Homework 2, due Mon 13/2-2006

1.) Apply the MacCormack scheme

$$\begin{aligned} u_j^* &= u_j^n - \lambda(f(u_{j+1}^n) - f(u_j^n)) \quad \text{Predictor step} \\ u_j^{n+1} &= \frac{1}{2}(u_j^n + u_j^*) - \frac{\lambda}{2}(f(u_j^*) - f(u_{j-1}^*)) \quad \text{Corrector step} \end{aligned}$$

to

$$u_t + f(u)_x = 0 \quad f(u) = \frac{u^2}{2} \quad x \in [-1, 1] \tag{1}$$

$$u(0, x) = -\sin(\pi x). \tag{2}$$

Here, $\lambda = \Delta t/\Delta x$. Use periodic boundary conditions, i.e. $u(1, t) = u(-1, t)$.

Run the problem until at least t=0.4 s in order to obtain a shock. Look at the numerical solution and study the resolution of the shock and explain the overshoots (oscillations) close to the shock. Hand in a plot of the numerical solution.

In order to damp the overshoots we add artificial viscosity. Apply the MacCormack method to the modified problem

$$u_t + f(u)_x = \epsilon u_{xx} \tag{3}$$

or written on conservative form

$$u_t + \underbrace{[f(u) - \epsilon u_x]}_{\tilde{f}}{}_x = 0 \tag{4}$$

where ϵ is a small constant of order Δx and $\tilde{f}$ is a modified flux function.

The artificial viscosity term should be implemented in conservative form, as in equation (4). The u_x term in the modified flux function has to be evaluated numerically. When u^* is computed by the predictor step, u_x should be approximated by a backward difference and when u^{n+1} is computed by the corrector step, u_x should be approximated by a forward difference. This will lead to a correct implementation of the artificial viscosity term.

Run the problem again with artificial viscosity and comment on the resolution of the shock and the overshoots.

The total variation,

$$TV(u^n) = \sum_{j=-\infty}^{\infty} |\Delta_+ u_j^n|, \quad \Delta_+ u_j^n = u_{j+1}^n - u_j^n$$

is a quantity that is used as a measure for oscillations.

A difference method is called total variation decreasing, (TVD), if it produces a solution satisfying $TV(u^{n+1}) \leq TV(u^n), \ \ \forall n$. If the solution is TVD it will not

have any oscillations close to the shock. Examine numerically if the MacCormack scheme produces a solution where the total variation is non-increasing.

Hint: plot the total variation as function of time.

Also, try to determine how large ϵ you need in order to obtain a solution where the total variation is decreasing for all times.

2.) Run a shock-tube problem for the 1D barotropic gas dynamics equations, i.e the Euler equations and the assumption that the pressure is only a function of the density:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \quad p = K\rho^\gamma \end{aligned} \tag{5}$$

where K is a constant determined by the initial conditions and $\gamma = 1.4$.

A shock-tube is a tube, closed at both ends, with a diaphragm separating a region with high pressure gas and a region with low pressure gas.

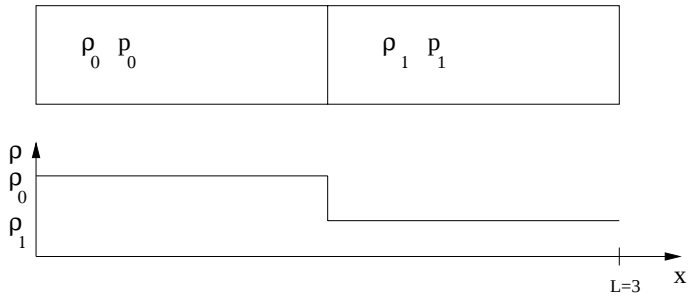

Figure 1: Initial conditions

Initial condition for the density, see figure 1, is

$$\rho(x,0) = \begin{cases} \rho_0 & \text{if } x \le L/2 \\ \rho_1 & \text{if } x > L/2 \end{cases} \tag{6}$$

and the velocity is $u(x,0) = 0$. $L = 3$ is the length of the tube.

