Source localization on graphs via l1-recovery and spectral graph theory

Rodrigo Pena, Xavier Bresson, and Pierre Vandergheynst
École Polytechnique Fédérale de Lausanne (EPFL), LTS2. Switzerland

Goal

Locate the **source(s)** of an outbreak from a **single snapshot**.
Examples:

- Disease spread
- Diffusion processes (wind, heat, etc.)
- Rumors in social networks

Graph Signal Processing

Data lies on graph $G = (\mathcal{V}, \mathcal{E}, w)$

Normalized graph Laplacian:

$$\mathcal{L} = I - D^{-1/2} W D^{-1/2} = U \Lambda U^T$$

Graph filtering:

$$b = g(\mathcal{L})x = Ax$$

Example: **heat kernel**

$$g_\theta(\mathcal{L}) = exp(-\theta \mathcal{L})$$

Error Measure

Hop distance

$$e(x, x^\star) = \sum_{i \in \mathcal{A}} \frac{\sum_{j \in \mathcal{N}_i} |x^\star(j)|\, h(i,j)}{\sum_{j \in \mathcal{N}_i} |x^\star(j)|}$$

"Center of mass" around every true source

Problem

Assumption: observation is **graph-low-pass** filtering of the **sparse** source signal.

$$(x^\star, \theta^\star) = \underset{x,\theta}{\arg\min} \underbrace{\left\{ \gamma \, ||x||_1 + \frac{\alpha}{2} \, ||A_\theta x - b||_2^2 \right\}}_{E(x,\theta)}$$

Non-convex.

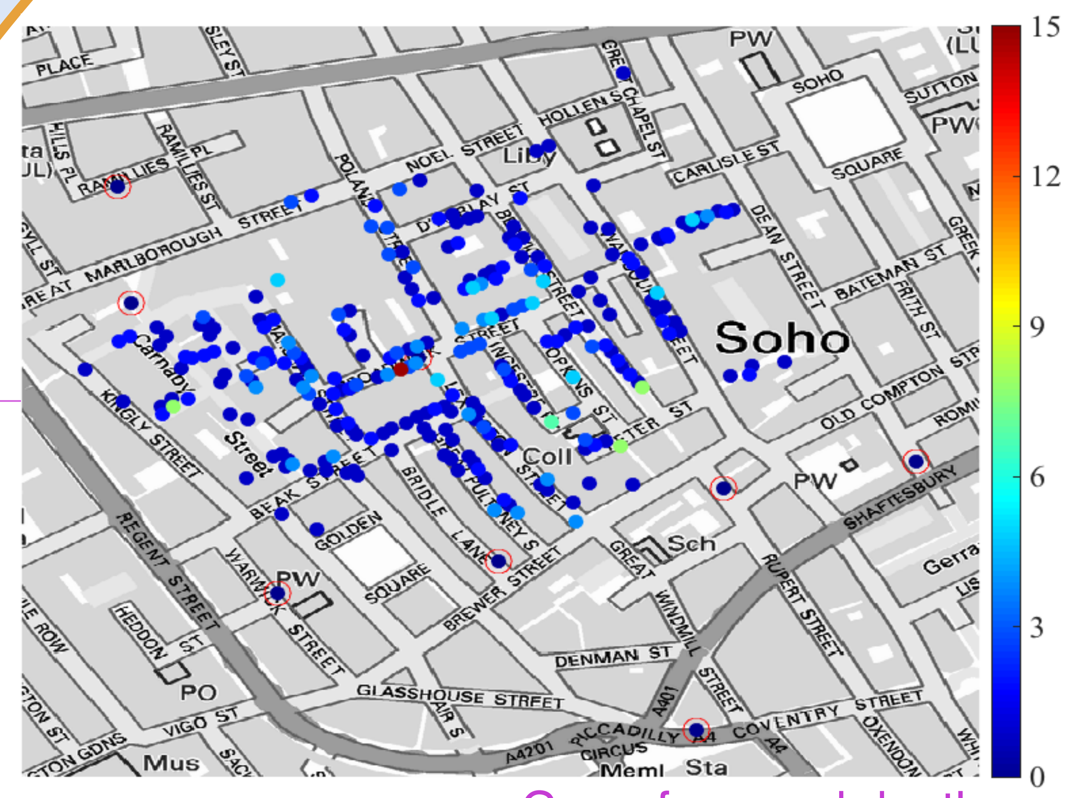
Alternating scheme:

