed with general computations to control the norms and variation of the iterates.

Assume that $n \in \mathcal{N}$ with $n \ge 9\cdot 18 = 162$ so that each of the characteristic functions used below lie entirely under single linear branches of the map T.

By direct computation we establish the following (See Figure 3.1):

$$\Psi_n = \frac{1}{31}\left\{\chi_{[\frac{1}{6},\frac{1}{6}+\frac{1}{2n}]} - \chi_{[\frac{1}{6}-\frac{1}{2n},\frac{1}{6}]}\right\}.$$

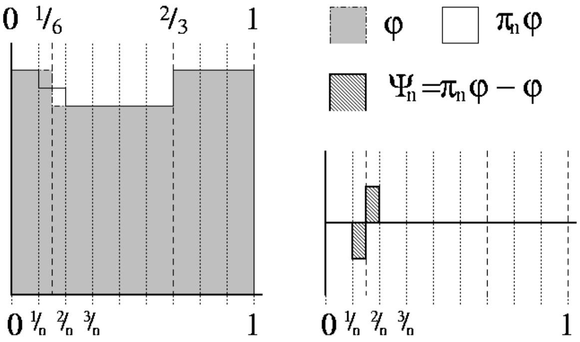

Figure 3.1: Affect of π_n on ϕ when $n = 9$.

Next $P\Psi_n = \frac{1}{31}\left\{\frac{1}{3}\chi_{[\frac{2}{3},\frac{2}{3}+\frac{3}{2n}]} - \frac{1}{4}\chi_{[\frac{2}{3}-\frac{4}{2n},\frac{2}{3}]}\right\}$ which, followed by projection yields

$$P_n\Psi_n = \frac{1}{31}\left\{\frac{1}{3}\chi_{[\frac{2}{3},\frac{2}{3}+\frac{1}{n}]} + \frac{1}{6}\chi_{[\frac{2}{3}+\frac{1}{n},\frac{2}{3}+\frac{2}{n}]} - \frac{1}{4}\chi_{[\frac{2}{3}-\frac{2}{n},\frac{2}{3}]}\right\}.$$

A similar computation gives,

$$\begin{aligned} P_n^2\Psi_n = \frac{1}{31}\bigg\{ & \frac{1}{12}\chi_{[\frac{1}{6}+\frac{1}{2n},\frac{1}{6}+\frac{7}{2n}]} + \frac{1}{18}\chi_{[\frac{1}{6}+\frac{7}{2n},\frac{1}{6}+\frac{9}{2n}]} \\ & +\frac{1}{24}\chi_{[\frac{1}{6}+\frac{9}{2n},\frac{1}{6}+\frac{15}{2n}]} + \frac{1}{48}\chi_{[\frac{1}{6}+\frac{15}{2n},\frac{1}{6}+\frac{17}{2n}]} \\ & -\frac{1}{12}\chi_{[\frac{1}{6}-\frac{11}{2n},\frac{1}{6}-\frac{1}{2n}]} - \frac{1}{24}\chi_{[\frac{1}{6}-\frac{13}{2n},\frac{1}{6}-\frac{11}{2n}]}\bigg\}. \end{aligned}$$

