Problem 2

The distribution function is given by

$$F(x) = \begin{cases} \frac{(x-2)^2}{4}, & 2 \leq x \leq 3 \\ -\frac{x^2}{12} + x - 2, & 3 \leq x \leq 6 \end{cases}$$

The inverse to which is given by

$$F^{-1}(y) = \begin{cases} 2(\sqrt{y} + 1), & 0 \leq y \leq \frac{1}{4} \\ 6 - 2\sqrt{3(1-y)}, & \frac{1}{4} \leq y \leq 1 \end{cases}$$

1000 samples generated with inverse transform algorithm yield the following result.

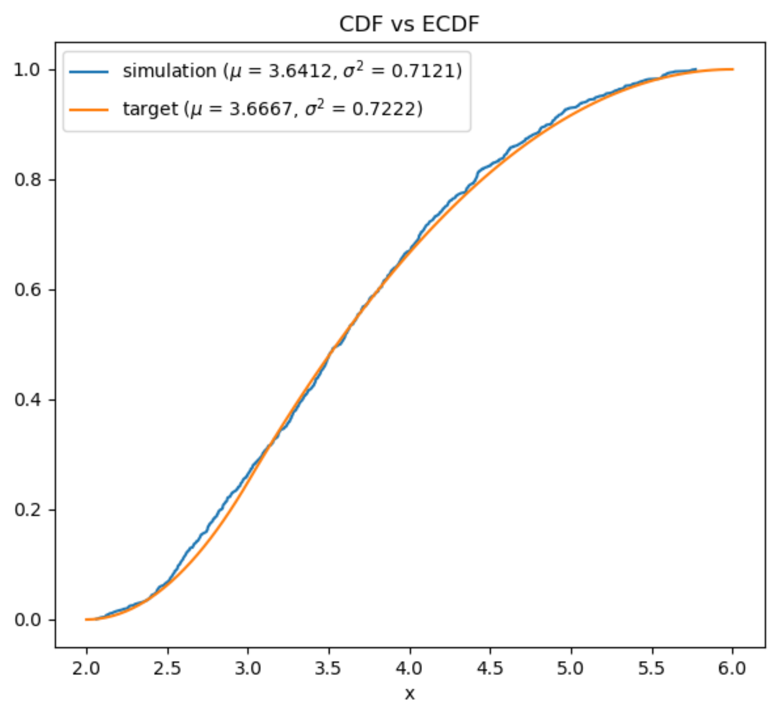

Figure 1: distribution comparison

Problem 4

$$F^{-1}(y) = \left[-\frac{\log(1-y)}{\alpha}\right]^{\frac{1}{\beta}}$$

The pdf is given by

$$f(x) = \alpha\beta\, x^{\beta-1}\exp(-\alpha x^{\beta})$$

The mean is $\frac{1}{\sqrt[\beta]{\alpha}}\Gamma\left(1+\frac{1}{\beta}\right)$ and the variance is $\frac{1}{\sqrt[\beta]{\alpha^2}}\left[\Gamma\left(2+\frac{1}{\beta}\right)-\Gamma\left(1+\frac{1}{\beta}\right)^2\right]$. 1000 samples generated with inverse transform algorithm yield the following result.

