M833 Maple

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0 Shortuts

ctrl+K - add new line before ctrl+J - add new line after ctrl+F S - save worksheet F5 - changes input mode F4 - joins input lines together F3 - separates input lines

1 Introduction to Maple

solve solves one or more equations or inequalities for their unknowns.

```
solve(x^2=1,x);
```

If it cant give an explicit answer it will give a *Rootof*

allvalues - used to find all values of solve if *Rootof* is given

assign Once a solution has been found, this can assign the answers t the relevant variables

```
assign(sol);
x,y;
x=a, y=b
```

int integrate. With this ensure the function has free variables, if you have assigned these variables values beforehand it will not work

subs this is used to give variables values which are not stored. more than one substitution can be made, nesting these would be best practice.

```
y:=4x:
subs(x=1,y);
4
y;
4x
```

eval a more generalised version of subs

evaln -evaluate to a name, this assigns values to a variable

```
z:=evln(2);
2
z:evaln(z)
z:='z'
```

other ways of doing this is

```
z:='z':
unassign('z,a')
```

evalc used when the function is complex

sum used to symbolically sum series. If k has been used before we can get around this by using single quotation marks around every k in the sum

```
sum(2*k,k=1..N);
(N + 1)^2 - N - 1
k:=1:
sum(2*'k','k'=1..N);
(N + 1)^2 - N - 1
k;
1
```

factor -factors the equation

seq - produces a sequence from an expression

```
seq(k^5,k=1..5);
0,1,32,243,1024
```

assigned see if something is assigned to the variable

```
assigned(s1);
false
```

anames finds assigned variables

```
anames();
m,n,z,x
```

evalf evaluates to float (10 digits)

digits change the number of digits that are evaluated

printf prints a statement. quotation marks or backquotes can be used.

```
 printf("This is a string!\n");  #statement +newline
 printf("The value of a is %d\n",a) #statement + value of 'a', an
                                     integer,+newline
 %f = float, %d = int,%s = string

printf("The value of g is %.13f \n", g);
#prints value of 'g' to 13dp
```

fnormal with calculations normally done at a higher precision than needed this sets any point less than 10^{-d} to zero, where fnormal(e,d), e= expression

```
a:=evalf(Pi):
fnormal(a,3);
3.14
```

plot -plots a simple line graph. More complex plots can be used using the package
with(plots):
using a semi colon here will show you all types of plots within the package. be aware that using restart will remove the with(plots) package and you need to use this command again.

If statements set out in the following format

```
if expression then
elif
else
fi;
```

While loop much like the python while loop

```
while expression do;
od;
```

for loop this is a usual for loop as below

```
for k from 1 to 5 do;
something;
od:

for k from 1 to 10 by 2 do; #by increments of 2
something;
od:
```

cat concatenate string or expressions

```
n:=5: y:=add(a.k*x^k,k=0..n);

y:=add(cat(a,k)*x^k,k=0..n);
```

isprime find if a number is prime or not

Lists are created using []. The form [[x,y],[x,y]] is used for plotting

```
nops(list); #number of elements
list[4] #4th otem in list (there is no 0 element)
L:=[op(4..7,list)]; #creates a subgroup of list from elements 4 to
                                7
list[4]:=1 # changes 4th element in list to the value 1
L:=[op(list1),op(list2)]; creates a new list from list1 and list2
                                concatenated
L:=[op(list1),Anna];
```

Lists {} order and repetition is not preserved thus {a,b,c}={b,a,a,c}

```
{1,2,3} union {a,b,c};
{1,2,3,a,b,c}

x:={a,b,c}:
op(2,x);
b
```

has finds if expression has something in it

select selects certain items from expression

```
expr:=cos(x)+sin(y):
select(has,expr,cos);

expr2:=1+x+x^2:
select(has,expr2,x^2)
x^2
```

diff - differentiate. Double differentiation is done by using the $ symbol

```
diff(x^2,x);
diff(x^3+y,x$2
```

2 Simplification

Expand - is used to expand expressions, only 1 argument can be used at a time. When applied to trig forms it uses trig rules to remove multiple angles, replacing them with powers. Manipulation is needed to get it into a form you want.

```
expand((x+1)*x+z)^2);
```

Factor - is in inverse of expand for polynomials, and can factor both numerator and denominator

```
factor(f);
```

Normal simplifies rational functions of the form f/g it finds the gcd of the polynomials f and g and factors it out, leaving the remainders unfactored. It is faster than Factor function so is normally preferred. It may also be needed to factor the numerator and denominator first to allow Normal to find simple common factors.

```
normal(r);
```

If only the gcd is needed then we can use the following command

```
gcd(f,g);
```

Simplify This is a general purpose command, although not always the most appropriate as it can be unpredictable. It normally, however, produces a simpler result by applying the following rules

$$\begin{aligned}
\sin^2 x &\rightarrow 1-\cos^2 x \\
\sinh^2 x &\rightarrow -1+\cosh^2 x \\
\exp(x)\exp(y) &\rightarrow \exp(x+y) \\
u^x u^y &\rightarrow u^{x+y} \\
u^{a/b}u^{c/d} &\rightarrow u^{(ad+bc)/bd}
\end{aligned}$$

```
simplify(f);

eqns:={sin(x)^2+cos(x)^2=1}:
e:=sin(x)^3-11sin(x)^2cos(x):
simplify(e,eqns);

simplify(f1,{x^4:=0});
```

The 2nd code allows us to simplify in an alternative form. The 3rd code can truncate $f1$ to a third-order polynomial

Combine allows the combination of functions, usually with an optional argument. This is in some respects the opposite of expand

```
combine(x^u*x^v,power);
  x^{u+v}

combine(ln(x)+ln(y),ln);
ln(x)+ln(y)

combine(ln(x)+ln(y),ln,symbolic);
ln(xy)

assume(x>0): combine(ln(x)+ln(y),ln);
ln(xy)
combine(e^x*e^y,power);
```

A useful feature of combine is trig simplification because it transforms powers of the sine and cosine functions into their multiple angle expansions, although sometimes it is first necessary to use the expand function

```
combine(sin(x)^4,trig);
```

convert and Series Convert has many expressions such as

```
list=[1,2,3,4]: convert(list,set);
{1,2,3,4}

convert(3.142,confrac); #confrac = continued fraction

convert(tan(x),sincos); #convets into sin and cos

convert(sin(x),exp); # converts trig into exponential
```

One important use is with the "series" command, a procedure returning the power series expansion of a function or expression.

the series command is normally used with 3 arguments, series(expr,x=a,n) in which an expression is expanded about $x = a$ up to $(n-1)$ powers (n being the Order value), for example:

```
X:=series(arctan(x),x=0,10);
```

typing **whattype(X)** gives the type *series*. This cannot always be used until it is converted into a polynomial, e.g. it cannot be used to plot a graph, thus we need to convert it.

```
Xs:=convert(X,polynom):
```

To remove the Order from a power series we can use the eval (or subs) command

```
Y:=eval(Y,O=0);
```

Text in Graphs - With $f(x, a)$ it is often helpful to show the value of a in text. This is done using

```
convert(expr,string);

NB. parse() is the opposite
```

this can be then used with concatenate.

```
t1:=cat("Value of a is ",convert(evalf(b,4),string)):

plot(eval(f,a=b),x=0..Pi,title=t1,titlefont=tfont);
```

This technique is very useful in conjunction with the animation package. Instead of using title, we can use textplot, which is more flexible.

tickmarks - adding tickmarks to the graph axis can show values better. this can be in the form

```
plot(y=cos(x),x=0..5,y=0..10,tickmarks=[5,10]); #1 to 5 tickmarked
                                                  on x axis, 1 to 10 on y

xtick:=seq(k/2,k=0..6): ytick:=seq(k,k=0..6):
ticks:=xtickmarks=xticks,ytickmarks=yticks:
plot(y=cos(x),x=0..5,y=0..10,ticks);
```

collect - this collects all the coefficients of the same power in a series.

```
f:=(x+y+1)^3*(x+3*y+5);
fx:=collect(f,x);
fy:=collect(f,y,factor);
```

coeff - Extracts the coefficient of x^n from a polynomial. Does not work on non integer power functions.

```
p:=3*y^3+2*x^2-5\\
coeff(p,x,2)\\
2
```

coeffs as above but shows all the coefficients in the polynomial. Collect may need to be used prior to this. Does not work on non integer power functions. Not in order so need to show the power command to find its order, as below

```
X=series(arctan(x),x=0,10):
Xs:=convert(X,polynom):
coeffs(Xs,x,'pow');
pow;
```

map - map(fcn, expr, arg1, ..., argN). pg85, doesnt seem to give any use at moment. but can do a lot of things not mentioned so far

3 Functions and Procedures

?inifcn - shows all the defined functions in Maple

$->$ - allows user defined functions. E.g if $f(x) = ax^3 + 2x + 1$

```
f:=x->a*x^3+2*x+1;
f(1);
a+3
f(u);
a*u^3+2*u+1
g:=w->collect(f(w+1),w);

diff(f(w),w);
int(f(w),w);
```

The last two procedures do not need to have $f(w)$ defined this way, as it assumed that $f(w)$ is a function (but assumed f on its own is a constant), so these do not have to be defined prior. Use the easiest method when using diff and int. Use $->$ only when necessary.

Valuable for use with graphs

```
h:=x->sin(5*sin(Pi*x)):
tick:=xtickmarks[0.5,1.0,1.5,2.0],ytickmarks=[-1,-0.5,0,0.5,1]:
plot(h(z),z=0..2,title="Graph of h(z)",tick,colour=black);
```

Can also be used for more than one expression

```
H(x,y)->y^2/2+x^2/2:
H(1,1);
```

functions n be add, subtracted and composed whre $(f+g)(x)$ means $f(x)+g(x)$ and $(-f)(x)$ means $-f(x)$
functions can also be used for lists

```
h:=x->[2*x,3*x^3,4*x^4]:
h(1);
```

we may also use lists as arguments

```
f:=ls->sum(ls[k],k=1..nops(ls));
ls:=[x,x^2,x^3,x^4,x^5];
f(ls);
   x+x^2+x^3+x^4+x^5
```

The map command can also be incorporated with defining functions, the examples below will square all the elements of A, and will convert all numbers in A to floating point 4sf. This is useful as evalf(x,N) is one of the operations that does not work with map

```
map(x->x^2,A)

mao(x->evalf(x,4),A);
```

unapply - With the function above, if we tried to assign $f(x) := x->diff(\exp(x^2), x\$2)$ and trying to find $f(1)$ we will get an error as Maple substituted 1 for x. To over come this problem by using the unapply command.

```
expr:=x^2+b:
F:=unapply(expr,x);
   F:=x->x^2+b
F:=unapply(expr,x,b);
   F:=(x,b)->x^2+b

f:=unapply(diff(\exp(x^2),x\$2),x)
f(1);
6e
```

Another use is to define a function from a solution which has multiple answers. Below is an expression which has only one real answer, and multiple complex answers. This is then used to graph the real answer

```
sol:=[solve(x^3-(a-1)*x^2+a^2*x-a^3=0,x)];
r:=unapply(sol[1],a);
plot(Re(r(a)),a=0..1);
```

Composition of Functions pg100

$$f@g \text{ means } f \circ g \tag{1}$$
$$(f@g)(x) \text{ means } f(g(x)) \tag{2}$$

(2) is the same as (1) but in Leibniz notation.
For (1) we use the following code

```
f:=x->b*x*(1-x):
g:=x->x^2:
gf:=g@f:
gf(x);
fg:=f@g:
fg(x);
```

As you can see we need to define the function $f \circ g$ first, whereas using (2) simply returns the expression

```
(g@f)(x);
```

Multiple compositions can be done using @@, for instance $(f \circ f \circ f)(x)$ is

```
  (f@@3)(x);
```

asympt computes the asymptotic expansion of f with respect to x approaching infinity, where n is the truncation order (degrees of, or less)

```
asympt(f,x,n);
```

piecewise functions

$$h(x) = \begin{cases} exp(-x^2), & x \leq 0, \\ \cos(x), & x > 0 \end{cases}$$

is written as

```
  h:=x->piecewise(x<=0,exp(-x^2),cos(x));
```

This can be differentiated, and $h \circ h$ can also be used
It can also be integrated symbolically, provided that the constituent parts can be evaluated, thus we need to use the unapply method.

