M821 Non Linear Ordinary Differential Equations

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10/10/2019

1 Chapter 1

1.1 Second-order differential equations

Implicit methods find a solution by solving an equations involving both the current state of the system and the later one.

$$G(Y(t), Y(t + \Delta t)) = 0$$

compared to explicit which calculates the state of a system at a later time only.

Simple Pendulum that is allowed to swing in a vertical plane with no friction, has the equation

$$\ddot{x} + \omega^2 \sin x = 0 \tag{1}$$

where x is horizontal distance. and $\omega^2 = g/a$.
With

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right)$$

where $\ddot{x}$ is called the energy transformation. Eq 1 then becomes

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) + \omega^2 \sin x = 0 \tag{2}$$

By integrating with respect to x we get

$$\frac{1}{2}\dot{x}^2 - \omega^2 \cos x = C \tag{3}$$

This is an equation for conservation of energy during any particular motion, multiplying by ma^2 gives

$$\frac{1}{2}ma^2\dot{x}^2 - mga\omega^2 \cos x = E$$

that is

$$E = \text{kinetic energy of } P + \text{potentional energy of} P$$

Writing in terms of $\dot{x}$ we get

$$\dot{x} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2} \tag{4}$$

With $\dot{x} = y$ we can set up a caresian plane, we can set up a phase plane and plot a family of curves from (4) using different values of C.

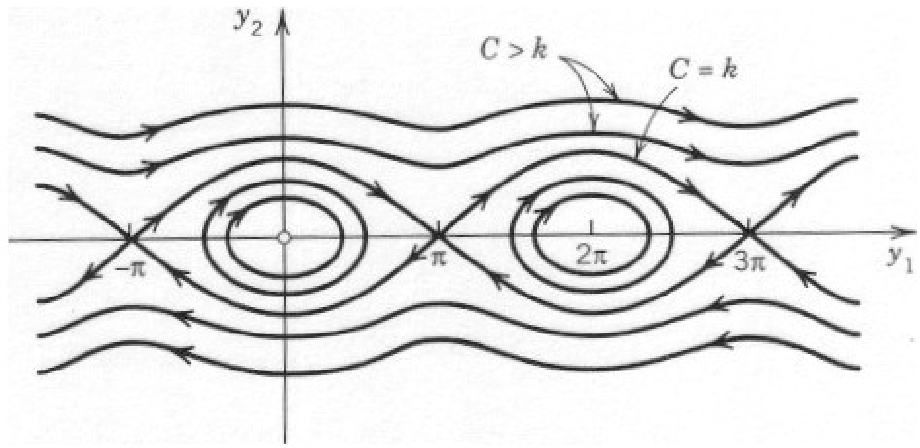


Figure 1: A simple pendulum phase diagram

The diagram above shows the position $x = y_1$ and velocity $\dot{x} = y_1$. At the origin $x = \dot{x} = 0$ therefore it has not velocity and no displacement. At $x = \pi, \dot{x} = 0$ the pendulum is balanced at the top and therefore can either stay there or move left or right.
We can use C as a constant (related to energy) to identify various types of phase path. On the paths joining $(-\pi, 0)$ and $(\pi, 0)$ $C = \omega^2$. for paths within these curves $\omega^2 > C > -\omega^2$ and for paths outside $C > \omega^2$.
A point P on the diagram is called the state of the system, and gives the angular velocity $\dot{x}$ at a particular inclination x. these serve as a pair of initial conditions for the original differential equation (1) which allows us to determine all subsequent states.
The directions in which we must proceed along the phase paths for increasing time is indicated by the arrow heads. These **always** go left to right in the upper plane, and right to left in the lower plane.

The points at $\dot{x} = 0, x = 0, x = \pi, x = -\pi$ represent states of physical equilibrium, they are as **equilibrium points**.
The family of closed curves surround the origin represent **periodic motion**. in which th pendulum swings to and fro about the vertical. The **amplitude** of the swings is the maximum value of x encountered on the curve.
The wavy lines represent the whirling motion of the pendulum, where the fluctuations about $\dot{x}$ are due to gravitational influences.

periodic motion - There are 2 types of periodic motion **liberation motion** where the pendulum oscillates (swings back and forth) thus the phase curve $y = \dot{x}$ changes sign, and **rotational motion** where the pendulum has enough energy to rotate around the pivot, thus $y = \dot{x}$ does not change sign. Both are distinct and regions of phase space containng each type are examples of **invariant regions** meaning that an orbit with initial conditions remains in this region for all time. The boundary between invariant regions is known as the **seperatrix**.

1.2 Constructing a phase Diagram

construct a phase diagram for $\ddot{x} - \omega^2 x = 0$
With

$$\dot{x} = y \qquad \dot{y} = f(x, y)$$

we have

$$\begin{aligned} \ddot{x} &= \omega^2 x \\ \dot{y} &= \omega^2 x \end{aligned}$$

When For equilibrium we need $\dot{x} = 0$ that is, $y = 0$ and $f(x, y) = 0$, only when $x = 0$ thus we have 1 equilibrium point at (0,0).
The phase path takes on the equation

$$\begin{aligned} \frac{dy}{dx} &= \frac{f(x, y)}{y} \\ \frac{dy}{dx} &= \omega^2 \frac{x}{y} \end{aligned}$$

Integrating gives

$$\begin{aligned} y =& \omega^2 \frac{x^2}{y} + C \\ y^2 =& \omega^2 x^2 + C \\ y^2 - \omega^2 x^2 =& C \end{aligned}$$

These paths are hyperbolas, together with their asymptotes $y = \pm\omega x$
This can be solved explicitly for x in terms of t. Eg, the general solution for $\ddot{x} + \omega^2 x = 0$ is:

$$a(t) = A\cos(\omega t) + B\sin(\omega t)$$

where A and B are arbitary constants. This can be written in another form by using the ordinary trigonometric identities. Put:

$$\kappa = (A^2 + B^2)^(1/2)$$

and let ϕ satisfy the equation:

$$\frac{A}{\kappa} = \cos\phi \ , \quad \frac{B}{\kappa} = \sin\phi$$

Then the general solution becomes:

$$x(t) = \kappa \cos(\omega t - \phi)$$

where the amplitude κ and the phase angle ϕ are arbitrary.
The period of ever oscillation is $2\pi/\omega$ which is independent of initial conditions (known as isochronous oscillation).

For $\ddot{x} - \omega^2 x = 0$ the general solution is

$$Ae^{\omega t} + Be^{-\omega t}$$

Thus

$$y = \dot{x}(t) = A\omega e^{\omega t} - B\omega e^{-\omega t}$$

With $\omega > 0$ and all solutions approach to infinity as $x \to \infty$, except when $A = 0$ where:

$$x = Be^{-\omega t} \quad , \quad y = -B\omega e^{-\omega t}$$

giving the path

$$\frac{y}{x} = -\omega$$

similarly if $B = 0$

$$x = Ae^{\omega t} \quad , \quad y = A\omega e^{\omega t}$$

where $y = \omega x$

1.3 Mechanical Analogy

Consider the conservative system

$$\ddot{x} = f(x) = 1 + x^2$$

then $f(x) > 0$ always, so with f say acting left to right, there will be no equilibrium points of the system, so we expect that for whatever initial conditions, the point will be carried away to infinity and there will be no oscillatory behaviour.
Next consider

$$f(x) = -\lambda x, \quad \lambda > 0$$

the equation of motion is $\ddot{x} = -\lambda x$. This can be thought of as a spring with stiffness λ, in which $\lambda = \omega^2$. The causes of the oscillations is that $f(x)$ is a restoring force, meaning that its direction is always trying to drive point P to the origin.
Now consider a spring having a non linear relation between tension and extension, where tension $= -f(x)$. Then we have

$$\begin{aligned} &f(x) > 0 \ \text{ for } x < 0 \\ &f(0) = 0, \\ &f(x) < 0 \ \text{ for } x > 0 \end{aligned}$$

When the particle P moves from position x to a nearby position $x + \delta x$, then work done $\delta\mathcal{W}$ is given by:

$$\delta\mathcal{W} = f(x)\delta x$$

This work goes into the kinetic energy $\mathcal{T} = \frac{1}{2}\dot{x}^2$,from the equation for conservation of energy (3),of the unit particle giving

$$\delta\mathcal{T} = \delta\mathcal{W} = f(x)\delta x$$

diving by δx and let $\delta x \to 0$ we obtain

$$\frac{d\mathcal{T}}{dx} = f(x)$$

Now define the function $\mathcal{V}$ by

$$\frac{d\mathcal{V}}{dx} = -f(x)$$

where $\mathcal{V}(x)$ is the potential function for f(x)

$$\mathcal{V}(x) = -\int f(x)dx$$

We have

$$\frac{d}{dt}(\mathcal{T} + \mathcal{V}) = 0$$

That is

$$\mathcal{T} + \mathcal{V} = \text{constant}$$

or explicitly

$$\frac{1}{2}\dot{x}^2 - \int f(x)dx = C$$

With

$$\frac{1}{2}\dot{x}^2 + \mathcal{V}dx = C$$

we can transpose to get

$$\dot{x} = \pm\sqrt{2(C - \mathcal{V}(x))} = y$$

Example , to find all the solutions to the equation $\ddot{x} + x^3 = 0$, $f(x) = -x^3$ so

$$\mathcal{V} = -\int f(x)dx = \frac{1}{4}x^4$$

thus

$$y = \pm\sqrt{2(C - \frac{1}{4}x^4)}$$

For any real values $C \geq 0$. When $C = 0, x = 0$ producing 1 point, the equilibrium point.

Equilibrium points occur when $y = 0, f(x) = 0$, or alternatively

$$y = 0, \quad \frac{d\mathcal{V}}{dx} = 0$$

A **minimum** of $\mathcal{V}(x)$ generates a **centre** (stable)
A **maximum** of $\mathcal{V}(x)$ generates a **saddle** (unstable)
A **point of inflection** leads to a **cusp**

1.4 Dampened Linear Oscillator

The simplest system is a linear oscillator with linear damping

$$\ddot{x} + k\dot{x} + cx = 0 \tag{5}$$

In a spring mass system this is

$$m\ddot{x} = -mcx - mk\dot{x}$$

The procedure to solving (5) is to look for solutions in the form

$$x(t) = e^{pt}$$

substituting this in to (5) gives

$$\begin{aligned}
\ddot{x} + k\dot{x} + cx &= 0 \\
\frac{d}{dt}\frac{d}{dt}x + k\frac{d}{dt}x + cx &= 0 \\
\frac{d}{dt}\frac{d}{dt}e^{pt} + k\frac{d}{dt}e^{pt} + ce^{pt} &= 0 \\
p^2 e^{pt} + kpe^{pt} + ce^{pt} &= 0 \\
p^2 + kp + c &= 0
\end{aligned}$$

which has solutions

$$\frac{1}{2}\left(-k \pm \sqrt{k^2 - 4c}\right) \tag{6}$$

which may be both real or complex conjugates. For $k^2 - 4ac \neq 0$ we have found two solutions, $e^{p_1 t}$ and $e^{p_2 t}$, and the general solution

$$x(t) = Ae^{p_1 t} + Be^{p_2 t} \tag{7}$$

where A and B are arbitrary constants and real if $k^2 - 4ac > 0$, complex conjugates if $k^2 - 4ac < 0$.

If $k^2 - 4ac = 0$ we have only one solution of the form $e^{-\frac{1}{2}kt}$. we create a second by using $te^{-\frac{1}{2}kt}$, giving the general solution

$$x(t) = (A + tB)e^{-\frac{1}{2}kt} \tag{8}$$

Putting

$$k^2 - 4c = \Delta$$

where Δ is called the discriminant of the characteristic equation (6). The physical character of motion depends upon the nature of the parameter Δ

Strong Damping $(\Delta > 0)$
p_1, p_2 are real, distinct and negative, and the general solution is:

$$x(t) = Ae^{p_1 t} + Be^{p_2 t}; \;\; p_1 > 0, p_2 > 0$$

There is no oscillation and the system converges quickly towards the origin

Weak Damping $(\Delta < 0)$
p_1, p_2 are complex with negative real part. To extract real solutions, allow A and B to be arbitrary and complex, and put:

$$A = \frac{1}{2}Ce^{i\alpha}$$

where α is real and $C = 2|A|$; and

$$B = \bar{A} = \frac{1}{2}Ce^{-i\alpha}$$

where $barA$ is the complex conjugate of A (i.e. the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. For example, (if a and b are real, then) the complex conjugate of $a + bi$ is $a - bi$). Then (7) reduces to

$$x(t) = Ce^{-\frac{1}{2}kt}cos\left(\frac{1}{2}\sqrt{(-\Delta)t} + \alpha\right)$$

where C and α are real and arbitrary, and $C > 0$ This type of damping will oscillate infinitely, with the amplitude deceasing over time

Critical Damping $(\Delta = 0)$
$p_1 = p_2 = -\frac{1}{2}k$ and the solutions are given by (8). This resembles the strong damping.

Negative Damping $(k < 0, c < 0)$
Instead of energy being lost to the system due to friction or resistance, energy is generated within the system. The node or spiral is then unstable and the directions are outwards. A slight disturbance from equilibrium leads to the system being carried far away from the equilibrium state.

Negative Stiffness $(c < 0, k$ takes any value$)$
The phase diagram shows a saddle point since p_1, p_2 are real but of opposite signs.

1.5 Nonlinear Damping

Consider the autonomous system

$$\ddot{x} = f(x, \dot{x})$$

where f is a nonlinear function and takes the form

$$f(x, \dot{x}) = -h(x, \dot{x}) - g(x)$$

so the differential equation becomes

$$\ddot{x} + h(x, \dot{x}) + g(x) = 0 \tag{9}$$

Giving the equivalent pair of 1st order equations for phase paths

$$\dot{x} = y \quad \dot{y} = -h(x, \dot{x}) - g(x) \tag{10}$$

We shall also assume that there is a single equilibrium point at origin thus

$$h(0, 0) + g(0) = 0$$

giving the only solution of $-h(x, 0) - g(0)$ is $x = 0$
Assuming that $g(0) = 0, \ \ h(0, 0) = 0$ we can say

$$\ddot{x} + g(x) = -h(x, \dot{x}) \tag{11}$$

which can be modelled by a particle unit on a spring whose free motion is governed by the equation $\ddot{x} + g(x) = 0$ (a conservative system, where $g(x)$ is the restoring force) but is also acted upon by an external force $-h(x, \dot{x})$ which either supplies or absorbs energy.

Define a potential energy function for the spring system

$$\mathcal{V} = \int g(x) dx \tag{12}$$

and the kinetic energy of the spring

$$\mathcal{T} = \frac{1}{2}\dot{x}^2 \tag{13}$$

Giving

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \frac{1}{2}\dot{x}^2 + \int g(x)dx \tag{14}$$

so that the rule of change of energy is (using implicit differentiation and the chain rule):

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2\right) + \frac{d}{dt}\left(\int g(x)dx\right) && (15)\\
&= \frac{1}{2}\frac{d}{d\dot{x}}\dot{x}^2\frac{d\dot{x}}{dt} + \frac{d}{dx}\int g(x)dx\frac{dx}{dt} && (16)\\
&= \dot{x}\ddot{x} + g(x)\dot{x} && (17)
\end{aligned}$$

Therefore, using (9) we get

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \dot{x}(-g(x) - h(x,\dot{x}) + g(x) = -\dot{x}h(x,\dot{x}) && (18)\\
&= -yh(x,y) && (19)
\end{aligned}$$

In some connected region $\mathcal{R}$ of the phase plane which contains the equilibrium point (0,0), $d\mathcal{E}/dt$ is negative

$$\frac{d\mathcal{E}}{dt} = -yh(x,y) < 0 \tag{20}$$

(except on $y = 0$). With any phase path which lies on $\mathcal{R}$ after a certain time, $\mathcal{E}$ continuously decreases along the path . The effect of h resembles damping or resistance, and energy is decreasing until it runs out, in which the path approaches the equilibrium point. If $\frac{d\mathcal{E}}{dt} > 0$ in $\mathcal{R}$ ($y \neq 0$) the energy increases along every such path, and h has the effect of negative damping

Example - examine the following for damping effects

$$\ddot{x} + |\dot{x}|\dot{x} + x = 0$$

The free motion is governed by $\ddot{x} + g(x) = 0$, that is $\ddot{x} + x = 0$ and the external force $-h(x,\dot{x}) = -|\dot{x}|\dot{x}$
Therefore the rate of change of energy is

$$\frac{d\mathcal{E}}{dt} = -|\dot{x}|\dot{x}^2 = -|y|y^2 < 0$$

Another example
Examine the following for energy input and damping effects

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$$

This gives

$$h(x,\dot{x}) = (x^2 + \dot{x}^2 - 1)\dot{x} = h(x,y)$$

putting $y = \dot{x}$

$$\frac{d\mathcal{E}}{dt} = -yh(x, y) = -(x^2 + y^2 - 1)y^2$$

giving

$$\frac{d\mathcal{E}}{dt} > 0 \text{ along the paths is the region } x^2 + y^2 < 1$$
$$\frac{d\mathcal{E}}{dt} < 0 \text{ along the paths is the region } x^2 + y^2 > 1$$

This produces an **isolated closed path**, isolated in the sense that there are no other closed paths in its immediate neighbourhood. An isolated path is called a **limit cycle** and when it exists it is always one of the most important features of a physical system. These can only occur in non linear systems. A stable limit cycle is one where any paths will be drawn in to the limit cycle (circle on x y plot) which ever side the point is. Unstable limit cycle is the opposite.

The above can be done using polar coordinates where:

$$x = r\cos\theta, \quad y = r\sin\theta = \dot{x}$$

and

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

differentiating these with respect to t

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}, \quad \dot{\theta}\sec^2\theta = \frac{x\dot{y} - \dot{x}y}{x^2} \tag{21}$$

so that

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}, \quad \dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2} \tag{22}$$

We have

$$\begin{aligned}
\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x &= \ddot{x} + (r^2\cos^2\theta + r^2\sin^2\theta - 1)r\sin\theta + r\cos\theta \\
&= \ddot{x} + (r^2(\cos^2\theta + \sin^2\theta) - 1)r\sin\theta + r\cos\theta \\
&= \ddot{x} + (r^2 - 1)r\sin\theta + r\cos\theta = 0 \\
\dot{y} &= -\ddot{(}r^2 - 1)r\sin\theta + r\cos\theta
\end{aligned}$$

Now we have x, y and $\dot{y}$ we can substitute them into equation (20) to get

$$\begin{aligned}
\dot{r} &= -r(r^2 - 1)\sin^2\theta \\
\dot{\theta} &= -1 - (r^2 - 1)\sin\theta\cos\theta
\end{aligned}$$

this gives

$$\begin{aligned}
&\dot{r} > 0 \text{ when } 0 < r < 1, \\
&\dot{r} < 0 \text{ when } r > 1
\end{aligned}$$

showing the path approaches the limit cycle $r = 1$ from both sides. The equation for $\dot{\theta}$ also shows a steady clockwise spiral motion for the representative points around the limit cycle.

