1. 概率基础

$Var[X] = E[X^2] - E[X]^2$

$E[X] = \int_0^\infty [1 - F_X(x)]dx - \int_{-\infty}^0 F_X(x)dx$

$Cov[X,Y] = E[XY] - E[X]E[Y]$

Var[X+Y] = Var[X]+Var[Y]+2Cov[X,Y]

分布	期望	方差
Be(p)	p	pq, q=1-p
Bi(n,p)	np	npq
Geo(p)	q/p	q/p^2
$Poi(\lambda)$, inifi Bi(n,$p = \frac{\lambda}{n}$), $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ
$Exp(\lambda)$, x>0, $f = \lambda e^{-\lambda x}$, $F = 1 - e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang(k,λ), k $Exp(\lambda)$ sum. x>0, $f = \lambda e^{-\lambda x}\frac{(\lambda x)^{k-1}}{(k-1)!}$, $F = 1 - e^{-\lambda x}[1 + \frac{\lambda x}{1!} + .. + \frac{(\lambda x)^{k-1}}{(k-1)!}]$	k/λ	k/λ^2

Exp distri memoryless: P[X>t+h | X>t] = P[X>h]

Erlang pdf: $f(x;n,\mu) = \frac{\mu(\mu x)^{n-1}}{(n-1)!}e^{-\mu x}$

Distri of max, min

- Define Y=max$\{X_1, X_2, \ldots, X_n\}$, then

$$F_Y(y) = F_{X_1}(y)F_{X_2}(y)\cdots F_{X_n}(y)$$

- Define Y=min$\{X_1, X_2, \ldots, X_n\}$, then

$$F_Y(y) = 1-(1-F_{X_1}(y))(1-F_{X_2}(y))\cdots(1-F_{X_n}(y))$$

Distri of sum (conv): Z=X+Y $f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$

g.f. of $\{p_k\}$: $\Sigma p_k z^k = E[z^k]$

$\{a^i\}$: $\frac{1}{1-az}$

g(1)=1, $g^{(1)}(1) = E$, $g^{(i)}(1) = E[X..(X-i+1)] = F_i$

$M_1 = F_1, M_2 = F_2 + F_1, Var = M_2 - M_1^2$

2. Stochastic Processes

Poi proc: N(t)~Poi(λt)

PASTA: $P[N(t) = n | a(t, t+\Delta t) = 1] = P(N(t) = n)$

Little's Law: $L - L_q = \lambda/\mu$

G G 1: $\rho = \lambda/\mu$ = L-Lq = 1-p_0 = p_b

Birth-death proc

Global Balance Equations(BE)解： $p_n = p_0 \Pi_{i=1}^n \frac{\lambda_{i-1}}{\mu_i}$

Markov Chain(MC)

$p_{ij}(u,s) = Pr[X(u) = i, X(s) = j | X(u) = i]$

Chapman-Kolmogorov eqs (CKE): $p_{ij}^{(m)} = \Sigma_r p_{ir}^{(m-k)} p_{rj}^{(k)}$, $0 < k < m$

使用：

- Theorem: For an irreducible, aperiodic Markov chain

$$\pi_j = \lim_{n\to\infty} \pi_j^{(n)} = \frac{1}{m_{jj}}, \forall j$$

(m_{jj}: mean recurrence time of state j)

1. the MC is positive recurrent **if and only if** a stationary distribution exists (it is then also unique and equal to the steady-state distribution):
$$\boldsymbol{\pi} = \boldsymbol{\pi} \cdot \boldsymbol{P}, \quad \boldsymbol{\pi} \cdot \mathbf{1}^T = 1, \quad \boldsymbol{\pi} \geq 0$$
2. the MC is not positive recurrent (null recurrent or transient) **if and only if** $\forall j$: $\pi_j = 0$ $(m_{jj} = \infty) \rightarrow$ no stationary/steady-state distribution exists

How to use:

- Show irreducible and aperiodic (e.g., state 0 can be reached from any state, and, e.g., a self-transition exists)
- Compute the stationary distribution (implies also positive recurrence)

