ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Convex recovery under linear measurements

Rodrigo Pena
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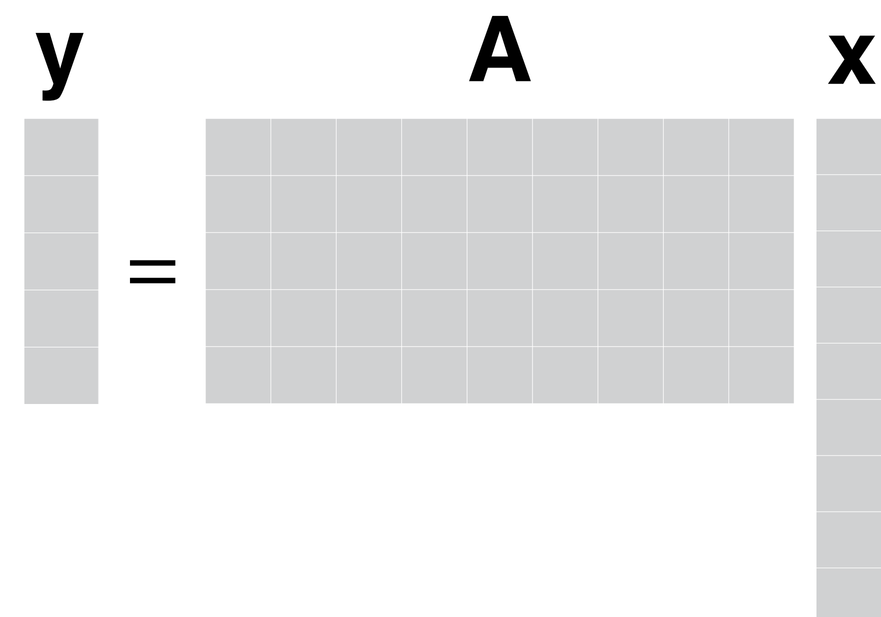
Model

Ill-posed!

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{y} \in \mathbb{R}^m, m < n$$



