22.520 Numerical Methods for PDEs
Homework 4

Instructor : Prof. D. Willis
Email: david_willis@uml.edu
Due: May 6th FIRM Deadline

1 Background

In this homework you will use finite element methods to examine the potential flow around NACA 4-digit airfoils and elliptical shapes. The geometry discretization routines have been provided for your convenience, in addition to a skeleton Matlab FEM code. For simplicity we will be studying symmetric shapes and flows. An example two dimensional geometry is shown below:

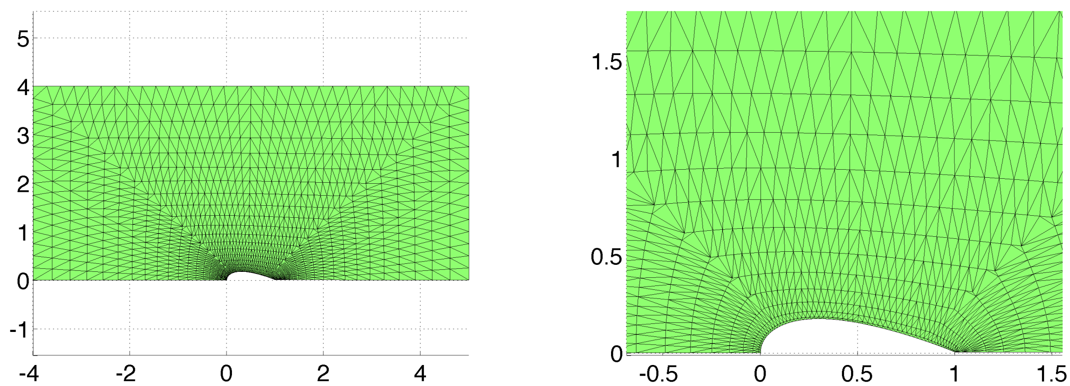

Figure 1: (left) The complete discrete geometry (right) zoom of the geometry in question)

2 Theory

You will use potential flow theory to explore the flow around the various objects in question.

Consider the PDE that describes the conservation of mass for an incompressible fluid:

$$\nabla \cdot \vec{u} = 0. \quad (1)$$

We can introduce a scalar potential function, ϕ, such that:

$$\nabla \phi = \vec{u} \tag{2}$$

By definition the resulting flow is irrorational ($\nabla \times \vec{u} = 0$). The conservation of mass equation (equation 1) can be combined with the scalar potential definition (equation 2) to yield:

$$\begin{aligned} \nabla \cdot \vec{u}(x,y) &= \nabla \cdot (\nabla \phi(x,y)) && (3) \\ &= \nabla^2 \phi(x,y) && (4) \\ &= 0 && (5) \end{aligned}$$

The potential flow equation ($\nabla^2 \phi = 0$), will be solved using FEM in this homework project. Once solved on the half two-dimensional domain, the full domain solution will be reconstructed by mirroring the domain as shown below:

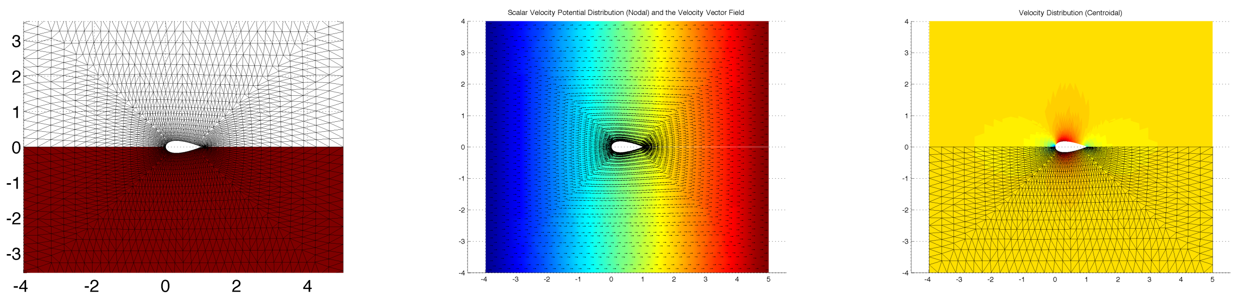

Figure 2: (left) The mirrored domain (middle) the scalar potential (right) the flow velocity)

The geometry discretization and skeleton code is provided for you in the download packet.

2.1 Boundary Conditions

The upper and lower boundaries (top and bottom boundaries) have a prescribed no-normal flow boundary condition. This includes the airfoil or the ellipsoidal shape being examined. This corresponds to the following boundary condition expression:

$$\vec{u} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = 0. \tag{6}$$

At the left and right hand sides, we will apply a prescribed potential value. Based on potential flow theory, we can set the value at these boundaries equal to the value of the x-coordinate at those locations. This effectively sets up a uniform freestream flow entering and exiting the domain:

$$\phi(x = X_{LHS}, y) = X_{LHS} \tag{7}$$

$$\phi(x = X_{RHS}, y) = X_{RHS} \tag{8}$$

These boundary conditions should be easy to apply in your implementation. The geometry routine infat only hands back the locations where the Dirichlet condition will be applied (hint).

