Li, Li, Huo - Optimal In-Place Suffix Sorting

Patrick Indri

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DSSC - Algorithmic Design Exam

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Problem Setting

Suffix Arrays: a space-saving alternative to suffix trees.

Definition

Given a string $T = T[0, \ldots, n-1]$ where each $T[i] \in \Sigma$ integer alphabet, the *suffix array* SA contains the indices of all suffixes of T which are sorted in lexicographical order.

Problem Setting

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Example

$T =$ "1120", the suffixes are $\{1120, 220, 20, 0\}$. Since $suf(3) < suf(0) < suf(2) < suf(1)$, then $SA = [3021]$.

Problem Setting

Problem

Construct SA for a given string T.

Main Theorem

There is an in-place linear time algorithm for suffix sorting over integer alphabets, even if the input string T is read only and the size of the alphabet $|\Sigma|$ is $O(n)$.

Naive Solution

- Get all the suffixes and sort them using *Quicksort*, while retaining their original indices. $O(n \log n)$ comparisons for sorting, $O(n)$ to compare suffixes: worst case is $O(n^2 \log n)$.
- Build a suffix tree in $O(n)$ and perform a depth-first traversal on it in $O(n)$.

Preliminary Notions

Notations:

- $suf(i)$ is said to be S-suffix if $suf(i) < suf(i+1)$. Otherwise, it is L-suffix;
- $suf(i)$ is said to be LMS-suffix if $suf(i)$ is S-suffix and $suf(i-1)$ is L-suffix;

Note

Types (S or L, and LMS) can be computed by a linear scan of T.

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Note

Types (S or L, and LMS) can be computed by a linear scan of T.

Example: $T =$ "3122120"

Index	0	1	2	3	4	5	6	7
T	3	1	2	2	1	1	2	0
Type	L	S	L	L	S	S	L	S
LMS		*			*			*

Suffix Sorting for Read-only Integer Alphabets

Definitions and assumptions:

- n_S (n_L) denotes the number of S-suffixes (L-suffixes);
- $n_L \leq n_S$;
- n_1 denotes the number of LMS-suffixes;
- $SA[0, \ldots, n-1]$ will store the result;
- Bucket: set of suffixes with the same first character.

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- $SA[0, \ldots, n-1]$ will store the result;
- Bucket: set of suffixes with the same first character.

Algorithm:

1. Sort all LMS-characters of T;
2. Induced sort all LMS-substrings from sorted LMS-characters;
3. Construct the reduced problem T_1 from sorted LMS-substrings;
4. Sort the LMS-suffixes by recursively solving T_1;
5. Induced sort all suffixes of T from the sorted LMS-suffixes.

1. Sort all *LMS*-characters of T

The *LMS*-characters can be sorted with *Counting Sort*, using $SA[0, \ldots, n/2]$ as the counting array and storing the result in $SA[n - n_1, \ldots, n - 1]$. AUX

Index	0	1	2	3	4	5	6	7
T	3	1	2	2	1	1	2	0
Type	L	S	L	L	S	S	L	S
LMS		*			*			*

SA								
						7	1	4

The sorting step takes $O(n)$ time and uses $O(1)$ workspace.

2. Induced sort all *LMS*-substrings from sorted *LMS*-characters

This step is analogous to step 5.

After this step, indices of the ordered *LMS*-substrings are stored in $SA[n - n_1, \ldots, n - 1]$.

Index	0	1	2	3	4	5	6	7
T	3	1	2	2	1	1	2	0
Type	L	S	L	L	S	S	L	S
LMS		*			*			*

SA						7	1	4

$\downarrow$

SA						7	4	1

LMS-substrings are $\{1121, 1120, 0\}$.

3. Construct the reduced problem T_1

Using the lexicographical order of the *LMS*-substrings, build the reduced problem T_1 and store it in $T[0, \ldots, n_1]$. T_1 can be obtained by a liner scan of SA, thus using $O(n)$ time and $O(1)$ workspace.

