Solutions to Hw 1

Problem 1:

a) By definition of conditional expectation, and $X \to Y \to Z$:

$P[Z = z, X = x|Y = y] = \frac{P[Z=z,Y=y,X=x]}{P[Y=y]} = \frac{P[Z=z|Y=y,X=x]P[Y=y,X=x]}{P[Y=y]} = \frac{P[Z=z|Y=y]P[Y=y,X=x]}{P[Y=y]} =$
$P[Z = z|Y = y]P[X = x|Y = y]$

Note that the reverse holds true (i.e., a) implies the Markov property), since $P[Y = y] > 0$.

b) condition a) is symmetric in X, Z whence also equivalent to $Z \to Y \to X$.

Problem 2:

a) From the equivalent characterization of the Poisson process via Exponential interarrival times, the result follows since the min of independent Exp r.v.'s is Exp, with rate equal to the sum.

b) It is clear that each of the split processes satisfy the Poisson property with the given rates. To establish independence, we show this for a splitting in two processes (the general case is analogous, using the multinomial distribution in place of binomial):

Let $N \sim Poi(\lambda)$ and $X + Y = N$, where X, Y are produced by splitting N with probabilities $p, 1 - p$, respectively. We compute the joint distribution $p_{X,Y}(x, y)$. Note that given $N = x + y$, the joint distribution is Binomial(p), i.e., $p_{X,Y}(x, y|N = x + y) = \binom{x+y}{x} p^x (1 - p)^y$, therefore from Bayes' rule:

$$p_{X,Y}(X = x, Y = y) = p_{X,Y}(x, y|N = x + y) \cdot p(N = x + y) = \binom{x+y}{x} p^x (1-p)^y \frac{e^{-\lambda}\lambda^{x+y}}{(x+y)!}$$
$$= \left(\frac{(\lambda p)^x}{x!} e^{-\lambda p}\right) \cdot \left(\frac{(\lambda(1-p))^y}{y!} e^{-\lambda(1-p)}\right) = p_X(x) \cdot p_Y(y),$$

which shows that $X \sim Poi(\lambda p), Y \sim Poi(\lambda(1 - p))$ are independent.

-- For the last part: the sum of independent Poisson r.v.'s is Poisson with rate equal to the sum of rates. Sample paths may be different with the merged process, since two or more events from different processes could happen at the exact same time; this has zero probability.

Problem 3:

a) same as in C-K derivation:

$$\begin{aligned} f_{ij}^{(n)} &:= \Pr\{X_n = j, X_r \neq j (r = 1, 2, \dots, n - 1)|X_0 = i\} \\ &= \textstyle\sum_{k \neq j} \Pr\{X_n = j, X_r \neq j (r = 2, \dots, n - 1), X_1 = k|X_0 = i\} \\ &= \textstyle\sum_{k \neq j} \Pr\{X_n = j, X_r \neq j (r = 2, \dots, n - 1)|X_1 = k, X_0 = i\} \Pr\{X_1 = k|X_0 = i\} \\ &= \textstyle\sum_{k \neq j} \Pr\{X_n = j, X_r \neq j (r = 2, \dots, n - 1)|X_1 = k\} \Pr\{X_1 = k|X_0 = i\} \\ &= \textstyle\sum_{k \neq j} f_{kj}^{(n-1)} p_{ik} \end{aligned}$$

The first step is by the law of total probability; the second uses Bayes' rule; the third uses the Markov property; the fourth uses the time homogeneity of the MC (along with the definition of $p_{ik}, f_{kj}^{(n-1)}$).

b) From the definition of mean hitting times / mean recurrence times & conditioning; i.e., define the random variables $\{\tau_{ij}\}$ that represent the hitting time of state j starting at state i. It follows that:

$\tau_{ij} = \begin{cases} 1 + \tau_{kj}, & w.p. \ p_{ik} \ (k \neq j) \\ 1, & w.p. \ p_{ij} \end{cases}$ and the property follows as an application of the law of total expectation:

$m_{ij} := E[\tau_{ij}] = 1 + \sum_{k \neq j} p_{ik} E[\tau_{kj}] = 1 + \sum_{k \neq j} p_{ik} m_{kj}$

(alternatively, you may use part a) and sum over n).

c) same as above by replacing $1 \to \frac{1}{q_i}$ (expected time the chain stays at state i), and $p_{ij} \to \frac{q_{ij}}{q_i}$ (from the embedded MC).

Problem 4:

Application of Little's law for $\lambda \to \lambda_{eff} = 1 \cdot (1 - 0.1) = 0.9$ $\left(\rho_{eff} = \frac{\lambda_{eff}}{\mu} = 0.9\right)$

Little's law: $5 = 0.9W \to W = 50/9$

$W = W_q + \frac{1}{\mu} \to W_q = 41/9$

$p_0 = 1 - \rho_{eff}$=0.1

Yes, Poisson is needed because PASTA is used to define λ_{eff} (the blocking probability is defined at the steady-state).

