M840 Dissertation in mathematics

The Sound of Eigenvalues

R. May

X8477943

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Abstract

This dissertation investigates the eigenvalues for different domains of the problem $\Delta u + \lambda u = 0$. It begins with a brief historical overview and methods for calculating simple models such as a string, rectangular and circular domains, before reaching the main topic of modelling domains using the methods of calculus of variations, along with the Rayleigh quotient, the Rayleigh-Ritz method and the maximum-minimum principles.

These methods and proofs where applied to model the asymptotic behaviour for arbitrary two-dimensional domains for which calculations of eigenvalues too complex are investigated, giving a firm foundation to discuss the paper "Can One Hear the Shape of a Drum" by Mark Kac. A brief look at work conducted after this famous question in which was answered by Gordon, Webb and Wolpert where it was found that generally you cannot but for more specific domains such as triangles and isothermal drums the answer Kac's question is answered in the affirmative.

Contents

Chapter 1

Introduction

A plucked string will give a particular sound composing of many periodic waves of differing frequencies, known as "pure-tones". Disregarding the material and thickness of the strings, these "pure tones", more commonly known as eigenvalues, are dependent solely on the length of the strings. If the eigenvalues can be easily calculated by knowing the string length, does the reverse hold true? Is it possible to calculate the length just by knowing the eigenvalues? For a simple string the answer is yes, but what about more complicated instruments such as drums? This dissertation will investigate how the shape of a two-dimensional domain and its boundary can affect its eigenvalues, and whether you can assert the shape of the domain just from knowing these eigenvalues.

The reader will be given a brief history of Calculus of Variations, before finding the eigenvalues and respective eigenfunctions for a one-dimensional, and two-dimensional domains. It quickly becomes apparent that for any domain other than simple shapes, finding the eigenvalues other than the first few is very difficult. Another approach sees the problem as a variational one for which it can be minimised by using certain constraints. This will lead to the Rayleigh quotient, Rayleigh-Ritz method and the maximum-minimum principle which improves on these methods by allowing any eigenvalue to be calculated without the need of knowing the previous eigenfunctions, and thus reducing the complexity of calculations. In theory, these methods can evaluate any eigenvalue and respective eigenfunction for any domain, but in practice finding a suitable equation for the domain and its constraints is difficult, and the calculations too complicated for anything other than simple shapes. Although it may not be possible to find all the eigenvalues for many cases, the behaviour of a vibrating system is of importance to many fields. This leads to understanding the asymptotic distribution and domain decomposition, along with proofs that the eigenvalues are positively unbounded and that the set of eigenfunctions are complete. This result allows a method for finding asymptotic distribution of eigenvalues, first by modelling the domain as a simple rectangle, then a finite number of square domains before modelling an arbitrary domain with a continuous smooth boundary.

The last section discusses Mark Kac's paper which came to the conclusion that the eigenvalues depend upon the area and perimeter of the domain by using the theory of diffusion, leading to the question and the title of the paper "Can one hear the shape of a drum?". A brief outline of work done since Kac's paper namely Gordon, Webb and Wolpert [3], Grieser and Maronna [17], and Antunes and Freitas [18] completes the dissertation by answering Kac's question for different domains.

Chapter 2

History of Calculus of Variations

The field of Calculus of Variations has vast applications through mathematics, from topology to partial differential equation and in other sciences, from quantum fields to spacecraft landings. It help solve any problem that needs an optimal solution as long as the said problem can be mathematically modelled. Probably the first consideration to the field was from Hero of Alexandria who studied optics of reflection around 150BC-300AD and hypothesised that light reflected always travelled in a path that minimises its time [28]. It was not until 1662 that Pierre de Fermat solved this conundrum with his "Principle of least time" [28].

Newton and Leibniz The concept of what we now know as calculus was created in the late 17th century by Sir Isaac Newton and Gottfried Wilhelm Leibniz. Originally known as the calculus of infinitesimals, it has two major branches, differential and integral calculus. These were based on the ideas of working with very small quantities, which today we denote by ϵ or δ. The concept of infinitesimals and was so advanced in terms of mathematical thinking, it was the subject of political and religious controversies in Europe [29]. By the late 17th century Newton's notion of Fluxions, that is the instant rate of change, and Leibniz' differential, Δx, were used by mathematicians and engineers alike.

While both Newton and Leibniz where involved in the process of creating a mathematical system to deal with variable quantities, their elementary base was differed. For Newton, change was a variable quantity over time and for Leibniz, it was the difference ranging over a sequence of infinitely close values. Notably, the descriptive terms each system created to describe change was also different [30].

Leibniz employed integral calculus for the first time in history when he found the area under a graph for $y = f(x)$, introducing the integral sign $\int$ to denote the Latin word *summa*, and the d symbol used for differentials, from the word *differentia*. He showed the relations between differentials and integrals, which in time became known as the fundamental theorem of calculus, in his 1693 paper "Supplementum geometriae dimensoriae" [30].

The Bernoulli Brothers The idea of finding a shape that could move through a fluid with the least resistance was of great importance to naval constructors [31]. This seemed a natural progression for differential calculus with the solutions of the problem involved the maxima and minima. The first of these problems to be addressed was proposed by Newton in the "Philosophiae naturalis principia mathematica" (1687) where he asked of what solid, moving through a fluid, offers the least resistance. The methods of maxima and minima of quantities were also of great importance to the Bernoulli's brothers.

Johann Bernoulli asked for the curve of quickest decent from point A to point B, in the "Leipzig Acts" (1696). This became the famous puzzle, the Brachistochrone problem. His brother Jakob Bernoulli found the answer of the question to be a cycloid, although the result was independent of the starting position, along with Newton, Leibniz, and l'Hôpital, . Naturally the question of which cycloid would a body descend in the least time, which was proven by Johann. In 1697 Jakob proposed the problem to find the greatest area for a curve with a common base [30].

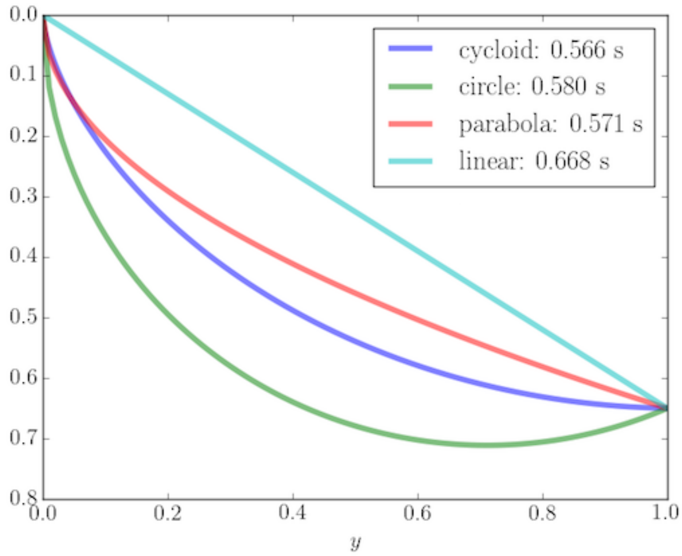

Figure 2.1: Finding the path of shortest time for which a bead will slip from rest between two points due to gravity[34].

Johann Bernoulli published the answer in "The Memoirs of the Academical Sciences" (1718) in which the methods used were not too dissimilar to the Brachistochrone problem, for which if one property is involved in the problem then the variation of two elements was needed, and for two properties, three elements needed. This made a considerable advancement in the calculus of Isoperimetrical problems creating the foundations for future methods of solutions. Although the methods were at the time seen as complete, they did not apply to problems involving three or more properties, nor did they allow for differentials of orders higher than one. For instance, with the Brachistochrone curve, it did not allow for forces acting upon the body outside the x-y plane [30].

Euler and Lagrange Euler, a close friend of the Bernoulli's, led the way in 1733 by finding a more compact solution to the Brachistochrone problem which hinted at the general solution to similar problems. By the end Euler could now solve problems involving three or more properties and had reduced the number of dependent equations down to two, as well as solutions to the problems involving the first class and some higher classes (the classification, according to Euler, is how many properties it involves), but it was still without some faults. It could not find solutions involving the differentials of x or y of an order higher than the second, nor when the integrals were not of a constant, for instance finding a curve with the lowest centre of gravity [30].

Euler's "Methodus inveniendi lineas curvas maximi minimive proprietate gaudente" (1744), reduced the maxima and minima problems to a dependence on as many equations as the properties involved. An important result was that the reductions of all questions of relative maxima and minima to those of absolute, each solution was now expressed neither in determinants nor integrals, but purely in the forms of differential equations allowing the study of generalised problems [30][28].

Lagrange reduced the inconveniences in Euler's work by creating the δ-function [30], thus small quantities were now denoted by the likes of δy. With this change, along with other processes, Lagrange established an underlying concept, which Euler named "The Calculus of Variations", preferring Lagrange's methods over his own. The fundamental formula is known as the Euler-Lagrange equation. In 1788 "Mechanique Analytique" was published by Lagrange which contained the famous rule of multipliers. This allowed problems which had conditions or constraints to be reformulated as a problem in which the conditions or constraints did not need to be solved explicitly.

Riemann, Dirichlet and Weierstrass By the 19th century mathematical modelling had advanced considerably with the invention of Newtonian mechanics, thermodynamics and Maxwell's equations. The search for initial and general boundary problems for partial differential equations were undertaken by mathematicians such as Bernhard Riemann and Gustave Lejeune Dirichlet. Riemann used principles from calculus of variations to create proof for Dirichlet's problem $\Delta u = 0$ for the domain Ω with boundary conditions $u = f$, on the boundary $\partial\Omega$ [28]. After their deaths

Weierstrass pointed out the problem in this theory that it was possible for a minimising function to come arbitrarily close to the lower bound without ever reaching it. This led to a complete overhaul of how calculus of variations was applied, stressing the importance of the domain which was to be minimised [28].

Courant and Hilbert Hilbert continued where Weierstrass left off, developing fundamental ideas in mathematics with Hilbert space, a infinite dimensional euclidean space, Hilbert's program in which he laid down the principles for a complete logical foundation for mathematicians to work from, and his influential 23 problems in which the last problem set was to advance the field of calculus of variations [32].

Hilbert's pupil and friend Richard Courant created the now famous books "Methods of Mathematical Physics Vol. 1&2" (1937) awarding Hilbert co-authorship, due to the significant influence of his papers and lectures. Much of the research for this dissertation has come from the first volume of these books.

Chapter 3

Eigenvalues for One and Two Dimensional Domains

3.1 The Wave Equation

A vibrating string of finite stiffness resonates when the string is displaced from its resting state, stretching the string which in turn causes standing waves of many different frequencies to be produced. These waves can be described using the second order partial differential wave equation

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}$$

This wave equation for one dimension can be derived in the form of a function $v(x,t)$ for x in $[0,L]$ and $t \geq 0$ by modelling the string as a series of masses m in line with each other, held together by a spring of spring constant k [23]. Suppose each mass is h distance apart we have have modelled the spring as Figure 3.1

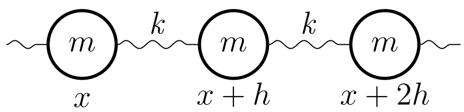

Figure 3.1: String modelled as spring

The force exerted on the mass m at distance $x + h$ can be written in terms of

Newton's second law of motion[8] and Hooke's law for linear springs[9]

$$F_{Newton} = ma(t) = m\frac{\partial^2}{\partial t^2}v(x+h,t)$$
$$F_{Hooke} = F_{x+2h} - F_x = k[v(x+2h,t) - u(x+h,t)] - k[v(x+h,t) - v(x,t)]$$

Equating both terms

$$m\frac{\partial^2}{\partial t^2}v(x+h,t) = k[v(x+2h,t) - 2v(x+h,t) + v(x,t)]$$

Now by "seeing" the string as N number of masses m evenly spaced over the length $L = Nh$, with the total mass being $M = Nm$ and the total spring constant being $K = k/n$ then the equation can be written as

$$\frac{\partial^2}{\partial t^2}v(x+h,t) = \frac{KL^2}{M}[v(x+2h,t) - 2v(x+h,t) + v(x,t)]$$

Taking the limits $N \to \infty, h \to 0$ and using the L'Hopital rule. Thus giving the one-dimensional wave equation

$$\frac{\partial^2 v(x,t)}{\partial t^2} = c^2\frac{\partial^2 v(x,t)}{\partial x} \qquad \text{where } c^2 = \frac{KL^2}{M} \tag{3.1}$$

3.2 Eigenvalues of a Plucked String

One method of solving (3.1) is by finding the stationary solutions in the form $v(x,t) = u(x)f(t)$ which dissects the function into one that has a basic standing waveform, $u(x)$, in which its amplitude varies in time, $f(t)$.

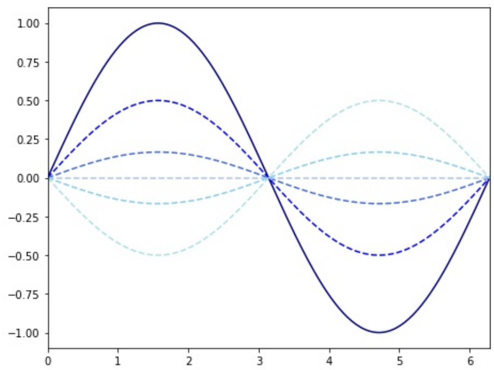

Figure 3.2: sinusoidal wave with amplitude varying in time

Using the method of separation of variables, the wave equation can be rewritten by substitution of the functions so that

$$u(x)f''(t) = c^2 u''(x) f(t)$$

or equivalently

$$\frac{f''(t)}{c^2 f(t)} = \frac{u''(x)}{u(x)}$$

The left side is independent of x so the right side cannot depend on x, and similarly the right side is independent of t meaning the left hand cannot depend on t, therefore both equations are equal to some constant, which will be denoted by $-\lambda$, where λ is greater than 0.