Differentiation of Functions This an be done using D. the top method assigns the derivative, whereas as the bottom method simply returns the expression. This should not be used if a function is required

```
f:=x->x^ne^(cos(x));
g:=D(f);

g(x)=D(f)(w);
```

This should only be used on functions and not expressions, where these will be differentiated using 'diff'.

Total Derivatives - The D operator can be used on expressions containing several free expressions and it returns the total derivative with respect to a dummy variable, for instance, with $f = xy/(x^2 + y^2)$, then df/ds equals

```
f:=x*y/(x^2+y^2):
D(f);
```

where x and y are treated as functions, so $D(x) = dx/ds$, $D(y) = dy/ds$
This use of D only works on expressions, if we need to compute, say, $\sin(f^2)$ then we use the composition rule $\sin \circ f^2$

```
g:=D(sin@(f^2));
```

Implicit differentiation can also be carried out using D

```
eq:=x^3+y^3=c:
eq1:=D(eq);
dy:=solve(eq1,D(y)) # solves for D(y)
#but c is constant so D(c)=0, and by definition D(x)=1 thus
yp:=subs(D(c)=0,D(x)=1,dy);
```

alternatively we could use

```
x:=x->x; #defines variable x as the function x
assume(c,constant)
eq1:=D(eq);
yp:=solve(eq1,D(y))

eq2:=(D@@2)(eq); # 2nd derivative
y2p:=solve(eval(eq2,D(y)=yp),(D@@2)(y)); #solving for y
```

alias - can be used to shorten defined function names, eg. alias(J=BesselJ):

differentiation of Piecewise functions The D operator can e used to differentiate continuous piecewise functions

```
h:=x->piecewise(x<=0,exp(-x^2),cos(x));
hp:=D(h);
h2p:=(D@@2)(h);
plot([hp(x),h2p(x)],x=-2..2*Pi);
```

Partial derivatives If

```
  f:=(x,y,z)->1/sqrt(x^2+y^2+z^2):
  #df/dx is given by
  D[1]f;
  #and d^2f/dx^2 is given by
  D[1,1]f;
  #or
  D[1](D[1]f);

  D[1,2,2]=d^3f/dxdy^2
```

although the D operator can be used to obtain any partial derivatives it does not lnw about the chain rule for partial derivatives.

Differentiation of arbitrary functions pg109-111 Say we had $\frac{d}{dx}f(x)^2$ we can use the chain rule to find the answer for the arbitrary function $f(x)$

$$\frac{d}{dx}f(x)^2 = \frac{d}{du}u^2\frac{du}{dx} = 2u\frac{f(x)}{dx} = 2f(x)f'(x) \quad \text{u=f(x)}$$

Example - If we have $z(x,y)$ defined implicitly as $x+y+z = f(x-z)$ show that $\frac{\delta z}{\delta x} - 2\frac{\delta z}{\delta y} = 1$
first we put everything to the LHS so LHS=0. then differentiate wrt x, and differentiate wrt y. we then solve for dz(x,y)/dx and dz(x,y)/z)

```
#1st define the equation
eq:=(x,y)->x+y+z(x,y)-f(x-z(x,y));
#diff respect to x and y
fx:=D[1](eq);
fy:=D[2](eq);

zx:=solve(fx(x,y)=0,D[1](z)(x,y));
zy:=solve(fy(x,y)=0,D[2](z)(x,y));

simplify(zx - 2*zy -1);
```

Procedures These can be used for functions in which need to be repeated, or if it is too complicated to be written in the normal Maple form. Starting off easy we have

```
  f:=proc(x)
  local y;
  y:=2+sin(x);
  y^2+1/(1+y^2)+sin(y);
  end;

f(1);

fp:=D(f);
plot(fp,-2..2*Pi)
```

Above has the global variable x, and local variable y. f(1) will give us the value when x=1, and fp is the differential wrt x. Checking the output shows that

Maple has added another local variable yx, the derivative of y(x), for more complicated procedures it may not be possible to find the derivative on this manner.

```
h:=proc(x);
if x<=0 then exp(-x^2) else cos(x) fi;
end:
plot('h(x)',x=-2..6);
```

above is a piecewise procedure. To get it to plot the h(x) must be encaptured by single quotes, or by using the evaln(h(x)) in its place.

```
np:=proc(n::integer) local a,b;
a:=nextprime(n);
b:=nextprime(a);
[a,b];
end:

np(1000);
```

above finds the next 2 primes after an integer 'n'

RETURN this immediately returns the procedure, useful in conditional staements to return the 1st match

```
first_even:=proc(L::list) local el;
for el in L do;
if type(el,even) the RETURN(el) fi;
od;
end:

L:=[1,3,5,7,2,6,9]:
first_even(L);
2
```

pg115 onwards, global v local v option.
save load procedures.

4 Sequences, series and limits

Some sequences converge to a certain value, other diverge to $\pm\infty$ others are neither such as $\sin x$
If a series converges then $S_n = S$ where S_n is a sequence, S is a limit.
If it diverges then we say that $R_n = S - S_n$ where R_n is the remainder

In order for a series of real or complex terms

$$S = \sum_{k=1}^{\infty} z_k$$

converge, it is sufficient that the series

$$S = \sum_{k=1}^{\infty} |z_k|$$

converges. In this case, then it is said to be **absolutely convergent**. Series that are convergent but not absolutely are said to be **conditionally convergent**.

If 2 series are absolutely convergent, then the product of these series, written in any order, is also absolutely convergent, and its product is $S_1 S_2$. The product of 2 conditionally convergent sequences is not unique, and its value depends upon the order of summation, and may diverge.

The sum of 2 absolutely convergent series is unaffected by the order of the terms, but for a conditionally convergent real series the terms can be arranged to converge to any real number.

Order of This is slightly different to M823 as we now look for the order of a series when $x \to 0$, not to infinity. For this we use the Taylor series if its not a series already.

$$\begin{aligned}
f(x) =& \sinh(\exp(\sqrt{x}) - 1) \\
& \quad \text{With } e^x = 1 + x + O(x^2) \\
f(x) =& \sinh(1 + \sqrt{x} O(x) - 1) = \sinh(\sqrt{x} + O(x)) \\
& \quad \text{With } \sinh x = x + \frac{x^3}{3!} + O(x^5) \\
f(x) =& \sqrt{x} + O(x) \\
f(x) =& O(\sqrt{x}) \text{ as } x \to 0
\end{aligned}$$

$$f(x) = \exp\left(\frac{\exp(x)-1}{\sinh x}\right)$$
$$= \exp\left(\frac{1+x+O(x^2)-1}{x+O(x^3)}\right) = \exp\left(\frac{x+O(x^2)}{x+O(x^3)}\right)$$

$$\text{Now}\frac{x+O(x^2)}{x+O(x^3)} = (x+O(x^2))(x+O(x^3))^{-1}$$
$$= (x+O(x^2))\frac{1}{x}(1+O(x^2))^{-1}$$
$$= (1+O(x))(1+O(x^2) \text{ where the } ()^{-1} \text{ gets absorbed into the } x^2 \text{ on the RHS}$$
$$= 1+O(x) \text{ thus}$$

$f(x) = \exp(1+O(x)) = \exp(1) = O(1)$

For $f(x) = sin^{-1}(\tanh x) - \tanh(sin^{-1}x)$ we use maple

```
series(f(x),x=0,10);
-1/90 x^7-1/756 x^9 +O(x^10)
```

giving $f(x) = O(x^7)$

Geometric series is defined as

$$S = a + az + az^2 + ... + az^n + ...$$

with

$$S_{n+1} = a(1+z+z^2+....+z^n = \frac{1-z^{n+1}}{1-z}$$

If $|z| < 0$ then as $n \to \infty$

$$S = \frac{a}{1-z}$$

Zeta function and Harmonic Series pg142

4.1 Tests for Convergence

Comparison Test - The series $S = z_1 + z_2 + ..$ is absolutely convergent if there exists an absolutely convergent series $v_1 + v_2 + ...$ and a constant C such that $|z_n| \leq C \leq |v_n|$.
e.g

$$S + \cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + ..., \text{ x is real}$$

$$V + 1 + \frac{1}{2^2} + \frac{1}{3^3} + ...$$

With $V = \zeta(2) = \pi^2/6$ this shows that S is also absolutely convergent since

$$|z_k| = \left|\frac{\cos kx}{k^2}\right| \leq \frac{1}{k^2} = v_k$$

Cauchy's Test If

$$\lim_{n\to\infty} |z_n|^{1/n} < 1, \text{ the series} \sum_{n=1}^{\infty} z_n$$

converges absolutely
But if

$$\lim_{n\to\infty} |z_n|^{1/n} > 1$$

the series doesnt not tend to 0 thus is does not converge. If the limit is unity (=1) this test does not provide any information.

Example If α, a and x are positive real numbers, show that the series converges if $x < 1$

$$\sum_{n=0}^{\infty} \frac{x^n}{a + n^\alpha}$$

With

$$\begin{aligned} z_n =& \frac{x^n}{a + n^\alpha} \\ |z_n|^{1/n} =& \frac{x}{(a + n^\alpha)^{1/n}} \end{aligned}$$

but $(a + n^\alpha)^{1/n} = n^{\alpha/n}(1 + a/n^\alpha)^{1/n}$ in which the bracketed term tends to 1 as $n \to \infty$ so we have

$$|z_n|^{1/n} = \frac{x}{n^{\alpha/n}}$$

With $\alpha/n \to 0$ as $n \to \infty$ therefore $|z_n|^{1/n}$ tends to x as $n \to \infty$. Therefore by Cauchy's Test this will converge if $x < 1$

D'Alembert's Ratio Test $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| < 1$$

and divergent if > 1, again no information if the limit is unity If

$$\lim_{n\to\infty} n \left\{\left|\frac{z_{n+1}}{z_n}\right| - 1\right\} = -1 - c$$

for some positive constant c then the series is absolutely convergent

Example Determine the values of p in which the series converges.

$$S = \sum_{n=1}^{\infty} p^n n^p$$

We have $z_n = p^n n^p$ and $z_{n+1} = p^{n+1}(n+1)^p$ giving

$$\frac{z_{n+1}}{z_n} = \frac{p(n+1)^p}{n^p} = p(1+1/n)^p = p + \frac{p^2}{n} + ..$$

Therefore the series converges when $p < 1$

Raabe's Test following on from D'Alembert's Ratio Test

$$\left| \frac{z_{n+1}}{z_n} \right| = 1 + \frac{a}{n} + O(n^{-1-\lambda}), \quad \lambda > 0$$

where a is independent of n, then the series is absolutely convergent if $a < -1$. If $a \geq -1, c < 0$ the series diverges.

intgeral test This gives a lower bound and upper bound for the series, much like dividing the area of the curve into small rectangles then finding the integrals of the lowest points, and highest points of the rectangle.

$$\int_1^{n+1} dx f(x) \leq s_n \leq f(1) + \int_1^n dx f(x)$$

It may be shown that the sequence s_n and the intgral $\int_1^X dx f(x)$ both converge or both diverge as $n \to \infty$ and $X \to \infty$. In either case

$$\lim_{n\to\infty} \left(s_n - \int_1^n dx f(x) \right)$$

Example

$$\sum_{k=3}^{\infty} \frac{1}{k \ln k \ln \ln k}$$

consider

$$S_n = \sum_{k=3}^{\infty} u_k \text{ where } u_k = \frac{1}{k \ln k \ln \ln k} = f(k)$$

For $k \geq 3, u_k > 0$ and decreasing and they are bounded by

$$0 < u_k < \frac{1}{3 \ln 3 \ln \ln 3}$$

as the largest term is when $k = 3$.
Using the integral test

$$\int_3^{n+1} dk f(k) \leq S_n \leq u_3 + \int_3^n dk f(k)$$

Now

$$\int \frac{dk}{k\ln k\ln\ln k} = \int \frac{d}{dk}(\ln\ln\ln k)dk = \ln\ln\ln k$$

which increases as $k \to \infty$ so

$$[\int_3^{n+1} dkf(k) \to \infty \text{ as } n \to \infty$$

Hence S_∞ diverges.