Poi proc

CKE: $q_i(t) = \Sigma_{j\neq i} q_{ij}(t) \equiv 1 - p_{ii}$

Transient analysis: $\frac{dp_j(t)}{dt} = -p_j(t)q_j + \Sigma_{r\neq j} p_r(t)q_{rj}$

mat form: $p'(t) = p(t)Q$, $\mathbf{Q} = \begin{pmatrix} -q_0 & q_{01} & q_{02} & \cdots \\ q_{10} & -q_1 & q_{12} & \cdots \\ q_{20} & q_{21} & -q_2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$

Stationary: $0 = \pi Q$, 即$p'(t) = 0$, p(t)变成π。不变的： $\pi\mathbf{1}^T = 1$.

$$\pi_j = \frac{1}{q_j \cdot m_{jj}},$$

3 简单QT模型

M M 1

$p_n = (1-\rho)$

$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$, $L_q = L - \rho = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$

Given queue is not empty, $L_q' = \frac{1}{1-\rho} = \frac{\mu}{\mu-\lambda} = L_q/\rho^2$

$W = \frac{L}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu-\lambda}$, $W_q = \frac{L_q}{\lambda} = \frac{\rho^2}{\lambda(1-\rho)} = \frac{\rho}{\mu-\lambda}$

$q_n = p_n$ (PASTA)

$W_q(t) = 1 - \rho e^{-\mu(1-\rho)t}$, $W(t) = 1 - e^{-(\mu-\lambda)t}$

M M c

$$p_n = \begin{cases} \dfrac{\lambda^n}{n!\mu^n} p_0 & (0 \le n < c) \\ \dfrac{\lambda^n}{c^{n-c} c! \mu^n} p_0 & (n \ge c) \end{cases} \qquad p_0 = \left(\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!(1-\rho)} \right)^{-1}$$

$$L_q = \sum_{n=c+1}^{\infty} (n-c) p_n = \left(\frac{r^c \rho}{c!(1-\rho)^2} \right) p_0 \quad W_q = \frac{L_q}{\lambda} = \left(\frac{r^c}{c!(c\mu)(1-\rho)^2} \right) p_0$$

$$W_q(0) = P[T_q = 0] = \sum_{n=0}^{c-1} p_n = p_0 \sum_{n=0}^{c-1} \frac{r^n}{n!} = 1 - \frac{r^c p_0}{c!(1-\rho)}, \quad W_q(t) = 1 - \frac{r^c p_0}{c!(1-\rho)} e^{-(c\mu-\lambda)t}$$

W(t): second part is conv of Exp($c\mu - \lambda$), Exp(μ)

$$P[T \le t | T_q > 0] = \frac{c(1-\rho)}{c(1-\rho)-1}(1-e^{-\mu t}) - \frac{1}{c(1-\rho)-1}(1-e^{-(c\mu-\lambda)t})$$

- Combining both cases:
  $W(t) = W_q(0)(1-e^{-\mu t}) + (1-W_q(0))P[T \le t | T_q > 0]$

$$\boxed{W(t) = \frac{c(1-\rho) - W_q(0)}{c(1-\rho)-1}(1-e^{-\mu t}) - \frac{1-W_q(0)}{c(1-\rho)-1}(1-e^{-(c\mu-\lambda)t})}$$

Choosing #servers: Square-Root Law (SRL): $c \approx r + \beta\sqrt{r}$, 保持QoS不变: $1 - W_q(0)$

M M c K

$$p_n = \begin{cases} \dfrac{\lambda^n}{n!\mu^n} p_0 & (0 \le n < c) \\ \dfrac{\lambda^n}{c^{n-c} c! \mu^n} p_0 & (c \le n \le K) \end{cases}, \quad p_0 = \begin{cases} \left(\sum_{n=0}^{c-1} \dfrac{r^n}{n!} + \dfrac{r^c}{c!} \dfrac{1-\rho^{K-c+1}}{1-\rho} \right)^{-1} & \rho \ne 1 \\ \left(\sum_{n=0}^{c-1} \dfrac{r^n}{n!} + \dfrac{r^c}{c!}(K-c+1) \right)^{-1} & \rho = 1 \end{cases},$$

$$L_q = \frac{p_0 r^c \rho}{c!(1-\rho)^2} \left[1 - \rho^{K-c+1} - (1-\rho)(K-c+1)\rho^{K-c} \right]$$