There are now two separate equations that can be solved, first the spatial variable x,

$$u(x)'' + \lambda u(x) = 0$$

Setting $\lambda = \omega^2$, and using the boundary conditions $u(0) = u(L) = 0$ we arrive at the eigenfunctions $u_n(x)$ and eigenvalues λ_n

$$u_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x, \qquad n = 1, 2, ...$$

$$\lambda_n = \frac{(n\pi)^2}{L^2} \tag{3.2}$$

Where the coefficient B has been calculated so that u_n is normalized, that is when

$$\int_0^L dx \; u_n(x)^2 = 1$$

The other equation involving time t

$$f''(t) + \omega f(t) = 0, \qquad \omega = \lambda c^2$$

gives the solution expressed in complex form $f(t) = c_n e^{i\omega_n t}$ for $n = 1, 2, ...$, completing the overall solution to $v(x,t)$.

$$v(x,t) = \sum_{n\geq 1} c_n e^{i\omega_n t} u_n(x)$$

in which the real values correspond to physical solutions. The angular frequencies are known as pure tones, where ω_1 is the dominant frequency, known as the *fundamental tone* and the others $\{\omega_n\}_{n>1}$ are known as the *overtones*.

The values of λ_n depend only on L, the length of the string, proving that if the value of λ_1 is known, the length of the string can be found.

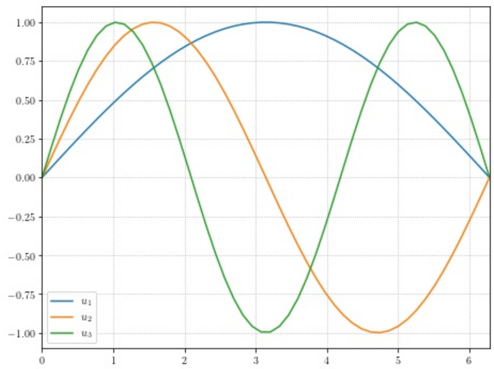

Figure 3.3: First three eigenfunctions of a plucked string of length 2π

3.3 Two-dimensional Euclidean Domains

The wave equation for a two-dimensional Euclidean domain is

$$\frac{\partial^2 v(\mathbf{x})}{\partial t^2} = c^2 \Delta v(\mathbf{x}), \qquad \mathbf{x} = (x,y)$$

where Δ is the d-dimensional second-order Laplacian operator

$$\Delta v(\mathbf{x}) = \frac{\partial^2 v(\mathbf{x})}{\partial x} + \frac{\partial^2 v(\mathbf{x})}{\partial y}$$

with solution

$$v(\mathbf{x},t) = \sum_{n\geq 1} c_n e^{i\omega t} u_n(\mathbf{x})$$

where ω_n, the angular frequencies, are given by $\omega_n = c\sqrt{\lambda_n}$. The eigenvalues λ_n and eigenfunctions $u_n(\mathbf{x})$ are given by

$$\Delta u(\mathbf{x}) + \lambda u(\mathbf{x}) = 0 \tag{3.3}$$

where the domain for which this problem will act upon will be denoted by Ω, along with the domains boundary, $\partial\Omega$, which has the following general condition.

$$\frac{\partial u(\mathbf{x})}{\partial n} + \sigma u(\mathbf{x}) = 0 \tag{3.4}$$

If the boundary $\partial\Omega$ is free to vibrate, then $\sigma = 0$ in (3.4) and $\partial u(\mathbf{x})/\partial n = 0$, known as the Neumann boundary condition, where $\partial u/\partial n$ is the derivative along the outward normal to $\partial\Omega$ at a specific point. If the boundary is fixed so that it cannot vibrate then σ tends to infinity in (3.4) such that $u(\mathbf{x}) = 0$, which is known as the Dirichlet boundary condition.

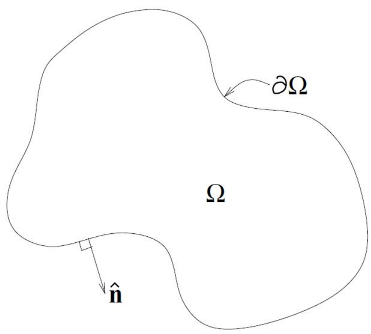

Figure 3.4: An open region Ω bounded by $\partial\Omega$ [6]

Extending the information found from the one-dimensional domain to two will be conveyed by investigating simple shapes such as a rectangle, triangle and circle, allowing the generation of all eigenvalues λ_n from the corresponding eigenfunctions $u(\mathbf{x})$.

3.3.1 Rectangular Domains

For a rectangle of sides $0 < x < a$, $0 < y < b$ for the Laplacian problem in (3.3) with the Dirichlet boundary conditions such that [6, p.11-12]

$$
\begin{aligned}
u(0,y) = u(a,y) =& 0 \qquad \text{for all } 0 \leq y \leq b \in \partial\Omega \\
u(x,0) = u(x,b) =& 0 \qquad \text{for all } 0 \leq x \leq a \in \partial\Omega
\end{aligned}
$$

The method of separation of variables can be used as before such that $u(x,y) = X(x)Y(y)$, giving the two equations.

$$X''(x) + p^2 X(x) = 0, \qquad Y''(y) + q^2 Y(y) = 0, \qquad \lambda = p^2 + q^2$$

Upon solving the boundary conditions to produce the eigenvalues

$$\lambda_{j,k} = \pi^2 \left(\frac{j^2}{a^2} + \frac{k^2}{b^2} \right), \qquad j,k = 1,2,3,... \tag{3.5}$$

with the corresponding normalised eigenfunction

$$u_{j,k}(\mathbf{x}) = \frac{2}{\sqrt{ab}} \sin\left(\frac{j\pi}{a} x \right) \sin\left(\frac{k\pi}{b} y \right) \tag{3.6}$$

To keep to convention we rename $\lambda_{j,k}$ as λ_n so that the first eigenvalue has the lowest frequency of all eigenvalues, and all others being at least or larger than its predecessor, such that, when ordered, $\lambda_n \leq \lambda_{n+1}$.

By choosing the side a to be less than or equal to side b, it can be seen once again that the eigenvalues are dependent on the length of a and b, and each differently sized rectangle will produce unique eigenvalues. Conversely, if the eigenvalues are known then although slightly more difficult than the string it is entirely possible to calculate the lengths of the rectangular domain.

Note that for when the sides a and b are equal, as in a square domain, two of the eigenfunctions will have the same eigenvalues due to symmetry, that is $\lambda_{j,k}$ will equal $\lambda_{k,j}$. This is known as *double degeneracy.*

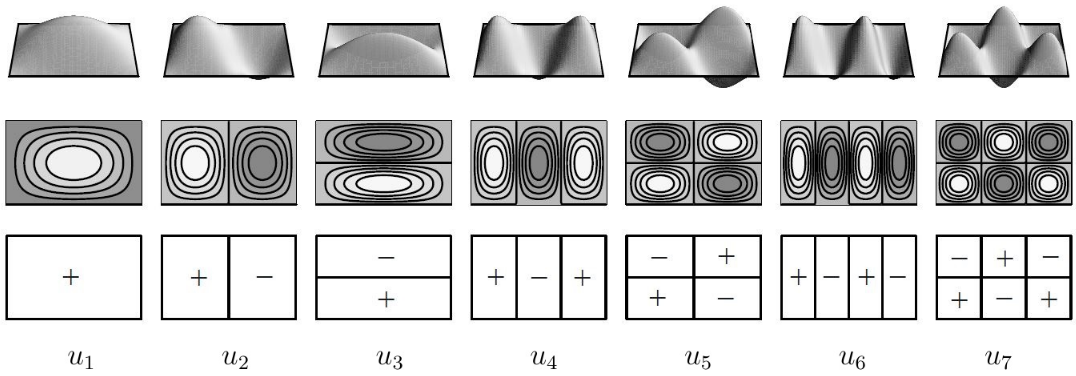

Figure 3.5: The first seven eigenfunctions of a rectangular domain of side lengths $a = (1 + \sqrt{5})l/2,\ b = l$ [19]

3.3.2 Nodes

The last set of diagrams of fig 3.5 represent the nodes of the domain for each eigenfunction u_n. The nodal set N is defined as all the points $\mathbf{x}$ in the domain Ω, but not including the points on the boundary $\partial\Omega$, in which the eigenfunctions $u(\mathbf{x}) = 0$. The nodal set allows a great visualisation of the positive and negative areas of the domain, especially for higher values of n when the nodes can become intricate. Small changes to the eigenfunction can lead to extremely different nodal sets. For example, the eigenfunction $\sin 12x \sin y + v \sin x \sin 12y$ for the square domain of side lengths $(0 < x < \pi,\ 0 < y < \pi)$ has 12 subregions when $v = 1$, but only two subregions when v is near one [22, p.281].

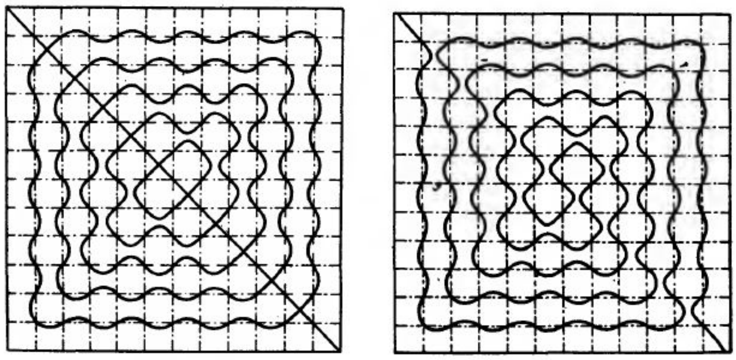

Figure 3.6: The nodal curves of (i) the eigenfunction when $v = 1$, and (ii) the eigenfunction when v is near one [20, p.456]

The first eigenfunction u_1 is either positive everywhere or negative everywhere and therefore does not have any nodes [20, p.451-452]. Due to orthogonality, the eigenfunctions u_n, $n \geq 2$ must have at least one node, dividing the domain into two pieces. It can also be proved [20, p.452-454] they can at most have $n - 1$ nodes, dividing the domain into n pieces.

3.3.3 Triangular Domains

Restricting to right-angled isosceles triangles of height and base length a with Dirichlet boundary conditions, the eigenvalues and eigenfunctions of the problem (3.3) can be found by noting that the triangle and its symmetry have the same eigenvalues for all $x \neq y$ of the square domain due to its double degeneracy for all eigenfunctions $j \neq k$. This information can be used to find the values for the triangular domain by restricting the values for j, k so that $k > j$ to give only unique eigenvalues. The triangular domain boundary $x = y$ relate to the square eigenfunctions $u_{j,k}(x, y) = u_{k,j}(x, y)$, which the eigenvalues can of course be removed along this line by the equation $u_{j,k}(x, y) - u_{k,j}(x, y) = 0$, leading to the equation [6, p.13]

$$u_{j,k}(\mathbf{x}) = \frac{2}{a}\left[\sin\left(\frac{j\pi}{a}x\right)\sin\left(\frac{k\pi}{a}y\right) - \sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{j\pi}{a}y\right)\right] \tag{3.7}$$

Although finding the eigenvalues and respective eigenfunctions of a right-hand triangle was very easy, analysing the eigenstructure in other types of triangles is much more difficult [21]

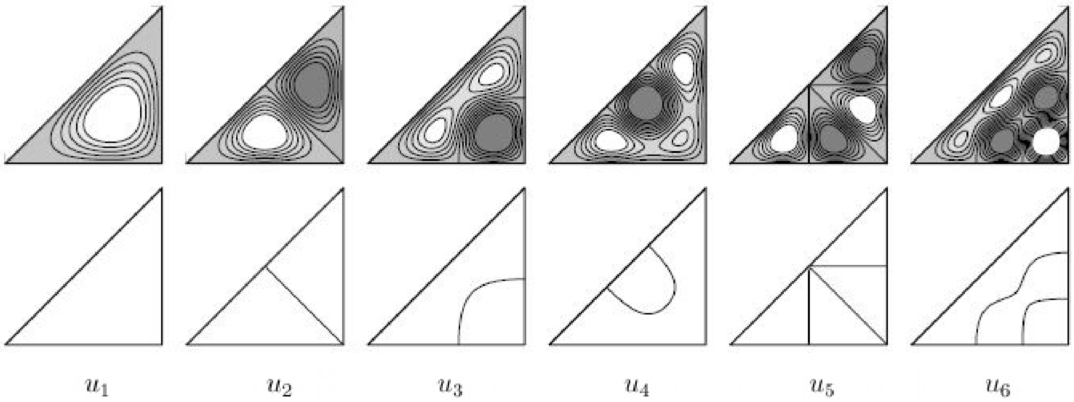

Figure 3.7: The first six eigenfunctions of a right-angled isosceles triangular domain of side lengths l [19]

3.3.4 Circular Domains

For a circle domain Ω with radius a such that $x^2 + y^2 < a^2$, again with Dirichlet boundary conditions on $\partial\Omega$. By converting to polar coordinates so that $u = (r, \theta)$, where $x = r\cos\theta$ and $y = r\sin\theta$ to ease calculations, the problem $\Delta u + \lambda u = 0$ becomes [6, p.14-15]

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0, \qquad 0 \leq r \leq a, \qquad 0 \leq \theta < 2\pi$$

such that the boundary conditions are $u(a, \theta) = 0$ for all $0 \leq \theta < 2\pi$. The additional condition that $u(r, 2\pi + \theta) = u(r, \theta)$, ensures that u is periodic. Applying the separation of variables in which $u(r, \theta) = R(r)\Theta(\theta)$ gives the equations

$$\Theta''(\theta) + p^2\Theta(\theta) = 0$$

$$r^2R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0$$

and the solution

$$\Theta(\theta) = A\cos n\theta + B\sin n\theta$$

where $n = 0, 1, 2, ...$ satisfies the boundary condition $\Theta(\theta + 2\pi) = \Theta(\theta)$.

Solving the equation requires substituting $r = z/\sqrt{\lambda}$ and $R(r) = F(z)$ so that the equation is in the form of the Bessel differential equation[1]

$$z^2 F''(z) + zF'(z) + (z^2 - n^2)F(z) = 0$$

which has two linearly independent solutions, $J_n(z)$ and $Y_n(z)$.