Leibnitz's Criterion The infinite alternating sequence

$$S = \sum_k^\infty (-1)^k a_k$$

formed by the sequence of real numbers a_k which for sufficiently large k are positive and decreasing converges provided that $\lim_{k\to\infty} a_k = 0$. Ot can be shown that

$$S < s_n + a_{n+1}$$

Example state whether the series converge or diverge, if the converge is it absolute or conditional?

$$\sum_{k=4}^\infty (-1)^k \frac{\ln\ln k}{k^{1/5}}$$

Set $S_1 = \sum_{k=4} u_k$ where

$$u_k = (-1)^k \frac{\ln\ln k}{k^{1/5}} = (-1)^k a_k$$

Knowing that $\ln(k) > 0$ for large enough k, therefore $a_k > 0$ for large enough k. This fits with Leibnitz criterion, now we have to find if a_k is decreasing. Differentiating a_k gives

$$\frac{d}{dk}a_k = \frac{1}{k\ln k^{6/5}} - \frac{1\ln\ln k}{5k^{6/5}} = \frac{1}{k^{6/5}\ln k}\left(1 - \frac{1}{5}\ln k \cdot \ln\ln k\right)$$

Using the prduct rule and the chain rule. The term outside the brackets is always positive for large k, the term inside the brackets is dominated by the 2nd term, which is always negative for large k, therefore $\frac{da_k}{dk} < 0$ for large values k and thus is descending as $k \to \infty$

Now we need to check that $\lim_{k\to\infty} a_k = 0$

$$a_k = \frac{\ln\ln k}{k^{1/5}} < \frac{\ln k}{k^{1/5}} \text{ as } \ln k < k$$

$$\lim_{k\to\infty}\frac{\ln k}{k^{1/5}} = \lim_{k\to\infty}\frac{1/k}{\frac{1}{5}k^{-4/5}} = \lim_{k\to\infty}\frac{5}{k^{1/5}} = 0$$

Using L'Hopital's Rule.

Therefore, by Leibnitz, S is convergent.

Now

$$S_2 = \sum_{k=4}^{\infty}\frac{\ln\ln k}{k^{1/5}} = \frac{k^{4/5}\ln\ln k}{k} > \frac{1}{k}$$

for large enough k.
Comparison with the harmonic series ($\sum_{k=1}^{\infty}\frac{1}{k}$) which diverges shows that S_2 diverges, hence S_1 is conditionally convergent.

Euler-Maclaurin Expansion

$$\sum_{k=n}^{m} f(k) = \int_n^m f(z) + \frac{1}{2}[f(n)+f(m)] - \sum_{k=1}^{N}\frac{B_{2k}}{(2k)!}[f^{(2k-1)}(n) - f^{(2k-1)}(m)] + R'_{N+1}$$

where $N = \lfloor\frac{q-1}{2}\rfloor$ where q is the number of times $f(k)$ has to be differentiated until it becomes a constant. R'_{N+1} is the remainder,which doesn't need to be calculated, B_k are the Bernoulli numbers. $f^{(2k-1)}$ gives the number of times the function has to be differentiated. Since B_k increases rapidly and without bound as k increases, this approximation is generally useful only for small values of N, unless the sum terminates as when $f(z)$ is a polynomial.

Example - Find the expression for

$$\sum_{k=1}^{m} k^2$$

$$\begin{aligned}\sum_{k=n=1}^{m} k^2 &= \int_{n=1}^{m} z^2 + \frac{1}{2}[1^2 + m^2] - \sum_{k=1}^{1}\frac{1}{12}[2-2m] + R'_{N+1}\\ &= \frac{m^3-1}{3} + \frac{m^2+1}{2} - \frac{1}{6}(1-m)\\ &= \frac{m}{6}(2m+1)(m+1)\end{aligned}$$

4.2 Radius of Convergence

A series of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ... = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + O(z^{n+1})$$

where a_k is independent to z^k, is known as the power series. The convergence of such series may depend on the values of z.
The radius of convergence is a circle of all the values of z in which the values converge, and excludes all the values in which it diverges. At the the circumference the series does not necessarily converge. In general f is unbounded at any root of the equation. e.g $f(z) = 1 + z^2 + z^4 + z^8 + .. + z^{2^n}$ has roots at $z = 1, z^2 = 1, z^4 = 1, z^8 = 1..z^{2^n} = 1..$

With

$$\sum_{k=0}^{\infty} a_k z^k$$

The radius of convergences may be shown to be given by either of the limits (if they exist)

$$r = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|, \qquad r = \lim_{k\to\infty} |a_k|^{-1/k}$$

Example Determine the radius of convergence of the series

$$S(x) = \sum_{k=0}^{\infty} \frac{(1+bk)^{ak}}{(1+k)!^2} x^k, a > 0, b > 0$$

Hint: $(1 + x/n)^n = e^x(1 + O(n^{-1}))$ as $n \to \infty$

Using

$$r = \lim_{k\to\infty} \left|\frac{u_k}{u_{k+1}}\right|$$

So u_k is the terms before x^k giving

$$u_k = \frac{(1+bk)^{ak}}{(1+k)!^2}$$

and

$$u_{k+1} = \frac{(1+b(k+1))^{a(k+1)}}{(1+(k+1))!^2}$$

so

$$\frac{u_{k+1}}{u_k} = \frac{(1+(k+1))!^2(1+bk)^{ak}}{(1+b(k+1))^{a(k+1)}(1+k)!^2} = \frac{(k+2)^2}{(bk+b+1)^a}\left(\frac{(1+bk)}{bk+b+1)}\right)^{ak}$$

Looking at the first term

$$\begin{aligned}\frac{(k+2)^2}{(bk+b+1)^a} =& \frac{(k+2)^2}{(bk^a)((b+1)/bk+1)^a} \\ =& \frac{(k^2+4k+4}{(bk)^a}\left(1+(-a)\left(\frac{1+b}{bk}\right)+O(\frac{1}{k^2})\right) \text{ via Taylor expansion} \\ =& \frac{k^2}{(bk)^a} + O(k^{1-a})\end{aligned}$$

Now looking at the second term, starting with the numerator and looking at the hint
$$(1+bk)^{ak} = [(1/bk+1)^k]^a = [e^{1/b}(1+O(k^{-1})]^a$$
where $n = k, x = 1/b$ and the denominator
$$(bk+b+1)^{ak} = [(1+(1+b)/bk)^k]^a = [e^{(1+b)/b}(1+O(k^{-1})]^a$$
therefore we have
$$\left[\frac{e^{1/b}(1+O(k^{-1})}{e^{(1+b)/b}(1+O(k^{-1})}\right]^a = e^{-a}(1+O(k^{-1}))$$
so overall we have
$$\frac{u_k}{u_{k+1}} = \left(\frac{k^2}{b^a k^a} + O(k^{1-a})\right) e^{-1}(1+O(k^{-1}) = \frac{k^{2-a}}{(eb)^a} + O(k^{1-a})$$
the order is found by $k^{2-a}.O(k^{-1} = O(k^{2-a-1}) = O(k^{1-a})$ which also holds for the other order.

So as $k \to \infty$ we have.
$r = 0$ with $a > 2$
$r = (eb)^a = (eb)^2$ with $a = 2$
$r = \infty$ with $0 < a < 2$

4.3 Taylor's Series

Differentiation and integration of power series A convergent power series may be differentiated and integrated term by term to give another convergent series having the same radius of convergence as the original series. If
$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$
is a power serie with radius of convergence r then
$$f_1(z) = \sum_{k=0}^{\infty} k a_k z^{k-1} \text{ and } f_2(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$$
are also convergent series with the same radius of convergence r for $|z| < r$ for the derivative and integral of $f(z)$.

examples Expand the integrand and integrating term by term, derive Gregory's expansion for $tan^{-1}x$
$$\tan^{-1} = \int_0^x dt \frac{1}{1+t^2}$$

With Maclaurin series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

we can say that

$$\frac{1}{1+t^2} = \frac{1}{1-(-1)(t)^2} = \sum_{k=0}^{\infty}(-1)^k t^{2k}$$

thus

$$\begin{aligned}\int_0^x dt \frac{1}{1+t^2} =& \int_0^x dt \sum_{k=0}^{\infty}(-1)^k t^{2k} \\ =&(-1)^k \int_0^x dt \sum_{k=0}^{\infty} t^{2k} \\ =&\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \\ =&1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...\end{aligned}$$

With Gregorys series being

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

replacing $x = 1$ with give the necessary result

Manipulation of the power series This is best done using Maple and the powerseries package

```
with(powseries):
```

In order to create a power series in the form

$$\sum_{k=0}^{N} f_k x^k$$

it is necessary to define the coefficient f_k. We can also add values for f which will overwrite the power series values, for instance if that value does not exist

```
powcreate(f(k)=1/k!);
e:=tpsform(f,x,6);

powcreate(g(n)=(-1)^(n-1)/n,g(0)=0
L:=tpsform(g,x,6)
```

These can be transformed into standard series by convert(e,polynom); However as the powseries is very powerful it is worth leaving this.

```
a:=powadd(f,g):tpsform(a,x,8);
m:=multiply(f,g):tpsform(m,x,8)
i:=inverse(f):tpsform(i,x,6):  #where i=1/f
c1:=compose(f,g)tpsform(c1,x,10); # C1(x)=(f(g(x))
r:=reversion(g)z:=tpsform(r,y,6) #y=ln(1+x) then x=e^y-1
```

4.4 Accelerated convergence

Two types of accelerating the converge of a series to the same limit are.

Aitken's Δ^2 process / Shanks Transformation

$$b_n = S(A_n) = A_{n+1} - \frac{(A_{n+2} - A_{n+1})(A_{n+1} - A_n)}{(A_{n+2} - A_{n+1}) - (A_{n+1} A_n)}$$

where n is the nth partial sum of the series. We can write a new sequence $(A_n^{(1)} = S(A_n)$.

Finding the Value for $S^{(m)}(A_{(n)})$ involves computing determinants, this was bypassed by Wynn by iterating the sequence

$$T_n^{(k+1)} = T_{n+1}^{(k-1)} + \frac{1}{T_{n+1}^{(k)} - T_n^{(k)}}$$

where

$$T_n^{(-1)} = 0, \quad T_n^{(0)} = A_n$$

and

$$S^{(m)}(A_n) = T_n^{(2m)}$$

Richardson's Extrapolation Aitken's process will fail or be less efficient if the elements of the sequence depend upon n in other ways as was presumed. Richardson's extrapolation determines an approximation to the limit when

$$A_n \simeq A + \frac{B_1}{n} + \frac{B_2}{n^2} + \frac{B_3}{n^3} + ... \quad \text{for large } n$$

This type of convergence is common when computing the radius of convergence when using the ratio test.

the solution for A is

$$A = \sum_{k=0}^{N} (-1)^{k+N} \frac{A_{n+k}(n+k)^N}{k!(N-k)!}$$

4.5 Properties of Power Series

Find the series expansion of

$$f(x) = \int_0^1 \frac{t^a}{(1-xt^3)^b} dt, \quad a > 0, \quad b > 1$$

and show that the radius of convergence of the series is 1, and that, for $x \simeq 1$,

$$f(x) \sim \frac{A}{(1-x)^{b-1}}$$

Using EX4.48

$$(1-z)^{-b} = \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+b)z^k}{k!}, \quad |z| < 1$$

With the series given, $(1-z)^{-b} = (1-xt^3)^{-b}$ so $z = xt^3$ giving

$$(1-xt^3)^{-b} = \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+b)x^k t^{3k}}{k!}$$

Putting this into the original series gives

$$f(x) = \int_0^1 t^a \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+b)x^k t^{3k}}{k!} dt$$

Taking any values that do not rely upon t from the integral and any values that do not rely upon k outside the summation gives

$$f(x) = \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+b)x^k}{k!} \int_0^1 t^a t^{3k} dt = \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+b)x^k}{k!} \int_0^1 t^{3k+a} dt$$

Integrating

$$\begin{aligned} f(x) = & \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+b)x^k}{k!} \left[\frac{t^{3k+a+1}}{3k+a+1} \right]_0^1 \\ = & \frac{1}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+b)x^k}{k!(3k+a+1)} \end{aligned}$$

giving the expansion of the series.

