

Problem 2

The distribution function is given by

$$F(x) = \begin{cases} \frac{(x-2)^2}{4}, & 2 \leq x \leq 3 \\ -\frac{x^2}{12} + x - 2, & 3 \leq x \leq 6 \end{cases}$$

The inverse to which is given by

$$F^{-1}(y) = \begin{cases} 2(\sqrt{y} + 1), & 0 \leq y \leq \frac{1}{4} \\ 6 - 2\sqrt{3(1-y)}, & \frac{1}{4} \leq y \leq 1 \end{cases}$$

1000 samples generated with inverse transform algorithm yield the following result.

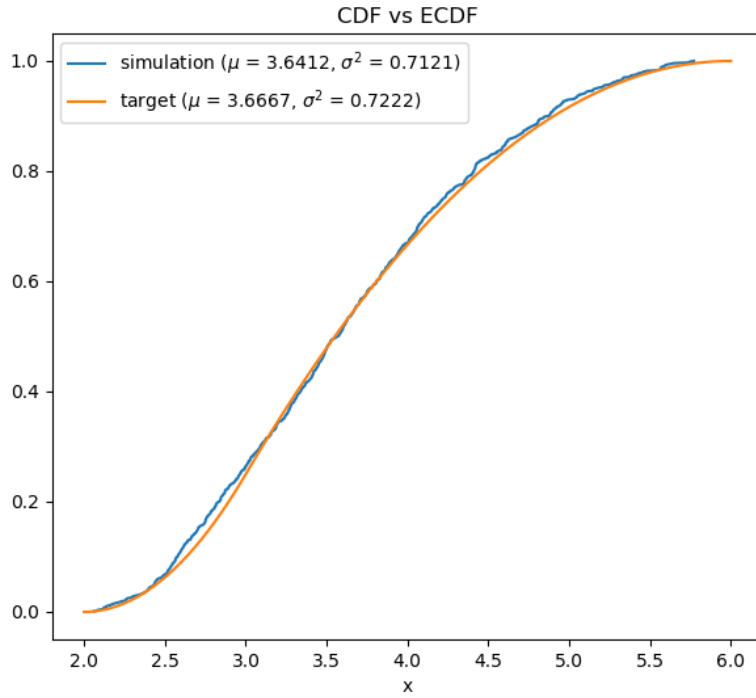


Figure 1: distribution comparison

Problem 4

$$F^{-1}(y) = \left[-\frac{\log(1-y)}{\alpha} \right]^{\frac{1}{\beta}}$$

The pdf is given by

$$f(x) = \alpha \beta x^{\beta-1} \exp(-\alpha x^{\beta})$$

The mean is $\frac{1}{\sqrt[\beta]{\alpha}} \Gamma\left(1 + \frac{1}{\beta}\right)$ and the variance is $\frac{1}{\sqrt[\beta]{\alpha^2}} \left[\Gamma\left(2 + \frac{1}{\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}\right)^2 \right]$.
1000 samples generated with inverse transform algorithm yield the following result.

