Lecture 2: The Relation of Mathematics and Physics

[BBC TV Film Leader]
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The Relation of Mathematics and Physics

Hi. I can see the audience tonight, so I can see also from the size of it (which last time was a black splot in front of my eyes), that there must be many of you here who are not thoroughly familiar with physics, and also a number that are not too versed in mathematics—and I don't doubt that there are some who know neither physics nor mathematics very well. That puts a considerable challenge on a speaker who is going to speak on the relation of physics and mathematics—a challenge which I, however, will not accept: I published the title of the talk in clear and precise language, and didn't make it sound like it was something it wasn't—it's the relation of physics and mathematics—and if you find that in some spots it assumes some minor knowledge of physics or mathematics, I cannot help it. It was named so.

In thinking of the applications of mathematics to physics, it's perfectly natural that mathematics will be useful when large numbers are involved in complex situations. Although, for example, if we took biology, the action of a virus on a bacterium is, if you watch it under the microscope, unmathematical: a jiggling virus finds some spot on this odd—shape bacterium—they're all different shapes—and finds some spot; maybe it pushes its DNA in, and maybe it doesn't, and so forth. And yet, if we do the experiment with millions and billions of bacteria and viruses, then we can learn a great deal about the virus—it's like taking averages and working with large numbers: we can use the mathematics involved in the averaging; we can see whether the viruses develop in the bacteria some new strains, and what percentage, so we can study the genetics, the mutation, and so forth.

To take another—more trivial—example, imagine an enormous board—a checkerboard—to play checkers (or draughts): if the board is very large, the actual operation of any one step is not mathematical—or very simple, if it is mathematical: it either goes one side or the other on the diagonal, or it reaches [the end] and becomes a king, and can go backwards when it reaches the end. In other words, the statement of the rules are very simple, and do not really involve any mathematics. But you could imagine that on an enormous board with lots and lots of pieces, some analysis of the best move—or good moves or bad moves—might be made by a deep kind of reasoning, which would involve somebody having gone off first and thought about it in great depth-and that becomes mathematics, this abstract reasoning.

Another example is switching, in computers. If you have one switch, it's either on or off, and there's nothing very mathematical about that (although mathematicians like to start there with their mathematics). But with all the interconnections and wires to figure out what a very large system will do, one requires a mathematics.

I would like immediately to say that a mathematics has its primary application—it has a tremendous application—in physics in the discussion of the detailed phenomena in complicated situations, granting the fundamental rules of the game. And that is something which, if I were talking only about the relation of mathematics and physics, I would spend most of my time discussing. But since this is part of a series of lectures on the character of physical law, I do not have the time to discuss the applications of mathematics and physics to calculating what happens in complicated situations. But we'll go immediately to another question, which is the character of the fundamental law.

If we go back to our checker game, the fundamental laws are the rules by which the checkers move. The mathematics may be applied in a complex situation—to figure out what happens in the next move, what's a good move to make in a complicated situation—but very little mathematics is needed in the fundamental, simple character of the basic laws.

Now, the strange thing about physics is, that for the fundamental laws, we still need mathematics. For example-well, I give two examples: one in which we really do not, and one in which we do.

There's a law in physics called Faraday's Law, which says that in electrolysis the amount of material which is deposited is proportional to the current, and to the time that the current is acting—and that means that the amount of material is proportional to the charge which goes through the system. Well, it sounds very mathematical. But what's actually happening is that electrons going through the wire each carry one charge. To take an example, a particular example, it may be that to deposit one atom requires one electron to come, and so the number of atoms that are deposited are necessarily proportional to the number of electrons that flow, and thus to the charge that goes through the wire. So the mathematically-appearing law has as its basis nothing very deep, requiring no real knowledge of mathematics, that one electron is needed for each atom in order for it to deposit itself. That's not a deep-that's mathematics, if I had to say, number one, but it's not the kind of mathematics I'm talking about tonight.

Now, if we take, on the other hand, Newton's law for gravitation (which has these aspects which I wrote down last time just to impress you with the speed at which mathematical symbols can convey—can carry—information), we said that the force was proportional to the product of the masses of two objects, and inversely as the square of the distance between them, and that bodies react to forces by changing their speeds—or changing their motions—in a direction of the force by amounts proportional to the force, and inversely proportional to their masses. Now that's words all right, and I didn't have to write the equation, but nevertheless it's kind of mathematical, and we would wonder: how can this be a fundamental law? How can this planet out there look—what does it do? It looks at the sun, and it sees how far away it is, and it decides to calculate on its internal adding machine the inverse of the square of the distance-and that tells it how much to move? This is certainly no explanation of the machinery of gravitation! So you might want to look further, and various people have tried to look further: Newton was originally asked, "It doesn't mean anything; it doesn't tell us anything!" He says, "It tells you how it moves. It should be enough—I told you how it moves, not why."