The Bessel function of the first kind, denoted $J_n(x)$, is visually similar to the oscillating sine or cosine wave, and is bounded at $z = 0$. Although it decays proportionally to $z^{-\frac{1}{2}}$, their roots are not generally periodic, except asymptotically for large z. These roots are labelled $j_{n,m}$ where the subscripts denote the m^{th} zero of $J_n(z)$, $m \geq 1$. The zeros $y_{n,m}$ of the Bessel function of the second kind, $Y_n(z)$, although similar to the sine wave are unbounded as z tends towards 0.

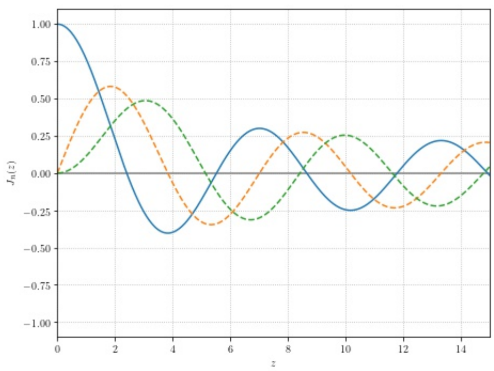

Figure 3.8: The first three first order Bessel J_n functions denoted by the colours blue, orange and green respectively.

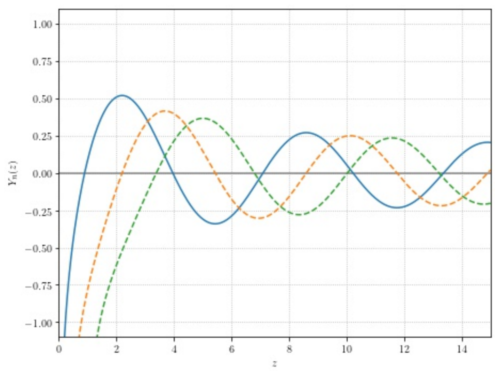

Figure 3.9: The first three second order Bessel Y_n functions denoted by the colours blue, orange and green respectively.

[1]Friedrich Wilhelm Bessel was the first to systematically investigate what are now known as the Bessel functions in 1817 [11] when determining the motion of three bodies moving under mutual gravitation, and developed these more fully when working on planetary perturbations in 1824[10], extending on Lagrange's work on elliptical orbits. It is now widely used for problems that involve circular or cylindrical symmetry, and wave propagation.

With the condition that the solution must be periodic, and due to $Y_m(z)$ diverging at $z = 0$, $J_n(z)$ is the only admissible function. By reverting back to $R(r)$ with the boundary condition $r = a$ the solution to the second equation becomes

$$J_n(a\sqrt{\lambda}) = 0$$

which is satisfied only when

$$\lambda_{n,m} = \frac{j_{n,m}^2}{a^2}, \qquad n = 0, 1, 2, ..., \qquad m = 1, 2, 3, ...$$

with the corresponding eigenfunction for a circular domain

$$u(r, \theta) = J_n(a\sqrt{\lambda}_{n,m})[A_n \cos(n\theta) + B_n \sin(n\theta)] \tag{3.8}$$

It is noteworthy that for $n \geq 1$ there are two linearly independent solutions, thus each eigenvalue is at least doubly degenerate.

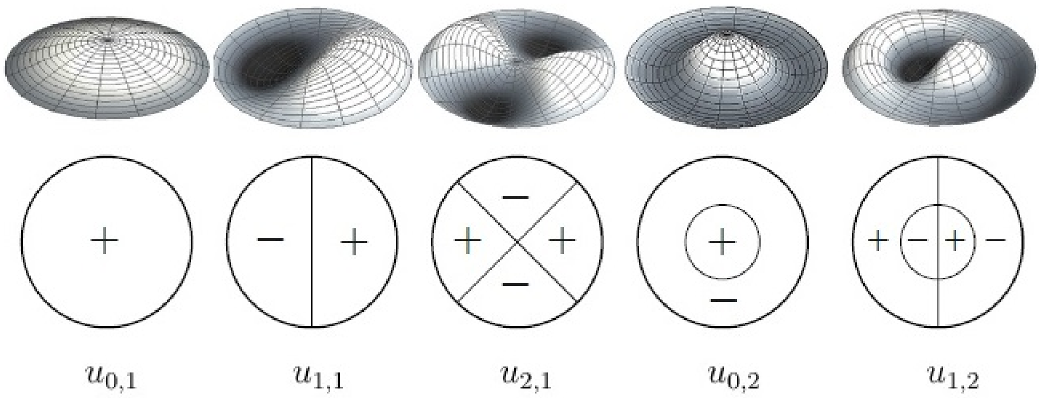

Figure 3.10: The first five eigenfunctions of a circular domain [19][35]

The Bessel function is not quite periodic and therefore cannot be solved explicitly. The power series for $J_n(z)$ when n is a non-negative integer is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}$$

where even for moderate x is likely to become numerically unstable, caused by proximity to almost colliding eigenvalues. Without knowing the eigenvalues for large enough λ_n, knowing if circular domains are isospectral becomes much more difficult.

Chapter 4

Calculus of Variations

4.1 Euler-Lagrange Equation

The eigenvalues and eigenfunctions of the problem $\Delta u + \lambda u = 0$ for any two-dimensional shape other than the rectangular and triangular domains are much more difficult to calculate. Knowing that a differentiable functional is stationary at the local extrema, the Euler-Lagrange equation uses this knowledge to solve optimisation problems where one wants to minimise or maximise the problem. By minimising the problem in hand, the minimum solution will be the first eigenvalue λ_1. Constraining the problem to exclude the first eigenvalue, the second eigenvalue λ_2 can be found using the same methods, this can then be repeated until all the necessary eigenvalues are known.

If the problem can be expressed as the functional

$$I[\varphi] = \int_{\Omega} d\mathbf{x}\ F(\mathbf{x}, \varphi(\mathbf{x}), \nabla\varphi(\mathbf{x})) + \int_{\partial\Omega} d\mathbf{s}\ F_1(\mathbf{s}, \varphi(\mathbf{s})) \tag{4.1}$$

for functions $\varphi(\mathbf{x})$ on the domain Ω in $\mathbb{R}^d$ with the boundary $\partial\Omega$, where $\nabla\varphi(\mathbf{x}) = \partial\varphi(\mathbf{x})/\partial\mathbf{x}$.

For the stationary to be a minimum, the following must hold true

$$\nabla \cdot \frac{\partial F}{\partial \nabla\varphi} - \frac{\partial F}{\partial \varphi} = 0, \qquad \text{for all } \mathbf{x} \in \Omega \tag{4.2}$$

which is known as the Euler-Lagrange equation, along with boundary conditions

$$\hat{\mathbf{n}} \cdot \frac{\partial F}{\partial \nabla\varphi} + \frac{\partial F_1}{\partial \varphi} = 0, \qquad \text{for all } \mathbf{x} \in \partial\Omega \tag{4.3}$$

If the integral over the boundary in (4.1) equates to zero then the problem has a Dirichlet boundary condition.

4.2 Sturm-Liouville

The problem $\Delta u+\lambda u = 0$ is a type of Sturm-Liouville equation. The Sturm-Liouville equation is an important class of second order differential equations that consist of the constant λ and a weight function $\rho(x)$. If a second order differential problem can be reduced to a Sturm-Liouville equation where u is continuously differentiable on the interval $[a, b]$ then the eigenfunctions will be self-adjoint, this leads to the properties that the eigenvalues are real, the eigenfunctions are orthogonal, and lastly the eigenvalues form a complete set.

Any equation is a type of Strum-Liouville if it can be rearranged to fit in the form

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) + \lambda\rho(x)u(x) = 0, \qquad a \leq x \leq b$$

with separated boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0$$

where $|\alpha_1|+|\alpha_2| > 0$ and $|\beta_1|+|\beta_2| > 0$. When $\alpha_1 = \beta_1 = 0$ the boundary conditions are Neumann, and with $\alpha_2 = \beta_2 = 0$ the conditions are Dirichlet. Using vector form for the general-dimension in which variable $\mathbf{x}$ has been be dropped to aid ease of reading, the Strum-Liouville equation is denoted as

$$\nabla \cdot (p\nabla u) - qu + \lambda\rho u = 0 \qquad \mathbf{x} \in \Omega \tag{4.4}$$

with boundary conditions as in (3.4).

The Sturm-Liouville operator $L[u]$ is

$$L[u] = \nabla \cdot (p\nabla u) - qu \tag{4.5}$$

allowing the general Sturm-Liouville equation to be expressed as

$$L[u] + \lambda\rho u = 0 \tag{4.6}$$

where $\rho > 0$, known as the weight function, $p > 0$, for all $\mathbf{x}$ in the domain Ω. An example of a Sturm-Liouville equation is the Bessel function

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

which can be re-written in the form of a Sturm-Liouville problem simply by dividing the equation by x and reducing the first two terms into one so that the equation becomes

$$(xy')' + \left(x - \frac{v^2}{x}\right) y = 0$$

4.2.1 Self-Adjointness

When L^* is the complex conjugate of L, then it is defined as being adjoint, when L^* equals L it is known as being self-adjoint. To prove that the eigenvalues of a Sturm-Liouville problem are real, and that the eigenfunctions are orthogonal, we first need to show that the Sturm-Liouville equation is self adjoint.

From [25, p.354], for any admissible functions u, v with the operator L being defined as (4.5), using the inner product notation

$$(f, g) = \int_a^b dx\ f(x)^* g(x)$$

and Lagrange's identity on the functions u and v, which may be complex, then

$$(Lu, v) - (u, Lv) = \left[p(x)\left(v(x)\frac{du^*}{dx} - u(x)^*\frac{dv}{dx}\right)\right]_a^b$$

For self-adjointness $[u\nabla v - (\nabla u)v]_a^b$ must equate to zero. With the boundary conditions (3.4) this reduces to zeros, proving any Sturm-Liouville problem which have these boundary conditions to be self-adjoint.

4.2.2 Orthogonal Eigenfunctions and Real Eigenvalues

Orthogonality can help to attack differential calculations that are difficult to directly calculate by providing properties of the solution without requiring the solution itself. A set of functions $\{u_1, u_2, ..., u_d\}$ are classed as orthogonal if $(u_n, u_m) = 0$, for all $n = 1, 2, ...d$, where $n \neq m$. To prove orthogonality, let two distinct eigenvalues λ_n, λ_m where $(\lambda_n \neq \lambda_m)$ with corresponding eigenfunctions u_n, u_m. By using (4.6), Lagrange's identity and the conditions of linearity and conjugate symmetry, then

$$\begin{aligned}(Lu_n, u_m) - (u_n, Lu_m) = (\rho\lambda_n u_n, u_m) - (u_n, \rho\lambda_m u_m) =& 0\\ (\lambda_n - \lambda_m^*)(u_n, u_m) =& 0\end{aligned}$$

where ρ is real. As $\lambda_n \neq \lambda_m$ it must be that $(u_n, u_m) = 0$, proving that the eigenfunctions of a Sturm-Liouville equation are indeed orthogonal.

From the above equation, if n was indeed equal to m then by the normalisation condition $\int_\Omega |u_n|^2 = 1$, or using the inner product notation $(u_n, u_n) = 1$, therefore λ_n is equal to its complex conjugate λ_n^* and thus proving that every eigenvalue of the Stum-Liouville equation is real.

4.3 Minimising the Functional

Knowing that a Sturm-Liouville problem will result in real eigenvalues with orthogonal eigenfunctions, by considering the eigenvalue problem $L[u] + \lambda\rho u = 0$ with a general boundary as per (3.4), the quadratic functional eigenvalue problems can be minimised by defining the problem as [20, p.398-399]

$$\mathscr{D}[\varphi] = \int_\Omega d\mathbf{x}\ (p|\nabla\varphi|^2 + q\varphi^2) + \int_{\partial\Omega} d\mathbf{s}\ p\sigma\varphi^2 \tag{4.7}$$

With the subsidiary condition

$$H[\varphi] \int_\Omega d\mathbf{x}\ \rho\varphi^2$$

where $p > 0, \rho > 0$, and in polar form

$$\begin{aligned} \mathscr{D}[\varphi, \psi] &= \int_\Omega d\mathbf{x}\ (p\nabla\varphi \cdot \nabla\psi + q\varphi\psi) + \int_{\partial\Omega} d\mathbf{s}\ p\sigma\varphi\psi \\ H[\varphi, \psi] &= \int_\Omega d\mathbf{x}\ \rho\varphi\psi \end{aligned}$$

where ψ is continuous and has piecewise continuous first derivatives in Ω, and ψ has piecewise continuity in Ω, which satisfies the relations [20, p.399]

$$\mathscr{D}[\varphi + \psi] = \mathscr{D}[\varphi] + 2\mathscr{D}[\varphi, \psi] + \mathscr{D}[\psi] \tag{4.8}$$

$$H[\varphi + \psi] = H[\varphi] + 2H[\varphi, \psi] + H[\psi] \tag{4.9}$$

The fundamental eigenfunction u_1 is the admissible function that minimises $\mathscr{D}[\varphi]$ under the orthonormal condition

$$H[\varphi] = 1 \tag{4.10}$$

and the solution for the associated eigenvalue is the minimum value of $\mathscr{D}[u_1] = \lambda_1$. Successive eigenfunctions u_n can be solved by the additional orthogonality condition

$$H[\varphi, u_i] = 0, \qquad i = 1, 2, ..., u_{n-1} \tag{4.11}$$

To show why the minimising of this function, subject to the constraints (4.10) and (4.11), is a valid Sturm-Liouville equation, using the method from [6, p.38-39], first let

$$\mathscr{D}[u] = \int_\Omega d\mathbf{x}\, (p|\nabla u|^2 + qu^2) + \int_\Omega d\mathbf{s}\, p\sigma u^2, \qquad \text{for all } \mathbf{x} \in \Omega \in \mathbb{R}^d$$

subject to the boundary constraint

$$H[u] = 1, \qquad \text{for all } \mathbf{x} \in \partial\Omega$$

The use of Lagrangian Multipliers gives a method of finding stationary points for independent variables where one is constrained without forcing an arbitrary distinction between the variables. The procedure is equivalent to creating a new function F, in which $F(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$, so that problem is now in the form of an unconstrained functional

$$\mathscr{D}[u] - \lambda(H[u] - 1) = I[u] + \lambda$$

The unconstrained problem can now be expressed as

$$I[u] = \int_\Omega d\mathbf{x}\, p|\nabla u|^2 + qu^2 - \lambda\rho u^2 + \int_{\partial\Omega} d\mathbf{s}\, p\sigma u^2$$

The Euler-Lagrange equation (4.2) ensures that the solution is of a minimum value, giving the respective problem as

$$\nabla \cdot (p\nabla u) - qu + \lambda\rho u = 0, \qquad \text{for } \mathbf{x} \in \Omega$$

with the boundary condition found using (4.3)

$$\frac{\partial u}{\partial n} + \sigma u = 0, \qquad \text{for } \mathbf{x} \in \partial\Omega$$

showing that indeed (4.7) minimises the Sturm-Liouville equation (4.4) with boundary conditions (3.4).