Topographic Curves Example:- investigate $\ddot{x} + |\dot{x}|\dot{x} + x^3 = 0$
This system only has one equilibrium point at (0,0). Rewriting the equation and multiply through by $\dot{x}$ gives

$$\ddot{x}\dot{x} + x^3\dot{x} = -|\dot{x}|\dot{x}^2$$

In phase plane terms of x, y this becomes

$$\dot{y}y + x^3 y = -|y|y^2$$

Consider a phase path that passes through arbitrary points A and B and arrives at t_A, t_B respectively, by integration we obtain

$$\left[\frac{1}{2}y^2 + \frac{1}{4}x^4\right]_{t=t_A}^{t_B} = -\int_{t_A}^{t_B} |y|y^2 dt$$

As the right hand side is always negative we can say that

$$\left[\frac{1}{2}y^2 + \frac{1}{4}x^4\right]_{t=t_B} > \left[\frac{1}{2}y^2 + \frac{1}{4}x^4\right]_{t=t_A}$$

along the phase path, therefore the bracketed values constantly diminish along every phase path, but the family of curves given by the bracketed term is constant, i.e they are all the same. In mechanical terms they are curves of constant energy.

such families of closed curves, which can be used to track the paths to a certain extent are called topographic curves. The constant energy curves in this example constitute a special case

2 Chapter 2 - Plane autonomous systems and linearisation

Chapter 1 describes application of the phase-plane methods to $\ddot{x} = f(x, \dot{x})$ through the equivalent 1st order system $\dot{x} = y$ and $\dot{y} = f(x, y)$. More commonly it is need to be in the form

$$\dot{x} = X(x, y) \qquad \dot{y} = Y(x, y) \tag{23}$$

in which solutions are represented by curves $(x(t), y(t))$ where $x(t)$ and $y(t)$ are solutions. Constant solutions are represented by solving the equations $X(x, y) = 0$ and $Y(x, y) = 0$. Near the equilibrium points we make a linear approximation to $X(x, y)$ and $Y(x, y)$ solve the simpler equations obtained, and so determine the local character of the paths, and thus is a starting point for global investigations of the solutions.

The system is autonomous as the time variable t does not appear in the RHS.

The solutions $x(t)$ and $y(t)$ traces out a directed curve in the $x - y$ plane called a **phase path**. the appropriate for for the initial conditions of (23) is

$$x = x_0 \qquad y = y_0 \qquad \text{at } t = t_0 \tag{24}$$

using the uniqueness theorem (Appendix A) there is one and only one solution satisfying the condition when (x_0, y_0) is an ordinary point.

With

$$\frac{dx}{dt} = X(x,y) \quad \frac{dy}{dt} = Y(x,y) \tag{25}$$

the derived equation is

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)} \tag{26}$$

The solutions to equation (26) define curves in phase space and contain no information on direction or rate of flow. These lines in phase space are known as **phase curves / phase paths** Provided $X(x_0, y_0), Y(x_0, y_0) \neq 0$ there is a unique phase curve.
Solutions to equation (25) are function os t and thus contain information about the direction and rate of flow. These are called **orbits / trajectories**.
Points where $X(x,y) = Y(x,y) = 0$ are known as **equilibrium points** and if x_1, y_1 are a solution to that then $x(t) = X - 1$, $y(t) = y_1$ are constant solutions, and are degenerate phase paths.

Since $dy/dx = Y(x,y)/X(x,y)$ is the diff equation of the phase paths, phase paths which cut the curve where $Y(x,y) = cX(x,y)$ do so with the same slope c. These are known as **isoclines**. Two particular isoclines are when $X(x,y) = 0$ where the path cuts with infinite slope, and $Y(x,y) = 0$ where the path cuts with zero slope. Between the isoclines $X(x,y$ and $Y(x,y)$ must be of one sign. If both are positive, or both are negative then the slope will be positive, if they have opposite signs then the slope will be negative.

Example - Locate the equilibrium points, and sketch the phase path of

$$\dot{x} = y(1 - x^2), \qquad \dot{y} = -x(1 - y^2)$$

First we find the equilibrium points $X(x,y) = Y(x,y) = 0$ that is $y(1 - x^2) = 0, \ \ -x(1 - y^2) = 0$ giving $y = 0, x = \pm 1$ and $x = 0, y = \pm 1$. This gives the 5 points (0,0),(1,1),(1,-1),(-1,1),(-1,-1).
Next we find dy/dx

$$\frac{dy}{dx} = -\frac{x(1 - y^2)}{y(1 - x^2)}$$
$$-\int \frac{x}{(1 - x^2)} dx = \int \frac{y}{(1 - y^2)} dy$$
$$\frac{1}{2} \ln |1 - x^2| = -\frac{1}{2} \ln |1 - y^2|$$
$$(1 - x^2)(1 - y^2) = A \; (constant)$$

Notice that there are special solutions at $x = \pm 1$ and $y = \pm 1$ where $A = 0$ Starting at point, say $x = 0, y = 1$ and finding $Y(x, y)/X(x, y)$ gives the result 0/1 thus we know that there is a horizontal isocline between points (1-,1) and (1,1), in the positive direction (left to right) as $\dot{x} > 0$. This can be done at all other points.

2.1 Jacobian Linearisation

Linearisation makes it possible to use tools for studying linear systems to analyze the behaviour of a non-linear function near a given point.
With two functions, say $f(x, y)$ and $g(x, y)$, we can find linear approximations that will help create the phase diagram with regards to flow directions. First we use the Jacobi Matrix

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \tag{27}$$

Then substitute x, y with the equilibrium points, and find the eigenvalues by We then find λ by

$$\lambda = \frac{1}{2}\left(Tr(A) \pm \sqrt{Tr(A)^2 - 4\det(A)}\right) \tag{28}$$

where the trace and determinant are:

$$Tr(A) = a + d \quad \det(A) = ad - bc$$

In terms of the book this is

$$p = Tr(A), \quad q = \det(A) \quad \Delta = p^2 - 4q = TR(A)^2 - 4\det(A)$$

Eigenvalues are both real and distinct , where $\lambda_2 > \lambda_1$. That is $Tr(A)^2 > 4\det(A)$. We then get:
Stable Node - Eigenvalues are real and negative
Unstable Node - Eigenvalues are real and positive
Saddle - Eigenvalues are real and of different sign

This gives solutions

$$u_1 = \alpha e^{\lambda_1 t}, \quad u_2 = \alpha e^{\lambda_2 t}$$

Complex eigenvalues - $\lambda = \alpha + i\omega, \ \ \lambda* = \alpha - i\omega, \ \ sgn(\omega) = sgn(c)$
Stable spiral real part of eigenvalue negative
Unstable spiral real part of eigenvalue positive
Centre Eigenvalues imaginary

This gives solutions

$$r(t) = r_0 e^{\alpha t}, \quad \theta(t) = \omega t + \theta_0$$

Eigenvalues are real and equal That is

$$Tr(A)^2 = 4\det(A) \quad \text{that is} \quad (a-d)^2 + 4bc = 0$$

This an be satisfied in two ways, if **Stable star** A is diagonal, $b = c = 0$ and $Tr(A) < 0$
Unstable star A is diagonal, $b = c = 0$ and $Tr(A) > 0$
This has solutions

$$u_1 = \alpha e^{\bar{a}t}, \quad u_2 = (\beta + c\alpha t)e^{\bar{a}t}$$

for some constants $\alpha = u_1(0)$ and $\beta = u_2(0)$ **Stable improper node** A is *not* diagonal, $b = c = 0$ and $Tr(A) < 0$
Unstable improper node A is *not* diagonal, $b = c = 0$ and $Tr(A) > 0$

this has solutions

$$u_1 = \alpha e^{\bar{a}t}, \quad u_2 = c\alpha t e^{\bar{a}t}, \text{ or } u_2 = \frac{c}{|\bar{a}|} u_1 \ln(\alpha/u_1), \;\; \bar{a} < 0, \;\; \frac{u_1}{\alpha} > 0$$

Example Find and classify the equilibrium points for:

$$\dot{x} = \frac{1}{8}(x+y)^3 - y, \quad \dot{y} = \frac{1}{8}(x+y)^3 - x$$

Finding the values where $X(x,y) = Y(x,y) = 0$ we have (0,0), (1,1,) and (-1,-1).
Next we create the Jacobian Matrix

$$A(x,y) = \begin{bmatrix} \frac{3}{8}(x+y)^2 & \frac{3}{8}(x+y)^2 - 1 \\ \frac{3}{8}(x+y)^2 - 1 & \frac{3}{8}(x+y)^2 \end{bmatrix}$$

At the origin (0,0) we have

$$A(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Giving $Tr(A) = 0$ and $\det(a) = -1$. With $\lambda = 1, -1$ we have a saddle.

At (1,1) we have

$$A(1,1) = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$$

Giving $Tr(A) = 3$ and $\det(a) = 2$. With $\lambda = 1, 2$ we have an unstable node.

At (-1,-1) we have the same as above.

2.2 Hamiltonian Systems

A Hamiltonian system exists if a function $H(x, y)$ is in the form

$$X = \frac{\delta H}{\delta y}, \quad Y = -\frac{\delta H}{\delta x} \tag{29}$$

H is called the Hamiltonian function. A necessary and sufficient condition is that

$$\frac{\delta H}{\delta x} + \frac{\delta H}{\delta y} = 0 \tag{30}$$

Let $x(t), y(t)$ represent a particular time solution, then along the corresponding phase path

$$\begin{aligned} \frac{dH}{dt} &= X = \delta H \delta x \frac{dx}{dt} + \delta H \delta y \frac{dx}{dy} \\ &= -YX + XY \\ &= 0 \end{aligned}$$

therefore

$$H(x, y) = \text{ constant} \tag{31}$$

along any phase path, the phase paths are level curves, or contours, $H(x, y) = C$ of the surface $z = H(x, y)$ in three dimensions.

Suppose we have an equilibrium point so that

$$\delta H \delta x = \delta H \delta y = 0 \text{ at } (x_0, y_0) \tag{32}$$

Then $H(x, y)$ has a stationary point at (x_0, y_0) Sufficient conditions for the the 3 main types are by the following. Put

$$q_0 = \delta^2 H \delta x^2 \delta^2 H \delta y^2 - \left(\frac{\delta H}{\delta x \delta y}\right)^2 \tag{33}$$

evaluated at (x_0, y_0). Then it has:
a maximum or minimum if $q_0 > 0$, this will be a centre
a saddle if $q_0 < 0$, this will be a saddle point

there is no case corresponding to a node or spiral, a Hamiltonian system contains only various centres and saddle points.

Linearising the equations at equilibrium points gives

$$\dot{x} = a(x - x_0) + b(y - y_0), \quad \dot{y} = c(x - x_0) + d(y - y_0) \tag{34}$$

where the coefficients become

$$a = \frac{\delta^2 H}{\delta x \delta y}, \;\; b = \frac{\delta^2 H}{\delta y^2}, \;\; c = -\frac{\delta^2 H}{\delta x^2}, \;\; d = -\frac{\delta^2 H}{\delta x \delta y} \tag{35}$$

With $p = a + d = 0$ and $q = ad - bc = -\left(\frac{\delta^2 H}{\delta x \delta y}\right)^2 + \frac{\delta^2 H}{\delta x^2}\frac{\delta^2 H}{\delta y^2}$ at (x_0, y_0) Note that for a centre of $H(x, y)$ is conclusive, that is the fixed point is a centre of a non linear system and not just its linearisation.

Exercise Show that the system $\dot{x} = \ln y$, $\dot{y} = 4xy((x+1)(x+2)$ is not Hamiltonian.
show that the transformation $u = x+1$, $v = \ln y$ makes it Hamiltonian and find the Hamiltonian.
Find and classify the fixed points in (u, v) space.

We need

$$\frac{\delta X}{\delta x} + \frac{\delta Y}{\delta y} = 0$$

This gives

$$0 + 4x(x+1)(x+2) \neq 0 \text{ for general } x$$

therefore not Hamiltonian

Let

$$\begin{aligned}
u &= x + 1 \\
\dot{u} &= \dot{x} \\
&= \ln y \\
&= v
\end{aligned}$$

$$\begin{aligned}
v &= \ln y \\
\dot{v} &= \frac{d}{dt}\ln y \\
&= \frac{1}{y}\dot{y} \\
&= 4x(x+1)(x+2) \\
&= 4(u-1)u(u+1) \text{ where u=x-1} \\
&= 4u(u^2-1)
\end{aligned}$$

Giving

$$\begin{aligned}
\dot{u} &= v = U(u, v) \\
\dot{v} &= 4u(u^2 - 1) = V(u, v)
\end{aligned}$$

The Hamiltonian system is now

$$\frac{\delta U}{\delta u} + \frac{\delta V}{\delta v} = 0 \quad (36)$$

$$0 + 0 = 0 \quad (37)$$

Therefore it is now a Hamiltonian system.
We now need to seek H such that

$$U = \frac{\delta H}{\delta v} \quad V = -\frac{\delta H}{\delta u}$$

so we have

$$\frac{\delta H}{\delta v} = v \tag{38}$$

$$\frac{\delta H}{\delta u} = -4u(u^2 - 1) \tag{39}$$

integrate (38) wrt v

$$H = \int v dv = \frac{v^2}{2} + f(u)$$

where $f(u)$ is an arbitrary function due to the partial integration.
Then substitute into (39) to get

$$\frac{\delta}{\delta u}\left(\frac{v^2}{2} + f(u)\right) = -4u(u^2 - 1)$$

we now finf $f(u)$ by integrating wrt u

$$f(u) = \int -4u(u^2 - 1)du$$

$$f(u) - u^4 + 2u^2 + C$$

where we set $C = 0$ as H is used only in differentiation, so there is no loss of generality.
Thus we have

$$H(u, v) = \frac{v^2}{2} - u^4 + 2u^2$$

3 Geometrical Aspects of plane autonomous systems

Given the system

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y)$$

let Γ be any closed curve, going counter clockwise, consisting of only ordinary points (i.e does not pass through any equilibrium points). Let $\mathcal{Q}$ be a point of Γ then there is only one phase path passing through $\mathcal{Q}$. The phase paths belong to the family described by the equation

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)} \tag{40}$$

In time $\delta t > 0$ the coordinates of $\mathcal{Q}, (x_{\mathcal{Q}}, y_{\mathcal{Q}})$ will increase by $\delta x, \delta y$ respectively, where

$$\delta x \approx X(x_{\mathcal{Q}}, y_{\mathcal{Q}})\delta t, \quad \delta y \approx Y(x_{\mathcal{Q}}, y_{\mathcal{Q}})\delta t$$

Therefore the vector $\mathbf{S} = (\mathbf{X}, \mathbf{Y})$ is tangential to the phase paths through this point, and points in the direction of increasing t. Its inclination can be measured by the angle ϕ from the positive direction of the x axis, so that

$$\tan\phi = Y/X \tag{41}$$

Once one value of ϕ has been found on the closed curve Γ, then the value for the other points is settled as going around the curve the first value of ϕ must be the same $\mod 2\pi$. This **change in** ϕ, denoted as:

$$[\phi]_\Gamma = 2\pi I_\Gamma$$

where I_Γ is an integer, positive, negative or zero. It is called thee **Index of** Γ and tells us how many times the closed path loops through 2π
Suppose that the curve Γ is described by position vector r such that

$$\mathbf{r}(s) = (x(s), y(s)), \quad s_0 \le s \le s_1$$

$$[\phi]_\Gamma = \int_{s=s_0}^{s=s_1} d\phi = \int_{s_0}^{s_1} ds \frac{d\phi}{ds}$$

where $x(s_0) = x(s_1), y(s_0) = y(s_1)$ for a closed curve we write

$$[\phi]_\Gamma = \oint d\phi$$

From eq41 we have

$$\frac{d}{ds}\tan\phi = \frac{d}{ds}\left(\frac{Y}{X}\right)$$

upon reduction

$$\frac{d\phi}{ds} = \frac{XY' - YX'}{X^2 + Y^2}$$

Therefore

$$I_r = \frac{1}{2\pi}\oint_{\Gamma_r} \frac{XY' - YX'}{X^2 + Y^2} \tag{42}$$

Theorem 3.1 If X and Y are not simultaneously zero and their 1st derivatives are continuous therefore there is no equilibrium point there, then $I_r = 0$

3.1 Green's Theorem

If Γ is a closed non-self-intersecting curve, on which the functions $P(x, y)$ and $Q(x, y)$ have continuous first partial derivatives then:

$$\oint_\Gamma (Pdx + Qdy) = \int\int_{D_\Gamma} \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)$$

where D_Γ is the region interior to Γ

Let Γ be a simple closed curve, and Γ' a simple closed curve inside of Γ, then if the region between them has no equilibrium points and if X, Y and their 1st derivatives are continuous then $I_\Gamma = I'_\Gamma$

If Γ surrounds n equilibrium points $P_1, P_2..P_n$ then

$$I_\Gamma = \sum_{i=1}^{n} I_i \tag{43}$$

where I_i is the index point P_i,$i = 1, 2, ..., n$

Let p be the number of times $\frac{Y(x,y)}{X(x,y)}$ changes from $+\infty$ to $-\infty$ and q be the number of times it changes from $-\infty$ to $+\infty$ on Γ. Then $I_\Gamma = \frac{1}{2}(p - q)$
This is basically counting the number of times $\tan\phi$ the direction is vertically up or down.

Example - Find the index of the equilibrium point (0,0) of $\dot{x} = y^3$, $\dot{y} = x^3$

Let Γ be a square around (0,0) where $x = \pm 1, y = \pm 1$. Starting from (1,1) and going around counter-clockwise.
$y = 1$ we have $\tan\phi = x^3$ in which there are no vertical lines going to infinity
$x = -1$ we have $\tan\phi = -y^{-3}$ this starts at $+y$ and going to $-y$, so we have $\tan\phi = -\infty$ to $\tan\phi = +\infty$
$y = -1$ we have $\tan\phi = -x^3$ in which there are no vertical lines going to infinity
$x = 1$ we have $\tan\phi = y^{-3}$ this starts at $-y$ and going to $+y$, so we have $\tan\phi = -\infty$ to $\tan\phi = +\infty$
Thus we have $p = 0, q = 2$ giving $I_\Gamma = \frac{1}{2}(-2) = -1$.