But people are often unsatisfied without a mechanism, and I would like to describe one theory which has been invented (among others) of the type that you might want, that this is a result of large numbers—and that's why it's mathematical. And I give this theory—perhaps you've thought of it yourself; every once in a while some student comes running in; he suddenly explains gravitation.

Suppose that in the world, everywhere, there are, flying through us at very high speed, a lot of particles. They come equally in all directions; they're just shooting by, shooting by, shooting by-and once in a while hit us in a bombardment. We and the sun are practically transparent for them—nearly—but some hit, and so it's not completely transparent. And look what would happen: if the sun is here and the earth is here, then if the sun weren't here, there would be particles bombarding from all sides, giving little impulses by the rattle of these—bang, bang, the few that hit—which would not shake the earth in any particular direction, because there's as many coming from one side as the other, from top, from bottom.

However, when the sun is here, the particles which are coming in this direction are partly absorbed by the sun, because some of them hit the sun and don't go through. Therefore, the number that are coming from this direction toward the earth is less than the number that are coming from the other side, because here they have no opposition from the sun there.

And it's easy to see (after some mental effort) that the further the sun is away, the less (in proportion of all of the particles) are being taken out of the possible directions in which particles can come—I mean, the solar size appears smaller—in fact inversely as the square of the distance. So there will, therefore, be an impulse toward the sun on the earth that's inversely as the square of the distance, and is a result of large numbers of very simple operations; just hit one after the other from all directions.

And, therefore, the strangeness of the mathematical relation would be very much reduced, because the fundamental operation is much simpler than calculating the inverse of the square of the distance: this "machine" does the calculating—these particles bouncing.

The only trouble with it is that it doesn't work—for other reasons: every theory that you make up has to be analyzed against all the possible consequences, to see if it predicts anything else—and this predicts something else. If the earth is moving this way, more particles would hit it from the front than from the back. If you're running in the rain, more rain hits you in the front of the face than in the back of the head, because you're running into the rain. And so, as the earth is moving in this direction, it's running into the particles, rather, and running away from the ones that are chasing it from behind, so that more particles hit it from the front than from the back. And there would be a force also sideways whenever there was any motion [that way]. This sideways force would slow the earth up in the orbit and certainly would not have lasted the three or four billion years that it has been going around the sun. So that's the end of that theory.

Well, you say, that was a good one, though: it got rid of the mathematics for a while; maybe I can invent a better one. And maybe you can, because nobody knows the ultimate. But up to today, from the time of Newton, no one has invented another theoretical description of the mathematical machinery behind this law which does anything else but say the same thing over again, or make the mathematics harder, and at the same time doesn't produce some wrong phenomenon—I mean like this model does it—but it produces something which isn't true.

So there is no model of the theory of gravitation today other than the mathematical form. If this were the only law of this character, it would be interesting and rather annoying. But what turns out to be true is that the more we investigate, and the more laws we find, and the deeper we penetrate nature, this disease—that every one of our laws is a purely mathematical statement in rather complex and abstruse mathematics (this is relatively simple mathematics)—gets more and more abstruse, and more and more difficult, as mathematics.

Why? I haven't the slightest idea why. It is only my purpose in this lecture to tell you about this fact. In other words, it's my purpose in this lecture to explain, really, why I cannot satisfy you, if you do not understand mathematics too well, in trying to explain nature in any other way. It is the burden of this lecture, in fact, to just tell you the fact that it is impossible to answer—really, honestly—the challenge of explaining, in a way that a person can feel, the beauties of the laws of nature without their having some deep understanding of mathematics. I'm sorry, but it seems to be the case.

You might say, "All right, then, there's no explanation of the law; at least tell me what the law is—why not tell me in words instead of in the symbols? Mathematics is just a language, and I want to be able to translate the language." And, in fact, I can—and with patience, I think I partly did. I could go a little further and explain more detail—that this means if it's twice as far away the force is one—fourth as much, and so on—and can convert all these into words. I would be, in other words, kind to the layman, as they all sit, hopeful that you will explain something. Various different people get different reputations for their skill at explaining to the layman in layman's language these difficult and abstruse subjects.

The layman then searches for book after book with the hope that he will avoid the complexity which ultimately sets in, even by the best expositor of this type. He reads the things, hoping—he finds, as he reads, a generally increased confusion, one complicated statement after the other, one difficult—to—understand thing after the other, all apparently disconnected from one another—and it becomes a little obscure, and he hopes that maybe in some other book there's some explanation which avoids—I mean, the man almost made it, you see—maybe another fellow makes it right.