- For M/M/c/K, not every arrival can enter the system, an effective arrival rate λ_{eff} is needed $\lambda_{eff} = \lambda(1-p_K)$

$r_{eff} = \frac{\lambda_{eff}}{\mu} = r(1-p_K)$, $\rho_{eff} = \frac{\lambda_{eff}}{c\mu} = \rho(1-p_K)$

- Mean waiting time in system and mean system size:

$$\boxed{L = L_q + \frac{\lambda_{eff}}{\mu} = L_q + r(1-p_K)} \qquad \boxed{W = \frac{L}{\lambda_{eff}}}$$

- Mean waiting time in queue:

$$\boxed{W_q = W - \frac{1}{\mu} = \frac{L_q}{\lambda_{eff}}}$$

$q_n = \frac{p_n}{1-p_K}$, $W_q(t) = 1 - \sum_{n=c}^{K-1} q_n \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!}$

M M 1 K

$$p_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{1-\rho^{K+1}} & (\rho \ne 1) \\ \dfrac{1}{K+1} & (\rho = 1) \end{cases}, \quad L_q = \begin{cases} \dfrac{\rho}{1-\rho} - \dfrac{\rho(K\rho^K+1)}{1-\rho^{K+1}} & (\rho \ne 1) \\ \dfrac{K(K-1)}{2(K+1)} & (\rho = 1) \end{cases}$$

M G c c

$$p_n = \frac{\dfrac{(\lambda/\mu)^n}{n!}}{\sum_{i=0}^{c} \dfrac{(\lambda/\mu)^i}{i!}}, \quad (0 \le n \le c)$$

$$B(c,r) = p_c = \frac{r^c/c!}{\sum_{i=0}^{c} r^i/i!}$$

$$B(c,r) = \begin{cases} \dfrac{rB(c-1,r)}{c+rB(c-1,r)}, & c \ge 1 \\ 1, & c = 0 \end{cases}$$

M M ∞

$p_n = \frac{r^n}{n!} e^{-r}$, $L_q = 0$, $L = r$, $W = 1/\mu$, $W(t) = 1 - e^{-\mu t}$

Finite src

M srcs

$$p_n = \begin{cases} \binom{M}{n} r^n p_0 & (1 \le n < c) \\ \binom{M}{n} \dfrac{n!}{c^{n-c} c!} r^n p_0 & (c \le n \le M) \end{cases}, \quad r = \lambda / \mu$$,

$$p_0 = \left(1 + \sum_{n=1}^{c-1} \binom{M}{n} r^n + \sum_{n=c}^{M} \binom{M}{n} \frac{n!}{c^{n-c} c!} r^n \right)^{-1}$$

$$L = \sum_{n=1}^{M} n p_n$$

$$\lambda_{eff} = \sum_{n=0}^{M-1} (M-n)\lambda p_n = \lambda(M-L)$$

→ avg. breakdown rate is $(M-L)\lambda$

$$L_q = L - \frac{\lambda_{eff}}{\mu} = L - r(M-L)$$

$$W = \frac{L}{\lambda_{eff}} = \frac{L}{\lambda(M-L)}$$

$$W_q = \frac{L_q}{\lambda(M-L)}$$

$q_n = \frac{(M-n)p_n}{M-L}$

With spares

Y spares

- If $c \le Y$,

$$p_n = \begin{cases} \dfrac{M^n}{n!} r^n p_0 & (0 \le n < c) \\ \dfrac{M^n}{c^{n-c} c!} r^n p_0 & (c \le n < Y) \\ \dfrac{M^Y M!}{(M-n+Y)! c^{n-c} c!} r^n p_0 & (Y \le n \le Y+M) \end{cases}$$,