To understand how the stationary point for function $\mathscr{D}[u]$ will result in the eigenvalue λ, let $\mathscr{A}_0$ be a set of all admissible functions φ on Ω, if u is a solution to the eigenvalue problem then, using the fact that $p|\nabla u|^2 = \nabla \cdot (up\nabla u) - u\nabla \cdot (p\nabla u)$, the minimising function can be expressed as.

$$\mathscr{D}[u] = \int_\Omega d\mathbf{x}\ u\{-\nabla \cdot (p\nabla u) + qu\} + \int_{\partial\Omega} d\mathbf{s}\ up\left\{\frac{\partial u}{\partial n} + \sigma u\right\}$$

where the divergence theorem was used to find the second equation along with the fact that $\nabla \cdot (up\nabla u) = \hat{\mathbf{n}} \cdot (up\nabla u)$.

With $-\nabla \cdot (p\nabla u) - qu = \lambda \rho u$ (4.4) and $\partial u/\partial n + \sigma u = 0$ (3.4), it can be seen that

$$\mathscr{D}[u] = \int_\Omega d\mathbf{x}\ \lambda p u^2 = \lambda H[u]$$

that is

$$\mathscr{D}[u] = \lambda H[u] \tag{4.12}$$

With the constraint (4.10), if u_1 is a solution to the eigenvalue problem, minimising $\mathscr{D}[u]$ will indeed give the eigenvalue λ_1 as expected, thus proving that (4.7) is valid for $u = u_1$.

For all other eigenvalues λ_n and their respective eigenfunctions u_n, $n > 1$, from [20, p.400-402] if ζ is any function that satisfies φ and if ϵ is an arbitrary constant, then for every value ϵ and for $u = u_1$, from (4.12) then $\mathscr{D}[u + \epsilon\zeta] \geq \lambda H[u + \epsilon\zeta]$ and by (4.17)

$$\{\mathscr{D}[u] - \lambda H[u]\} + 2\epsilon \{\mathscr{D}[u, \zeta] - \lambda H[u, \zeta]\} + \epsilon^2 \{\mathscr{D}[\zeta] - \lambda H[\zeta]\} \geq 0$$

For the function to be minimised, by (4.12), the first and last of the inequalities must equal zero, and therefore for the equation to be valid the second inequality

$$\mathscr{D}[u,\zeta] - \lambda H[u,\zeta] \tag{4.13}$$

must also be zero.

When $u = u_1$,and $\lambda = \lambda_1$, since ζ is arbitrary, equating it to u_1 (4.13) is then obtained as expected. For the second minimum problem, there is the condition that $H[\varphi, u_1] = 0$, so to ensure that for $u = u_2$ and $\lambda = \lambda_2$, let $\mathscr{A}_1$ be all the admissible functions for φ. From (4.13) ζ must satisfy the relation

$$H[\zeta, u_1] = 0 \tag{4.14}$$

so that $\mathscr{A}_1$ is in the set $\mathscr{A}_0$ with the constraint above. If ζ is rewritten as $\zeta = \eta + tu_1$ where η is any piecewise second order continuous function. For t to satisfy (4.11),

$$t = -H[u_1, \eta] \tag{4.15}$$

Now, for when ζ is the second eigenfunction u_2, from (4.12) with $u = u_1$ and $\lambda = \lambda_1$

$$\mathscr{D}[u, u_2] - \lambda H[u, u_2] = 0 \tag{4.16}$$

and as the constraint $H[u, u_2] = 0$, $\mathscr{D}[u, u_2] = 0$ is also satisfied.

By replacing $\zeta = \eta + tu_1$ in (4.13) so that $\mathscr{D}[u, \eta + tu_1] - \lambda H[u, \eta + tu_1] = 0$, when expanded

$$\mathscr{D}[u,\eta] - \lambda H[u,\eta] + t\left\{\mathscr{D}[u,u_1] - \lambda H[u,u_1]\right\} = 0$$

With $u = u_2$ and $\lambda = \lambda_2$, the bracketed part reduces to zero, giving

$$\mathscr{D}[u_2,\eta] - \lambda_2 H[u_2,\eta] = 0$$

which holds for any arbitrary functions η or ζ, not just u_1, removing the constraint in (4.14). Thus the equation (4.7) is valid for $u = u_2$ and $\lambda = \lambda_2$

Likewise, For all other eigenvalues and eigenfunctions, using the same thought process for all admissible functions ζ that satisfies φ in the set $\mathscr{A}_n$ which is in the set $\mathscr{A}_0$ with constraints $H[\varphi, u_j] = 0$, $0 \leq j \leq n$

$$\zeta = \eta - \sum_{j=1}^{n-1} H[u_j, \eta]u_j$$

and from (4.13), replacing ζ with its new form

$$\mathscr{D}[u, \eta] - \lambda H[u, \eta] - \sum_{j=1}^{n-1} H[u_j, \eta]\left(\mathscr{D}[u, u_j] - \lambda H[u, u_j]\right)$$

but as the last inequality is equal to zero for all j, then for $u = u_n$ and $\lambda = \lambda_n$, $n \geq 1$

$$\mathscr{D}[u_n, \eta] - \lambda_n H[u_n, \eta] = 0$$

Proving that (4.7) is valid for all eigenvalues $\lambda = \lambda_n$ and respective eigenfunctions $u = u_n$. Thus

$$\mathscr{D}[u_i] = \lambda_i, \qquad \mathscr{D}[u_i, u_j] = 0, \qquad (i \neq j) \tag{4.17}$$

$$H[u_i] = 1, \qquad H[u_i, u_j] = 0, \qquad (i \neq j) \tag{4.18}$$

and the eigenvalues satisfies the inequality

$$\lambda_{n-1} \leq \lambda_n$$

since the set of admissible functions for λ_n is a subset of the previous admissible functions which included λ_{n-1} therefore the minimum λ_n can be is no smaller than the previous minimum λ_{n-1}.

In fact, the constraint $H[u] = 1$ can be omitted from the constraints by minimising the constraints $\mathscr{D}[u]/H[u]$. For proof, let any value φ be in the set $\mathscr{A}_n$, which is in the set $\mathscr{A}_0$ subject to the constraints $H[\varphi, u_j] = 0$ for $1 \leq j \leq n$. Minimising $\mathscr{D}[\varphi]$ gives

$$\lambda_n = \min_{\varphi \in \mathscr{A}_{n-1}} \mathscr{D}[\varphi]$$

subject to the constrain $H[\varphi] = 1$.

Creating a new function $J[\varphi]$, known as the Rayleigh Quotient, in which

$$J[\varphi] = \mathscr{D}[\varphi]/H[\varphi] \tag{4.19}$$

Minimising φ, gives

$$\lambda_n = \min_{\varphi \in \mathscr{A}_{n-1}} J[\varphi] \tag{4.20}$$

Importantly this has found the eigenvalue without imposing the constraint $H[u] = 1$ reducing the complexity of the calculation.

4.4 The Rayleigh-Ritz Method

The Rayleigh-Ritz method is a type of linear variation method and approximates the problem with a linear combination of independent functions allowing for an estimate of the eigenvalue via linear expansion. This is based off the Rayleigh quotient and was devised by Walter Ritz. [1]

By choosing a good approximation of a trial function ψ_N, for which the trial function is a linear combination of

$$\psi_N = \begin{cases} \varphi_0 & N = 0 \\ \varphi_0 + c_1\varphi_1 + ... + c_N\varphi_N & N \geq 1 \end{cases}, \qquad \{\varphi_j \in \mathscr{A}_{n-1}\}_{j=0}^{N}$$

substituting this trial function into the Rayleigh Quotient (4.19) and minimising implies that $J[\psi_0] \geq \lambda_n$ giving an upper bound to the eigenvalue, with the exact answer when $\psi_N = u_n$.

[1] Walter Ritz wrote his famous method in 1908 and in 1911. Lord Rayleigh wrote a paper congratulating Ritz on his achievements, saying that he himself had used the method many times before. Since then the method has been attributed to both mathematicians, but it has been subsequently found that Rayleigh's calculations used a more abstruse approach and this direct method had never been used [24].

For example, a circular membrane with a fixed boundary of length unity, the first eigenvalue will obey the zero order Bessel function

$$y'' + \frac{1}{x}y' + \lambda y = 0, \qquad 0 \leq x \leq 1$$

along with the boundary conditions $y(1) = 0$ and for y to be non-singular at $x = 0$. Putting the equation into the standard Sturm-Liouville form gives

$$-\nabla \cdot (x\nabla y) = \lambda xy$$

Due to the lowest eigenvalue having an eigenfunction with no nodes, it is expected to correspond to an even function allowing a trial solution where the lowest-order non trivial trial function which satisfies the simplest polynomial function.

$$\psi_0 = 1 + \alpha x^2$$

which satisfies the boundary $u(1) = 0$ as long as $\alpha = -1$. With $J[\psi_0] = \mathscr{D}[\psi_0]/H[\psi_0]$

$$J[\psi_0] = \frac{6\alpha^2}{\alpha^2 + 3\alpha + 3} = 6$$

therefore $\lambda_1 \leq J[\varphi_0] = 6$. The actual value of λ_1 is 5.78, giving a close approximation with a small trial function.

Finding a better estimate of λ_1 by increasing to $N = 1$ so that the new trial function is

$$\psi_1 = 1 + \alpha x^2 + c_1 x^4$$

which again satisfies the boundary $u(1) = 0$ as long as $1 + \alpha + c_1 = 0$. With $J[\psi_1] = \mathscr{D}[\psi_1]/H[\psi_1]$ and substituting $\alpha = -1 - c_1$

$$J[\psi_1] = \frac{20c_1^2 - 40c_1 + 60}{c_1^2 - 5c_1 + 10}$$

Minimising $J[\psi_1]$ by taking the derivative of c_1 and equating to zero

$$\frac{\partial}{\partial c_1} J[\psi_1] = \frac{-60c_1^2 + 280c_1 - 100}{(c_1^2 - 5c_1 + 10)^2} = 0$$

gives $c_1 = \frac{7}{3} \pm \frac{\sqrt{34}}{3}$. Substituting the values of c_1 into $J[\psi_1]$ of which the minimum equates to $\lambda_1 \leq 5.7841...$. The true value of the lowest eigenvalue is 5.7832... showing that the Rayleigh-Ritz method produces an extremely good estimate. A better estimation can of course be produced by adding more terms to the trial function, but this will create more complex calculations that even computers may struggle to calculate.

4.5 The Maximum-Minimum Principle

The Rayleigh Quotient has decreased the complexity of the calculations, but for λ_n to be solved, all previous eigenfunctions need to be known. This difficulty can be overcome by a method known as the maximum-minimum principle, enabling solutions to be found more easily by allowing the n-th eigenvalue and associated eigenfunction to be found without referencing previous eigenvalues and eigenfunctions.

Using the method from [20, p.405-407], substituting the constraint $H[\varphi, u_i] = 0$ for $(1 \leq i \leq n-1)$ by the new constraint

$$H[\varphi, v_i] = 0, \qquad \text{for } i = 1, 2, ...n-1$$

where $v_1, v_2, ..v_{n-1}$ are arbitrary chosen piecewise continuous functions on the domain Ω. By letting the functional $d\{v_1, v_2, ..., v_{n-1}\}$ be the greatest lower bound (infimum) of $\mathscr{D}[\varphi]$ such that $H[\varphi, v_i] = 0$ for all $1 \leq i \leq n-1$. Then

$$\lambda_n = \max_{v_i \in \mathscr{A}_0} d\{v_1, v_2, ..., v_{n-1}\}$$

that is, λ_n is equal to the largest value which this lower bound d assumes. The maximum-minimum is achieved when $v_1 = u_1,\ v_1 = u_2, ..., v_{n-1} = u_{n-1}$ and $\varphi = u_n$. and therefore this information can be used to find the properties of λ_n without knowing the previous $\lambda_{1 \leq i < n}$.