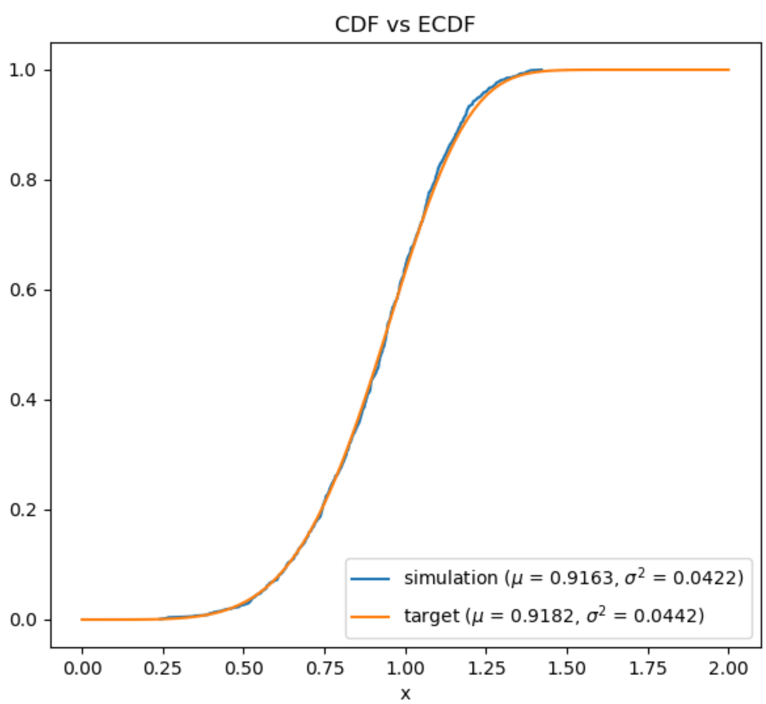

Figure 2: distribution comparison for $\alpha = 1$, $\beta = 5$

Problem 6

The inverse of the cdf is given by

$$F^{-1}(y) = -\log(1 - cy), \quad c = 1 - e^{-0.05}$$

Mean of the target distribution is given by,

$$\frac{1 - 1.05e^{-0.05}}{1 - e^{-0.05}}$$

1000 samples generated with inverse transform algorithm yield the following result.

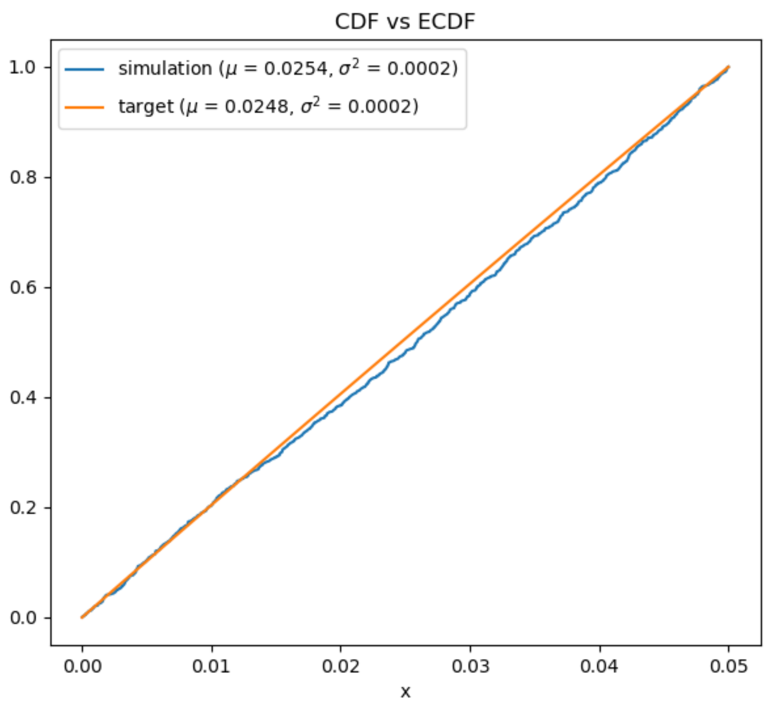

Figure 3: distribution comparison

Problem 8(a)

The inverse of $F_i(x)$ is given by

$$F_i^{-1}(y) = y^{\frac{1}{2i-1}}, \quad i = 1, 2, 3$$

1000 samples generated with composition method yield the following result.