- If $c > Y$,

$$p_n = \begin{cases} \dfrac{M^n}{n!} r^n p_0 & (0 \le n < Y) \\ \dfrac{M^Y M!}{(M-n+Y)! n!} r^n p_0 & (Y+1 \le n < c) \\ \dfrac{M^Y M!}{(M-n+Y)! c^{n-c} c!} r^n p_0 & (c \le n \le Y+M) \end{cases}$$

$$\lambda_{eff} = \sum_{n=0}^{Y-1} M\lambda p_n + \sum_{n=Y}^{Y+M} (M-n+Y)\lambda p_n = \lambda\left(M - \sum_{n=Y}^{Y+M} (n-Y) p_n \right)$$

$$L = \sum_{i=0}^{M+Y} i p_i, \quad W = \lambda_{eff} L$$

$$L_q = L - \frac{\lambda_{eff}}{\mu}, \quad W_q = \lambda_{eff} L_q$$

When $Y \to \infty$, M/M/c with arrival rate $M\lambda$

State-depedent service

$$\mu_n = \begin{cases} \mu_{(1)}, & 1 \le n < k \\ \mu, & n \ge k \end{cases}$$,

- From birth-death equations:

$$p_n = \begin{cases} \rho_1^n p_0 & (0 \le n < k) \\ \rho_1^{k-1} \rho^{n-k+1} p_0 & (n \ge k) \end{cases}$$

Define $\rho_1 = \lambda/\mu_{(1)}$, $\rho = \lambda/\mu$; only need $\rho<1$ for steady-state

$$p_0 = \begin{cases} \left(\dfrac{1-\rho_1^k}{1-\rho_1} + \dfrac{\rho\rho_1^{k-1}}{1-\rho} \right)^{-1} & (\rho_1 \ne 1, \rho < 1) \\ \left(k + \dfrac{\rho}{1-\rho} \right)^{-1} & (\rho_1 = 1, \rho < 1) \end{cases}$$

$$L = p_0 \left(\frac{\rho_1 \left[1 + (k-1)\rho_1^k - k\rho_1^{k-1} \right]}{(1-\rho_1)^2} + \frac{\rho\rho_1^{k-1} \left[k - (k-1)\rho \right]}{(1-\rho)^2} \right)$$,

$$L_q = L - (1 - p_0)$$

ueue: $W_q = L_q / \lambda$

ystem: $W = L / \lambda = W_q + \dfrac{1-p_0}{\lambda}$

te is $\frac{1-p_0}{}$

4 Advanced QT Models

$M^{[X]}/M/1$

Arrival rate for batch of n: λ_n, prob. $c_n = \lambda_n/\lambda$, $\lambda = \Sigma\lambda_i$

$$P(z) = \frac{\mu p_0 (1-z)}{\mu(1-z) - \lambda z[1 - C(z)]} = \frac{p_0}{1 - rz\,\tilde{C}(z)}$$

$$\left(r := \lambda/\mu, \ \tilde{C}(z) := \frac{1-C(z)}{1-z} \right)$$

$p_0 = 1 - rE[X] := 1 - \rho$, 因为$\lambda_{eff} = \lambda E[X]$

$$L_q = L - \rho$$

$$L = P'(1) = \frac{\rho + rE[X^2]}{2(1-\rho)}, \quad W = \frac{L}{\lambda E[X]}$$

eue: $W_q = \dfrac{L_q}{\lambda E[X]}$

$M^{[K]}/M/1$

$$X = K, \text{ deterministic constant } \left(\rho = \frac{\lambda E[X]}{\mu} = \frac{\lambda K}{\mu} \right)$$

- $L = \frac{\rho + K\rho}{2(1-\rho)} = \frac{K+1}{2} \frac{\rho}{1-\rho}$
- $L_q = L - \rho = \frac{2\rho^2 + (K-1)\rho}{2(1-\rho)}$