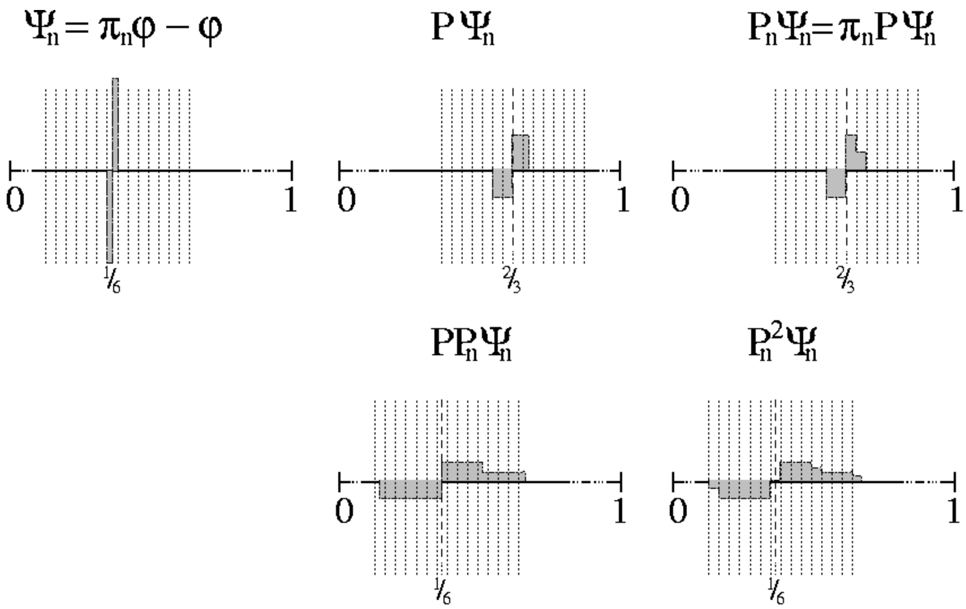

Figure 3.2: First two iterates of Ψ_n under P_n.

Note the effect of projection π_n on the part of $PP_n\Psi_n$ lying over the discretization interval $[\frac{1}{6}-\frac{1}{2n},\frac{1}{6}+\frac{1}{2n}]$ leading to a loss of a small amount of the mass from Ψ_n in the iteration $P_n^2\Psi_n$ (see Figure 3.2).

We collect the relevant observations about these first two iterates as straightforward calculations using the above expressions.

LEMMA 3.1

(i) $V_0^1\Psi_n = 4/31,\ ||\Psi_n|| = 1/31n$ *and*

$$\operatorname{supp}(\Psi_n)^- = \left[\frac{1}{6}-\frac{1}{2n},\frac{1}{6}\right],\ \operatorname{supp}(\Psi_n)^+ = \left[\frac{1}{6},\frac{1}{6}+\frac{1}{2n}\right].$$

(ii) $V_0^1 P_n\Psi_n = 7/186,\ ||P_n\Psi_n|| = 1/31n$ *and*

$$\operatorname{supp}(P_n\Psi_n)^- = \left[\frac{2}{3}-\frac{2}{n},\frac{2}{3}\right],\ \operatorname{supp}(P_n\Psi_n)^+ = \left[\frac{2}{3},\frac{2}{3}+\frac{2}{n}\right].$$

(iii) $V_0^1 P_n^2\Psi_n = 1/93,\ ||P_n^2\Psi_n|| = 1/31n - 1/372n = 11/372n$ *and*

$$\operatorname{supp}(P_n^2\Psi_n)^- = \left[\frac{1}{6}-\frac{13}{2n},\frac{1}{6}-\frac{1}{2n}\right],\ \operatorname{supp}(P_n^2\Psi_n)^+ = \left[\frac{1}{6}+\frac{1}{2n},\frac{1}{6}+\frac{17}{2n}\right].$$

3.3. Estimate on the Order of $N(n)$ – The Details.[4] We now wish to determine the number j of iterations which can be performed before the supports of $P_n^j(P_n^2\Psi_n)^+$ and $P_n^j(P_n^2\Psi_n)^-$ again overlap with a consequent loss of mass in the quantity $||P_n^j\, P_n^2\Psi_n||$.

To this end, define $N = N(n)$ to be the largest integer such that for $j = 0,1,\ldots N-2$ both[5]

$$\operatorname{supp}(P_n^j(P_n^2\Psi_n)^+)) \subseteq [1/6+1/2n, 5/18] \cup [2/3, 21/24]$$

$$\operatorname{supp}(P_n^j(P_n^2\Psi_n)^-)) \subseteq [0, 1/6-1/2n] \cup [11/18, 2/3].$$

LEMMA 3.2. *$N(n)$ as defined above we have the following*

(i) $N(n) \geq (\log 4)^{-1}\{\log n - \log(156)\} + 1 = O(\log n)$

(ii) For $j = 0,1,2\ldots,N(n)-2,\ (P_n^{j+2}\Psi_n)^{\pm} = P_n^j(P_n^2\Psi_n)^{\pm}$ *and*

$$\left\|\sum_{l=0}^{N(n)} P_n^l\,\Psi_n\right\| = \frac{11(N(n)-1)}{372n} + \frac{2}{31n}.$$

(iii) $V_0^1\, P_n^{N(n)+1}\Psi_n \leq \frac{204}{62n}$.

[4]We strike a balance in this section between optimal estimates on the various constants, and transparency of the exposition. It is clear that a number of these estimates could be considerably tightened with a more delicate analysis.

[5]This slightly awkward definition using $N-2$ is necessary in order to keep our exposition consistent with earlier sections.