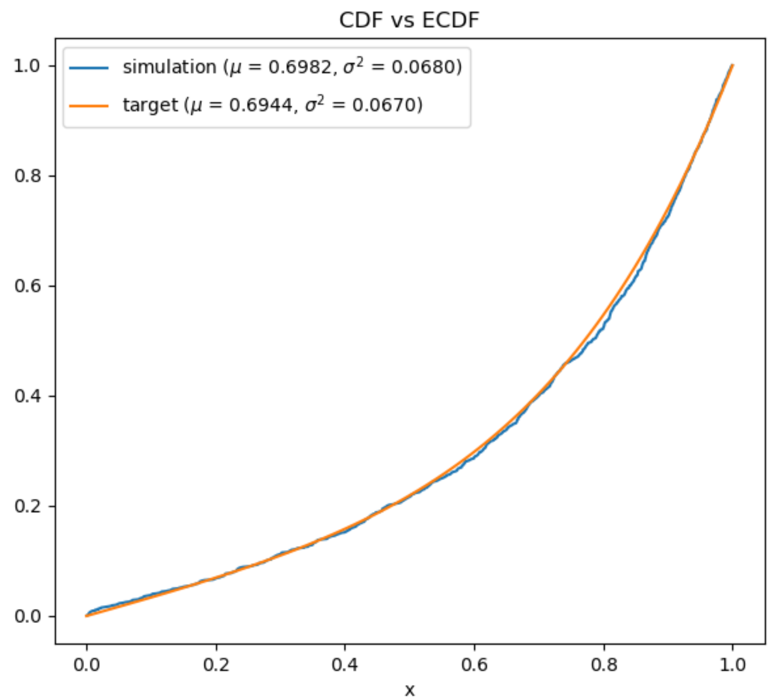

Figure 4: distribution comparison

Problem 8(b)

Our setup is as follows.

$$F_1(x) = 1 - e^{-2x}, \;\; 0 < x < \infty$$
$$F_2(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \le x < \infty \end{cases}$$
$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

1000 samples generated with composition method yield the following result.

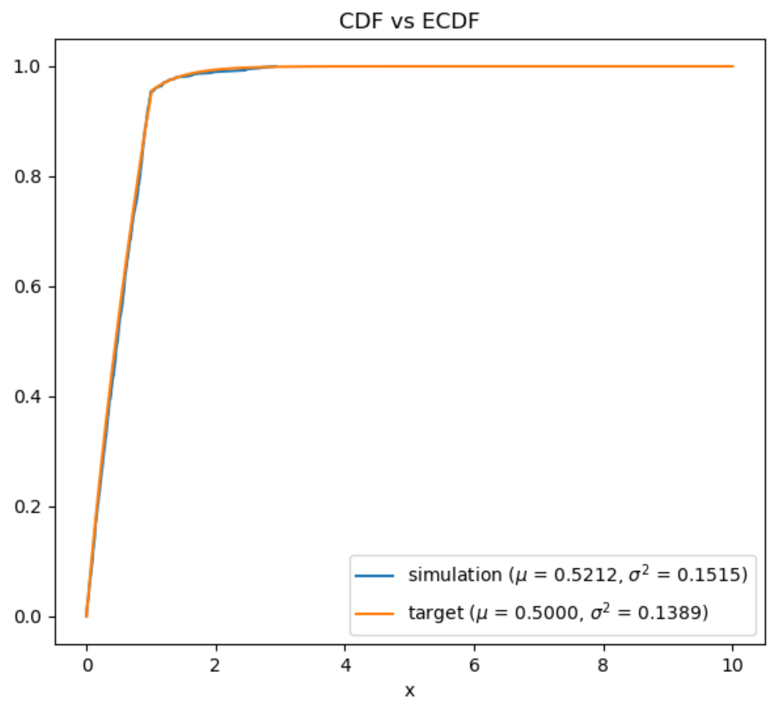

Figure 5: distribution comparison

Problem 8(c)

Our setup is as follows. The probability weights have been generated randomly.

$$\begin{aligned}\bar{\alpha} = (&0.17681302, 0.08941652, 0.13690479, 0.14830053, 0.15965611,\\ &0.17054088, 0.03269305, 0.00496038, 0.0506282, 0.03008653)\end{aligned}$$

1500 samples generated with composition method yield the following result.

