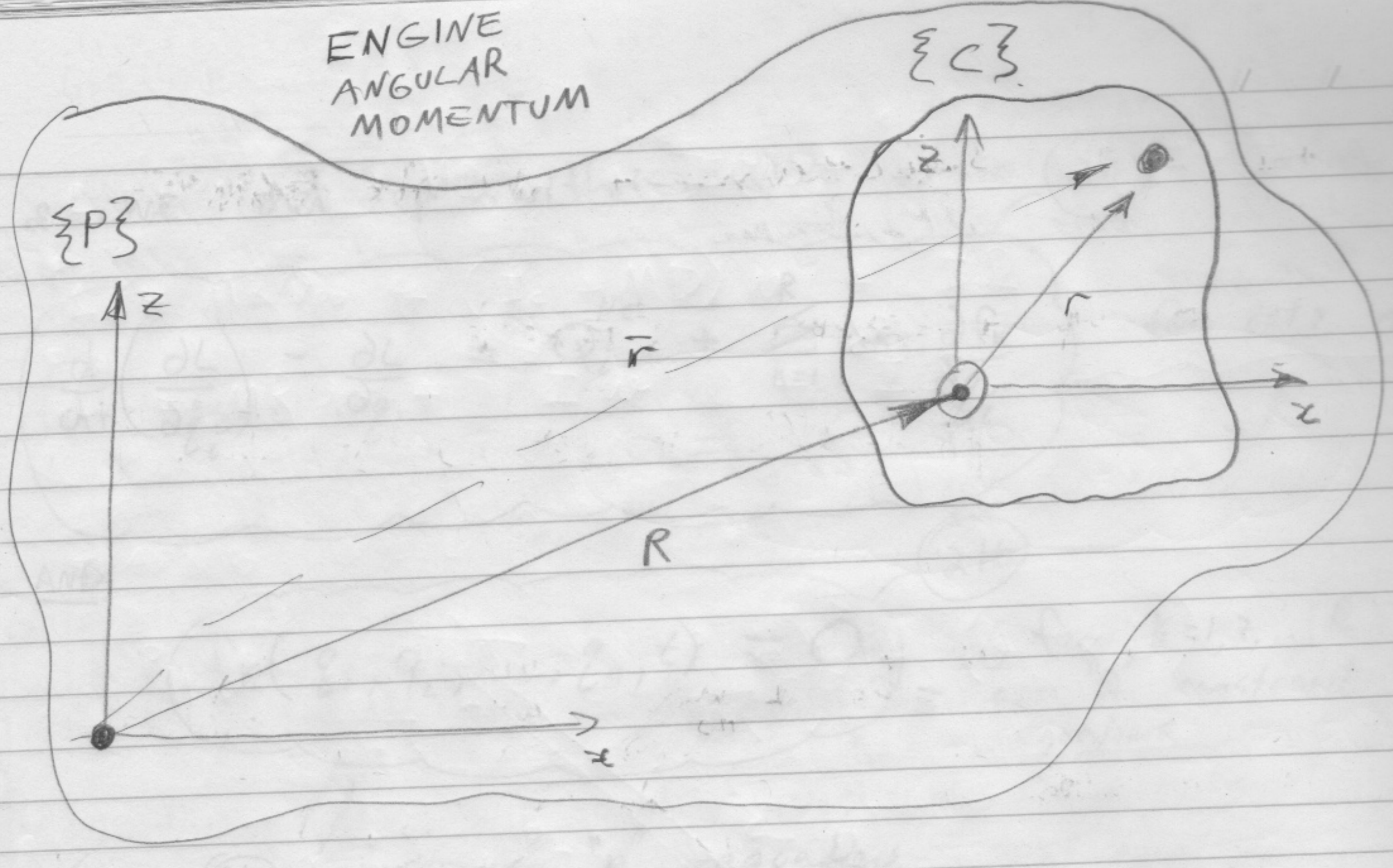
ENGINE ANGULAR MOMENTUM



Assumptions

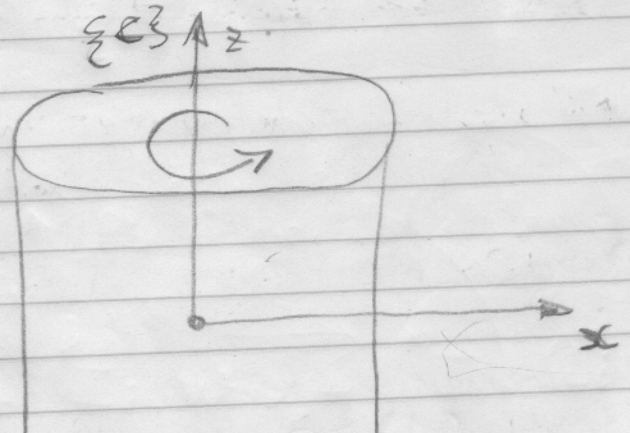
1.) {P} & {C} are body fixed to the aircraft

2.) {P} & {C} are Parallel

3.) ω is the angular velocity of the aircraft

4.) {P} is the com aircraft frame

5.) {C} is the com of a spinning engine



6.) Ω ≡ angular velocity of engine relative to fixed {C}

7.) {C} is at the com of the engine

8.) M = mass of engine

9.) ${}^{c}I$ = inertia of engine about {C}, ${}^{c}I$ is time invariant

GYROSCOPIC EFFECT

What is the ANGULAR MOMENTUM of the ENGINE relative to the {P}-frame.

$$ {}^{P}v_{dm} = v_{po} + (\omega \times \bar{r}) + (\Omega \times r) $$

$$ {}^{P}v_{dm} = v_{po} + (\omega \times R) + (\omega \times r) + (\Omega \times r) $$

So:-

$$ {}^{P}L = \int \bar{r} \times v \, dm $$
$$ = \int (R + r) \times (v_{po} + \omega \times R + \omega \times r + \Omega \times r) \, dm $$

$$ {}^{P}L = \left[\begin{array}{l} \overset{①}{\int R \times v_{po} \, dm} + \overset{②}{\int R \times (\omega \times R) dm} + \overset{③}{\int R \times (\omega \times r) dm} + \overset{④}{\int R \times (\Omega \times