The radius of convergence is given by u_k/u_{k+1} where

$$u_k = \frac{\Gamma(k+b)}{k!(3k+a+1)}$$

$$\begin{aligned}\frac{u_k}{u_{k+1}} =& \frac{\Gamma(k+b)(k+1)!(3(k+1)+a+1)}{k!(3k+a+1)\Gamma(k+b+1)} \\ =& \frac{\Gamma(k+b)(k+1)(3k+a+4)}{(3k+a+1)\Gamma(k+b+1)}\end{aligned}$$

Now,

$$\Gamma(1+z) = z\Gamma(z)$$

so replacing $z = (k+b)$, $\Gamma(1+(k+b)) = (k+b)\Gamma(k+b)$ giving

$$\frac{u_k}{u_{k+1}} = \frac{(k+1)(3k+a+4)}{(3k+a+1)(k+b)} = \frac{O(k^2)}{O(k^2)} \to 1 \text{ as } k \to \infty$$

so the radius of convergence is 1.

Lastly, p166 shows the ratio of coefficients as

$$\frac{c_n}{c_{n-1}}$$

This gives

$$\frac{u_k}{u_{k-1}} = \frac{\Gamma(k+b)}{k!(3k+a+1)} \frac{(k-1)!(3k+a-2)}{\Gamma(k+b-1)}$$

Using the same theory as above with $\Gamma(1+z) = z\Gamma(z)$ we have $z = (k+b-1)$ giving $\Gamma(k+b) = (k+b-1)\Gamma(k+b-1)$ thus

$$\frac{(k+b-1)\Gamma(k+b-1)}{k!(3k+a+1)} \frac{(k-1)!(3k+a-2)}{\Gamma(k+b-1)} = \frac{(k+b-1)(3k+a-2)}{k(3k+a+1)}$$

As we are interested in how the series converges we need to have expand the result above. The easiest way to do this is divide the 1st brackets of the numerator/denominator, then divide the 2nd brackets of the numerator/denominator by $3k$, so

$$\frac{k+b-1}{k} = (1+(b-1)/k)$$

and

$$\frac{(3k+a-2)}{(3k+a+1)} = \frac{1+(a-2)/3k}{1+(a+1)/3k}$$

and bringing the denominator up to give

$$(1+(b-1)/k)(1+(a-2)/3k)(1+(a+1)/3k)^{-1}$$

Now all we are bothered about are the first order of terms, that is anything larger than $O(1/k^2)$. Expanding the 1st two brackets out gives

$$\left(1 + \frac{b-1}{k} + \frac{a-2}{3k} + O\left(\frac{1}{k^2}\right)\right)((1+(a+1)/3k)^{-1})$$

Expanding the last bracket binomially will give $(1 - \frac{a+1}{3k} + O(\frac{1}{k^2}))$ Giving

$$\begin{aligned}
\left(1+\frac{b-1}{k}+\frac{a-2}{3k}+O\left(\frac{1}{k^2}\right)\right)\left(1-\frac{a+1}{3k}+O(\frac{1}{k^2})\right) =&1+\frac{b-1}{k}+\frac{a-2}{3k}-\frac{a+1}{3k}+O\left(\frac{1}{k^2}\right)\\
=&1+\frac{3b-3+a-2-a-1}{3k}O\left(\frac{1}{k^2}\right)\\
=&1+\frac{3b-6}{3k}+O\left(\frac{1}{k^2}\right)\\
=&1+\frac{b-2}{k}+O\left(\frac{1}{k^2}\right)
\end{aligned}$$

Looking at EQ 4.40 where

$$\frac{c_n}{c_{n-1}} - \frac{1}{a}\left(1-\frac{1+\alpha}{n}\right)$$

we need the terms we got above to be in the form $1 - \frac{1+\alpha}{n}$ thus we have

$$1-\frac{1+\alpha}{n} = 1 - \frac{(2-b)}{k}$$

so $1+\alpha = 2-b$ thus $\alpha = 2-1-b = 1-b$ Now

$$f(x) = g(x)\left(1+\frac{x}{a}\right)^{\alpha}$$

where $g(x)$ is some power series function with a radius of convergence exceeding a With $a \simeq 1$ and setting $g(x) = A$ where A is some arbitrary constant, we have

$$f(x) \sim A(1-x)^{1-b} \sim \frac{A}{(1-x)^{b-1}}$$

5 Asymptotic Expansion

Asymptotic (of a function) - approaching a given value as an expression containing a variable tending to infinity.

We are used to finding a Taylor series, that is a convergent power series, to represent a function near a point $x = a$. Now we generalise this so instead of expanding in ascending powers of $x-a$ where each x^{-k} $k = 0, 1, 2, ...$ is smaller than the last, we now expand in any sequence of functions $\phi_k(x)$, $k = 0, 1, 2, ...$ where each ϕ_k is smaller than the last. That is

$$\lim_{x\to a}\left\{\frac{\phi_{k+1}(x)}{\phi_k(x)}\right\} = 0$$

or

$$\phi_{k+1}(x) = o(\phi_k(x)) \text{ as } x \to a$$

That is, $\phi_{k+1}(x)$ is strictly smaller than $\phi_k(x)$

Letting

$$f_n(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

have an asymptotic expansion for $f(x)$ as $x \to a$. If, as x approaches a, the difference between the function, and the partial sums of the asymptotic expansion is smaller than the last included term for a fixed number of n, that is

$$\lim_{x \to a} \left\{ \frac{f(x) - f_n(x)}{\phi_n(x)} \right\} = 0 \text{ where } n \text{ is fixed}$$

then we write

$$f(x) \sim \sum_{k=0}^{\infty} a_k \phi_k(x)$$

which is a good approximation.

Taking it one further, we know from above this works well when x is close to a, but by dropping the requirement that as $n \to \infty$ that we always approach the original function. Giving:

$$\lim_{n \to \infty} \left\{ \frac{f(x) - f_n(x)}{\phi_n(x)} \right\} = \infty \text{ where } x \text{ is fixed}$$

that is the expansion diverges as $n \to \infty$ for fixed x if we take enough values of n. This can still be useful.

Example Find an asymptotic expansion of

$$f(x) = \int_x^{\infty} \frac{e^{-t}}{t} \, dt \text{ as } x \to \infty$$

With the integration/differentiation of the 2 terms always giving a minus sign, to make it easier we can write the intgrand as $(-1/t)(-e^{-t})$ and using integration by parts

$$\begin{aligned} &= \left[(-1/t)e^{-t}\right]_x^{\infty} - \int_x^{\infty} \frac{1}{t^2} e^{-t} \, dt \\ &= \frac{e^{-x}}{x} - \int_x^{\infty} \frac{1}{t^2} e^{-t} \, dt \end{aligned}$$

we can integrate by parts again to get the next part of the series, etc. In general, giving:

$$\int_x^{\infty} \frac{1}{t^k} e^{-t} \, dt = \frac{e^{-x}}{x^k} - k \int_x^{\infty} \frac{e^{-t}}{t^{k+1}} \, dt$$

Thus

$$\int_x^\infty \frac{e^{-t}}{t} \ dt \sim \frac{e^{-x}}{x} - \left[\frac{e^{-x}}{x^2} - 2\left[\frac{e^{-x}}{x^3} - 3\left[...\right]\right]\right]$$
$$\sim e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + ...\right]$$
$$\sim \sum_{k=0}^{\infty} \frac{(-1)^k k! e^{-x}}{x^{k+1}}$$

To demonstrate explicitly that this is an asymptotic expansion we use

$$\lim_{x \to a} \left\{ \frac{f(x) - f_n(x)}{\phi_n(x)} \right\} = 0 \text{ where } n \text{ is fixed}$$

where $\phi_k(x) = \frac{e^{-x}}{x^{k+!}}$
We define $f_n(x)$ to be the first $n+1$ terms With

$$f(x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k k! e^{-x}}{x^{k+1}}$$

we can expand the sum to show that

$$f(x) = \int_0^\infty \frac{e^{-t}}{t} dt = \sum_{k=0}^{n-1} \frac{(-1)^k k! e^{-x}}{x^{k+1}} + \sum_{k=n}^{\infty} \frac{(-1)^k k! e^{-x}}{x^{k+1}}$$

but as we've shown above we know that

$$\int_x^\infty \frac{1}{t^k} e^{-t} \ dt = \frac{e^{-x}}{x^k} - k \int_x^\infty \frac{e^{-t}}{t^{k+1}} \ dt$$

so we can write the expanded summation to be

$$\int_0^\infty \frac{e^{-t}}{t} dt = \sum_{k=0}^{n-1} \frac{(-1)^k k! e^{-x}}{x^{k+1}} + (-1)^n n! \left\{ \frac{e^{-x}}{x^{n+1}} - (n+1) \int_x^\infty \frac{e^{-t}}{t^{n+2}} \ dt \right\}$$

in which

$$f_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^k k! e^{-x}}{x^{k+1}} + (-1)^n n! \frac{e^{-x}}{x^{n+1}}$$

and the remainder being

$$(-1)^n n! \left\{ -(n+1) \int_x^\infty \frac{e^{-t}}{t^{n+2}} \ dt \right\} = (-1)^{n+1} (n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} \ dt$$

Thus

$$f(x) = f_n(x) + (-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} \ dt$$

Now with

$$\lim_{x \to a} \left| \frac{f(x) - f_n(x)}{\phi_n(x)} \right| = \frac{x^{n+1}}{e^{-x}}(n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} \ dt$$

Note, as we are taking the absolute value, $(-1)^{n+1} = 1$

As we need to show this quantity goes to 0 as $x \to \infty$ we can bound it by saying that within the integral $\frac{1}{t} < \frac{1}{x}$ thus

$$\lim_{x \to a} \left| \frac{f(x) - f_n(x)}{\phi_n(x)} \right| < \frac{x^{n+1}}{e^{-x}} \frac{(n+1)!}{x^{n+2}} \int_x^\infty e^{-t} \ dt \tag{3}$$

$$< \frac{(n+1)!}{xe^{-x}} [-e^{-t}]_x^\infty \tag{4}$$

$$< \frac{(n+1)!}{xe^{-x}} [0 + e^{-x}]_x^\infty \tag{5}$$

$$< \frac{(n+1)!}{x} \to 0 \text{ as } x \to \infty \tag{6}$$

so we do indeed have an asymptotic expansion

6 Continued fractions / Pade approximations

continued fractions are used to represent numbers as fractions, and functions as ratios of polynomials. 22/7 is one instance which approximates π. Euclids algorithm for GCD is another. This is represented in the form.

$$c = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_k ...}}} \tag{7}$$

where the values may be numbers,complex,real or integers, or functions of one or more variables. The continued fraction is simpler if $b_k = 1$ and this is what we will be dealing with.

