Project 2: One dimensional Euler equations with a Shock

- **Anand Dhariya**

ME 523: Computational Fluid Dynamics 1

November 21, 2007

The University of Michigan

Department of Mechanical Engineering

ME523/AE523 Computational Fluid Dynamics I

Project 2. Due Wednesday November 21, 2007

One-Dimensional Euler Equations with a Shock

Write a program to compute the solution to the one dimensional Euler equations:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0$$

where

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho\left(e + \frac{1}{2}u^2\right) \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u\left(\rho\left(e + \frac{1}{2}u^2\right) + p\right) \end{bmatrix}$$

In the above, the pressure is given by the ideal gas relation $p = (\gamma - 1)\rho e$. The sound speed is given by $c^2 = \gamma p / \rho$. Use $\gamma = 1.4$ for air. The initial conditions are as given on page 208 in Hirsch, Vol II, which consist of a shock tube with two constant states:

$$p_L = 10^5;\ p_R = 10^3;\ \rho_L = 1;\ \rho_R = 0.01;\ u_L = u_R = 0.$$

where subscripts *L* and *R* denote, respectively, the left and right side of the domain with respect to the initial discontinuity. The tube is 10 meter long and the initial separation between the two states is in the middle. The exact solution at 3.9msec is given on page 210 in Hirsch, Vol. II (see figure shown in the next page).

Solve the above system of equations to reproduce the solution as shown in the figure by

(a) Using the Lax (Lax-Friedrichs) method:

$$\mathbf{q}_j^{n+1} = \frac{1}{2}\left(\mathbf{q}_{j+1}^n + \mathbf{q}_{j-1}^n\right) - \frac{\Delta t}{2h}\left(\mathbf{F}_{j+1}^n - \mathbf{F}_{j-1}^n\right)$$

(b) Using the two-step Lax-Wendroff (LW-II) method:

$$\mathbf{q}_{j+1/2}^* = 0.5\left(\mathbf{q}_j^n + \mathbf{q}_{j+1}^n\right) - 0.5\frac{\Delta t}{h}\left(\mathbf{F}_{j+1}^n - \mathbf{F}_j^n\right)$$

$$\mathbf{q}_j^{n+1} = \mathbf{q}_j^n - \frac{\Delta t}{h}\left(\mathbf{F}_{j+1/2}^* - \mathbf{F}_{j-1/2}^*\right)$$

Run your program for several different spatial resolution and time steps. For (b), you are going to find that the solution is not very good. Large "wiggles" arise even when the time step is small, and if the "wiggles" become negative, bad things will happen! To improve the results, add artificial viscous fluxes of the form:

$$\mathbf{F}^* \to \mathbf{F}^* - \alpha h^2 \rho \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix} \left|\frac{\partial u}{\partial x}\right| \frac{\partial u}{\partial x}$$

where α is an adjustable coefficient. Experiment with the coefficient. In general, the stability requirement becomes more restrictive once we add artificial viscosity, so we want the coefficient to be as small as possible. Comment on the overall accuracy (qualitatively) between the solutions with the Lax and the LW-II scheme (with artificial viscosity) at the same grid resolution.

Your report should include a printout of your code and plots of the solution at one or more times for a few different resolutions. Discuss the solution behavior depending on different schemes and the magnitude of the artificial viscosity.

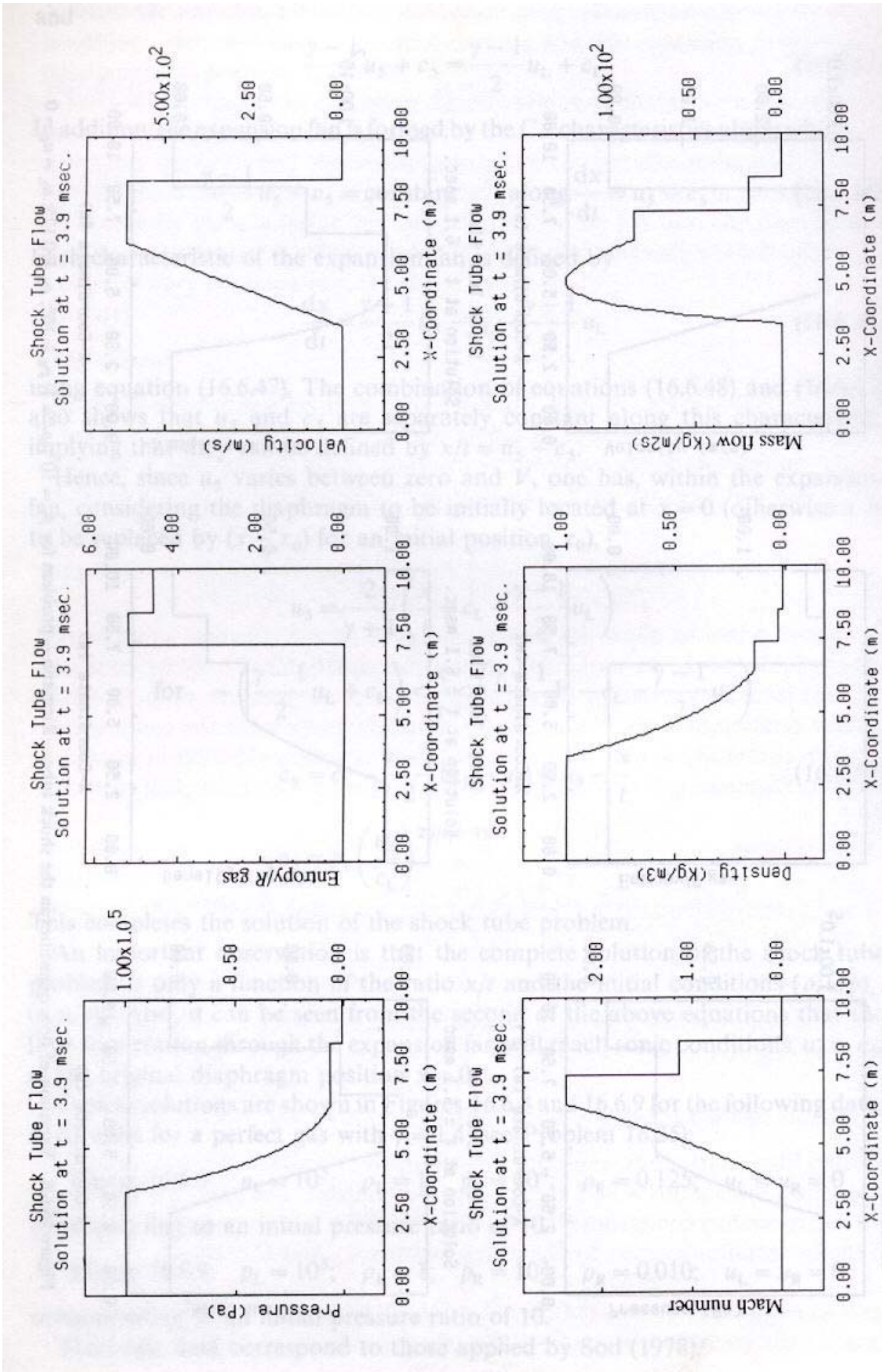


Figure 16.6.9 Variation of flow quantities in the shock tube—Riemann—problem for $p_L = 10^5$; $\rho_L = 1$; $p_R = 10^3$; $\rho_R = 0.010$; $u_L = u_R = 0$

One-Dimensional Euler Equations:

We have to write a code to solve the 1-D Euler equation given by:

$$\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{t}} + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}} = \mathbf{0}$$

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\rho u} \\ \boldsymbol{\rho}\left(\boldsymbol{e} + \frac{1}{2}\boldsymbol{u}^2\right) \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{\rho u} \\ \boldsymbol{\rho u}^2 + \boldsymbol{p} \\ \boldsymbol{u}\left(\boldsymbol{\rho}\left(\boldsymbol{e} + \frac{1}{2}\boldsymbol{u}^2\right) + \boldsymbol{p}\right) \end{bmatrix}$$

A typical 1-D shock tube problem is considered to check the validity of code. The problem involves a shock tube with both ends open and containing a diatomic gas (say air). In the centre of the tube, there is a wall. The conditions on either sides of wall at time t=0 are given as:

$$p_L = 10^5;\ p_R = 10^3;\ \rho_L = 1;\ \rho_R = 0.01;\ u_L = u_R = 0$$

At time t=0, the centre wall is broken and a shock-expansion waves are formed. MATLAB codes are written using Lax-Friedrichs method and Lax-Wendroff II method to solve the euler equations and get plots of the flow parameters. This plots are then compared with results obtained by Hirsch using exact Riemann solver.