For proof of this reasoning, by definition $d\{v_1, v_2, ., v_{n-1}\} = \lambda_n$ when $v_1 = u_1$, $v_2 = u_2$ up to $v_{n-1} = u_{n-1}$ inclusive, therefore it is sufficient to show that $d\{v_1, v_2, ..., v_{n-1}\}$ for all other choices of v_j will be less than λ_n. Let φ be an admissible function on

Ω such that it is a linear combination of the first n eigenfunctions with constants $c_1, c_2, ...c_n$

$$\varphi = \sum_{i=1}^{n} c_i u_i$$

with the new constraints

$$H[\varphi, v_i] = 0, \qquad 1 \leq i \leq n-1$$

this leads to the n quantities $c_1, c_2, ..., c_n$ with $n-1$ constraints, thus can always be satisfied. With

$$H[\varphi] = \sum_{i,j=1}^{n} c_i c_j H[u_i, u_j] = \sum_{i=1}^{n} c_i^2 = 1$$

where the latter is a normalising condition to determine a factor of proportionality, and

$$\mathscr{D}[\varphi] = \sum_{i,k=1}^{n} c_i c_j \mathscr{D}[u_i, u_j]$$

but as

$$\mathscr{D}[u_i, u_j] = \begin{cases} 0, & i \neq j \\ \lambda_i & i = j \end{cases}$$

along with the fact that as the eigenvalues are ordered $\lambda_1 < \lambda_2 \leq \lambda_3 \leq ...\lambda_n$, this implies that $\lambda_i \leq \lambda_n$.

$$\mathscr{D}[\varphi] = \sum_{i=1}^{n} c_i^2 \lambda_i \leq \lambda_n \sum_{i=1}^{n} c_i^2 = \lambda_n$$

Thus

$$\mathscr{D}[\varphi] \leq \lambda_n$$

showing that the minimum $d\{v_1, v_2, ...v_{n-1}\}$ is no greater than λ_n, allowing λ_n to be the maximum value the lowest bound can assume.

4.6 Domain Decomposition

Complex domains can be solved more easily if the domain is separated into several simpler non-overlapping subdomains. This can be achieved using a technique known as domain decomposition. For example, an irregularly shaped domain may be split into a union of square subdomains, for which can be solved. This enables the determination of spectral properties of the entire domain purely from the spectral properties of the subdomains. If the subdomain has eigenvalues and eigenfunctions associated with it the they will be unique to that subdomain. The totality of these eigenvalues and eigenfunctions will also be eigenvalues and eigenfunctions of the domain Ω.

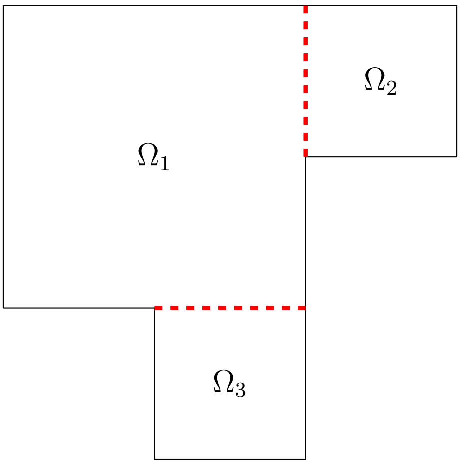

Figure 4.1: Domain Ω split into three subdomains

4.7 Theorems

To use the domain decomposition technique it is helpful to state some theorem beforehand which have been created from the information gained so far.[20][2]:-

[2]The symbols of the quoted theorems have been replaced to reflect the notation of this dissertation where necessary.

Theorem 1 [20, p.408]

If a system which is capable of vibrating is subjected to constraining conditions, the fundamental tone and every overtone become higher (at least not lower) in pitch. Conversely, if restraining conditions are removed, the fundamental tone and every overtone become lower (at least not higher).

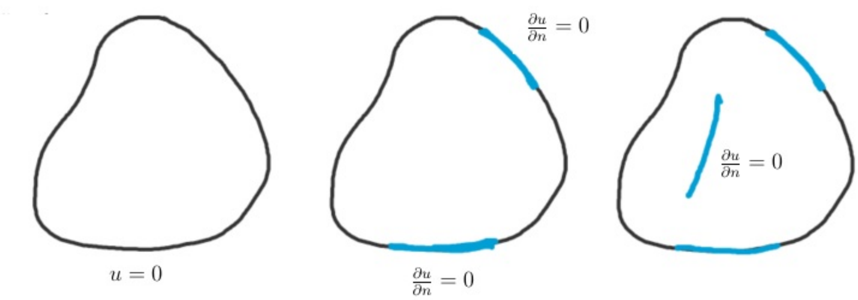

Figure 4.2: Three Domains. The first with $u = 0$ on $\partial\Omega$, the second with mixed boundary in which two parts have $\partial u/\partial n = 0$ on $\partial\Omega$, the rest being $u = 0$, the third with same boundary conditions as the second but with a "tear" in the membrane which has conditions $\partial u/\partial n = 0$

The first shape represents a fixed boundary with eigenvalues λ_n, whereas the second shape has eigenvalues λ_n^* and represents the condition in which two sections of $\partial\Omega$ are free to vibrate whereas the rest of $\partial\Omega$ is fixed. The latter membrane has less restrictions on the boundary $\partial\Omega$ than the first thus when ordered, the eigenvalues of the domains have the property $\lambda^* \leq \lambda$. The third membrane with eigenvalues λ^{**} has the same boundary conditions for $\partial\Omega$ as the second, but an area of the membrane itself has had the constraints relaxed, a physical representation would be a tear in a drum material or a crack in a metal plate, and like the real world the fundamental frequencies would sound lower in pitch, that is $\lambda^{**} \leq \lambda$.

Theorem 2 [20, p.408-409]

Suppose that $\Omega', \Omega", \Omega'", ...$ are a finite number of non-overlapping subdomains of the domain Ω. Let $N(\lambda)$ denote the number of eigenvalues less than λ of the differential equation $L[u] + \lambda\rho u = 0$ for Ω with the boundary condition $u = 0$. Then the total number of eigenvalues less than λ for all the separate subdomains with the same boundary condition does not exceed $N(\lambda)$.

Courant and Hilbert denote $N(\lambda)$ as

$$N(\lambda) = \sum_{\lambda_n < \lambda}^{n} 1 \tag{4.21}$$

By defining λ_n by the maximum-minimum theory, if a finite number of subdomains Ω^i, $i = 1, 2, ...$ are created on the domain Ω by setting the boundaries $\partial\Omega^i$ to by $\varphi = 0$, and for any area within Ω but not in Ω^i to also equate to $\varphi = 0$, by theorem 1 the new max-min value λ_n^* will be less or equal to λ_n

Theorem 3 [20, p.409]

Under the boundary condition $u = 0$ the n-th eigenvalue for a domain Ω never exceeds the n-th eigenvalue for a subdomain of Ω.

From Theorem 2 it can be seen that for $u = 0$ the n-th eigenvalue for the domain Ω is never greater than the n-th eigenvalue of any of its subdomains. This reasoning can be extended to $\partial u/\partial n = 0$ for which the statement holds.

Theorem 4 [20, p.409-410]

Suppose that $\Omega', \Omega'', \Omega''', ...$ are a finite number of non-overlapping subdomains which exhaust the domain Ω completely. Let $N(\kappa)$ denote the number of eigenvalues less than κ of the differential equation $L[u]+\lambda\rho u = 0$ for Ω with the boundary condition $\partial u/\partial n = 0$. Then the total number of eigenvalues less than κ for all the separate subdomains with the same boundary condition is at least as large as $N(\kappa)$.

This Theorem uses the max-min theory, and the knowledge from Theorem 1. Dividing the domain Ω, which has eigenvalues κ_n, into subdomains, with eigenvalues κ_n^*, with each subdomain boundary having Neumann conditions. This is much like the slit in 4.2 which decreases the frequency of each n-th eigenfunction, therefore $\kappa_n^* \geq \kappa_n$. Of course, this is only true if φ is continuous across all subdomain boundaries, in this case it is discontinuous, but at most it is a finite jump which will reduce the maximum-minimum so it still holds true.

Theorem 5 [20, p.410]

> *Let λ_n be the n-th eigenvalue of the differential equation $L[u] + \lambda\rho u = 0$ for the domain Ω under the boundary condition $u = 0$, and let μ_n be the n-th eigenvalue for the condition $\partial u/\partial n + \sigma u = 0$, or more generally for the condition $\partial u/\partial n + \sigma u = 0$ on a part $\partial\Omega'$ of the boundary $\partial\Omega$, $u = 0$ on the remaining part $\partial\Omega$" of the boundary. Then*
>
> $$\mu_n \leq \lambda_n$$

Theorem 6 [20, p.410]

> *If, in the boundary condition $\partial u/\partial n + \sigma u = 0$ on $\partial\Omega$, the function σ is either increased or diminished at every point, then each individual eigenvalue can change only in the same sense*

With σ increasing the elasticity of the membrane is decreasing. When σ is infinite the boundary condition becomes Dirichlet where the membrane completely fixed at the boundary and is at its most restrictive constraint, conversely the least restriction is when $\sigma = 0$ giving Neumann conditions.

Theorem 7 [20, p.411]

> *If in the differential equation $L[u] + \lambda\rho u = 0$, the coefficient ρ varies at every point in the same sense, then, for every boundary condition, the n-th eigenvalue changes in the opposite sense. If either of the coefficients*

p or q changes everywhere in the same sense, every eigenvalue changes in this same sense.

Referencing the Rayleigh Quotient (4.19), the functions p and q are linked to the functional $\mathscr{D}[\varphi]$, so if these change in value λ_n changes in value monotonically as well. With ρ being a function of $H[\varphi]$ an increase in this functions value will decrease the n-th eigenvalue monotonically, and vice-versa.

In fact, it can be seen [20, p.418] that the n-th eigenvalue depends continuously on q, and for sufficiently small changes in q, $\mathscr{D}[\varphi]$ also changed by a sufficiently small amount. The n-th eigenvalues are also continuously dependent on the functions ρ and p.

Chapter 5

Asymptotic Distribution of Eigenvalues

The content so far has been interested in finding the exact values, or at least a good approximation of the eigenvalues. The frequencies of the string, and the rectangular and triangular membranes are integer multiples and therefore n-th eigenvalue can be easily calculated, but when looking at the circular membrane the frequencies are the irregular spaced nodes of the m^{th} Bessel function J_n and finding these values is therefore complex and time-consuming. Instead, by understanding the asymptotic distribution of the eigenvalues as n tends towards infinity it can help establish the properties of the domain without the need for exact values. To be able to use asymptotic distribution, it needs to be seen that there are an infinite number of positive eigenvalues, and a limited amount of negative eigenvalues.

5.1 Boundedness of Eigenvalues

For the eigenvalue problem of a Sturm-Liouville system with Dirichlet boundary conditions

$$\mathscr{D}[u] = \int_\Omega p\nabla u^2 + qu^2 = \lambda$$

If $q > 0$ then λ is positive and not bounded above, but if $Q < q < 0$ for all the domain Ω and

$$\int_\Omega dx\ qu^2 \geq Q \int_\Omega u^2 \geq \frac{Q}{\min \rho}$$

showing that λ is bounded below [25, p.360-361]. When the boundary conditions are general, the Sturm-Liouville system can be rewritten as

$$\mathscr{D}[\varphi] = \int_a^b dx\ (p|\nabla\varphi|^2 + q\varphi^2) + h_1 p(a)\varphi(a)^2 + h_2 p(b)\varphi(b)^2 = \min$$

where by choosing the appropriate values for h_1, h_2 any homogenous boundary conditions can be obtained. In this case, if all the functions and values are positive then the number of non-negative eigenvalues is unbounded, but if $q < 0$, or even the numbers h_1, h_2 and the function σ are non positive in this variational problem then there can be negative eigenvalues, but this is also bounded [20, p.412-418]. Thus it can be concluded that

$$\lim_{n\to\infty} \lambda_n = \infty$$

5.2 Completeness of Eigenfunctions

For the proofs so far to hold true, it has been assumed that $\mathscr{D}[\varphi]$ is non-negative and vanishes only if $\varphi = 0$. Suppose that f is an admissible piecewise continuous function in $\mathscr{A}_0$ that can be written as an orthogonal set of functions $\{u_1, u_2, ..., u_n\}$ that are also admissible in $\mathscr{A}_0$, then the set is complete if the limit of the mean square error vanishes, known as the Cauchy convergence test [12], that is

$$\lim_{N\to\infty} \int_\Omega d\mathbf{x}\ \left[f - \sum_{n=1}^{N} c_n u_n\right]^2 = 0$$

where the expansion coefficient c_j is the generalised Fourier coefficient

$$c_n = \frac{(f, u_n)}{(u_n, u_n)}$$

From [22, p.311-313], let $r_N(\mathbf{x})$ be the remainder of

$$r_N = f - \sum_{n=1}^{N} c_n u_n$$

and by multiplying both sides by u_j so that

$$\begin{aligned}(r_N, u_j) = &\left(f - \sum_{n=1}^{N} c_n u_n, u_j\right)\\ =&(f, u_j) - c_j(u_j, u_j) = 0\end{aligned}$$

for $j = 1, 2, ...N$, showing that r_N is also an admissible function in $\mathscr{A}_0$. For the problem $\Delta u + \lambda u = 0$ in Ω with the general boundary condition on $\partial\Omega$, using Green's first identity

$$\begin{aligned}||\nabla r_N||^2 &= \int_\Omega \left|\nabla \cdot \left[f - \sum_{n=1}^{N} c_n u_n\right]\right|^2\\ &= \int_\Omega |\nabla f|^2 - \sum_{n=1}^{N} c_n \lambda_n (u_n, u_n)\\ ||\nabla r_N||^2 &\leq \int_\Omega |\nabla f|^2 = ||\nabla f||^2\end{aligned}$$

Now the Rayleigh Quotient rule shows that

$$\lambda_N \leq \frac{||\nabla \varphi_N||^2}{||\varphi_N||^2}$$

therefore

$$||r_N|||^2 \leq \frac{||\nabla r_N||^2}{\lambda_N} \leq \frac{||\nabla f||^2}{\lambda_N}$$

It has been proved that λ_N tends to infinity as N tends to infinity, thus the remainder $||r_N||^2 \to 0$ and thus proving that the system is complete for this problem. For the more general case $L[u] + \lambda\rho u = 0$ it has been proven that this is also complete [20, p.424-426].