r) dm} \\ + \overset{⑤}{\int r \times v_{po} \, dm} + \overset{⑥}{\int r \times (\omega \times R) \, dm} + \overset{⑦}{\int r \times (\omega \times r) dm} + \overset{⑧}{\int r \times (\Omega \times r) dm} \end{array} \right. $$

So we need to evaluate these 8 terms i.e. t_1, t_2, etc

$$ t_1 = \int R \times v_{po} \, dm = R \times v_{po} \int dm = M . R \times v_{po} $$
$$ \cancel{= M R \times v_{po}} $$

$$ t_2 = \int R \times \omega \times R \, dm = R \times \omega \times R \int dm = R \times \omega \times R . M $$

$$ t_2 = \begin{bmatrix} (Y^2+Z^2) & -YX & -ZX \\ -YX & (X^2+Z^2) & -YZ \\ -XZ & -YZ & (X^2+Y^2) \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = M . [D] . \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} $$

$$ t_3 = \int R \times (\omega \times r) \, dm = R \times \omega \times \underbrace{\int r \, dm}_{= 0} = 0 $$

$$ t_4 = \int R \times (\Omega \times r) \, dm = R \times \int (\Omega \times r) dm = R \times (\Omega \times \overbrace{\int r \, dm}^{= 0}) = 0 $$

$$ t_5 = \int r \times v_{po} \, dm = - \int v_{po} \times r \, dm = - v_{po} \times \overbrace{\int r \, dm}^{= 0} = 0 $$

