No Title

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Chapter 1

Preliminaries to Complex Analysis

1 Complex number and the complex plane

1.1 Basic properties

1.2 Convergence

Theorem 1.1. $\mathbb{C}$*, the complex numbers, is complete.*

Proof. For a Cauchy sequence of complex numbers $\{z_n\}$, then

$$|z_n - z_m| \to 0 \quad \text{as} \quad n, m \to \infty.$$

In other words, given $\epsilon > 0$ there exists an integer $N > 0$ so that $|z_n - z_m| < \epsilon$ whenever $n, m > N$. If assuming $z_n = x_n + iy_n, z_m = x_m + iy_m$, so we can get

$$|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}.$$

According to Cauchy's convergence theorem: every Cauchy sequence of real numbers converges to a real number. So we can get the Cauchy's convergence theorem of complex numbers. □

Theorem 1.2. *The set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.*

Proof. For a compact set Ω, then it is closed and bounded. □

Theorem 1.3. *A set Ω is compact if and only if every open covering of Ω has a finite subcovering.*

Proof. □

Proposition 1.4. *if $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that*

$$diam(\Omega_n) \to 0 \quad as\ n \to \infty,$$

then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

Proof. Choose a point z_n in each Ω_n. We prove $\{z_n\}$ is a Cauchy sequence. Because of the condition $\text{diam}(\Omega_n) \to 0$, so we can get

$$\forall \epsilon > 0, \exists N \Rightarrow \text{diam}(\Omega_n) < \epsilon.$$

We take two integers $m, n > N$, so $z_m, z_n \in \Omega_N$. We can get

$$|z_n - z_m| \leq \text{diam}(\Omega_n) < \epsilon.$$

$\{z_n\}$ is a Cauchy sequence, therefore this sequence converges to a limit that we call w. Next, we will prove $w \in \Omega_n$ for all n. Finally, w is the unique point satisfying this property, for otherwise, if w' satisfied the same property with $w' \neq w$ we would have $|w - w'| > 0$ and the condition $\text{diam}(\Omega_n) \to 0$ would be violated. □

2 Functions on the complex plane

2.1 Continuous functions

Theorem 2.1. *A continuous function on a compact set Ω is bounded and attains a maximum and minimum on Ω.*

Proof. □

Proposition 2.2. *if f and g are holomorphic in Ω, then:*

(i) $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$.

(ii) fg is holomorphic in Ω and $(fg)' = f'g + fg'$.

(iii) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z) \textit{for all} z \in \Omega.$$

Chapter 2

Cauchy's Theorem and Its Applications

1 Goursat's theorem

Theorem 1.1. *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$*