$$x_{k+1} = \underset{x}{\arg\min}\, E(x, \theta_k) \text{ (FISTA)}$$

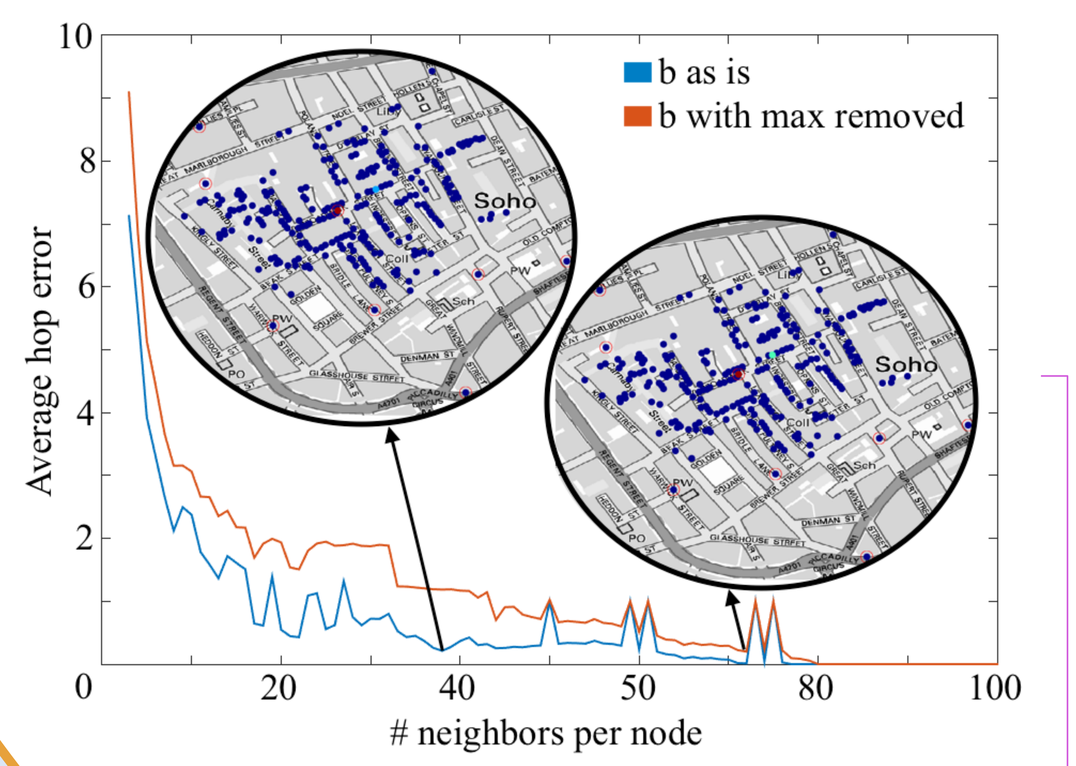
$$\theta_{k+1} = \underset{\theta}{\arg\min}\, E(x_{k+1}, \theta) \text{ (Newton)}$$

Real data 1

Cholera outbreak, Soho, London (1854)