If the fraction is finite, then it terminates at a_n. This is called a continued fraction of order n, and has $n+1$ elements. If the elements are positive integers then it is a rational number. It is convenient to write the continued fractions as

$$c = [a_0; a_1, a_2, ...]$$

Another way is

$$c = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} + ...$$

Maple can deal with continued fractions

```
    convert(evalf(exp(1),12),confrac);
    [2,1,2,1,1,4,1,1,6,1,1,8,1,1,10]
    #1st digit = a_0)
```

For more serious use it is better to use 'cfrac'

```
with(numtheory):
cfrac(eval(x),n,'quotients','con')

evaln(con)
```

which finds the 1st $n+1$ values of x. 'con' is in quotes so it stay unevaluated. it may be evaluated by using the evaln(con) as above. If it is bounded it will produce an error, for which con will need to be unassigned first. Another way is

```
cfrac(eval(Pi,11),7,'quotients',evaln(con))

con[2]=22/7
```

The quotients puts it in the form as the convert does, otherwise it stays in a continued fraction, which can get unwieldy quickly.

The reverse is

```
with(numtheory):
c:=[seq(k,k=1..10)]
c2:=nthconver(c,2): c2=evalf(c2); #convets n+1,ie [1;2,3]
10/7 = 1.239682529
```

To do these manually we first look at

$$c = [1; 2, 2, 2] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}} \tag{8}$$

and work backwards

$$= 1 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{2}}} = 1 + \frac{1}{2 + \frac{2}{5}} = 1 + \frac{1}{\frac{12}{5}} = 1 + \frac{5}{12} = \frac{17}{12} \tag{9}$$

For the other way, the 1st element is clearly the integer part, then we use

$$a_{k+1} = \text{ floor}(r_{k+1}), \text{ where } r_{k+1} = \frac{1}{r_k - a_k}$$

where floor = trunc (use either for maple) which gives the integer part of the sum only.

For $e = 2.71828183$ on the calculator, perform the steps
(1) subtract the integer part ($a_0 = 2$) to form the next number $x_1 = e - 2.0 \leq x_1 < 1$;
(2) define $r_1 = 1/x_1 > 1$ and subtract the integer part ($a + 1 = int(r_1) = 1$) to form the next number $x_2 = r_1 - a_1 0 \leq x_2 < 1$
(3)define $r_2 = 1/x_2 > 1$ and subtract the integer part to form the next number, etc.

In practice, due to the rounding error of calculators, it is only accurate to a few decimal places (10-12) although Maple can be much more accurate.

7 Greens Function

With

$$\frac{d^4y}{dx^4} = f(x), \quad y(0) = y'(0) = y''(0) = 0, \quad y'(1) = 0 \tag{10}$$

We set $y = G(x, u)$, the Green's function, where u is an artificial variable which is used to help solve the problem. We say that G, $\frac{\delta G}{\delta x}$ and $\frac{\delta^2 G}{\delta x^2}$ are continuous at $x = u$, but the last derivative d^{n-1}/dx^{n-1} we write as

$$\lim_{\epsilon \to 0} \left[\frac{\delta^3 G}{\delta x^3} \right]_{x=u-\epsilon}^{u+\epsilon} = 1 \tag{11}$$

This shows that there is a 'jump' just below u to just above u of magnitude 1 as x passes through u.

We now split G into separate parts, G_1 which equates to the boundary $0 \leq x \leq u$ and G_2 in which $u \leq x \leq 1$.

Now let (10) satisfy the homogenous equation, that is

$$\frac{d^4 G_1}{dx^4} = 0 \tag{12}$$

and using the boundary conditions

$$G_1(0, u) = \left. \frac{\delta G_1}{\delta x} \right|_{x=0} = \left. \frac{\delta^2 G_1}{\delta x^2} \right|_{x=0} = 0 \tag{13}$$

Integrating (12) and applying the boundary conditions

$$\frac{\delta^3 G_1}{\delta x^3} = a(u) \tag{14}$$

$$\frac{\delta^2 G_1}{\delta x^2} = a(u)x + b \tag{15}$$

But at $G_1''(0,u) = 0$ so we have $b = 0$ therefore $G_1'' = ax$

$$\frac{\delta G_1}{\delta x} = \frac{a(u)x^2}{2} + c \tag{16}$$

Again at $G_1'(0,u) = 0$ so we have $c = 0$ therefore $G_1' = ax^2/2$

$$G_1 = \frac{a(u)x^3}{6} + d \tag{17}$$

Again at $G_1(0,u) = 0$ so we have $d = 0$ therefore $G_1' = ax^3/6$.
To make the maths simpler we absorb the 6 into a, thus giving

$$G_1 = a(u)x^3, \;\; x \leq u \tag{18}$$

to show that we havent forgotten about the 6 either use the above statement or use $\tilde{a}$ for all the a previous to the solution.

Now let G_2 be the Green's function for $u \leq x \leq 1$ so

$$\frac{d^4 G_2}{dx^4} = 0 \tag{19}$$

and using the boundary conditions

$$\left.\frac{\delta G_2}{\delta x}\right|_{x=1} = 0 \tag{20}$$

Integrating (19)

$$\frac{\delta^3 G_2}{\delta x^3} = b(u) \tag{21}$$

$$\frac{\delta^2 G_2}{\delta x^2} = b(u)(x-1) + c(u) \tag{22}$$

Note that we have used $(x-1)$ rather than x as this makes the maths easier when coming to apply the boundary conditions, if we differentiate (22) we get (21) so all is good

$$\frac{\delta G_2}{\delta x} = \frac{b(u)(x-1)^2}{2} + c(u)(x-1) + d(u) \tag{23}$$

Applying the boundary condition on the shows $d = 0$

$$G_2 = \frac{b(u)(x-1)^3}{6} + \frac{c(u)(x-1)^2}{2} + E(u) \tag{24}$$

so

$$G_2 = b(u)(x-1)^3 + c(u)(x-1)^2 + E(u) \tag{25}$$

Using (10) and (11), at $x = u$ we have

$$\begin{aligned} G_1 =& G_2 && (26)\\ \frac{\delta G_1}{\delta x} =& \frac{\delta G_2}{\delta x} && (27)\\ \frac{\delta^2 G_1}{\delta x^2} =& \frac{\delta^2 G_2}{\delta x^2} && (28)\\ 1 + \frac{\delta^3 G_1}{\delta x^3} =& \frac{\delta^3 G_2}{\delta x^3} && (29)\\ & && (30) \end{aligned}$$

giving us 4 equations for 4 unknowns.
Using Maple we can find these unknowns easily

```
restart:
G1:=a*x^3:
G2:=b*(x-1)^3+c*(x-1)^2+E:
eq1:=subs(x=u,G2-G1)):
eq2:=subs(x=u,diff(G2,x)-diff(G1,x)):
eq3:=subs(x=u,diff(G2,x$2)-diff(G1,x$2)):
eq4:=subs(x=u,diff(G2,x$3)-diff(G1,x$3)):
sol:=solve({eq1=0,eq2=0,eq3=0,eq4=1},{a,b,c,d});
assign(sol);
```

So

$$G(x,u) = \begin{cases} -\frac{1}{6}x^3(1-u)^2, & 0 \le x \le u, \\ \frac{1}{6}x^3 u(2-u) - \frac{1}{2}x^2 u + \frac{1}{2}u^2 x - \frac{1}{6}u^3, & u \le x \le 1 \end{cases}$$

For the second screencast we carry on with the above but with

$$\frac{d^4 y}{dx^4} = e^{\alpha x}, \quad y(0) = y'(0) = y''(0) = 0, \quad y'(1) = 4, \quad \alpha > 0$$

The first screencast was the general case (RHS f(x)), and the exact same boundary conditions except that they were homogenous (equal 0), whereas this one, we have 2 boundary conditions that don't equal 0.

The solution $y = y_1 + y_2$ where y_1 satisfies the homogenous diff equation, but the boundary conditions are exactly as given. y_2 is found by the integral

$$y_2 = \int_0^1 G(x,u) f(u)\, du$$

where $f(u) = f(x)$. In this case $f(u) = e^{\alpha u}$.

To find y_1

$$
\begin{aligned}
\frac{d^4y}{dx^4} =& 0 && (31)\\
y_1 =& Ax^3 + Bx^2 + Cx + D && (32)\\
y'1 =& 3Ax^2 + 2Bx + C && (33)\\
y''1 =& 6Ax + 2B && (34)\\
& && (35)
\end{aligned}
$$

$y_1(0) = 0 \Rightarrow D = 0$, $y_1'(0) = 1 \Rightarrow C = 1$, $y_1''(0) = 0 \Rightarrow B = 0$, $y_1'(1) = 4 \Rightarrow 3A + 1 = 4 \Rightarrow A = 1$ so

$$y_1 = x^3 + x$$

Finding y_2

$$y_2 = \int_0^1 G(x,u)e^{\alpha u}\, du \tag{36}$$

but from the first equation.

$$G(x,u) = \begin{cases} G_1, & 0 \le x \le u, \\ G_2 & u \le x \le 1 \end{cases}$$

we can break the integrand up, from 0 to x and x to 1, where u integrated between these limits. Now looking at the limits above we see that for G_1, $u > x$, so thats the integrand x to 1, and G_2 $u < x$ so thats the integrand 0 to x, so

$$y_2 = \int_0^x G_2(x,u)e^{\alpha u}\, du + \int_x^1 G_1(x,u)e^{\alpha u}\, du \tag{37}$$

which can be found by integrating by parts, this can be done using Maple to make it easier. Use the assigned variables from previous calculations rather than typing it out yourself like below

```
restart;
G1:=-1/6*x^3*(1-u)^2:
G2:=1/6*x^3*u*(2-u)-1/2*x^2*u+1/2*u^2*x-1/6*u^3:
y2:=int(G2*exp(a*u),u=0..x)+int(G1*exp(a*u),u=x..1);
simplify(y2);
```

It may be worth factoring by hand. Hence,

$$y = y_1 + y_2 = \ldots$$

8 Fourier Series

A Taylor series approximation is most accurate near the point of expansion, $x = a$, and its accuracy generally decreases as $|x - a|$ increases, so this type of approximation suffers from the defect that it is not usually uniformly accurate over the required range of x, although the Pade approximant of the Taylor series can often improve the accuracy.

Fourier series eliminates these problems by approximating functions in quite a different manner.Suppose we have a set of functions $\phi_k(x)$, $k = 1, 2, 3$...which may be complex defined on some interval $a \le x \le b$ i.e $\phi_k(x) = x^k$ or $\phi_k(x) = \sin kx$. Then we approximate the function as

$$f(x) \simeq f_N(x) = \sum_{k=1}^{N} c_k \phi_k(x) \tag{38}$$

by choosing the coefficient c_k to minimise the square difference

$$E_N = \int_a^b dx \left| f(x) - \sum_{k=1}^{N} c_k \phi_k(x) \right|^2$$

8.1 Orthogonal systems of functions

Here we consider complex functions of the real variable x The inner product of two such functions $f(x)$ and $g(x)$ is denoted by (f, g) and is defined by

$$(f, g) = \int_a^b dx\ f^*(x) g(x) \tag{39}$$

where f^* denotes the complex conjugate of f. Note that $(g, f) = (f, g)^*$. The inner product of a function with itself is real, positive and $\sqrt{(f, f)}$ is named the norm. A function whose norm is unity is said to be normalised.
The two functions f and g and orthogonal if their inner product is zero, $(f, g) = 0$.
An orthogonal system is one where

$$(\phi_r, \phi_s) = \delta_{rs} = \begin{cases} 1, & r = s \\ 0, & r \neq s \end{cases}$$

where δ_{rs} is known as the Kronecker Delta

8.2 Expansions in terms of Orthogonal functions

Suppose $\phi_k(x), k = 1, 2, ...$ is a system of orthogonal functions on the interval $a \le x \le b$ we can expand a given real function $f(x)$ in terms of these functions by (38) and choose the N coefficients c_k, which may be complex, to minimise the square norm of $(f_N - f)$, that is

$$\begin{aligned} F(\mathbf{c}) &= \int_a^b dx \left| \sum_{k=1}^{N} c_k \phi_k(x) - f(x) \right|^2 \\ &= \sum_{k=1}^{N} |c_k - (\phi_k, f)|^2 - \sum_{k=1}^{N} |(\phi_k, f)|^2 + (f, f) \end{aligned}$$

$F(\mathbf{c})$ has its only minimum when the first term of the last expression equals zero by choosing

$$c_j = (\phi_j, f), \quad j = 1, 2, .., N$$

The numbers $c_j = (\phi_j, f)$ are the expansion coefficients of f with respect to the orthogonal system $\{\phi_1, \phi_2, ...\}$ This type of approximation is called an approximation in the mean.