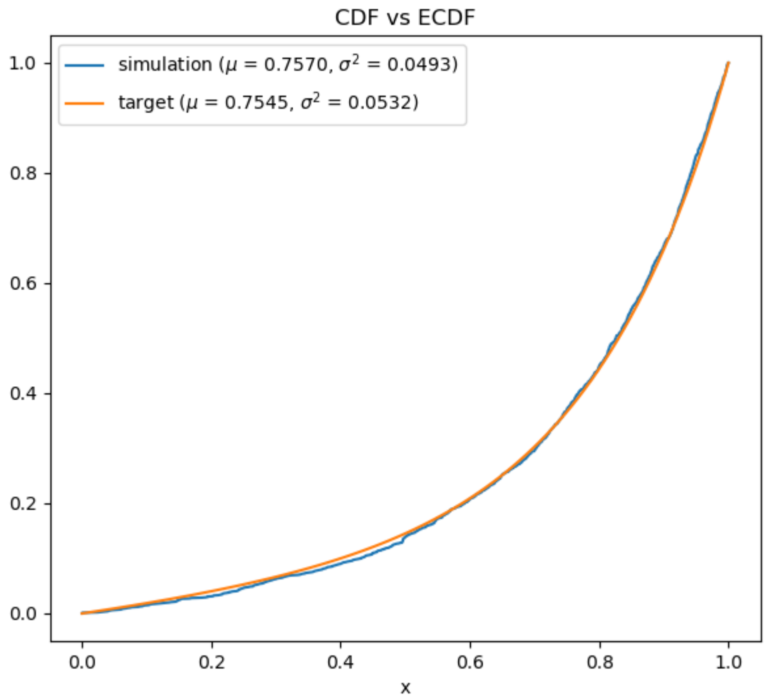

Figure 6: distribution comparison

Problem 10

Our notation is as follows.
$n = 1000 =$ number of policyholders
$s_i =$ indicator variable for i-th policyholder presenting a claim
$p = E[s_i] = 0.05\ \forall i$
$X_i =$ amount of claim presented by i-th policyholder
$\mu = E[X_i] = \$\,800\ \forall i$
$Y = \sum_{i=1}^{n} s_i X_i =$ amount of total claim presented by all policyholders
$F(x) =$ cdf of Y
$A = \$\,50,000$
$M =$ set of all binary n-tuples
$M_k =$ set of binary n-tuple with exactly k-entries being 1
$P(Y > A|m) =$ probability conditioned on $s_i = m_i\ \forall i$ where $m \in M$.
$P(m) =$ probability that $s_i = m_i\ \forall i$ where $m \in M$.
$\Gamma(x, k, \mu) =$ cdf of gamma random variable that is the sum of k identical exponential random variables with mean μ
With our notations we need to compute the following.

$$\begin{aligned}
F(A) = P(Y \le A) &= \sum_{m \in M} P(Y \le A|m)P(m) \\
&= \sum_{k=1}^{n} \sum_{m \in M_k} P(Y \le A|m)P(m) \\
&= \sum_{k=1}^{n} \sum_{m \in M_k} P(Y \le A|m)p^k(1-p)^{n-k} \\
&= \sum_{k=1}^{n} |M_k| p^k (1-p)^{n-k} P(Y \le A|m_k), \qquad m_k \in M_k \\
&= \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} P(Y \le A|m_k) \\
&= \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} \Gamma(A, k, \mu) \\
&=: \sum_{k=1}^{n} p_{n,k} \Gamma(A, k, \mu)
\end{aligned}$$

$\sum_{k=1}^{n} p_{n,k} = 1$ and $p_{n,k} > 0$ which implies we can sample Y using composition technique because we can already sample gamma distributions. 1000 samples generated using composition technique sets the required probability at $12\% - 13\%$. We can also sample Y by simulating n Bernoulli trials and then simulating s exponential random variables where $s =$ the number of successes in the Bernoulli trials. This straight-forward method sets the required probability at $10\% - 12\%$. The actual value of the required probability $\approx 10.7\%$.

Problem 12

$\max_{1\le i\le n} X_i$ has the distribution $\prod_{i=1}^n F_i(x)$ and $\min_{1\le i\le n} X_i$ has the distribution $1 - \prod_{i=1}^n (1 - F_i(x))$.

Problem 14

(a)

Assuming $x \in [a, b]$,

$$\begin{aligned} P(X \le x | a \le X \le b) &= \frac{P(X \le x | a \le X \le b)}{P(a \le X \le b)} \\ &= \frac{P(a \le X \le x)}{P(a \le X \le b)} \\ &= \frac{G(x) - G(a)}{G(b) - G(a)} = F(x) \end{aligned}$$

(b)

Let $Y = X | a \le X \le b$. Density of Y is given by

$$f(y) = \begin{cases} \frac{g(y)}{G(b)-G(a)}, & \text{if } a \le y \le b \\ 0, & \text{otherwise} \end{cases}$$

So in this case we can choose $c = \frac{1}{G(b)-G(a)}$. And a sample X drawn from G will be selected if the generated uniform random number U satisfies

$$U \le \begin{cases} \frac{f(X)}{cg(X)} = 1, & \text{if } a \le X \le b \\ 0, & \text{otherwise} \end{cases}$$

which implies X is selected iff it lies in $[a, b]$.