If we already know the nature of the equilibrium point, the index is readily found by simply drawing a figure and following the angle around.

3.2 Index at Infinity

Example - Express the equations $\dot{x} = 2xy, \dot{y} = x^2 + y^2$ in terms of x_1, y_1 Find the index at infinity of the system.

we have

$$Z = X + iY$$

and

$$Z_1 = X_1 + iY_1$$

now

$$\frac{dz_1}{dt} = -z_1^2 Z = Z_1$$

where

$$z = x + iy, \quad z_1 = x_1 + iy_1$$

Thus

$$\begin{aligned} \frac{dz_1}{dt} = -z_1^2 Z =& Z_1 \\ -z_1^2(2xy + i(x^2 + y^2)) =& Z_1 \end{aligned}$$

With $x = r\cos\theta,\ y = r\sin\theta,\ x_1 = r_1\cos\theta_1,\ y_1 = r_1\sin\theta_1,\ r_1 = 1/r,\ \theta_1 = -\theta.$
Let $z_1 = r_1 e^{i\theta_1}$

$$\begin{aligned} Z_1 =& -z_1^2(2xy + i(x^2 + y^2)) \\ =& -r_1^2 e^{2i\theta_1}(2r^2 cos\theta \sin\theta + i(r^2 cos^2\theta + r^2\sin^2\theta)) \\ =& -r_1^2 e^{2i\theta_1}(r^2\sin 2\theta + ir^2) \\ =& -e^{2i\theta_1}(\sin 2\theta + i) \\ =& e^{2i\theta_1}(\sin 2\theta_1 - i) \\ =& (\cos(2\theta_1) + i\sin(2\theta_1))(\sin 2\theta_1 - i) \\ =& \sin(2\theta_1) + \frac{1}{2}\sin(4\theta_1) + \frac{1}{2}i\,(1 - 2\cos(2\theta_1) - \cos(4\theta_1)) = X_1 + Y_1 \end{aligned}$$

Now with

$$I_\infty = \frac{1}{2\pi}\int_{s_0}^{s_1} \frac{XY' - YX'}{X^2 + Y^2}$$

$$X^2 + Y^2 = \frac{3}{2} - \frac{1}{2}\cos(4\theta_1)$$

and

$$X_1\frac{dY_1}{d\theta_1} - Y_1\frac{dX_1}{d\theta_1} = 3 + 2\cos(2\theta_1) - \cos(4\theta_1)$$

giving

$$\begin{aligned} I_\infty =& \frac{1}{2\pi}\int_0^{2\pi} d\theta_1 \frac{3 + 2\cos(2\theta_1) - \cos(4\theta_1)}{\frac{3}{2} - \frac{1}{2}\cos(4\theta_1)} \\ =& \frac{1}{\pi}\int_0^{2\pi} d\phi \frac{3 + 2\cos(\phi) - \cos(\phi)}{3 - \cos(\phi)} \\ =& 2 + \frac{1}{\pi}\int_0^{2\pi} d\phi \frac{\cos\phi}{1 + sin^2\phi} = 2 \end{aligned}$$

due to the last integral being zero by symmetry

Limit Cycles and other closed paths The index of a limit cycle is 1 since the vector (X,Y) is tangential to $\mathcal{C}$ at every point on it, and the change in ϕ is 2π by (43). This result applies to any closed paths.

A closed path cannot surround a region containing no equilibrium points, nor one containing only a saddle.

Bendixson's Negative Criterion states that there are no closed paths in a simply connected region of the phase plane on which $\frac{\delta X}{\delta x} + \frac{\delta Y}{\delta y}$ is of one sign.

example -

$$\dot{x} = yx^2 - x^3 - 3xy^2 - 2y + y^2 = X$$
$$\dot{y} = 2xy^2 - x^2 + 4x^3 - 2y = Y$$

$$\begin{aligned}\frac{\delta X}{\delta x} + \frac{\delta Y}{\delta y} =& 2yx - 3x^2 - 3y^2 + 4xy - 2 \\ =& -3x^2 - 3y^2 + 6xy - 2 \\ =& -3(x^2 + y^2 - 2xy) - 2 \\ =& -3(x-y)^2 - 2 < 0\end{aligned}$$

Thus there are no periodic solutions in the phase plane.

example-

$$X = y, \quad Y = x - y + x^2 + y^2$$
$$\frac{\delta X}{\delta x} + \frac{\delta Y}{\delta y} = 0 - 1 + 2y$$

Therefore there *may* be a periodic solution in the phase pane.

Dulacs Test - is the general case of Bendixsons, where ρ is a suitable value to achieve the result (it will be given for you). In the example above, setting $\rho = e^{-2x}$ and

$$\frac{\delta(\rho X)}{\delta x} + \frac{\delta(\rho Y)}{\delta y}$$

gives

$$\begin{aligned}&= -2e^{-2x}y + e^{-2x}(-1 + 2y) \\ &= -e^{-2x} < 0\end{aligned}$$

Thus there are no periodic solutions in the phase plane.

3.3 Homoclinic and Heteroclinic Paths

A seperatrix is a phase path that seperates two distinct regions.
Any phase path which joins an equilibrium point to itself is known as a **homoclinic path** - these can only be associated with saddle points since both outgoing and incoming path is needed.
A **heteroclinic path** is one which joins an equilibrium point to another. These two hyperbolic equilibrium points (eigenvalues with nonzero real parts) can be a saddle point, a node or a spiral.

example Find the homoclinic phase paths for the system

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

Also find the solutions for x in terms of t.

Equilibrium points at (0,0) and $(\pm 1, 0)$ the origin is a saddle point whilst $x = \pm 1$ are both centres. The phase path satisfies the seperable differential equation

$$\frac{dy}{dx} = \frac{x(1 - x^2)}{y}$$

integrating gives

$$y^2 = x^2 - \frac{1}{2}x^4 + C$$

Homoclinic paths can only be associated with the saddle point at origin, the phase paths approach the origin only if $C = 0$
That is

$$y^2 = x^2 - \frac{1}{2}x^4$$

There are two paths, one over the interval $0 \leq x \leq \sqrt{2}$ and $-\sqrt{2} \leq x \leq 0$

the time solutions can be found be integrating the equation above

$$y^2 = \left(\frac{dx}{dt}\right)^2 = x^2 - \frac{1}{2}x^4$$

$$\int \frac{dx}{x\sqrt{1 - \frac{1}{2}x^2}} = \int_{t_0}^{t} dt = (t - t_0)$$

substituting $x = \pm\sqrt{2}\,\mathrm{sech}\,u$ gives

$$u = \pm(t - t_0)$$

Hence the homoclinic solutions are $x = \pm\sqrt{2}\,\mathrm{sech}(t - t_0)$ for any t_0 since $x \to 0$ as $t \to \infty$

Homoclinic bifurication is a transition from hetroclinic spiral-saddle connection to homoclinic saddle connection via the effect of dampening. In the example given, there is a saddle at (0,0) with an equilibrium point either side. These are unstable spirals when $\epsilon < 0$, centres at $\epsilon = 0$, where the saddle at (0,0) goes around these and attaches back to itself, and stable spirals at $\epsilon > 0$

4 Energy balance method for limit cycles

With

$$\ddot{x} + \epsilon h(x, \dot{x}) + x = 0$$

then on the phase plane we have

$$\dot{x} = y \quad \dot{y} = -\epsilon h(x, y) - x$$

Assume that $|\epsilon| << 1$, so that non linearity is small, and that $h(0,0) = 0$ so that the origin is an equilibrium point. When $\epsilon = 0$, the equation becomes $\ddot{x} + x = 0$ - called the **linearised equation**. Its general solution is

$$x(t) = a\cos(t + \alpha)$$

where a and α are arbitrary constants. Restricting to the cases $a > 0$, $\alpha = 0$, the family of phase paths is given parametrically by

$$x = a\cos t, \quad y = -a\sin t$$

which is the family of circles $x^2 + y^2 = a^2$, the period f motion is equal to 2π.

For small enough ϵ we expect that any limit cycle, or periodic motion, will behave likeone of the circular motions as $\epsilon \to 0$, therefore for some value a

$$x(t) \approx a\cos t, \quad y(t) \approx -a\sin t, \quad T \approx 2\pi$$

From 1.53 we get

$$\mathcal{E}(t) = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t)$$

over one period this is, over the closed path (thus =0)

$$\mathcal{E}(T) - \mathcal{E}(0) = -\epsilon\int_0^T h(x(t)), y(t))y(t)dt = 0$$

Inserting the approximations into this equation gives

$$\mathcal{E}(2\pi) - \mathcal{E}(0) = \epsilon a\int_0^{2pi} h(a\cos t, -a\sin t)\sin t dt = 0$$

Reducing to

$$\int_0^{2pi} h(a\cos t, -a\sin t)\sin t dt = 0$$

example Find the approximate amplitude of the limit cycle of the van der Pol equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

This gives $h(x, y) = (x^2 - 1)y$ placing in the integral gives

$$\int_0^{2\pi} \left[(x(t)^2 - 1)y(t)\right] y(t)dt = 0$$

Assuming $x = a\cos t$ this gives $y = -a\sin t$. Putting this into the integral

$$\begin{aligned}
\int_0^{2\pi} \left[(a^2\cos^2 t - 1)(-a\sin t)\right](-a\sin t)dt =& 0 \\
a^2 \int_0^{2\pi} (a^2\cos^2 t - 1)\sin^2 t dt =& 0 \\
\int_0^{2\pi} a^2\cos^2 t\sin^2 t - \sin^2 t dt =& 0 \\
\int_0^{2\pi} a^2\left(\frac{1 - \cos(4t)}{8}\right) - \frac{1 - \cos(2t)}{2} dt =& 0 \\
\left[\frac{a^2}{8}\left(t - \frac{-\sin(4t)}{4}\right) - \frac{t}{2} + \frac{\sin(2t)}{2}\right]_0^{2\pi} =& 0 \\
\frac{2\pi a^2}{8} - \frac{2\pi}{2} =& 0 \\
\frac{1}{4}a^2 - 1 =& 0
\end{aligned}$$

with a positive solution $a = 2$

By an extension of this argument the stability of a limit cycle can also be determined, we should expect that *unclosed* paths near enough to the limit cycle, spiralling gradually , will also be given by $x \sim a(t)\cos t, y \sim -a(t)\sin t$ where $a(t)$ is nearly constant over a time interval (not now an exact period) of $0 \leq t \leq 2\pi$.

denoting the energy balance equation by $g(a)$

$$g(a) = \epsilon a \int_0^{2pi} h(a\cos t, -a\sin t)\sin t dt = 0$$

and let $a \sim a_0 (> 0)$ on the limit cycle, then $g(a_0) = 0$
If the limit cycle is stable then along the interior spiral segments $(a < a_0)$, energy is gained, and along the exterior segments $(a > a_0)$ energy is lost. That is to say, for some value of $\delta > 0$

$$g(a) > 0 \text{ when } a_0 - \delta < a < a_0$$

$$g(a) < 0 \text{ when } a_0 < a < a_0 + \delta$$

If the signs of the inequalities are both reversed then the limit cycle is unstable. Therefore the existence and stability of a limit cycle of amplitude a_0 are determined by the conditions

$$g(a_0) = 0, \quad \text{stable if } g'(a_0) < 0, \quad \text{unstable if } g'(a_0) > 0$$

Example From the example above we have

$$-\epsilon a^2 \int_0^{2\pi} \left[(a^2 \cos^2 t - 1)(-a \sin t)\right] (-a \sin t) dt = -\epsilon a^2 \pi \left(\frac{1}{4} a^2 - 1\right) = 0$$

Note that we have kept the $-\epsilon a^2$ and π in the equation.
Therefore

$$g'(a) = -\epsilon \pi a(a^2 - 2)$$

Putting $a_0 = 2$ from the example gives

$$g'(2) = -4\epsilon\pi$$

Therefore the cycle in stable when $\epsilon > 0$, unstable when $\epsilon < 0$.

Direct integration equation is given on pg 129

4.1 Amplitude and frequency estimates: Polar Coordinates

With the same equations as before

$$\ddot{x} + \epsilon h(x, \dot{x}) + x = 0, \quad \dot{x} = y, \quad \dot{y} = -\epsilon h(x, y) - x$$

and suppose it has at least one periodic time solution, corresponding to aa closed path. Let it be represented by polar coordinates $a(t), \theta(t)$, by eqn 1.58 we have

$$\dot{a} = -\epsilon h \sin\theta$$

$$\dot{\theta} = -1 - \epsilon a^{-1} h \cos\theta$$

and the differential equation for the phase paths is therefore

$$\frac{da}{d\theta} = \frac{\epsilon h \sin\theta}{1 + \epsilon a^{-1} h \cos\theta}$$

where $h = h(a\cos\theta, a\sin\theta)$ These equations generally hold whether ϵ is small or not.

Let its time period be T, and that θ has time period of 2π exactly, thus meaning $a(t_0 + T) = a(t_0)$ for every t_0. t is measured in the clockwise direction, θ is measured in the anticlockwise direction such that

$$a = a_0 \quad \theta = 2\pi \text{ at } t = 0$$

$$a = a_0 \quad \theta = 0 \text{ at } t = T$$

so as t increases θ decreases.

The circular frequency of the periodic oscillation is

$$\omega = \frac{2\pi}{T} \approx 1 + \frac{\epsilon}{2\pi a_0} \int_0^{2\pi} h(a_0 \cos\theta, a_0 \sin\theta) \cos\theta d\theta$$

Example Obtain the frequency of the limit cycle of the van der Pol equation, correct to order ϵ

Since $h(x;\dot{x}) = (x^2 - 1)\dot{x}$ and the amplitude $a_0 = 2$ as found in the example before. We have $(x^2 - 1)\dot{x} = (a^2 \cos^2\theta - 1)(a \sin\theta) = (4\cos^2\theta - 1)(2\sin\theta)$ so

$$\begin{aligned} \omega =& 1 + \frac{\epsilon}{2\pi a_0} \int_0^{2\pi} h(a_0 \cos\theta, a_0 \sin\theta) \cos\theta d\theta \\ =& 1 + \frac{\epsilon}{4\pi} \int_0^{2\pi} (4\cos^2\theta - 1)(2\sin\theta) \cos\theta d\theta \\ =& 1 + 0 \end{aligned}$$

Giving the frequency equal to 1, with error $O(\epsilon^2)$

Example 2 Obtain the relation between frequency and amplitude of the swings of the pendulum

$$\ddot{x} + \sin x = 0$$

Expanding $\sin x \approx x - \frac{1}{6}x^3$ gives the equation

$$\ddot{x} - \frac{1}{6}x^3 + x = 0$$

which is the member of the $\ddot{x} + \epsilon x^3 + x = 0$ family, with $\epsilon = -\frac{1}{6}$. Thus $h(x, \dot{x}) = x^3$.
The amplitude is therefore

$$\begin{aligned} \int_0^{2\pi} h(a_0 \cos\theta, a_0 \sin\theta) \sin\theta d\theta =& 0 \\ a_0^3 \int_0^{2\pi} \cos^3\theta \sin\theta d\theta =& 0 \end{aligned}$$

The integrand will equal 0 (as it's an odd function) so the equation is satisfied identically for all a_0. This is to be expected since the origin $x = 0, y = 0$ in the phase plane is a center.

The frequency ω becomes

$$\omega = 1 + \frac{\epsilon}{2\pi a_0}\int_0^{2\pi} a_0^3 \cos^4\theta d\theta$$

for $\epsilon = -\frac{1}{6}$. Carrying out the integration gives

$$\omega = 1 - \frac{1}{16}a_0^2$$

4.2 Averaging method for spiral phase paths

Find the approximate time solution for the van der Pol's equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

for small positive ϵ. We use the equation

$$\begin{aligned}
p_0(a) =& \frac{1}{2\pi}\int_0^{2\pi} h(a\cos u, a\sin u)\sin u \;\; du \\
=& \frac{a}{2\pi}\int_0^{2\pi} (a^2\cos^2 u - 1)\sin^2 u \;\; du \\
=& \frac{1}{2}a\left(\frac{1}{4}a^2 - 1\right)
\end{aligned}$$

We have the approximate equation for the radial coordinate

$$\begin{aligned}
\frac{da}{dt} =& -\epsilon p_0(a) \\
=& -\frac{1}{2}\epsilon a\left(\frac{1}{4}a^2 - 1\right)
\end{aligned}$$

with constant solution $a = 2$ corresponding to the limit cycle. The equation seperates giving

$$\begin{aligned}
\int \frac{da}{a(a^2-1)} =& -\frac{1}{8}\epsilon(t + C) \\
-\frac{1}{4}\log a + \frac{1}{8}\log|a^2 - 4| =& -\frac{1}{8}\epsilon(t + C)
\end{aligned}$$

With initial condition $a(0) = a_1$ the solution is

$$a(t) = \frac{2}{\{1 - (1 - (4/a_1^2))e^{-\epsilon t}\}^{1/2}}$$

which tends to 2 as $t \to \infty$ With

$$r_0(a) = \frac{1}{2\pi}\int_0^{2\pi} h\{a\cos u, a\sin u\}\cos u \;\; du = 0$$

we have the equation

$$\begin{aligned}\frac{d\theta}{dt} =& -1 - \epsilon a^{-1} r_0(a) + O(\epsilon^2)\\ =& -1\\ \theta(t) =& -t + \theta_1\end{aligned}$$

where $4\theta_1$ is the initial polar angle. The frequency of the spiral motion is therefore the same as that of the limit cycle to our degree of approximation. Finally the approximate time solutions are given by

$$\begin{aligned}x =& a\cos\theta\\ x(t) =& a(t)\cos\theta(t) = \qquad \frac{2\cos(t-\theta_1)}{\{1-(1-(4/a_1^2))e^{-\epsilon t}\}^{1/2}}\end{aligned}$$

example Find the approximate phase paths for the equation

$$\ddot{x} + \epsilon(|\dot{x}| - 1)\dot{x} + x = 0$$

We have $h(x,y) = (|y|-1)y)$ Therefore

$$\begin{aligned}p_0(a) =& \frac{a}{2\pi}\int_0^{2\pi}(a|a\sin\theta| - 1)\sin^2\theta \;\; d\theta\\ =& \frac{a}{2\pi}\left(\int_0^{\pi}(a\sin\theta - 1)sin^2\theta \;\; d\theta + \int_{\pi}^{2\pi}(-a\sin\theta - 1)sin^2\theta \;\; d\theta\right)\\ =& \frac{a}{2\pi}2\int_0^{\pi}(a\sin\theta - 1)sin^2\theta \;\; d\theta\\ =& \frac{1}{\pi}a\left(\frac{4}{3}a - \frac{1}{2}\pi\right)\end{aligned}$$

There's a limit cycle when $p_0(a) = 0$ that is when $a = \frac{3}{8}\pi = a_0$.
We can either find the solution as a function of t before or use the equation

$$\begin{aligned}\frac{da}{d\theta} =& \epsilon p_0(a)\\ =& \frac{4\epsilon}{3\pi}a(a-a_0)\end{aligned}$$

This can be easily integrated at $a = a_1,\ \theta = \theta_1$ to give

$$a(\theta) = a_0 / \left(1 - (1 - (a_0/a_1))\exp\left[\epsilon\frac{4a_0}{3\pi}(\theta - \theta_1)\right]\right)$$

4.3 Periodic solutions:Harmonic Balance

Find an approximation to the amplitude and frequency of the limit cycle of van der Pol's equation $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$.