As boundary conditions we use the fact the tube is closed, so $u(0,t) = u(0,L) = 0$ and the value of ρ at the boundaries is extrapolated from the value of the density inside the tube.

Your job is to complete a MATLAB code to solve the 1D barotropic gas dynamics equations, (5), numerically using the MacCormack scheme.

(The files can be downloaded from the course home page)

The following files need to be completed

The file `flux_function.m` defines the flux function. Here the flux function for the system of equations (5) must be coded.

`boundary_cond.m` sets the boundary conditions, i.e. $(\rho)_1, (\rho)_n, (\rho u)_1$ and $(\rho u)_n$, (n is the number of grid points). In this case the velocity is equal to zero, hence

$$(\rho u)_1 = (\rho u)_n = 0$$

and the density is extrapolated yielding

$$\rho_1 = \rho_2 \qquad \rho_{n-1} = \rho_n$$

Artificial viscosity

See also Anderson, chapter 6.6.

To damp oscillations in the solution artificial viscosity has to be added. Instead of (5), we solve

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x + (K\boldsymbol{U}_x)_x = 0$$

where $\boldsymbol{U} = (\rho, \rho u)$ and $\boldsymbol{F}(\boldsymbol{U}) = (\rho u, \rho u^2 + p)$. The term $(K\boldsymbol{U}_x)_x$ is the artificial viscosity model. Here we use a density switch model where the density is used to localize the shock.

$$K = -\Delta x V_{scal}(C_2 sw(\rho) + C_0)$$

where $\Delta x V_{scal}$ is a velocity scaling in order to obtain the right size of the viscous term. Δx is given by the mesh size and the parameter V_{scal} is related to the convective speed (characteristic speed) of the problem and is chosen as

$$V_{scal} = max(abs(u + c), abs(u - c))$$

where c is the speed of sound.

The density switch, $sw(\rho)$ is computed in the following way

$$sw(\rho) = |\frac{\partial^2 \rho}{\partial x^2}|/\bar{\rho}$$

The second order derivative of ρ is approximated by a second order central difference and $\bar{\rho}$ is a mean value of ρ computed in grid point j as $\bar{\rho}_j = \rho_{j+1} + 2\rho_j + \rho_{j-1}$.

When there are large gradients in ρ, $sw(\rho)$ will be of order 1, and when ρ is smooth $sw(\rho)$ will be approximately zero. C_2 is a parameter and should be chosen to obtain sufficiently much viscosity to damp the oscillations. C_0 is a "background" diffusion parameter and should also be chosen. Both C_0 and C_2 should be of order 1 or less. The optimal values of C_0 and C_2 is usually determined after some experimentation with different values, see iii) below.

The artificial viscosity model is implemented in the file `artificial_visc.m`.

i) The CFL stability condition is guaranteed by taking $\Delta t = CN\Delta x/u_{max}$ with CN (the Courant number) < 1. Here, u_{max} is the maximal absolute value of the characteristic speeds. From the quasi-linear form of the equations,

$$\begin{pmatrix}\rho\\ u\end{pmatrix}_t + \begin{pmatrix}u & \rho\\ K\gamma\rho^{\gamma-2} & u\end{pmatrix}\begin{pmatrix}\rho\\ u\end{pmatrix}_x = 0 \tag{7}$$

show that the characteristic speeds are $u \pm c$ with $c^2 = K\gamma\rho^{\gamma-1} = \gamma p/\rho$ where c is the speed of sound. Use this to set the time step in the main program `shocktube.m`.

ii) Run the problem as set up. The initial jump breaks up into a rarefaction wave moving left and a shock moving right. Measure the shock speed. Check that it is correct by computing the shock speed s from the jump relation (Rankine-Hugoniot condition)

$$s(\rho_l - \rho_r) = \rho_l u_l - \rho_r u_r$$

Here, ρ_l and u_l are the computed left hand states. The right hand ones, ρ_r and u_r are given by the initial conditions.

iii) Run the code with different Courant numbers and different values of the artificial viscosity parameters C_2 and C_0 and comment on the solution. Try to find the optimal (no oscillations and a good resolution of the shock) choice. (Note that C_0 should be smaller than C_2.)