I don't think it's possible, because there's another feature: mathematics is not just a language; mathematics is a language plus reasoning; it's like a language plus logic. Mathematics is a tool for reasoning. It's in fact a big collection of the results of some person's careful thought and reasoning. By mathematics it is possible to connect one statement to another. For instance, I can say that the force is directed toward the sun. I can also tell you, as I did before, that the planet moves—so if I draw a line from the sun to the planet, and draw another one at some definite period like three weeks later, the area that is swung out by the planet is exactly the same as it will be the next three weeks, and the next three weeks, and so on, as it goes around the sun.

Now, I could explain both of those statements to you carefully, but I cannot explain why they're both the same—so that if you don't appreciate the mathematics, you cannot see that the great variety of facts, the enormous apparent complexity of nature—with all its funny laws and rules, each of which have been carefully explained to you—are really very closely interwoven, that logic permits you to go from one to the other.

It may be unbelievable that I can demonstrate that equal times will be swept out if the forces are directed toward the sun. If I may try, I will show you one demonstration, to show you that those two things really are equivalent, so that you can appreciate that there's more to merely the statement of the two laws; that the two laws are connected such that reasoning alone will bring you from one to the other; that mathematics is this organized reasoning, and that it's good to know how to do that-so you will appreciate the beauty of the relationship of the statement. So I'm going to prove, if I may, the relationship that if the forces are directed toward the sun, that the equal areas are swept out in equal time.

So, we start. Here's the sun, and we imagine at a certain time, let's say the planet is here, and is moving in such a way that let's say one second later—or one hour, pick any time, say one second later—it's moved in such a manner that it has gotten to the position 2. Now, if the sun did not exert any force on it, then by Galileo's principle, it would keep right on going in a straight line. So in the same interval of time—later, the next second—it'll move exactly the same distance in the same straight line to the position 3, were there no force. All right. Now first we're going to show that if there's no force, equal areas are swept out in equal time.

I remind you that the area of a triangle is half the base times the altitude, and that the altitude is the vertical distance to the base. And that if the triangle is sort of cockeyed (there's a name for it which I forget—obtuse! obtuse!), then the altitude is this vertical height here. (I know about the triangles; I just don't know their names.) Now, let us draw the lines to these two points in the case that there was no motion whatsoever.

The question is-it doesn't draw very well; I'm not accurate, but these two distances are equal, remember. The question is: are these two areas equal? Well, consider the triangle made from the sun and the two points 1 and 2; it's this one. What's its area? It's this base multiplied by this height. And what about the other triangle, which is the triangle in motion from 2 to 3? It's this base times the same altitude: the two triangles have the same altitude, and as I indicated, the same base and have the same area. So, so far so good. If there were no force from the sun, equal areas would be swept out in equal times; the two triangles have equal areas. But there is a force from the sun, and during this interval from 1 to 2 to 3 the sun is pulling and changing the motion in various directions—this way, this way, that way.

To get a good approximation to that, we'll take the central position, or average position, here, and say that the whole effect during this interval was to change the motion by some amount in this direction, toward the sun. That means that although the particles were moving this way, and would have moved this way in the next second; because of the influence of the sun the motion is altered by an amount that's poking in this direction, that's parallel to this, exactly parallel—these lines are parallel. It's the direction in which this new motion—there's a new motion, a compound of what I wanted to do and the change that's been induced by the action of the sun. So it doesn't really end up at position 3, but rather at position 4.

So now we would like to compare (it's getting complicated in the diagram) the triangle 2—4—S and 2-3-S-I show you that those are equal. Because they have the same base [2—S] those two triangles—this one here, and the one that happened when we had no force; the one with force, and with no force have the same base. And do they have the same altitude? Sure—because they're included between parallel lines [2—S and 3—4], and so they have the same altitude. And thus the area of the last triangle I drew [S—2—4] is the same as the second one I drew, this [S]—2—3. And that, I had proved earlier, was the same as the first one [S—1—2]. So in the actual orbital motion of the planet, the areas—you have the first in the first second, and in the second second—are equal.

So by reasoning we can see a connection between the the fact that that the force is toward the sun, and that the areas are equal. Ingenious, no? I borrowed this from Newton: it comes right out of the Principia, diagram and all. The letters are different, that's all—because he wrote in Latin. (These are Arabic numerals.)

