Problem 2

The distribution function is given by

$$F(x) = \begin{cases} \frac{(x-2)^2}{4}, & 2 \le x \le 3 \\ -\frac{x^2}{12} + x - 2, & 3 \le x \le 6 \end{cases}$$

The inverse to which is given by

$$F^{-1}(y) = \begin{cases} 2(\sqrt{y} + 1), & 0 \le y \le \frac{1}{4} \\ 6 - 2\sqrt{3(1-y)}, & \frac{1}{4} \le y \le 1 \end{cases}$$

1000 samples generated with inverse transform algorithm yield the following result.

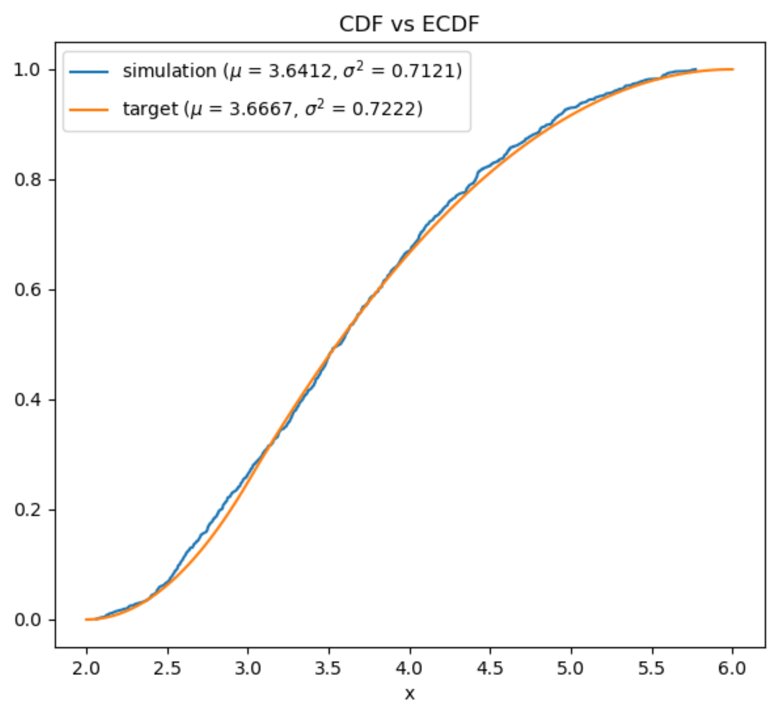

Figure 1: distribution comparison

Problem 4

$$F^{-1}(y) = \left[-\frac{\log(1-y)}{\alpha}\right]^{\frac{1}{\beta}}$$

The pdf is given by

$$f(x) = \alpha\beta\, x^{\beta-1} \exp(-\alpha x^{\beta})$$

The mean is $\frac{1}{\sqrt[\beta]{\alpha}}\Gamma\left(1+\frac{1}{\beta}\right)$ and the variance is $\frac{1}{\sqrt[\beta]{\alpha^2}}\left[\Gamma\left(2+\frac{1}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2\right]$. 1000 samples generated with inverse transform algorithm yield the following result.