Once the q and F matrix are formed, we integrate in time by step 'dt' and update these matrices at each time step. The time step is evaluated using the CFL number and spatial grid size h as:

$$dt = \frac{CFL \times h}{\max\,(a)}$$

where a=speed of sound=$\frac{\gamma p}{\rho}$

Once the final time is reached, the density, velocity, pressure and energy can be obtained from the q and F matrices. However, the Mach number, Mass flow rate and Entropy needs to be calculated. The equations for these variables are as follows:

Mass flow rate, $Q = \rho u$

Mach number, $Ma = \frac{u}{a}$

The entropy for a reversible process with respect to a reference condition is given as:

$$\Delta s = \frac{1}{\gamma - 1}\ln\left(\frac{p}{p_{ref}}\right) + \frac{\gamma}{\gamma - 1}\ln\,\left(\frac{\rho_{ref}}{\rho}\right)$$

Here I have taken the reference state as standard atmospheric pressure and density at sea level. So, $p_{ref} = 101325\ {}^{N}/_{m^2}$ and $\rho_{ref} = 1.225\ {}^{kg}/_{m^3}$

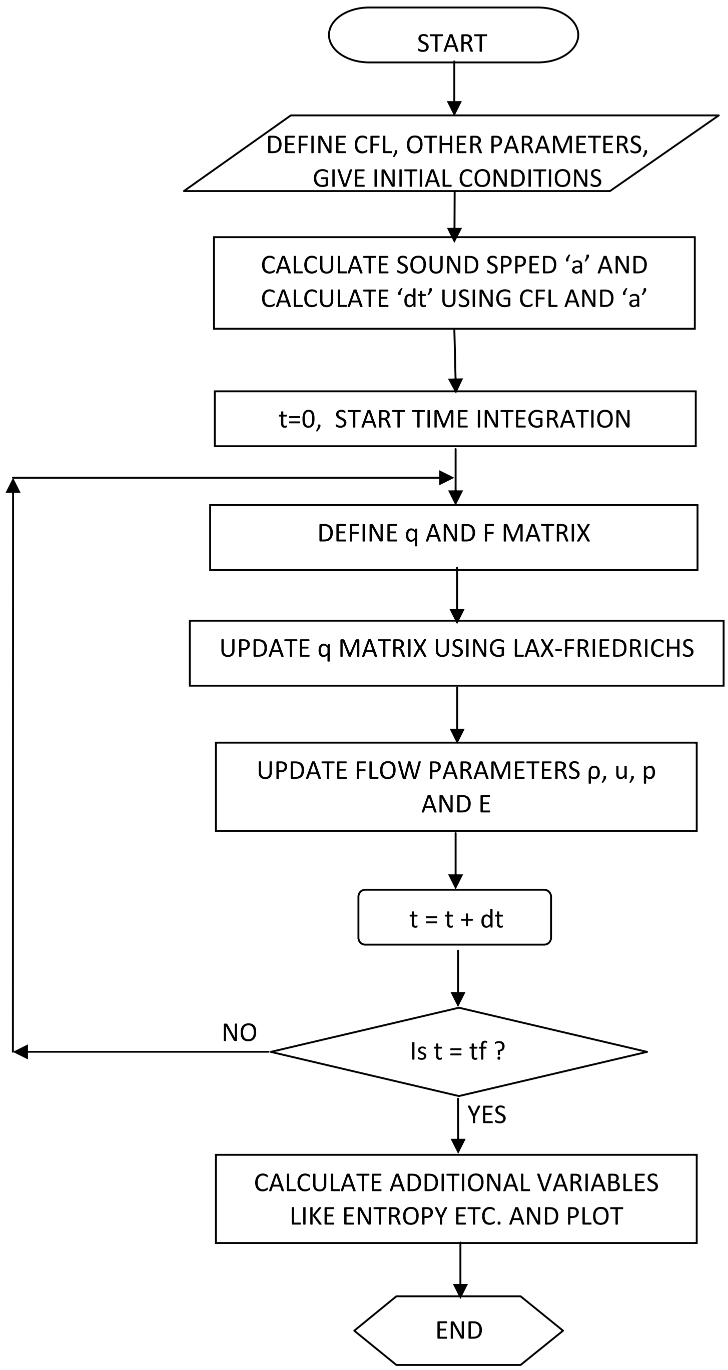
The following pages explain the results of Lax-Friedrichs and Lax-Wendroff-II methods.

Part A: Lax-Friedrichs method:

In this method we update the q matrix in each time step as:

$$\boldsymbol{q}_j^{n+1} = \frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{q}_{j+1}^n + \boldsymbol{q}_{j-1}^n\right) - \frac{\Delta \boldsymbol{t}}{\mathbf{2}\boldsymbol{h}}\left(\boldsymbol{F}_{j+1}^n - \boldsymbol{F}_{j-1}^n\right)$$

The flowchart shows the various steps of the algorithm.



Effect of changing spatial grid resolution:

The results of this method are plotted for 3 spatial resolutions, viz. n=201, n=401 and n=601 keeping the CFL number fixed at 0.34. It is observed that refining the grid size enables to better capture the steep gradients of the solution. Thus a higher grid resolution is favorable.

Effect of changing CFL number:

Next the grid resolution is fixed at n=601 and the CFL number is varied. Since the grid size is fixed, changing the CFL number amounts to changing the time step dt. The results are seen only for the velocity, u plot since the fine variations due to change in CFL are best seen in this plot. The solutions are plotted for 3 different CFL numbers viz

CFL=0.05 (dt=2.2272e-006); CFL= 0.15 (dt=6.6815e-006); CFL= 0.35 (dt= 1.5590e-005).

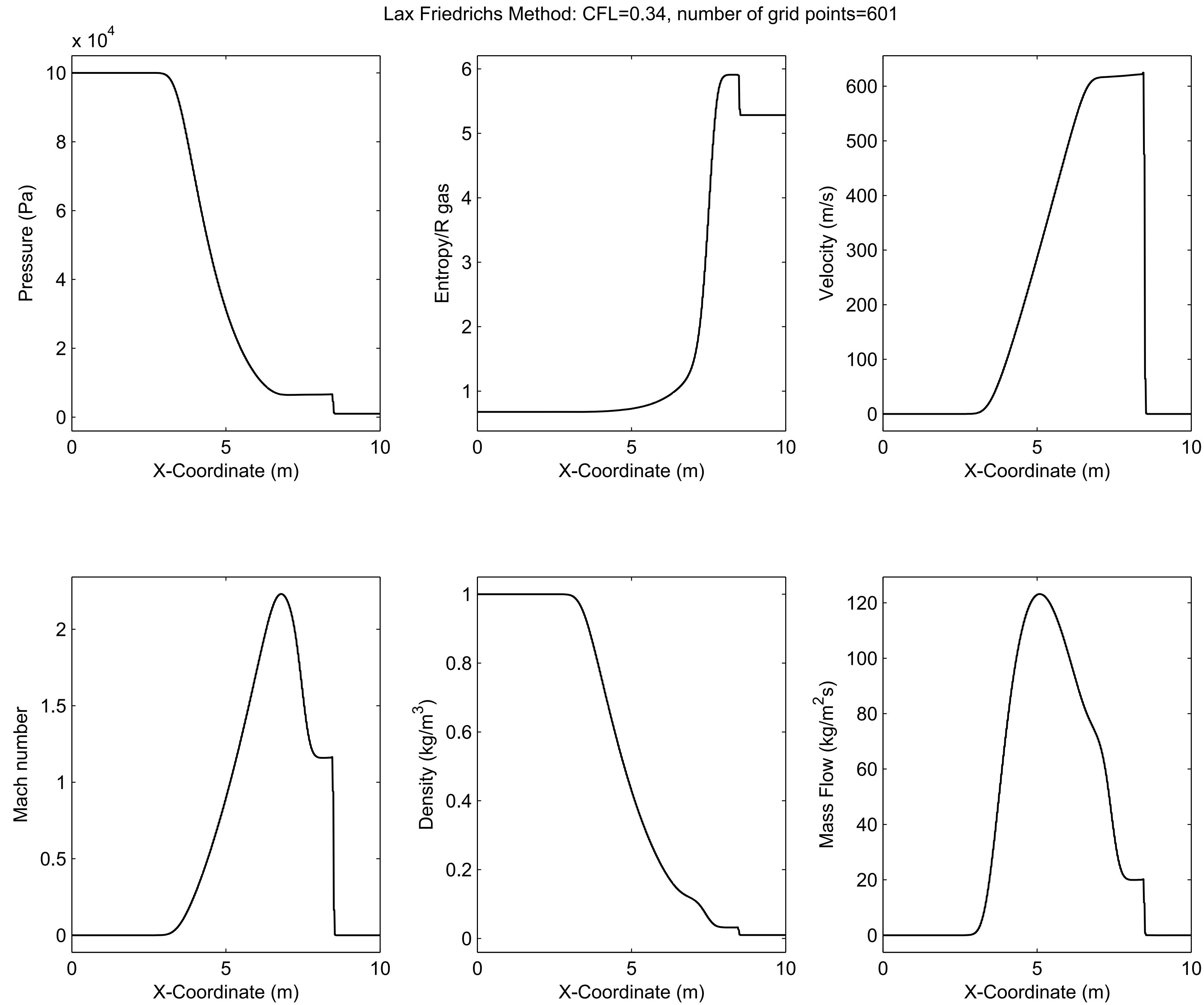
It is observed that reducing the CFL number or dt leads to a more diffusive curve. This is counterintuitive but can be explained by taking into consideration the accumulation of rounding errors in MATLAB. At large CFL numbers, wiggles are observed in the solution as shown in attached figure.