Assume an approximate solution $x = a\cos\omega t$. We expect the angulat frequency to be close to 1‘for small $|\epsilon|$.

$$\ddot{x} + x = -\epsilon(x^2 - 1)\dot{x}$$

Upon substituting the assumed form of the solution we get

$$\begin{aligned}(-\omega^2 + 1)a\cos\omega t =& -\epsilon(a^2\cos^2\omega t - 1)(-a\omega\sin\omega t)\\ =&\epsilon a\omega(\frac{1}{4}a^2 - 1)\sin\omega t + \frac{1}{4}\epsilon a^3\omega\sin 3\omega t\end{aligned}$$

The RHS is just the Fourier series for $\epsilon h(x,\dot{x})$ which is the easiest way to work with the equation compared to last chapter. Now ignoring the higher harmonic term involving $\sin 3\omega t$. When $\sin\omega t = 0$ we have

$$1 - \omega^2 = 0$$

and when $\cos\omega t = 0$ we have

$$\frac{1}{4}a^2 - 1 = 0$$

The first equation gives $\omega = 1$ (sign doesnt matter due to symmetry) and the 2nd equation gives $a = 2$ as expected.

For the general equation

$$\ddot{x} + \epsilon h(x,\dot{x}) + x = 0$$

suppose there is a periodic solution close to $a\cos\omega t$ and that $h(a\cos\omega t, -a\omega\sin\omega t)$ has a Fourier series, the constant term (which is its mean value over a cycle) being zero:

$$\begin{aligned}h(x,\dot{x}) =& h((a\cos\omega t, -a\omega\sin\omega t)\\ =& A_1(a)\cos\omega t + B_1(a)\sin\omega t + \text{ higher harmonics}\end{aligned}$$

where

$$A_1(a) = \frac{\omega}{\pi}\int_0^{2\pi/\omega} h(a\cos\omega t, -a\omega\sin\omega t)\cos\omega t \;\; dt$$

$$B_1(a) = \frac{\omega}{\pi}\int_0^{2\pi/\omega} h(a\cos\omega t, -a\omega\sin\omega t)\sin\omega t \;\; dt$$

The general equation becomes

$$(1-\omega^2)a\cos\omega t + \epsilon A_1(a)\cos\omega t + \epsilon B_1(a)\sin\omega t + \text{ higher harmonics} = 0$$

This can hold for all t only if

$$(1-\omega^2)a + \epsilon A_1(a) = 0. \quad B_1(a) = 0$$

which determines a and ω

5 Perturbation Methods

Diffential equations of the form

$$\ddot{x} = f(x, \dot{x}, t)$$

and first order systems having the general form

$$\dot{x} = X(x, y, t), \quad \dot{y} = Y(x, y, t)$$

in which t appears explicitly are called **non-autonomous** and are the main subject of this chapter.

Perturbation theory is a collection of methods for obtaining approximate solutions to equations involving a small parameter, which we normally denote by ϵ or δ
The main assumption underlying perturbation theory is that the solution is a well behaved function of ϵ, the small parameter. A **regular perturbation** problem is defined as one whose perturbation series is a power series in ϵ with a non-vanishing radius of convergence; for these problems the perturbed solution smoothly approaches the solution of the unperturbed equation, that is the equation with $\epsilon = 0$, as $\epsilon \to 0$.
A singular perturbation problem is one whose perturbation series either does not take the form of a power series or, if it does, the power series has a vanishing radius of convergence; for example it could be an asymptotic expansion. In these problems the equation normally has a different character when $\epsilon = 0$ than when $\epsilon \neq 0$.

In general we wish to solve equations in the form $F(x, \epsilon) = 0$. It is assumed that the unperturbed equation $f(x, 0) = 0$ obtained by setting $\epsilon = 0$ has a known unperturbed solution x_0 and that the perturbed equation is close to this solution, and that it can be expanded as a power series in ϵ

Finding an approximation to

$$x^2 - ax + \epsilon = 0, \quad |\epsilon| << 1$$

The unperturbed equation is obtained by piutting $\epsilon = 0$

$$x^2 - ax = 0$$

which has roots $x = x_0$ with $x_0 = 0, a$. When $|\epsilon| << 1$ we should expect roots to be close to this, we express this assumption by writing them as a power series in ϵ

$$\sum_{n=0}^{\infty} x_n \epsilon^n = x_0 + x_1 \epsilon + x_2 \epsilon^2 + a_3 \epsilon^3 + ...$$

where $x_0 = 0$ or a but $x_k, k = 1, 2...$ are unknown coefficients, depending upon the value taken by x_0, but independent of ϵ. We assume that the series is sufficiently well behaved so that all operations can be carried out.

Substituting the power expansion up to $O(\epsilon^2)$ into the original equation gives:

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2)^2 - a(x_0 + \epsilon x_1 + \epsilon^2 x_2) + \epsilon + O(\epsilon^3) = 0$$

which can be rearranged to give

$$x_0^2 - ax_0 + \epsilon(2x_0x_1 - ax_1 + 1) + \epsilon^2(2x_0x_2 + x_1^2 - ax_2) + O(\epsilon^3) = 0$$

The parameter ϵ is considered to be a variable, not a fixed constant. Therefore all coefficients of $\epsilon^k, k = 0, 1, ..$ must be identically zero. thus giving us

$$\begin{aligned} x_0^2 - ax_0 &= 0 \\ 2x_0x_1 - ax_1 + 1 &= 0 \\ 2x_0x_2 + x_1^2 - ax_2 &= 0 \end{aligned}$$

Giving

$$x_1 = \frac{1}{a - 2x_0}, \quad x_2 = \frac{x_1^2}{a - 2x_0} = \frac{1}{(a - 2x_0)^3}$$

For $x_0 = 0, a$ substituting the values above back into the power series gives

$$x = \frac{\epsilon}{a} + \frac{\epsilon^2}{a^3} + O(\epsilon^3), \quad x = a - \frac{\epsilon}{a} - \frac{\epsilon^2}{a^3} + O(\epsilon^3)$$

5.1 Singular Pertubations

In many interesting cases the 'small' perturbation may change the nature of the solution. For instance $\epsilon x^2 + x - 1 = 0$ has one solution, $x - 1$, when $\epsilon = 0$ but two solutions when $\epsilon \neq 0$. This is an example of a **singular** perturbation. In this example the product of the roots is $-1/\epsilon$ so if $0 < |\epsilon| << 1$ one roots is near 1, the other must be approximately $-1/\epsilon$.
The 1st root $x_1 \to 1 as \epsilon \to 0$. The second root is $O(1/\epsilon)$ and tends to infinity as $\epsilon \to 0$ and is not a root of the unperturbed equation and cannot therefore be given by a direct applicartion of perturbation theory.

5.2 Duffing's Equation

The equation of motion for an dampened pendulum with a harmonic forcing term is an important family of nonlinear equations called the Duffing's equations. In standard form it is written as

$$\ddot{x} + k\dot{x} + \omega_0^2 \sin x = F \cos \omega t$$

where x is the angular displacement from the vertical, $k = \alpha/ma^2$ where m is the mass and a the length, and $\alpha\dot{x}$ is the moment of friction about the support, $\omega_0^2 = g/a$ and $F = M/ma^2$ where M is the amplitude of the driving moment about the support.

Consider the undamped Duffings equation $k = 0$ We can suppose that $\omega_0 > 0, \omega > 0$ and $F > 0$, putting

$$\sin x \approx x - \frac{1}{6}x^3$$

then the original equation becomes

$$\ddot{x} + \omega_0^2 x - \frac{1}{6}\omega_0^2 x^3 = F \cos \omega t$$

Standardizing the form by putting $\tau = \omega t$, $\Omega^2 = \omega_0^2/\omega^2$, $(\Omega > 0)$ and $\Gamma = F/\omega^2$, we then obtain

$$x'' + \Omega^2 x - \frac{1}{6}\Omega^2 x^3 = \Gamma \cos \tau$$

where the dashes represent differentiation wrt τ. We assume that the nonlinear terms, that is $\frac{1}{6}\Omega^2$, to be small and we write

$$\frac{1}{6}\Omega^2 = \epsilon_0$$

Then we have

$$x'' + \Omega^2 x - \epsilon_0 x^3 = \Gamma \cos \tau$$

Instead of taking this as it stands, we consider the family of differential equations

$$x'' + \Omega^2 x - \epsilon x^3 = \Gamma \cos \tau$$

where ϵ is a parameter occupying an interval I_ϵ. When $\epsilon = \epsilon_0$ we recover the equation, when $\epsilon = 0$ we obtain the linearised equation. Expanding using the power series

$$x(\epsilon, \tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + ...$$

whose coefficients $x_i(\tau)$ are functions only of τ. As before all coefficients of powers of ϵ must be balanced. (not shown). Here we are only interested in periodic solutions that have a period of 2π, that is

$$x(\epsilon, \tau + 2\pi) = x(\epsilon, \tau)$$

5.3 Forced oscillations far from resonance

Suppose that

$$\Omega \neq \text{ an integer}$$

and that it is subjected to the the periodicity condition defined above, then we can obtain an approximation to the forced response, of period 2π. An example

$$x'' + \frac{1}{4}x + 0.1x^3 = \cos \tau$$

Here we have

$$x'' + \Omega^2 x + \epsilon x^3 = \cos\tau$$

where $\Omega^2 = 1/2$ and $\epsilon = 0.1$. Now expanding ϵ in the power series gives the two equations

$$x_0'' + \frac{1}{4}x_0 = \cos\tau$$
$$x_1'' + \frac{1}{4}x_1 = -\,x_0^3$$

For the 1st equation, setting $x_0 = a\cos\tau$ we get

$$-a\cos\tau + \frac{1}{4}a\cos\tau = \cos\tau$$

$$-\frac{3}{4}a = 1$$

giving $a = -4/3$, thus $x_0(\tau) = -4/3a\cos\tau$
The second becomes

$$x_1'' + \frac{1}{4}x_1 = \frac{16}{9}\cos\tau + \frac{16}{27}\cos 3\tau$$

giving

$$x_1(\tau) = -\frac{67}{27}\cos\tau - \frac{64}{945}\cos 3\tau$$

Therefore

$$x(\epsilon, tau) = -\frac{4}{3}\cos\tau - \epsilon(\frac{67}{27}\cos\tau + \frac{64}{945}\cos 3\tau) + O(\epsilon^2)$$

With $\epsilon = 0.1$

$$x(\epsilon,\tau) \approx -2.570\cos\tau - 0.007\cos 3\tau$$

5.4 Forced oscillations near resonance with weak excitation

Methods in previous sections fail to give good representation near certain critical values of Ω. Also, they don't reveal all the 2π solutions of the forcing term. With the approximate pendulum equation

$$\ddot{x} + k\dot{x} + \omega_0^2 x - \frac{1}{6}\omega_0^2 x^3 = F\cos\omega t$$

corresponding to

$$x'' + Kx' + \Omega^2 x - \epsilon_0 x^3 = \Gamma\cos\tau$$

where $\tau = \omega t$, $\Omega^2 = \omega_0^2/\omega^2$, $\epsilon_0 = \frac{1}{6}\Omega^2$, $K = k/\omega$, $\Gamma = F/\omega^2$.
Assume that Γ is small (weak excitation) and K is small (small damping) thus,

$$\Gamma = \epsilon_0\gamma, \quad K = \epsilon_0\kappa, \quad (\gamma, \kappa > 0)$$

Suppose that Ω is close to one of the critical resonance values 1,3,5,... We can write $\Omega^2 = 1 + \epsilon_0\beta$ thus the corresponding equation becomes

$$x'' + x = \epsilon_0(\gamma\cos\tau - \kappa x' - \beta x + x^3)$$

Now consider the family of equations

$$x'' + x = \epsilon(\gamma\cos\tau - \kappa x' - \beta x + x^3)$$

Now when $\epsilon = 0$ we have the linearised equation $x'' + x = 0$ which has infinitely many solutions with period 2π.
Now expanding

$$x\epsilon, \tau) = x_0(\tau) + \epsilon x_1(\tau) + ...$$

where by the same argument as previous

$$x_i(\tau + 2\pi) = x_i(\tau), \quad i = 0, 1, 2..$$

substituting into the family of equations gives

$$\begin{aligned}
x_0'' + x_0 =& 0 \\
x_1'' + x_1 =& \gamma\cos\tau - \kappa x_0' - \beta x_0 + x_0^3 \\
x_2'' + x_2 =& -\kappa x_1' - \beta x_1 + 3x_0^2 x_1
\end{aligned}$$

The solution to the first equation is

$$x_0(\tau) = a_0\cos\tau + b_0\sin\tau$$

for every value a_0, b_0.
Now substituting into the second equation gives

$$\begin{aligned}
x_1'' + x_1 =& \{\gamma - \kappa b_0 + a_0[-\beta + \frac{3}{4}(a_0^2 + b_0^2)]\}\cos\tau \\
& + \{\kappa a_0 + b_0[-\beta + \frac{3}{4}(a_0^2 + b_0^2)]\}\sin\tau \\
& + \frac{1}{4}a_0(a_0^2 + 3b_0^2)\cos 3\tau + \frac{1}{4}b_0(3a_0^2 - b_0^2)\sin 3\tau
\end{aligned}$$

The solution $x_1(\tau)$ is required to have a period of 2π but unless the coeeficients of $\cos\tau, \sin\tau$ are zero, there are no periodic solutions, since any solution would contain terms of the form $\tau\cos\tau, \tau\sin\tau$. We eliminate these secular terms by requiring the coefficients to be zero, that is

$$\kappa b_0 + a_0\{\beta - \frac{3}{4}(a_0^2 + b_0^2)\} = \gamma \tag{44}$$

$$\kappa a_0 - b_0\{\beta - \frac{3}{4}(a_0^2 + b_0^2)\} = 0 \tag{45}$$

To solve these values for a_0, b_0, let r_0 be the amplitude of the generating solution

$$r_0 = \sqrt{a_0^2 + b_0^2} > 0$$

by squaring and adding the two equations together we obtain the cubic **amplitude equation** for r_0^2

$$r_0^2\left\{\kappa^2 + \left(\beta - \frac{3}{4}r_0^2\right)^2\right\} = \gamma^2$$

There may be as many as 3 positive values for r_0^2 , therefore 3 distinct 2π-periodic solutions. With the appropriate values of a_0, b_0 chosen, we can solve the equation to give

$$x_1(\tau) = a_1 \cos\tau + b_1 \sin\tau - \frac{1}{32}a_0(a_0^2 - 3b_0^2)\cos 3\tau - \frac{1}{32}b_0(a_0^2 - 3b_0^2)\sin 3\tau$$

5.5 Amplitude equation for undamped pendulum

Suppose that the damping coefficient is zero, then

$$k = K = \kappa = 0$$

then (44)(45) becomes

$$\begin{aligned} b_0 &= 0 \\ a_0(\beta - \frac{3}{4}a_0^2) &= \gamma \end{aligned}$$

etc on pg 160

Example Investigate the forced periodic solution of

$$x'' + (9 + \epsilon\beta)x - \epsilon x^3 = \Gamma\cos\tau$$

With $\Gamma^2 = 9 + \epsilon\beta$, using equation JS5.23 would cause it to fail so we rewrite the equation as

$$x'' + 9x = \Gamma\cos\tau + \epsilon(x^3 - \beta x)$$

power series, where $x_i,\ i = 0, 1, ..$ having 2π period giving

$$x_0'' + 9x_0 = \Gamma\cos\tau \tag{46}$$

$$x_1'' + 9x_1 = x_0^3 - \beta x_0 \tag{47}$$

and so on. The 1st equation has the form

$$\begin{aligned} x_0\tau &= a_0\cos 3\tau + b_0 \sin 3\tau + \frac{1}{8}\Gamma\cos\tau \\ &= A_0 e^{3i\tau} + \bar{A}_0 e^{-3i\tau} + \frac{1}{16}\Gamma e^{i\tau} + \frac{1}{16}\Gamma e^{-i\tau} \end{aligned}$$

where $A_0 = \frac{1}{2}a_0 - \frac{1}{2}ib_0$ the second equation gives

$$x_0^3 - \beta x_0 = \left[\left(\frac{\Gamma^3}{16^3} + \frac{6\Gamma^2}{16^3}A_0 + 3A_0^2\bar{A}_0 - \beta A_0\right)e^{3i\tau} + complexconjugate\right] + higherharmonics$$

Therefore we require

$$A_0\left(3A_0\bar{A}_0-\beta+\frac{6\Gamma^2}{16^2}\right)=-\frac{\Gamma^3}{16^3}$$

this implies A_0 is real; $b_0=0, a_0=\frac{1}{2}a_0$ and the equation for a_0 is

$$\frac{1}{2}a_0\left(\frac{3}{4}a_0^2-\beta+\frac{6\Gamma^2}{16^2}\right)+\frac{\Gamma^3}{16^3}=0$$