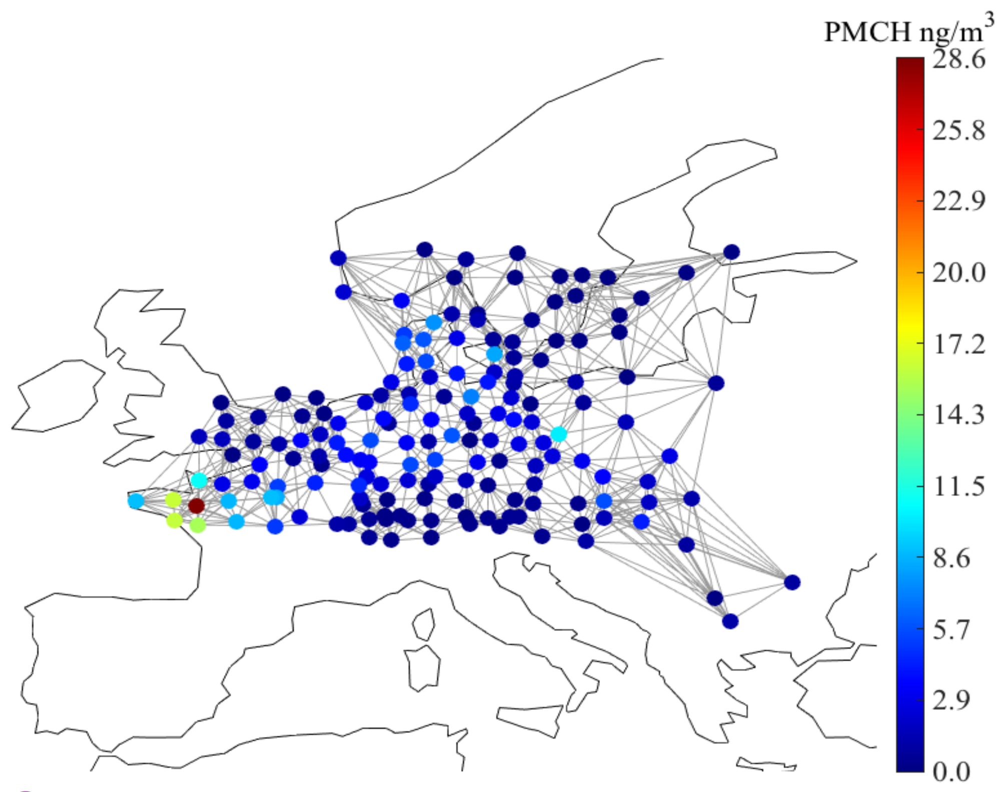
Georeferenced deaths



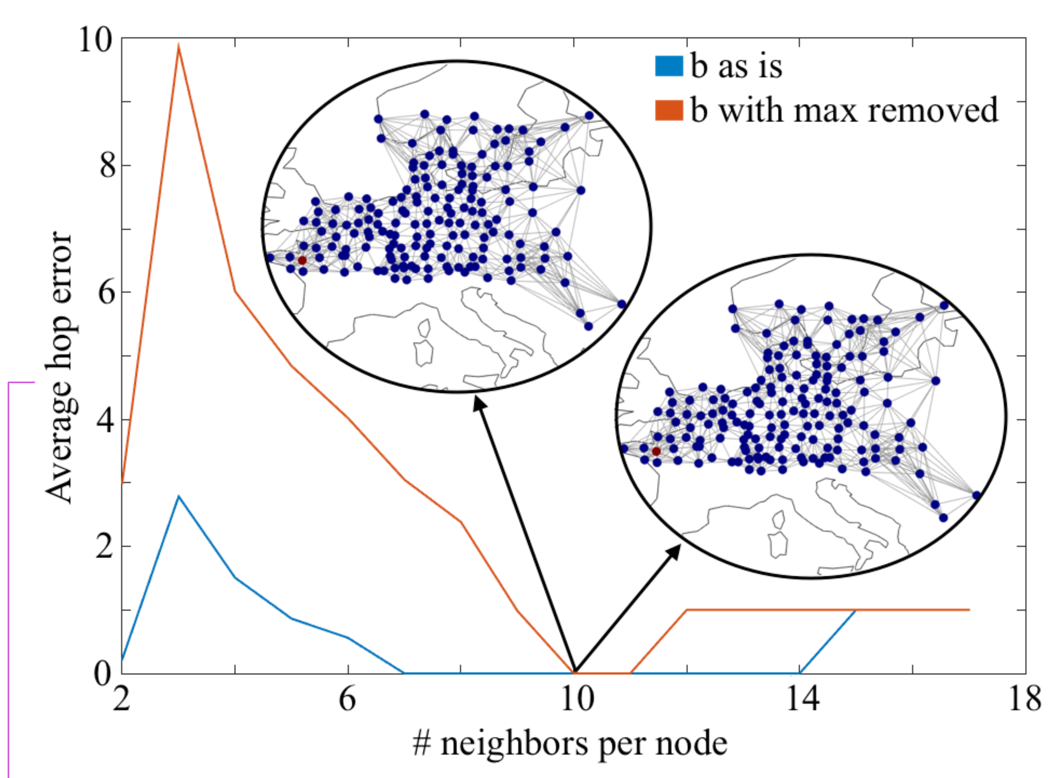
Recovery error as a function of graph density

Real data 2

European Tracer Experiment (ETEX, 1994)



Cummulative tracer concentration



Recovery error as a function of graph density

Graph construction

k-Nearest Neighbors

$$W_{ij} = exp\left(\frac{-d(i,j)^2}{\sigma^2}\right)$$

Euclidean distance

Control data

Simulations:

- ✓ Noise
- ✓ Source separation
- x Bad recovery if snapshot far away in time

Summary

- ✓ Linear model + sparsity: it works!
- ✓ General framework
- ✓ Robust to noise and source separation
- x Initialization plays big role
- x Graph construction as well
- x Improvement if diffusion model is application-specific?

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