Problem 16

We can use rejection method with reference exponential distribution with mean 1 and $c = 4$ or reference exponential distribution with mean 2 and $c = 3$. 1000 samples generated with the first method yield the following result.

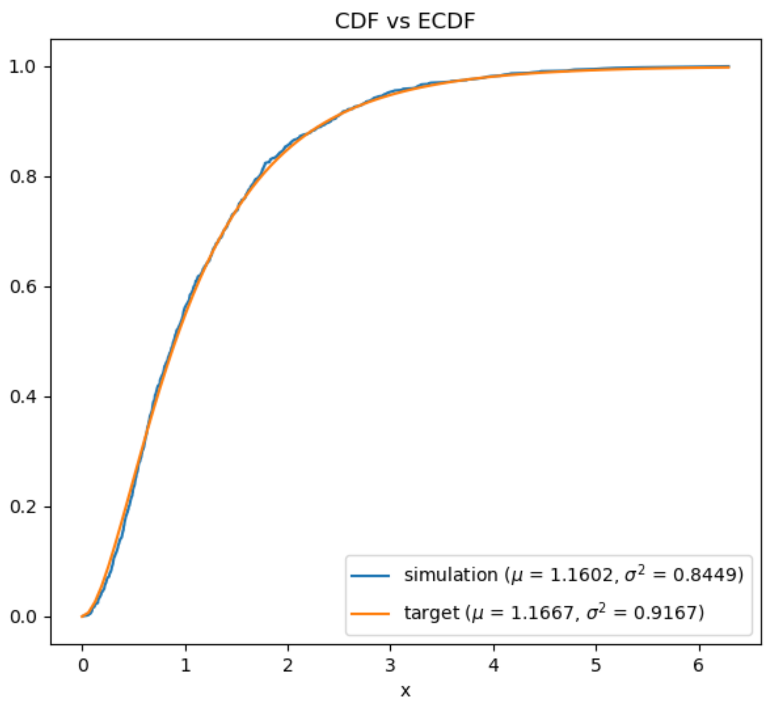

Figure 7: distribution comparison

1000 samples generated with the second method yield the following result.

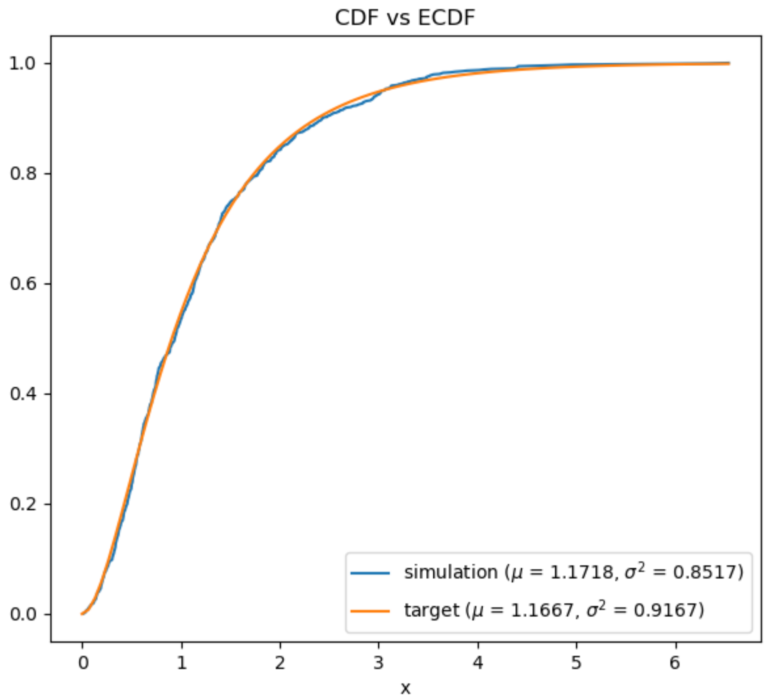

Figure 8: distribution comparison