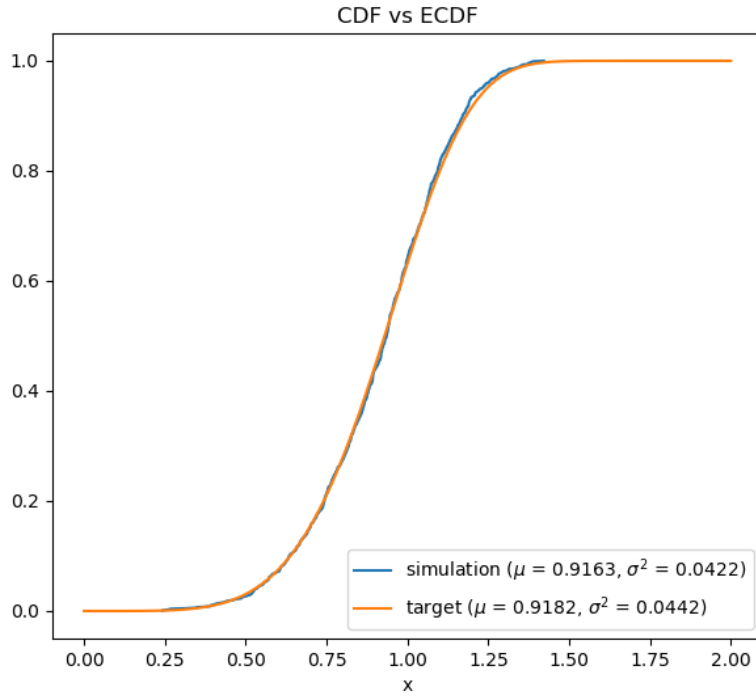


Figure 2: distribution comparison for $\alpha = 1, \beta = 5$

Problem 6

The inverse of the cdf is given by

$$F^{-1}(y) = -\log(1 - cy), \quad c = 1 - e^{-0.05}$$

Mean of the target distribution is given by,

$$\frac{1 - 1.05e^{-0.05}}{1 - e^{-0.05}}$$

1000 samples generated with inverse transform algorithm yield the following result.

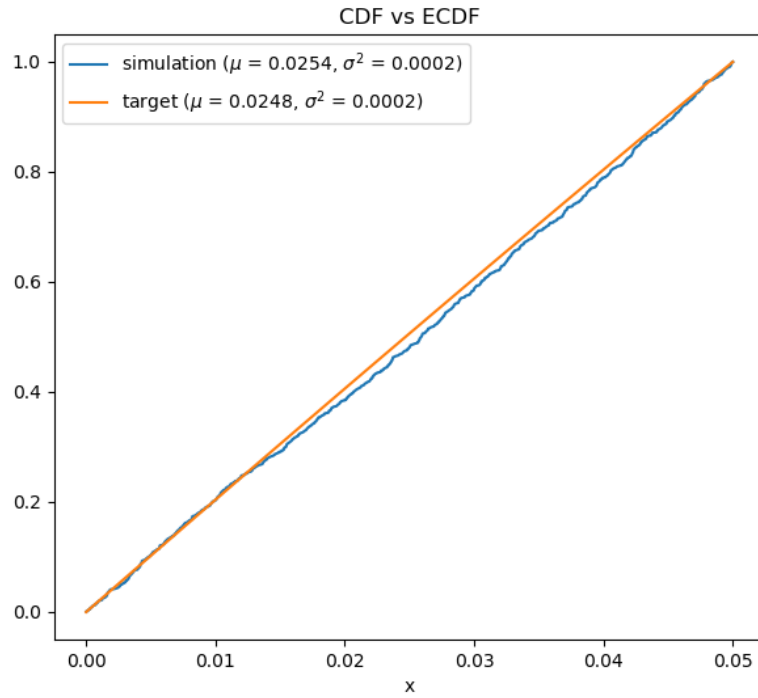


Figure 3: distribution comparison

Problem 8(a)

The inverse of $F_i(x)$ is given by

$$F_i^{-1}(y) = y^{\frac{1}{2i-1}}, \quad i = 1, 2, 3$$

1000 samples generated with composition method yield the following result.

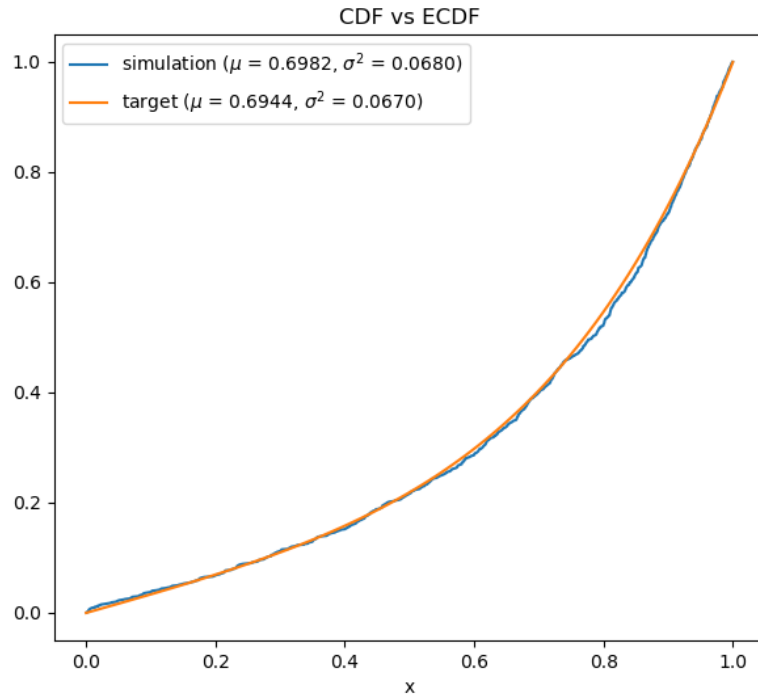


Figure 4: distribution comparison

Problem 8(b)