Index	0	1	2	3	4	5	6	7
T	3	1	2	2	1	1	2	0
Type	L	S	L	L	S	S	L	S
LMS		*			*			*

SA						7	4	1
SA	2	1	0			7	4	1

LMS-substrings are $\{1121, 1120, 0\}$.

4. Sort the *LMS*-suffixes by recursively solving T_1

T_1 can be solved iteratively in linear time[1] with no additional workspace. It is stored at the beginning of *SA*. The *LMS*-suffixes are sorted using linear scans and the solution of T_1.

Index	0	1	2	3	4	5	6	7
T	3	1	2	2	1	1	2	0
Type	L	S	L	L	S	S	L	S
LMS		*			*			*

SA	2	1	0					
SA	2	1	0			1	4	7
SA	7	4	1			1	4	7
SA						7	4	1

LMS-suffixes are $\{1221120, 1120, 0\}$.

The complexity of this step is $O(n)$ in time and $O(1)$ in workspace.

[1] $\mathcal{T}(n) = \mathcal{T}(n/2) + n = n(1 + 1/2 + 1/4 + 1/8 + \cdots + 1/\log_2 n) \in \Theta(n)$

5. Induced sort all suffixes of T from the sorted LMS-suffixes

It can be demonstrated that sorting the n_L L-suffixes from the sorted LMS-suffixes is symmetrical as sorting the n_S S-suffixes from the sorted L-suffixes. Suppose L-suffixes are sorted.

Index	0	1	2	3	4	5	6	7
T	3	1	2	2	1	1	2	0
Type	L	S	L	L	S	S	L	S
LMS		*			*			*

SA						7	4	1
SA	6	3	2	0				

L-suffixes are $\{31221120, 221120, 21120, 20\}$.

5. Induced sort all suffixes of T from the sorted LMS-suffixes

Pointer Data Structure

Built in linear time, indicates the bucket tails of a S-suffix in constant time. Occupies at most $c_P = cn/\log n$ words, placed at the end of SA. AUX

Interior Counter Trick

Dynamically maintain the RF-pointers (rightmost free pointers) for each bucket. AUX

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The ordering of the S-suffixes proceeds in two steps:

1. Construct a *pointer data structure* $\mathcal{P}$ and, combining it with the *interior counter trick* induce the first $n_S - c_P$ S-suffixes;
2. Use *Binary Search* and the *Interior Counter Trick* on the last c_P S-suffixes.

5. Induced Sort Algorithm

Suppose L-indices are already sorted in $SA[0, \ldots, n_L]$.

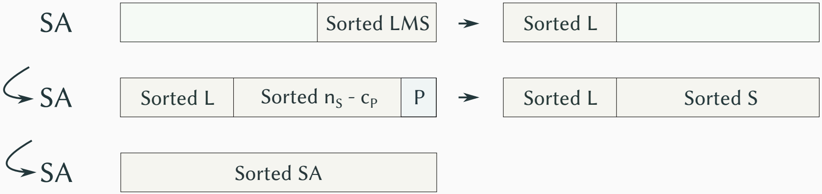
```
def InducedSort(T, SA):
  for i = nL downto 0:
    j = SA[i] - 1 # Indicizes the bucket.
    if T[j] is S-type:
      *RF[j] = j
      RF[j] = next_free_entry
    endif
  endfor
enddef
```

For each query of RF, the tail of the bucket is provided in constant time by the pointer data structure, from that the interior counter trick indicates the RF entry.

5. Induced sort all suffixes of T from the sorted LMS-suffixes

The first $n_S - c_P$ S-suffixes can be ordered in linear time with no additional workspace. The remaining suffixes can be sorted without $\mathcal{P}$, using binary search to find the tails of the buckets: since $c_P \log n = O(n)$, time linearity is preserved.

A stable, in place linear time merging can be used to merge the sorted S- and L-suffixes.



