Module 3: *An Introduction to Conservation Laws and Finite Volume Methods*

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Announcements

- Assignment # 3 Posted this week (Hyperbolic Conservation Laws)
- Today and next class – Finite Volume
- Finite Element Methods (FEM) starting next week (2 HW, 1 x 1D, 1 x 2D)

References and Acknowledgements

The following materials were used in the preparation of this lecture:

1. Randall J. LeVeque *Numerical Methods for Conservation Laws*.
2. 16.920, lecture 11,12 Notes
3. Sethian, J.A., Level Set Methods – Evolving interfaces in Geometry, Fluid Mechanics
4. Versteeg and Malalasekera, *An introduction to computational fluid dynamics – the finite volume method*

The author of these slides wishes to thank these sources for making the current lecture.

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Conservation of Mass Example

- Consider a simple conservation of mass representation:

Control Volume Mathematics

- We could recall from fluid dynamics that there is a control volume representation for this situation:

$$0 = \frac{\partial}{\partial t} \iiint_{\Omega} \rho dV + \iint_{\Gamma} \rho \vec{u} \cdot n dS \tag{1}$$

- This expression relates the change in mass inside the element with the rate at which it is entering/leaving the element. We can impose the divergence theorem to transform the second integral (boundary integral):

$$0 = \frac{\partial}{\partial t} \iiint_{\Omega} \rho dV + \iiint_{\Omega} \nabla \cdot (\rho \vec{u}) dV \tag{2}$$

- If we then combining the two integrals:

$$0 = \iiint_{\Omega} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \right] dV \tag{3}$$

Control Volume Mathematics

- The result is the familiar *conservation of mass* in differential form
$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \tag{4}$$
- Idea: What if we had the above conservation of mass equation, can we introduce a "finite volume" or control volume approach? If so, how do we do so?

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General Conservation Laws

- In general, many physical phenomena that change with respect to time are conserved.
  - Conservation of mass (ρ is mass per unit volume)
  - Conservation of Momentum ($\rho\vec{u}$ is momentum per unit volume $\rightarrow$ Newtons Law)
  - Conservation of Energy ($\rho u \cdot u^T$ is energy per unit volume $\rightarrow$ Energy equation)
- The underlying conservation form is:

$$\frac{\partial u}{\partial t} + \frac{\partial(f(u))}{\partial x} = 0 \tag{5}$$

- Where:
  - $u =$ the unknown property under consideration (in fluids, it is the per unit volume of the quantity)
  - $f(u) =$ The flux of the quantity through the control volume

General Conservation Laws

- The basic conservation form:

$$\frac{\partial u}{\partial t} + \frac{\partial(f(u))}{\partial x} = 0 \tag{6}$$

- FYI, this can be re-written or expressed as:

$$\frac{\partial u}{\partial t} + \underbrace{\frac{df}{du}}_{a(u)} \frac{\partial u}{\partial x} = 0 \tag{7}$$

- or the (non-linear) advection equation. We saw the linear version of this last class (hyperbolic PDE).

Finite Volume Method

- Let's define a 1-D finite volume geometry and discretization:
- We will need to define 1-D "volumes" and "boundaries":

Finite Volume Method

- We are going to integrate the conservation law across a volume/area/ **line** element. Our original conservation law in 1-Dimension is:

$$\frac{\partial u}{\partial t} + \frac{\partial (f(u))}{\partial x} = 0 \tag{8}$$

- Let us take an integral across each of our finite line segments, and claim that our solution is the addition of all of the segments:

$$\sum_i \int_{x_L}^{x_R} \frac{\partial u_i}{\partial t} + \frac{\partial (f(u_i))}{\partial x} dx = 0 \tag{9}$$

- It turns out that each segment must satisfy the following integral expression:

$$0 = \frac{\partial}{\partial t} \int_{x_L}^{x_R} u dx + \int_{x_L}^{x_R} \frac{\partial f(u)}{\partial x} dx \tag{10}$$

- Discussion: By taking this integral, have we changed the nature of the conservation law at all?