Incidentally, Newton made his proof geometrical like this, and made all his proofs in his book geometrical, of this type. Today, we don't use that kind of reasoning; we use a kind of analytic reasoning with symbols. This [geometrical] kind of reasoning requires an ingenuity—to draw the right triangles (the correct triangles, I mean) to notice about the areas—and to figure out how to do it, you have to be clever. But there have been improvements in the methods of analysis, so that one can be quite more stupid—and I write much faster, and much more efficient, then. I want only to show what it looks like in

the notation of the more modern mathematics, where you don’t do anything but write a lot of symbols to figure it out.
First, we would like to talk about how fast the area changes, and we represent that by "area dot". And the area changes, because when the radius is swinging, it’s the component of velocity at right angles to the radius, times the radius, that tells how fast the area changes. So this is the component of the radial distance, multiplied by the velocity, or rate of change of the distance.
Now, the question is, whether the rate of change of area itself changes. The principle, is it’s not supposed to change; the rate of change of area is not supposed to change. So we differentiate (so called) this again; it means some little trick about putting dots in the right place. And that’s all; you have to learn the tricks—it’s just a series of rules that people have found out that are very powerful for such a thing. And this says, the component of the velocity at right angles to the velocity: it is none-there is none; the velocity is in the same direction as itself. And the acceleration, which is this thing, the second derivative, or the derivative of velocity, is the force divided by the mass.
So this says that the rate of change of the rate of change of the area is the component of force at right angles to the radius. But if the force is in a direction of the radius, as Newton said, then there’s no force at right angles to the radius and that means that the rate of change of area doesn’t change.
I just wanted to illustrate the different kinds of notation.
Now, Newton knew how to do this, more or less, with slightly different notation, but he wrote everything this way because he tried to make it possible for people to read his papers. He invented the calculus, which is this kind of mathematics, and is a good illustration of the relation of mathematics to physics.
When the problems in physics get difficult, we may often look to the mathematicians who have already studied such a thing, and have reasoned about such an item before, and have prepared a line of reasoning for us to follow. On the other hand, they may not have—in which case we have to invent our own line of reasoning, which will then pass back to the mathematicians, because everybody who reasons carefully about anything is making a contribution to the knowledge of what happens when you think about something. And if you abstract it away, and send it to the department of mathematics, they put it in the books as a branch of mathematics.
Mathematics, then, is a way of going from one set of statements to another. It’s evidently useful in physics, because we have all these different ways that we could speak of things, and it permits us to develop consequences, and analyze situations, and change the laws in different ways, and to connect all the various statements. So that, as a matter of fact, the total amount that a physicist knows is very little: he has only to remember the rules for getting from one place to another, and he’s able to do it then.
In other words, all of the various statements—about equal times, the forces in a direction of the radius, and so on—are all interconnected by reasoning. Now, an

interesting question comes up: is there some pattern to it? Is there a place to begin—fundamental principles—and deduce the whole works? Or, is there some particular pattern, or order in nature, in which we can understand that these are more fundamental statements, and these are more consequential statements?
There are two kinds of ways of looking at mathematics, which for the purpose of this lecture I will call the Babylonian tradition, and the Greek tradition. In Babylonian schools in mathematics, the student would learn something by doing a large number of examples, until he caught onto the general rule. Also, a large amount of geometry was known—a lot of properties of circles, Theorem of Pythagoras, formulas for the areas of cubes and triangles and everything else—and some degree of argument was available to go from one thing to another. Tables of numerical quantities were available so that you could solve elaborate equations and so on-everything was prepared for calculating things out. But Euclid discovered that there was a way in which all of the theorems of geometry could be ordered from a set of axioms that were particularly simple—and you're all familiar with that much geometry, I'm sure.
But the Babylonian attitude was—if I make my way of talking what I call Babylonian mathematics—is that you know all these various theorems and many of the connections in between, but you've never really realized that it could all come up from a bunch of axioms.
Modern mathematics, the most modern mathematics, concentrates on axioms and demonstrations within a very definite framework of conventions of what's acceptable and not acceptable as axioms. For example, in geometry, it takes something like Euclid's axioms (modified to be made more perfect), and then to show the deduction of the system. For instance, it would not be expected that a theorem like Pythagoras'—that the sum of the squares (of the areas of squares) put on the sides of the triangle will equal the area of a square on a hypotenuse—should be an axiom.
On the other hand, from another point of view of geometry—that of Descartes-the Pythagorean Theorem is an axiom. So the first thing we have to worry about is that, even in mathematics, you can start in different places. Because all these various theorems are interconnected by reasoning, there isn't any real way to say, "Well, these on the bottom, here, are the bottom, and these are connected through logic." Because if you were told this one instead, or this one, you could also run the logic the other way, if you weren't told that one, and work out that one. It's like a bridge with lots of members, and it's overconnected: if pieces have dropped out, you can reconnect it another way. The mathematical tradition of today is to start with some particular ones, which are chosen by some kind of convention to be axioms, and then to build up the structure from there. The "Babylonian" thing that I'm talking about (which I know is really not Babylonian) is to say, "Well, I happen to know this, and I happen to know that, and maybe I know that, and I work everything out from there. And then tomorrow I forgot that this was true, but I remembered that this was true, and then I reconstruct it again,

and so on—I'm never quite sure of where I'm supposed to begin, and where I'm supposed to end; I just remember enough all the time so that as the memory fades, and the pieces fall out, I put this thing back together again every day.