Signal to be recovered

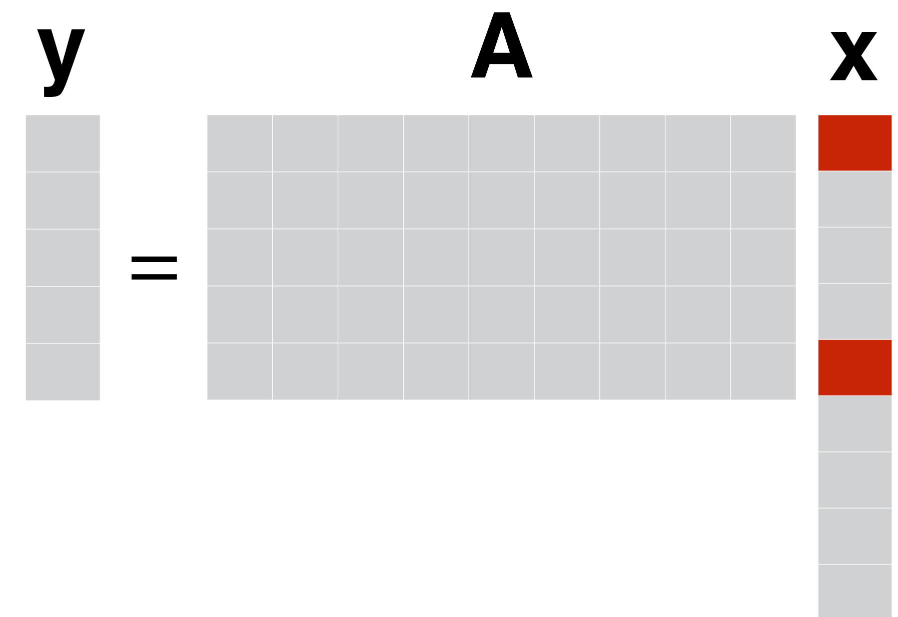
(Random) measurement matrix

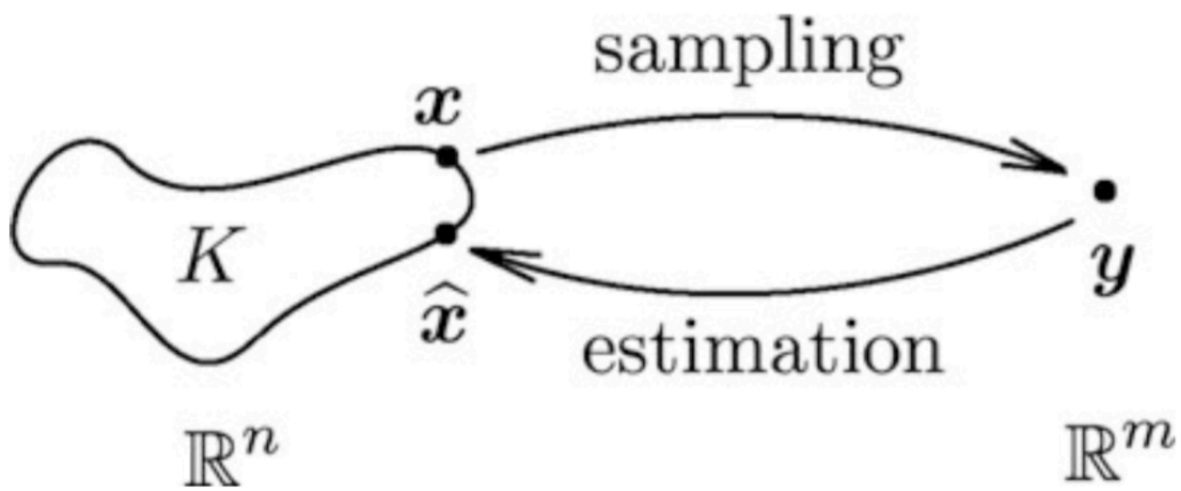
Linear measurements

Goal: recover the signal by inverting the measurement process

Assumptions

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{y} \in \mathbb{R}^m, m < n$$





[Vershynin, 2015]

$$\min_{\mathbf{z} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{z} - \mathbf{y}\|_2^2 \text{ s.t. } \mathbf{z} \in K$$

$$\boxed{\min_{\mathbf{z} \in \mathbb{R}^n} f(\mathbf{z}) \text{ s.t. } \mathbf{A}\mathbf{z} = \mathbf{y}}$$

Unique recovery

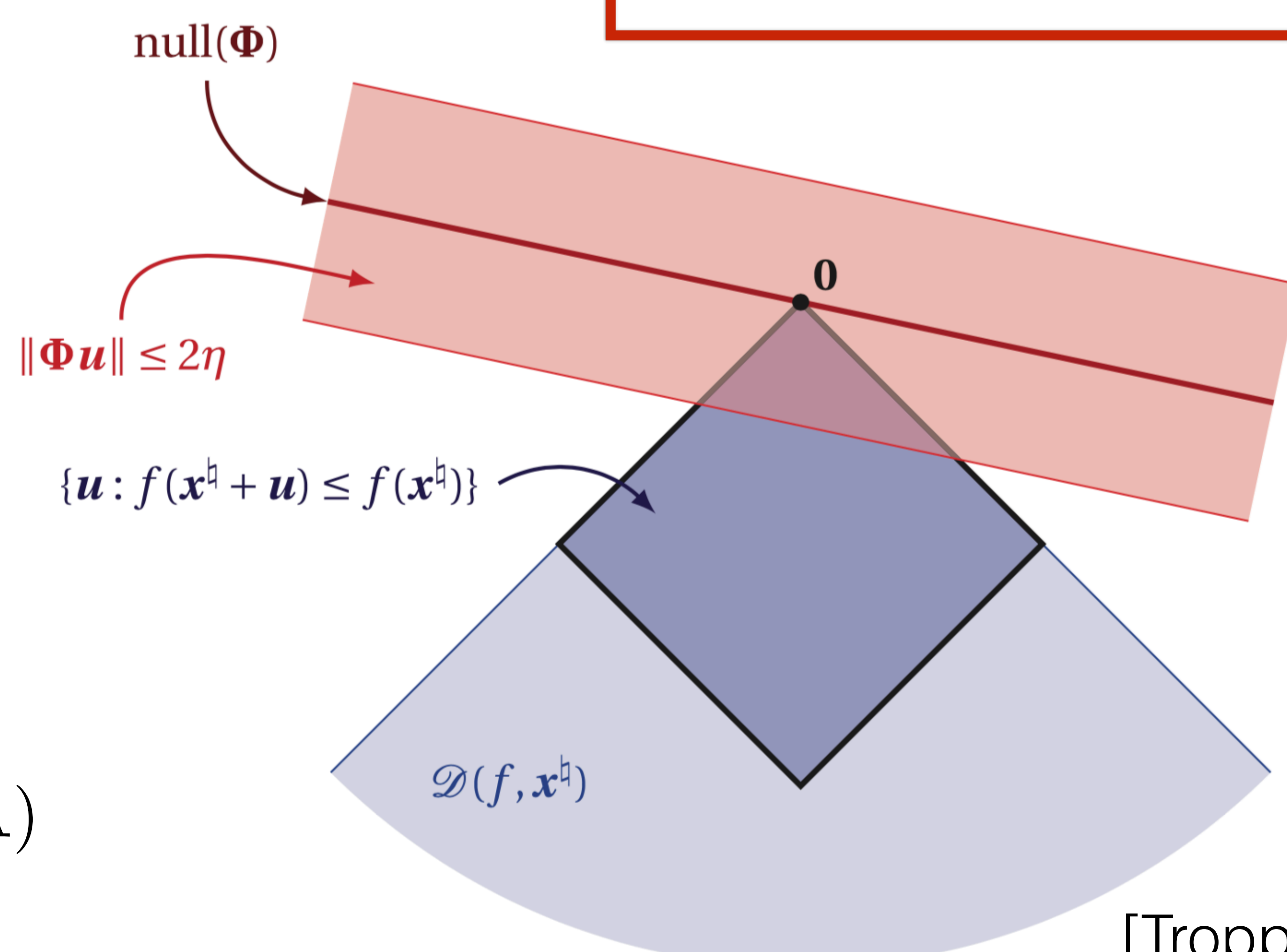
$$\mathbf{y} = \mathbf{A}\mathbf{x} \longrightarrow \min_{\mathbf{z}\in\mathbb{R}^n} f(\mathbf{z}) \text{ s.t. } \mathbf{A}\mathbf{z} = \mathbf{y} \longrightarrow \mathbf{z}^\star \stackrel{?}{=} \mathbf{x}$$