Conclusion:

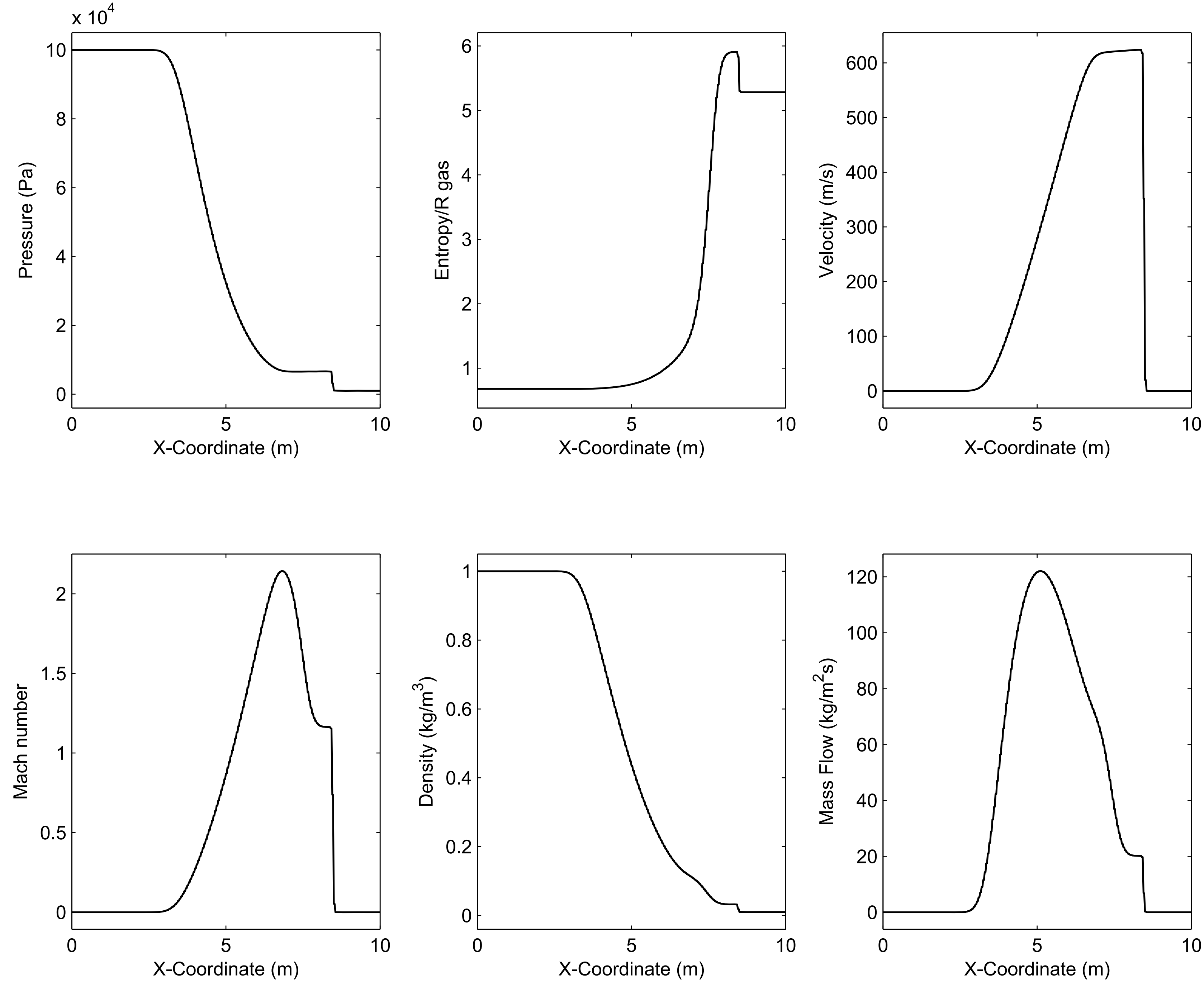
The Lax Friedrichs method is diffusive due to presence of numerical viscosity which is inherent to the finite-difference equation used. Due to this diffusive nature, monotonicity is maintained however the scheme fails to represent the steep gradients faithfully.

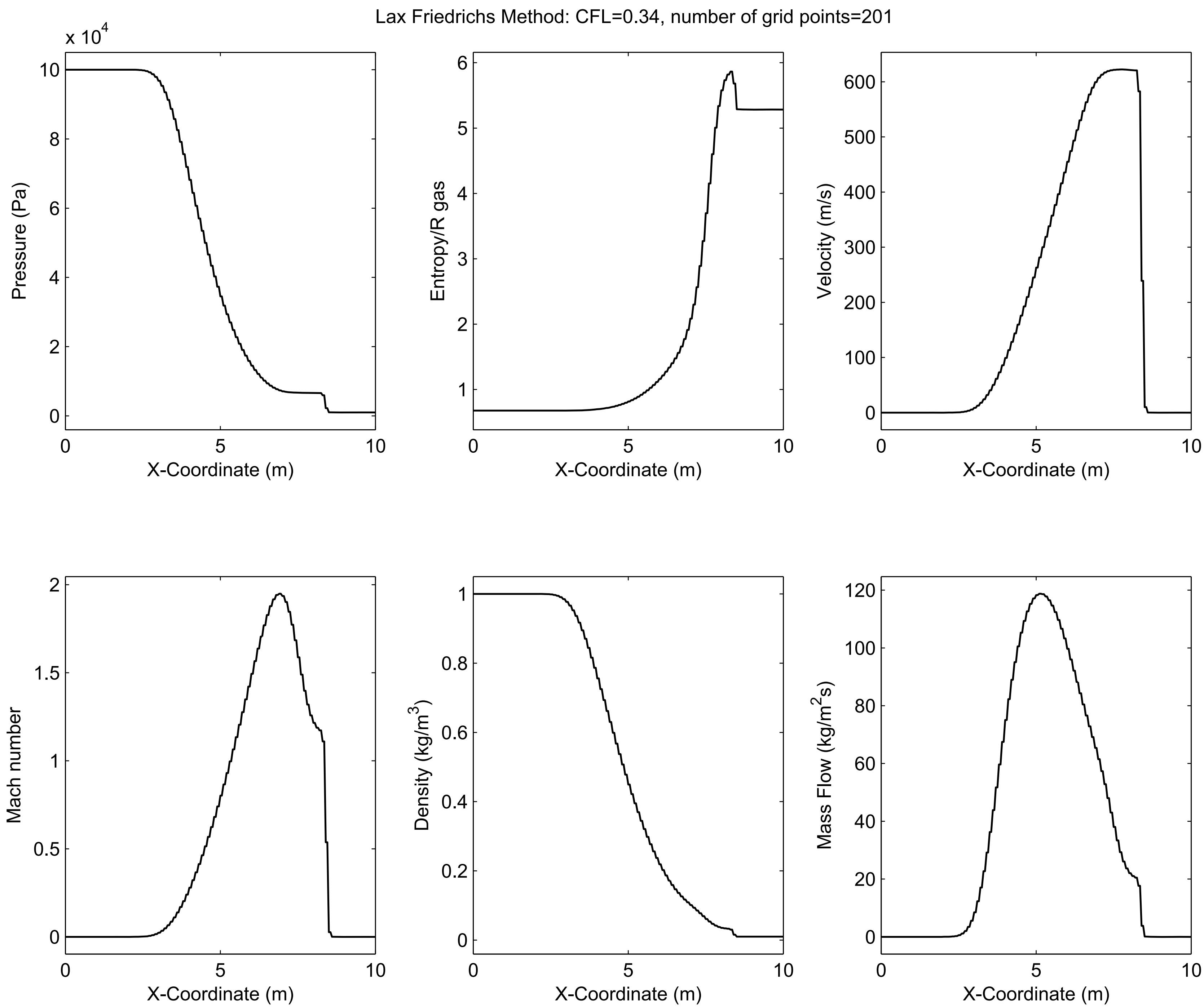
```
%-------------------------------------------------------------------------%
% Project 2, Part (a):Lax-Friedrichs method to solve 1-D Euler equations
% ME-523 Computational Fluid Dynamics - 1
% by- Anand Dhariya
% Date: 21/11/2007
%-------------------------------------------------------------------------%
clear;
clf;
n=601;                          %Number of grid points
L=10;                           %Length of domain
h=L/(n-1);                      %Spatial step size
CFL=0.34;                       %CFL number for stability
t_final=3.9e-3;                 %Final time
x=0:h:L;
gamma=1.4;                      %Ratio of specific heats for ideal di-atomic gas
%-------------------------------------------------------------------------%
% Define intial conditions
%-------------------------------------------------------------------------%
p_l=1e5;                        %Pressure in left side of shock tube at t=0
p_r=1e3;                        %Pressure in right side of shock tube at t=0
rho_l=1;                        %Density at left side of shock tube at t=0
rho_r=0.01;                     %Density at right side of shock tube at t=0
u_l=0;                          %Velocity in left side of shock tube at t=0
u_r=0;                          %Velocity in right side of shock tube at t=0
p(1:1:(n+1)/2)=p_l;
p((n+3)/2:1:n)=p_r;
rho(1:1:(n+1)/2)=rho_l;
rho((n+3)/2:1:n)=rho_r;
u(1:1:(n+1)/2)=u_l;
u((n+3)/2:1:n)=u_r;
E=p./((gamma-1)*rho)+0.5*u.^2;  %Total Energy
a=sqrt(gamma*p./rho);           %Speed of sound
dt=CFL*h/max(abs(u+a));         %Time step based on CFL number
step=0;
%-------------------------------------------------------------------------%
% Time integration begins
%-------------------------------------------------------------------------%
for t=dt:dt:t_final
    %Define q & F matrix
    q=[rho; rho.*u; rho.*E];
    F=[rho.*u; rho.*u.^2+p; u.*(rho.*E+p)];
    %Update q matrix and flow parameters
    q(1:3,2:n-1)=0.5*(q(1:3,3:n)+q(1:3,1:n-2))-dt/(2*h)*(F(1:3,3:n)-F(1:3,1:n-2));
    rho=q(1,1:n);
    u=q(2,1:n)./rho(1:n);
    E=q(3,1:n)./rho;
    p=(gamma-1)*rho.*(E-0.5*u.^2);
    step=step+1;
end
%Calculation of flow parameters
a=sqrt(gamma*p./rho);
M=u./a;
```

```
p_ref=101325;                     %Reference air pressure (N/m^2)
rho_ref=1.225;                    %Reference air density (kg/m^3)
s=1/(gamma-1)*(log(p/p_ref)+gamma*log(rho_ref./rho));   %Entropy w.r.t reference condition
Q=rho.*u;                         %Mass Flow rate per unit area
%--------------------------------------------------------------------------%
% Plot the variables
%--------------------------------------------------------------------------%
offset=0.05;
subplot(231);plot(x,p,'k');xlabel('X-Coordinate (m)');ylabel('Pressure (Pa)');
ylim([min(p)-(offset)*max(p) (1+offset)*max(p)]);
subplot(232);plot(x,s,'k');xlabel('X-Coordinate (m)');ylabel('Entropy/R gas');
ylim([min(s)-(offset)*max(s) (1+offset)*max(s)]);
subplot(233);plot(x,u,'k');xlabel('X-Coordinate (m)');ylabel('Velocity (m/s)');
ylim([min(u)-(offset)*max(u) (1+offset)*max(u)]);
subplot(234);plot(x,M,'k');xlabel('X-Coordinate (m)');ylabel('Mach number');
ylim([min(M)-(offset)*max(M) (1+offset)*max(M)]);
subplot(235);plot(x,rho,'k');xlabel('X-Coordinate (m)');ylabel('Density (kg/m^3)');
ylim([min(rho)-(offset)*max(rho) (1+offset)*max(rho)]);
subplot(236);plot(x,Q,'k');xlabel('X-Coordinate (m)');ylabel('Mass Flow (kg/m^2s)');
ylim([min(Q)-(offset)*max(Q) (1+offset)*max(Q)]);
%--------------------------------------------------------------------------%
%Hooray!!! code ends here ;D
```