The method of starting from the axioms is not efficient in obtaining the theorems; in working something out in geometry, you're not very efficient if each time you have to start back at the axioms. But if you have to remember a few things in the geometry, you can always get somewhere else; it's much more efficient to do it the other way: what the best axioms are, are not exactly the same—in fact, are not ever the same—as the most efficient way of getting around in the territory. In physics, we need the Babylonian method, and not the Euclidean or Greek method, and I would like to say why.

The problem in the Euclidean method is to make something about the axioms a little bit more interesting or important. But the question that we have is, in the case of gravitation, is it more important—is it more basic, is it more fundamental, is it a better axiom—to say that the force is directed toward the sun, or to say that equal areas are swept in equal time?

Well, from one point of view, the force is better, because once I state what the forces are, I can deal with a system with many particles, in which the orbits are no longer ellipses—because of the pull of one on the other-and the theorem about equal areas fails. Therefore, I think that the force law ought to be an axiom, instead of the other.

On the other hand, the principle that equal times is swept out in equal—equal areas are swept out in equal times—can be generalized, when there's a system of a large number of particles, to another theorem (which I had prepared to explain, but I see I'm running out of time): there's another statement, which is a little more general than equal areas in equal time.

Well, I have to state what it is. It's rather complicated to say, and it's not quite as pretty as this one, but it's obviously the son of this one—I mean it's the offspring. If you look at all these particles—Jupiter, Saturn, the sun, and all these things going around; lots of stars, or whatever they are, all interacting with each other—and look at it from far away, and project it on a plane like this picture, then everything is moving—this moving this way, and that moving that way, and so on.

Then take any point at all—say this point—and then calculate how much each one is changing its area-how much area is being swept out by the radius to every particle, and add them all together. But wait: those masses which are heavier—this is twice as heavy as this one; then this area counts twice as much. So you count each of the areas that are being swept out in proportion to the mass that's doing the sweeping, and the total of all of that is not changing in time-that's a generalization, obviously, of the other one.

Incidentally, the total of that is called the angular momentum, and this is called the law of conservation of angular momentum. ("Conservation" just means that it doesn't change.)

Now, one of the consequences of this is—just to show what it's good for—imagine a lot of stars falling together to form a nebula or a galaxy. As they come closer in—if they were very far out, and moving slowly so there was a little bit of area being generated but on very long arms (distances from the center), then if the thing falls in, the distances to the center are shorter now (if all the stars are now close in), then these radii are smaller. And in order to sweep out the same area, they have to go a lot faster. So as the things come in, they swing—swirl around—and thus we can roughly understand the qualitative shape of the spiral nebulae.

You can also understand, in the same way—exactly the same way—the way a skater spins: when you start with a leg out, moving slowly, and as you pull the leg in, it spins faster (because when the leg is out, it's contributing, when it's moving slowly, a certain amount of area per second), and then, when it comes in, to get the same area you have to go around faster. But I didn't prove it for the skater: the skater uses muscle force; gravity is a different force—yet it's proof for the skater.

Now we have a problem: we can deduce often from one part of physics, like the law of gravitation, a principle which turns out to be much more valid than the derivation! This doesn't happen in mathematics—that theorems come out in places where they're not supposed to be. In other words, if we were to say that the postulates of physics were this law of gravitation, we could deduce the conservation of angular momentum, but only for gravitation. But we discover experimentally that the conservation of angular momentum is a much wider thing.

Now, Newton had other postulates by which he could get the more general conservation law of angular momentum, but Newtonian laws were wrong: there's no forces; it's all a lot of baloney; the particles don't have orbits, and so on. Yet, the analog-the exact transformation of this principle about the area as the conservation of angular momentum-is true for the atomic motions in quantum mechanics-and is still, as far as we can tell today, exact.

So we have these wide principles which sweep across all the different laws, and if one takes too seriously his derivations, and feels that this is only valid because this is valid, you cannot understand the interconnections of the different branches of physics.

Someday, when physics is complete, then maybe with this kind of argument we know all the laws, then we could start with some axioms—and no doubt somebody will figure out a particular way of doing it, and then all of the deductions will be made. But while we don't know all the laws, we can use some to make guesses at theorems, which extend beyond the proof.

So in order to understand the physics, one must always have a neat balance and contain in his head all of the various propositions and their interrelationships, because the laws often extend beyond the range of their deductions. This will only have no importance when all the laws are known.

Another thing that's interesting in the relation of mathematics to physics is this—a very strange thing: that by mathematical arguments you can show that you can start from very many different apparent starting points, and come to the same thing. That's pretty clear, if you have axioms; you can use some of the theorems. But actually, the physical laws are so delicately constructed that the statements of them have such qualitatively different characters, that it's very interesting.