Problem 5:

a) The reverse process is Markov following the same reasoning as in problem 1 above. Check explicitly; for any positive integer n and times $t_1 < t_2 < \cdots < t_n$, the Markov property is $P\left[X_{t_{n+1}} \middle| X_{t_n}, X_{t_{n-1}}, \ldots, X_{t_1}\right] = P\left[X_{t_{n+1}} \middle| X_{t_n}\right]$.

From the definition of conditional probability, the Markov property, and Bayes' rule:

$$P\left[X_{t_1} \middle| X_{t_2}, \ldots, X_{t_n}, X_{t_{n+1}}\right] = \frac{P[X_{t_1}, \ldots, X_{t_n}, X_{t_{n+1}}]}{P[X_{t_2}, \ldots, X_{t_n}, X_{t_{n+1}}]} = \frac{P[X_{t_{n+1}}|X_{t_n}] \cdot P[X_{t_n}|X_{t_{n-1}}] \cdots P[X_{t_2}|X_{t_1}] \cdot P[X_{t_1}]}{P[X_{t_{n+1}}|X_{t_n}] \cdot P[X_{t_n}|X_{t_{n-1}}] \cdots P[X_{t_3}|X_{t_2}] \cdot P[X_{t_2}]} = \frac{P[X_{t_2}|X_{t_1}] \cdot P[X_{t_1}]}{P[X_{t_2}]} = P\left[X_{t_1} \middle| X_{t_2}\right]$$

b) Let $\boldsymbol{P}^r$ be the transition matrix of the reverse chain. By Bayes' rule and the assumption that the MC is stationary:

$$p_{ij}^r = P[Y_{t+1} = j | Y_t = i] = P[X_{-t-1} = j | X_{-t} = i] = P[X_{-t} = i | X_{-t-1} = j] \frac{P[X_{-t-1} = j]}{P[X_{-t} = i]} = p_{ji} \frac{\pi_j}{\pi_i}$$

Detailed balance equations are equivalent to $p_{ij}^r = p_{ij}$.

c) For part a) showing that the reverse process is Markov is the same as above. Note that holding times are exponential r.v.'s (time-reversal does not affect holding times) so it is a continuous MC.

For second part, use the same derivation as in part b) to obtain:

$p_{ij}^r(t, t + \Delta t) = p_{ji}(-t - \Delta t, -t) \frac{\pi_j}{\pi_i}$, whence: $q_{ij}^r = q_{ji} \frac{\pi_j}{\pi_i}$. Again, it becomes evident that reversibility is equivalent to local balance equations.

Problem 6:

Use symmetry! Since probabilities are 50%, doors (3,5), (2,6) are equivalent (have the same mean hitting time) in terms of the mouse being caught. Doors 1 & 7 are absorbing states. From the mean hitting time recursions (problem 3b) we get the linear system:

$$\begin{aligned}
m_4 &= 1 + m_{(3,5)} \\
m_{(3,5)} &= 1 + 0.5m_4 + 0.5m_{(2,6)} \\
m_{(2,6)} &= 1 + 0.5m_{(3,5)}
\end{aligned}$$

$\Longrightarrow m_{(2,6)} = 5, \qquad m_{(3,5)} = 8, \qquad m_4 = 9$

The mouse has 9 days on average, meow!

For the second case, even simpler system:

$$\begin{aligned}
m_3 &= 1 + m_{(2,4)} \\
m_{(2,4)} &= 1 + 0.5m_3
\end{aligned}$$

$\Longrightarrow m_{(2,4)} = 3, \qquad m_3 = 4$

The mouse has 4 days on average, meow!

Note: another way (slightly more tedious) would be to define variables for all doors (noting that for first and last door these are 0; absorbing states) → 5x5 linear system in first case, 3x3 in the second.

Note: this generalizes problem 3b), since here the goal is to compute the time it takes to reach *either absorbing state*. The derivation is identical for this case.

Problem 7:
Define as state the # of umbrellas at the current location (home of office). It is clear that this number depends entirely on the # of umbrellas on the previous location and whether it rained or not during the last transition. Since the rain events are independent, we get a MC. We consider the general case where the student has K umbrellas in total and compute the stationary distribution. The transition probabilities are as follows:

$$p_{ij} = \begin{cases} 1, & \text{if } i = 0 \ \& \ j = K \\ 1-p, & \text{if } i = 1, \dots, K \ \& \ j = K - i \\ p, & \text{if } i = 1, \dots, K \ \& \ j = K - i + 1 \end{cases}$$

The Markov chain is finite, irreducible, and aperiodic. Therefore, it admits a unique stationary distribution. We try a solution to the local (detailed) balance equations. If it exists, it is the unique stationary distribution, and the MC is reversible.