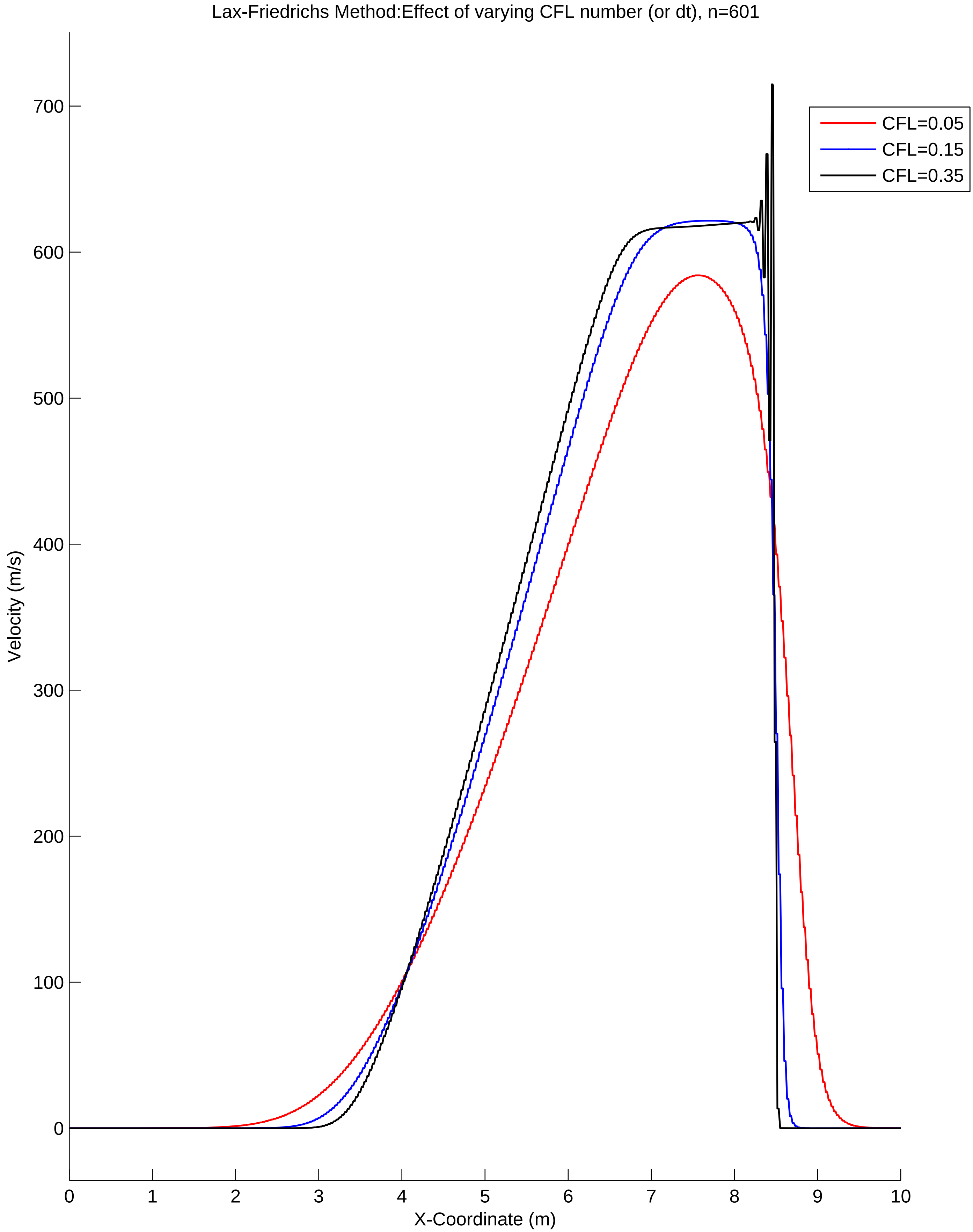

Lax Friedrichs Method: CFL=0.34, number of grid points=601
x 10^4
Pressure (Pa)
Entropy/R gas
Velocity (m/s)
Mach number
Density (kg/m^3)
Mass Flow (kg/m^2s)
X-Coordinate (m)

Lax Friedrichs Method: CFL=0.34, number of grid points=401





Lax Friedrichs Method: CFL=0.34, number of grid points=201
x 10^4
Pressure (Pa)
Entropy/R gas
Velocity (m/s)
Mach number
Density (kg/m^3)
Mass Flow (kg/m^2s)
X-Coordinate (m)


Lax-Friedrichs Method:Effect of varying CFL number (or dt), n=601
CFL=0.05
CFL=0.15
CFL=0.35
Velocity (m/s)
X-Coordinate (m)
0
100
200
300
400
500
600
700
0
1
2
3
4
5
6
7
8
9
10

Part B: Lax-Wendroff:

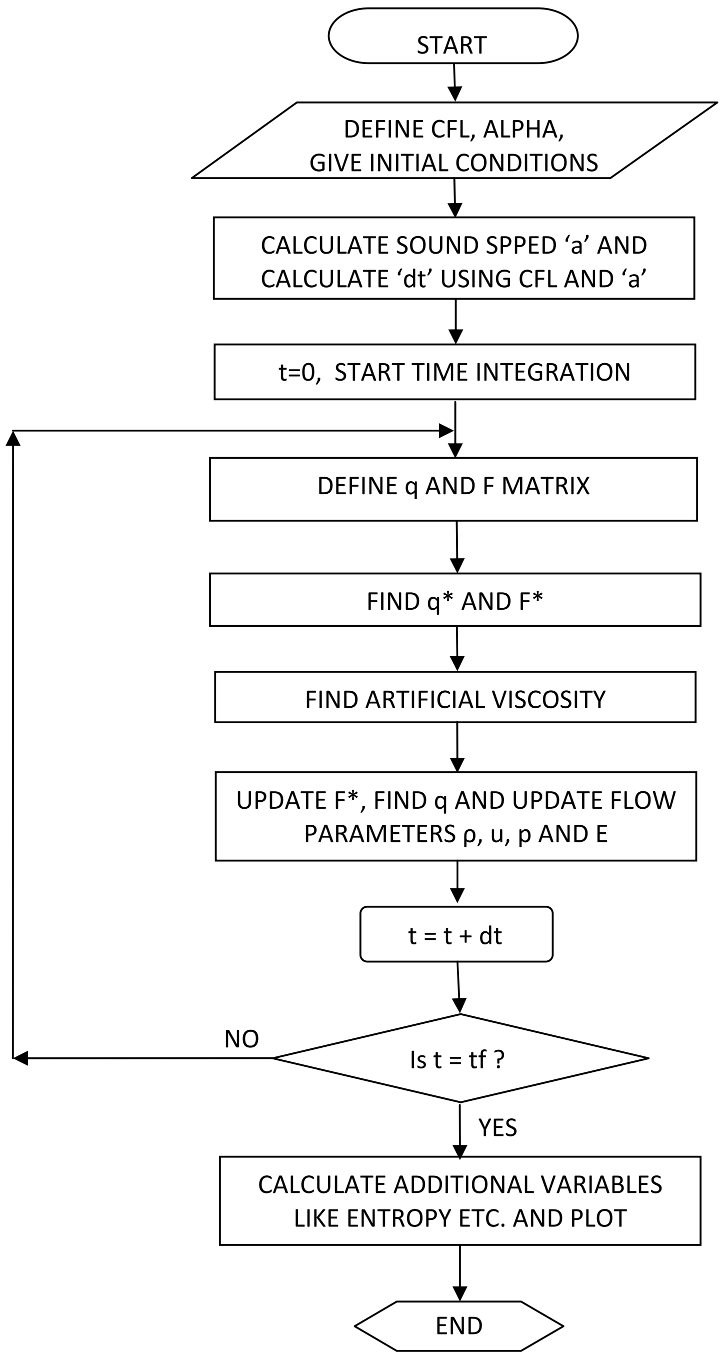
Here we update the q matrix in 2 steps as follows:

$$\boldsymbol{q}^{*}_{j+1/2} = \frac{1}{2}\left(\boldsymbol{q}^{n}_{j} + \boldsymbol{q}^{n}_{j+1}\right) - \frac{\Delta t}{2h}\left(\boldsymbol{F}^{n}_{j+1} - \boldsymbol{F}^{n}_{j}\right)$$

$$\boldsymbol{q}^{n+1}_{j} = \boldsymbol{q}^{n}_{j} - \frac{\Delta t}{h}\left(\boldsymbol{F}^{*}_{j+1/2} - \boldsymbol{F}^{*}_{j-1/2}\right)$$

$$\boldsymbol{F}^{*} \rightarrow \boldsymbol{F}^{*} - \alpha h^{2} \rho \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix} \left|\frac{\partial u}{\partial x}\right| \frac{\partial u}{\partial x}$$

The flowchart for Lax-Wendroff method is similar to Lax-Friedrichs, but we have few additional steps due to artificial viscosity.



Effect of changing spatial grid resolution:

The results of this method are plotted for 3 spatial resolutions, viz. n=201, n=401 and n=601 keeping the CFL number fixed at 0.30 and α=0.40. The effect of varying grid size is similar to that in Lax-Friedrichs. It is observed that refining the grid size enables to better capture the steep gradients of the solution. Thus a higher grid resolution is favorable.

Effect of changing CFL number:

Next the grid resolution and alpha are fixed at n=601 and α=0.40 respectively. The CFL number is varied. Since the grid size is fixed, changing the CFL number amounts to changing the time step dt. The results are seen only for the velocity, u plot since the fine variations due to change in CFL are best seen in this plot. The solutions are plotted for 3 different CFL numbers viz.

CFL=0.01 (dt= 4.4544e-007); CFL= 0.10 (dt= 4.4544e-006); CFL= 0.30 (dt= 1.3363e-005).

Unlike the Lax-Friedrichs method, changing CFL or dt does not produce any appreciable change in most part of the solution. However it was observed that for small values of CFL number very large wiggles were formed near the discountinuity. On increasing the CFL number the amplitude of these wiggles reduced significantly and at CFL=0.30 the plot resembled closely to the required solutions.

Effect of varying viscosity parameter α:

If α is set to zero, it is observed that very large wiggles are produced not only at discontinuity but also at other steep gradients as shown in attached figure. Increasing α reduces these fluctuations however, the range of values over which the CFL number and hence dt can be varied is very limited. When α is increased we need to reduce the time step dt and vice versa.

The attached velocity plot shows the effect of varying α keeping CFL number fixed at CFL=0.10 and spatial grid size, n=601. The observations were made at three different values of α viz. α=0.01, α=0.40 and α=0.95. The amplitude of wiggles is largest for smallest α and vice versa. Only velocity is plotted to show this effect since the variations can be easily seen without zooming in; however the nature is same for all other flow variables as well.

Conclusion:

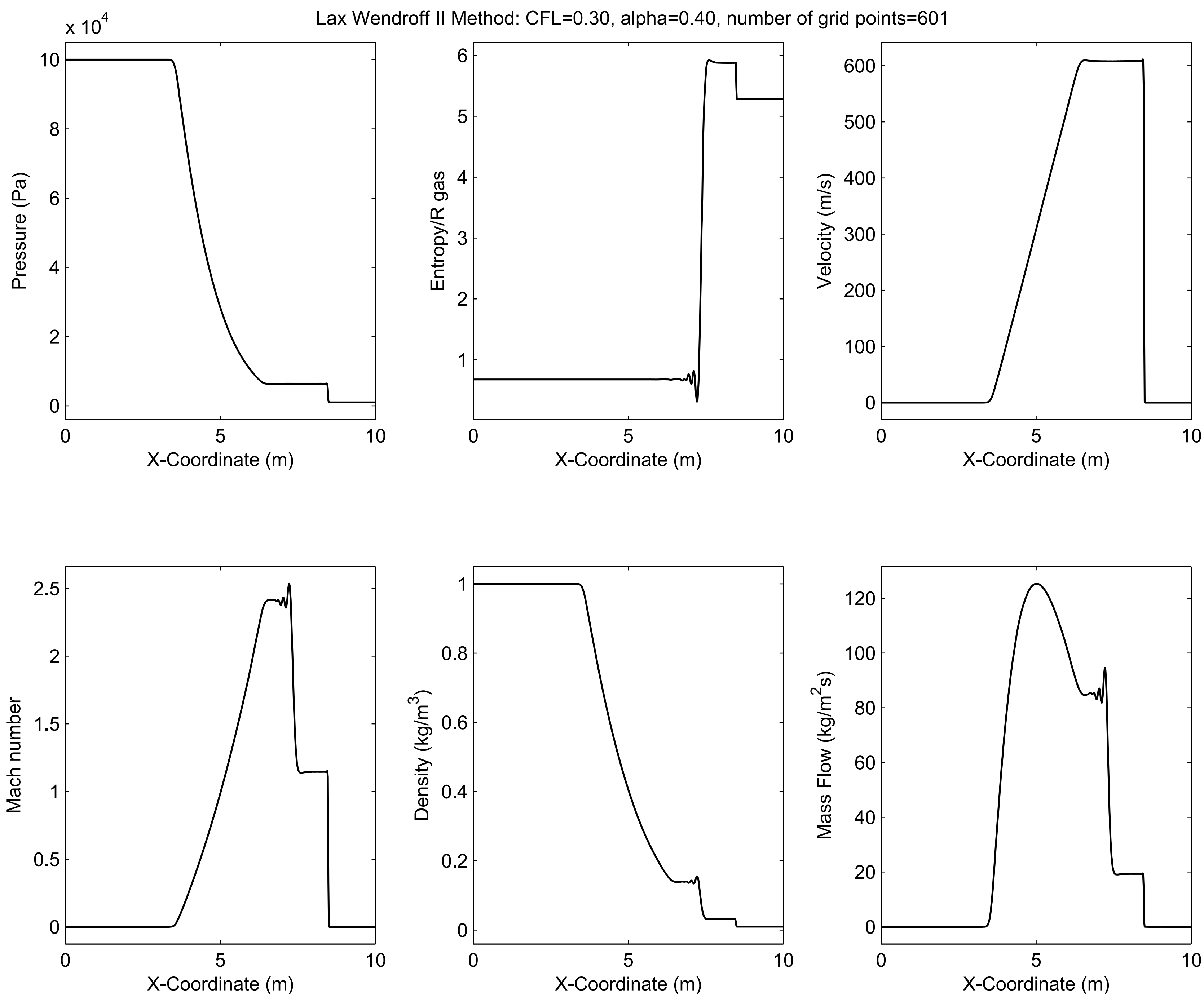
The Lax-Wendroff II method is much better than the Lax-Friedrichs method at resolving the shock. However, the method being dispersive produces large wiggles near the discontinuity. By introducing the artificial second order viscous term, these wiggles can be contained to a certain extent, however it also imposes limitations on the temporal step size dt.