Additional results and conclusion

(Read-only) Integer Alphabets

Considering all the steps, it follows that the algorithm takes $O(n)$ time and $O(1)$ workspace to compute the suffix array of a string T over integer alphabets Σ, where T is read-only and $|\Sigma| = O(n)$.

The result trivially holds for non-read-only integer alphabets.

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Read Only General Alphabets

For read-only general alphabets (i.e., only comparisons allowed on T) there is an in-place $O(n \log n)$ time algorithm for suffix sorting.

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Read Only General Alphabets

For read-only general alphabets (i.e., only comparisons allowed on T) there is an in-place $O(n \log n)$ time algorithm for suffix sorting.

References

Li, Zhize, Jian Li, and Hongwei Huo. "Optimal in-place suffix sorting." *International Symposium on String Processing and Information Retrieval*. Springer, Cham, 2018.

Auxiliary Material

AUX: Sort all LMS-characters of T

Since $|\Sigma| = O(n)$, assume $\exists d \in \mathbb{N}$ s.t. $|\Sigma| \leq dn$. Divide LMS-characters in $2d$ partitions, where partition i contains elements in $\left[\frac{i|\Sigma|}{2d} + 1, \frac{(i+1)|\Sigma|}{2d}\right]$. Since

$$\frac{|\Sigma|}{2d} \leq \frac{dn}{2d} = \frac{n}{2},$$

$SA[0, \ldots, n/2]$ can be used as a counting array. ↩

AUX: Interior counter trick - 1

Consider a bucket of size m, indexing it as $SA_S\{0, \ldots, m-1\}$. Define the special symbols B_H, B_T, E, R_1 and R_2. `Index(i)` denotes the index of the i-th S-suffix of the bucket. The position of the tail, i.e., `m-1`, is given by the pointer data structure.

```
def InteriorCounterTrick(SAs):
  SAs[0]=BH, SAs[m-2]=E, SAs[m-1]=BT
  # O(m) time.
  if SAs[m-1]=BT and
    (SAs[m-2]=E or SAs[m-SAs[m-2]-3]!=BH):
    for i=1 upto m-3:
      SAs[m-2-i]=Index(i)
      SAs[m-2]++    # Acts as a counter.
    endfor
  endif
```

AUX: Interior counter trick - 2

```
# O(m) time.
if SAs[m-1]=BT and SAs[m-SAs[m-2]-3]=BH:
  shift SAs[1,...,m-3] to SAs[2,...m-2]
  SAs[1]=Index(m-2)
  SAs[m-1]=R2
endif
# O(m) time.
if SAs[m-1]=R2;
  shift SAs[1,...,m-2] to SAs[2,...,m-1]
  SAs[1]=Index(m-1)
  SAs[0]=R1
endif
```

AUX: Interior counter trick - 3

```
    # O(m) time, need to scan from tail
    # backwards to find R1.
    else:
        SAs[0]=Index(m)
enddef
```

The function consists of four steps, each $O(m)$ time, assuming that the tail of a bucket is known. It uses $O(1)$ workspace and, for all the buckets, results in $O(n)$ time. ↩

AUX: Pointer data structure - 1

Assuming $|\Sigma| \leq dn$, divide the S-suffixes of T in $4d$ parts, according to their first character. Let D_i denote the pointer data structure of the i-th part. D_0 can be constructed as follows (analogously for the others). For brevity, $\mathtt{b} = |\Sigma|/4d$.

```
def PointerDataStructure(T, SAs):
  SAs[i]=1 forall i in [1,b]
  for i=n-1 downto 0:
    if T[i] is S-type and in [1,b]: SAs[T[i]]++
  endfor
```

AUX: Pointer data structure - 2

```
  sum=-1
  for i=1 upto b:
    sum+=SAs[i]
    SAs[i]=sum
  endfor
enddef
```

For every S-suffix for which $T[i] \in SA_S[i, |\Sigma|/4d]$, $SA_S[T[i]] - T[i]$ indicates the tail of the bucket $T[i]$: the tail of a bucket can be obtained in constant time. ↩