8.3 Fourier Series

This is an expansion of a function in terms of a set of trigonometric functions

$$1, \cos nx, \sin nx$$

or their complex equivalents

$$\phi_k = e^{-ikx}, \quad k = 0, \pm 1, \pm 2...$$

Any sufficiently well behaved function $f(x)$ may be approximated by the trigonometric series

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{-ikx}, \quad c_k = \frac{(\phi_k, f)}{\phi_k, \phi_k)} = \frac{1}{2\pi} \int_{-\pi}^{pi} dx\ e^{ikx} f(x) \tag{40}$$

It is often written in the real form

$$F(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \tag{41}$$

where

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} dx\ f(x), \quad \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \begin{pmatrix} \cos kx \\ \sin kx \end{pmatrix} f(x) \tag{42}$$

Screencast 1 Calculate the Fourier coefficients a_k for the Fourier series

$$\sin \alpha x = \sum_{k=1}^{\infty} a_k \sin kx, \quad -\pi \le x \le \pi$$

where α is not an integer. What happens as $\alpha \to n$ where n is a positive integer?

As $\sin \alpha x$ is an odd function we can ignore the even function cos, so

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} dx\ \sin \alpha x \sin kx$$

Using

$$\sin p \sin q = \frac{1}{2} [\cos(p - q) - \cos(p + q)]$$

As cos is an even function it doesnt matter which variable is chosen for p and q, so setting $p = \alpha x$, $q = kx$.

$$\begin{aligned} a_k =& \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \ [\cos((\alpha-k)x) - \cos((\alpha+k)x)] \\ =& \frac{1}{\pi} \int_{0}^{\pi} dx \ [\cos((\alpha-k)x) - \cos((\alpha+k)x)] \\ =& \frac{1}{\pi} \left[\frac{1}{\alpha-k} \sin((\alpha-k)x) - \frac{1}{\alpha+k} \sin((\alpha+k)x) \right]_0^{\pi} \end{aligned}$$

As cos is an even function, we can integrate from 0 to π and multiply by 2 which makes the maths easier. Also, note that as α isn't an integer $\alpha - k$ is not 0, which stops dividing by zero.

$$\begin{aligned} a_k =& \frac{1}{\pi} \left[\frac{1}{\alpha-k} \sin((\alpha-k)\pi) - \frac{1}{\alpha+k} \sin((\alpha+k)\pi) \right] \\ =& \frac{1}{\pi} \left[\frac{1}{\alpha-k} (\sin\alpha\pi \cos k\pi - \sin k\pi \cos\alpha\pi) - \frac{1}{\alpha+k} (\sin\alpha\pi \cos k\pi - \sin k\pi \cos\alpha\pi) \right] \end{aligned}$$

As k is an integer, $\sin k\pi = 0$ and $\cos k\pi = (-1)^k$ giving

$$\begin{aligned} a_k =& \frac{1}{\pi} \left[\frac{1}{\alpha-k} (-1)^k \sin\alpha\pi - 1 \frac{1}{\alpha+k} (-1)^k \sin\alpha\pi \right] \\ =& \frac{1}{\pi} (-1)^k \sin\alpha\pi \left[\frac{\alpha+k-\alpha+k}{\alpha^2-k^2} \right] \\ =& \frac{2k(-1)^k \sin\alpha\pi}{\pi(\alpha^2-k^2)} \end{aligned}$$

So

$$\begin{aligned} \sin\alpha x =& \sum_{k=1}^{\infty} a_k \sin kx \qquad |x| < \pi \\ =& \frac{2}{\pi} \sin\alpha\pi \sum_{k=1}^{\infty} \frac{k(-1)^k}{\alpha^2-k^2} \sin kx \end{aligned}$$

Giving the Fourier series representation of $\sin\alpha x$

For the 2nd part of the question, if $\alpha \to n$ we need to consider two separate cases, $k \neq n$ and $k = n$. The first case, substituting $\alpha = n$ in the RHS we have

$$\lim_{\alpha \to n} \sin\alpha\pi = \frac{2}{\pi} \sin n\pi \sum_{k=1}^{\infty} \frac{k(-1)^k}{n^2-k^2} \sin kx$$

in which $\sin n\pi = 0$ for all $k \neq n$.
In the second case where $k = n$ we substitute $\alpha = n$ and $k = n$, but the

denominator gives $n^2 - n^2$ which means dividing by zero, so instead we write

$$\frac{2}{\pi} n(-1)^n \sin nx \lim_{\alpha \to n} \frac{\sin \alpha\pi}{\alpha^2 - n^2}$$

where we have replaced $k = n$ but left α as is. As the numerator and denominator both equal 0, it is a candidate for L'Hopitals Rule, so differentiating both by α gives

$$\begin{aligned}
&= \frac{2}{\pi} n(-1)^n \sin nx \lim_{\alpha \to n} \frac{\pi \cos n\pi}{2n} \\
&= (-1)^{2n} \sin nx \\
&= \sin nx
\end{aligned}$$

so

$$\sin \alpha x \to \sin nx \text{ as } \alpha \to n$$

as expected

Screencast 2 Use the Fourier series

$$x = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin kx, \quad -\pi \leq x \leq \pi$$

to find a Fourier series representation of $\sin \alpha x$ where α is not an integer. Hence derive a Fourier series representation of $\cos \alpha x$

Following on from above, If we look at the term

$$\frac{k(-1)^k}{\alpha^2 - k^2}$$

as $k \to \infty$ we have a_k decaying like $O(k^-1)$, which is quite slow. The reason for this is that the periodic extension is not continuous at its ends $(\pm\pi)$, when α is not an integer. So we are going to use a different Fourier series representation which will allow for a faster decay.

Let the function $g(x)$ be the difference between the function we want, and a constant multiple of the function we have

$$g(x) = \sin \alpha x - \beta x \tag{43}$$

where β is such that $g(\pm\pi) = 0$ This removes the discontinuity from both ends, making it a superior function. Therefore we have

$$\beta = \frac{\sin \alpha\pi}{\pi}$$

From (43) we take the Fourier series of both parts, where $\sin\alpha x$ is the series we found in the previous screencast, and x is the one given in this screencast.

$$\begin{aligned}
G(x) =&\frac{2}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}\frac{k(-1)^k}{\alpha^2-k^2}\sin kx-\frac{2}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sin kx\\
=&\frac{2}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}(-1)^k\left[\frac{k}{\alpha^2-k^2}+\frac{1}{k}\right]\sin kx\\
=&\frac{2}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}(-1)^k\left[\frac{k^2+\alpha^2-k^2}{k(\alpha^2-k^2)}\right]\sin kx\\
=&\frac{2}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}(-1)^k\left[\frac{\alpha^2}{k(\alpha^2-k^2)}\right]\sin kx
\end{aligned}$$

Hence from (43)

$$\sin\alpha x = G(x) = \frac{2}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}(-1)^k\left[\frac{\alpha^2}{k(\alpha^2-k^2)}\right]\sin kx+\frac{x}{\pi}\sin\alpha\pi \tag{44}$$

If we look at the coefficients $a_k \sim O(k^{-3})$ which is much faster than previous representation.
For the last part we need to differentiate to get $\cos\alpha x$. Although sometimes this can be fraught, and care is needed (Richards p282). As it is $O(k^{-3})$ we are fine.

$$\frac{dG(x)}{dx} = \alpha\cos\alpha x = \frac{2}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}\frac{(-1)^k\alpha^2}{k(\alpha^2-k^2)}k\cos kx+\frac{1}{\pi}\sin\alpha\pi$$

So

$$\cos\alpha x = \frac{1}{\alpha\pi}\sin\alpha\pi+\frac{2\alpha}{\pi}\sin\alpha\pi\sum_{k=1}^{\infty}\frac{(-1)^k}{(\alpha^2-k^2)}\cos kx$$

8.4 Differentiation

Example

Differentiate

$$x^3 = 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k^2\pi^2-6}{k^3}\sin kx$$

This gives $f(x) = x^3$, $a_k = 0$,

$$b_k = \frac{2}{\pi}\int_0^{\pi}dx x^3\sin kx = 2(-1)^{k-1}\frac{k^2\pi^2-6}{k^3}$$

For b we have

$$b = \lim_{k\to\infty}(-1)^k kc_k = \lim_{k\to\infty}2(-1)^{2k-1}\frac{k^3\pi^2-6}{k^3} = -2\pi^2$$

Therefore

$$\begin{aligned}
\frac{df}{dx} = 3x^2 =& \frac{-b}{2} + \sum_{k=1}^{\infty} \left\{ (kb_k - (-1)^k b) \cos kx - ka_k \sin kx \right\} \\
=& \pi^2 + \sum_{k=1}^{\infty} \left(2(-1)^{k-1} \frac{k^2\pi^2 - 6}{k^2} + (-1)^k 2\pi^2 \right) \cos kx \\
=& \pi^2 + \sum_{k=1}^{\infty} \left(2(-1)^k \left(-\pi^2 + \frac{6}{k^2} + \pi^2 \right) \right) \cos kx \\
=& \pi^2 + 12 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx \\
x^2 =& \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx \\
x^2 =& \frac{\pi^2}{3} - 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos kx
\end{aligned}$$

Integration to be done later..................

8.5 Fourier series on arbitrary ranges

With $f(x)$ $a \leq x \leq b$ different from $(-\pi, \pi)$ Define a new variable

$$w = \frac{2\pi}{b-a} \left(x - \frac{b+a}{2} \right), \quad x = \frac{b+a}{2} + \frac{b-a}{2\pi} w$$

which maps $-\pi \leq w \leq \pi$ onto $a \leq x \leq b$. Then it follows $f(x(w))$ can be expressed as

$$f(x(w)) = \sum_{k=-\infty}^{\infty} c_k e^{ikw}, \quad \text{and so} \quad f(x) = \sum_{k=-\infty}^{\infty} d_k \exp\left(i \frac{2\pi k}{b-a} x \right)$$

where

$$d_k = \frac{1}{b-a} \int_a^b dx\ f(x) \exp\left(\frac{2\pi k}{b-a} x \right)$$

8.6 Sine and Cosine series

Fourier series of only even functions $f(x) = f(-x)$ compromise only of cosine terms, odd functions $f(x) = -f(x)$ only of sine terms. This fact can be used to produce cosine or sine series of any function $(like f(x) = x^3)$ over a given range, which we shall take to be $(0, \pi)$.

For any function $f(x)$ an odd function $f_o(x)$ may be produced as follows

$$f_o(x) = \begin{cases} -f(-x), & x < 0 \\ f(x), & x \geq 0 \end{cases}$$

we would usually use this extension when $f(0) = 0$ so that it is continuous at $x = 0$. As the series only contains sine functions we have

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx, \quad 0 < x \leq \pi$$

This series is called the half-range Fourier sine series.