A trade-off being necessary between the diffusive and dispersive trends, we see the significance of Godunov's theorem that monotonicity cannot be assured if discontinuities are to be represented faithfully in the solution.

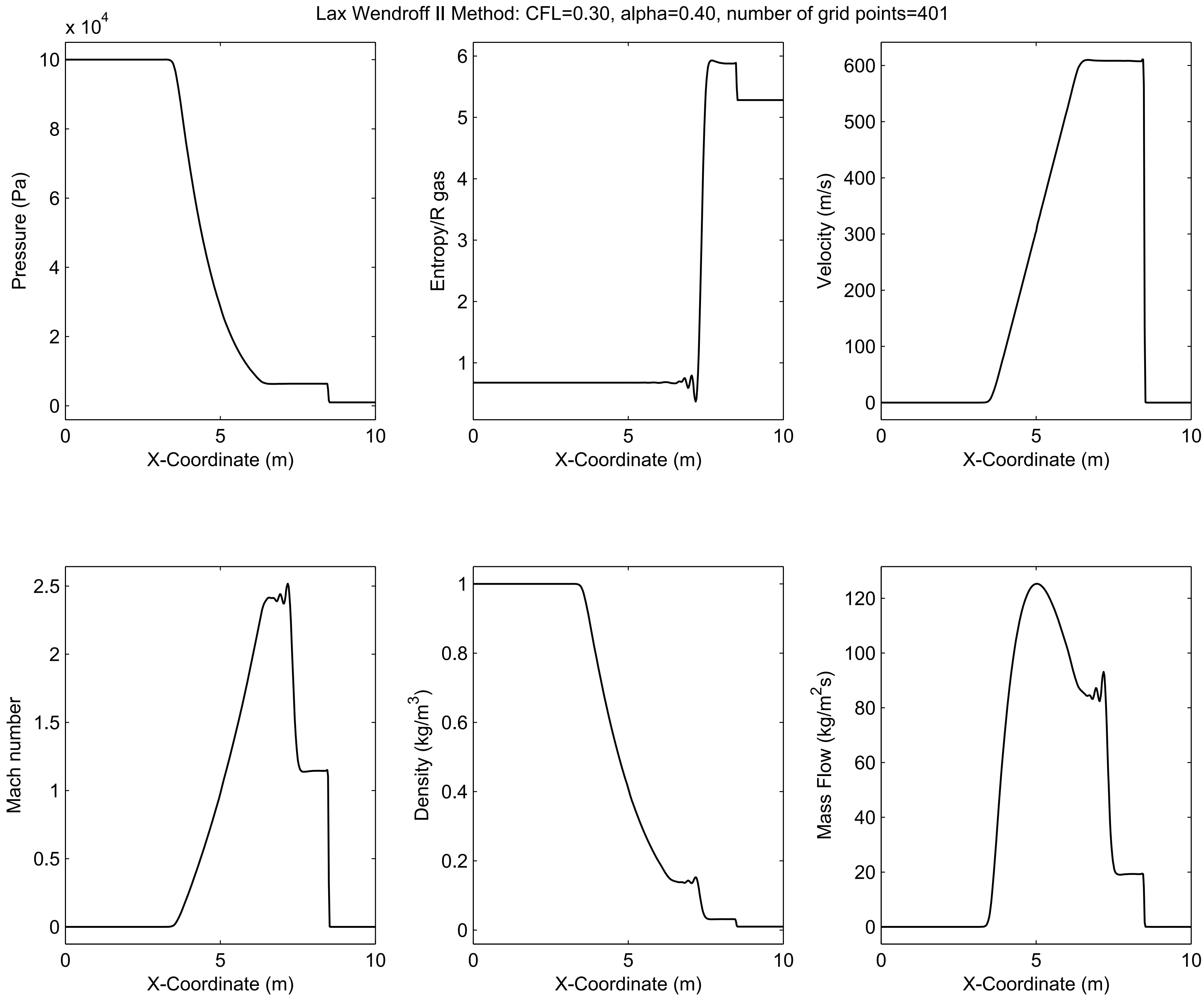
This signifies that alternative schemes like implicit and spectral methods need to be implemented to capture shock accurately.

```
%-----------------------------------------------------------------------------%
% Project 2, Part (b):Lax-Wendroff II method to solve 1-D Euler equations
% ME-523 Computational Fluid Dynamics - 1
% by- Anand Dhariya
% Date: 21/11/2007
%-----------------------------------------------------------------------------%
clear;
clf;
n=601;                          %Number of grid points
L=10;                           %Length of domain
h=L/(n-1);                      %Spatial step size
CFL=0.30;                       %CFL number for stability
alpha=0.40;                     %Parameter for artificial viscosity
t_final=3.9e-3;                 %Final time
x=0:h:L;
gamma=1.4;                      %Ratio of specific heats for ideal di-atomic gas
%-----------------------------------------------------------------------------%
% Define intial conditions
%-----------------------------------------------------------------------------%
p_l=1e5;                        %Pressure in left side of shock tube at t=0
p_r=1e3;                        %Pressure in right side of shock tube at t=0
rho_l=1;                        %Density at left side of shock tube at t=0
rho_r=0.01;                     %Density at right side of shock tube at t=0
u_l=0;                          %Velocity in left side of shock tube at t=0
u_r=0;                          %Velocity in right side of shock tube at t=0
p(1:1:(n+1)/2)=p_l;
p((n+3)/2:1:n)=p_r;
rho(1:1:(n+1)/2)=rho_l;
rho((n+3)/2:1:n)=rho_r;
u(1:1:(n+1)/2)=u_l;
u((n+3)/2:1:n)=u_r;
E=p./((gamma-1)*rho)+0.5*u.^2;  %Total Energy
a=sqrt(gamma*p./rho);           %Speed of sound
dt=CFL*h/max(a);                %Time step based on CFL number
step=0;
%-----------------------------------------------------------------------------%
% Time integration begins
%-----------------------------------------------------------------------------%
for t=dt:dt:t_final
    %Define q & F matrix
    q=[rho; rho.*u; rho.*E];
    F=[rho.*u; rho.*u.^2+p; u.*(rho.*E+p)];
    %Calculate q* and flow parameters
    q_star(1:3,1:n-1)=0.5*(q(1:3,1:n-1)+q(1:3,2:n))-dt/(2*h)*(F(1:3,2:n)-F(1:3,1:n-1));
    rho(1:n-1)=q_star(1,1:n-1);
    u(1:n-1)=q_star(2,1:n-1)./rho(1:n-1);
    E(1:n-1)=q_star(3,1:n-1)./rho(1:n-1);
    p(1:n-1)=(gamma-1)*rho(1:n-1).*(E(1:n-1)-0.5*u(1:n-1).^2);
    %Calculate F*
    F_star(1:3,1:n-1)=[rho(1:n-1).*u(1:n-1); rho(1:n-1).*u(1:n-1).^2+p(1:n-1); u(1:n-1).*
(rho(1:n-1).*E(1:n-1)+p(1:n-1))];
    mx=[zeros(1,n-1);ones(1,n-1);u(1:n-1)];
```