A.: Yes, but iff

$$\boxed{\mathcal{D}(f, \mathbf{x}) \cap \operatorname{null}(\mathbf{A}) = \{\mathbf{0}\}}$$

$$\mathcal{D}(f, \mathbf{x}) := \operatorname{cone}\{\mathbf{u} = \mathbf{z} - \mathbf{x} : f(\mathbf{z}) \leq f(\mathbf{x})\}$$

"descent cone"
vs.
"null space"



$$\begin{aligned} \mathbf{A}\mathbf{z} &= \mathbf{y} = \mathbf{A}\mathbf{x} \\ &\Longrightarrow \mathbf{A}(\mathbf{z} - \mathbf{x}) = \mathbf{0} \\ &\Longrightarrow (\mathbf{z} - \mathbf{x}) \in \operatorname{null}(\mathbf{A}) \end{aligned}$$

[Tropp, 2012]

Equivalent condition

$$\mathbf{y} = \mathbf{A}\mathbf{x} \longrightarrow \min_{\mathbf{z} \in \mathbb{R}^n} f(\mathbf{z}) \text{ s.t. } \mathbf{A}\mathbf{z} = \mathbf{y} \longrightarrow \mathbf{z}^\star \stackrel{?}{=} \mathbf{x}$$

$$\mathcal{D}(f, \mathbf{x}) \cap \mathrm{null}(\mathbf{A}) = \{\mathbf{0}\} \longrightarrow \min_{\mathbf{z} \in \mathcal{D}(f,\mathbf{x}) \cap \mathbb{S}^{n-1}} \|\mathbf{A}\mathbf{z}\|_2^2 > \mathbf{0}$$

Beauty of randomness: (**A** sub-Gaussian or "small-ball")

$$\|\mathbf{A}\mathbf{z}\|_2^2 \gtrsim \sqrt{m} - w(\mathcal{D}(f, \mathbf{x}) \cap \mathbb{S}^{n-1}) - t$$

with probability at least $1 - \exp(-t^2)$, $\forall \mathbf{z} \in \mathcal{D}(f, \mathbf{x}) \cap \mathbb{S}^{n-1}$

"Gaussian width"

[Tropp, 2012]