$M/M^{[K]}/1$

Partial-batch mode

$$\boxed{p_n = (1-r_0)r_0^n, \quad (n \ge 0, r_0 \in (0,1))}$$

- r_0 is the only root in $(0,1)$ of: $\mu r^{K+1} - (\lambda+\mu)r + \lambda = 0$

$$\boxed{L = \frac{r_0}{1-r_0}, \qquad L_q = L - \frac{\lambda}{\mu}}$$

$$\boxed{W = \frac{r_0}{\lambda(1-r_0)}, \qquad W_q = W - \frac{1}{\mu}}$$

Full-batch mode

$$p_n = \begin{cases} \dfrac{p_0(1-r_0^{n+1})}{1-r_0} & (1 \le n \le K-1) \\ \dfrac{p_0 \lambda r_0^{n-K}}{\mu} & (n \ge K-1) \end{cases}$$ r_0 is the only root in $(0,1)$ of: $\mu r^{K+1} - (\lambda+\mu)r + \lambda = 0$,

$p_0 = \frac{1-r_0}{K}$

Erlangian Models

$Erlang(k, k\mu)$: E[T]=1/μ, Var[T]=1/$k\mu^2$

$M/E_k/1$

等价到 $M^{[k]}/M/1$ with service rate 1/$k\mu$

W_q = mean # of phases in system × phase service time

Use $M^{[k]}/M/1$: $L = \frac{k+1}{2}\frac{\rho}{1-\rho}$ $\left(\rho = \frac{\lambda k}{k\mu} = \frac{\lambda}{\mu}\right) \rightarrow$

$$\boxed{W_q = \frac{1+1/k}{2} \cdot \frac{\rho}{\mu(1-\rho)}}$$

- From Little's law:

$$\boxed{L_q = \lambda W_q = \frac{1+1/k}{2k} \cdot \frac{\rho^2}{1-\rho}}$$

$$\boxed{L = L_q + \rho, \qquad W = W_q + \frac{1}{\mu}}$$

$M/E_k/1$极限是M D 1

$E_k/M/1$

inter-arrival time ~ Erlang(k. $k\lambda$)，等价到M $M^{[k]}$ 1, full-batch mode with arrival rate $k\lambda$, mean service time还是1/μ

$\boxed{p_n = \rho(1-r_0^k)(r_0^k)^{n-1}}$, $r_0 \in (0,1)$: $\mu r^{k+1} - (k\lambda+\mu)r + k\lambda = 0$

$$\boxed{L = \sum_{n=0}^{\infty} np_n = \rho(1-r_0^k)\sum_{n=1}^{\infty} n\left(r_0^k\right)^{n-1} = \frac{\rho}{1-r_0^k}}$$

$$L = L_q + \rho, \ W = L/\lambda, \ W_q = W - 1/\mu$$

$q_n = (1-r_0^k)r_0^{kn}$, $W_q(t) = 1 - r_0^k e^{-\mu(1-r_0^k)t}$

$M/M/1/\infty/PR$

two priorities, prio-1 higher than prio-2; no preemption

$\lambda_1, \lambda_2, \mu$

Recall: M/M/1 (and related models)

- derivations of steady-state distribution and measures of effectiveness are independent of queue discipline
- waiting time distribution depends on queue discipline
  - FCFS was used (*stochastically dominates* any other scheme)

pn, measurements都和M M 1同

$\rho_1 = \lambda_1/\mu, \rho_2 = \lambda_2/\mu, \rho = \rho_1 + \rho_2$, $L = \frac{\rho}{1-\rho} \rightarrow L^{(2)} = \frac{\rho}{1-\rho} - L^{(1)}$

$$\boxed{\begin{aligned} L_q^{(1)} &= \frac{\rho\rho_1}{1-\rho_1} \\ L_q^{(2)} &= \frac{\rho\rho_2}{(1-\rho)(1-\rho_1)} \\ L_q &= \frac{\rho^2}{1-\rho} \end{aligned}}, \quad W_q = \frac{\lambda_1}{\lambda}W_q^{(1)} + \frac{\lambda_2}{\lambda}W_q^{(2)} = \frac{\rho^2}{\lambda-\mu}$$