5.6 Periodic solutions of Autonomous equations - Lindstedt's Method

With

$$\frac{d^2x}{dt^2}+x-\epsilon x^3=0$$

The system is conservative and method from JS1.3 can be used to show that all motions of small enough amplitude are periodic. Assume that

$$\omega=1+\epsilon\omega_1+...$$

$$x(\epsilon,t)=x_0(t)+\epsilon x_1(t)+...$$

Now, writing

$$\omega t=\tau$$

Then the original equation becomes

$$\omega^2x''+x-\epsilon x^3=0$$

By this substitution we have replaced an equation with unknown period, with one that has a known period of 2π This equation then becomes

$$(1+\epsilon 2\omega_1+...)(x_0''+\epsilon x_1''+...)+(x_0+\epsilon x_1+...)=\epsilon(x_0+\epsilon x_1+...)^3$$

and assembling the powers oe ϵ we have

$$x_0''+x_0=0$$

$$x_1''+x_1=-2\omega_1x_0''+x_0^3$$

etc.
To simplify the calculations we can impose the conditions

$$x(\epsilon,0)=a_0,\quad x'(\epsilon,0)=0$$

without loss of generality (when dealing with autonomous case). This implies that

$$x_0(0)=a_0,\quad x_0'(0)=0$$

and

$$x_i(0) = 0, \;\; x_i'(0) = 0, \;\; i = 1, 2, ...$$

This gives

$$x_0 = a_0 \cos\tau$$

and

$$x_1'' + x_1 = (2\omega_1 a_0 + \frac{3}{4}a_0^3)\cos\tau + \frac{1}{4}a_0^3 \cos 3\tau$$

which will only be periodic if

$$\omega_1 = -\frac{3}{8}a_0^2$$

This gives

$$x_1'' + x_1 = \frac{1}{4}a_0^3 \cos 3\tau$$

The general solution is $a_1 \cos\tau + b_1 \sin\tau$. The particular solution (setting $x_1 = A\cos 3\tau$ is

$$\begin{aligned} -9A\cos 3\tau + A\cos 3\tau =& \frac{1}{4}a_0^3 \cos 3\tau \\ -8A =& \frac{1}{4}a_0^3 \\ A =& -\frac{1}{32}a_0^3 \end{aligned}$$

giving

$$x_1'' + x_1 = a_1 \cos\tau + b_1 \sin\tau - \frac{1}{32}a_0^3 \cos 3\tau$$

Again, with

$$x_i(0) = 0, \;\; x_i'(0) = 0, \;\; i = 1, 2, ...$$

this implies $b_1 = 0$, $a_1 = \frac{1}{32}a_0^3$. Therefore

$$x_1(\tau) = \frac{1}{32}a_0^3(\cos\tau - \cos 3\tau)$$

Finally putting it together with regards to the power series of x

$$x(\epsilon, \tau) \approx a_0 \cos\tau + \frac{1}{32}\epsilon a_0^3(\cos\tau - \cos 3\tau) + O(\epsilon^2)$$

With $\tau = \omega t$ this gives

$$x(\epsilon, t) \approx a_0 \cos\omega t + \frac{1}{32}\epsilon a_0^3(\cos\omega t - \cos 3\omega t) + O(\epsilon^2)$$

Again, with $\omega = 1 + \epsilon\omega_1 + ...$

$$\omega = 1 - \frac{3}{8}a_0^2$$

this gives the dependence of frequency on amplitude

5.7 Homoclinic bifurcation

optional -pg 175-179

5.8 secular terms

From the screencast With

$$\ddot{x} + x = f(t), \quad x(0) = 0, dotx(0) = 1$$

The solution is a complimentary function $x(t) = A\cos(t) + B\sin(t)$ and a particular integral.
secular behaviour occurs if $f(t)$ forces the natural oscillation of the complimentary function, that is, the solution becomes large preventing a perturbation expansion from being valid.
for example, if $f(t) = \cos t$ we have

$$x = At\cos(t) + Bt\sin(t)$$

$$\ddot{x} = -A(2\sin t + t\cos t) + B(2\cos t - t\sin t)$$

Then

$$-2B\sin(t) + 2A\cos t =, \cos t$$

$$B = 0, \quad A = \frac{1}{2}$$

Giving a general solution

$$x = A\cos t + b\sin t + \frac{t}{2}\sin t$$

At $x(0)$ $a = 0$ and at $\dot{x}(0) = 1$, $B = 1$ thus

$$x = \left(1 + \frac{t}{2}\right) sint$$

but as $t \to \infty$ $x \to \infty$ so $|x(t)|$ is unbounded.

Looking at another problem

$$f(t) = \frac{4}{3}\cos^3(\frac{t}{3}) - \alpha\cos t$$

we need $f(t)$ to be expressed as pure harmonics. So

$$f(t) = \frac{1}{3}\cos t + \cos\frac{t}{3} - \alpha\cos t$$

For a bounded solution, we need to avoid $\cos t$ terms, forcing $\alpha = \frac{1}{3}$ Giving

$$\ddot{x} + x = \cos(\frac{t}{3})$$

setting $x = C\cos\frac{3}{3}$

$$\ddot{x} + x = -\frac{C}{9}\frac{t}{3} + C\cos\frac{t}{3} = \cos\cos(\frac{t}{3})$$

$$C(1 - \frac{1}{9}) = 1$$

Giving $C = 9/8$ Therefore

$$x = A\cos t + B\cos t + \frac{9}{8}\cos\frac{t}{3}$$

now at $x(0) = 0$ $a + 9/8 = 0$ giving $A = -9/8$ and at $\dot{x}(0) = 1$ $B = 1$ thus the particular solution is

$$x = \frac{9}{8}\left(\cos\frac{t}{3} - \cos t\right) + \sin t$$

Drawing the graph shows that this has $T = 6\pi$

6 Singular Perturbation Methods

This chapter is concerned with approximating solutions of differential equations containing ϵ in which straightforward expansion is unobtainable or unusable.

Most standard perturbation solutions are only valid for short periods of time ($t << 1/\epsilon$), we introduce a notion of slow time, denoted by η, which accounts explicitly for slow changes which is valid all the way up to $t = 1/\epsilon$ as $\epsilon \to 0$.

$$\eta = \epsilon t$$

which is a series solution for $X(t,\eta)$ in ϵ. If trajectory approaches a stable point or limit cycle, solution is valid for all t.

With

$$x(t) = X(t,\eta) = X_0(t,\eta) + \epsilon X_1(t,\eta) + \epsilon^2 X_2(t,\eta) + ...$$

where $X_0, X_1, X_2... = O(1)$ Using the chain rule we have

$$\frac{dx}{dt} \equiv \frac{d}{dt}X(t,\eta) = \frac{\delta t}{\delta t}\frac{\delta X}{\delta t} + \frac{\delta \epsilon t}{\delta t}\frac{\delta X}{\delta \epsilon t} = \frac{\delta X}{\delta t} + \epsilon\frac{\delta X}{\delta \eta}$$

and

$$\frac{d^2x}{dt^2} = \frac{\delta^2 X}{\delta t^2} + 2\epsilon\frac{\delta X}{\delta\eta\delta t} + \epsilon^2\frac{\delta^2 X}{\delta\eta^2}$$

Consider

$$\ddot{x} + \epsilon(x^2\dot{x}^2 - 1)\dot{x} + x = 0, \;\; x(0) = a, \;\; \dot{x}(0) = 0$$

The boundaries give $X(0,0) = a$, $\frac{\delta X(0,0)}{\delta t} + \epsilon \frac{\delta X(0,0)}{\delta \eta} = 0$. Now,

$$\ddot{x} + \epsilon(x^2\dot{x}^2 - 1)\dot{x} + x = \frac{\delta^2 X}{\delta t^2} + 2\epsilon \frac{\delta X}{\delta \eta \delta t} + \epsilon^2 \frac{\delta^2 X}{\delta \eta^2} + \epsilon \left(X^2 \left(\frac{\delta X}{\delta t} + \epsilon \frac{\delta X}{\delta \eta} \right)^2 - 1 \right) \left(\frac{\delta X}{\delta t} + \epsilon \frac{\delta X}{\delta \eta} \right) + X$$

$$= \left\{ \frac{\delta^2 X}{\delta t^2} + X \right\} + \epsilon \left\{ 2\frac{\delta X}{\delta \eta \delta t} + \left[X^2 \left(\frac{\delta X}{\delta t} \right)^2 - 1 \right] \frac{\delta X}{\delta t} \right\} + O(\epsilon^2) = 0$$

Expanding $X(t, \eta)$ gives

$$= \left\{ \frac{\delta^2 X_0}{\delta t^2} + X_0 \right\} + \epsilon \left\{ 2\frac{\delta X_0}{\delta \eta \delta t} + \left[X_0^2 \left(\frac{\delta X_0}{\delta t} \right)^2 - 1 \right] \frac{\delta X_0}{\delta t} \right\} + \epsilon X_1 + O(\epsilon^2) = 0$$

therefore

$$\left\{ \frac{\delta^2 X_0}{\delta t^2} + X_0 \right\} = 0 \quad O(1) \tag{48}$$

$$2\frac{\delta X_0}{\delta \eta \delta t} + \left[X_0^2 \left(\frac{\delta X_0}{\delta t} \right)^2 - 1 \right] \frac{\delta X_0}{\delta t} + \frac{\delta^2 X_1}{\delta t^2} + \epsilon X_1 = 0 \quad O(\epsilon) \tag{49}$$

With the initial conditions $x(0,0) = a$

$$X(0,0) = X_0(0,0) + \epsilon X_1(0,0) = a$$

giving $X_0(0,0) = a$, $X_1(0,0) = 0$, and with $\frac{\delta X(0,0)}{\delta t} + \epsilon \frac{\delta X(0,0)}{\delta \eta} = 0$

$$\frac{\delta X(0,0)}{\delta t} + \epsilon \frac{\delta X(0,0)}{\delta \eta} = \frac{\delta X_0(0,0)}{\delta t} + \epsilon \frac{\delta X_0(0,0)}{\delta \eta} + \epsilon \frac{\delta X_1(0,0)}{\delta t} + \epsilon^2 \frac{\delta X_1(0,0)}{\delta \eta} = 0$$

$$= \frac{\delta X_0(0,0)}{\delta t} + \epsilon \left(\frac{\delta X_0(0,0)}{\delta \eta} + \frac{\delta X_1(0,0)}{\delta t} \right) = 0$$

giving $\frac{\delta X_0(0,0)}{\delta t} = 0$ and $\frac{\delta X_0(0,0)}{\delta \eta} + \frac{\delta X_1(0,0)}{\delta t} = 0$
giving the necessary PDE's from the initial conditions.

As usual we will start with finding a solution for $X_0(t, \eta)$, from (48)

$$\frac{\delta^2 X_0}{\delta t^2}(t, \eta) + X_0(t, \eta) = 0$$

notice that the above PDE only has derivatives wrt t so we can treat this as a 2nd order ODE where η is an arbitrary constant. The solutions for $\alpha e^{\lambda t}$ is $\lambda^2 + 1 = 0$, giving $\lambda = \pm i$. The algebra is easier if we keep as an explicitly complex function rather than the usual sin / cos, so the general solution is

$$A(\eta)e^{it} + B(\eta)e^{-it}$$

Now for X_0 to be real it must equal its complex conjugate, that is $X_0 = \bar{X}_0$, therefore

$$A(\eta)e^{it} + B(\eta)e^{-it} = \bar{A}(\eta)e^{-it} + \bar{B}(\eta)e^{it}$$

for all t, so $A = \bar{B}$
So we have

$$X_0(t, \eta) = A(\eta)e^{it} + \bar{A}(\eta)e^{-it}$$

Now, using the boundary conditions

$$X_0(0, 0) = a \quad \text{so} \quad A(0) + \bar{A}(0) = a$$

and

$$\frac{\delta X_0}{\delta t} = 0 \quad \text{so} \quad iA(0) - i\bar{A}(0) = 0$$

Giving

$$A = \bar{A} = \frac{a}{2}$$

Now to find X_1, from (49) we have

$$\frac{\delta^2 X_1}{\delta t^2} + X_1 = -2\frac{\delta^2 X_0}{\delta t \delta \eta} - X_0^2 \left(\frac{\delta X_0}{\delta t}\right)^3 + \frac{\delta X_0}{\delta t}$$

With

$$\frac{\delta X_0}{\delta t} = iAe^{it} - i\bar{A}e^{-it}$$

and

$$\frac{\delta^2 X_0}{\delta t \delta \eta} = iA'e^{it} - i\bar{A}'e^{-it}$$

Now

$$X_0^2 \left(\frac{\delta X_0}{\delta t}\right)^3 = \left[A^2e^{2it} + 2A\bar{A} + \bar{A}^2e^{-2it}\right]\left[-iA^3e^{3it} + 3iA^2\bar{A}e^{it} - 3iA\bar{A}^2e^{-it} + i\bar{A}^3e^{-3it}\right]$$

Now as we want the series expansion of $X(t, \eta)$ to be valid with $X_1(t, \eta)$ being bounded we need to avoid any secular terms, that is we require no e^{it} or e^{-it}. So from (49) collecting all the e^{it} terms we have $-2\frac{\delta^2 X_0}{\delta t \delta \eta} = -2iA'$ and $\frac{\delta X_0}{\delta t} = iA$. The middle part is a bit more difficult. We need to find pairs that will multiple to give e^it terms, so

$$(A^2e^{2it})(-3iA\bar{A}^2e^{it}) = -\,3iA^3\bar{A}^2e^{it}$$
$$(2A\bar{A})(3iA^2\bar{A}e^{it} = 6iA^3\bar{A}^2e^{it}$$
$$(\bar{A}^2e^{-2it})(-iA^3e^{3it}) = -\,iA^3\bar{A}^2e^{it}$$

thus altogether giving us

$$-2iA' - \left[-3iA^3\bar{A}^2 + 6iA^3\bar{A}^2 - iA^3\bar{A}^2\right] + iA = 0$$

$$
\begin{aligned}
-2A' - 2A^3\bar{A}^2 + A = 0\\
\frac{\delta A}{\delta \eta} = A\left(\frac{1}{2} - A^2\bar{A}^2\right)
\end{aligned}
$$

As we have found a solution for $e^{it} = 0$, as we are dealing with real values the complex conjugate $e^{-it} = 0$ is also satisfied. Finding the value for A we have

$$
\begin{aligned}
\eta =& \int dA \, \frac{1}{A\left(\frac{1}{2} - A^2\bar{A}^2\right)}\\
=& 2\log A - \log(1 - 2A^2\bar{A}^2) + C\\
Ce^{\eta} =& \frac{A^2}{1 - 2A^2\bar{A}^2}
\end{aligned}
$$

————————————————————-Need to find out how to solve screencast

7 Forced Oscillations

7.1 General forced periodic solutions

Consider the Duffing's equation

$$\ddot{x} + k\dot{x} + \alpha x + \beta x^3 = \Gamma \cos \omega t \tag{50}$$

Suppose that $x(t)$ is a periodic solution with period $2\pi/\lambda$. Then $x(t)$ can be represented by a Fourier series for all t

$$x(t) = a_0 + a_1 \cos \lambda t + b_1 \sin \lambda t + a_2 \cos 2\lambda t + ... \tag{51}$$

If we substitute this into the Duffings equation the x^3 is periodic, and so generates another Fourier series. The Duffings equation can be represented as

$$A_0 + A_1 \cos \lambda t + B_1 \sin \lambda t + A_2 \cos 2\lambda t + ... = \Gamma \cos \omega t \tag{52}$$

for all t, where the coefficients are functions of $a_0, a_1, b_1, a_2...$ Matching the two sides gives an infinite set of equations for $a_0, a_1, a_2...$ and enables λ to be determined. The obvious matchings are $\lambda = \omega$, $A_1 = \Gamma$, $A_0 = B_1 = A_2 = ... = 0$.

A less obvious matching can sometimes be achieved when

$$
\begin{aligned}
\lambda =& \omega/n \;\; (n \text{ an integer})\\
A_n =& \Gamma\\
A_i = 0 \; (i \neq n);&\\
B_i = 0&
\end{aligned}
$$

If a solution of this set exists, the system will deliver responses of period $2\pi n/\omega$ and angular frequency ω/n . These are called **subharmonics** of order $\frac{1}{2}, \frac{1}{3}, ...$

Not all of these actually occur.

Going back to x^3 if we look at the terms, there is a something called a **throw-back** in which higher order terms contribute to terms of low order. For instance

$$x^3(t) = a_0 + ...a_{11}\cos 11\omega t + ... + a_{21}\cos 21\omega t + ...)^3$$

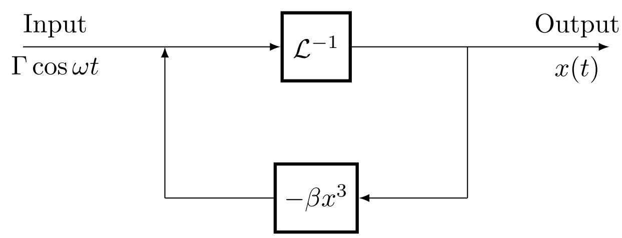
$$3a_{11}^2 a_{21}\cos^2 11\omega t\cos 21\omega t = 3a_{11}^2 a_{21}\left(\frac{1}{2}\cos 21\omega t + \frac{1}{4}\cos 43\omega t + \frac{1}{4}\cos\omega t\right)$$

which contains a $\cos\omega t$ term. Normally we assume that the terms in (51) above a certain small order are negligible, that is the coefficients are small. Then hopefully the combined throwback effect , as measured by the modification of coefficients by high-order terms will be small.

An alternative way of looking at the effect of a nonlinear term is as a feed-back loop. Consider the undamped form of (50) (with $k = 0$) written as

$$\mathcal{L}(x) \equiv \left(\frac{d^2}{dt^2} + \alpha\right)x = -\beta x^3 + \Gamma\cos\omega t$$

Regard $\Gamma\cos\omega t$ as the input to the system , and $x(t)$ as the output. We get



Where $\mathcal{L}^{-1}$ solves the linear equation $\ddot{x} + \alpha x = f(t)$ for a given input f and for assigned initial conditions. Here the input to $\mathcal{L}^{-1}$ is equal to the current value of $-\beta x^3 + \gamma\cos\omega t$ Its output is $x(t)$. Suppose the output is assumed to be simple, containing only a few harmonic components. The $-\beta x^3$ generates a shower of harmonics of higher and possibly lower orders, that are then fed back into the system. The higher orders are attenuated (reduced in force) much like n^{-2} when the order is n. It is therefore expected that a satisfactory consistency between the inputs might be obtained by a representation of $x(t)$ in terms only of the lower harmonics present. The low-order approximation should be most adequate when the lowest harmonic present is **amplified** by $\mathcal{L}^{-1}$, that is, when $\omega^2 \approx n^2\alpha$, being a condition of near resonance.