Likewise,for cosine series, we can have

$$f_e(x) = \begin{cases} f(-x), & x < 0 \\ f(x), & x \geq 0 \end{cases}$$

to give the half-range Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx, \quad 0 < x \leq \pi$$

8.7 Gibbs phenomenon

This occurs in any discontinuous function. With the Heavyside function (square wave), looking on a small interval centred on the discontinuity at $x = 0$, where the Nth partial sum is

$$H_N(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{N} \frac{\sin(2k-1)x}{2k-1}$$

pg230 At N=1 we see that the sine wave is approximately 45 degrees from the vertical axis, with amplitude error on both sides. As N increases, the sine wave compresses to look more like the square wave as expected, but the amplitude remains constant. To understand this we use the expression

$$f_N = \sum_{k=-N}^{N} c_k e^{ikx}$$

Replacing the Fourier coefficients by their integral definition (8.13 pg 273) and change the order of summation and integration to obtain

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dy \, f(y) \sum_{k=-N}^{N} e^{ik(y-x)}$$

The sum integral is just a geometric series, giving

$$s_N(z) = \sum_{k=-N}^{N} e^{ikz} = \frac{sin(N+1/2)z}{sin(z-2)}$$

This shows that $s_N(0) = 2N+1$, that the width of the maximum at $z = 0$ is about π/N and that for most other values of z, $s_N(z)$ is relatively small.

9 Perturbation Theory

collection of methods to obtain approximate solutions involving a small parameter

A regular perturbation is one defined as a power series in ϵ with a non vanishing radius of convergence, eg an asymptotic expansion

A singular perturbation either does not take on the form of a power series, or does not converge. eg a power series that has a vanishing radius of convergence. This will have a different character when $\epsilon = 0$ as opposed to when $\epsilon \to 0$

9.1 Perturbation with Maple

Consider the general equation

$$f(x) = \epsilon g(x)$$

we shall assume
(a) $f(x) = 0$ has a simple root at $x = 0$, so $f(0) = 0$, $f'(0) \neq 0$
(b) $g(0) \neq 0$
(c) both $f(x), g(x)$ have Taylor expansions about the origin, with a finite radius of convergence

If the root of $f(x) = 0$ is at $x = a \neq 0$ then we define a new variable $y = x - a$ and replace the original equation by $F(y) = \epsilon G(x)$ where $F(y) = f(y + a)$ and $G(y) = g(y + a)$
There are two ways to tackle this problem using Maple, first is implementing pg308, the other is to invert series, which is easier to use but less general and cant be used to solve coupled equations.

Consider

$$\sin x = \epsilon e^{-x}$$

In order to find the root near $x = \pi$ we define $y = \pi - x$so that $|y|$ is small, since $\sin x = \sin y$ the equation satisfied by y is

$$\sin y = \delta e^{y}, \quad \delta = \epsilon e^{-\pi}$$

with $\epsilon = 0$, $\sin y = 0$ thus $y = 0$ therefore we assume the solution can be represented by the series

$$y = \delta r_1 + \delta^2 r_2 + ...$$

where the parameter r_n is determined from the previous $n-1$ parameters $(r_1, r_2, ...)$ and where $r_1 = g(0)/f'(0)$. Since $y << 1$ we may replace $\sin y$ by y and e^y by 1 to give $y \simeq \delta$ and hence $r_1 = 1$

9.2 Screencast 1 - Algebraic Expressions

With

$$\cosh \epsilon x \cos x = (1-\epsilon), \quad |\epsilon| << 1$$

Show that an approximate solution is

$$x = \pm\sqrt{2\epsilon}\left(1 + \frac{\epsilon}{12} + O(\epsilon^2)\right)$$

A sketch of this shows that $\cosh \epsilon x \cos x$ is even (upside down x^2 like) and $1-\epsilon$ is a horizontal line very close to 1. Now if $\epsilon < 0$ then it lifts the line up far enough that there are no roots, if $\epsilon > 0$ then it drops it down low enough that there are 2 roots, and if $\epsilon = 0$ then there is 1 root.

The question shows that we need to roots, so we go with $\epsilon > 0$ Working with $O(x^4)$ and $O(\epsilon^4)$ and using Power series to expand cos, cosh and then multiply out gives

$$\frac{x^4}{24}\left(\epsilon^4 - 6\epsilon^2 + 1) + \frac{x^2}{2}(\epsilon^2 - 1) + \epsilon\right) = 0$$

then using quadratic rule to find the roots (up to ϵ^2). The bottom part goes to the top using $(...)^{-1}$ which then can be changed with the binomial theorem. The square root part is also expanded using the binomial theorem, hence

$$x^2 = 2\epsilon + \frac{1}{3}\epsilon^2 + O(\epsilon^3) = 2\epsilon(1 + \frac{1}{6}\epsilon + O(\epsilon^2))$$

giving

$$x = \pm\sqrt{2\epsilon}(1 + \frac{1}{12}\epsilon + O(\epsilon^2))$$

9.3 screencast 2 - differential equations

Find a first order approximate solution to

$$\frac{dx}{dt} = x^2 + \epsilon x^{5/2}, \quad x(0) = A > 0, \quad 0 < \epsilon << 1$$

Let

$$x = x_0 + \epsilon x_1 + O(\epsilon^2)$$

so

$$\frac{dx_0}{dt} + \epsilon\frac{dx_1}{dt} = (x_0^2 + 2\epsilon x_0 x_1 + \epsilon x_0^{5/2})$$

the latter being done via binomial expansion. For the $O(1)$ terms

$$\frac{dx_0}{dt} = x_0^2$$

$$\int x_0^{-2}\, dx = \int dt$$

$$x_0 = c + t$$

Now $x(0) = A$, we can assume $x_0(0) = A$ and $x_n(0) = 0$ for all other values. So

$$x_0(0) = 1/k = A, \quad x_0(t) = \frac{1}{\frac{1}{k} - t} = \frac{A}{1 - At}$$

For $O(\epsilon)$

$$\frac{dx_1}{dt} = 2x_0 x_1 + x_0^{5/2}$$

This is a linear first order differential equation. Replace x_0 to give

$$\frac{dx_1}{dt} - \frac{2A}{1-At}x_1 = \left(\frac{A}{1-At}\right)^{5/2}$$

Integrating factor method where $P = \exp(\int -\frac{2A}{1-At}dt = (1-At)^2$ gives

$$(1-At)^2 x_1 = \int (1-At)^2 \left(\frac{A}{1-At}\right)^{5/2} dt$$

$$x_1 = A^{5/2}\frac{(1-At)^{1/2}2}{-A} + C$$

now at $x_1(0) = 0 \Rightarrow 0 = -2A^{3/2} + C$ hence

$$x_1 = \frac{2A^{3/2}}{(1-At)^2}(1 - \sqrt{1-At}$$

thus $x = x_0 + \epsilon x_1$..with values put in.

10 Sturm-Liouville system

10.1 Introduction

This allows the generalisation of the idea of trigonometric series and extends the use of Green's functions to a wider range of differential equations. With the general linear second order differential equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda w(x))\, y = 0$$

where the real variable is confined to $a \leq x \leq b$ which may be the whole real line or just $x \geq 0$. $p(x), q(x), w(x)$ are real functions. λ is a constant, which are named eigenvalues, and the solution for $y_\lambda(x)$ is called the eigenfunction for the eigenvalue λ.
Strum-Liouville problems are important as they arise in diverse circumstances as well because the properties of eigenvalues and eigenfunctions are well understood, i.e they can be used to form generalised Fourier series.

10.2 A simple example

Consider

$$\frac{d^2y}{dx^2} + \lambda y = 0, \;\; y(0) = y(\pi) = 0$$

Here $p = 1, \; q = 0$ and the interval can be taken as $(0, \pi)$. If $\lambda < 0$ the general solution is $y = A\sinh(\sqrt{-\lambda}x) + B\cosh(\sqrt{-\lambda}x)$ and we can see that only one boundary condition can be satisfied. When $\lambda = 0$ only the trivial solution of $y = 0$ satisfies both the equation and the boundary conditions. If $\lambda = \omega^2, \; \omega > 0$ the general solution is $A \sin \omega x + b \cos \omega x$ which can be made to fit in the boundary conditions at $x = 0$ by choosing $B = 0$ and the condition $x = \pi$ by making ω an integer > 0. Because the equation is linear the value of the constant A is indeterminate, and we take as the solutions

$$y_n(x) = \sin nx, \;\; \lambda_n = n^2, \;\; n = 1, 2, ..$$

The eigenvalues are real, $\lambda_n = n^2$ and the eigenfunctions are $y_n(x) = \sin nx$ In this example each eigenvalue is associated with a unique eigenfunction. The boundary conditions determine that there are a countable number of eigenvalues and their dependence upon the index; thus for instance $y'(0) = y'(\pi) = 0$ will give different eigenfunctions and eigenvalues.

There are several important properties of the eigenvalues and eigenfunctions that are common to the solutions of many Sturm-Liouville systems

1. The eigenvalues are real
2. The smallest eigenvalue is unity, but there is no largest eigenvalue. $\lambda_n/n^2 = O(1)$ as $n \to \infty$, that is λ_n increases as n^2 for large n
3. For each eigenvalue there is one eigenfunction. (although this is not universally true)
4. The nth eigenfunction has $n-1$ zeros in $0 < x < \pi$. For the general problem with $[a, b]$ the nth eigenfunction has $n-1$ zeros in $a < x < b$.
5. The zeros interlace, so there is one and only one zero of $y_{n+1}(x)$ between adjacent zeros of $y_n(x)$.

6. The inner product of two eigenfunctions with distinct eigenvalues is zero, that is

$$\int_0^\pi dx\; y_n(x)y_m(x) = \int_0^\pi dx\;\; \sin nx \sin mx = 0, \quad n \neq m$$

   though in general it is necessary to use a modified inner product (shown later on)

7. The eigenfunctions $y_n(x) = \sin(nx)$ form a complete orthogonal system, so any sufficiently well behaved function $F(x)$ may be approximated in the mean by the series

$$F(x) = \sum_{k=1}^{\infty} a_k y_k(x)$$

   for some coefficients a_k. This property generalises the theory of Fourier series developed in Chapter 8, and many of the properties of Fourier series carry over to these more general series.

10.3 Examples of Strum-Liouville Equations

Not essential

10.4 Sturm-Liouville systems

A regular Sturm-Liouville system is

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda w(x))\, y = 0 \tag{45}$$

defined on a finite interval of the real axis $a \leq x \leq b$ together with homogenous boundary conditions

$$A_1 y(a) + A_2 y'(a) = 0, \quad B_1 y(b) + B_2 y'(b) = 0 \tag{46}$$

where

1. the functions $p(x), w(x), q(x)$ are real and continuous for $a \leq x \leq b$
2. $p(x), w(x)$ are positive for $a \leq x \leq b$
3. $p'(x)$ exists and is continuous for $a \leq x \leq b$
4. A_1, A_2, B_1, B_2 are real constants and the trivial case $A_1 = A_2 = 0$ and $B_1 = B_2 = 0$ are excluded.

If $p(x)$ is positive for $a < x < b$ but vanishes at one or both ends, then a **singular Sturm-Liouville system** comprising of the original equation with $w(x), q(x)$ satisfying the same conditions as for a regular system, and

1. the solution is bounded for $a < x < b$
2. at an end point at which $p(x)$ does not vanish $y(x)$ satisfies a boundary condition of the type above.

Example The system

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right)+\left(r\lambda-\frac{1}{r}\right)y=0,\ \ 0\leq r\leq a$$

with Boundary conditions $B_1y(a)+B_2y'(a)=0$, $|B_1|+|B_2|\neq 0$ and $y(x)$ bounded is a singular Sturm-Liouville system. On defining $x=r\sqrt{\lambda}$ it becomes

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1)y=0$$

which is just the Bessel's equation with the two linearly independent solutions $J_1(x)$ and $Y_1(x)$. But $Y_1(x)$ is unbounded - near the origin $Y_1(x)\sim\ln x$ - thus $J_1(x)$ is the only bounded solution. This is typical Sturm-Liouville behaviour and will often have two linearly independent solutions only one of which is bounded.