unequal service rates: μ_1, μ_2

$$L_q^{(1)} = \frac{\lambda_1(\rho_1/\mu_1 + \rho_2/\mu_2)}{1-\rho_1},$$
$$L_q^{(2)} = \frac{\lambda_2(\rho_1/\mu_1 + \rho_2/\mu_2)}{(1-\rho_1)(1-\rho)},$$
$$L_q = L_q^{(1)} + L_q^{(2)}.$$

no prio, unequal srv rates

$$\boxed{\begin{aligned} L_q^{(1)} &= \frac{\lambda_1(\rho_1/\mu_1 + \rho_2/\mu_2)}{1-\rho}, \\ L_q^{(2)} &= \frac{\lambda_2(\rho_1/\mu_1 + \rho_2/\mu_2)}{1-\rho}, \\ L_q &= \frac{\lambda(\rho_1/\mu_1 + \rho_2/\mu_2)}{1-\rho}. \end{aligned}}$$

Higher L_q than an equivalent M/M/1 queue -- arrival rate $\lambda = \lambda_1 + \lambda_2$, service rate $\mu = \left(\frac{\lambda_1}{\lambda}\mu_1 + \frac{\lambda_2}{\lambda}\mu_2\right)^{-1}$

SPT rule: serv rates不同时，优先服务serv rate大的，使L_q小（小于no priority）。

With preemption：e.g. $\lambda_1, \lambda_2; \mu_1, \mu_2$

$$L^{(1)} = \frac{\rho_1}{1-\rho_1},$$
$$L^{(2)} = \frac{\rho_2 - \rho_1\rho_2 + \rho_1\rho_2(\mu_2/\mu_1)}{(1-\rho_1)(1-\rho_1-\rho_2)}$$

Multiple priority classes: r classes of prio

$\rho_k = \frac{\lambda_k}{\mu_k}, \sigma_k = \sum_{i=1}^{k}\rho_i, \sigma_r \equiv \rho,$ $\boxed{W_q^{(i)} = \frac{\sum_{k=1}^{r}\rho_k/\mu_k}{(1-\sigma_{i-1})(1-\sigma_i)}}$,

$L_q = \sum_{i=1}^{r} L_q^{(i)} = \sum_{i=1}^{r} \frac{\lambda_i \sum_{k=1}^{r}\rho_k/\mu_k}{(1-\sigma_{i-1})(1-\sigma_i)}$

General serv distri

$$W_q^{(i)} = \frac{\lambda \mathrm{E}[S^2]/2}{(1-\sigma_{i-1})(1-\sigma_i)}$$

$M/M/c/\infty/PR$

r个λ_k; 但都是μ

$$W_q^{(i)} = \frac{E[S_0]}{(1-\sigma_{i-1})(1-\sigma_i)} = \frac{[c!(1-\rho)(c\mu)\sum_{n=0}^{c-1}(c\rho)^{(n-c)} / n! + c\mu]^{-1}}{(1-\sigma_{i-1})(1-\sigma_i)}$$
$$W_q = \sum_{i=1}^{r}\frac{\lambda_i}{\lambda}W_q^{(i)}$$

Retrial queue

M M 1： m,n: m in serv, n in orbit; γ: retrial rate

$$W = \frac{1}{\mu-\lambda}\cdot\frac{\lambda+\gamma}{\gamma}$$
$$L = \lambda W = \frac{\rho}{1-\rho}\cdot\frac{\lambda+\gamma}{\gamma}$$

product of $M/M/1$ metrics and factor $(\lambda+\gamma)/\gamma$

- as $\gamma\to\infty$, $M/M/1/\infty/RSS$ (no orbit, random service selection)

5 Networks

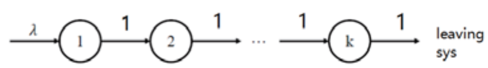
each node arrival rate from outside: γ_i ;

$r_{ij} := \Pr\{i \to j\}, r_{i0} := \Pr\{\text{leave at } i\}$

Close: $\gamma_i, r_{i0} = 0$

Series

special case of open Jackson network，中间不离开sys



T: inter-departure time; C(t) = Pr[T<= t] = 1-$e^{-\lambda t}$

Open Jackson networks

Each node has one server，k nodes

Traffic eqs: $\lambda_i = \gamma_i + \sum_{j=1}^{k}\lambda_j r_{ji}$, mat form $\boxed{\boldsymbol{\lambda} = \boldsymbol{\gamma} + \boldsymbol{\lambda R}}$, $\lambda = \gamma(I-R)^{-1}$