Our setup is as follows.

$$F_1(x) = 1 - e^{-2x}, \quad 0 < x < \infty$$

$$F_2(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \leq x < \infty \end{cases}$$

$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

1000 samples generated with composition method yield the following result.

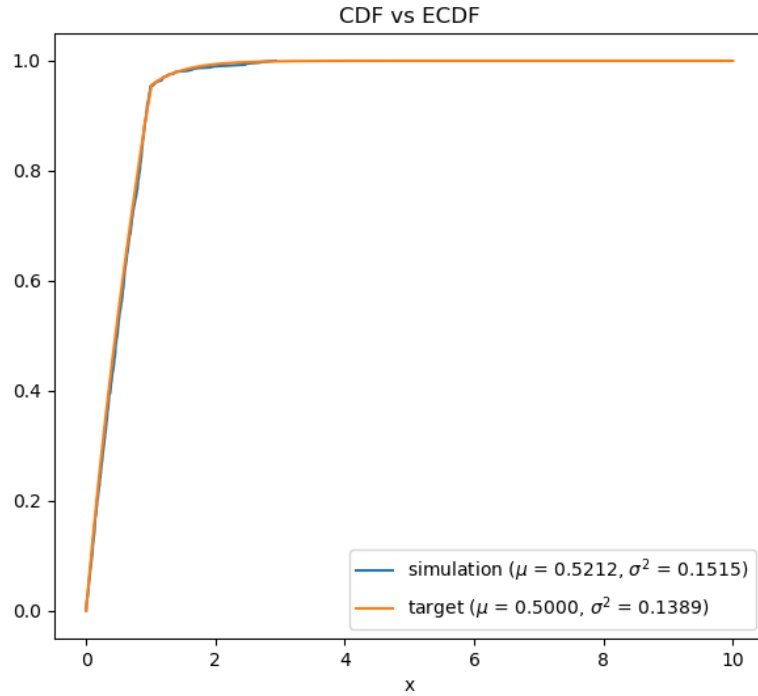


Figure 5: distribution comparison

Problem 8(c)

Our setup is as follows. The probability weights have been generated randomly.

$$\bar{\alpha} = (0.17681302, 0.08941652, 0.13690479, 0.14830053, 0.15965611, \\ 0.17054088, 0.03269305, 0.00496038, 0.0506282, 0.03008653)$$

1500 samples generated with composition method yield the following result.

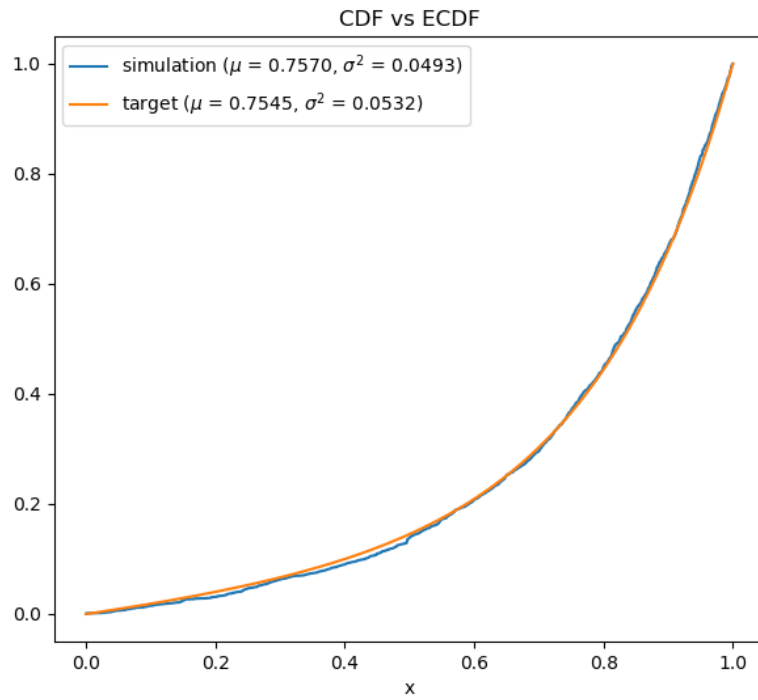


Figure 6: distribution comparison

Problem 10

Our notation is as follows.