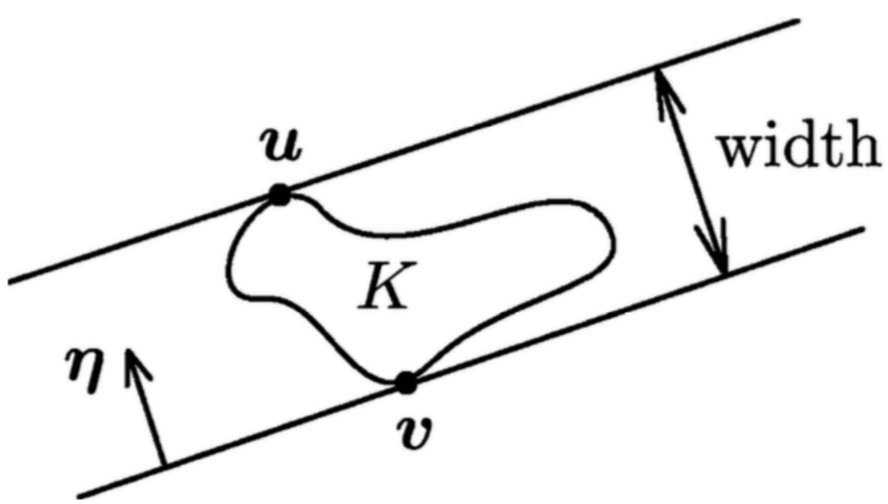
[Liaw et al., 2017]

Gaussian mean width

$$\mathbf{y} = \mathbf{A}\mathbf{x} \longrightarrow \min_{\mathbf{z}\in\mathbb{R}^n} f(\mathbf{z}) \text{ s.t. } \mathbf{A}\mathbf{z} = \mathbf{y} \longrightarrow \mathbf{z}^\star \stackrel{?}{=} \mathbf{x}$$

$$\|\mathbf{A}\mathbf{z}\|_2^2 \gtrsim \sqrt{m} - w(\mathcal{D}(f,\mathbf{x}) \cap \mathbb{S}^{n-1}) - t$$

$$w(K) = \mathbb{E}\sup_{\mathbf{u}\in K} \langle \mathbf{u}, \eta \rangle, \quad \eta \sim \mathcal{N}(0, I)$$



So what?

- Singular values of random matrices

$K = \mathbb{S}^{n-1}$ $\qquad$ $\sigma(\mathbf{A}) \in [\sqrt{m} - C\sqrt{n}, \sqrt{m} + C\sqrt{n}]$

- Johnson-Lindenstrauss lemma

$K = X - X, |X| < \infty$ $\qquad$ $\left| \frac{\|\mathbf{Ax}\|_2}{\sqrt{m}} - 1 \right| \leq \frac{C\sqrt{\log |X|}}{\sqrt{m}}$

- Intersection of a set K by a random subspace L

$L = \mathrm{null}(\mathbf{A})$ $\qquad$ $\mathrm{rad}(L \cap K) \leq \frac{Cw(L)}{\sqrt{m}}$

- Exact recovery in compressed sensing

$K = B_{\ell_1}$ $\qquad$ $m \geq Cs\log(n/s)$

[Liaw et al., 2017]

Summary

$$\mathbf{y} = \mathbf{A}\mathbf{x} \longrightarrow \min_{\mathbf{z}\in\mathbb{R}^n} f(\mathbf{z}) \text{ s.t. } \mathbf{A}\mathbf{z} = \mathbf{y} \longrightarrow \mathbf{z}^\star \stackrel{?}{=} \mathbf{x}$$

Sub-Gaussian? "Small-ball"?

Then,

$$\boxed{m \gtrsim \left(w(\mathcal{D}(f,\mathbf{x}) \cap \mathbb{S}^{n-1}) + t\right)^2}$$

Guarantees unique recovery with probability at least

$$1 - \exp(-t^2)$$

Atomic norms are great!
[Chandrasekaran et al., 2012]

Thank you

Q & A

References

[1] R. Vershynin, “Estimation in high dimensions: a geometric perspective,” in Sampling theory, a renaissance, Birkhauser/Springer, Cham, 2015, pp. 3–66.

[2] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, “The Convex Geometry of Linear Inverse Problems,” Found Comput Math, vol. 12, no. 6, pp. 805–849, Oct. 2012.

[3] J. A. Tropp, “Convex recovery of a structured signal from independent random linear measurements,” in Sampling theory, a renaissance, no. 2, Cham: Birkh\"auser/Springer, Cham, 2015, pp. 67–101.

[4] Y. Plan and R. Vershynin, “The generalized lasso with non-linear observations,” IEEE Transactions on Information Theory, vol. 62, no. 3, pp. 1528–1537, Mar. 2016.