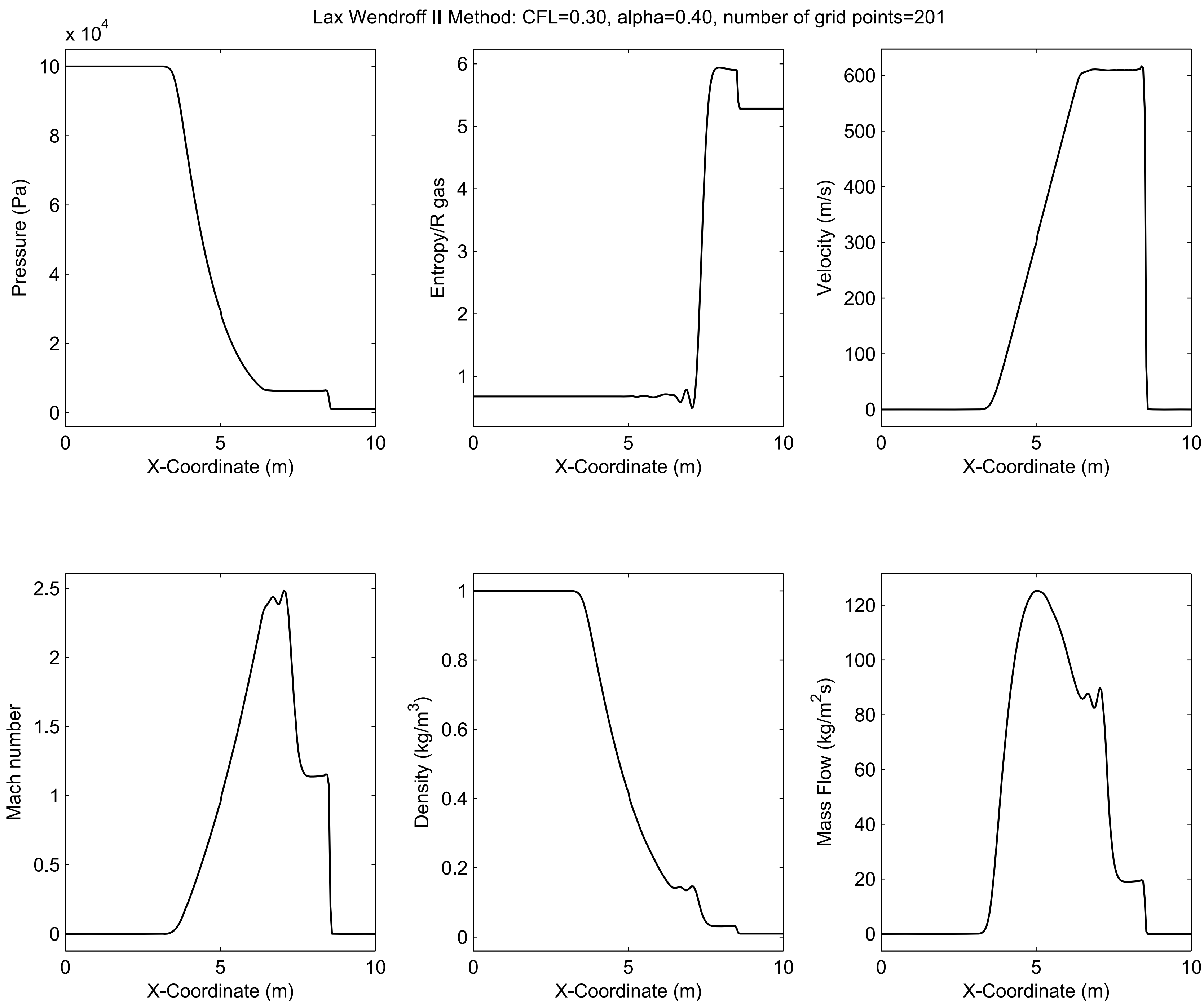
```
    %Calculate artificial viscosity
    for i=1:3
        visc(i,1:n-1)=alpha*h^2*rho(1:n-1).*abs((u(2:n)-u(1:n-1))/h).*((u(2:n)-u(1:n-1))↙
/h).*mx(i,1:n-1);
    end
    %update F* and q matrix
    F_star=F_star-visc;
    q(1:3,2:n-1)=q(1:3,2:n-1)-dt/h*(F_star(1:3,2:n-1)-F_star(1:3,1:n-2));
    rho=q(1,1:n);
    u=q(2,1:n)./rho(1:n);
    E=q(3,1:n)./rho;
    p=(gamma-1)*rho.*(E-0.5*u.^2);
    step=step+1;
end
%Calculation of flow parameters
a=sqrt(gamma*p./rho);
M=u./a;
p_ref=101325;                  %Reference air pressure (N/m^2)
rho_ref=1.225;                 %Reference air density (kg/m^3)
s=1/(gamma-1)*(log(p/p_ref)+gamma*log(rho_ref./rho));   %Entropy w.r.t reference values
Q=rho.*u;                      %Mass Flow rate per unit area
%--------------------------------------------------------------------------%
% Plot the variables
%--------------------------------------------------------------------------%
offset=0.05;
subplot(231);plot(x,p,'k');xlabel('X-Coordinate (m)');ylabel('Pressure (Pa)');
ylim([min(p)-(offset)*max(p) (1+offset)*max(p)]);
subplot(232);plot(x,s,'k');xlabel('X-Coordinate (m)');ylabel('Entropy/R gas');
ylim([min(s)-(offset)*max(s) (1+offset)*max(s)]);
subplot(233);plot(x,u,'k');xlabel('X-Coordinate (m)');ylabel('Velocity (m/s)');
ylim([min(u)-(offset)*max(u) (1+offset)*max(u)]);
subplot(234);plot(x,M,'k');xlabel('X-Coordinate (m)');ylabel('Mach number');
ylim([min(M)-(offset)*max(M) (1+offset)*max(M)]);
subplot(235);plot(x,rho,'k');xlabel('X-Coordinate (m)');ylabel('Density (kg/m^3)');
ylim([min(rho)-(offset)*max(rho) (1+offset)*max(rho)]);
subplot(236);plot(x,Q,'k');xlabel('X-Coordinate (m)');ylabel('Mass Flow (kg/m^2s)');
ylim([min(u)-(offset)*max(Q) (1+offset)*max(Q)]);
%--------------------------------------------------------------------------%
%Hooray!!! code ends here ;D
```



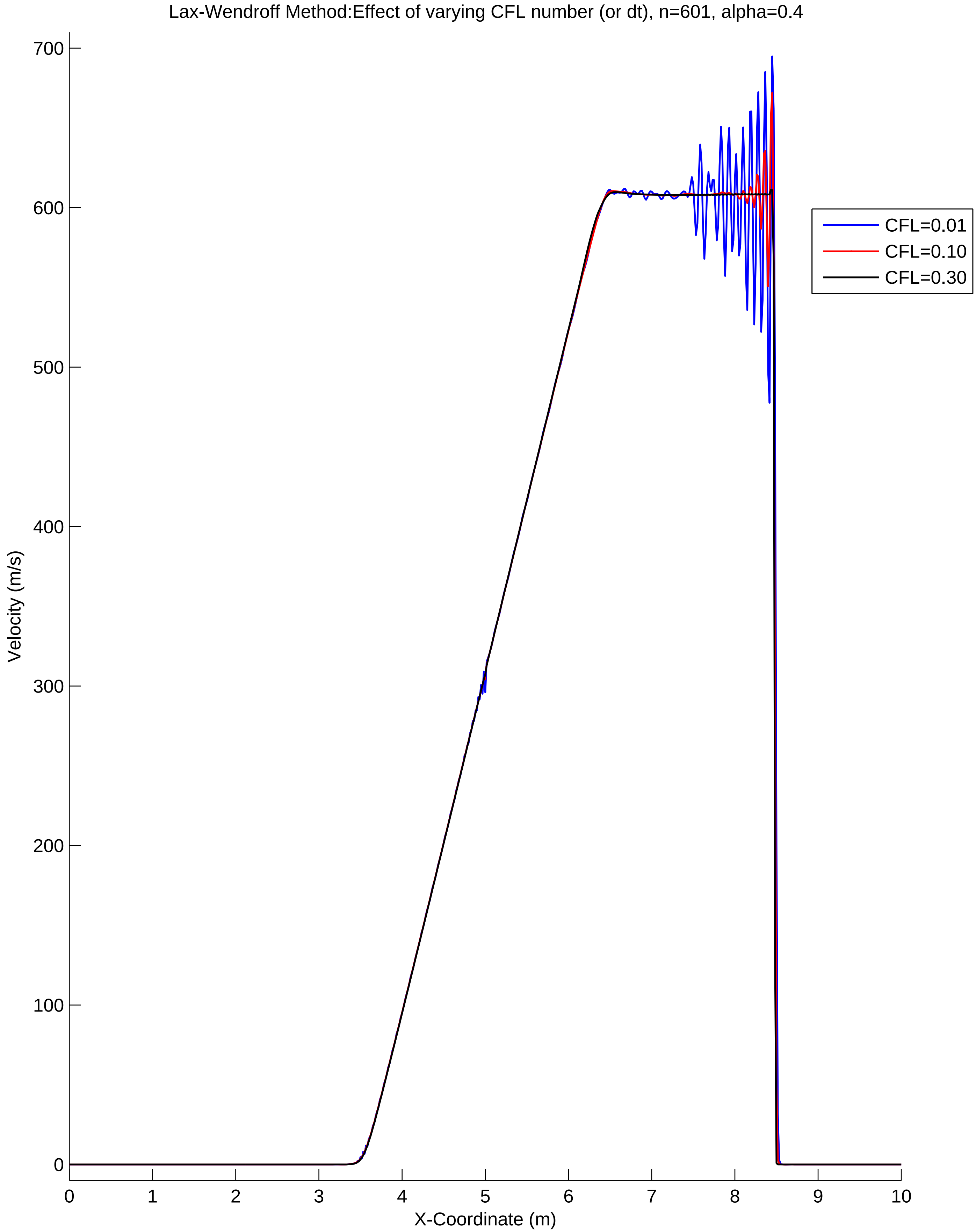
Lax Wendroff II Method: CFL=0.30, alpha=0.40, number of grid points=601
x 10^4
Pressure (Pa)
Entropy/R gas
Velocity (m/s)
Mach number
Density (kg/m^3)
Mass Flow (kg/m^2s)
X-Coordinate (m)