5.3 Asymptotic Behaviour of Eigenvalues

To determine the order of magnitude of the n-th eigenvalue and find its asymptotic value, by [20, p.414-415] the Sturm-Liouville differential equation is rewritten by Liouville's Transformation into

$$z'' - rz + \lambda z = 0, \qquad 0 \leq t \leq l$$

where $r(t)$ is a continuous function, and

$$t = t(x) = \int_0^x dx' \sqrt{\frac{\rho(x')}{p(x')}}, \quad l = t(L), \quad z(t) = (p(x)\rho(x))^{1/4} y(x) \qquad [6, \text{p.26}]$$

Considering the case $y(0) = y(l) = 0$, which by transformation gives the boundary conditions $z(0) = z(l) = 0$ and

$$\mathscr{D}[\varphi] = \int_0^L dx\ (z'^2 + rz^2) \tag{5.1}$$

Ignoring the rz^2 term so that $\mathscr{D}[\varphi] = \int_0^l dx\ z'^2 = 1$, this expression differs from the previous by no more than a maximum of the absolute of r. This integral is the problem $z'' + \mu z = 0$ for the interval $(0, l)$ and is known to have the eigenvalues $\mu_n = n^2\pi^2/l^2$, therefore the eigenvalues of (5.1) can be written as

$$\lambda_n = \mu_n + O(1) = \frac{n^2\pi^2}{l^2} + O(1)$$

Thus as $n \to \infty$, returning to the original form, the one-dimensional Stum-Liouville problem has eigenvalues asymptotic to

$$\lim_{n\to\infty} \lambda_n = \frac{n^2\pi^2}{\left(\int_0^L dx \sqrt{\frac{\rho(x)}{p(x)}}\right)^2} \tag{5.2}$$

where L is the length if the domain Ω in the subset of $\mathbb{R}$.

This reasoning can be extended to singular differential equations for example the Bessel function where the zeros $\sqrt{\lambda_{m,n}}$ of J_m which are asymptotically equal to [20, p.415-416]

$$\lim_{n\to\infty} \lambda_n = n^2\pi^2$$

Finding the asymptotic distribution for any general two-dimensional domain can now be attempted. It is still complicated to model an arbitrary shaped membrane, but by breaking the domain down into many subdomains, the problem can be evaluated more easily. By letting each subdomain take on the shape of a rectangle, for which the eigenvalues are known and increasing the number of subdomains, the domains boundary can be more accurately modelled. Even with this approach there will be sections near the boundary $\partial\Omega$ that do not fit into a whole rectangle, for these areas it is appropriate to model with right-angled triangles for a more accurate representation of the domain. The following subsections uses the techniques from [20].

5.4 Asymptotic Distribution of Eigenvalues for a Rectangle

For the rectangular domain with sides of length a and b, the eigenfunctions and eigenvalues of the problem $\Delta u + \lambda u = 0$ with Neumann boundary conditions are, up to a normalising factor,

$$u_{j,k}(x, y) = \sin\left(\frac{j\pi}{a}x\right)\sin\left(\frac{k\pi}{b}y\right), \qquad \lambda_{j,k} = \pi^2\left(\frac{j^2}{a^2} + \frac{k^2}{b^2}\right), \quad j, k = 1, 2, 3, ...$$

By denoting the asymptotic number of ordered eigenvalues less than a bound λ by $N(\lambda)$ it is possible to model this as a number of integer lattice points inside an area of the ellipse. [20, p.429-431]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\lambda}{\pi^2}, \qquad x \geq 0, y \geq 0$$

Joining the lattice points together, the ellipse can now be modelled as a number of squares of unit length for which each has its own lattice point, for instance the bottom right corner point. These squares will lie either entirely within the ellipse or partially within the ellipse. Concentrating on one quadrant of the ellipse, for example the top right, increasing λ increases the number of lattice points within the ellipse. For large enough λ the ratio between the area of the ellipse and number of lattice points increases towards one [22, p.323-324]. By denoting the number of squares by $N(\lambda)$ this value can will be at least equal to the area of the quadrant, where equality holds in the case of the Neumann boundary,

$$N(\lambda) \geq \frac{ab}{4\pi}\lambda$$

To find the lower bound, letting $R(\lambda)$ be the number of incomplete squares, so that $N(\lambda) - R(\lambda)$ are all the complete squares within the ellipse quadrant, giving

$$N(\lambda) - R(\lambda) \leq \lambda \frac{ab}{4\pi} \leq N(\lambda)$$

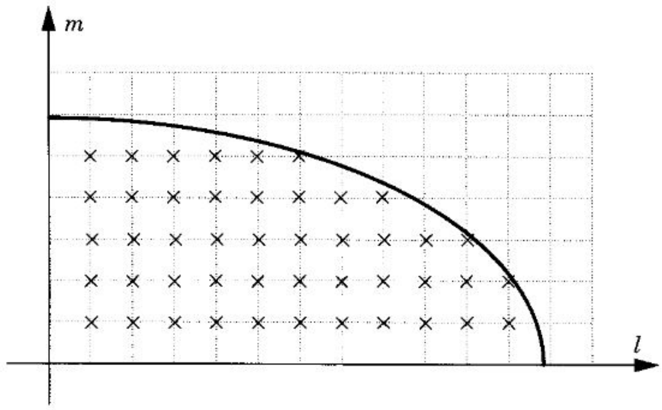

Figure 5.1: Grid points of an ellipse's upper right quadrant[22]

For large λ, it can be seen that $R(\lambda)$ differs only by the length of the perimeter, which is of $O(\sqrt{\lambda})$, therefore as λ increases asymptotically

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{ab}{4\pi}$$

That is

$$N(\lambda) \sim \lambda \frac{ab}{4\pi}$$

If the remainder $R(\lambda)$ is taken into account a more precise measurement can be shown as

$$N(\lambda) = \lambda \frac{ab}{4\pi} + c\sqrt{\lambda}$$

where c is a constant independent of λ.

For the Dirichlet boundary condition, the lattice points lying on the boundary curve is asymptotically equal to $(a+b)\sqrt{\lambda}/\pi$ and therefore the formulas are still valid [20, p.431].

Due to the eigenvalues being ordered, it is possible to set $N(\lambda) = n$, and thus the asymptotic value of the n-th eigenvalue for the rectangular domain is

$$\lambda_n \sim \frac{4\pi}{ab} n$$

5.5 Asymptotic Distribution of Eigenvalues for a Finite Number of Square Domains

The problem $\Delta u + \lambda u = 0$ with a two-dimensional domain Ω for which it is possible to model the domain by a finite number, h, of non-overlapping square subdomains $Q_1, Q_2, ...Q_h$ of side length a, the area of Ω is simply $f = ha^2$.
If the domain has the boundary condition $u = 0$ then using the technique from [20, p.431-434] let $A(\lambda)$ be the number of eigenvalues less than a bound λ, then the number of square subdomains with boundary $u = 0$, which will be denoted as $A_{Q_i}(\lambda),\ (i = 1, 2, ...h)$ is

$$A_{Q_i}(\lambda) = \frac{a^2}{4\pi}\lambda + ca\sqrt{\lambda}$$

Likewise, if the domain boundary condition is $\partial u/\partial n = 0$, and the number of eigenvalues of less than a bound λ can be denoted as $B(\lambda)$, and the square subdomains also with the Neumann boundary conditions are then $B_{Q_i}(\lambda),\ (i = 1, 2, ...h)$ giving the similar formula

$$B_{Q_i}(\lambda) = \frac{a^2}{4\pi}\lambda + Ca\sqrt{\lambda}$$

where C is a constant independent of n. With Theorem 2 stating that the n-th eigenvalue for all the subdomains of Ω will be at most equal to the n-th eigenvalue of the domain, a lower bound can be found. Theorem 4 states that for $\partial u/\partial n = 0$ the n-th eigenvalue for all the subdomains of Ω will be at least equal to the n-th eigenvalue of the domain, and along with Theorem 5 which states that for the domain Ω the n-th eigenvalue for the boundary $u = 0$ will be at most equal to the n-th eigenvalue of the domain with general boundary conditions, the upper bound can be deduced, so that

$$\sum_{i=1}^{h} A_{Q_i}(\lambda) \leq A(\lambda) \leq \sum_{i=1}^{h} B_{Q_i}(\lambda)$$

The only difference in the numbers A_{Q_i}, B_{Q_i} is the small variance between c and C so it is safe to conclude that asymptotically

$$A(\lambda) = \frac{f}{4\pi}\lambda + ca\sqrt{\lambda}$$

for large λ, and with the summation being the number h of subdomains we therefore have

$$N(\lambda) \sim \frac{f}{4\pi}\lambda$$

that is

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{f\lambda} = \frac{1}{4\pi} \tag{5.3}$$

Extending the problem to the differential equation $L[u] + \lambda\rho u = 0$ [20, p.431-434], again let Ω be a domain consisting of finite number, h, of squares of side length a. Let $N(\lambda)'$ be the number of eigenvalues less than a bound λ for the general boundary condition $\partial u/\partial n + \sigma u = 0$ where $\sigma \geq 0$. For the boundary condition $u = 0$, with the domain consisting of a finite number of squares of side length a, the number of subdomains can be denoted as $A'_{Q_i} < \lambda$ using the same reasoning as before. For the Neumann boundary condition $B'_{Q_i} < \lambda$ likewise. From the result from [20, p.412], where by denoting p_M, ρ_M, q_M as the maximum values of these functions, and p_m, ρ_m, q_m by their minimum values,

$$\frac{p_m + q_m}{\rho_M} \leq \lambda \leq \frac{p_M + q_M}{\rho_m}$$

Reducing the bounds by halving the length of a repeatedly, the largest and smallest respective values of p, q, ρ become closer until the difference is less than an a arbitrary small positive number ϵ. Small changes in q make very little difference to $\mathscr{D}[\varphi]$, so by setting this to zero it does not affect the asymptotic distribution, thus

$$\frac{p_m}{\rho_M} \leq \lambda \leq \frac{p_M}{\rho_m}$$

From Theorems 2,4 and 5

$$\sum_{i=1}^{h} A'_{Q_i}(\lambda) \leq N'(\lambda) \leq \sum_{i=1}^{h} B'_{Q_i}(\lambda)$$

and from Theorem 7

$$A'_{Q_i}(\lambda) \geq \frac{\rho_m^{(i)}}{p_M^{(i)}} A_{Q_i}(\lambda), \qquad B'_{Q_i}(\lambda) \leq \frac{\rho_M^{(i)}}{p_n^{(i)}} B_{Q_i}(\lambda)$$

where $\rho_m^{(i)}$ and $p_M^{(i)}$ are the minimum and maximum values of their respective functions Q_i.

Substituting $\rho_m^{(i)}$ and $p_M^{(i)}$ into the differential equation $L[u] + \lambda\rho u = 0$ and diving by p_M so that

$$\Delta u + \lambda \frac{\rho_m(i)}{p_M^{(i)}} u = 0$$

The eigenvalues for this problem are the eigenvalues of $\Delta u + \lambda u = 0$ multiplied by a factor of $\frac{p_M^{(i)}}{\rho_m(i)}$, a similar expression can be made for p_m and ρ_M. Since the functions p and ρ are continuous,

$$\int_\Omega d\mathbf{x}\ \frac{\rho}{p} = a^2 \sum_{i=1}^{h} \frac{\rho_m^{(i)}}{p_M^{(i)}} + \delta = a^2 \sum_{i=1}^{h} \frac{\rho_m^{(i)}}{p_M^{(i)}} + \delta'$$

where $|\delta|$ and $|\delta'|$ are arbitrary small constants. Piecing it all together

$$N(\lambda) = \frac{\lambda}{4\pi} \int_\Omega d\mathbf{x}\ \frac{\rho}{p} + \lambda\delta'' + c\sqrt{\lambda}$$

where $|\delta''|$ is arbitrary small constant. Therefore by taking λ to infinity,

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{1}{4\pi} \int_\Omega d\mathbf{x}\ \frac{\rho}{p} \tag{5.4}$$

5.6 Asymptotic Distribution of Eigenvalues for an Arbitrary Domain

For an arbitrary domain which has a continuous smooth boundary, restricting the problem to $\Delta u + \lambda u = 0$ the boundary condition $u = 0$ can be calculated as it has been before, but for other boundary conditions described the boundary shape needs to be taken into account. First by letting Ω be a right angled isosceles triangle, every eigenvalue within the triangle is also an eigenvalue in its reflection along the hypotenuse. With the boundary $\partial u/\partial n = 0$ the eigenvalues along $\partial\Omega$ are included so for every eigenvalue $\lambda'_n < \lambda$ for the triangle domain is also an eigenvalue $\lambda_n < \lambda$ for its related square domain [20, p.436-443].

Next let Ω be a right-angled triangle with sides $b \leq a$ from [20, p.437] it is shown that the eigenvalues $\lambda'_n < \lambda$ are no larger than that of the related square. Using these results it is possible to model any incomplete square on the boundary by either a triangle, or an area that can be divided up into a triangle and a rectangle

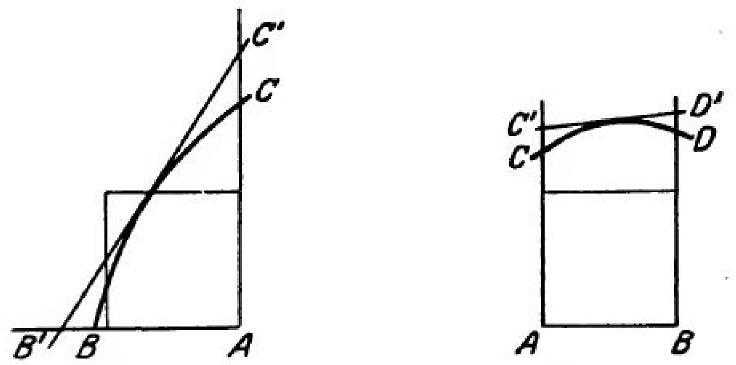

Figure 5.2: Modelling boundary with triangles and rectangles[20]

Using these results it is possible to model the incomplete squares at the boundary (per figure 5.2) as either a right-angled triangle of sides smaller than $4a$, or a combination of a right-angled triangle and a rectangle of sides smaller than $3a$ [20, p.439].