So, if you'll permit me, I'm going to state the law of gravitation in three different ways—all of which are exactly equivalent, it turns out—but they sound completely different.

One: there's the forces between the objects as described before, and each object, when it sees the force on it, accelerates—or changes its motion, rather—at a certain amount per second, as I've described before—the regular way; I call it Newton's Law.

Now, there's a completely different way: that law says that the force depends on something at a finite distance away. See, it has what we call an "unlocal" quality: the force on this depends on where that one is over there. Now, you may not like the idea of action at a distance—that it can "know" what's going on over there-so then there's another way of stating the laws, which are very strange, and it's called the field way of representing the laws—and it's so very hard to explain, but I just want to give you some rough idea of what it's like.

And it says a different thing, a completely different thing: that there's a number at every point in space—I know it's a number; it's not a mechanism; it's the trouble with this whole physics, that it must be mathematical. There's a number at every point in space—here's a number, here's a number, and so on. And the number is changing—it changes, rather—when you go from place to place. If an object is placed at one of these points somewhere in space, the force on it is in a direction in which that number—I'll call it the name it's given, called a potential—is in the direction in which that potential changes as quick as it can. And the force is proportional to how fast it changes as you move. That's one statement; that's not enough, yet, because I have to tell you, now, how to determine how the potential varies. I could say the potential varies as 1 over the distance from each object, but that's back to the action—at—a—distance idea.

However—the force is at a distance, but—you can state the law in another way, and it says the following: you don't have to know what's going on anywhere outside of a little ball; if you want to know what the potential is here, you tell me what it is on the surface of any ball, no matter how small—you don't have to look outside, you just tell me what it is in the neighborhood—and how much mass there is in the ball.

The rule is this: that the potential at the center is equal to the average of the potential on the little ball surface, minus this constant (that's over there on the other equation), divided by twice the radius of the ball (let's suppose the radius of the ball is called a), and then multiplied by the mass that's inside the ball—if the ball is small enough.

Now you see that this law is different than the other one, because it only tells what happens at one point in terms of what happens very close by. Newton's laws tell what happens at one time in terms of what happens at another instant—it skips from instant to instant, how to work it out—but in space it leaps from place to place. But this thing is both local in time, and also local in space, because it depends only on what's in the neighborhood.

And [so] there's another way of representing—that's another way—a completely different way. See, there's a difference in the philosophy, in the qualitative ideas involved. You don't like action at a distance? You can get away without it.

Now I'll show you one which is philosophically the exact opposite, in which there's no discussion at all about how the thing works its way from place to place, in which the whole thing is an overall statement, and goes as follows: when you have other particles around, and you want to know how this one moves from one place to another, you do it as follows.

You calculate a certain quantity for—you invent a possible motion that gets from one given place to some other place that you're interested in, in a given amount of time. Say it wants to go from here to here in an hour, and you want to know by what route it can get from there to there in an hour—by what curve.

So what you do is, you calculate a quantity, guessing the curve; if you try this curve, you calculate a certain number for this quantity. (I don't want to just tell you what the quantity is, but for those who have heard of these terms, this quantity on this route is the average of the difference between the kinetic and potential energy.) Now, if you calculate this quantity for this route, then for another route, you'll get, of course, different numbers for the answer. But there's one route which gives the least possible number for that, and that's the route that the particle takes.

Now we're describing the actual motion—the ellipse—by saying something about the whole curve. We have lost the idea of causality—that the particle is here, it sees the pull, it moves to here. Instead of that, in some grand fashion, it "smells" all the curves around here—all the possibilities—and "decides" which one to take.

This is an example of the wide range of beautiful ways of describing nature. And that when people talk that nature must have causality—well, you could talk about it this way; nature must be stated in terms of a minimum principle—well, you can talk about it this way; nature must have a local field—so it can do that, and so on. And the question is, which one is right?

Now, if these various alternatives are mathematically not exactly equivalent, and if for certain ones there will be different consequences than for others, then that's perfectly all right, because we have only to do the experiments to find out which way nature actually chooses to do it. Mostly, people come along and they argue philosophically they like this one better than that one, but we have learned from much experience that all intuitions about what nature's going to do philosophically fail—it never works. One just has to work out all the possibilities, and just try all the alternatives. Now, in this particular case that I'm talking about here, these theories are exactly equivalent. The mathematical consequences in every one of the different formulations of the three formulations—Newton's laws, the local field method, and this minimum principle—give exactly the same consequences. What do we do then? You will read in all the books that we therefore cannot decide scientifically on one or the other. That's true.

