**University of Science and Technology of China**
**School of Computer Science and Technology**

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**CS05136** **Final Examination** **June 11, 2020**
**Queueing Theory** **7:00pm - 10:15pm**

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**Time allowed: 3 hours 15 minutes**

Problems carry the weights indicated. Show clearly the steps to solve the problems.
The total number of points for the exam is 100 + 10 extra credit.

A cheat sheet of two A4 pages (front and back) is allowed. Scientific calculators are allowed.
No books, notes, or electronic devices are allowed.

**Directions for online exam:**
Use Classin in your laptop for front camera (your face, hands, and exam papers need to be visible at all times).
Use WeChat in your phone for side/back camera (the phone has to be placed so that your computer screen and hands are visible at all times).

***Exam papers have to be submitted in Classin before 10:15pm***

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**Name:** ______________________ **Student Number:** ____________

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| Problem 1 | 15 |
| Problem 2 | 15 |
| Problem 3 | 15 |
| Problem 4 | 10 |
| Problem 5 | 10 |
| Problem 6 | 15 |
| Problem 7 | 10 |
| Problem 8 | 20 |
| **Total** | **100** |

1. [15 points] For an irreducible, aperiodic, finite-state discrete Markov chain with $n$ states and transition matrix $\boldsymbol{P}$, you are given that the columns of $\boldsymbol{P}$ sum to one (i.e., $\boldsymbol{P}$ is *column-stochastic*).
    a) Calculate the steady-state distribution.
    b) Repeat for an irreducible, finite-state continuous Markov chain with $n$ states and generator matrix $\boldsymbol{Q}$, with columns that sum to zero.

2. [15 points] Explain the *Shortest Processing Time* (SPT) rule.
    a) For a multi-class $M/M/1/\infty/PR$, with unequal arrival and service rates, show that the minimum value for the waiting time in the queue
$$W_q = \sum_{i=1}^{r} \frac{\lambda_i}{\lambda} \cdot \frac{\sum_{k=1}^{r} \rho_k/\mu_k}{(1-\sigma_{i-1})(1-\sigma_i)}$$
is achieved when $\rho_1 \le \rho_2 \le \cdots \le \rho_r$ [recall: $\sigma_k \coloneqq \sum_{i=1}^{k} \rho_i$].

3. [15 points] In a Jackson network (open or closed), how to handle the case that the service rate in one node, say node $i$, goes to infinity?
    a) Illustrate in a 2-node series queue with arrival rate $\lambda$, service rates $\mu_1, \mu_2$, single server at both nodes, and $\mu_1 \to \infty$: write the stability condition along with the mean number of customers in the system.

4. [10 points] Consider an $M/M/c$ system, with $r = \frac{\lambda}{\mu} = 10$. The system designer requires that the probability of waiting in the queue is less than 0.05. How many servers $c$ should the designer purchase? Show your calculations or pseudocode.

5. [10 points] For an $M/M/c/c$ system with $r = \frac{\lambda}{\mu} = 0.9$, compute the minimum number of servers $c$ so that the blocking probability is less than 3%. Show your calculations.

6. [15 points] Compute the steady-state distribution for an $M^{[2]}/M/1$ queue with parameters $\lambda = 0.5, \mu = 3$.

7. [10 points] Prove the Pollaczek-Khintchine formula for an $M/G/1$ system: $L_q = \frac{\rho^2 + \lambda^2 Var^2[S]}{2(1-\rho)}$; $\lambda$ is the arrival rate, $S$ is the r.v. of service time, and $\rho \coloneqq \lambda E[S]$.
    a) Which service distribution minimizes $L_q$? Justify your answer.

8. [20 points] A *slotted* queueing system is modeled as a discrete-time stochastic process. In each time slot, an arrival occurs with probability $\lambda$ and a service completion (when the system is not empty) with probability $\mu$. Both processes are i.i.d. over time, and uncorrelated. Multiple arrivals/service completions in a given slot occur with zero probability, and there is no bound on the queue size. Finally, when the system is empty we assume that an arrival followed by service of the arriving customer in one slot is possible (this happens with probability $\lambda\mu$).
    a) Show that the system can be described by a Markov chain that is irreducible and aperiodic.
    b) Derive the necessary and sufficient condition for positive recurrence and compute the stationary distribution for the case that this condition is met.
    c) Compute the measures of effectiveness $L, L_q, W, W_q$.