By denoting these as $E_i = AB'C'$ or $ABC'D'$ then it is possible to show that

$$B_{E_i}(\lambda) < c_1 a^2 \lambda + c_2 a\sqrt{\lambda}$$

where c_1, c_2 are constants. Using the same process as before, resulting in the asymptotic equation (5.3) for general boundary conditions is reached even without the necessity that $\sigma \geq 0$. With the same reasoning it is also possible to show for $L[u] + \lambda \rho u = 0$ the asymptotic equation is equivalent to (5.4).

In fact, the general $d-$dimension asymptotic result for $N(\lambda)$ for an arbitrary shaped domain, known as Weyl's law of asymptotic distribution[13], is

$$\lim_{n\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{v_d}{(2\pi)^d} \int_\Omega d\mathbf{x} \left(\frac{\rho(\mathbf{x})}{p(\mathbf{x})}\right)^{d/2} \tag{5.5}$$

for a general shaped domain Ω in $\mathbb{R}^d$ dimension for $d \geq 1$, where v_d is the volume of a unit sphere in d-dimensions, that is

$$v_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)} = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}$$

where $\Gamma(z)$ is the Gamma function.

A more precise estimate can be found by for the problem $\Delta u + \lambda u = 0$ by the use of finite squares of a suitable size of side length 1 which fit completely within the boundary [20, p.443-445]. The area left between these squares and the boundary can be estimated with finite squares of size 1/2 which fit outside of the original squares but inside the boundary. The gaps left are then filled with finite squares of side length 1/4 and so on. Denoting the number, h_t, of squares by

$$Q_1^t, Q_2^t, ..., Q_{h_t}^t, \qquad t \geq 0$$

of sides $1/2^t$. After the t-th approximation the remaining strips of the boundary will consist of r number of subdomains $E_1, E_2, ..., E_r$. in which the relations $h_i < 2^i c$ and $r < 2^t c$, where c is a constant independent of i and t, and only dependent on the length of the boundary, are satisfied. This allows the domain's eigenvalues less than λ to be denoted as $A^i_m(\lambda)$ and $A_{E_m}(\lambda)$ for the Dirichlet boundary and $B^i_m(\lambda)$ and $B_{E_m}(\lambda)$ for the Neumann boundary condition. Following the prior techniques, by choosing t to be the largest integer less than $\log\lambda/\log 4$ for sufficiently large λ

$$N(\lambda) \leq \frac{f}{4\pi}\lambda + C\sqrt{\lambda}\log\lambda$$

The lower bound of $N(\lambda)$ is the same as above with negative C, and thus a sharper form of the error in which $N(\lambda) - \frac{f}{4\pi}\lambda$ as λ tends to infinity, is no larger than $O(\lambda\log\lambda)$.

Chapter 6

Mark Kac's Hearing the Shape of a Drum, and Beyond

Mark Kac's famous paper "Can One Hear the Shape of a Drum?" [1] asks if two domains in $\mathbb{R}^2$ are isospectral, are they necessarily isometric? Kac discusses the problem of the wave equation for a two-dimensional membrane Ω with a fixed smooth boundary $\partial\Omega$. By the time this paper was published Weyl's law was known [13] and John Milnor had constructed a pair of 16 dimension flat tori with the same eigenvalues [33]. The paper itself was not written to give the answer to this question but to explore and collate what was already known so that others could attempt to find the answer.

Kac constructed the problem as a physical one using diffusion theory with a concentration of matter, starting at a point $\mathbf{x}_0$ on the two-dimensional membrane at time $t = 0$ and exploring the probability to find a specified diffusing particle at another point $\mathbf{x}$ in time $t > 0$. To ensure that the boundary is of Dirichlet form, any matter that arrives at the boundary is absorbed by the boundary.
In mathematical terms, let the conditional probability density
$P = P(\mathbf{x}|\mathbf{x_0}; t)$ be the probability density for the position $\mathbf{x}$ of the diffusing particle after time t, given its initial position at the centre of the membrane $\mathbf{x}_0 = (x_0, y_0)$, allowing P to change according to the diffusion equation

$$\frac{\partial P}{\partial t} = \Delta P \tag{6.1}$$

For the domain Ω, the matter is being absorbed by the boundary, so for the diffusion process to hold [1, p.7-8]

$$P_\Omega(\mathbf{x}_0|\mathbf{x};t) \to 0 \text{ as } \mathbf{x} \text{ approaches a boundary point} \tag{6.2}$$

with the initial condition

$$P_\Omega(\mathbf{x}_0|\mathbf{x};t) \to \delta(\mathbf{x}-\mathbf{x}_0) \text{ as } t \to 0 \tag{6.3}$$

which shows that all the matter is concentrated at the initial point $\mathbf{x_0}$. Here $\partial(\mathbf{x}-\mathbf{x}_0)$ is the Dirac delta function with value ∞ if $\mathbf{x}=\mathbf{x}_0$ and 0 otherwise.

Posing this in terms of the wave equation, the problem becomes

$$P_\Omega(\mathbf{x}_0|\mathbf{x};t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} u_n(\mathbf{x}_0) u_n(\mathbf{x})$$

of which the eigenvalues are of the differential equation

$$\begin{aligned} \frac{1}{2}\nabla^2 u + \lambda u =& 0 \text{ in } \Omega \\ u =& 0 \text{ on } \partial\Omega \end{aligned}$$

Using a physical interpretation creates the difficulty of the boundary influencing the particle. By restricting the time to small t, another model P_0 can be used, which is the probability of finding matter at the point $\mathbf{x}$ with the same initial condition as P_Ω but no boundary conditions. As t is infinitesimal the matter would not have had time to diffuse very far, so it would still be within the region Ω, and therefore it is acceptable to model the behaviour of the particle close to the membrane's boundary P_Ω by how the particle would behave at P_0. This simplifies matters as the equation for P_0 was already known to be [1, p.9]

$$P_0(\mathbf{x}_0|\mathbf{x};t) = \frac{1}{2\pi t} \exp\left[-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{2t}\right]$$

where $|\mathbf{x}-\mathbf{x}_0|$ is the euclidean distance between the two points. For some small $t>0$, $\mathbf{x}$ can be expected to be approximately in the same position as $\mathbf{x}_0$, and by normalising the wave equation for P_Ω Kac arrived at [1, p.9]

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{1}{2\pi t}$$

Which is valid only for small t, so the symbol $\sim$ can be taken to mean "asymptotic to".

For the region Ω with the boundary condition, the matter can be absorbed at the boundary $\partial\Omega$ so there is potentially less matter at any time $t > 0$ for this case compared to the region with no boundary condition, thus the inequality has given the upper limit to this problem.

$$P_\Omega(\mathbf{x}_0|\mathbf{x};t) \leq P_0(\mathbf{x}_0|\mathbf{x};t) \sim \frac{1}{2\pi t}$$

To find a lower bound, let Q be a square with sides a that is fully contained in Ω with Dirichlet boundary conditions, where $\mathbf{x}_0$ is at the centre of the square.

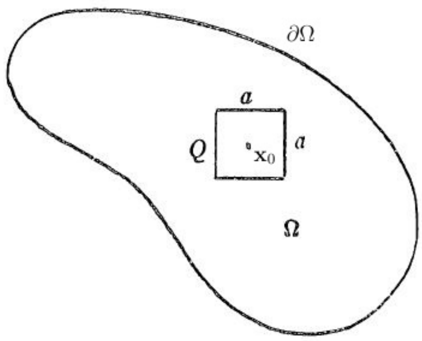

Figure 6.1: The domain Ω and domain Q with the Dirichlet boundary $\partial\Omega$ where $\mathbf{x}_0$ is at the centre of both domains [1]

The conditional probability density for the diffusing particle at $\mathbf{x}$ in the domain Q at time t will be P_Q, and will have the same boundary and initial conditions as P_Ω. With Q being contained within Ω it is expected that even more particles were lost to the boundary ∂Q as far as P_Q is concerned, but it needs to be stated that it is not lost in the view of P_Ω, so $P_Q \leq P_\Omega$. The eigenvalues of λ_n for a square and eigenfunctions are well known (3.5) (3.6), so with $\mathbf{x}_0 = \mathbf{x}$ as before, the inequality becomes [1, p.11]

$$P_Q = \frac{4}{a^2} \sum_{\substack{m,n \\ \text{odd integers}}} \exp\left[-\frac{(m^2+n^2)\pi^2}{2a^2}t\right] \leq P_\Omega \leq \frac{1}{2\pi t} \tag{6.4}$$

but, as t approaches to 0, P_Q is asymptotic to $\frac{1}{2\pi t}$. With the lower and upper bounds being asymptotically equal, $P_\Omega \sim \frac{1}{2\pi t}$ and thus arriving at Carleman's theory [1, p.11].

$$\sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n^2(\mathbf{x}_0) \sim \frac{1}{2\pi t}$$

To prove Weyl's theorem using diffusion theory, Kac segmented the domain into squares with sides of length a, letting $N(a)$ be the number of full squares within the domain, and $\Omega(a)$ being the total area of the number of squares.

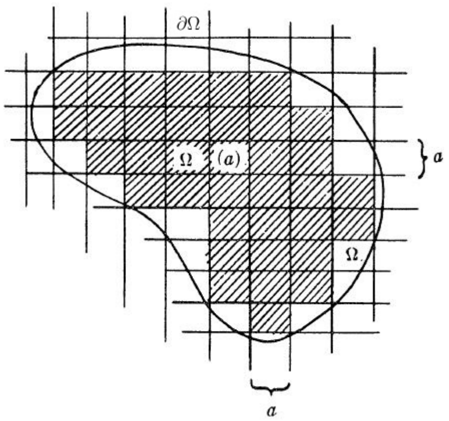

Figure 6.2: The area of the domain Ω with the Dirichlet boundary $\partial\Omega$, along with $\Omega(a)$ being the total area of complete squares within the boundary [1]

This allowed Kac to see the area of the membrane as being greater or equal to the area of Q multiplied by number squares within the domain. Since the size of Q is known by integrating the value of P_Q in (6.4), a lower limit is found. The upper limit to the size of P_Ω is also known within the same equation, therefore by integrating the upper limit over Ω a maximum area is found. By sufficiently reducing the size a so that the area $\Omega(a)$ is comparable to Ω (denoted as $|\Omega|$) which led Kac to arrived at

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t}, \qquad \text{as } t \to 0 \tag{6.5}$$

Proving Weyl's theorem and thus concluding that we can hear the area of the drum at least [1, p.11-14].

The above equation was based on the assumption that the particles are not influenced by the boundary which of course is not true, if $\mathbf{x}_0$ is close enough to the boundary then some of the matter diffusing from this point would indeed be affected by the boundary. By taking this into account Kac is able to create a better estimate for the diffusion problem. To simplify the mathematics, the domain was restricted to a convex drum.

By letting $\mathbf{q}$ be a point on the boundary $\partial\Omega$ closest to the initial starting position $\mathbf{x}_0$, and let $l(\mathbf{x}_0)$ be a line perpendicular to the line joining $\mathbf{x}_0$ to $\mathbf{q}$. Using the assumption that for some short time, t, the matter would not have travelled far enough about to have felt the curvature of the boundary. Therefore it is possible for $P_{\partial\Omega}$ to be approximated by $P_{l(\mathbf{x}_0)}$, which again satisfies the diffusion equation (6.1) and the initial condition (6.3) but with the boundary condition $P_{l(\mathbf{x}_0)}$ approaching 0 as $\mathbf{x}$ approaches a point on $l(\mathbf{x}_0)$ [1, p.14-15].

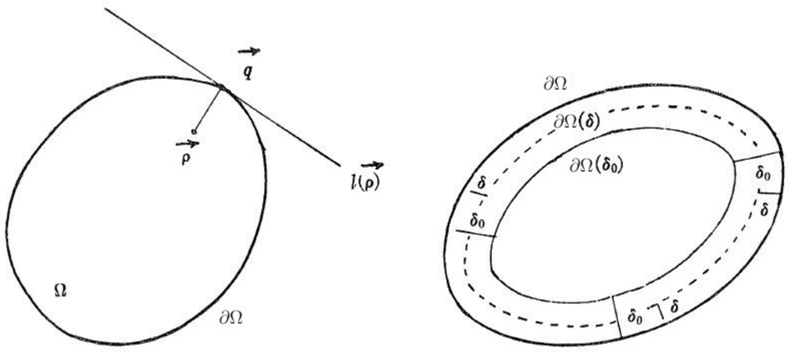

Figure 6.3: $P_{\partial\Omega}$ being approximated by $P_{l(\mathbf{x}_0)}$[1]

The reason for using this approach is that $P_{l(\mathbf{x}_0)}$ is well known [1, p.15].

$$P_{l(\mathbf{x}_0)}(\mathbf{x}_0|\mathbf{x};t) \sim \frac{1-e^{-2\partial^2/t}}{2\pi t}$$

where $\partial = ||\mathbf{q} - \mathbf{x}_0||$, the minimal distance from $\mathbf{q}$ to the boundary $\partial\Omega$. From (6.5) this can now be better estimated by [1, p.15]

$$\int_\Omega d\mathbf{x}_0\ P_\Omega(\mathbf{x}_0|\mathbf{x};t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{1}{2\pi t}\int_\Omega d\mathbf{x}_0\ e^{-2\partial^2/t}$$

To calculate the integral asymptotically by considering a curve whose distance from $\partial\Omega$ is δ. If δ is small enough the perimeter for this curve is well defined, thus Kac was able to show that by rearranging the equation to include the length of the perimeter, neglecting an exponentially small term, and any terms of order t [1, p.16], that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}}, \qquad t \to 0$$

Proving that not only could you hear the area of the drum, but also its perimeter. With $L^2 \geq 4\pi|\Omega|$ holding equality only for a circle, Kac determined that each circular drum has a unique frequency spectrum.

Pleijel showed that for a drum with a smooth boundary, a closer approximation could be found using Green's function [15].

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + \frac{1}{6}, \qquad t \to 0$$

Although Kac could not find a workable mathematical expression for a solution when Ω is a circular region using diffusion theory, he was able to solve the problem for polygonal drums, by setting the perimeter angles to be an infinite wedge of angle θ_0. He Kept the angle within the limits $\pi/2 < \theta_0 < \pi$ so that the drum's perimeter contained only obtuse angles and any matter diffusing from point $\mathbf{x}_0$ would "see" either a straight line, or an infinite wedge. By taking the number of sides N to infinity, so that θ_0 tends towards π, Kac was able to arrive at the same result [1, p.23].

