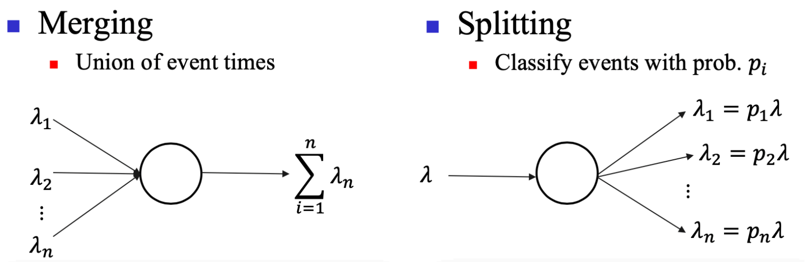
Homework 1 (due April 9 @ 7:30pm)

1. Let random variables X, Y, Z, and events $\{X = x\}$, $\{Y = y\}$, $\{Z = z\}$ of positive probability (so that conditional probabilities below are well-defined).
   Denote the Markov property $P[Z = z|Y = y, X = x] = P[Z = z|Y = y]$ as $X \to Y \to Z$.
   Show that it is equivalent with the following two properties:
   a) *Conditional independence*: the future & past are independent given the present (i.e., X, Z are independent given Y):
   $$P[X = x, Z = z|Y = y] = P[X = x|Y = y]P[Z = z|Y = y].$$
   b) *Reversed* Markov property ($Z \to Y \to X$), i.e.,
   $$P[X = x|Y = y, Z = z] = P[X = x|Y = y].$$

2. Prove that the following processes are Poisson with rates as illustrated in the figures
   a) The merged process (union of event times).
   b) The split process (events split according to the time-invariant p.m.f. p). Further show that the split processes are independent from one another.



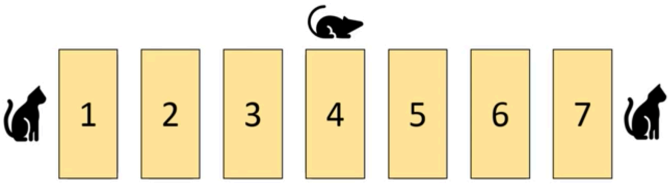
   In addition, show that the merged process has the same distribution as the sum process $\sum_{i=1}^{n} N_i(t)$ (where $\{N_i(t)\}$ are independent Poisson(λ_i) processes).
   Why is it possible that the sum process and merged process may have different sample paths (but this happens with zero probability)?

3. In a discrete MC with transition matrix $\mathbf{P}$, the *hitting time* (also known as *first passage time*) from state i to state j is defined as the number of transitions made by the process in going from state i to state j. Let $f_{ij}^{(n)}$ be the probability that the first passage time from state i to state j is equal to n, i.e., $f_{ij}^{(n)} := \Pr\{X_n = j, X_r \neq i\ (r = 1, 2, \dots, n-1)|X_0 = i\}$.
   a) Show that for all $0 < m < n$:
   $$f_{ij}^{(n)} = \sum_{k \neq j} p_{ik}^{(m)} f_{kj}^{(n-m)}$$
   b) Define the *mean hitting time* as $m_{ij} := \sum_{n=0}^{\infty} n\, f_{ij}^{(n)}$ (note that for $i = j$ you obtain the mean recurrence time as special case). Show that the mean hitting times satisfy a system of linear equations:
   $$m_{ij} = 1 + \textstyle\sum_{k \neq j} p_{ik} m_{kj}, \quad \forall\, i, j.$$
   c) Write the analogous formulas for a continuous MC with generator matrix $\mathbf{Q}$.

4. An M/G/1/K queueing system with $\lambda = \mu = 1$ has steady-state blocking probability (i.e., the probability that an arrival customer reneges due to finding a full system) $p_b = 0.1$, and average number of customers $L = 5$. Find $\lambda_{eff}, \rho_{eff}, W, W_q$, and p_0.
   Is the property of Poisson arrivals needed for your derivation?

5. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a discrete MC, with transition matrix **P**. Define the reverse-time process as $\{Y_t\} \equiv \{X_{-t}\}$.
   a) Show that $\{Y_t\}$ is a MC.
   b) Assume that $\{X_t\}$ is stationary with distribution $\boldsymbol{\pi}$ (i.e., $\boldsymbol{\pi}(t) = \boldsymbol{\pi}$, for all t).
      i. Compute the transition matrix of the reverse chain $\{Y_t\}$ as function of $\boldsymbol{P}, \boldsymbol{\pi}$.
      ii. Show that the MC is *reversible*, i.e., the transition matrix for $\{Y_t\}$ is the same as for $\{X_t\}$, *if and only if* the detailed balance equations are satisfied.
   c) Repeat parts a), b) for a continuous MC with generator matrix **Q**.

6. A mouse starts in the middle at door 4. At doors 1 and 7 are cats. Every day the mouse moves one door, left or right, at random, with probability 0.5. If the mouse reaches a door with a cat (door 1 or door 7), the mouse is caught. What is the expected number of days until a cat catches the mouse? What if there were only 5 doors?



Day 0

Disclaimer: no animals where harmed during this problem!

7. A student moves from home to office in the morning and from office to home in the evening. The student has 5 umbrellas and takes an umbrella only when it rains. If there is no umbrella at the present location, the student gets wet. Assume that during each of the student's transitions it rains independently with probability p.
   a) What is the stationary probability that the student carries an umbrella at least once (i.e., from home→office, office→ home, or both) during a given day?
   b) If the rain probability is $p = 3/4$, how many extra umbrellas does the student need to buy, so that the student stays dry with probability at least 95% (in steady-state)?
   c) Assume now that the student with 5 umbrellas changes his strategy, and whenever there are 5 umbrellas either at home or in the office, the student takes one regardless of whether it rains or not. What is the steady-state probability that the student gets wet? What if the student had only one umbrella, but follows the same strategy?