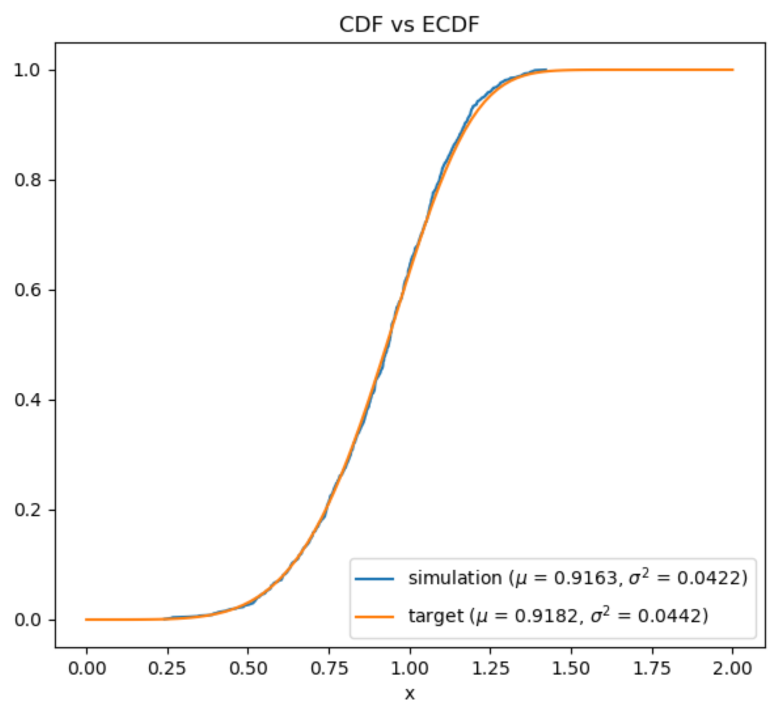

Figure 2: distribution comparison for $\alpha = 1$, $\beta = 5$

Problem 6

The inverse of the cdf is given by

$$F^{-1}(y) = -\log(1 - cy), \quad c = 1 - e^{-0.05}$$

Mean of the target distribution is given by,

$$\frac{1 - 1.05e^{-0.05}}{1 - e^{-0.05}}$$

1000 samples generated with inverse transform algorithm yield the following result.

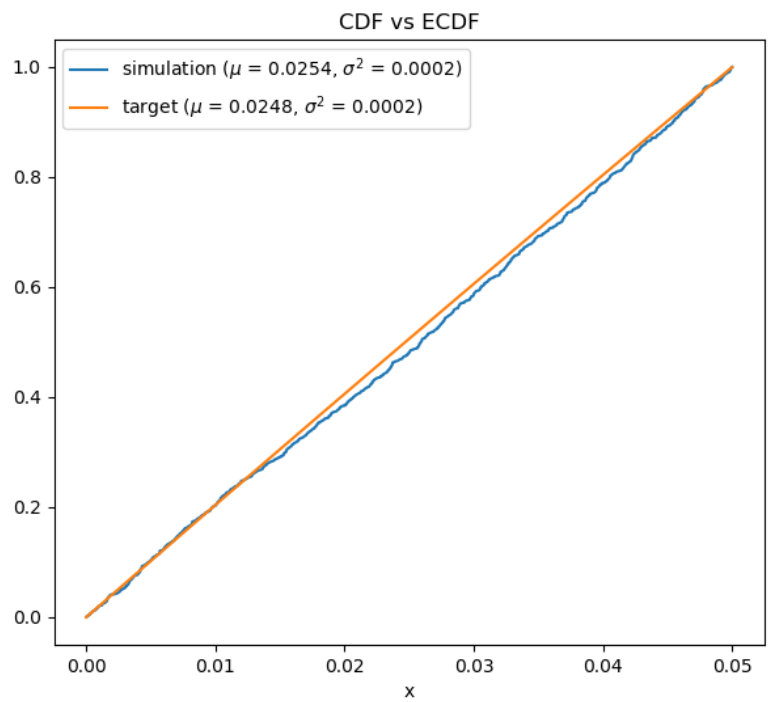

Figure 3: distribution comparison

Problem 8(a)

The inverse of $F_i(x)$ is given by

$$F_i^{-1}(y) = y^{\frac{1}{2i-1}}, \quad i = 1, 2, 3$$

1000 samples generated with composition method yield the following result.