$n = 1000$ = number of policyholders

s_i = indicator variable for i -th policyholder presenting a claim

$p = E[s_i] = 0.05 \forall i$

X_i = amount of claim presented by i -th policyholder

$\mu = E[X_i] = \$800 \forall i$

$Y = \sum_{i=1}^n s_i X_i$ = amount of total claim presented by all policyholders

$F(x)$ = cdf of Y

$A = \$50,000$

M = set of all binary n -tuples

M_k = set of binary n -tuple with exactly k -entries being 1

$P(Y > A|m)$ = probability conditioned on $s_i = m_i \forall i$ where $m \in M$.

$P(m)$ = probability that $s_i = m_i \forall i$ where $m \in M$.

$\Gamma(x, k, \mu)$ = cdf of gamma random variable that is the sum of k identical exponential random variables with mean μ

With our notations we need to compute the following.

$$\begin{aligned}
F(A) &= P(Y \leq A) = \sum_{m \in M} P(Y \leq A|m)P(m) \\
&= \sum_{k=1}^n \sum_{m \in M_k} P(Y \leq A|m)P(m) \\
&= \sum_{k=1}^n \sum_{m \in M_k} P(Y \leq A|m)p^k(1-p)^{n-k} \\
&= \sum_{k=1}^n |M_k|p^k(1-p)^{n-k}P(Y \leq A|m_k), \quad m_k \in M_k \\
&= \sum_{k=1}^n \binom{n}{k} p^k(1-p)^{n-k}P(Y \leq A|m_k) \\
&= \sum_{k=1}^n \binom{n}{k} p^k(1-p)^{n-k}\Gamma(A, k, \mu) \\
&=: \sum_{k=1}^n p_{n,k}\Gamma(A, k, \mu)
\end{aligned}$$

$\sum_{k=1}^n p_{n,k} = 1$ and $p_{n,k} > 0$ which implies we can sample Y using composition technique because we can already sample gamma distributions. 1000 samples generated using composition technique sets the required probability at 12% – 13%. We can also sample Y by simulating n Bernoulli trials and then simulating s exponential random variables where s = the number of successes in the Bernoulli trials. This straight-forward method sets the required probability at 10% – 12%. The actual value of the required probability $\approx 10.7\%$.

Problem 12

$\max_{1 \leq i \leq n} X_i$ has the distribution $\prod_{i=1}^n F_i(x)$ and $\min_{1 \leq i \leq n} X_i$ has the distribution $1 - \prod_{i=1}^n (1 - F_i(x))$.

Problem 14

(a)

Assuming $x \in [a, b]$,

$$\begin{aligned} P(X \leq x | a \leq X \leq b) &= \frac{P(X \leq x | a \leq X \leq b)}{P(a \leq X \leq b)} \\ &= \frac{P(a \leq X \leq x)}{P(a \leq X \leq b)} \\ &= \frac{G(x) - G(a)}{G(b) - G(a)} = F(x) \end{aligned}$$

(b)

Let $Y = X | a \leq X \leq b$. Density of Y is given by

$$f(y) = \begin{cases} \frac{g(y)}{G(b) - G(a)}, & \text{if } a \leq y \leq b \\ 0, & \text{otherwise} \end{cases}$$

So in this case we can choose $c = \frac{1}{G(b) - G(a)}$. And a sample X drawn from G will be selected if the generated uniform random number U satisfies

$$U \leq \begin{cases} \frac{f(X)}{cg(X)} = 1, & \text{if } a \leq X \leq b \\ 0, & \text{otherwise} \end{cases}$$

which implies X is selected iff it's in $[a, b]$.