7.2 Harmonic solutions, transients, and stability for Duffing's equation

Consider the Duffing equation

$$\ddot{x} + x + \beta x^3 = \Gamma\cos\omega t \tag{53}$$

where $\Gamma > 0$. As approximation to the solution we use the truncated Fourier series

$$x(t) = a\cos\omega t + b\sin\omega t \tag{54}$$

This form allows for a possible phase difference between the forcing term and the solution. The omission of the constant term is due to the knowledge that for reasonably small amplitude the solutions will have a zero mean value. Now

$$\ddot{x} = -a\omega^2\cos\omega t + b\omega^2\sin\omega t \tag{55}$$

and

$$x^3(t) = \frac{3}{4}a(a^2+b^2)\cos\omega t + \frac{3}{4}b(a^2+b^2)\sin\omega t + \text{ higher orders} \tag{56}$$

substituting these into (54) gives

$$b\left\{(\omega^2-1) - \frac{3}{4}\beta(a^2+b^2)\right\} = 0 \tag{57}$$

$$a\left\{(\omega^2-1) - \frac{3}{4}\beta(a^2+b^2)\right\} = -\Gamma \tag{58}$$

The only solution for (57) is when $b = 0$, giving

$$\frac{3}{4}\beta a^3 - (\omega^2-1)a - \Gamma = 0 \tag{59}$$

giving roots

$$z = \Gamma, \quad z = \frac{3}{4}\beta a^3 - (\omega^2-1)a$$

when $\beta < 0$ we have 3 solutions for Γ small and one for Γ larger when $\omega^2 < 1$, and that when $\omega^2 > 1$ there is only one solution. The oscillations are in phase with the forcing term when the critical value of a is positive, and out of phase by half a cycle when a is negative. It is necessary to decide whether these oscillations are stable or otherwise, as unstable oscillations will not occur in practice. To investigate this we need to look at the transient states, which lead to or away from periodic states, by supposing that the coefficients a and b are slowly varying functions of time, at any rate near to periodic states. Assume that

$$x(t) = a(t)\cos\omega t + b(t)\sin\omega t \tag{60}$$

giving

$$\dot{x}(t) = (\dot{a} + \omega b)\cos\omega t + (\dot{b} - \omega a)\sin\omega t \tag{61}$$

and neglecting $\ddot{a}, \ddot{b}$

$$\ddot{x} \approx (-\omega^2 a + 2\omega\dot{b})\cos\omega t + (-2\omega\dot{a} - \omega^2 b)\sin\omega t \tag{62}$$

lastly

$$x^3 = \frac{3}{4}a(a^2+b^2)\cos\omega t + \frac{3}{4}b(a^2+b^2)\sin\omega t + \text{ higher orders} \tag{63}$$

Substituting these into (53) gives the autonomous system

$$\dot{a} = -\frac{1}{2\omega} b \left[(\omega^2 - 1) - \frac{3}{4}\beta(a^2 + b^2) \right] \equiv A(a, b) \tag{64}$$

$$\dot{b} = -\frac{1}{2\omega} a \left[(\omega^2 - 1) - \frac{3}{4}\beta(a^2 + b^2) \right] + \frac{\Gamma}{2\omega} \equiv B(a, b) \tag{65}$$

With the initial conditions given as the terms of the original equation

$$a(0) = x(0), \quad b(0) = \dot{x}(0)/\omega$$

The phase plane for a, b above is called the van der Pol plane. The equilibrium points given by $A(a, b) = B(a, b) = 0$ represent the steady periodic solutions already obtained.

Consider the case where we have 3 equilibrium points

$$\omega^2 < 1, \quad \beta < 0$$

let a_0 represent any one of the values $a = a_1, a_2$ or a_3 Putting

$$a = a_0 + \xi \tag{66}$$

The local linear approximation to (64) (65) is

$$\dot{\xi} = A_1(a_0, 0)\xi + A_2(a_0, 0)b \tag{67}$$

$$\dot{b} = B_1(a_0, 0)\xi + B_2(a_0, 0)b \tag{68}$$

where $A_1(a, b) = \delta A(a, b)/\delta a$ and so on. Giving

$$A_1(a_0, 0) = B_2(a_0, 0) = 0 \tag{69}$$

$$A_2(a_0, 0) = B_1(a_0, 0) = -frac\omega^2 - 12\omega + \frac{3\beta a_0^2}{8\omega} = -\frac{z_0}{2\omega\alpha_0} \tag{70}$$

and that

$$B_1(a_0, 0) = s_0/(2\omega)$$

where s_0 is the slope of the curve at a_0 Therefore we can rewrite (67), (68) as

$$\dot{\xi} = -\frac{z_0}{2\omega a_0} b \tag{71}$$

$$\dot{b} = \frac{s_0}{2\omega}\xi \tag{72}$$

By considering the signs of a_0, z_0, s_0 we can see if the equilibrium points are centres or saddles, although saddles will never be observed exactly.

Introducing a damping term

$$\ddot{x} + k\dot{x} + x + \beta x^3 = \Gamma \cos \omega t, \quad k > 0 \tag{73}$$

The equilibrium points are given by

$$b\left\{\omega^2 - 1 - \frac{3}{4}\beta(a^2+b^2)\right\} + k\omega a = 0 \tag{74}$$

$$a\left\{\omega^2 - 1 - \frac{3}{4}(a^2+b^2)\right\} - k\omega b = -\Gamma \tag{75}$$

so that after squaring and adding these equations we have

$$r^2\left\{k^2\omega^2 + \left(\omega^2 - 1 - \frac{3}{4}\beta r^2\right)^2\right\} = \Gamma^2, \quad r = \sqrt{a^2+b^2} \tag{76}$$

and the closed paths become spirals

7.3 The Jump Phenomenon

Equation (73) for the Damped Duffing oscillator has periodic solutions which are approximately of the form

$$a\cos\omega t + b\sin\omega t$$

in which the amplifier r satisfies (76) where $k > 0, \omega > 0, \Gamma > 0$.
Suppose $\beta < 0$ and we put $\rho = -\beta r^2$, $\gamma = \Gamma\sqrt{-\beta}$ then the amplitude equation can be expressed as

$$\gamma^2 = G(\rho) = \rho\left\{k^2\omega^2 + (\omega^2 - 1 - \frac{3}{4}\rho)^2\right\} \tag{77}$$

Interesting phenomena are generally associated with variations in the forcing frequency ω and the forcing amplitude Γ, so we simplify the variables by assuming the damping parameter k is specified. Thus we can rewrite the amplitude function as a cubic in ρ

$$G(\rho) = \frac{9}{16}\rho^3 + \frac{3}{2}(\omega^2-1)\rho^2 + \{k^2\omega^2 + (\omega^2-1)^2\}\rho \tag{78}$$

Since $G(0) = 0$ and $G(\rho) \to \infty$ as $\rho \to \infty$, $G(\rho) = \gamma^2$ must have at least one positive root. There will be three roots for some parameter values if $G'(\rho) = 0$ has two distinct solutions for $\rho \geq 0$. Thus

$$G'(\rho) = \frac{27}{16}\rho^2 + 3\left(w^2 - 1\right)\rho + k^2\omega^2 + (\omega^2-1)^2 = 0$$

has two real roots

$$\rho_1, \rho_2 = \frac{8}{9}(1-\omega^2) \pm \frac{4}{9}\sqrt{[(1-\omega^2)^2 - 3k^2\omega^2]} \tag{79}$$

etc pg232 onwards

Exercise Find the forcing amplitude γ in terms of k of the Duffing equation at the cusp where the frequency is $\omega = \frac{1}{2}(\sqrt{(3k^2+4)} - k\sqrt{3}$. Compute how the amplitude varies with k.

7.4 Changing order of Integrals

$$\int_0^t \left(\int_0^u f(u,s) ds \right) du = \int_0^t \left(\int_s^t f(u,s) du \right) ds$$

!!!!!!!!!!!!!!!!Write down in glossary!!!!!!!!!!!!!!!! This allows us to proceed further for some integrals compared to leaving it as is.

Example
Given

$$\frac{dz_{n+1}(t)}{dt} = e^{\lambda t} \int_0^t e^{-\mu s} z_n(s)\ ds$$

where $z_{n+1}(0) = 0$. Find $Z_{n+1}(t)$.
We fist rewrite the equation replacing t by u (a dummy variable) which helps making mathematics easier to see.

$$\frac{dz_{n+1}(u)}{du} = e^{\lambda u} \int_0^u e^{-\mu s} z_n(s)\ ds$$

Integrating both sides wrt u from 0 to t gives

$$[z_{n+1}(u)]_{u=0}^t = \int_{u=0}^t \left[e^{\lambda u} \int_0^u e^{-\mu s} z_n(s)\ ds \right] du$$

With the boundary condition given we have

$$z_{n+1}(t) - z_{n+1}(0) = z_{n+1}(t) = \int_{u=0}^t \left[\int_0^u e^{\lambda u} e^{-\mu s} z_n(s)\ ds \right] du$$

ensuring that you can bring the expression $e^{\lambda u}$ into the second integral. Now this gives us an expression that is equal to $f(u,s)$ as we need for the equation at the start. So

$$z_{n+1}(t) = \int_{s=0}^t e^{-\mu s} z_n(s) \left[\frac{e^{\lambda t}}{\lambda} \right]_{u=s}^t ds$$

$$z_{n+1}(t) = \int_{s=0}^t \frac{e^{-\mu s}}{\lambda} z_n(s) \left[e^{\lambda t} - e^{\lambda s} \right]\ ds$$

which is as far as we can go without knowing $z_n(s)$

8 Stability

This chapter is about how to distinguish stability, and how to define if one system is more stable than another. We are concerned with regular systems throughout and the treatment is not restricted to merely second order systems.

The general autonomous system in n dimensions can be written as

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ is a column vector with n components. A time solution $\mathbf{x} = \mathbf{x}(t)$ defines a phase path in the $x_1, x_2..., x_n$ phase space, and a set of phase paths defines a phase diagram for the system. A point x_0 is an equilibrium point if the constant vector $\mathbf{x}(t) = \mathbf{x}_0$ is a solution for all t, thus equilibrium points are given by solutions $\mathbf{X}(\mathbf{x}) = \mathbf{0}$. The general n-th order non-autonomous system can be represented by

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t)$$

8.1 Poincare stability (stability of paths)

Lets define $\mathbf{x}^*(t)$ as a point on a phase curve, which we will call the standard path. We are interested in only the path from a particular point a^* onwards, that is for all $t \geq t_0$, then we have a positive half-path $\mathcal{H}^*$ with initial point $\mathbf{a}^*$. The solution which $\mathcal{H}^*$ represents is

$$\mathbf{X}^*(t), \;\; t \geq t_0$$

where

$$\mathbf{a}^*(t_0) = \mathbf{a}^*$$

and since the system is autonomous all choices of t_0 lead to the same $\mathcal{H}^*$.

The solution $\mathbf{x}^*(t)$ is Poincaré stable (or orbitally stable) if all sufficiently small disturbances of the initial value $\mathbf{a}^*$ lead to half-paths that are a small distance away from $\mathcal{H}^*$. If a small strip was drawn either side of $\mathcal{H}^*$ at an arbitrary value $\epsilon > 0$, then for stability it must be shown that we can find a δ such that all paths starting within a distance δ of a $\mathbf{a}^*$ where $\delta \leq \epsilon$ (and where δ is the radius of a circle centred at point $\mathbf{a}^*$) remain permanently within the strip.

Exercise Find the equation of phase paths of $\dot{x} = 1 + x^2, \dot{y} = -2xy$ Show that the path for $y = 0$ all paths which start in $(x+1)^2 + y^2 = \delta^2$ subsequently remain in a circle of radius $\delta[1 + (1+\delta)^2]$ centred on $y = 0$.

First we see that $\dot{x}$ is monotonically increasing ,i.e $\dot{x} > 0$ for all x, and that for large x, $\dot{x} \simeq x^2$. so that $x(t) \to \infty$ at some finite time, regardless of the initial value of x or y. Specifically, if $x(0) = x_0$ we have

$$t = \int_{x_0}^{x} dx \frac{1}{1+x^2} = [\tan^{-1} x]_{x_0}^{x} = \tan^{-1} x - \tan^{-1} x_0$$

giving

$$x(t, x_0) = \tan(t + \tan^{-1} x_0)$$

thus as $t \to t_\infty$ $x \to \infty$ now

$$t_\infty = \tan^{-1}\infty - \tan^{-1} x_0 = \frac{\pi}{2} - \tan^{-1} x_0 = \tan^{-1}\frac{1}{x_0}$$

For y

$$\frac{dy}{dx} = \frac{Y}{X} = \frac{2xy}{1+x^2} \text{ and hence } y = y_0 \frac{1+x_0^2}{1+x^2}$$

Consider the phase curve starting at $y_0 = 0$, so $y(t) = 0$ for all $t \geq 0$. If $x_0 = -1 + \epsilon$ then $t_\infty = \pi/2 - \tan^{-1}(1-\epsilon) = 3\pi/4 - 2\epsilon + O(\epsilon^2)$. Thus the distance between two initially close solutions

$$\Delta x = x(t, -1+\epsilon) - x(t, -1)$$

tends to infinity as $t \to 3\pi/4 - 2\epsilon + O(\epsilon^2)$. Phase curves starting in the circle $(x+1)^2 + y^2 = \delta^2$ will have initial values $x_0, y_0 = (-1 + \delta\cos\phi, \delta\sin\phi)$, $-\pi < \phi \leq \pi$. We have shown that if $\phi = 0$ or π the separation between initially close phase points increases without bound, so the system is not Liapunov stable. Notes pg 215 for the rest as it seems more complicated than what the question is asking for

8.2 Paths and solution curves for general systems

Suppose that we have a pendulum, if the path slightly larger by distance δ, then we can see that this will always be Poincare stable but due to its larger path, after a certain time the two pendulums will no longer be in sync i.e out of phase. This could cause problems for such things as timekeeping. So we need a more demanding criterion for stability

8.3 Stability of time solutions: Liapunov stability

Consider a regular dynamic system, not necessarily autonomous, in n dimensions, written in vector form

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t)$$

or in component form

$$\begin{aligned}
\dot{x}_1 &= X_1(x_1, x_2, ..., x_n, t) \\
&... \\
\dot{x}_n &= X_n(x_1, x_2, ..., x_n, t)
\end{aligned}$$

Suppose that we have two real or complex solution vectors with components

$$\mathbf{x}^*(t) = [x_1^*(t), x_2^*(t), ..., x_n^*(t)]^T$$
$$\mathbf{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$$

The separation between them at any time t is $||\mathbf{x}(t) - \mathbf{X}^*(t)||$ where

$$||\mathbf{x}(t) - \mathbf{X}^*(t)|| = \left(\sum_{i=1}^{n} |x_i(t) - x_i^*(t)|2 \right)^{1/2}$$

where $|..|$ denotes the modulus in the complex sense, and magnitude when x_i, x_i^* are real solutions. Such a measure is called a metric or distance function, and satisfies the triangle inequality

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

where the norm of vector $\mathbf{u}$ is defined by

$$||\mathbf{u}|| = \left(\sum_{i=1}^{n} |u_i|^2 \right)^{1/2}$$

$\mathbf{x}^*(t)$ is Liapunov stable for $t \geq t_0$ iff to each value of $\epsilon > 0$, however small, there corresponds a value of $\delta > 0$ (where δ may depend only on ϵ and t_0 such that

$$||\mathbf{x}(t_0) - \mathbf{x}^*(t_0)|| < \delta \Rightarrow ||\mathbf{x}(t) - \mathbf{x}^*(t)|| < \epsilon$$

If the system is autonomous then the solution $\mathbf{x}^*(t)$ is either Liapunov stable or not for all t_0, which also implies Poincaré stability.

example Show that all solutions of the $n-$dimensional system $\dot{\mathbf{x}} = -\mathbf{x}$ are stable for $t \geq 0$ in the Liapunov sense.

the general solution is given by

$$\mathbf{x}(t) = \mathbf{x}(0)e^{-t}$$

Consider the stability of $x^*(t)$ for $t > 0$ where

$$\mathbf{x}^*(t) == \mathbf{x}(0)e^{-t}$$

For any $x^*(0)$ we have

$$||\mathbf{x}(t) - \mathbf{x}^*(t)|| \leq ||\mathbf{x}(0) - \mathbf{x}^*(0)||e^{-t} \leq ||\mathbf{x}(0) - \mathbf{x}^*(0)||, \;\; t \geq 0$$

therefore for any $\epsilon > 0$

$$||\mathbf{x}(0) - \mathbf{x}^*(0)|| < \epsilon \Rightarrow ||\mathbf{x}(t) - \mathbf{x}^*(t)|| < \epsilon, \;\; t > 0$$

thus $\delta = \epsilon$

Uniform Stability If a solution is stable for $t \geq t_0$ and the δ is independent of t_0. For instance, any stable solution for autonomous systems are uniformly stable since the system is invariant with respect to time translation.

Asymptotic Stability The system $\dot{\mathbf{x}} = 0$ has the general solution

$$\mathbf{x}(t) = \mathbf{x}(t_0) = \mathbf{C}$$

where $\mathbf{C}$ is an arbitrary constant vector, and t_0 is any value of t.
These are stable on $t \geq t_0$ for any t_0 (and also uniformly stable) however a disturbed solution shows no tendency to return to the original solution:it remains a constant distance away. On the other hand solutions that do re-approach the undisturbed solution after being disturbed, thus returning to its original operating curve are known as **asymptotically stable** which can be uniformly and non uniformly stable.

Let $\mathbf{x}^*$ be a stable (or non uniformly stable) solution for $t \geq t_0$. if additionally there exists $\eta(t_0) > 0$ such that

$$||\mathbf{x}(t_0) - \mathbf{x}^*(t_0)|| < \eta \Rightarrow \lim_{t\to\infty} ||\mathbf{x}(t) - \mathbf{x}^*(t)|| = 0$$

then the solution is said to be asymptotically stable (or uniformly and asymptotically stable).

8.4 Lipunov stability of plane autonomous systems

We consider the first stability of constant solutions (equilibrium point) of 2-D constant-coefficient system as in chapter 2.
We can place the equilibrium point at the origin by modifying the solutions by additive constants. We then have

$$\dot{x}_1 + ax_1 + bx_2, \quad \dot{x}_2 = cx_1 + dx_2$$

where a, b, c, d are constants.
The stability properities pf the constant solutions $\mathbf{x}^* = [x_1^*(t), x_2^*(t)]^T = 0, \; t \geq t_0$ can sometimes be read off of a phase diagram, for example a saddle is obviously unstable.