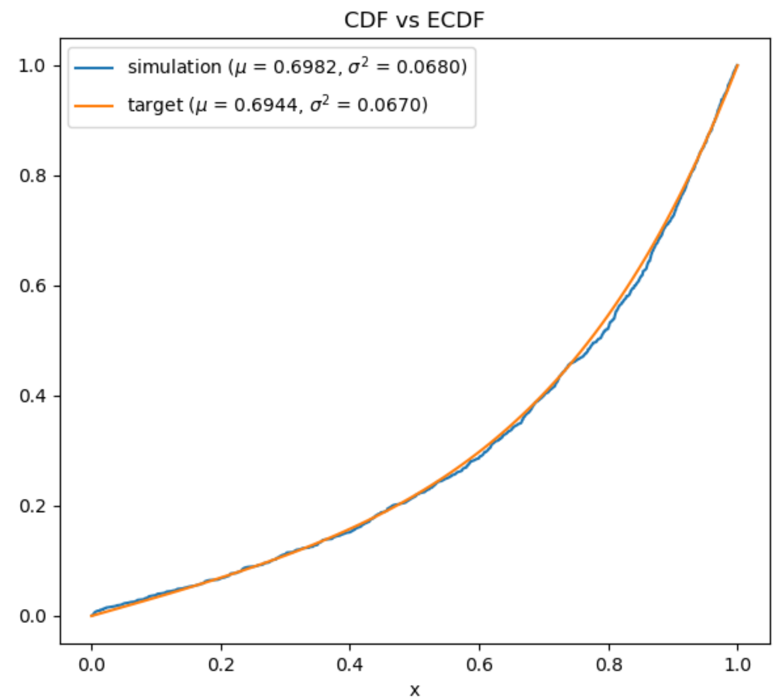

Figure 4: distribution comparison

Problem 8(b)

Our setup is as follows.

$$F_1(x) = 1 - e^{-2x}, \;\; 0 < x < \infty$$
$$F_2(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \leq x < \infty \end{cases}$$
$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

1000 samples generated with composition method yield the following result.

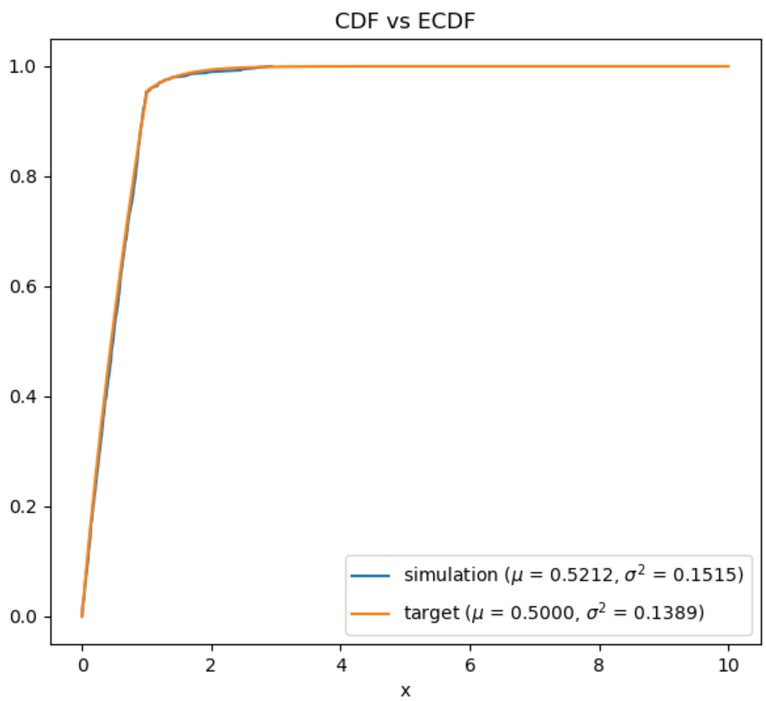

Figure 5: distribution comparison

Problem 8(c)

Our setup is as follows. The probability weights have been generated randomly.

$$\bar{\alpha} = (0.17681302, 0.08941652, 0.13690479, 0.14830053, 0.15965611,\\ 0.17054088, 0.03269305, 0.00496038, 0.0506282, 0.03008653)$$

1500 samples generated with composition method yield the following result.

