Homework 3 (due April 30 @ 7:30pm)

1. Show that the *characteristic polynomial* $P(z) = \mu z^{K+1} - (\lambda + \mu)z + \lambda$ has a unique root r_0 in the interval (0,1), when $\frac{\lambda}{\mu K} < 1$.
   [Hint: Use Rouché's theorem (page 100 in the reference book) with $f(z) = -\left(\frac{\lambda}{\mu} + 1\right)z + \frac{\lambda}{\mu}$, $g(z) = z^{K+1}$]

2. For the partial-batch $M/M^{[K]}/1$ model, show that $L_q = r_0^K L = L - \lambda/\mu$.

3. Show that $M/D/1$ can be obtained as the limit of $M/E_k/1$ when $k \to \infty$.

4. Show that a single-server queue with two classes of customers (arrival rates λ_1, λ_2 and service rates μ_1, μ_2) and *no priorities* has higher L_q than an $M/M/1$ queue with arrival rate $\lambda = \lambda_1 + \lambda_2$ and service rate $\mu = \left(\frac{\lambda_1}{\lambda} \cdot \frac{1}{\mu_1} + \frac{\lambda_2}{\lambda} \cdot \frac{1}{\mu_2}\right)^{-1}$.

5. For a two-class $M/M/1/\infty/PR$ queue, show that the imposition of priorities decreases the mean number of priority-l customers in the queue and increases the mean number of priority-2 customers in the queue.

6. Use induction on i to show that the solution to the linear system of equations for a multi-class $M/M/1/\infty/PR$ queue:
   $$W_q^{(i)} = W_q^{(i)} \sum_{k=1}^{i-1} \rho_k + \sum_{k=1}^{i} \rho_k W_q^{(k)} + E[S_0] \qquad (i \in \{1,2,\dots,r\})$$
   is given by ($\sigma_i \equiv \sum_{k=1}^{i} \rho_k$, $i = 1,2,\dots,r$)
   $$W_q^{(i)} = \frac{E[S_0]}{(1 - \sigma_{i-1})(1 - \sigma_i)}$$

7. For a multi-class $M/M/1/\infty/PR$, show that the mean waiting time $W_q = \sum_{i=1}^{r} \frac{\lambda_i/\lambda \sum_{k=1}^{r} \rho_k/\mu_k}{(1-\sigma_{i-1})(1-\sigma_i)}$ ($\sigma_i \equiv \sum_{k=1}^{i} \rho_k$, $i = 1,2,\dots,r$) is the same as for an $M/M/1$ queue *if and only if* $\mu_1 = \mu_2 = \cdots = \mu_r \equiv \mu$.