Stability properties of the zero solutions of $\dot{\mathbf{x}} = \mathbf{ax} + \mathbf{by}, \;\; \dot{\mathbf{y}} = \mathbf{cx} + \mathbf{dy}$

Phase plane feature	Poncare Stability	Liapuov stability property
Centre	Stable	uniformly stable
Stable Spiral	Stable;paths approach zero	Uniformly,asymptotically stable
Stable Node	Stable;paths approach zero	Uniformly,asymptotically stable
Saddle	unstable	unstable
Unstable spiral and Node	unstable	unstable

The most generl linear system is the non autonomous and non homogeneous equation in n variables given by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

where $\mathbf{A}(t)$ is an n x n matrix. To investigate the stability of solution $\mathbf{x}^*(t)$. Let $\mathbf{x}(t)$ represent any other solution, and define $\boldsymbol{\xi}(t)$ by

$$\boldsymbol{\xi}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$$

Then $\boldsymbol{\xi}(t)$ tracks the difference between the 'test' solution and a solution having a different value at time t_0. The initial condition for $\boldsymbol{\xi}$ is

$$\boldsymbol{\xi}(t_0) = \mathbf{x}(t_0) - \mathbf{x}^*(t_0)$$

also, $\boldsymbol{\xi}$ satisfies the homogenous equation

$$\dot{\boldsymbol{\xi}} - \mathbf{A}(t)\boldsymbol{\xi}$$

$\boldsymbol{\xi}(t)$ is called a perturbation of the solution $\mathbf{x}^*(t)$.

Theorem 8.1 All solutions of the regular linear system $\dot{\mathbf{x}} = \mathbf{A}(t) + \mathbf{f}(t)$ have the same Liapunov stability property. This is the same as that of the zero (or any other) solution of the homogenous equation $\dot{\boldsymbol{\xi}} = \mathbf{A}(t)\boldsymbol{\xi}$
Notice that the stability of time solutions of linear systems does not depend upon the forcing term $f(t)$

Example solve the forced linear system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

and state why all solutions are asymptotically stable.

The equations are

$$\dot{x} = -x + y, \quad \dot{y} = -2y + e^t$$

Using the Integrating Factor method, the latter can be rewritten as

$$\begin{aligned}
\frac{dy}{dt} + P(t)y =& Q(t) \\
\frac{dy}{dt} = 2y =& e^t \\
\text{With } IF = e^{\int P(x)dt} = e^{2t} & \\
\frac{d}{dt}(IFy) =& IFQ(x) \\
\frac{d}{dt}ye^{2t} =& e^{3t} \\
ye^{2t} =& a + \frac{1}{3}e^{3t} \\
y =& ae^{-2t} + \frac{1}{3}e^t
\end{aligned}$$

where a is a constant. The equation for x becomes

$$\begin{aligned}
\frac{d}{dt}xe^t =& y = ae^{-2t} + \frac{1}{3}e^t \\
x =& be^{-t} - ae^{-2t} + \frac{1}{6}e^t
\end{aligned}$$

where b is another constant. Different initial conditions give different values of the constants, so the difference between the solutions tend to zero as $t \to \infty$

The eignenvalues of the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

are

$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = \left| \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{bmatrix} -1-\lambda & 0 \\ 0 & -2-\lambda \end{bmatrix} = ad - bc = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

are -1 and -2, so the homogeneous system is also asymptotically stable, in accordance with JS theorem 8.1

8.5 Structure of n-D solutions

The general homogenous first order linear system of $n-$dimensions is

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$$

where $\mathbf{A}(t)$ is an nxn matrix whose elements $a_{ij}(t)$ are a function of time, and $\mathbf{x}$(t) is a column vector of the n dependent variables. The explicit appearance of t indicates that the system may be non-autonomous.

$$\dot{x}_i = \sum_{j=0}^{n} a_{ij} x_j; \quad i = 1, 2, .., n$$

or

$$\begin{bmatrix} \dot{x}_1 \\ . \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & ... & a_{1n} \\ . & & \\ . & & \\ . & & \\ a_{n1} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

We shall assume that each a_{ij} is continuous on $\mathbf{x}(t_0) = \mathbf{x}_0$ and that there is a unique solution satisfying this condition (not true in general of nonlinear systems).

If $\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_m(t)$ are real or complex solutions, then so is $+\alpha_2 \mathbf{x}_2(t) + ...\alpha_m \mathbf{x}_m(t)$ where α_m are constants, real or complex.

Linearly dependent vector functions Let $\boldsymbol{\psi}_1(t), \boldsymbol{\psi}_2(t), ..., \boldsymbol{\psi}_m(t)$ be vector functions (real or complex) , none being identically zero. If there exists constants, not all zero such that

$$\alpha_1 \boldsymbol{\psi}_1(t) + \alpha_2 \boldsymbol{\psi}_2(t) + ... + \alpha_m \boldsymbol{\psi}_m(t) = 0$$

the functions are linearly dependent, otherwise they are linearly independent.

So the vector functions $[1, 1]^T, [t, t]^T$ are linearly independent, although the constant vectors $[1, 1]^T, [t_0, t_0]^T$ are linearly dependent.

Example $\cos t$ and $\sin t$ are linearly independent on $-\infty < t < \infty$.

In amplitude/phase form

$$\alpha_1 \cos(t) + \alpha_2 \sin(t) = \sqrt{(\alpha_1^2 + \alpha_2^2)} \sin(t + \beta)$$

where β is given by

$$\alpha_1 = \sqrt{(\alpha_1^2 + \alpha_2^2)} \sin(\beta), \quad \alpha_2 = \sqrt{(\alpha_1^2 + \alpha_2^2)} \cos(\beta)$$

There is no choice α_1, α_2 except for $\alpha_1 = \alpha_2 = 0$ which makes this function zero for all t

Example 2 The functions $\cos t,\ \sin t,\ 2\sin t$ are linearly dependent.

For example
$$0(\cos t) + 2(\sin t) - 1(2\sin t) = 0$$
for all t, the results follows. Here $\alpha_1 = 0,\ \alpha_2 = 2,\ \alpha_3 = -1$

If functions are linearly dependent then at least one of them can be expressed as a linear combination of the others.

Given any set of n solution (column) vectors, $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ..\boldsymbol{\phi}_n$ real or complex, where $\boldsymbol{\phi}_j$ has elements $\phi_1 j, \phi_2 j, .., \phi_n j$ we have

$$\mathbf{\Phi}(t) = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ..\boldsymbol{\phi}_n] = \begin{bmatrix} \phi_{11} & \phi_{12} & ... & \phi_{1n} \\ \phi_{21} & & & \\ ...\phi n1 & \phi_{n2} & ... & \phi_{nn} \end{bmatrix}$$

for the matrix of these solutions, since $\dot{\boldsymbol{\phi}}_j = \mathbf{A}(t)\boldsymbol{\phi}_j(t)$.

$$\dot{\mathbf{\Phi}}(t) = [\dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2, ..\dot{\boldsymbol{\phi}}_n] = \mathbf{A}(t) = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ..\boldsymbol{\phi}_n] = \mathbf{A}(t)\mathbf{\Phi}(t)$$

where $\dot{\mathbf{\Phi}}(t)$ represents the matrix $\mathbf{\Phi}(t)$ with all its elements differentiated.

It may be convenient to be able to choose a matrix of solutions for

$$\dot{x}_1 = x_2, \ \ \dot{x}_2 = -x_1$$

of the form
$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}$$

where the constants occur in linear dependence, and te eigenvalues and eigenvectors may all be complex.

Theorem 8.2 Any $n+1$ nonzero solutions of the homogenous system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ are linearly dependent.

Theorem 8.3 There exists a set of n linearly independent solutions of $\mathbf{x} = \mathbf{A}(t)\mathbf{x}$

Theorem 8.4 Let $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), ..\boldsymbol{\phi}_n(t)$ be any set of linearly independent vector solutions of the homogenous system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. Then every solution is a linear combination of these solutions

8.6 Stucture of n-dimensional inhomogeneous linear systems

The general inhomogeneous linear system is

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

where $\mathbf{f}(t)$ is a column vector. The associated homogeneous system is

$$\dot{\boldsymbol{\phi}} = \mathbf{A}(t)\boldsymbol{\phi}$$

Let $\mathbf{x} = \mathbf{x}_P(t)$ be the particular solution of the 1st equation, $\boldsymbol{\phi} = \boldsymbol{\phi}_C(t)$ be the complimentary solution of the second function, then $\mathbf{x}(t) = \mathbf{x}_P(t) + \boldsymbol{\phi}_C(t)$ is a solution of the first equation.

The strategy for finding all solutions is therefore to obtain any one solution, then to find all the solutions.(really?)

example Find all solutions of the system $\dot{x}_1 = x_2, \;\; \dot{x}_2 = -x_1 + t$
We have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

The corresponding homogenous system is $\dot{\phi}_1 = \phi_2$, $\dot{\phi}_2 = \phi_1$. With $\ddot{\phi}_1 = \dot{\phi}_2 = -\phi_1$ we therefore have $\ddot{\phi}_1 + \phi_1 = 0$. The linearly independent solutions $\phi_1 = \cos t, \sin t$ correspond relatively to $\phi_2 = -\sin t, \cos t$. Therefore all solutions of the homogenous system are the linear combinations of

$$\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

which are given in matrix form

$$\boldsymbol{\phi}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where a_1, a_2 are arbitrary constants. Setting $\dot{x}_2 = -x_1 + t = 0$ we see $x_1 = t$, and $\dot{x}_1 = 1 = x_2$ giving the particular solution $x_1 = t, \; x_2 = 1$ Therefore all solutions are given by

$$\mathbf{x}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} + \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Theorem 8.8 The solution of the system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{f}(s)\, ds$$

Example Find the solution of

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ te^{-t} & te^{-t} & 1 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} e^t \\ 0 \\ 1 \end{bmatrix}$$

which satisfies the initial conditions $\mathbf{x}(0) = [0, 1, -1]^T$ answer JS pg281, but basically find the general solutions to $\dot{\boldsymbol{\phi}} = \mathbf{A}(t)\boldsymbol{\phi}$ then use the equation in Th8.8

8.7 Stability and Boundedness for linear systems

The following theorem requires a suitable norm for a matrix (denoted by double vertical lines, i.e $||\mathbf{x}||$). For any matrix $\mathbf{A}$ we define

$$|| = \left[\sum_{i,j} |a_{ij}|^2 \right]^{1/2}$$

The norm serves as a measure of magnitude for $\mathbf{A}$ which has the inequality

$$||\mathbf{A}\mathbf{a}|| \leq ||\mathbf{A}||\ ||\mathbf{a}||$$

if $\mathbf{a}$ is a vector of dimension n.

Theorem 8.9 For the regular linear system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ the zero solution, and hence by theorem 8.1, all solutions, are Liapunov stable on $t \geq t_0$ iff every solution is bounded as $t \to \infty$. If $\mathbf{A}$ is constant and every solution is bounded, the solutions are uniformly stable.

Exercise Find the norm of

$$\mathbf{A}(t) = \begin{bmatrix} e^{-t} & -1 \\ 1/(t^2+1) & \sin t \end{bmatrix}$$

and show that $||\mathbf{A}(t)|| \leq 2$

The square of the norm $||\mathbf{A}||^2$ is

$$||\mathbf{A}||^2 = e^{-2t} + 1 + (t^2+1)^{-2} + \sin^2 t$$

so $||\mathbf{A}|| \to \infty$ as $-t \to \infty$.
For $t > 0$
$$||\mathbf{A}||^2 \leq |e^{-2t}| + |1| + |(t^2+1)^{-2}| + |\sin^2 t| \leq 4$$
hence $||\mathbf{A}|| \leq 2$

8.8 Stability of linear systems with constant coefficients

When coefficients $a_{ij}(t)$ are functions of t it will usually be impossible to construct an explicit fundamental matrix for the system. When the coefficients are all constants then it is much easier. see section 2.4 where a system of 2 dimensions is solved.

Consider
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
We look for solutions of the form
$$\mathbf{x} = \mathbf{r}e^{\lambda t}$$
where λ is a constant, and $\mathbf{r}$ is a constant column vector. We therefore get
$$\mathbf{A}\mathbf{r}e^{\lambda t} - \lambda\mathbf{r}e^{\lambda t} = (\mathbf{A}{-}\lambda\mathbf{I})\mathbf{r}e^{\lambda t} = 0$$
for all t, or
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{r} = 0$$
This has non trivial solutions iff
$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
In component form this becomes
$$\begin{vmatrix} a_{11}-\lambda & a_{12} & ... & a_{1n} \\ a_{21} & a_{22}-\lambda & ... & a_{2n} \\ ... & & & \\ a_{n1} & -a_{n2} & ... & a_{nn}-\lambda \end{vmatrix}$$
in which the polynomial equation has a degree of n for λ, called the characteristic equation. It therefore has n roots, real of complex, some of which may be repeated roots. If λ is complex then so is $\bar{\lambda}$ since $\mathbf{A}$ is a real matrix. The values of λ are the eigenvalues.

Now suppose that the eigenvalues are all different, so that there are exactly n distinct eigenvalues. For each λ_i there exists nonzero solutions $\mathbf{r} = \mathbf{r}_i$. These are eigenvectors corresponding to λ_i, these eigenvectors are simply multiples of each other, therefore we have essentially n solutions of
$$\mathbf{r}_1 e^{\lambda_1 t}, \mathbf{r}_2 e^{\lambda_2 t}, ..., \mathbf{r}_n e^{\lambda_n t}$$
where $\mathbf{r}_i$ is any one of the eigenvectors of λ_i.

Theorem 8.10 For the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with $\mathbf{A}$ a real constant matrix whose eigenvalues λ_n $n = 1, 2..$ are all different.

$$\mathbf{\Phi}(t) = [\mathbf{r}_1 e^{\lambda_1 t}, \mathbf{r}_2 e^{\lambda_2 t}, ..., \mathbf{r}_n e^{\lambda_n t}]$$

is a fundamental matrix (complex in general) where $\mathbf{r}_i$ is any eigenvector corresponding to λ_i.

Example p285-287

Theroem 8.11 Corresponding to an eigenvalue of $\mathbf{A}$, $\lambda = \lambda_i$, of multiplicity $m \leq n$ there are m linearly independent solutions of the system $\mathbf{x} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a constant matrix. These are of the form

$$\mathbf{p}_1(t)e^{\lambda_i t}, ..., \mathbf{p}_m(t)e^{\lambda_i t}$$

where the $\mathbf{p}_j(t)$ are vector polynomials of degree less than m.
If λ_i is a complex eigenvalue then $\bar{\lambda}_i$ is another.

Theorem 8.12 Let $\mathbf{A}$ be a constant matrix in the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with eigenvalues λ_i, $i = 1, 2.., n$

1. if the system is stable then $\mathrm{Re}\{\lambda_i\} \leq 0, i = 1, 2, .., n$
2. If either $Re\{\lambda_i\} < 0, i = 1, 2, ..., n$ or if $Re\{\lambda_i\} \leq 0, i = 1, 2, ..., n$ and there is no zero repeated eigenvalue, the system is uniformly stable.
3. The system is asymptotically stable iff $Re\{\lambda_i\} < 0, i = 1, 2, ..., n$ (and also uniformly stable by (2).
4. if $Re\{\lambda_i\} > 0$ for any i, the solution is unstable

8.9 Linear approximation at equilibrium points for first order system in n variables

The equilibrium points of the autonomous 1st order system in n variables is

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$$

occur at solutions given by $\mathbf{X}(\mathbf{x}) = \mathbf{0}$ For non linear systems there are no general methods of solving this equations.

Example Find all equilibrium points of the third order system

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \mathbf{X}(\mathbf{x}) = \begin{bmatrix} x_1^2 - x_2 + x_3 \\ x_1 - x_2 \\ 2x_2^2 + x_3 - 2 \end{bmatrix}$$

we require that all simultaneous solutions of $\mathbf{X}(\mathbf{x})$ =0 that is

$$\begin{aligned} x_1^2 - x_2 + x_3 =& 0 \\ x_1 - x_2 =& 0 \\ 2x_2^2 + x_3 - 2 =& 0 \end{aligned}$$

eliminating gives $x_1 = -2$ or 1, and the coordinates of the two equilibrium points are (-2,-2,-6) and (1,1,0).

We can investigate the nature of an equilibrium point by examining its linear approximation. Suppose that the system has an equilibrium point at $\mathbf{x} = \mathbf{x}_c$. consider a perturbation $\mathbf{x} = \mathbf{x}_c + \boldsymbol{\xi}$ about the equilibrium point. Substitution into the original equation gives

$$\dot{\boldsymbol{\xi}} = \boldsymbol{X}(\boldsymbol{x}_c + \boldsymbol{\xi}) = \boldsymbol{X}(\boldsymbol{x}_c) + \mathbf{J}\boldsymbol{\xi} + o(||\boldsymbol{\xi}||) = \mathbf{J}\boldsymbol{\xi} + o(||\boldsymbol{\xi}||)$$

where $\mathbf{J}$ is the $n \times n$ Jacobian matrix of $\mathbf{X} = [\mathbf{X_1}(\mathbf{x}), \mathbf{X_2}(\mathbf{x}), ..., \mathbf{X}_n(\mathbf{x})]^T$ evaluated at the equilibrium point $\mathbf{x}_c$, namely the matrix with elements J_{ij} given by

$$\mathbf{J} = [j_{ij}] = \left[\frac{\delta X_i(\mathbf{x}}{\delta x_j}\right]_{\mathbf{x}=\mathbf{x}_c}$$

Then the linear approximation is $\dot{\boldsymbol{\xi}} = \boldsymbol{J}\boldsymbol{\xi}$ Using the example above we have

$$\begin{aligned} \mathbf{X_1} =& x_1^2 - x_2 + x_3 \\ \mathbf{X_2} =& x_1 - x_2 \\ \mathbf{X_3} =& 2x_2^2 + x_3 - 2 \end{aligned}$$

giving the Jacobian matrix

$$\mathbf{JX} = \begin{bmatrix} 2x_1 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 4x_2 & 1 \end{bmatrix}$$

Hence at (-2,-2,-6) the linear approximation is

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} -4 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & -8 & 1 \end{bmatrix} \boldsymbol{\xi}$$

and at (1,1,0)

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \boldsymbol{\xi}$$

Finding the stability of the system as discussed before is done by finding the eigenvalues. With the eigenvalues satisfying the characteristic equation.

$$det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

Example find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & -3 & 5 \\ -1 & 0 & 0 \end{bmatrix}$$

We have

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{bmatrix} -3-\lambda & 0 & 2 \\ -1 & -3-\lambda & 5 \\ -1 & 0 & -\lambda \end{bmatrix} = -(1+\lambda)(2+\lambda)(3+\lambda) = 0$$

Giving the eigenvalues ($\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$) and the corresponding eigenvectors

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{bmatrix} -3 & 0 & 2 \\ -1 & -3 & 5 \\ -1 & 0 & 0 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 5 \\ -1 & 0 & 1 \end{bmatrix}$$

Now solving

$$\begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 5 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x = z$ and $y = 2x$ so we have $\mathbf{r}_1 = [1, 2, 1]^T$, and carrying out for the other eigenvalues to give $\mathbf{r}_2 = [2, 3, 1]^T$ and $\mathbf{r}_3 = [0, 1, 0]^T$. Thus the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-t} + \beta \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e^{-2t} + \gamma \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-3t}$$

Since all eigenvalues are negative the origin will be asymptomatically stable.