3 Sample Solutions

Below are some sample solutions:

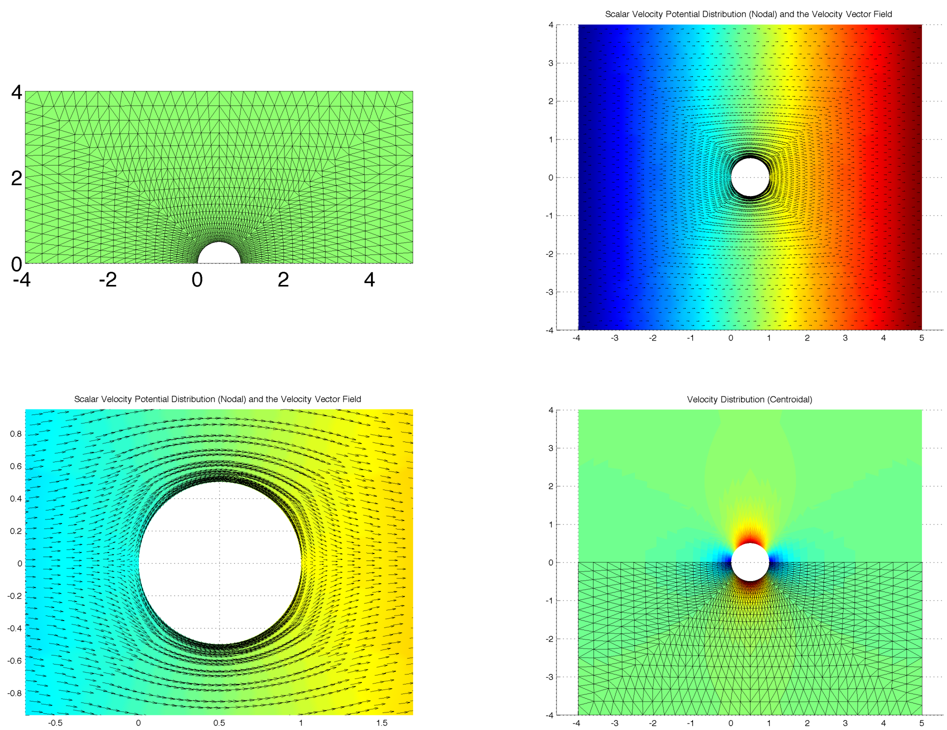

Figure 3: (upper-left) The grid (upper right) the scalar potential value, (lower left) zoom in of the scalar potential and the velocity vectors (lower right) The velocity in the domain.)

4 Tasks and Deliverables

1. Starting from the strong form of the partial differential equation, derive the relevant weak form that will be used as the basis of the finite element method.

2. Show how the weak form that you have derived represents the minimization of the kinetic energy in the potential flow representation.

3. You will use linear-hat nodal basis functions to represent the solution and the test functions in your solver. Describe/draw how these linear hat basis functions look in two dimensions for:

   (a) For a node i in the triangulation, show the "tent" basis function (nodal hat basis function)

   (b) For an element k, show the three linear basis functions on a triangular element.

4. Write the discrete form of the weak form of the equation for a weighting function/node i in the domain.

5. Write the discrete form of the weak form of the equation for a node i on the boundary of the domain where you have Dirichlet BCs.

6. Write the discrete form of the weak form of the equation for a node i on the boundary of the domain where you have Neumann BCs.

7. Derive the elemental matrices that will be used to build the overall A-matrix (stiffness matrix). Also derive the elemental load vectors that will be used to determine the RHS. Include how you will calculate the gradient values using the equation for a plane surface.

8. Describe the Stamping procedure that will be used to construct the global matrix using the elemental matrices. You may use both words and example matrices.

9. For each boundary condition, describe the implementation in your FEM code.

10. Write pseudo code describing the finite element method implementation that you will use.

11. Add the elemental matrices and boundary conditions to the provided pseudo code and complete the FEM solver so that it works (currently the solver has some random numbers in places to allow it to "work"). You may optionally develop your own two-dimensional FEM solver. If you choose to use the matlab version provided, please read through and understand the comments and structure. If you choose to develop your own solver, please use the geometry definition tool provided.

12. Solve the potential flow problem and plot the scalar potential using a minimum grid refinement value of 3.

13. Show how the value of your scalar potential solution at the front of the object (location 0,0) converges with grid refinement.