$$ t_6 = \int r \times (\omega \times R) \, dm = - \int (\omega \times R) \times r \, dm = -(\omega \times R) \times \overbrace{\int r \, dm}^{= 0} = 0 $$

$$ t_7 = \int r \times (\omega \times r) \, dm = {}^{c}I . \omega $$
$$ t_8 = \int r \times (\Omega \times r) \, dm = {}^{c}I . \Omega $$

$$\therefore \quad {}^{P}L = M R \times v_{po} + MD.\omega + {}^{c}I\omega + {}^{c}I\Omega$$

$${}^{P}L = M R \times v_{po} + \underbrace{(I_c + MD)}\omega + {}^{c}I\Omega$$

$$\boxed{{}^{P}L = M R \times v_{po} + {}^{P}I\omega + {}^{c}I\Omega}$$

$${}^{P}I = {}^{c}I + MD$$

As before we can apply the DERIVATIVE TRANSFORMATION formula:-

$$\boxed{{}^{P}M = \frac{{}^{G}d}{dt}\left({}^{P}L\right) = \frac{{}^{P}d}{dt}\left({}^{P}L\right) + \left(\omega \times {}^{P}L\right)}$$

$$\therefore \quad {}^{P}M = \left(M R \times \dot{v}_{po}\right) + \boxed{\left({}^{P}I\dot{\omega}\right)} + \left({}^{c}I.\dot{\Omega}\right)$$

$$+ M\left(\omega \times \left(R \times v_{po}\right)\right) + \boxed{\omega \times \left({}^{P}I\omega\right)} + \omega \times \left({}^{c}I\Omega\right)$$

$$\boxed{\begin{aligned} \therefore \quad {}^{P}M = {}^{P}I\dot{\omega} + \omega \times \left({}^{P}I\omega\right) &+ \underline{{}^{c}I.\dot{\Omega} + \omega \times \left({}^{c}I\Omega\right)} \\ &+ M\, R \times {}^{P}\dot{v}_{po} \\ &+ M\left(\omega \times \left(R \times v_{po}\right)\right) \end{aligned}}$$

Equation (1) is the angular momentum of the engine, where the momentum is taken about the COM of the vehicle ie: the body fited {P} frame. Let's look at it again:-

$${}^{P}L_i = M_i . R_i \times v_{po} + {}^{P}I_i . \omega + {}^{c}I_i . \Omega i$$

Now iff we say that the vehicle is made up of a collection of these components ... and sure NOT all of these components will be an engine, then we can write the TOTAL vehicle Angular Momentum as:-

$$ {}^{P}L_V = \sum_{i=1}^{N} {}^{P}L_i = \sum_{i=1}^{N} M_i . R_i \times v_{P_0} + {}^{P}I_i . \omega + {}^{C}I_i . \Omega_i $$

Now since $\{P\}$ is the vehicles COM, then we know that:- $\boxed{\sum_{i=1}^{N} M_i R_i = 0}$ $\therefore \left(\sum_{i=1}^{N} M_i R_i\right) \times v_{P_0} = 0$ since the cross product is distributive

$$ \therefore {}^{P}L_V = \sum_{i=1}^{N} {}^{P}I_i . \omega + {}^{C}I_i . \Omega $$

$$ \therefore \boxed{ {}^{P}L_V = {}^{P}I_V . \omega + \sum_{i=1}^{N} {}^{C}I_i . \Omega_i } $$

<u>where</u>:- ${}^{P}I_V \equiv$ the TOTAL INERTIA of the vehicle about the vehicle's COM

If we now apply the DERIVATIVE TRANSFORMATION formula to ②, we get

$$ {}^{P}M = \left({}^{P}I_V \overset{\bullet P}{\omega}\right) + \left(\sum_{i=1}^{N} {}^{C}I_i \overset{\bullet P}{\Omega}_i\right) + \omega \times \left[{}^{P}I_V . \omega + \sum_{i=1}^{N} {}^{C}I_i . \Omega_i \right] $$