$\rho_i \equiv \lambda_i/\mu_i$; $p_{\bar{n}} = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}\cdots(1-\rho_k)\rho_k^{n_k}$

$L_i = \rho_i/(1-\rho_i)$, $W_i = L_i/\lambda_i$, $W = \frac{\sum_i L_i}{\sum_i \gamma_i}$

Close Jackson networks

$p_{n_1,n_2,\ldots,n_k} = \frac{1}{G(N)}\prod_{i=1}^{k}\frac{\cancel{\rho_i}^{n_i}}{a_i(n_i)}$, pi $\quad G(N) = \sum_{n_1+\cdots+n_k=N}\prod_{i=1}^{k}\frac{\cancel{\rho_i}^{n_i}}{a_i(n_i)}$, pi

$$a_i(n_i) = \begin{cases} n_i! & (n_i < c_i) \\ c_i^{n_i-c_i}c_i! & (n_i \ge c_i) \end{cases}$$

求ρ例:

$$\boldsymbol{R} = \begin{pmatrix} 0 & r_{12} & 1-r_{12} \\ 1-r_{23} & 0 & r_{23} \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda\rho_1 = \mu_2(1-r_{23})\rho_2 + \mu_3\rho_3 \\ \mu_2\rho_2 = \lambda r_{12}\rho_1 \\ \mu_3\rho_3 = \lambda(1-r_{12})\rho_1 + \mu_2 r_{23}\rho_2 \end{cases}$$

to remove redundancy, set $\rho_i = 1$

Buzen's alg for G(N): $G(N) = g_k(N)$, k nodes

求解: $g_m(0)=1$; $f_i(n_i) = \rho_i^{n_i}/a_i(n_i)$, $g_1(n) = f_1(n)$; $g_m(n) = \sum_{i=0}^{n} f_m(i)g_{m-1}(n-i)$

- Illustrate via the preceding example:

$$f_1(0)=1, f_1(1)=\frac{2}{3}, f_1(2)=\frac{2}{9}, \quad f_2(0)=f_2(1)=f_2(2)=1,$$
$$f_3(0)=1, f_3(1)=\frac{2}{9}, f_3(2)=\frac{4}{81}$$

$$G(2) = g_3(2) = f_3(0)g_2(2) + f_3(1)g_2(1) + f_3(2)g_2(0) = 2.3806$$
$$g_2(2) = f_2(0)g_1(2) + f_2(1)g_1(1) + f_2(2)g_2(0) = 17/9$$
$$g_2(1) = f_2(0)g_1(1) + f_2(1)g_1(0) = 5/3$$
$$g_1(1) = f_1(1) = 2/3, \quad g_1(2) = f_1(2) = 2/9$$
$$g_1(0) = g_2(0) = 1$$

Mean-Value Analysis (MVA)

to compute $L_i(N), W_i(N)$ in a k-node, single server closed network with R：下面step1就是 $v = vR$

1. Solve $v_i = \sum_{j=1}^{k} v_j r_{ji}$, setting $v_l = 1$ (l arbitrary)
2. Initialize $L_i(0) = 0 \quad (i = 1, 2, \dots, k)$
3. For $n = 1$ to N, calculate:
   a) $W_i(n) = \frac{1+L_i(n-1)}{\mu_i} \quad (i = 1,2,\dots,k)$
   b) $\lambda_l(n) = \frac{n}{\sum_{i=1}^{k} v_i W_i(n)}$ (assume $v_l = 1$)
   c) $\lambda_i(n) = \lambda_l(n) v_i \quad (i = 1,2,\dots,k, i \neq l)$
   d) $L_i(n) = \lambda_i(n) W_i(n) \quad (i = 1,2,\dots,k)$ -> Wi(n+1)