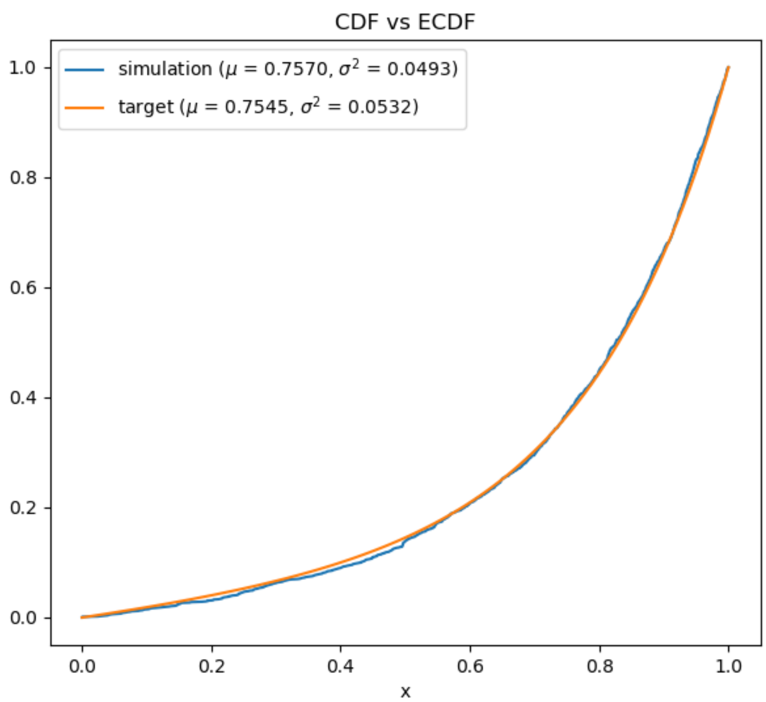

Figure 6: distribution comparison

Problem 10

Our notation is as follows.
$n = 1000 =$ number of policyholders
$s_i =$ indicator variable for i-th policyholder presenting a claim
$p = E[s_i] = 0.05\ \forall i$
$X_i =$ amount of claim presented by i-th policyholder
$\mu = E[X_i]\ \forall i$
$Y = \sum_{i=1}^{n} s_i X_i =$ amount of total claim presented by all policyholders
$F(x) =$ cdf of Y
$A =$ a certain amount of money, for our purposes $A = \$\,50,000$
$M =$ set of all binary n-tuples
$M_k =$ set of binary n-tuple with exactly k-entries being 1
$P(Y > A|m) =$ probabilty conditioned on $s_i = m_i\ \forall i$ where $m \in M$.
$P(m) =$ probabilty that $s_i = m_i\ \forall i$ where $m \in M$.
$\Gamma(x, k, \mu) =$ cdf of gamma random variable that is the sum of k identical exponential random variables with mean μ
With our notations we need to compute the following.

$$\begin{aligned}
F(A) = P(Y \leq A) &= \sum_{m \in M} P(Y \leq A|m)P(m) \\
&= \sum_{k=1}^{n} \sum_{m \in M_k} P(Y \leq A|m)P(m) \\
&= \sum_{k=1}^{n} \sum_{m \in M_k} P(Y \leq A|m)p^k(1-p)^{n-k} \\
&= \sum_{k=1}^{n} |M_k| p^k (1-p)^{n-k} P(Y \leq A|m_k), \qquad m_k \in M_k \\
&= \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} P(Y \leq A|m_k) \\
&= \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} \Gamma(A, k, \mu) \\
&=: \sum_{k=1}^{n} p_{n,k} \Gamma(A, k, \mu)
\end{aligned}$$

$\sum_{k=1}^{n} p_{n,k} = 1$ and $p_{n,k} > 0$ which implies we can sample Y using composition technique because we can already sample gamma distributions. 1000 samples generated using composition technique sets the required probability at $12\% - 13\%$. We can also sample Y by simulating n Bernoulli trials and then simulating s exponential random variables where $s =$ the number of successes in the Bernoulli trials. This straight-forward method sets the required probability at $10\% - 12\%$. The actual value of the required probability $\approx 10.7\%$.