22.520 Numerical Methods for Partial Differential Equations

Homework # 3 : Due Monday April 8th

In this problem you will first examine the numerical implementation of the traffic flow problem using several different numerical schemes for approximating the flux across cell boundaries. Once you have implemented the 4 different flux functions, the goal will be to solve an optimization problem for maximizing the traffic flow along a particular road.

For the entirety of this problem, you may assume that the road is defined by a line segment between $x = -1$ and $x = 1$. The cars will travel with a direction from left to right, and will have a density of $\rho = 0.8$ at $x = -1$. The initial conditions for the density of cars on the road is always:

$$\rho_{t=0} = \begin{cases} 0.8 \text{ if } x \leq 0 \\ 0.0 \text{ if } x > 0 \end{cases} \tag{1}$$

The equation for the conservation of mass of the cars at any location along the road is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u(\rho))}{\partial x} = 0 \tag{2}$$

The cars are assumed to have a non-dimensional velocity between $0 \leq u \leq 1$. The density of cars s also $0 \leq \rho \leq 1$. The relationship between the density and velocity is linear and is given as:

$$u(\rho) = 1 - \rho \tag{3}$$

The result is the following scalar, hyperbolic conservation law:

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho - \rho^2)}{\partial x} = 0 \tag{4}$$

The one-dimensional finite volume method, as covered in class, will be used to solve this problem:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+0.5}^n - F_{i-0.5}^n\right) \tag{5}$$

The right and left boundary fluxes $F_{i+0.5}^n$ and $F_{i-0.5}^n$ respectively, will be determined using approximate flux functions.

For the numerical implementation, you should use:

$$\Delta x = \frac{1}{100} \tag{6}$$

$$\tag{7}$$

$$\Delta t = 0.8\Delta x \tag{8}$$

Questions

For the first 4 questions, assume that there is a traffic light at $x = 0$ that turns red ($flux = F = 0$) when $t = 1, 3, ..., 2n + 1$, and green when $t = 0, 2, 4, ..., 2n$.

1. Implement in Matlab (or other programming language of choice) the Godunov flux scheme (you should **not** use the finite volume traffic flow matlab code posted online as your solution to this question – you should code your own solution independently, either using your in-class effort or starting again):

$$F_{i+0.5}^{Godunov} = \begin{cases} \min_{(\rho_i \le \rho \le \rho_{i+1})} f(\rho) & \text{if } \rho_i \le \rho_{i+1} \\ \max_{(\rho_i \le \rho \le \rho_{i+1})} f(\rho) & \text{if } \rho_i > \rho_{i+1} \end{cases}$$

2. In the same matlab code, implement a naive averaged flux scheme shown below:

$$F_{i+0.5}^{avg} = 0.5\left[f(\rho_i) + f(\rho_{i+1})\right]$$

   How does this naive average scheme compare with the Godunov scheme? Suggest a reason for the poor performance of this scheme.

3. In the same matlab code, implement the Lax-Friedrich's flux scheme as an alternative case. The L-F scheme is shown below:

$$F_{i+0.5}^{LF} = 0.5\left[f(\rho_i) + f(\rho_{i+1})\right] - 0.5\frac{\Delta x}{\Delta t}\left(\rho_{i+1} - \rho_i\right);$$

   How does the L-F scheme compare with the Godunov scheme? Be sure to zoom into the plot of the car density along the x-axis.

4. In the same matlab code, implement the Richtmyer flux scheme as a third alternative case. The Richtmyer scheme is shown below:

$$\begin{aligned} \rho_{i+0.5}^{Richt} &= 0.5\frac{\Delta x}{\Delta t}\left[f(\rho_i) - f(\rho_{i+1})\right] + 0.5(\rho_{i+1} + \rho_i) \\ F_{i+0.5} &= \rho_{i+0.5}^{Richt}(1 - \rho_{i+0.5}^{Richt}) \end{aligned}$$

   How does this Richtmyer scheme compare with the previous two?

5. Of the four flux functions you examined, which appears to be (a) the most accurate for this problem, (b) the most dissipative and (c) the most efficient?

6. Using the Godunov scheme, maintain the traffic light at $x = 0$ and add a second traffic light at $x = 0.25$. Assuming that the second light alternates between red for 1 unit of time and then green for 1 unit of time, determine the optimal time lag between the two lights to maximize the *average* flow of cars along the road. *Hint:* Run the code for a sufficiently long time so that a 'periodic steady state' solution is achieved. Once a periodic steady state is reached, you can determine the average flux of cars by a discrete averaging formula:

$$Flux_{avg} = \frac{1}{2}\sum_{t=2n}^{t=2n+2} f(\rho_i^t) * \Delta t$$