6 General pattern

M/G/1

svc time CDF B(t), and mean $\mu = 1/E[S]$

Derive on arrivals: PK formula: $\boxed{W_q = \frac{1 + C_B^2}{2} \cdot \frac{\rho}{1-\rho} \cdot E[S]}$, $C_B^2 = Var[S]/E[S]^2$

Derivation on departures: $\boxed{L^{(D)} = \rho + \frac{\rho^2 + \lambda^2 \sigma_B^2}{2(1-\rho)}}$, $L^{(D)} = L$

$k_i = Pr$[A=i] = $= \frac{\lambda^i}{i!} \cdot \int_0^\infty e^{-\lambda t} t^i dB(t)]$, $P = \{p_{ij}\} = \begin{pmatrix} k_0 & k_1 & k_2 & k_3 & \cdots \\ k_0 & k_1 & k_2 & k_3 & \cdots \\ 0 & k_0 & k_1 & k_2 & \cdots \\ 0 & 0 & k_0 & k_1 & \cdots \\ 0 & 0 & 0 & k_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

g.f. $\Pi(z) = \frac{(1-\rho)(1-z)K(z)}{K(z) - z}$

使用：k_i -> K(z) -> $\Pi(z)$ -> π

M/G/1/K

Derive on departures: K capacity破坏了arrival，trans mat变为

$$P = \begin{array}{c} \\ 0 \\ \\ \\ \\ \\ \text{k-1} \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & & & & \text{k-1} \end{array} \\ \begin{pmatrix} k_0 & k_1 & k_2 & \cdots & 1 - \sum_{n=0}^{K-2} k_n \\ k_0 & k_1 & k_2 & \cdots & 1 - \sum_{n=0}^{K-2} k_n \\ 0 & k_0 & k_1 & \cdots & 1 - \sum_{n=0}^{K-3} k_n \\ 0 & 0 & k_0 & \cdots & 1 - \sum_{n=0}^{K-4} k_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 - k_0 \end{pmatrix} \end{array}, \quad L^{(D)} = \sum_{i=0}^{K-1} i\pi_i$$

Derive on arrivals:

q_n'：n可达K，为K时new arrival不进入sys。$q_n' = p_n = \frac{\pi_n}{\pi_0 + \rho}$

推导过程产生的公式：$q_K' = \frac{\rho - 1 + p_0}{\rho}$ ，$p_0 = \frac{\pi_0}{\pi_0 + \rho}$

$M/G/\infty$

q_t = an arrival t时仍在sys = $\frac{1}{t}\int_0^t [1 - B(x)]dx$,

$\Pr\{N(t) = n \mid X(t) = i\} = \binom{i}{n} q_t^n (1-q_t)^{i-n}$

Pr[N(t)=n] $\frac{(\lambda q_t t)^n e^{-\lambda q_t t}}{n!}$, $\lim_{t\to\infty}(\lambda q_t t) = \lambda \int_0^\infty [1 - B(x)]dx = \frac{\lambda}{\mu} \to$ steady-state is $Poisson(\lambda/\mu)$ [same as $M/M/\infty$]

$\Pr\{Y(t) = n\} = \frac{[\lambda(1-q_t)t]^n e^{-\lambda(1-q_t)t}}{n!}$

[nonhomogeneous Poisson process with rate $\lambda(1-q_t)$]

- Since $\lim_{t\to\infty} q_t = 0 \to$ it tends to a Poisson process with rate λ–same as the arrival process [as for $M/M/c$]

G/M/1

Derive on arrivals: b_k = Pr[相邻arrival间服务掉k客] = $\int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!} dA(t)$

$P = \{p_{ij}\} = \begin{pmatrix} 1 - b_0 & b_0 & 0 & 0 & \cdots \\ 1 - \sum_{k=0}^{1} b_k & b_1 & b_0 & 0 & \cdots \\ 1 - \sum_{k=0}^{2} b_k & b_2 & b_1 & b_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, Stationary: q=qP,即

$z = \sum_{n=0}^{\infty} b_n z^n =: \beta(z)$, 有unique root r_0 in (0,1)，$q_n = (1 - r_0) r_0^n$