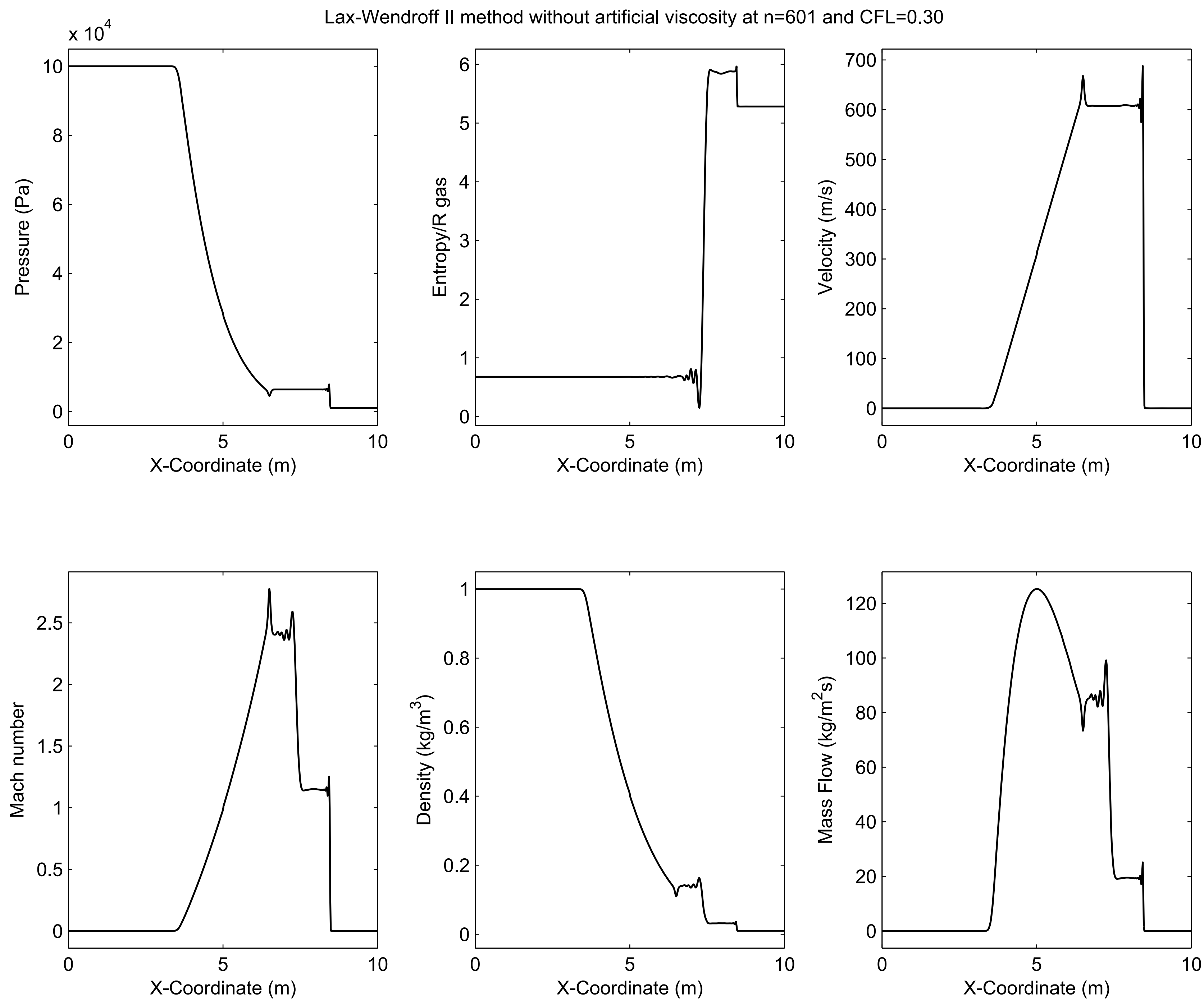
Lax Wendroff II Method: CFL=0.30, alpha=0.40, number of grid points=401
x 10^4
Pressure (Pa)
Entropy/R gas
Velocity (m/s)
Mach number
Density (kg/m^3)
Mass Flow (kg/m^2s)
X-Coordinate (m)

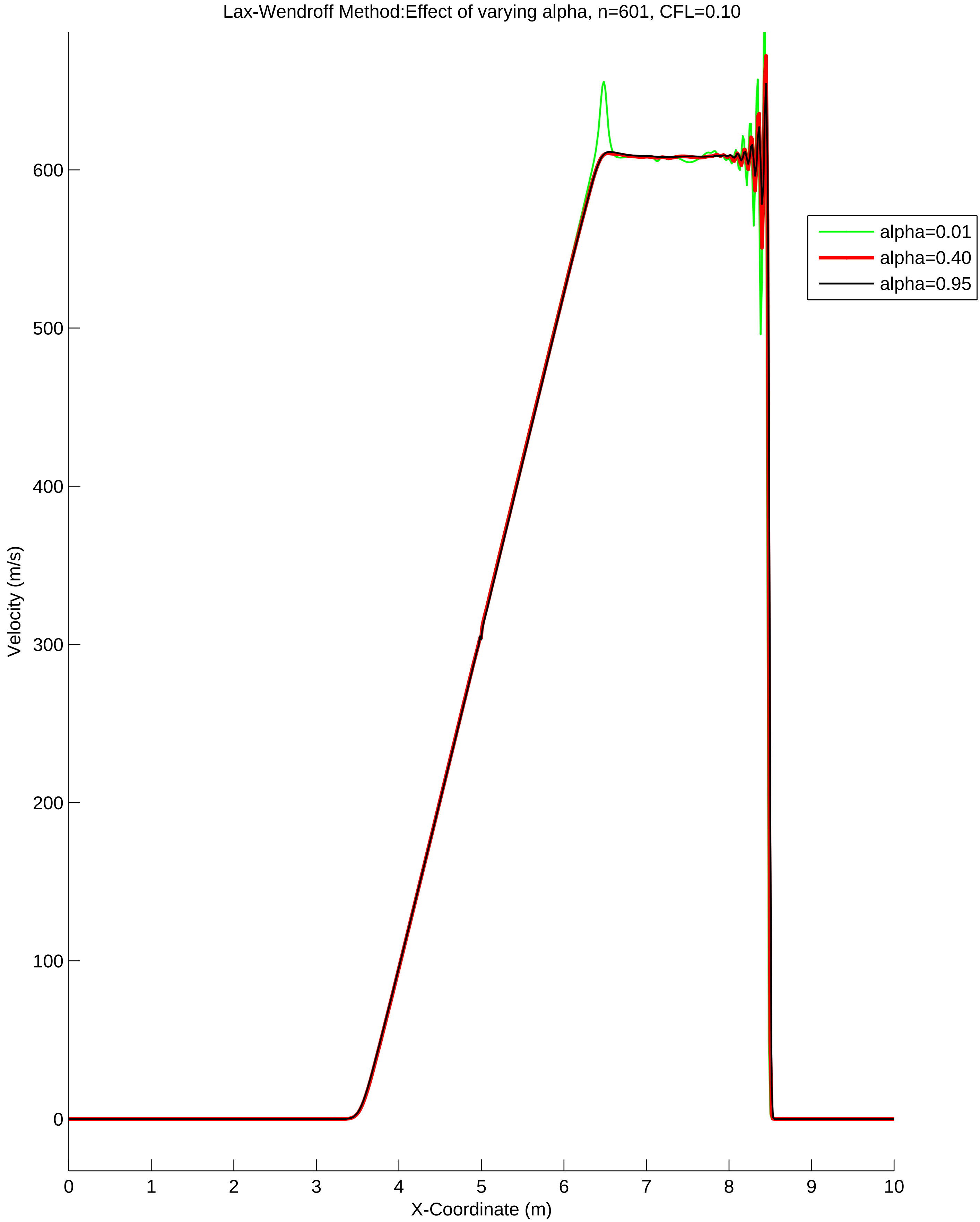


Lax Wendroff II Method: CFL=0.30, alpha=0.40, number of grid points=201
x 10^4
Pressure (Pa)
Entropy/R gas
Velocity (m/s)
Mach number
Density (kg/m^3)
Mass Flow (kg/m^2s)
X-Coordinate (m)



Lax-Wendroff Method:Effect of varying CFL number (or dt), n=601, alpha=0.4
Velocity (m/s)
X-Coordinate (m)
CFL=0.01
CFL=0.10
CFL=0.30
0
100
200
300
400
500
600
700
0
1
2
3
4
5
6
7
8
9
10


Lax-Wendroff II method without artificial viscosity at n=601 and CFL=0.30
x 10^4
Pressure (Pa)
Entropy/R gas
Velocity (m/s)
Mach number
Density (kg/m^3)
Mass Flow (kg/m^2s)
X-Coordinate (m)
X-Coordinate (m)
X-Coordinate (m)
X-Coordinate (m)
X-Coordinate (m)
X-Coordinate (m)


Lax-Wendroff Method:Effect of varying alpha, n=601, CFL=0.10
Velocity (m/s)
X-Coordinate (m)
0
100
200
300
400
500
600
0
1
2
3
4
5
6
7
8
9
10
alpha=0.01
alpha=0.40
alpha=0.95