In order to establish these estimates on the quantity $N(n)$ we will need some additional notation. Let $f_j^{\pm} = P_n^j(P_n^2\Psi_n)^{\pm}$, let $s_j^{\pm}$ denote the number of discretization intervals of length $1/n$ occupied by the support of the corresponding functions $f_j^{\pm}$ and let $g_j^{\pm}$ denote the number of discretization intervals (possibly $\frac{1}{2}$) lying between the periodic points and the support of the $f_j^{\pm}$. To illustrate, the support of f_j^+ is $\left[\frac{1}{6}+\frac{g_j^+}{n}, \frac{1}{6}+\frac{g_j^++s_j^+}{n}\right]$ if j is an even integer and $\left[\frac{2}{3}+\frac{g_j^+}{n}, \frac{2}{3}+\frac{g_j^++s_j^+}{n}\right]$ if j is odd. Continuing, define $m_j^{\pm} = ||f_j^{\pm}||_{\infty}$. Finally, for $0 \le j \le N-2$ the supports of the $f_j^{\pm}$ lie under linear branches of the map T so we may unambiguously define the constants $t_j^{\pm} = T'$ over the supports of $f_j^{\pm}$ respectively.

Again, we illustrate all this by an example. For $j = 0$, $f_0^{\pm} = (P_n^2\Psi_n)^{\pm}$, $s_0^+ = 8$, $s_0^- = 6$, $g_0^{\pm} = 1/2$, $m_0^{\pm} = \frac{1}{31}\frac{1}{12} = \frac{1}{372}$, $t_0^+ = 3$ and $t_0^- = 4$. Note that the quantity $g_j^{\pm} + s_j^{\pm}$ upper bounds the 'drift' of the supports away from the periodic orbit, and $N(n)$ is defined to keep this drift in the selected monotonicity intervals for T.

Our notation now established, each of the following statements follows by induction (on j) provided $0 \le j \le N(n)-2$

LEMMA 3.3.

(i) *Each $f_j^{\pm}$ is a unimodal step function: the value of f_j^+ rises monotonically from 0 to m_j^+ then monotonically back down to 0. Similarly for f_j^-.*

(ii) $g_{j+1}^{\pm}+s_{j+1}^{\pm} = t_j^{\pm}(g_j^{\pm}+s_j^{\pm})+d_j^{\pm}$ *The term $d_j^{\pm}$ is necessary to cover the additional spread of the support arising from the application of π_n after the expansive (ie. $t_j^{\pm}$) effect from P. $d_j^{\pm}$ satisfies $d_j^{\pm} \in \{0, \frac{1}{2}\}$.*

(iii) $s_{j+1}^{\pm} = t_j^{\pm}s_j^{\pm}+e_j^{\pm}$ *where as in (ii) above, the term $e_j^{\pm}$ arises from the projection and satisfies $e_j^{\pm} \in \{0, \frac{1}{2}, 1\}$.*

(iv) $m_{j+1}^{\pm} = (t_j^{\pm})^{-1}m_j^{\pm}$.

(v) $||f_j^{\pm}|| = 11/744n$. *(Observe that the projection π_n annihilates no mass after the iteration $P_n^2\,\Psi_n$ since* $\operatorname{supp} f_j^+$, $\operatorname{supp} f_j^-$ *are separated by $g_j^++g_j^-$ (≥ 1) discretization intervals.)*

We are now in a position to estimate the order of N. Using property Lemma 3.3*(ii)*, the trivial observations $t_j^{\pm} \le 4$, $d_j^{\pm} \le 1/2$ and the fact that there are at least $\frac{n}{18}$ full intervals of length $1/n$ lying to the left and right of $1/6$ (*resp.* $2/3$) under a single linear piece of T we find

$$\begin{aligned} g_{N-1}^{\pm}+s_{N-1}^{\pm} &\le 4(4(4\ldots(4(g_0^{\pm}+s_0^{\pm})+d_0^{\pm})\cdots+d_{N-4}^{\pm})+d_{N-3}^{\pm})+d_{N-2}^{\pm} \\ &\le 4^{N-1}\left\{(g_0^{\pm}+s_0^{\pm})+\frac{1}{6}\left(1-4^{-(N-1)}\right)\right\} \le 4^{N-1}\frac{26}{3}. \end{aligned}$$

Since $j = N-1$ is the first such that the support of some $f_j^{\pm}$ fails to lie under a single linear branch of T, we have one of $g_{N-1}^{\pm}+s_{N-1}^{\pm} \ge \frac{n}{18}$ and hence the estimate

$$N = N(n) \ge (\log 4)^{-1}\{\log n - \log(156)\} + 1.$$

Part *(i)* of Lemma 3.2 has been established.