They're not equivalent—they are equivalent, scientifically; it is impossible to make a decision, because there's no experimental way to distinguish if all the consequences are the same. Psychologically they're very different in two ways. First, philosophically, you like them or don't like them-training is the only thing you can do to beat that disease. Second, psychologically they're different because they're completely unequivalent when you go to guess at a new law. As long as physics is incomplete, and we're trying to find out the other laws, and to understand the other laws, then the different possible formulations give clues as to what might happen in another circumstance. And they become not equivalent in psychologically suggesting to us to guess as to what the laws might look like in a wider situation.

For instance, Einstein noticed that the law of gravity—he said that he realized that signals couldn't propagate faster than the speed of light for light, [or] for electricity. He guessed that it was a general principle—the same guessing game as taking this angular momentum, and extending it from one case where you proved it, to the rest of the universe. He guessed that it was true of everything, and he guessed that it would be true of gravitation. If the signals can't go any faster than the speed of light, it turns out that the method of describing the forces instantaneously is very poor. And in the Einstein generalization of gravitation, this method of describing physics is hopelessly inadequate and enormously complicated, whereas, this one is neat and simple—and so is this one—so we haven't decided between those two yet.

In fact, it turns out that the quantum mechanics says that, exactly as I stated them, neither is right—but the fact that a minimum principle exists turns out to be a consequence of the fact that, on a small scale, particles obey quantum mechanics. And the fact is, the best laws, as presently understood, are really a combination of the two, in which we use minimum principles plus local force—local laws. And present day believes that the laws of physics have to have the local character, and also the minimum principle—but we don't really know.

So, it's this way: if you have a structure that's only partly accurate, and something is going to fail, if you write it with just the right axioms, maybe only one axiom fails—and the rest remain; you [only have to] change one little thing. But if you write it with another set of axioms, they all collapse, because they all lean on that. But we can't tell ahead of time, without some intuition and guesswork, as to which is the best way to write it, until we find out the new situation. So we must, therefore, always keep all of the alternative

ways of looking at the thing in our heads; so the physicists do "Babylonian" mathematics, and pay little attention to the precise reasoning from fixed axioms.
One of the amazing characteristics of nature is this variety of interpretational schemes which is possible. It turns out that it’s only possible because the laws are just so, and special, and delicate. For instance, that the law is the inverse square is what permits it to become local-if it were the inverse cube, it couldn’t be done this way. That the other end of the equation—that the force is related to the rate of change of the velocity-that permits this kind of a way of writing the laws (the minimum principle), because if, for instance, the force were proportional to the rate of change of position instead of velocity, you couldn’t write it in that way. If you try to modify the laws much, you find you can only write them in very much fewer ways. I always find that mysterious—and I don’t understand the reason why it is—that the laws of physics always seem to be possible to be expressed in such a tremendous variety of ways-they seem to be able to get through several wickets at the same time.
Now, I would like to make a number of remarks on the relation of mathematics and physics, which are a little more general. The first is, that the mathematicians only are dealing with the structure of the reasoning, and they do not really care about what they’re talking. They don’t even need to know what they’re talking about—or, as they themselves say, whether what they say is true. Now, I explain that.
If you state the axioms—you say: such—and—such is so, and such—and—such is so, and such—and—such is so; what then?—then the logic can be carried out without knowing what the "such—and—such" words mean. That is, if the statements about the axioms are carefully formulated, and are complete enough, it is not necessary for the man who is doing the reasoning to have any knowledge of the meaning of these words. And he’ll be able to deduce, in the same language, new conclusions: if I use the word "triangle" in one of the axioms, there’d be some statement about "triangles" in the conclusion. Whereas, the man who’s doing the reasoning might not know what the "triangle" is. But then I can read his thing back and say, "Oh, a triangle—well, that’s just a three—sided what—have—you that’s a so—and—so"—and so I know this new fact. In other words, mathematicians prepare abstract reasoning that’s ready to be used if you will only have a set of axioms about the real world. But the physicist has meaning to all the phrases.
And there’s a very important thing that the people who—a lot of people who—study physics that come from mathematics don’t appreciate: that physics is not mathematics, and mathematics is not physics; one helps the other.
But [In physics] you have to have some understanding of the connection of the words with the real world: it’s necessary, at the end, to translate what you figured out into English-into the world, into the blocks of copper and glass that you’re going to do the experiments with—to find out whether the consequences are true. This is a problem which is not a problem with mathematics at all.

I’ve already mentioned the only other relationship that—of course, it’s obvious how the mathematical reasonings which have been developed are of great power and use for physicists. On the other hand, sometimes the physicist’s reasoning is useful for mathematicians. Mathematicians also like to make their reasoning as general as possible. If you say I have a three dimensional space—an ordinary space, I want to talk about ordinary space, you know, you’re in it, you measure distances and there are three numbers you need to tell where something is, you go breadth, width and height, three dimensional space—and you begin to ask him about theorems. Then they say, "Now look, if you had a space of n dimensions, then here are the theorems." "Well, yeah, but I only want the case 3." "Well, substitute n = 3"—and then it turns out that very many of the complicated theorems they have are much simpler, because it happens to be a special case.