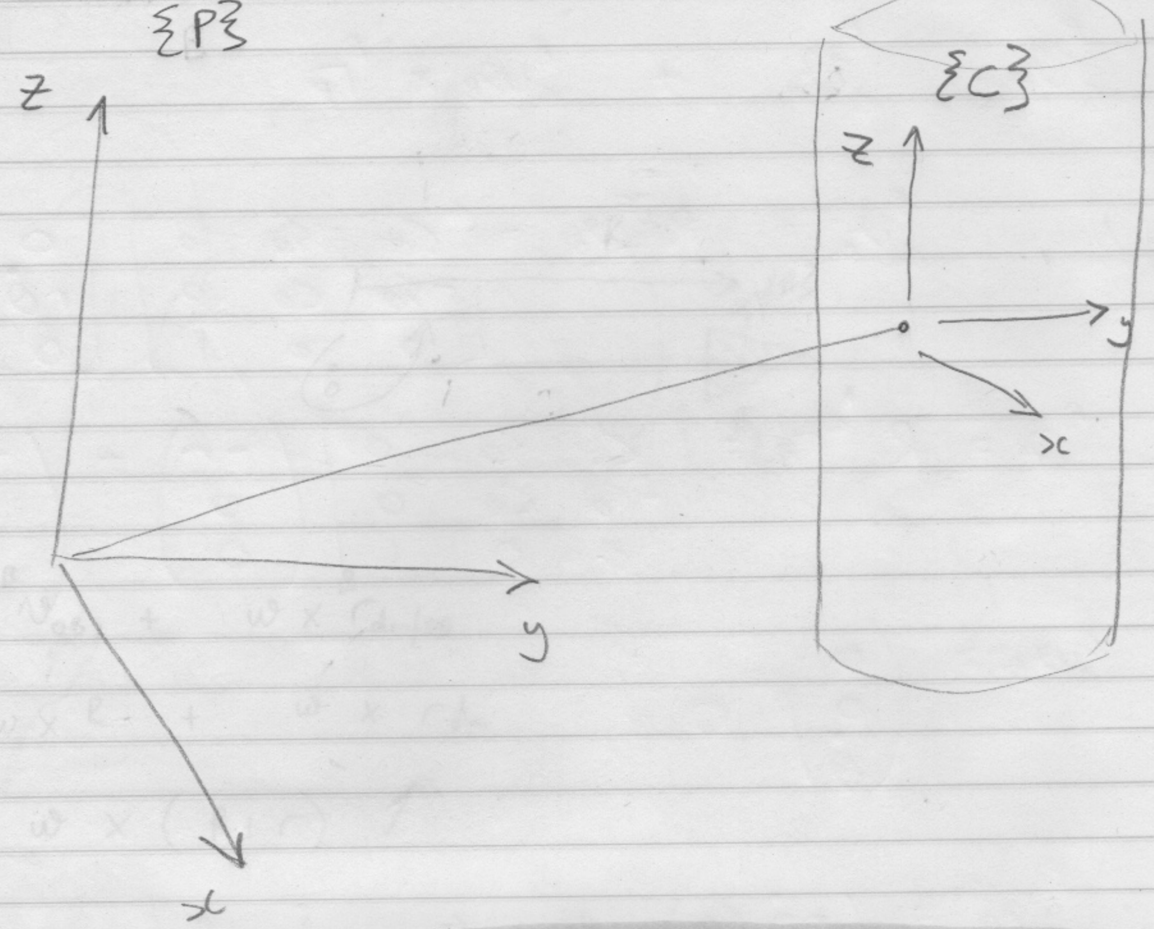
$$ {}^{P}M = \left({}^{P}I_V . \overset{\bullet P}{\omega}\right) + \left(\sum_{i=1}^{N} {}^{C}I_i . \overset{\bullet P}{\Omega}_i\right) + \left(\omega \times {}^{P}I_V . \omega\right) + \omega \times \left(\sum_{i=1}^{N} {}^{C}I_i . \Omega_i\right) $$

$$ \boxed{ {}^{P}M = \left({}^{P}I_V . \overset{\bullet P}{\omega}\right) + \left(\omega \times {}^{P}I_V . \omega\right) + \sum_{i=1}^{N} \left({}^{C}I_i . \overset{\bullet P}{\Omega}_i + \omega \times {}^{C}I_i . \Omega_i \right) } $$

<u>Note</u>:- we have assumed (#7) that ${}^{C}I$ is time invariant
<u>eg</u>:- a cylinder.