Part *(ii)* of Lemma 3.2 follows from the disjointness of the f_j^+ and f_j^- and Lemma 3.3*(v)* .

It remains to estimate the variation of $P_n^{N(n)-1}(P_n^2\Psi_n) = P_n^{N(n)+1}\Psi_n$.

First, we compare $s_j^{\pm}$ to $g_j^{\pm}$. An easy induction on j, using Lemma 3.3*(ii)* and *(iii)* and the observation $d_j^{\pm} \le e_j^{\pm}$ yields

$$g_j^{\pm} \le s_j^{\pm} \; j = 1, 2 \dots, N-2,$$

in other words, the support of the $f_j^{\pm}$ covers at least half of the 'drift' of the supports away from the periodic orbit.

Next, we derive two expressions from Lemma 3.3*(iii)* and *(iv)*:

$$s_{N-1}^{\pm} = t_{N-2}(t_{N-3}(\dots(t_0^{\pm}s_0^{\pm} + e_0^{\pm})\dots) + e_{N-3}^{\pm}) + e_{N-2}^{\pm}$$
$$m_{N-1}^{\pm} = (\Pi_{j=0}^{N-2} t_j^{\pm})^{-1} m_0^{\pm}.$$

Recalling the estimates $t_j^{\pm} \ge 3$, $e_j^{\pm} \le 1$ and $s_0^{\pm} \le 8$ we obtain from the first expression above

$$s_{N-1} \le \frac{17}{2} \Pi_{j=0}^{N-2} t_j^{\pm}$$

which, combined with $m_0^{\pm} = 1/372$ and the second expression yields

$$m_{N-1}^{\pm} \le \frac{17}{744 s_{N-1}^{\pm}}$$

Since $g_{N-1}^{\pm} < s_{N-1}^{\pm}$ and $g_{N-1}^{\pm} + s_{N-1}^{\pm} \ge n/18$ (the supports have for the first time exceeded the designated intervals defining N), we have $s_{N-1}^{\pm} \ge n/36$. Putting this all together yields

$$V_0^1 P_n^{N+1}\Psi_n = V_0^1 P_n^{N-1}(P_n^2\Psi_n) = 2(m_{N-1}^{+} + m_{N-1}^{-}) \le 4\frac{51}{62n} = \frac{204}{62n},$$

which is the estimate required for part *(iii)* of Lemma 3.2.

3.4. Application–The Lower Estimate. We pick up the argument from Section 3.1, incorporating the estimates above. First, by Lemma 3.2*(i)* and *(ii)* and for $n \ge 162$ we lower bound the first sum in (3.1) as follows:

$$\begin{aligned}
\left\| \sum_{l=0}^{N} P_n^l \Psi_n \right\| &= \frac{11(N(n)-1)}{372n} + \frac{2}{31n} \\
&\ge \frac{11}{372n\log 4}(\log n - \log 156) + \frac{2}{31n} \\
&\ge \left(\frac{11}{372\log 4} - \left(\frac{11\log 156}{372\log 4} - \frac{2}{31} \right)(\log 162)^{-1} \right) \frac{\log n}{n} \\
&\ge 0.0128 \frac{\log n}{n}.
\end{aligned}$$

The second sum in (3.1) is majorized (*cf.* Section 2.3) using Lemma 2.1 (i), Lemma 2.4 (spectral gap), Lemma 2.1 (iii) and Lemma 3.2*(iii)*:

$$\begin{aligned}
\left\| \sum_{l=N+1}^{\infty} P_n^l \, \Psi_n \right\| &\le 2 \left\| \sum_{l=N+1}^{\infty} P_n^l \, \Psi_n \right\|_{a=3/2} \\
&\le 2\frac{1}{1-2/3} \left\| P_n^{N+1}\Psi_n \right\|_{a=3/2} \\
&\le 6 \max\left\{ \frac{||P_n^{N+1}\Psi_n||}{2}, \frac{V_0^1 \, P_n^{N+1}\Psi_n}{3/2} \right\} \\
&\le 6 \max\left\{ \frac{11}{744n}, \frac{2}{3}\frac{204}{62n} \right\} = \frac{408}{31n}.
\end{aligned}$$

Putting these two estimates together gives the lower bound in Theorem 1, Section 1, provided $n \in \mathcal{N}$ is chosen sufficiently large.

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E-mail address: cbose@math.uvic.ca
E-mail address: r.murray@math.waikato.ac.nz

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