Now, the physicist is always interested in the special case; he’s never interested in the general case. He’s talking about something; he’s not talking abstractly about anything; he knows what he’s talking about: he wants to discuss the gravity law; he doesn’t want the arbitrary force case; he wants the gravity law. And so there’s a certain amount of reducing because the mathematicians have prepared these things for a wide range of problems, which is very useful. And later on, it always turns out that the poor physicist has to come back and say, "Excuse me, when you wanted to tell me about the four dimensions..."

Now, another item that’s interesting in this relationship is the question of how to do new physics. Is it important to have a feeling, a kind of intuition—oh, I must mention one other item: when you know what it is you’re talking about—that these things are forces, and these are masses, and this is inertia, and this is so on—then you can use an awful lot of commonsense seat—of—the—pants feeling about the world: you’ve seen various things; you know more or less how the phenomenon is going to behave.

Well, the poor mathematician, he translates it into equations, and the symbols don’t mean anything to him, and he has no guide but precise mathematical rigor and care in the argument. Whereas the physicist, who knows more or less how the answer is going to come out, can sort of guess part way and go right along rather rapidly—the mathematical rigor of great precision is not very useful in the physics, nor is the modern attitude in mathematics to look at axioms.

Now, mathematicians can do what they want to do. One should not criticize them, because they are not slaves to physics. It is not necessary that, just because this would be useful to you, they have to do it that way; they can do what they will—it’s their own job—and if you want something else, then you work it out yourself.

The next point is the question of whether we should guess, when we try to get a new law, whether we should use a seat—of—the—pants feeling, and philosophical principle—I don’t like a minimum principle, or I do like a minimum principle; or I don’t like action at a distance, or I do like action—the question is, to what extent models help. It’s a very interesting thing: very often, models help—and most physics teachers try to teach how to use these models and get a good "physical feel" for how things are going to work out. But the greatest discoveries, it always turns out, abstract away from the model; it never did any good. Maxwell's discovery of electrodynamics was first made with a lot of imaginary wheels and idlers and everything else in space; if you got rid of all the idlers and everything else in space, the thing was okay. Dirac discovered the correct laws of quantum mechanics—for relativity quantum mechanics—simply by guessing the equation. The method of guessing the equation seems to be a pretty effective way of guessing new laws. This shows, again, that mathematics is a deep way of expressing nature, and attempts to express nature in philosophical principles, or in seat-of-the-pants mechanical feelings, is not an efficient way.

I must say, that it is possible—and I've often made the hypothesis—that physics ultimately will not require a mathematical statement, that the machinery ultimately will be revealed—it's just a prejudice, like one of these other prejudices. It always bothers me that, in spite of all this "local" business, what goes on—in no—matter—how—tiny a region of space, and no—matter—how—tiny a region of time, according to the laws as we understand them today—takes a computing machine an infinite number of logical operations to figure out. Now, how could all that be going on in that tiny space? Why should it take an infinite amount of logic to figure out what one stinky, tiny bit of space-time is going to do? So I made the hypothesis often that the laws are going to turn out to be, in the end, simple like the checkerboard, and that all the complexity is from size—but that is of the same nature as the other speculations that other people make that say, "I like it; you don't like it"-it's not good to be too prejudiced about the thing.

To summarize, I would use the words of J. H. Jeans, who said that the great architect seems to be a mathematician—and for you who don't know mathematics, it's really quite difficult to get a real feeling across as to the beauty, the deepest beauty, of nature. C.P. Snow talked about two cultures. I really think that those two cultures are people who do, and people who do not—people who have had, and people who have not had this experience of understanding mathematics well enough to appreciate nature once. It's too bad that it has to be mathematics, and that mathematics for some people is hard. When one of the [kings] (it's reputed, I don't know if it's true, that when one of the kings) was trying to learn geometry from Euclid, he complained that it was difficult. And Euclid said, "There's no royal road to geometry." There's no royal road. It's not the job-we cannot, as people who look at this thing, the physicists, cannot convert this thing to any other language.

You have-if you want to discuss nature, to learn about nature, and to appreciate nature, it's necessary-to find out the language that she speaks in. She offers her information only in one form. We are not so unhumble as to actually demand that she change before we pay any attention. It seems to me that it's like: all the intellectual arguments that you can make would not in any way—or very, very little—communicate to deaf ears what the experience of music really is; all the intellectual arguments in the world will not convince those of "the other culture"—the philosophers who try to teach you by telling you qualitatively about this thing. [People like] me, who's trying to describe it to you (but is not getting it across, because it's impossible), we're talking to deaf ears. It's perhaps that their horizons are [so] limited, which permit such people to imagine that the center of the universe of interest is man.

Thank you.