Kac was also able to show that for multiple connected drums, i.e drums with holes that are both polygonal, even if the diffusion particles will see an obtuse angle at the holes boundary, the principle still works and thus for a smooth drum with a number, r, of smooth holes, one can hear the connectivity of the drum [1, p.23]

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + \frac{1}{6}(1-r), \qquad t \to 0$$

6.1 Since Kac's Paper

6.1.1 General Domain

Gordon, Webb and Wolpert (1992)[3][2] finally proved Kac's question in the negative, one cannot hear the shape of a drum. They were able to show the existence of a pair of convex domains in the Euclidean plane which are isospectral but not isometric.

By looking at the drumhead as a special case of a Riemann manifold, where any small piece of a curved surface can be projected as a slightly warped piece of an euclidean plane, even though the space and plane as a whole are remarkably different. This allowed the team to solve an easier but related problem to Kac's question and as any Riemann manifold has an associated wave equation, they instead were able to ask "Can one hear the shape of a Riemann Manifold?"[4]

A geodesic, a curve representing the shortest path between two points on a surface, can, when on a Riemann manifold, be thought of as a straight line in the Euclidean plane, that is, a curve that does not deviate from the direction in which it is travelling. Unlike the straight line, if there is only one point on the surface, a geodesic can travel all the way around the surface and meet back at the starting point. The list of vibration frequencies of a vibrating manifold is closely related to the list of lengths of closed geodesics [4].

In 1984, Toshikazu Sunada [36] found that isospectral manifolds could be constructed using an idea from group theory. Group theory was well established by this time and naturally has symmetry, thus the idea of creating the manifolds using permutation and linear representations of groups could be used. The authors created a group G that contains three special elements, α, β and γ, and all the elements of G can be obtained by taking products of these. Using the group on the set $X = \{1, 2, ..., 7\}$ they were able to represent the permutations as a Cayley graph. A second Cayley graph was also created that was identical in shape but had a different permutation representation of G.

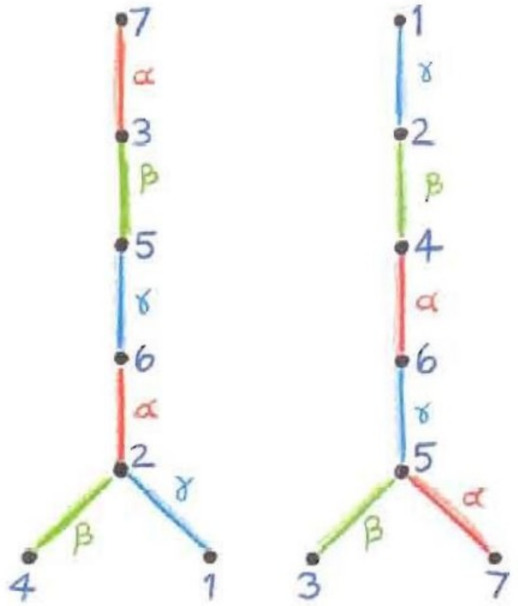

Figure 6.4: Two two cayley graphs with different permutation representations [4]

By representing the model triangle T, which has edges α, β, γ the team could create two different drums, D_1 and D_2 made of triangles, where each triangle was an element from the set X following the permutations of the Cayley graph. Using Pierre Bérard's proof of Sunada's theorem provided a method of "transplanting" a waveform on the drum D_1 onto a waveform of the same frequency on D_2, proving that the two non-isometric domains did indeed vibrate at the same frequency [4].

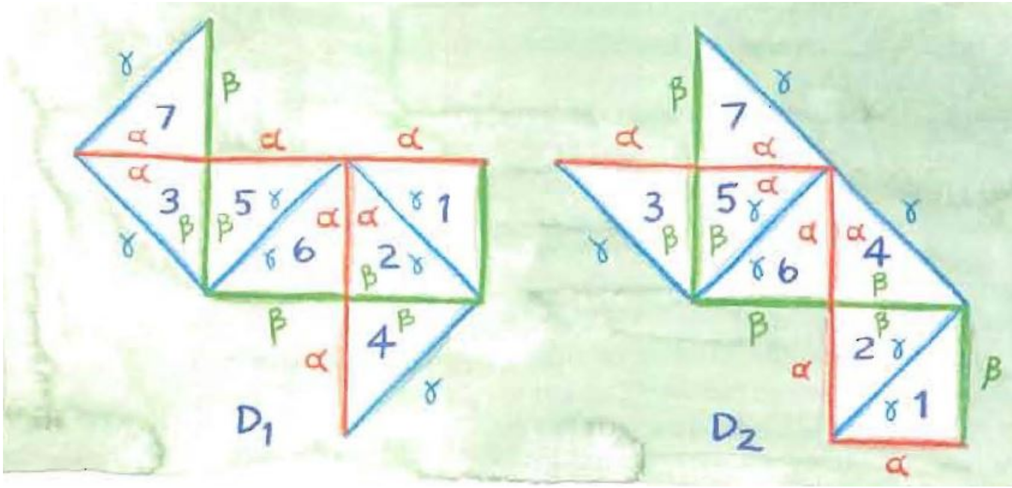

Figure 6.5: Two domains which have the same spectral properties [4]

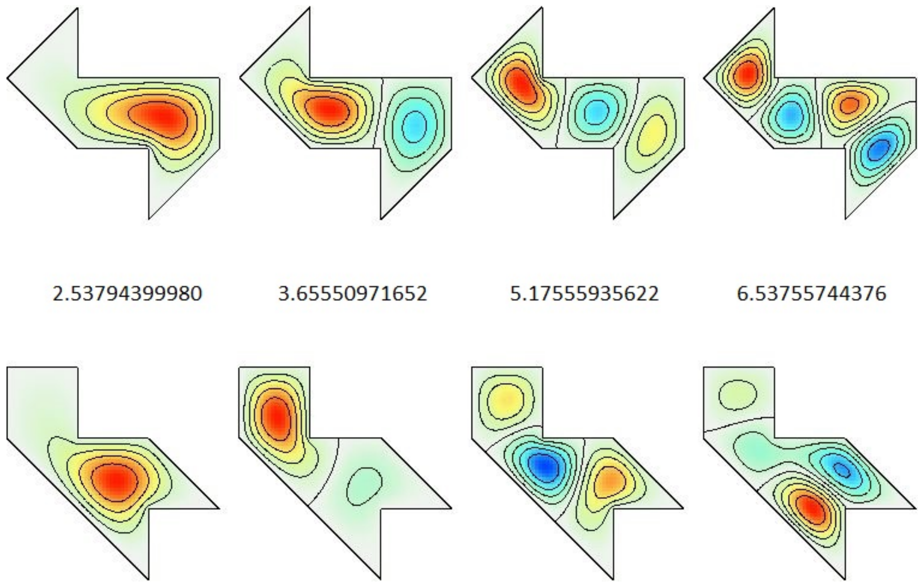

Figure 6.6: Eigenvalues that have been found analytically to 12 degrees accuracy. Note that although the eigenvalues for each membrane have the same values, the corresponding amplitudes differ. [27]

Although Gordon and Webb were able to show that not all domains were isospectral, Wolpert has proved that this is generally not the case and that most geometric surfaces are spectrally solitary [4].

Gordon et al. did not completely answer Kac's question as the domains were not convex. In 1994 Gordon and Webb [5] modified the shape of the fundamental tile to show pairs of convex domains with Dirichlet and Neumann boundary conditions in the hyperbolic domain were isospectral.

Kac's paper asked the question "Are the regions Ω_1 and Ω_2 congruent in the sense of Euclidean geometry?" [1, p.3] which of course has been answered above by Gordon et al. but if the title "Can One Hear the Shape of a Drum?" is taken in the literal sense, then the intensity of each eigenvalue must also be the same. Although this has not been answered either way, Zuluaga and Fonseca [7] have shown that the shapes in 6.1.1, the eigenvalues do have differing intensities and therefore can be distinguished by their sound.

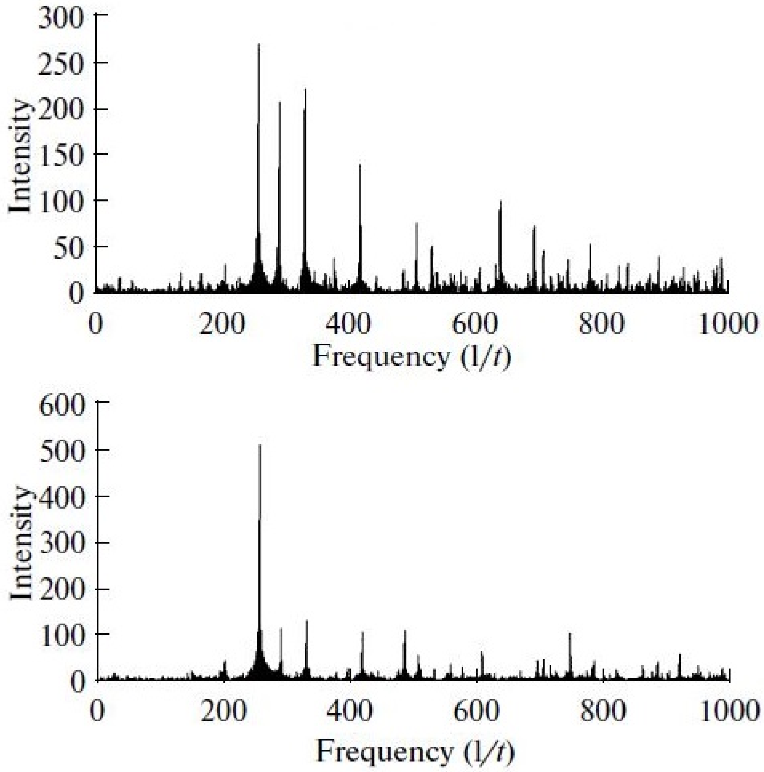

Figure 6.7: The intensities of the eigenvalues for the two domains D_1 and D_2, the second shape being the sharper of the two. [7]

6.1.2 Triangular Domain

By restricting the domains to have an analytical boundary and some symmetry Zelditch (2009) showed that the full spectrum was able to uniquely distinguish the shape of the domain. For Euclidean triangles, the area and perimeter would be unique and calculated, as long as the height was known. The difficulty was the calculation of the height purely based on the eigenvalues. Catherine Durso[16] was able to distinguish different the types of triangles by their spectral information. The unique shortest closed path for acute triangle already known, but Durso showed by using the wave trace equation, using the refractive properties of the triangles that for an obtuse or right angled triangle the shortest closed path is the shortest altitude traversed up and down, and this was unique.

Daniel Grieser and Svenja Maronna also proved the above by using the heat trace equation [17].

> *One can hear the shape of a triangle among all triangles. That is, if we know that Ω is a triangle, then the spectrum of Ω determines which triangle it is.*

Using this different path to produce the proof, they also proved that the first three eigenvalues determined the shape of a triangle, and stated without proof that $\lambda_1, \lambda_2, \lambda_4$. This was later proved by Pedro R. S. Antunes and Pedro Freitas in 2010[18] using numerical evidence that $\lambda_1, \lambda_2, \lambda_3$ does determine uniqueness of a triangle, but $\lambda_1, \lambda_2, \lambda_4$ does not determine shape of triangle.

6.1.3 Isothermal Drums

Lastly, with Kac use of diffusion theory for the basis of his paper, it would be interesting to know if diffusion itself determines the domain. If two domains in L^2 space had the same set of eigenvalues, by mapping the domains via the unitary operator, U, so that the set of eigenvalues A_1, A_2 for the domains Ω_1, Ω_2 respectively has the relation

$$Ue^{tA_1} = e^{tA_2}U, \qquad (t \geq 0)$$

Kac's question was answered in the positive for Dirichlet, Neumann and Robin Boundary conditions, but a counterexample for mixed boundaries in which Dirichlet or Neumann conditions for part of the boundary, and periodic boundary conditions imposed on the rest [26].

Chapter 7

Conclusion

In this dissertation it has been found that for one-dimensional systems, and two-dimensional system that have simple domains, the eigenvalues are periodic and can be be easily calculated. A circle membrane on the other hand has eigenvalues following the non-periodic Bessel zeros.

Techniques to find eigenvalues started by minimising the functional with certain constraints, the Rayleigh quotient simplified calculations by removing the need for the constraint $H[\varphi] = 1$. This helped create a simple way of estimating eigenvalues, especially for lower values of λ_n, from a trial function by using a method known as Rayleigh-Ritz. Used in engineering for approximating resonance in large structures such as skyscrapers and bridges, to molecular and atomic theory it has a vast array of uses in the physical world. The maximum-minimum principle showed it was possible to find eigenvalues without the need for knowing previous eigenfunctions, simplifying matters further.

Although these methods allowed individual eigenvalues to be found, in most cases it was too difficult, if not impossible to use for much larger λ_n. The method of finding asymptotic behaviour of eigenvalues, in spite of the fact that it cannot find specific values of λ_n would show how non-periodic systems behave over a large range, as n tends to infinity. This helps simplify the modelling of complex dynamical systems to allow the understanding of behaviour over large values of n. Weyl's Theorem showed that an arbitrary shaped domain's asymptotic behaviour relied upon its area and Pleijel extended this to give a closer approximation showing that the asymptotic behaviour also involved the domain's boundary length. Although Kac came to much the same conclusion he was able to do so by modelling the system using a more physical representation of diffusion theory. His famous question "Can one hear the shape of a Drum?" spurred on mathematicians to find an answer. Although it is now known that generally two isospectral membranes may have the same eigenvalues, it is not always the case, elliptical membranes of course are always isospectral as long as they ae not isometric, it has also been found to be the same for triangles, and for isothermal drums under certain domain conditions. Although these specific cases have been answered for large classes of domains the question remains open.

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