8.10 Stability of a class of non-autonomouslinear systems in n dimensions

The system considered is

$$\dot{\mathbf{x}} = \{\mathbf{A} + \mathbf{C}(t)\}\mathbf{x}$$

Under quite general conditions the stability is determined by the stability of the solutions of the autonomous system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Lemma let $\mathbf{\Phi}(t)$ be any fundamental matrix of the system $\dot{\boldsymbol{\phi}} = \boldsymbol{A\phi}, \boldsymbol{A}$ constant. Then for any two parameters s, t_0

$$\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0) = \mathbf{\Phi}(t-s)\mathbf{\Phi}^{-1}(0)$$

Theorem 8.13 The solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t-s+t_0)\mathbf{\Phi}^{-1}(t_0)\mathbf{f}(s)\ ds$$

where $\mathbf{\Phi}(t)$ is any fundamental matrix of the system. In particular, if $\mathbf{\Psi}(t_0)$ is any fundamental matrix satisfying $\mathbf{\Psi}(t_0) = mathbfI$ then

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Psi}(t-s)\mathbf{f}(s)\ ds$$

Example pg 294

Theorem 8.14 - Gronwall's Lemma If, for $t \geq t_0$

1. $u(t)$ and $v(t)$ are continuous and $u(t) \geq 0, v(t) \geq 0$
2. $u(t) \leq K + \int_{t_0}^{t} u(s)v(s)\ ds,\ k > 0$
   Then
   $$u(t) \leq K \exp\left(\int_{t_0}^{t} v(s)\ ds\right), \quad t \geq t_0$$

Theorem 8.15 Suppose that

1. $\mathbf{A}$ is a constant $n \times n$ matrix whose eigenvalues have negative real parts
2. For $t_0 \leq t \leq \infty, mathbfC(t)$, is continuous and
   $$\int_{t_0}^{t} ||mathbfC(t)||dt \quad \text{is bounded}$$

Then all solutions of the linear, homogenous system $\dot{\mathbf{x}} = \{\mathbf{A} + \mathbf{C}(t)\}\mathbf{x}$ are asymptotically stable.

Collary 8.15 If $\mathbf{C}$ satisfies the conditions of the theorem but all solutions of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are merely bounded, then all solutions of $\dot{\mathbf{x}} = \{\mathbf{A} + \mathbf{C}(t)\}\mathbf{x}$ are therefore stable.

9 Mathieu's Equation

The stability or instability of nonlinear systems can often be tested by an approximate procedure which leads to a linear equation describing the growth of the difference between the test solution and its neighbours. This variational equation (th8.9) often turns out to have a periodic coefficient (Mathieu's equation) and the properties of such equations are derived in this chapter.

9.1 The stability of forced oscillations by solutions perturbation

Consider the general n-dimension autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)$$

The stability of a solution $\mathbf{x}^*(t)$ can be reduced to a zero solutio of a related system. Let $\mathbf{x}(t)$ be any other solution then

$$\mathbf{x}(t) = \mathbf{x}^*(t) + \boldsymbol{\xi}(t)$$

Then $\boldsymbol{\xi}$ represents a perturbation, or disturbance, of the original solution. We can see how stable the solution is whether such small disturbances grow or not. The original equation can be rewritten as

$$\dot{\mathbf{x}}^* + \dot{\boldsymbol{\xi}} = \mathbf{f}(\mathbf{x}^*, t) + \{\boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\xi}, t) - \mathbf{f}(\mathbf{x}^*,t)\}$$

which becomes

$$\dot{\boldsymbol{\xi}} = \boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\xi}, t) - \boldsymbol{f}(\boldsymbol{x}^*, t) = \boldsymbol{h}(\boldsymbol{\xi}, t)$$

The stability properties of $\mathbf{x}^*(t)$ are the sa,e as those of the zero solution of the above equation, $\boldsymbol{\xi}(t) \equiv 0$. More on pg306-308

The Mathieu equation takes on the form

$$\ddot{y} + (a - 2q\cos 2x)y = 0$$

where a and q are parameters

Exercise Show that the damped equation

$$\ddot{x} + k\dot{x} + (\gamma + \beta\cos t)x = 0$$

can be transformed into a Mathieu equation by the change of variable $x = ze^{\mu t}$ for a suitable choice for μ.

Substituting for x gives

$$(\ddot{z} + 2\mu\dot{z} + \mu^2 z) + k(\dot{z} + \mu z) + (\gamma + \beta\cos t)z = 0$$

The coefficient of $\dot{z}$ is $2\mu + k$ and setting this to zero gives, $\mu = -k/2$, substituting this into the equation

$$\ddot{z} + (\mu^2 + k\mu + \gamma + \beta\cos t)z = 0 \text{ that is } \ddot{z} + \left(\gamma - \frac{1}{4}k^2 + \beta\cos t\right)z = 0$$

9.2 Equations with periodic coefficients (Floquet theory)

With the general n-dimensional first order system

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x} \tag{80}$$

where $\mathbf{P}(t)$is periodic with minimal period T, that is T is the smallest positive number for which

$$\mathbf{P}(t+T) = \mathbf{P}(t), \quad -\infty < t < \infty$$

The solutions are not necessarily periodic for example

$$\dot{x} = \mathbf{P}(t)x = (1+\sin t)x$$

the coefficient $P(t)$ has period 2π, but all solutions are given by

$$x = ce^{t-\cos t}$$

where c is any constant, so only the solution $x = 0$ is periodic.

Theroem 9.1 - Floquet's Theorem The regular system $\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}$ where $\mathbf{P}$ is an $n \times n$ matrix function with minimal period T, has at least one non-trivial solution $\boldsymbol{x} = \boldsymbol{\chi}(t)$ such that

$$\boldsymbol{\chi}(t+T) - \mu\boldsymbol{\chi}(t), \quad -\infty < t < \infty$$

Fundamental matrix - Screencast Consider

$$\ddot{x} + \omega(t)^2 x = 0, \quad \omega(t)^2 = \begin{cases} 0, & 0 \le t < \tau \\ a^2, & \tau \le t < T, \ a > 0 \end{cases}$$

where $\omega(t)$ is a $T-$periodic function.
find the two solutions with initial conditions $(1,0)^T$ and $(0,1)^T$ and hence find a fundamental matrix at $t = T$ associated with these solutions.

For $0 \le t < \tau$ we have $\ddot{x} = 0$ giving the general solution $x = A + Bt$. for the first condition, $(1,0)^T$ this means that $x(0) = 1$ and $\dot{x}(0) = 0$, and the second condition we have $x(0) = 0, \ \dot{x}(0) = 1$.
Let

$$\begin{aligned} \phi_{11}(t) =& A_1 + B_1 t \\ \phi_{12}(t) =& A_2 + B_2 t \end{aligned}$$

The first equation refers to the first condition, giving $A_1 = 1, \ B_1 = 0$, thus $\phi_{11} = 1$ and the second equation along with the second condition gives $A_2 = 0$ and $B_2 = 1$ thus $\phi_{12} = t$. So for $0 \le t < \tau$

$$\Phi(t) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \dot{\phi}_{11} & \dot{\phi}_{12} \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

For the next range of t values

$$\ddot{x} + a^2 x = 0$$

giving

$$x = A\cos at + B\sin at$$

To make things easier we use

$$x = A\cos a(t-\tau) + B\sin a(t-\tau)$$

so that

$$\begin{aligned}\phi_{11}(t) =& A_{11}\cos a(t-\tau) + B_{11}\sin a(t-\tau)\\ \phi_{12}(t) =& A_{12}\cos a(t-\tau) + B_{12}\sin a(t-\tau)\end{aligned}$$

As we cant use $t = 0$ as it is not in the range $\tau \leq t < T$ we have to use the continuity of x and $\dot{x}$ at $t = \tau$. So looking back at the values we have from the previous range. At $t = \tau$

$$\begin{aligned}\phi_{11}(\tau) =& 1\\ \phi_{12}(\tau) =& \tau\end{aligned}$$

Therefore

$$\begin{aligned}\phi_{11}(\tau) =& 1 = A_{11}\\ \phi_{12}(\tau) =& \tau = A_{12}\\ \dot{\phi}_{11}(\tau) =& 0 = aB_{11}\\ \dot{\phi}_{12}(\tau) =& 1 = aB_{12}\end{aligned}$$

Giving $B_{11} = 0$ and $B_{12} = 1/a$
So for $\tau \leq t < T$

$$\Phi(t) = \begin{bmatrix}\cos a(t-\tau) & \tau\cos a(t-\tau) + \frac{1}{a}\sin a(t-\tau)\\ -a\sin a(t-\tau) & -a\tau\sin a(t-\tau) + \cos a(t-\tau)\end{bmatrix}$$

Although the range does not go up to T, by continuity of $x, \dot{x}$ and periodicity of ω we can substitute above with $t = T$ to get

$$\Phi(T) = \begin{bmatrix}\cos u & \tau\cos u + \frac{1}{a}\sin u\\ -a\sin u & -a\tau\sin u + \cos u\end{bmatrix}$$

where $u = a(T-\tau)$

Theorem 9.2 The constants μ in Th9.1 are independent of the choice of $\mathbf{\Phi}$

Definition 9.1 A solution to $\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}$ is called a normal solution

Definition 9.2 Let μ be a characteristic number, real or complex, of the Floquet's Theorem equation, corresponding to the minimal period T of $\mathbf{P}(t)$ then ρ, defined by

$$e^{\rho T} = \mu$$

is called a characteristic exponent of the system. Note that ρ is defined only to an additive multiple of $2\pi i/T$.

Theorem 9.3 Suppose that the matrix $\mathbf{E} = \mathbf{\Phi}^{-1}(t_0)\mathbf{\Phi}(t_0 + T)$ has n distinct eigenvalues, μ_i, $i = 1, 2, ..., n$ then (80) has n linearly independent normal solutions of the form

$$\mathbf{x}_i = \mathbf{p}_i(t)e^{\rho_i t}$$

where $\mathbf{p}_i(t)$ are vector functions with period $T2$

Example Identify the periodic vectors $\mathbf{p}_i(t)$ in the solution of the periodic diff equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathbf{P}(t)\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & h(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $h(t) = (\cos t + \sin t)/(2 + \sin t - \cos t)$.

The fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{bmatrix} -2 - \sin t & e^t \\ 2 + \sin t - \cos t & 0 \end{bmatrix}$$

from the columns we can identify the 2π periodic vectors

$$\mathbf{p}_1 = a \begin{bmatrix} -2 - \sin t \\ -2 + \sin t - \cos t \end{bmatrix}, \ \mathbf{p}_2 = b \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where a, b are any constants. In terms of normal solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} -2 - \sin t \\ -2 + \sin t - \cos t \end{bmatrix} e^0 + b \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

Definition 9.4 Let $[\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), ..., \boldsymbol{\phi}_n(t)]$ be a matrix whose columns are any solutions of the n-dimensional system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ Then

$$W(t) = \det[\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), ..., \boldsymbol{\phi}_n(t)]$$

Theorem 9.4 For any t_0, the Wronskian of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ is

$$W(t) = W(T_0) \exp\left(\int_{t_0}^{t} tr\{\mathbf{A}(s)\}\, ds \right)$$

where $tr\{\mathbf{A}(s)\}$ is the trace of $\mathbf{A}(s)$ (the sum of the elements of its principle diagonal.

Theorem 9.5 For the system
$mathbf\dot{x} = P(t)\mathbf{x}$ where $\mathbf{P}(t)$ has a minimal period T, let the numbers of the system by $\mu_1, \mu_2, ..., \mu_n$. Then

$$\mu_1\mu_2..\mu_n = \exp\left(\int_0^T tr\{\mathbf{P}(s)\}\ ds\right)$$

a repeated characteristic number being counted according to its multiplicity.

Example If $T = 2\pi$ and

$$\mathbf{P}(t) = \begin{bmatrix} 1 & 1 \\ 0 & (\cos t + \sin t)/(2 + \sin t - \cos t) \end{bmatrix}$$

Then

$$\int_0^{2\pi} tr\{\mathbf{P}(s)\}\ ds = \int_0^{2\pi} \left[1 + \frac{\cos s + \sin s}{2 + \sin s - \cos s}\right] ds = \int_0^{2\pi} \left[1 + \frac{d(\sin s - \cos s)/ds}{2 + \sin s - \cos s}\ ds\right]$$

$$= [s + \log(2 + \sin s \cos s)]_0^{2\pi} = 2\pi$$

Therefore

$$\exp\left[\int_0^{2\pi} tr\{\mathbf{P}(s)\}\ ds\right] = e^{2\pi} = e^0 \times e^{2\pi} = \mu_1\mu_2$$

Exercise Find the matrix E for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & \cos t - 1 \\ 0 & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The equations are

$$\begin{aligned} \dot{x}_1 =& x_1 - (1 - \cos t)x_2 \\ \dot{x}_2 =& x_2 \cos t \end{aligned}$$

For x_2 we have

$$\int 1/x dx = \int \cos t dt$$

$$\log x = \sin t + c$$

$$x = e^{\sin t} + e^c = ae^{\sin t}$$

substituting into $\dot{x}_1$ and using the IF method

$$\begin{aligned} \dot{x} - x =& - a(1 - \cos t)e^{\sin t} \\ \dot{x}e^{-t} =& - a(1 - \cos t)e^{\sin t - t} \\ xe^{-t} =& b - a\int dt\ (1 - \cos t)e^{\sin t - t} = b + ae^{-t - \cos t} \end{aligned}$$

No idea how they got the above, probably as the cos term cant be differentiated it was taken out?
Hence the general solution is

$$x_1 = be^t + ae^{\sin t}, \;\; x_2 = ae^{\sin t}$$

To find the fundamental matrix $\mathbf{\Phi(0)} = \boldsymbol{I}$ we have

$$\begin{aligned} x_1(0) = 1 = b + a \\ x_2(0) = 0 = a \end{aligned}$$

so $a = 0, \; b = 1$ giving

$$x_1 = e^t, \; x_2 = 0$$

and

$$\begin{aligned} \dot{x}_1(0) = 0 = be^t + a\cos t e^{\sin t} = a + b \\ \dot{x}_2(0) = 1 = a\cos t e^{\sin t} = a \end{aligned}$$

so $a = 1, \; b = -1$ giving

$$\dot{x}_1 = e^{\sin t} - e^t, \;\; \dot{x}_2 = e^{\sin t}$$

thus

$$\boldsymbol{\phi}(t) = \begin{bmatrix} e^t & e^{\sin t} - e^t \\ 0 & e^{\sin t} \end{bmatrix}$$

Not that this time we have put the x_1, x_2 in columns not rows. It doesn't matter which way you write this.

The characteristic numbers are the eigenvalues of $\mathbf{E} = \mathbf{\Phi}(2\pi)$ so we need to calculate $\mathbf{E} = \mathbf{\Phi}^{-1}(0)\mathbf{\Phi}(2\pi)$

$$\mathbf{\Phi}(0) = \begin{bmatrix} 1 & 1-1 \\ 0 & 1 \end{bmatrix}, \;\; \mathbf{\Phi}(0)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbf{\Phi}(2\pi) = \begin{bmatrix} 1 & 1-e^{2\pi} \\ 0 & 1 \end{bmatrix}$$

so

$$\mathbf{E} = \begin{bmatrix} A_{11} * B_{11} + A_{12} * B_{21} & A_{11} * B_{12} + A_{12} * B_{22} \\ A_{21} * B_{11} + A_{22} * B_{21} & A_{21} * B_{12} + A_{22} * B_{22} \end{bmatrix} = \begin{bmatrix} e^{2\pi} & 1-e^{2\pi} \\ 0 & 1 \end{bmatrix}$$

Now to find the roots we must satisfy

$$\begin{bmatrix} e^{2\pi} - \mu & 1-e^{2\pi} \\ 0 & 1-\mu \end{bmatrix} = 0$$

that is the det of the matrix $ad - bc = 0$, so $(e^{2\pi} - \mu)(1-\mu) - 0 = 0$ giving the roots $e^{2\pi}$ and 1. Verifying Th9.5 where $\mu_1\mu_2..\mu_n = \exp\left(\int_0^T tr\{\mathbf{P}(s)\}\, ds\right)$ and trace of a matrix is the sum of the diagonals, and $\mathbf{P}(s)$ is the original matrix ($\dot{\mathbf{x}} = \mathbf{P}(s)\mathbf{x}$).

$$\exp\left(\int_0^{2\pi} 1 + \cos s \, ds\right) = \exp(2\pi) = 1 \times e^{2\pi} = \mu_1\mu_2$$

as expected.

9.3 Mathieu's Equation arising from a Duffing Equation

With Mathieu's equation

$$\ddot{x} + (\alpha + \beta\cos t)x = 0 \tag{81}$$

As a first order system it can be expressed as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha - \beta\cos t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In the notation in the previous section

$$\mathbf{P}(t) = \begin{bmatrix} 0 & 1 \\ -\alpha - \beta\cos t & 0 \end{bmatrix}$$

which is clearly periodic, with minimal period 2π. From the above equation $Tr\{\mathbf{P}(r)\} = 0$ so $\mu_1\mu_2 = e^0 = 1$ ... a lot of matrices etc not worth writing out again. pg 316-322

9.4 Transition curves for Mathieu's equation by perturbation

For small values $|\beta|$ a perturbation method can be used to establish the transition curves.With

$$\ddot{x} + (\alpha + \beta\cos t)x = 0$$

suppose the transition curves are given by

$$\alpha = \alpha(\beta) = \alpha_0 + \beta\alpha_1 + \beta^2\alpha_2...$$

and that the corresponding solutions have the form

$$x(t) = x_0(t) + \beta x_1(t) + \beta^2 x_2(t)...$$

when substituted into each other and the powers of β are equated to zero, this gives

$$\begin{aligned} \ddot{x}_0 + \alpha_0 =& 0 \\ \ddot{x}_1 + \alpha_1 =& -(\alpha_1 + \cos t)x_0 \ddot{x}_2 + \alpha_2 = && -\alpha_2 x_0 - (\alpha_1 + \cos t)x_1 \\ \ddot{x}_3 + \alpha_3 =& -\alpha_3 x_0 - \alpha_2 x_1 - (\alpha_1 + \cos t)x_2 \end{aligned}$$

From section 9.3 we are searching for solutions with a minimum period 2π if $a_0 = n^2, \ n = 0, 1, 2...$ and for solutions of minimum period 4π if $a_0 = (n + \frac{1}{2})^2, \ n = 0, 1, 2....$

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