See back of this book for continued discussion
– the section is called "GYROSCOPIC EFFECT continued.

GYROSCOPIC EFFECT Cont'd



$$\boxed{{}^{P}M = {}^{P}I \cdot {}^{\bullet P}\omega + \omega \times ({}^{P}I\,\omega) + \sum_{i=1}^{N} {}^{c}I_i \, {}^{c}\dot{\Omega}_i^{P} + \omega \times ({}^{c}I_i \cdot {}^{c}\Omega_i)}$$

Let:- $\boxed{{}^{c}\Omega = \begin{pmatrix} {}^{c}\Omega_x \\ {}^{c}\Omega_y \\ {}^{c}\Omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ {}^{c}\Omega_z \end{pmatrix}}$ and:- $\boxed{{}^{c}\dot{\Omega}^{P} = \begin{pmatrix} 0 \\ 0 \\ \dot{\Omega}_z \end{pmatrix}}$

$${}^{c}I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$\therefore {}^{c}I_i \cdot \dot{\Omega}_i^{P} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\Omega}_{iz} \end{pmatrix} = \begin{bmatrix} I_{xz} \cdot \dot{\Omega}_{iz}^{P} \\ I_{yz} \cdot \dot{\Omega}_{iz}^{P} \\ I_{zz} \cdot \dot{\Omega}_{iz}^{P} \end{bmatrix}$$

IFF $\{C\}$ is at the COM of the engine AND if the engine has symmetry such that

$$^{C}I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\therefore \quad ^{C}I_i \cdot \overset{\circ P}{\Omega}_i = \begin{pmatrix} 0 \\ 0 \\ I_{zz} \overset{\circ P}{\Omega}_{zi} \end{pmatrix}$$

Similarly :-

$$^{C}I_i \cdot \Omega_i = \begin{pmatrix} 0 \\ 0 \\ I_{zz} \cdot \Omega_{iz} \end{pmatrix}$$

$$\therefore \quad \omega \times \left(^{C}I_i \cdot {}^{C}\Omega_i \right) = \begin{vmatrix} \omega_x & \omega_y & \omega_z \\ 0 & 0 & I_{zz} \cdot \Omega_{iz} \end{vmatrix} = \begin{pmatrix} \omega_y I_{zz} \Omega_{iz} \\ -\omega_x I_{zz} \Omega_{iz} \\ 0 \end{pmatrix}$$

$$\therefore \quad ^{P}M = {}^{P}I \overset{\circ P}{\omega} + \omega \times \left(^{P}I \omega \right) + \sum_{i=1}^{N} \begin{pmatrix} 0 \\ 0 \\ ^{C}I_{zz} \dot{\Omega}_{zi} \end{pmatrix} + \begin{pmatrix} \omega_y I_{zz} \Omega_{iz} \\ -\omega_x I_{zz} \Omega_{iz} \\ 0 \end{pmatrix}$$

$$\boxed{\therefore \quad ^{P}M - \sum_{i=1}^{N} \begin{pmatrix} \omega_y {}^{C}I_{zzi} \cdot \Omega_{iz} \\ -\omega_x {}^{C}I_{zzi} \cdot \Omega_{iz} \\ ^{C}I_{zzi} \overset{\circ P}{\Omega}_{iz} \end{pmatrix} = {}^{P}I \cdot \omega + \omega \times \left(^{P}I \cdot \omega \right)}$$