Finite Volume Method

- The second integral simply becomes the flux evaluated at the left and right boundaries of the element/cell:

$$0 = \frac{\partial}{\partial t}\int_{x_L}^{x_R} u dx + \int_{x_L}^{x_R} \frac{\partial f(u)}{\partial x} dx = \frac{\partial}{\partial t}\int_{x_L}^{x_R} u dx + f(u)_R - f(u)_L \quad (11)$$

- We can re-arrange this a little as:

$$\frac{\partial}{\partial t}\int_{x_L}^{x_R} u dx = -[f(u)_R - f(u)_L] \quad (12)$$

- This expression says: The change in the property integrated across our control volume is balanced by the flux in and out of the LHS and RHS boundaries of the element.

Finite Volume Method

- Let's focus on the left hand side now. We may notice that the average value of the property u across the cell is:

$$\frac{1}{\Delta x}\int_{x_L}^{x_R} u dx = u_{avg} \tag{13}$$

- Or:

$$\int_{x_L}^{x_R} u dx = \Delta x u_{avg} \tag{14}$$

- We can re-write our previous expression now as:

$$\frac{\partial}{\partial t}(\Delta x u_{avg}) = -[f(u)_R - f(u)_L] \tag{15}$$

Finite Volume Method

- Pictorially, the average value vs. the actual value. What we have effectively done, is assumed a constant value function across each element (most crude approximation we could make, but lets go with it).

Finite Volume Method

- Only the time derivative is left:

$$\frac{\partial}{\partial t}(\Delta x u_{avg}) = -[f(u)_R - f(u)_L] \tag{16}$$

- Discretizing with a forward Euler equation:

$$\frac{(\Delta x u_{avg})^{k+1} - (\Delta x u_{avg})^k}{\Delta t} = -[f(u)_R - f(u)_L] \tag{17}$$

- Rearranging to find the new u:

$$u^{k+1} = u^k - \frac{\Delta t}{\Delta x}[f(u)_R - f(u)_L] \tag{18}$$

- Now we can find the updated u value at some future time.

Finite Volume Method

- Let's Recap:

Finite Volume Method

Traffic Flow Example (Leveque):

- ρ : Density of cars on the road
- u : velocity of the cars
- We are going to claim there is a relationship between the velocity of the cars and the density:

$$u(\rho) = u_{max}(1 - \frac{\rho}{\rho_{max}}) \tag{19}$$

- Let's take a look at this function (assuming that $u_{max} = 1$ and $\rho_{max} = 1$, ie – normalized flow)

Finite Volume Method

Finite Volume Method

- The conservation of cars (mass) equation is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} \tag{20}$$

- Based on the above equation, and the expression for the velocity as a function of density, the **flux** $= \rho u$ is:

$$f(\rho) = \rho u_{max}(1 - \frac{\rho}{\rho_{max}}) = \rho - \rho^2 \tag{21}$$

- Let's plot the flux as a function of ρ now:

Finite Volume Method

Finite Volume Method

Let's try to implement this for a particular problem:

- Let's choose $\rho = a$ when $x > 0$
- Let's choose $\rho = b$ when $x < 0$
- From before, we have:

$$u^{k+1} = u^k - \frac{\Delta t}{\Delta x}[f(u)_R - f(u)_L] \tag{22}$$

- Let's try to put this together for a couple of control volumes.
- Notice, at the boundaries, the flux of cars from one element, into another could be different!

Finite Volume Method

Finite Volume Method

Let's try to implement this for a particular problem:

- Godunov came up with the following flux function:

$$F(u_l, u_r) = \begin{cases} \min_{(u_l \leq u \leq u_r)} f(u) & \text{if } u_l \leq u_r \\ \max_{(u_r \leq u \leq u_l)} f(u) & \text{if } u_l > u_r \end{cases} \tag{23}$$

- What does this mean?