- For $i = 1, \dots, K \ \& \ j = K - i$:
$$\pi_i p_{ij} = \pi_j p_{ji} \leftrightarrow \pi_i(1-p) = \pi_j(1-p) \leftrightarrow \pi_i = \pi_j$$

- For $i = 1, \dots, K \ \& \ j = K - i + 1$:
$$\pi_i p_{ij} = \pi_j p_{ji} \leftrightarrow \pi_i p = \pi_j p \leftrightarrow \pi_i = \pi_j$$

- For $i = 0 \ \& \ j = K$:
$$\pi_0 p_{0K} = \pi_K p_{K0} \leftrightarrow \pi_0 = (1-p)\pi_K$$

We need $\pi_i = \pi_K$ for $i = 1, \dots, K$ & $\pi_0 = (1-p)\pi_K$. Using $\sum_{j=0}^{K} \pi_i = 1$, we get $\pi_K = \frac{1}{K+1-p}$.
Therefore, the stationary distribution is:

$$\pi_i = \begin{cases} \frac{1-p}{K+1-p}, & i = 0 \\ \frac{1}{K+1-p}, & i = 1, \dots, K \end{cases}$$

For part a) we need to compute $\sum_{i=1}^{K} \pi_i p = \frac{Kp}{K+1-p}$

$$\text{For } K = 5 \rightarrow \quad \Pr(\text{carry umbrella}) = \frac{5p}{6-p}$$

Note that the above is the probability to carry an umbrella during any transition (home→office, or office →home)—*<u>only 1 point will be subtracted if you answered the probability to carry an umbrella in a given transition, instead</u>*—the probability to carry umbrella at least once in a given day is $\frac{Kp(2-p)}{K+1-p}$ (this can be justified in view of ergodicity, as the term $(2-p)p = 1-(1-p)^2$ is the probability that it rains at least once in a given day).

$$\text{For } K = 5 \rightarrow \quad \Pr(\text{carry umbrella at least once in a day}) = \frac{5p(2-p)}{6-p}$$

For part b) probability that student gets wet is equal to $\pi_0 p = \frac{p(1-p)}{K+1-p}$.

Need $\frac{p(1-p)}{K+1-p} \leq q; p = \frac{3}{4}, q = 0.05 \rightarrow \ K \geq \left\lceil \frac{14}{4} \right\rceil = 4$ umbrellas needed → student has enough umbrellas

For part c) the answer is 0: the student will always have an umbrella available (in fact, regardless of the number of umbrellas).
Note: this is at steady-state—the student can get wet at most once (if it rains the first morning and all the umbrellas are at the office).
For example, if there is only one umbrella, the student simply carries the umbrella w.p. 1, during all transitions.

Here is an alternative formulation. Define instead the state as the # of umbrellas in the morning at home. The transition probabilities become:

$$P(X_{n+1} = j | X_n = i) = \begin{cases} 1-p, & \text{if } i = 0 \ \& \ j = 0 \\ p, & \text{if } i = 0 \ \& \ j = 1 \\ p(1-p), & \text{if } i = 1, \ldots, K-1 \ \& \ j \in \{i-1, i+1\} \\ 1-2p(1-p), & \text{if } i = 1, \ldots, K-1 \ \& \ j = i \\ p(1-p), & \text{if } i = K \ \& \ j = K-1 \\ 1-p(1-p), & \text{if } i = K \ \& \ j = K \end{cases}$$

and it is easy to verify from the local balance equations that this MC has the exact same stationary distribution as the previous one (this should be expected by ergodicity).
Then for part a) need to compute $\pi_0 p + \sum_{i=1}^{K-1} \pi_i (1 - (1-p)^2) + \pi_K p$
Explanation: for first term, student can carry umbrella on the way back, if it rains (prob. p); for second term, umbrella is carried at least once if it rains at least once (i.e., the complement of no rain, which has probability $1 - (1-p)^2$); last term, umbrella can be carried if it rains on the way to office (but not otherwise), w.p. p.
Final answer is:

$$\frac{p(1-p) + (K-1)(1-(1-p)^2) + p}{K+1-p} = \frac{Kp(2-p)}{K+1-p}$$

For $K = 5 \rightarrow$ Pr (carry umbrella at least once in a day) $= \frac{5p(2-p)}{6-p}$

For part b) probability that student gets wet at least once in a given day is equal to:

$$\pi_0 p + \pi_K (1-p)p = \frac{2p(1-p)}{K+1-p}.$$

Explanation: first term reflects that student will be wet if it rains on the way out; for second term, $\pi_r(1-p)p$, student will be wet if it does not rain on the way to the office, and rains on the way back. This is the probability that the student gets wet at least once in a given day—*only 1 point will be subtracted if you compute this one instead*.

For this one to be smaller than 5% need $\frac{2p(1-p)}{r+1-p} \le q$, where $p = \frac{3}{4}, q = 0.05 \rightarrow$
$r \ge \left\lceil \frac{29}{4} \right\rceil = 8$ umbrellas are needed → student needs to buy 3 more umbrellas.

Note that we can still compute the probability that the student gets wet (in general), by ergodicity, as ½ of the above, i.e., $\frac{p(1-p)}{K+1-p}$ (since a day has two transitions).