Many of the nice properties of the eigenvalues and eigenfunctions of a Sturm-Liouville system, regular and singular, stem from the special form of the differential equation (45). The first beging the Lagrange's identity.

$$vLu-uLv=\frac{d}{dx}\left[p(x)\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]$$

where L is the real differential operator

$$Lf=\frac{d}{dx}\left(p(x)\frac{df}{dx}\right)+q(x)f$$

and u and v are any functions for which both sides of the identity exist.
The inner product of two functions f and g was defined in chapter 8 to be the integral

$$(f,g)=\int_a^b dx\ f(x)^*g(x)$$

It follows from Lagrange's identity that

$$(Lu,v)=(u,Lv)$$

where $u(x)$ and $v(x)$ are two functions, that may be complex, for which Lu and Lv exist and which satisfy any boundary conditions. Note u,v need not be solutions of the Sturm-Liouville equation.

The operator satisfying the above equation is said to be **self-adjoint**.

10.5 Sturm-Liouville Theorem

Solutions to the regular Sturm-Liouville system (45),(46) have the following properties

1. There is an infinite sequence of real eigenvalues λ_k, $k = 1, 2, ...$ and $\lambda_k \to \infty$ as $k \to \infty$; this means that there can be at most a finite number of negative eigenvalues.

2. For each eigenvalue there is a unique - up to a multiplicative constant - eigenfunction $\phi_k(x)$, that is, the system is non degenerative.

3. The eigenfunctions are orthogonal with respect to the weight function $w(x)$

$$\int_a^b dx\ w(x)\phi_k(x)\phi_j(x) = 0, \quad k \neq j$$

   and form a complete set of functions, which means that any function $f(x)$ for which $\int_a^b dx\ |f(x)|^2$ exists, can be represented by the infinite series

$$f(x) = \sum_{k=1}^{\infty} a_k \phi_k(x), \quad a_k = \frac{\int_a^b dx\ w(x)\phi_k(x)^* f(x)}{\int_a^b dx\ w(x)|\phi_k(x)|^2} = \frac{(\phi_k, f)}{(\phi_k, \phi_k)}$$

   which converges in the mean to $f(x)$

$$\lim_{N\to\infty} \int_a^b dx \left| f(x) - \sum_{k=1}^{N} a_k \phi_k(x) \right|^2 = 0$$

   For most well behaved functions the series is also pointwise convergent. The series converges pointwise to $f(x)$ at the endpoint $x = a$ if either $A_2 \neq 0$ or $f(a) = 0$; similarly $x = b$

4. The nth eigenfunction $\phi_n(x)$ has precisely $n - 1$ zeros in the interval $a < x < b$ and each zero of $\phi_{n+1}(x)$ lies between adjacent zeros of $\phi_n(x)$, this is known as the oscillation theorem.

The last sentence doesn't mean that the eigenfunctions are evenly distributed, nor that they cannot coincide.

10.6 Green's Function

Recalling the Green's function by finding the solution of a linear homogenous boundary value problem

$$\frac{d^2y}{dx^2} = f(x) \;\; y(a) = A, \;\; y(b) = B$$

The 1st stage is to write the solution as the sum of to functions $y(x) = y_1(x) + y_2(x)$ where $y_1(x)$ satisfies the homogeneous equation with inhomogeneous boundary conditions

$$\frac{d^2y_1}{dx^2} = 0 \;\; y_1(a) = A, \;\; y_1 b) = B$$

and $y_2(x)$ satisfies the inhomogeneous equation with homogeneous boundary conditions

$$\frac{d^2y_2}{dx^2} = f(x) \;\; y_2(a) = 0, \;\; y_2(b) = 0$$

The distinctions between these two types of equations is important as only the second is a self-adjoint system.
The solution to the homogeneous equation is

$$y_1(x) = A + \frac{B-A}{b-a}(x-a)$$

and the solution of the homogeneous equation is obtained with the help of the Greens function $G(x, \xi)$

$$\frac{dG}{dx^2} = \delta(x-\xi), \;\; G(a,\xi) = 0, \;\; G(b,\xi) = 0$$

For $x < \xi$ and $x > \xi$ the right hand side of the equation is zero, and the solution has form $\alpha(\xi) + \beta(\xi)x$. The functions α and β, which are different for $x < \xi$, are obtained by using the fact that $G(x, \xi)$ is continuous at $x = \xi$.

Now consider the more general linear differential equation

$$Ly = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = f(x)$$

with the unmixed homogeneous boundary conditions

$$A_1 y(a) + A_2 y'(a) = 0, \;\; B_1 y(b) + B_2 y'(b) = 0$$

The Greens function for this system is the function $G(x, \xi)$ satisfying

$$LG = \frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) + q(x)G = \delta(x-\xi), \;\; a < \xi < b$$

Again, LG=0 for $x < \xi, \; x > \xi$ and the conditions.

1. $G(x, \xi)$ is continuous at $x = \xi$
2. The first derivative is discontinuous at $x = \xi$

The Green's function can be written as

$$G(x,\xi) = \frac{H(\xi - x)u_2(\xi)u_1(x) + H(x-\xi)u_1(\xi)u_2(x)}{p(\xi)W(\xi)}$$

where u_1, u_2 are two linearly independent functions ($u_1 u_2' - u_2 u_1' \neq 0$), $W(\xi) = u_1(\xi)u_2'(\xi) - u_2(\xi)u_1'(\xi)$ and the solution satisfying the boundary conditions is

$$y(x) = u_2(x)\int_a^x d\xi \, \frac{f(\xi)u_1(\xi)}{p(\xi)W(\xi)} + u_1(x)\int_x^b d\xi \, \frac{f(\xi)u_2(\xi)}{p(\xi)W(\xi)}$$

Exercise 10.20 Write the solution of the equation

$$\frac{d^2y}{dx^2} + \omega^2 y = f(x), \quad y(0) = y(\pi) = 0$$

The general solution is

$$A \sin \omega x + B \sin \omega x = 0$$

at $x = 0$ this gives $B = 0$ therefore the solution is

$$u_1 = \sin \omega x$$

With regards to the boundary conditions we can now set u_2 to be

$$u_2 = \sin \omega(\pi - x)$$

With

$$\begin{aligned}
W(\xi) =& u_1(\xi)u_2'(\xi) - u_2(\xi)u_1'(\xi) \\
=& \sin \omega(x)\omega \cos \omega(\pi - x) - \sin \omega(\pi - x)\omega \cos \omega(x) \\
=& \omega \sin \omega(x - (\pi - x)) = \sin(-\pi) \\
=& -\omega \sin \omega\pi
\end{aligned}$$

With $p(x) = 1$ we have the Greens function

$$\begin{aligned}
p(\xi)W(\xi)G(x, \xi) =& H(\xi - x)u_2(\xi)u_1(x) + H(x - \xi)u_1(\xi)u_2(x) \\
\omega \sin \omega\pi G(x, \xi) =& H(\xi - x) \sin \omega(\pi - \xi) \sin \omega x + H(x - \xi) \sin \omega\xi \sin \omega(\pi - x)
\end{aligned}$$

and the solution

$$\begin{aligned}
y(x) =& u_2(x) \int_a^x d\xi \, \frac{f(\xi)u_1(\xi)}{p(\xi)W(\xi)} + u_1(x) \int_x^b d\xi \, \frac{f(\xi)u_2(\xi)}{p(\xi)W(\xi)} \\
y(x) =& -\frac{\sin \omega(\pi - x)}{\omega \sin \omega\pi} \int_0^x d\xi \, f(\xi) \sin \omega\xi - \frac{\sin x}{\omega \sin \omega\pi} \int_x^\pi d\xi \, f(\xi) \sin \omega(\pi - \xi)
\end{aligned}$$

Two functions are linearly independent whenever their Wronskian determinant

$$W(x) = y_1 \frac{dy_2}{dx} - \frac{dy_1}{dx} y_2$$

is not identically zero (that is, zero for all x), although the converse is not true.

10.7 Eigenfunction expansions

Assuming all eigenfunctions are real, we shall now find an alternative representation of the Green's function in terms of eigenfunctions of L which provides a method of dealing with when u_1, u_2 are linearly dependent.

Suppose that $\phi_k(x)$ are eigenfunctions of the Sturm-Liouville system $L\phi_k + \lambda_k w\phi_k = 0$, with eigenvalues $\lambda_k, k = 1, 2, ..$ and that none of the λ_k are zero. Then it follows from the eigenfunctions representation of the delta function (10.25 pg 359) that the Green's function may be expressed in terms of the series

$$G(x,\xi) = -\sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k(\xi)}{(\phi_k,\phi_k)\lambda_k} \tag{47}$$

Exercise Use the eigenfunctions of the Strum-Liouville system

$$\frac{d^2y}{dx^2} + \lambda y = 0, \;\; y(0) = y(\pi) = 0$$

to construct the Green's function $\frac{d^2G}{dx^2} = \delta(x-\xi)$.

With

$$LG = \frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) + q(x)G = \delta(x-\xi), \;\; a < \xi < b$$

we take $Ly = \frac{d^2y}{dx^2}$ so the eigenfunctions of L are $\sin nx$, and the eigenvalues are λ_n, $n = 1, 2, ...$
Now

$$\begin{aligned} G(x,\xi) &= -\sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k(\xi)}{(\phi_k,\phi_k)\lambda_k} \\ &= -\sum_{k=1}^{\infty} \frac{\sin kx \sin k\xi}{(\phi_k,\phi_k)\lambda_k} \end{aligned}$$

For the denominator, remember that $\lambda = \omega^2$, so with the eigenfunction $\sin kx$ we can say that $k = \omega$, thus $\lambda_k = k^2$. Now

$$\begin{aligned} (f,g) &= \int_a^b f(x)^* g(x)\; dx \left(= \int_a^b f(x)g(x)\; dx \text{ if } f, g = \mathbb{R} \right) \\ (\phi_k,\phi_k) &= \int_0^{\pi} \sin^2 kx\; dx \\ &= \left[\frac{x}{2} - \frac{\sin 2kx}{4k}\right]_0^{\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

so

$$G(x,\xi) = -\frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin kx \sin k\xi}{k^2}$$

Using eq(10.32)

$$G(x,\xi) = \begin{cases} -\frac{(x-a)(b-\xi)}{b-a} & x \le \xi \\ -\frac{(b-x)(\xi-a)}{b-a} & x \ge \xi \end{cases}$$

$$-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin kx \sin k\xi}{k^2} = \begin{cases} -\frac{x(\pi-\xi)}{\pi} & x \le \xi \\ -\frac{(\pi-x)\xi)}{\pi} & x \ge \xi \end{cases}$$

The expansion (47) for the Green's function is valid provided zero is not an eigenvalue, that is there are no non-trivial solution $u_0(x)$ of the equation $Lu_0 = 0$. Recall that we required the functions u_1, u_2 to be linearly independent, and if they were linearly dependent then there exists a solution of $Lu = 0$ satisfying both boundary conditions.

The zero-eigenvalue problem is connected with the fact that the equation $Lu = f$ has a unique solution iff $u_0(x)$, the solution of $Lu_0 = 0$ is orthogonal with weight function unity to $f(x)$, that is

$$(u_0, f) = \int_a^b dx\ u_0(x) f(x) = 0$$

Also has generalised Greens function pg367

10.8 Asymptotic Expansions

The solutions of the Sturm-Liouville equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda w(x))y = 0, \quad a \le x \le b$$

are oscillatory when λ is large, so its natural to attempt to approximate the solutions with a function of the form

$$\psi(x) = \sin(\sqrt{\lambda} g(x) + \alpha)$$

where $g(x)$ is some unknown functions of x, and α is a constant. The eigenvalue appears in the form $\sqrt{\lambda}$ because of the need to differentiate $\psi(x)$ twice, giving

$$g(x) = \int_a^x du \sqrt{\frac{w(u)}{p(u)}}$$

This provides the first estimate to the eigenfunctions and eigenvalues if the appropriate solution is made to fit the boundary conditions.

More on pg372-373 on amplitude correction.

11 Special Functions